Solving Inverse Problems using Generative Prior

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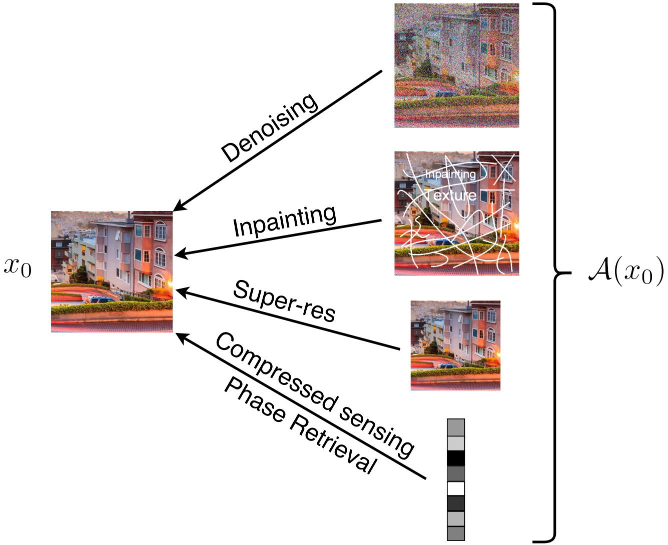
What is an Inverse Problem?



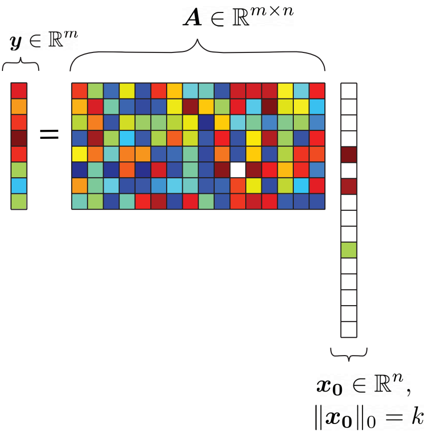
Natural Images are sparse wrt Wavelet Basis

Original image

Sparse approximation

Convex Relaxation promotes sparsity

Compressed Sensing:



$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{x}\|_0 \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

relaxation

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{x}\|_1 \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

Recovery Guarantee in Compressed Sensing

Fix $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $\|\boldsymbol{x}_0\|_0 = k$.
Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a random gaussian matrix with $m = \Omega(k \log n)$.

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{x}\|_1 \text{ s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}_0 \tag{1}$$

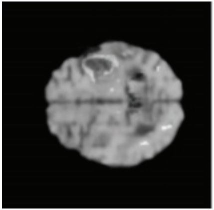
Theorem (Candes, Romberg, Tao [2004], Donoho [2004])

The minimizer of (1) *is* $\boldsymbol{x}_0$ *with high probability.*

Generative models impressively sample from complex signal classes

Faces

MRI



Bedrooms

Images from Karras et al. and Shin et al.

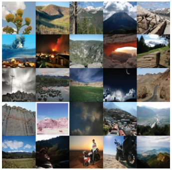
Main takeways

- Generative models provide SOTA performance in imaging inverse problems
- Generative models provide lower dimensional priors that can be directly and efficiently exploited
- Suggests new framework for experimental scientists in imaging

How are generative models used in inverse problems?

- Train generative models to output signal classes:

$$\mathcal{G} : \mathbb{R}^k \to$$



- Directly optimize over range of generative model via empirical risk:

$$\min_{\boldsymbol{z} \in \mathbb{R}^k} \left\| \mathcal{A}\left(\mathcal{G}(\boldsymbol{z})\right) - \mathcal{A}\left(\boldsymbol{x}^{\natural}\right) \right\|_2^2$$

Generative prior provides SOTA performance

- Denosing
- Inpainting
- Superresolution
- Compressed sensing
  - 5-10X less measurements than Lasso (Bora et al.)
  - 2 orders of magnitude speedup in MRI imaging (Mardani et al.)

Generative prior allow for direct optimization

Relaxation:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \; \|\boldsymbol{x}\|_0 \quad \text{s.t.} \;\; \mathcal{A}(\boldsymbol{x}) = \mathcal{A}(\boldsymbol{x}^\natural) \quad \longrightarrow \quad \min_{\boldsymbol{x} \in \mathbb{R}^n} \; \|\boldsymbol{x}\|_1 \quad \text{s.t.} \;\; \mathcal{A}(\boldsymbol{x}) = \mathcal{A}(\boldsymbol{x}^\natural)$$

Direct Optimization:

$$\min_{\boldsymbol{z} \in \mathbb{R}^k} \; \|\mathcal{A}(\mathcal{G}(\boldsymbol{z})) - \mathcal{A}(\mathcal{G}(\boldsymbol{z}^\natural))\|_2^2$$

Related work

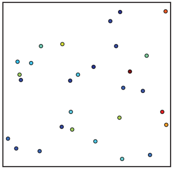
- Compressed Sensing
    - Local landscape (Bora et. al, 2017)
    - Global landscape for fully connected neural net (Hand et .al, 2017)
    - Global landscape for convolutional neural net (Ma et. al, 2018)
- Phase retrieval
    - Global landscape for fully connected neural net (Hand et. al, 2018)

What is a Bilinear Inverse Problem?

$$\mathcal{A} : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}^L$$

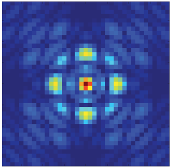
Blind Deconvolution:

$\boldsymbol{w}$



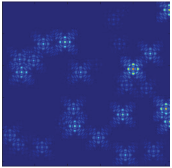
$*$

$\boldsymbol{x}$



$=$

$\boldsymbol{y}$



$= \mathcal{A}(\boldsymbol{w}, \boldsymbol{x})$

Find $(\boldsymbol{w}, \boldsymbol{x})$ given $\boldsymbol{y}$.

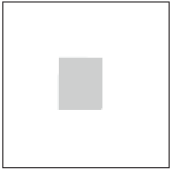
Image from paper "Structured Local Optima in Sparse Blind Deconvolution"

What is a Bilinear Inverse Problem?

$$\mathcal{A} : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}^L$$

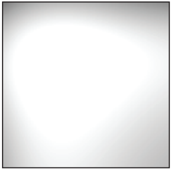
Blind Demodulation:

$\boldsymbol{w}$



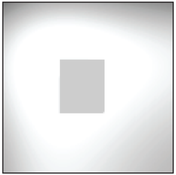
$\odot$

$\boldsymbol{x}$



$=$

$\boldsymbol{y}$



$= \mathcal{A}(\boldsymbol{w}, \boldsymbol{x})$

Find $(\boldsymbol{w}, \boldsymbol{x})$ given $\boldsymbol{y}$.

Ambiguities in Blind demodulation

Let $\boldsymbol{y},\ \boldsymbol{x}^\natural,\ \boldsymbol{w}^\natural \in \mathbb{R}^L$ such that

$$\boldsymbol{y} = \boldsymbol{w}^\natural \odot \boldsymbol{x}^\natural.$$

Structural ambiguity:

Both $(\boldsymbol{w}^\natural, \boldsymbol{x}^\natural)$ and $(\mathbf{1}, \boldsymbol{w}^\natural \odot \boldsymbol{x}^\natural)$ outputs the measurement $\boldsymbol{y}$.

Scaling ambiguity:

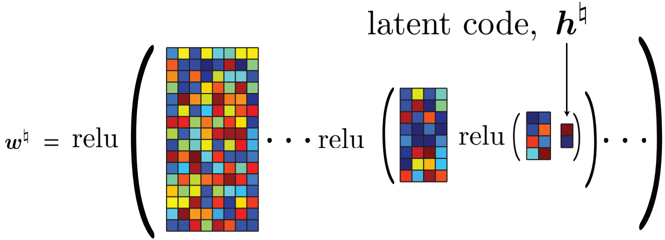
For any non-zero $c \in \mathbb{R}$, $\left(c\boldsymbol{w}^\natural, \frac{1}{c}\boldsymbol{x}^\natural\right)$ outputs the measurement $\boldsymbol{y}$.

Structural Ambiguity

- **Generative Prior:** Assume $\boldsymbol{w}^\natural$ and $\boldsymbol{x}^\natural$ are in the range of a generative model.

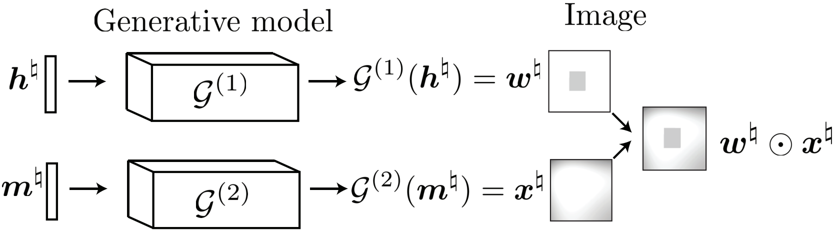
$$\boldsymbol{h}^\natural \mapsto \mathcal{G}^{(1)}(\boldsymbol{h}^\natural) = \boldsymbol{w}^\natural \qquad \boldsymbol{m}^\natural \mapsto \mathcal{G}^{(2)}(\boldsymbol{m}^\natural) = \boldsymbol{x}^\natural$$

$$\mathcal{G}^{(1)} : \mathbb{R}^n \to \mathbb{R}^L \qquad \mathcal{G}^{(2)} : \mathbb{R}^p \to \mathbb{R}^L$$



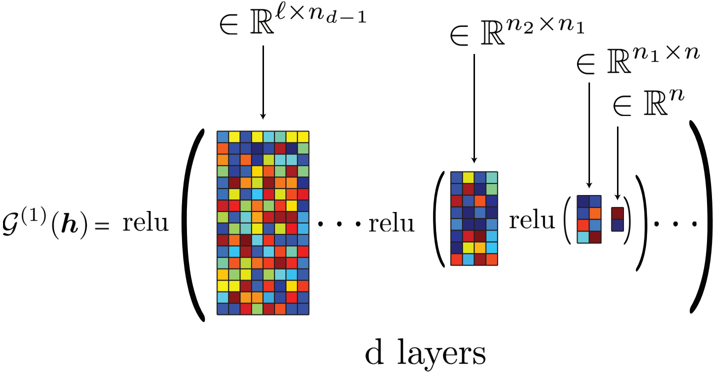
$$\mathrm{relu}(x) = \max(\boldsymbol{x}, \mathbf{0})$$

Our formulation: Deep Blind Demodulation



$$\min_{\boldsymbol{h}\in\mathbb{R}^n, \boldsymbol{m}\in\mathbb{R}^p} \left\| \mathcal{G}^{(1)}(\boldsymbol{h}) \odot \mathcal{G}^{(2)}(\boldsymbol{m}) - \mathcal{G}^{(1)}(\boldsymbol{h}^\natural) \odot \mathcal{G}^{(2)}(\boldsymbol{m}^\natural) \right\|_2^2$$

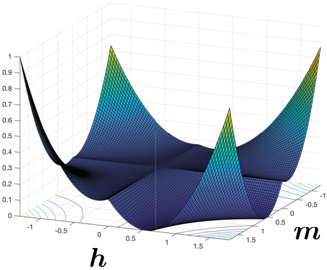
Setting of the Weights

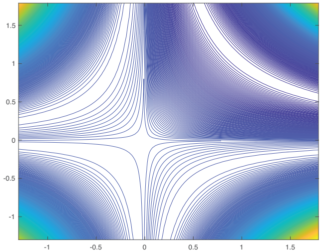


- Entries are random
- The network is sufficiently expansive:

$$n_i \gtrsim n_{i-1} \log n_{i-1}, \quad l \gtrsim n_{d-1}^2$$

Landscape of objective function





Recovery theorem

$$\min_{\boldsymbol{h}\in\mathbb{R}^n,\boldsymbol{m}\in\mathbb{R}^p} \left\| \mathcal{G}^{(1)}(\boldsymbol{h}) \odot \mathcal{G}^{(2)}(\boldsymbol{m}) - \mathcal{G}^{(1)}(\boldsymbol{h}^\natural) \odot \mathcal{G}^{(2)}(\boldsymbol{m}^\natural) \right\|_2^2 \tag{2}$$

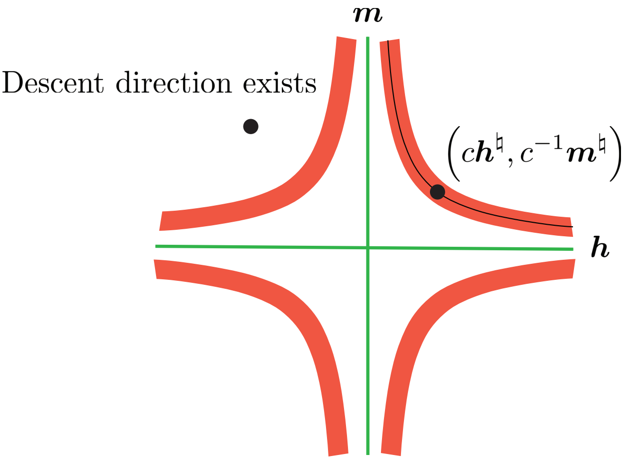
Assumptions:

- Network layers are sufficiently expansive
- Inner weights of G have i.i.d. Gaussian entries
- Outer weights of G have truncated i.i.d. Gaussian entries

Theorem (Hand, Joshi, [2019])

The objective function (2) has a strict descent direction outside of four hyperbolic neighborhoods of true signals and its sign permutations with high probability.

Characterization of stationary points

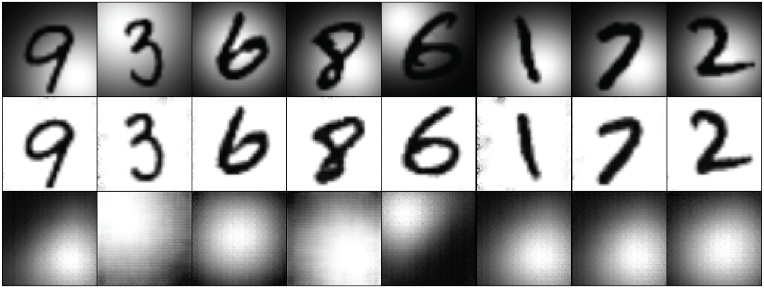


$$\mathcal{H} = \{(\boldsymbol{h}, \boldsymbol{0}) : \boldsymbol{h} \in \mathbb{R}^n\} \cup \{(\boldsymbol{0}, \boldsymbol{m}) : \boldsymbol{m} \in \mathbb{R}^p\}$$

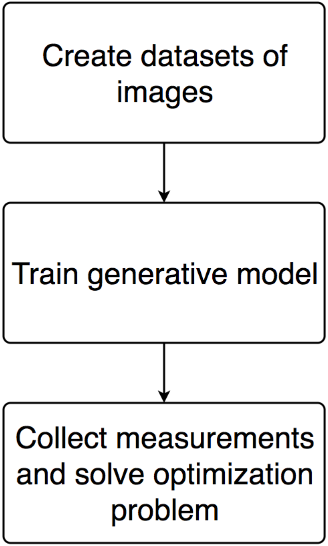
Outline of the Proof

- Let $\boldsymbol{g_h} = \nabla_{\boldsymbol{h}}$ objective$(\boldsymbol{h}, \boldsymbol{m})$ and $\boldsymbol{g_m} = \nabla_{\boldsymbol{m}}$ objective$(\boldsymbol{h}, \boldsymbol{m})$.
- Explicit formula for $\boldsymbol{v_h} = \mathbb{E}(\nabla_{\boldsymbol{h}}\text{objective}(\boldsymbol{h}, \boldsymbol{m}))$ and $\boldsymbol{v_m} = \mathbb{E}(\nabla_{\boldsymbol{m}}\text{objective}(\boldsymbol{h}, \boldsymbol{m}))$.
- Show $\boldsymbol{g_h} \approx \boldsymbol{v_h}$ and $\boldsymbol{g_m} \approx \boldsymbol{v_m}$ uniformly over all $(\boldsymbol{h}, \boldsymbol{m})$.
- Show $\boldsymbol{v_h} \neq \boldsymbol{0}$ and $\boldsymbol{v_m} \neq \boldsymbol{0}$ outside of the four hyperbolic neighborhoods and the set $\mathcal{H}$.

Simulation Results



New workflow for scientists



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