Monty Hall Meets Borat: A Bayesian Perspective

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1 Background on the Monty Hall Problem

The Monty Hall problem is a classic probability puzzle that gained popularity in the 1970s after being featured in the game show "Let's Make a Deal." It is named after the show's host, Monty Hall. The problem is deceptively simple but leads to a counter-intuitive result that often surprises people.

In this case study, we will explore the Monty Hall problem and demonstrate why switching your choice after Monty reveals a goat is the optimal strategy, using Bayes' theorem to analyze the probabilities involved.

2 Background on the Monty Hall Problem

Imagine you are a contestant on the game show "Let's Make a Deal", and you are presented with three doors: Door 1, Door 2, and Door 3. Behind one of these doors is a brand-new car, and behind the other two are goats.
The rules are as follows:

1. You choose one door (say, Door 1) without it being opened.
2. Monty, who knows what's behind each door, opens another door (say, Door 2) to reveal a goat.
3. You are given a choice: stick with your initial selection (Door 1) or switch to the remaining unopened door (Door 3).
4. The door you select after this choice will be opened, and whatever is behind it is your prize.

At first glance, one might think that since there are two doors left (one with a car and one with a goat), the probability of winning the car by sticking with your initial choice or switching would both be 50-50. However, as we will see, this is not the case. By working through Bayes Theorem, we can calculate the actual odds of winning the car if we stick with door 1, or switch to door 3.

3 Introduction to Bayes' Theorem

Before we delve into the probabilities of the Monty Hall problem, let's briefly discuss Bayes' theorem. Bayes' theorem is a fundamental concept in probability theory that allows us to calculate the probability of an event, given another event occurs.

In mathematical terms, Bayes' theorem is defined as follows:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \tag{1}$$

where:

- **posterior:** $P(A|B)$ is the probability of event A occurring given that event B has occurred.
- **likelihood:** $P(B|A)$ is the probability of event B occurring given that event A has occurred.
- **prior:** $P(A)$ is the probability of events A occurs, before we know if event B occurs.
- $P(B)$ is the probabilities of events B.

Let's apply Bayes' theorem to analyze the Monty Hall problem and understand why switching is the better strategy.

4 Applying Bayes' Theorem to the Monty Hall Problem

In order to gain a deeper understanding of the Monty Hall problem, we are going to examine a particular scenario. However, keep in mind that this is merely a specific case we're analyzing for its clarity and simplicity. Regardless of the initial choice and Monty's subsequent action, the underlying principles and probability calculations remain consistent across all scenarios.

Now, suppose that we pick Door 1, and Monty opens Door 2 to reveal a goat. Let's define the following events:

- A: The car is behind Door 1.
- B: Monty opens Door 2 with a goat.
- C: The car is behind Door 3.

Since Monty opens Door 2, we know that the car will be behind either Door 1 or Door 3. Therefore, to determine whether switching from Door 1 to Door 3 is a good decision. The two things we would like to know are:

- $\Pr(\mathrm{A} \mid \mathrm{B})$: The probability that the car is behind Door 1, given the fact that Monty opens Door 2.
- $\Pr(\mathrm{C} \mid \mathrm{B})$: The probability that the car is behind Door 3, given the fact that Monty opens Door 2.

Let's proceed with the calculations for each probability.

4.1 Probability of Winning the Car by Switching

Recall that we pick Door 1 and Monty opens Door 2 to reveal a goat. Therefore, the probability of winning the car by switching is the same as calculating the probability that the car is behind Door 3, given that Monty opens Door 2.

$$\Pr(\mathrm{C} \mid \mathrm{B}) = \frac{\Pr(\mathrm{B} \mid \mathrm{C}) \cdot \Pr(\mathrm{C})}{\Pr(\mathrm{B})} \tag{2}$$

Now, let's calculate each component of the above equation.

Prior: Pr(C)

Given that prizes are randomly arranged behind the three doors, the probability that the car is behind any door will be $\Pr(\mathrm{A}) = \Pr(\mathrm{B}) = \Pr(\mathrm{C}) = 1/3$.

Likelihood: Pr(B | C)

If we initially choose Door 1, and Monty knows that behind Door 3 is a car. Then Monty can only open Door 2 to reveal a goat. Thus, $\Pr(\mathrm{B} \mid \mathrm{C}) = 1$

Normalizing Constant: Pr(B)

Since Monty can open Door 2 when the car is behind either Door 1 or Door 3, the probability of Monty opening Door 2 $\Pr(\mathrm{B})$ can be calculated as follows.

$$\Pr(\mathrm{B}) = \Pr(\mathrm{B} \mid \mathrm{C}) \cdot \Pr(\mathrm{C}) + \Pr(\mathrm{B} \mid \mathrm{A}) \cdot \Pr(\mathrm{A}) \tag{3}$$

We've calculated that $\Pr(\mathrm{A}) = \Pr(\mathrm{C}) = 1/3$, and that $\Pr(\mathrm{B} \mid \mathrm{C}) = 1$. Now, let's calculate $\Pr(\mathrm{B} \mid \mathrm{A})$. If the

car is actually behind door A, then Monty can open Doors B or C. So the probability of opening either is 50%. Thus, $\Pr(\mathrm{B} \mid \mathrm{A}) = 1/2$.

Now we get:

$$\Pr(\mathrm{B}) = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2} \tag{4}$$

Posterior: Pr(C | B)

Now we are ready to calculate the probability of the car behind Door 3, given that Monty opens Door 2:

$$\Pr(C \mid B) = \frac{\Pr(B \mid C) \cdot \Pr(C)}{\Pr(B)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \tag{5}$$

Thus, the probability that the car is behind Door 3 is 2/3. In other words, since we chose Door 1 at the beginning, the probability of winning the car by switching to the unopened door (Door 3 in our case) will be 2/3.

4.2 Probability of Winning the Car by Not Switching

Similarly, to calculate the probability of winning the car by not switching, we need to know the probability that the car is behind Door 1:

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B)} \tag{6}$$

We've calculated from the previous section that:

- $\Pr(A) = \Pr(B) = \Pr(C) = 1/3$.
- $\Pr(B \mid A) = 1/2$.
- $\Pr(B) = 1/2$.

Therefore,

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3} \tag{7}$$

Thus, the probability that the car is behind Door 1 is 1/3. In other words, since we chose Door 1 at the beginning, the probability of winning the car by sticking with our initial choice (Door 1 in our case) will be 1/3. Knowing that the probability of winning the car by switching is 2/3, we prove that switching is the best strategy to win a car.

4.3 Probability of Winning the Goat by Switching

Now you may be more confident on how to utilize Bayes' theorem to win the game. But let's change the situation a bit. Assuming that you are Borat, who comes from a country where goats are amongst the most prized possessions a person can own. In Borat's country, cars have no value, mainly because there is a lack of roads. Therefore, as Borat, the only goal for you is to win the goat. Now, let's use Bayes' rule to determine what is the best strategy for winning a goat. Do you think that switching would still be the best decision?

Similarly to the previous setting, let's suppose that we still choose Door 1 as our initial decision, and Monty opens Door 2 to reveal a goat.

Since Monty will always open a door with a goat behind it, we know that there will be one goat and one car behind the two remaining doors. One is the door we initially picked (Door 1 in our case), and the other is the unopened door to which we have a chance to switch (Door 3). Then, to calculate the probability of winning the goat by switching, it is equivalent to calculating the probability that behind Door 3 is a goat, and behind Door 1 is the car.

As we've calculated in section 4.2,

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B)} = \frac{1}{3} \tag{8}$$

For Borat, the probability of winning the goat by switching is 1/3.

4.4 Probability of Winning the Goat by Not Switching

Similar to what we've illustrated in section 4.3, the probability of winning the goat by not switching would be the same as the probability that behind Door 1 is a goat and behind Door 3 is the car.

Thus, based on equation (5) we get:

$$\Pr(C \mid B) = \frac{\Pr(B \mid C) \cdot \Pr(C)}{\Pr(B)} = \frac{2}{3} \tag{9}$$

The probability of winning the goat by sticking with the initially chosen door (Door 1) is 2/3. Therefore, to win a goat, the best strategy for Borat would then be sticking with the initially chosen door!