Some Syntactical Observations on Linear Logic

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Abstract

The purpose of this note is to clarify some syntactical matters in linear logic. We present a detailed proof of the faithfulness of Girard's embedding of intuitionistic logic into *classical* linear logic (**CLL**) and characterize *intuitionistic* linear logic (**ILL**) as the logic obtained from **CLL** by imposing a restriction on the right-rule for linear implication while keeping the property of Cut elimination. Also it is shown that **CLL** is *not* conservative over **ILL**.

Keywords: Syntax, linear logic, intuitionistic logic, sequent calculus, cut elimination.

1. Introduction: standard logic

In a Gentzen-type sequent calculus logic is formalized by means of a set of rules for the manipulation of so-called *sequents*: two strings Γ, Δ of formulas separated by the symbol $\Rightarrow$ (so '$\Gamma \Rightarrow \Delta$' will be our typical example of a sequent). A distinction is made between two kinds of rules: those that are said to be *logical* and those that are denoted as *structural* rules. In Appendix A we give a version of sequent calculus for classical predicate logic **CL**. As is well known we obtain a sequent calculus for intuitionistic predicate logic (**IL**) by limiting all succedent sets to one-element sets. The resulting calculus is presented in Appendix B. A less standard version of intuitionistic sequent calculus is obtained by limiting succedent sets to one-element sets *only* for the rules $\rightarrow R$ and $\forall R$. We will denote the resulting system by $\mathbf{IL}^{>}$. It is presented in Appendix C.

One of the basic results of proof theory is that Cut can be eliminated from derivations in **CL** and **IL**. The usual proof of this fact proceeds by induction, on, e.g. the *weight* of an application of Cut in a derivation. One then goes through all possible cases to show that a given application of Cut may always be replaced by a derivation without Cut, or with applications of Cut of a lower weight.

The asymmetry caused by the restricted rules, though, gives rise to some difficulties when one tries to adapt this technique to the system $\mathbf{IL}^{>}$. Before explaining this in more detail, we list some of our conventions and terminology in dealing with sequents and derivations.

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DEFINITION 1.1
In a sequent $\Gamma \Rightarrow \Delta$ we take Γ and Δ to represent *multisets* of formulas: we hardly ever explicitly mention the use of exchange, but take the order of formulas in sequents in a way that suits the occasion.

Derivations are represented in the usual tree-form. In a (representation of a) derivation $\mathscr{D}$ we will use double bars to denote a succession of applications of weakening- and/or contraction-rules.

Given a derivation of some sequent $\Gamma \Rightarrow \Delta$ we say that a formula A is the *main formula* if A is main formula in the first application of a *logical* rule appearing above the conclusion $\Gamma \Rightarrow \Delta$. (An instance of) a formula A occurring in a derivation is said to be *primitive* if it has been introduced by means of an axiom.

The *length* $|\mathscr{D}|$ of a derivation $\mathscr{D}$ is defined as follows:

—If $\mathscr{D}$ is an axiom, then $|\mathscr{D}| = 0$;

—If $\mathscr{D}$ is obtained from $\mathscr{D}'$ by means of a rule, then $|\mathscr{D}| = |\mathscr{D}'| + 1$;

—If $\mathscr{D}$ is obtained from $\mathscr{D}_1$ and $\mathscr{D}_2$ by means of a rule, then $|\mathscr{D}| = \max(|\mathscr{D}_1|, |\mathscr{D}_2|) + 1$.

The *height* $h(\mathscr{D})$ of a derivation $\mathscr{D}$ is defined as follows:

—If $\mathscr{D}$ is an axiom, then $h(\mathscr{D}) = 0$;

—If $\mathscr{D}$ is obtained from $\mathscr{D}'$ through a structural rule, then $h(\mathscr{D}) = h(\mathscr{D}')$;

—If $\mathscr{D}$ is obtained from $\mathscr{D}_1$ and $\mathscr{D}_2$ by Cut, then $h(\mathscr{D}) = \max(h(\mathscr{D}_1), h(\mathscr{D}_2))$;

—If $\mathscr{D}$ is obtained from $\mathscr{D}'$ through a logical rule, then $h(\mathscr{D}) = h(\mathscr{D}') + 1$;

—If $\mathscr{D}$ is obtained from $\mathscr{D}_1$ and $\mathscr{D}_2$ through a logical rule, $h(\mathscr{D}) = \max(h(\mathscr{D}_1), h(\mathscr{D}_2)) + 1$.

A *highest instance of Cut* in a derivation $\mathscr{D}$ is an instance of Cut such that the sub-derivation of $\mathscr{D}$ ending with it does not contain any other instances of Cut.

Let an instance of Cut be given:

$$\dfrac{\begin{array}{c}\mathscr{D}_1\\ \Gamma \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\mathscr{D}_2\\ \Gamma', A \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathrm{Cut}$$

We call A the *cut-formula*. The *sub-derivations given by the instance of Cut* are the derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of the premisses. The *height of the instance of Cut* is the minimum of the heights of the sub-derivations given by it, i.e. $\min(h(\mathscr{D}_1), h(\mathscr{D}_2))$. □

Inspection shows that we get into trouble when we try to adapt the usual proof of Cut elimination to the case of $\mathbf{IL}^{>}$ precisely in those cases where the cut-formula A is main formula of the left premiss, whereas the first logical rule in the sub-derivation which has the right premiss of the instance of Cut as its conclusion is one of the restricted rules of $\mathbf{IL}^{>}$, and does *not* have A as main formula. We are then no longer able to perform the permutation of rule and Cut necessary to obtain instances of Cut in which one of the premisses is conclusion of a sub-derivation of lower height:

$$\dfrac{\dfrac{\dfrac{}{\Gamma_1 \Rightarrow A, \Delta_1}}{\Gamma \Rightarrow A, \Delta} \qquad \dfrac{\dfrac{\Gamma_1', A, C \Rightarrow D}{\Gamma_1', A \Rightarrow C \to D}}{\Gamma', A \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma_1 \Rightarrow A, \Delta_1}}{\Gamma \Rightarrow A, \Delta} \qquad \dfrac{\dfrac{\Gamma_1', A \Rightarrow A(a)}{\Gamma_1', A \Rightarrow \forall x A(x)}}{\Gamma', A \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Nevertheless it *is* true that use of Cut is superfluous in $\mathbf{IL}^>$-derivations. In fact a system equivalent to $\mathbf{IL}^>$, namely the *Beth-tableau system* (**B**), has already been studied quite extensively in the late sixties by M. C. Fitting, who in Fitting [1] proved **B** to be closed under Cut by showing the system **B** *without* Cut to be sound and complete for Kripke-semantics.

In what follows we will show the eliminability of Cut in $\mathbf{IL}^>$ in two slightly more direct ways, referring only to the given systems **IL** and $\mathbf{IL}^>$.

Cut elimination for $\mathbf{IL}^>$: first method

In this section we will sketch a method of establishing Cut elimination for $\mathbf{IL}^>$. The main point is, that problems arising because of the restricted rules can be overcome by the possibility of inversion of application of some of the rules in $\mathbf{IL}^>$-derivations. (Note that as usual the presence of contraction-rules causes problems, which, as in Gentzen's original proof can be overcome by actually showing the eliminability of a generalized (but derivable) Cut rule of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \Rightarrow A^m, \Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma', A^n \Rightarrow \Delta'\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

where A^m, A^n with $m, n \geq 1$ denote m occurrences, n occurrences of the formula A. The reader who wishes to do so will easily be able to fill in for her- or himself the details necessary for a full proof.)

LEMMA 1.2
Let $\mathcal{D}$ be a Cut-free derivation of $\Gamma \Rightarrow A \Box B, \Delta$ or $\Gamma, A \Box B \Rightarrow \Delta$ (with $\Box \in \{\wedge, \vee\}$) in **CL** or $\mathbf{IL}^>$. Then we can transform $\mathcal{D}$ into a Cut-free derivation $\mathcal{D}'$ that ends with an application of the relevant $\Box$-rule, or such an application followed by a contraction.

PROOF. A long induction on the *length* of Cut-free derivations in **CL**, $\mathbf{IL}^>$. To be more precise, one shows inductively the following:

- $\wedge(1)$ If there is a Cut-free derivation of $\Gamma \Rightarrow (A \wedge B)^n$, Δ (where $(A \wedge B)^n$ again stands for $n \geq 1$ occurrences of $A \wedge B$) then there are

Cut-free derivations of $\Gamma \Rightarrow A^n, \Delta$ and $\Gamma \Rightarrow B^n, \Delta$. So in particular a Cut-free derivation of $\Gamma \Rightarrow A \wedge B, \Delta$ can be transformed into a Cut-free derivation ending with

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

- $\wedge(2)$ If there is a Cut-free derivation of $\Gamma, (A \wedge B)^n \Rightarrow \Delta$, then there is a Cut-free derivation of $\Gamma, A^n, B^n \Rightarrow \Delta$.
- $\vee(1)$ If there is a Cut-free derivation of $\Gamma \Rightarrow (A \vee B)^n, \Delta$, then there is a Cut-free derivation of $\Gamma \Rightarrow A^n, B^n, \Delta$.
- $\vee(2)$ If there is a Cut-free derivation of $\Gamma, A \vee B \Rightarrow \Delta$, then there are Cut-free derivations of $\Gamma, A^n \Rightarrow \Delta$ and of $\Gamma, B^n \Rightarrow \Delta$. $\square$

(Note that in **CL** we also have that if there is a Cut-free derivation of $\Gamma \Rightarrow (A \rightarrow B)^n, \Delta$ then there is a Cut-free derivation of $\Gamma, A^n \Rightarrow B^n, \Delta$; if there is a Cut-free derivation of $\Gamma, (A \rightarrow B)^n \Rightarrow \Delta$, then there are Cut-free derivations of $\Gamma \Rightarrow A^n, \Delta$ and $\Gamma, B^n \Rightarrow \Delta$. Both are *not* true for $\mathbf{IL}^{>}$.)

Lemma 1.3
If there is a Cut-free derivation of $\Gamma, (\exists x A(x))^n \Rightarrow \Delta$ in $\mathbf{IL}^{>}$ or **CL**, then there is a Cut-free derivation of $\Gamma, A^n \Rightarrow \Delta$.

Proof. Another induction on the length of Cut-free derivations. $\square$

Lemma 1.4
Highest instances of Cut of height 0 are redundant (i.e. they can be removed).

Proof. Easy. $\square$

Lemma 1.5
Highest instances of Cut on primitive formulas are redundant.

Proof. By careful inspection of cases one shows that these instances can either be removed, or permuted upwards (i.e. replaced by instances of Cut of lower height). $\square$

Theorem 1.6
(Cut elimination for $\mathbf{IL}^{>}$) Any $\mathbf{IL}^{>}$-derivation of a sequent $\Sigma \Rightarrow \Pi$ can be transformed into a Cut-free derivation.

Proof. Let $\mathcal{D}$ be an $\mathbf{IL}^{>}$-derivation of $\Sigma \Rightarrow \Pi$. First apply (the proof of) lemmas 1.4 and 1.5 to obtain an $\mathbf{IL}^{>}$-derivation in which no highest instance of Cut is of height 0, and in which no highest instance of Cut has a primitive cut-formula. Now let

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma', A \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

be one of the remaining highest instances of Cut. Then $A \equiv A_1 \Box A_2$ or $A \equiv QxA(x)$ with $\Box \in \{\vee, \wedge, \rightarrow\}$, $Q \in \{\exists, \forall\}$. As in the usual proof we show that in all possible cases the instance of Cut can either be removed or replaced by instances of Cut on formulas of strict lower complexity or of strict lower height. First note that we may assume that A is not introduced by (left- or right-) weakening (for then we obtain $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ directly by structural rules from $\mathscr{D}_1$ or $\mathscr{D}_2$). Next let us sketch how to handle the 'problematic cases', where A is main formula in the left premiss, while in the right premiss we have as first logical rule one of the restricted rules.

For $A \equiv A_1 \rightarrow A_2$ or $A \equiv \forall x A(x)$ to be main formula, the derivation in the left premiss of the instance of Cut necessarily is e.g. as follows:

$$\begin{array}{c}\mathscr{D}'\\ \dfrac{\dfrac{\Gamma_1, A_1 \Rightarrow A_2}{\Gamma_1 \Rightarrow A_1 \rightarrow A_2}}{\Gamma \Rightarrow A_1 \rightarrow A_2, \Delta}\end{array} \qquad \begin{array}{c}\mathscr{D}'\\ \dfrac{\dfrac{\Gamma_1 \Rightarrow A(a)}{\Gamma_1 \Rightarrow \forall x A(x)}}{\Gamma \Rightarrow \forall x A(x), \Delta}\end{array}$$

Consequently we *can* perform the permutations of Cut and restricted rules, as all other formulas in the succedent are introduced by right-weakening.

For $A \equiv A_1 \wedge A_2$, $A \equiv A_1 \vee A_2$ or $A \equiv \exists x A(x)$ we can avoid the problematic situation by using (the proof of) lemmas 1.2 and 1.3: we can transform the derivation $\mathscr{D}_2$ into a Cut-free derivation in which A is *main* formula. As an example let us look at $A \equiv A_1 \wedge A_2$. We then have, e.g.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_1'\\ \Gamma_1 \Rightarrow A_1, \Delta_1\end{array} \quad \begin{array}{c}\mathscr{D}_2'\\ \Gamma_1 \Rightarrow A_2, \Delta_1\end{array}}{\Gamma_1 \Rightarrow A_1 \wedge A_2, \Delta_1}}{\Gamma \Rightarrow A_1 \wedge A_2, \Delta} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\varepsilon\\ \Gamma_1', A_1, A_2 \Rightarrow \Delta_1'\end{array}}{\Gamma_1', A_1 \wedge A_2, A_2 \Rightarrow \Delta_1'}}{\Gamma_1', A_1 \wedge A_2, A_1 \wedge A_2 \Rightarrow \Delta_1'}}{\Gamma_1', A_1 \wedge A_2 \Rightarrow \Delta_1'}}{\Gamma', A_1 \wedge A_2 \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

which can be transformed into

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_1'\\ \Gamma_1 \Rightarrow A_1, \Delta_1\end{array} \quad \begin{array}{c}\varepsilon\\ \Gamma_1', A_1, A_2 \Rightarrow \Delta_1'\end{array}}{\Gamma_1, \Gamma_1', A_2 \Rightarrow \Delta_1, \Delta_1'}\,\text{Cut} \quad \begin{array}{c}\mathscr{D}_2'\\ \Gamma_1 \Rightarrow A_2, \Delta_1\end{array}}{\Gamma_1, \Gamma_1, \Gamma_1' \Rightarrow \Delta_1, \Delta_1, \Delta_1'}\,\text{Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Thus we replaced the original instance of Cut by two instances of lower height (and on formulas of lower complexity). $A \equiv A_1 \vee A_2$ and $A \equiv \exists x A(x)$ are treated similarly. All the remaining cases are treated in the usual way.

Therefore a finite number of transformations results in a derivation of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ in which all instances of Cut are on primitive formulas and/or of height 0. Starting with the highest instances, we use (the proofs of) lemmas 1.4 and 1.5 to remove them all. This gives us a Cut-free $\mathbf{IL}^{>}$-derivation of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

We have shown that each highest instance of Cut in an $\mathbf{IL}^{>}$-derivation can be removed. Therefore *all* instances of Cut can be removed. $\square$

Cut elimination for $\mathbf{IL}^{>}$: second method

DEFINITION 1.7
We write $\bigvee \Delta$ for any formula representing the disjunction of *all* formulas in Δ. If Δ is empty we take $\bigvee \Delta \equiv \bot$. $\square$

From the following proposition it follows that the comma in succedent sets of $\mathbf{IL}^{>}$-derivable sequents is precisely the intuitionistic disjunction.

PROPOSITION 1.8
$\mathbf{IL}^{>} \vdash \Gamma \Rightarrow \Delta$ if and only if $\mathbf{IL} \vdash \Gamma \Rightarrow \bigvee \Delta$.

PROOF. ($\leftarrow$) Suppose $\mathbf{IL} \vdash \Gamma \Rightarrow \bigvee \Delta$. As $\bigvee \Delta \Rightarrow \Delta$ is (Cut-free) derivable in $\mathbf{IL}^{>}$, we obtain the desired derivation of $\Gamma \Rightarrow \Delta$ by an application of Cut.
($\rightarrow$) By induction on the length of derivations in $\mathbf{IL}^{>}$. $\square$

(Note that in Troelstra and van Dalen [4], Chapter 10, for the equivalent systems '*Kleene's calculus G*3' and '*Beth-tableau system*' the left-to-right part of proposition 1.8 is proved via a reduction to natural deduction for intuitionistic predicate logic.)

THEOREM 1.9
(Cut elimination for $\mathbf{IL}^{>}$, again) Any $\mathbf{IL}^{>}$-derivable sequent $\Gamma \Rightarrow \Delta$ is derivable without application of Cut.

PROOF. Suppose $\mathbf{IL}^{>} \vdash \Gamma \Rightarrow \Delta$. Then by proposition 1.8 and Cut elimination for $\mathbf{IL}$ we have a Cut-free $\mathbf{IL}$-derivation of $\Gamma \Rightarrow \bigvee \Delta$. One then shows by induction on Cut-free $\mathbf{IL}$-derivations that it is possible to transform this derivation into a Cut-free derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{IL}^{>}$.

The only cases that need some consideration are the axioms and applications of $\bigvee R$-rules. These are handled by right-weakening, which in $\mathbf{IL}^{>}$ acts as right-rule for 'disjunction written as a comma'. $\square$

2. From standard to linear logic

The distinction made in the sequent calculus formulation of standard logic between *logical* and so-called *structural* rules is a bit misleading, as especially the rules of weakening and contraction express important and non-trivial properties of the connectives $\wedge$, $\vee$ and $\rightarrow$, properties that on closer observation appear to be at the very heart of (standard) logic.

Let's take a look at the following minimal version of sequent calculus for classical propositional logic, say $\mathbf{CL}_\mu$:

Axioms:

$$A \Rightarrow A \qquad \Gamma, \bot \Rightarrow \Delta$$

Logical rules:

$$\to R\,\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \qquad \to L\,\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}$$

Structural rules:

$$wL\,\frac{\Gamma \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \qquad wR\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow B, \Delta} \qquad cL\,\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad cR\,\frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$eL\,\frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \qquad eR\,\frac{\Sigma \Rightarrow \Gamma, A, B, \Delta}{\Sigma \Rightarrow \Gamma, B, A, \Delta} \qquad \mathrm{Cut}\,\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Clearly this limited calculus enables us to obtain *all* of classical propositional logic (e.g. as given by the sequent calculus of Appendix A) by taking the connectives $\wedge$, $\vee$ as being *defined* in terms of $\to$ and $\bot$. Observe that the rules of *weakening* are crucial in showing that the appropriate rules for our defined disjunction and conjunction are derivable in this limited calculus. Also note the following:

PROPOSITION 2.1
$\mathbf{CL}_\mu$ enjoys Cut elimination.

PROOF. Straightforward. □

In our formulation of the calculus we have given the rule $\to L$ in what is called a *multiplicative* form. Another option would have been to use the so-called *additive* form:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$

One easily shows that in the presence of the structural rules of weakening and contraction the additive form is equivalent to the multiplicative form, in the sense that given one of both, the other becomes derivable. And in fact there is a converse to this observation: by adding rules for $\to$ in additive form to our calculus, we may delete the rules for weakening and contraction while still being able to obtain all of classical propositional logic, provided we keep the rule for right-weakening in the special case of our constant $\bot$. But for this there is a price to be paid: our calculus will no longer enjoy Cut elimination.

Let us denote the modified calculus by $\mathbf{CL}^*_\mu$. It is given by the following set of axioms and rules:

Axioms:

$$A \Rightarrow A \qquad \Gamma, \bot \Rightarrow \Delta$$

Logical rules:

$$\bot R \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \bot, \Delta}$$

$$\rightarrow R_m \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \qquad \rightarrow R_{a_1} \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \qquad \rightarrow R_{a_2} \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

$$\rightarrow L_m \frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2} \qquad \rightarrow L_a \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

Structural rules:

$$eL \frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \qquad eR \frac{\Sigma \Rightarrow \Gamma, A, B, \Delta}{\Sigma \Rightarrow \Gamma, B, A, \Delta} \qquad \text{Cut} \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Now we observe:

PROPOSITION 2.2
$\mathbf{CL}^*_\mu$ is equivalent to $\mathbf{CL}_\mu$, but does not enjoy Cut elimination.

PROOF. We leave it as an exercise to show that weakening and contraction are derivable rules in $\mathbf{CL}^*_\mu$, but obviously a sequent like, e.g. $A, B \Rightarrow A$ is not derivable without use of Cut. □

Some reflection will make it clear that it is precisely the derivability of weakening- and contraction-rules that stands in the way of a possible elimination of Cut in $\mathbf{CL}^*_\mu$-derivations. Now taking a closer look at those derivations of weakening and contraction, we observe that they seem to depend on two features:

- the identification of '$\rightarrow$' in the use of multiplicative rules, with '$\rightarrow$' appearing in the additive rules;
- the joined possibility of '*ex falso*' for $\bot$ as given by the ($\bot$)-axiom, and rule $\bot R$.

Therefore, in order to *regain* eliminability of Cut, it seems good strategy to consider additive '$\rightarrow$' as being different from multiplicative '$\rightarrow$', and distinguish a multiplicative '$\bot$' (which can be used for right-weakening) from the additive '$\bot$' (giving us '*ex falso*'). So let us introduce a splitting of notions, as follows:

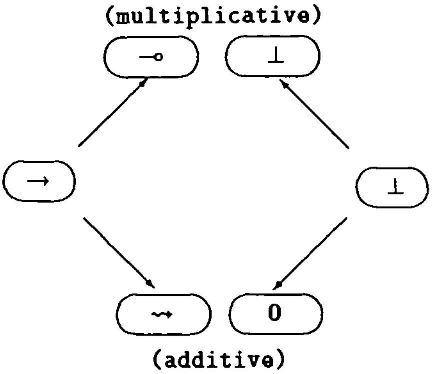

FIG. 1.

As we will see, the calculus obtained in this way enjoys Cut elimination, but of course again there is a price to pay: we have left the realm of standard classical logic, as clearly the logic obtained (we will denote it by $\mathbf{LL}_\mu$) can no longer be equivalent to $\mathbf{CL}_\mu$. It is given by the following set of axioms and rules:

Axioms:

$$A \Rightarrow A \qquad \Gamma, \mathbf{0} \Rightarrow \Delta \qquad \perp \Rightarrow$$

Logical rules:

$$\perp R\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \perp, \Delta}$$

$$\multimap R\,\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta} \qquad \multimap L\,\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Delta_1, \Delta_2}$$

$$\rightsquigarrow R_1\,\frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightsquigarrow B, \Delta} \qquad \rightsquigarrow R_2\,\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \rightsquigarrow B, \Delta} \qquad \rightsquigarrow L\,\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightsquigarrow B \Rightarrow \Delta}$$

Structural rules:

$$eL\,\frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \qquad eR\,\frac{\Sigma \Rightarrow \Gamma, A, B, \Delta}{\Sigma \Rightarrow \Gamma, B, A, \Delta} \qquad \text{Cut}\,\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

What we *did* obtain is a logic equivalent to Girard's so-called *classical linear* (*propositional*) *logic* [2], which we denote by **LL** and a sequent calculus formulation of which is given (by the propositional part of the calculus presented) in Appendix D. As a matter of fact, our formulation is 'minimal' in the same sense in which $\mathbf{CL}_\mu$ provided a minimal formulation for classical propositional logic: the additive connectives $\oplus$, & and their multiplicative companions $\wp$, $\otimes$ are definable from $\rightsquigarrow$, $\mathbf{0}$ and $\multimap$, $\perp$ in precisely the way we define $\vee$, $\wedge$ from $\rightarrow$, $\perp$ in standard logic. All this and more is contained in the following

THEOREM 2.3
$\mathbf{LL}_\mu$ enjoys Cut elimination and is equivalent to classical linear propositional logic **LL**.

PROOF. Cut elimination can be proved in the usual way, straightforwardly. For the equivalence of $\mathbf{LL}_\mu$ with **LL**, let us give the definitions of the various connectives and constants in Girard's logic in terms of our two arrows $\multimap$, $\rightsquigarrow$ and two constants $\bot$, $\mathbf{0}$:

- $[\wp]\ A \wp B := (A \multimap \bot) \multimap B$;
- $[\oplus]\ A \oplus B := (A \rightsquigarrow \mathbf{0}) \rightsquigarrow B$;
- $[\otimes]\ A \otimes B := (A \multimap (B \multimap \bot)) \multimap \bot$;
- $[\&]\ A \,\&\, B := (A \rightsquigarrow (B \rightsquigarrow \mathbf{0})) \rightsquigarrow \mathbf{0}$;
- $[\mathbf{1}]\ \mathbf{1} := \bot \multimap \bot$;
- $[\top]\ \top := \mathbf{0} \rightsquigarrow \mathbf{0}$.

We leave it as an exercise to show that the rules for these connectives as given in the Appendix are derivable in $\mathbf{LL}_\mu$ for the defined connectives.

Conversely, observe that the arrow $\rightsquigarrow$ is definable in **LL** by putting $A \rightsquigarrow B := (A \multimap \bot) \oplus B$. We leave the details of verification again as an exercise. □

Clearly the additive connective $\rightsquigarrow$ can be seen as an implication only in a formal sense: it obviously lacks some of the very basic properties we would like logical arrows to have. E.g. we can *not* derive $A \rightsquigarrow A$. (It might amuse the reader to show that adding $A \rightsquigarrow A$ as an axiom is equivalent to adding an *additive* form of the Cut rule to the calculus.) Still, in this formal sense we *can* consider linear logic as being 'a logic of two arrows'. That with the arrows we get but one 'classical' (i.e. involutive) negation is the content of the following

PROPOSITION 2.4
Both $A \multimap \bot$ and $A \rightsquigarrow \mathbf{0}$ behave as a negation, and we can derive in $\mathbf{LL}_\mu$:

- $(A \multimap \bot) \multimap \bot \Leftarrow\Rightarrow A$;
- $(A \rightsquigarrow \mathbf{0}) \rightsquigarrow \mathbf{0} \Leftarrow\Rightarrow A$.

But also the following are derivable:

- $A \multimap \bot \Leftarrow\Rightarrow A \rightsquigarrow \mathbf{0}$.

PROOF. Exercise. □

3. Linear logic

Girard [2] showed how to obtain a powerful logic with interesting properties by adding to **LL** weakening and contraction 'controlled' by modalities, the so-called exponentials ! ('*of course*') and ? ('*why not*'). This logic, extended with the usual rules for first-order quantifiers, is known as 'classical linear logic' (**CLL**), and enjoys Cut elimination (see Roorda [3]). A sequent calculus for **CLL** is given in Appendix D. It is important to note that the rules for the exponentials are taken to be *logical* rules. In linear logic the only remaining *structural* rules are exchange and Cut.

Embedding IL into CLL

In Girard [2] a translation $(\cdot)^\star$ of **IL** into **CLL** is defined as follows:

$$\text{for atomic } A \text{ put } A^\star := A; \text{ then put}$$

$$\begin{aligned}
\bot^\star &:= \mathbf{0} \\
(A \wedge B)^\star &:= A^\star \mathbin{\&} B^\star \\
(A \vee B)^\star &:= !A^\star \oplus !B^\star \\
(A \to B)^\star &:= !A^\star \multimap B^\star \\
(\forall x A)^\star &:= \forall x A^\star \\
(\exists x A)^\star &:= \exists x !A^\star
\end{aligned}$$

The embedding thus defined is claimed to be both *correct* and *faithful,* which is the content of the following

THEOREM 3.1
$\mathbf{IL} \vdash \Gamma \Rightarrow A$ if and only if $\mathbf{CLL} \vdash !\Gamma^\star \Rightarrow A^\star$.
(Here $!\Gamma^\star$ denotes the multiset $\{!B^\star \mid B \in \Gamma\}$.)

A straightforward induction on the length of (Cut-free) derivations of $\Gamma \Rightarrow A$ in the version of sequent calculus for **IL** given in Appendix B suffices to proof *correctness*. The proof of *faithfulness*, on the other hand, seems to be a bit more involved. In Girard [2] it is justified, first by the remark that, due to Cut elimination, we may assume a derivation of $!\Gamma^\star \Rightarrow A^\star$ to be obtained within the fragment $\mathcal{F}$ of **CLL** containing solely rules for $\mathbf{0}$, $\multimap$, $\oplus$, $\&$, $!$, $\forall$ and $\exists$. (See Appendix G.) Secondly, Girard says, 'if we erase all symbols $!$, and replace $\oplus$, $\&$, $\multimap$ by $\vee$, $\wedge$, $\to$, then we get a proof of A in intuitionistic logic.'

This, however, is not obvious at all. The reader may convince her/himself of the fact that in a derivation of $!\Gamma^\star \Rightarrow A^\star$ the combined use of $\mathbf{0}$-axioms and $\multimap L$-rules allows the occurrence of sequents with more than one succedent. Using the above recipe for proof transformation, the result is *not* a derivation of $\Gamma \Rightarrow A$ in **IL** and it is not clear whether the resulting proof will be intuitionistically valid.

Nevertheless Girard's claim of faithfulness holds, as in what follows we will show that we may assume a derivation of $!\Gamma^\star \Rightarrow A^\star$ to be of such a form that application of the above recipe for proof transformation necessarily results in a derivation of $\Gamma \Rightarrow A$ within $\mathbf{IL}^{>}$, and therefore is intuitionistically correct.

For this it will be helpful to have a lemma on the invertibility of rules in linear logic. In fact, the reader will readily write down an exhaustive list of all **CLL**-rules that are invertible (in the sense that the conclusion is derivable if and only if the premiss(es) is(are)). For our purpose it is sufficient to have invertibility of the rules $\forall$, $\multimap$, $\& R$.

The following defines a measure on $\mathcal{F}$-derivations to which all rules *except* $\forall$, $\multimap$, $\& R$ contribute.

DEFINITION 3.2
The measure $r(\mathcal{D})$ on derivations $\mathcal{D}$ in $\mathcal{F}$ is given by:

—If $\mathcal{D}$ is an instance of an axiom, then $r(\mathcal{D}) = 0$.

—If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by means of one of the rules $\oplus R_i$, $\& L_i$, $!L_i$, $!R$, $!c$, $\forall L$, $\exists R$, $\exists L$, then $r(\mathcal{D}') = r(\mathcal{D}) + 1$.

—If $\mathcal{D}$ is obtained from $\mathcal{D}_1$, $\mathcal{D}_2$ by means of one of the rules $\oplus L$, $\multimap L$ then $r(\mathcal{D}') = \max(r(\mathcal{D}_1), r(\mathcal{D}_2)) + 1$.

—If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by means of one of the rules $\multimap R$, $\forall R$, then $r(\mathcal{D}') = r(\mathcal{D})$.

—If $\mathcal{D}'$ is obtained from $\mathcal{D}_1$, $\mathcal{D}_2$ by means of the rule $\& R$, then $r(\mathcal{D}') = \max(r(\mathcal{D}_1), r(\mathcal{D}_2))$. $\square$

Now let $\vdash_n$ denote '*derivable from atomic instances of axioms* $P \Rightarrow P$ *with* $r(\mathcal{D}) \leq n$'. Then the following is easily checked by induction on the length of such $\mathcal{F}$-derivations:

LEMMA 3.3
(a) $\mathcal{F} \vdash_n \Gamma, A \multimap B \Rightarrow \Delta$ iff $\mathcal{F} \vdash_n \Gamma, A \Rightarrow B, \Delta$.
(b) $\mathcal{F} \vdash_n \Gamma \Rightarrow A \& B, \Delta$ iff $\mathcal{F} \vdash_n \Gamma \Rightarrow A, \Delta$ and $\mathcal{F} \vdash_n \Gamma \Rightarrow B, \Delta$.
(c) $\mathcal{F} \vdash_n \Gamma \Rightarrow \forall x A, \Delta$ iff $\mathcal{F} \vdash_n \Gamma \Rightarrow A, \Delta$. $\square$

Lemma 3.3 tells us that we may assume that a derivation $\mathcal{D}$ of a sequent $\Gamma \Rightarrow \Delta$ in $\mathcal{F}$ ends with a (possibly empty) series of applications of $\multimap$, $\&$, $\forall R$ starting from a collection of derivations $\mathcal{D}_i$ of sequents $\Gamma_i \Rightarrow \Delta_i$, where each formula in Δ_i has been introduced by an axiom or is of one of the forms $A \oplus B$, $\exists x A$ or $!A$. Moreover $r(\mathcal{D}_i) = r(\mathcal{D})$:

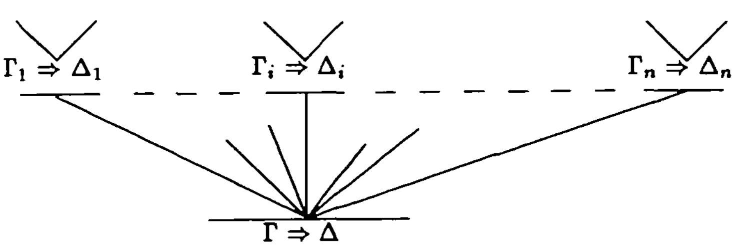

FIG. 2.

DEFINITION 3.4
In a derivation within $\mathcal{F}$ of a sequent $!\Gamma^\star, \Pi^\star \Rightarrow !\Lambda^\star, \Delta^\star$ we will call (an occurrence of) a formula $C^\star$ *f-primitive* if either it is primitive (i.e. has been introduced by an axiom) or has one of the forms $!A^\star \oplus !B^\star$ or $\exists x!A^\star$. □

We then have the following

LEMMA 3.5
Suppose in $\mathcal{F}$ a derivation is given of either

(a) $!\Gamma^\star, \Pi^\star \Rightarrow !\Lambda^\star$ or
(b) $!\Gamma^\star, \Pi^\star \Rightarrow !\Lambda^\star, B^\star$, where $B^\star$ is f-primitive.

Then we may assume the derivation to be such that all sequents having more than one succedent have one of the forms (i) or (ii):

(i) $!\Sigma^\star, \Delta^\star \Rightarrow !\Theta^\star, A^\star$, with $|\Theta| \geq 1$ and $A^\star$ f-primitive;
(ii) $!\Sigma^\star, \Delta^\star \Rightarrow !\Theta^\star$, with $|\Theta| \geq 2$.

PROOF. By induction on $r(\mathcal{D})$ of derivations $\mathcal{D}$ of (a), (b) in $\mathcal{F}$:

A sequent of the form (a) can be derived by means of a right-rule in $\mathcal{F}$ only if that rule is $!R$ and moreover $\Pi = \emptyset$, $|\Lambda| = 1$:

$$\frac{!\Gamma^\star \Rightarrow L^\star}{!\Gamma^\star \Rightarrow !L^\star}$$

Because of (the remarks following) lemma 3.3 we may assume that $!\Gamma^\star \Rightarrow L^\star$ is obtained solely through applications of $\multimap R$, $\& R$, $\forall R$ starting from derivations $\mathcal{D}_i$ of sequents $!\Gamma_i^\star \Rightarrow L_i^\star$, with $L_i^\star$ f-primitive. To these derivations we may apply the induction hypothesis for (b).

A sequent of the form (b) can be derived by means of a right-rule in $\mathcal{F}$ only if that rule is either $\oplus R_1$, $\oplus R_2$ or $\exists R$. In all these cases we can apply the induction hypothesis for (a) to the premiss of the rule.

Also if (a) or (b) has been obtained through application of a left-rule in $\mathcal{F}$ (including $!c$) the result follows directly by induction hypothesis.

Finally, notice that in case (a) or (b) is an axiom there is nothing to prove. □

PROPOSITION 3.6
Suppose the sequent $!\Gamma^\star \Rightarrow A^\star$ is derivable in $\mathcal{F}$. Then we may assume the derivation to be such that all applications of $\multimap R$, $\forall R$ only use sequents with precisely one succedent.

PROOF. Because of (the remarks following) lemma 3.3 we may assume that we have obtained $!\Gamma^\star \Rightarrow A^\star$ through a series of applications of $\multimap R$, $\& R$, $\forall R$ starting from a collection of sequents $!\Gamma_i^\star \Rightarrow A_i^\star$ with $A_i^\star$ f-primitive.

Lemma 3.5 then tells us that also we may assume the derivations of the sequents $!\Gamma_i^\star \Rightarrow A_i^\star$ to be such that all occurrences of sequents with more than one succedent have either the form (i) or (ii). Would there be, in any one of these derivations, an application of $\multimap R$ or $\forall R$ in which a sequent having more than one succendent occurs, then we would have a sequent of the form (i) or (ii) as a conclusion in an application of $\multimap R$ or $\forall R$. Obviously this is not possible. □

COROLLARY 3.7
Girard's embedding $\mathbf{IL} \hookrightarrow \mathbf{CLL}$ is faithful.

PROOF. Given the derivability of the sequent $!\Gamma^* \Rightarrow A^*$ in **CLL**, we know by Cut elimination that there is a derivation within $\mathcal{F}$. The previous proposition tells us that we may assume that applications of $\multimap R$, $\forall R$ only use sequents with precisely one succedent. Then, by erasing all !, and replacing occurrences of $\oplus$, &, $\multimap$ by $\wedge$, $\vee$, $\rightarrow$, we obtain a derivation of the sequent $\Gamma \Rightarrow A$ within $\mathbf{IL}^>$ (with left rule for $\rightarrow$ in multiplicative form). □

Intuitionistic linear logic

Intuitionistic linear logic **ILL** is defined in analogy to intuitionistic logic in the standard case as the logic obtained by restricting all succedent sets to one-element sets. As this means that we lose the rules for *par* ($\wp$) and the exponential ?, this connective and exponential are dropped altogether, as are both the axiom and rule for the 'neutral constant' corresponding to *par*, $\perp$. Thus we arrive at the calculus given in Appendix E.

If **CLL** were conservative over **ILL** we would obtain the faithfulness of Girard's embedding as a simple corollary to this conservativity. Therefore it is important to note that conservativity does *not* hold. In fact we have the following

PROPOSITION 3.8
Fragments of **CLL** in the language of **ILL** are conservative over **ILL** if and only if they do not include the constant **0**, or do not include linear implication $\multimap$.

PROOF. ($\Leftarrow$) Suppose the fragment does not include the constant **0**. Let $\mathcal{D}$ be a cut-free derivation of $\Gamma \Rightarrow A$. If there is in $\mathcal{D}$ a sequent with multiple succedents there is in $\mathcal{D}$ an instance of $\multimap L$ in which the right premiss has an empty succedent-set. We can then follow a branch upwards in the deduction tree consisting solely of sequents with empty succedent set. Such a branch has to end in an instance of an axiom, but that is impossible in a fragment without **0**.

Suppose the fragment does not include linear implication $\multimap$, and again let $\mathcal{D}$ be a cut free derivation of $\Gamma \Rightarrow A$. It is now straightforward by induction on the length of cut-free derivations that *all* sequents in $\mathcal{D}$ have precisely one succedent. Therefore, in both cases, $\mathcal{D}$ is in fact an **ILL**-derivation.

($\Rightarrow$) The following is a derivation in $\{\mathbf{0}, \multimap\}$:

$$\dfrac{\dfrac{\dfrac{C\Rightarrow C \quad \dfrac{\dfrac{\mathbf{0}\Rightarrow X, B}{\Rightarrow \mathbf{0}\multimap X, B} \quad A\Rightarrow A}{(\mathbf{0}\multimap X)\multimap A\Rightarrow B, A}}{C, C\multimap((\mathbf{0}\multimap X)\multimap A)\Rightarrow B, A}}{C\multimap((\mathbf{0}\multimap X)\multimap A)\Rightarrow C\multimap B, A} \quad \mathbf{0}\Rightarrow}{C\multimap((\mathbf{0}\multimap X)\multimap A), (C\multimap B)\multimap\mathbf{0}\Rightarrow A}$$

One easily checks that the final sequent is not cut-free derivable in **ILL**. Therefore it is not derivable in **ILL**. □

We will now go on to show that, as in the non-linear case, one gets a calculus equivalent to **ILL** by restricting the occurrence of one-element succedent sets to only *some* of the rules. In fact it turns out to be sufficient to impose this restriction on $\multimap R$. However, a consequence is that also the axiom ($\top$) has to be limited; this is because in **ILL** we can derive $\mathbf{0}\multimap A\Rightarrow \top$ as well as $\top\Rightarrow\mathbf{0}\multimap A$, for any A. So axiom ($\top$) in a way represents an instance of $\multimap R$.

We denote the resulting calculus by $\mathbf{ILL}^{>}$. It is given by the set of axioms and rules listed as Appendix F.

Remarks

1. Contrary to the non-linear case we do *not* need a restriction on $\forall R$.
2. When we insist on using the *full* axiom ($\top$), the resulting calculus can not enjoy Cut elimination; for then e.g. $A\Rightarrow\mathbf{0}\multimap A, A$ is derivable, as follows:

$$\dfrac{A\Rightarrow \top, A \qquad \dfrac{\top, \mathbf{0}\Rightarrow A}{\top\Rightarrow\mathbf{0}\multimap A}}{A\Rightarrow\mathbf{0}\multimap A, A}\,\text{Cut}$$

Clearly this sequent can *not* be derived without use of Cut in a calculus that has a restricted $\multimap R$-rule.

Definition 3.9

A sequent $\Gamma\Rightarrow\Delta$ is an *n-sequent* if the multiset Δ contains n formulas. □

Lemma 3.10

Any $\mathbf{ILL}^{>}$-derivation $\mathscr{D}$ of a sequent $\Gamma\Rightarrow$ contains at least one branch consisting solely of 0-sequents and ending in an instance $\Delta, \mathbf{0}\Rightarrow$ of axiom ($\mathbf{0}$). Moreover, for all Θ, Σ there exists an $\mathbf{ILL}^{>}$-derivation $\mathscr{D}'$ of $\Theta, \Gamma\Rightarrow\Sigma$ with $|\mathscr{D}'| = |\mathscr{D}|$.

Proof. By induction on the length of $\mathbf{ILL}^{>}$-derivations. □

The following proposition provides an interpretation for the non-singleton sets than can appear as succedents in $\mathbf{ILL}^{>}$-derivable sequents.

PROPOSITION 3.11
Let $\mathcal{D}$ be an $\mathbf{ILL}^{>}$-derivation of an n-sequent $\Gamma \Rightarrow \Delta$ with $n \neq 1$. Then there is an $\mathbf{ILL}^{>}$-derivation $\mathcal{D}'$ of $\Gamma \Rightarrow \mathbf{0}$ with $|\mathcal{D}'| \leq |\mathcal{D}|$.

PROOF. For 0-sequents this is a corollary to lemma 3.10. For $n > 1$ we again proceed by induction on the length of derivations. This is possible thanks to the restriction on $\multimap R$ and the fact that rules for right-weakening and left-par are lacking.

For the basis of induction we only need to consider axiom ($\mathbf{0}$), which trivially satisfies our demands. In the induction step most cases are more or less immediate by induction hypothesis. Consider, e.g. the rule $\otimes R$:

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma' \Rightarrow B, \Delta'}{\Gamma, \Gamma' \Rightarrow A \otimes B, \Delta, \Delta'}.$$

The induction hypothesis can be applied to at least one of the two premisses. In both cases we obtain our result by an application of Cut on $\mathbf{0}$.

For the rule $\multimap L$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma', A \multimap B \Rightarrow \Delta, \Delta'}$$

we have to distinguish two cases: if Δ is not empty we use the induction hypothesis on the left premiss and apply Cut on $\mathbf{0}$; otherwise we have a derivation of $\Gamma \Rightarrow A$ of strict lower length and obtain our result by induction hypothesis for the right premiss and application of $\multimap L$.

The same argument holds in case of Cut. □

THEOREM 3.12
$\mathbf{ILL}^{>} \vdash \Gamma \Rightarrow A$ iff $\mathbf{ILL} \vdash \Gamma \Rightarrow A$. (So $\mathbf{ILL}^{>}$ is conservative over $\mathbf{ILL}$.)

PROOF. Obviously only the left-to-right direction needs some attention, and for this we once more proceed by induction on the length of $\mathbf{ILL}^{>}$-derivations.

Clearly, for derivations of length 0 our claim holds. So suppose we already were able to give the proof for all sequents having an $\mathbf{ILL}^{>}$-derivation of length at most n. Then let a derivation of $\Gamma \Rightarrow A$ be given of length $n+1$. Now in most cases the result is more or less immediate by induction hypothesis and application of the same rule in $\mathbf{ILL}$. Let us check this in the case that $\Gamma \Rightarrow A$ has been obtained through application of $\multimap L$. For this there are two possibilities. Either we have

$$\frac{\Gamma_1 \Rightarrow C \quad \Gamma_2, B \Rightarrow A}{\Gamma_1, \Gamma_2, C \multimap B \Rightarrow A}$$

or the final step in the derivation has been

$$\frac{\Gamma_1 \Rightarrow C, A \quad \Gamma_2, B \Rightarrow}{\Gamma_1, \Gamma_2, C \multimap B \Rightarrow A}.$$

In the first case we are done by induction hypothesis and $\multimap L$ in **ILL**. In the second case, note that by proposition 3.11 we have an **ILL**$^>$-derivation of $\Gamma_1 \Rightarrow \mathbf{0}$ having at most the same length as the given derivation of $\Gamma_1 \Rightarrow C, A$. Therefore by induction hypothesis we have an **ILL**-derivation of $\Gamma_1 \Rightarrow \mathbf{0}$. The following then is an **ILL**-derivation:

$$\frac{\Gamma_1 \Rightarrow \mathbf{0} \quad \Gamma_2, \mathbf{0}, C \multimap B \Rightarrow A}{\Gamma_1, \Gamma_2, C \multimap B \Rightarrow A}\,\mathrm{Cut}$$

Cut is treated similarly. □

THEOREM 3.13
(Cut elimination for **ILL**$^>$) Cut can be eliminated from **ILL**$^>$-derivations.

PROOF. One may follow a procedure similar to the first method for Cut elimination described in the non-linear case. We encounter slight technical complications caused by the $!c$-rule, which again can be overcome by permitting a generalized (but derivable) rule of Cut (on !-formulas). For this we refer to Roorda [3], where an extensive description of the process of Cut elimination for **CLL**-derivations is given.

The 'problematic cases' can be handled by means of proposition 3.11 and theorem 3.12. As an example, let the following be some highest instance of Cut in an **ILL**$^>$-derivation, and suppose A is main formula in the left premiss.

$$\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \dfrac{\Gamma_2, A, B \Rightarrow C}{\Gamma_2, A \Rightarrow B \multimap C}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, B \multimap C}\,\mathrm{Cut}$$

As before, when $\Delta_1 \neq \emptyset$, we cannot permute Cut and application of $\multimap R$. But we know by (the proof of) 3.11 how to transform the derivation of $\Gamma_1 \Rightarrow A, \Delta_1$ into a derivation of $\Gamma_1 \Rightarrow \mathbf{0}$; by (the proof of) 3.12 we can transform this into an **ILL**-derivation of $\Gamma_1 \Rightarrow \mathbf{0}$, which, by applying the procedure of Cut elimination for **ILL**, may be changed into a *Cut-free* **ILL**-derivation.

Now replace the sub-derivation ending with the given highest instance of Cut by

$$\frac{\Gamma_1 \Rightarrow \mathbf{0} \quad \mathbf{0}, \Gamma_2 \Rightarrow \Delta_1, B \multimap C}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, B \multimap C}\,\mathrm{Cut}$$

In the derivation obtained this is a highest instance of Cut of height 0, and can be removed. □

Note that by theorem 3.12 any further restriction of rules to one-element succedent sets in $\mathbf{ILL}^{>}$ will result in some calculus that is also conservative over **ILL**. On the other hand, dropping the restriction on either $\multimap R$ or axiom $(\top)$ results in a calculus that no longer enjoys Cut elimination, while dropping the restriction on both gives us a calculus that is no longer conservative over **ILL**, e.g. by the non-conservativity result above. So we might call $\mathbf{ILL}^{>}$ '*minimally restricted*'. In fact we have the following

THEOREM 3.14
$\mathbf{ILL}^{>}$ is the unique minimally restricted sequent calculus in the language of **ILL** obtainable from **CLL** that is conservative over **ILL** and enjoys Cut elimination.

PROOF. First note that restricting *only* on **1**-, !-, quantifier- or structural rules, we would obtain a calculus that is no longer equivalent to **ILL**, again by proposition 3.8. The same proposition tells us that restricting only on $\oplus$ –, & or $\otimes$-rules results in a calculus not equivalent to **ILL**.

If we want to keep Cut elimination, a restriction on axiom $(\mathbf{0})$ forces a restriction on axiom $(\top)$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \top, \Delta \quad \mathbf{0} \Rightarrow \mathbf{0}}{\Gamma, \top \multimap \mathbf{0} \Rightarrow \mathbf{0}, \Delta} \qquad \mathbf{0} \Rightarrow \top \multimap \mathbf{0}}{\Gamma, \mathbf{0} \Rightarrow \mathbf{0}, \Delta}\,\text{Cut}$$

But a restriction on both $(\mathbf{0})$ and $(\top)$ gives us precisely **ILL**, i.e. it forces restriction on *all* rules.

As we already saw above, a restriction on $\multimap R$ forces a restriction on $(\top)$. Conversely, a restriction on $(\top)$ forces a restriction on either $\multimap R$ or $(\mathbf{0})$:

$$\dfrac{\dfrac{\mathbf{0} \Rightarrow A, B}{\Rightarrow \mathbf{0} \multimap A, B} \qquad \mathbf{0} \multimap A \Rightarrow \top}{\Rightarrow \top, B}\,\text{Cut}$$

Finally, a restriction on $\multimap L$ forces a restriction on $(\mathbf{0})$:

$$\dfrac{\mathbf{0} \Rightarrow B, C \qquad \dfrac{A \Rightarrow A \quad \mathbf{0} \Rightarrow \mathbf{0}}{A, A \multimap \mathbf{0} \Rightarrow \mathbf{0}}}{A, A \multimap \mathbf{0} \Rightarrow B, C}\,\text{Cut} \qquad \square$$

Also $\mathbf{ILL}^{>}$ is in some sense *maximal* as a sequent-calculus:

- we might consider extending $\mathbf{ILL}^{>}$ with the exponential ? and its rules, but then note that we would necessarily have to restrict rules $?R$ in order to keep eliminability of Cut, e.g. because of the following:

$$\dfrac{?\mathbf{0} \Rightarrow \top \qquad \dfrac{X \Rightarrow X}{X \Rightarrow ?\mathbf{0}, X}}{X \Rightarrow \top, X}\,\text{Cut}$$

With this restriction the introduction of ? becomes harmless; but also quite useless.

- extending $\mathbf{ILL}^{>}$ with the rules for *par* ($\wp$) results in a calculus in which Cut is not eliminable, as follows from the next example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{0}\Rightarrow A, B \qquad C\Rightarrow C}{\mathbf{0}\wp C\Rightarrow A, B, C}}{\mathbf{0}\wp C\Rightarrow A, C, B}}{\mathbf{0}\wp C\Rightarrow A\wp C, B} \qquad \dfrac{\dfrac{\dfrac{A\Rightarrow A \qquad \mathbf{0}\Rightarrow}{A, A\multimap\mathbf{0}\Rightarrow} \qquad C\Rightarrow C}{A\wp C, A\multimap\mathbf{0}\Rightarrow C}}{A\wp C\Rightarrow (A\multimap\mathbf{0})\multimap C}}{\mathbf{0}\wp C\Rightarrow (A\multimap\mathbf{0})\multimap C, B}\ \text{Cut}$$

We leave it to the reader to convince her/himself of the fact that $\mathbf{0}\wp C\Rightarrow (A\multimap\mathbf{0})\multimap C, B$ is *not* Cut-free derivable in $\mathbf{ILL}^{>} + par$.

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Appendix A: Classical predicate logic CL

Axioms:

$$A\Rightarrow A$$
$$\Gamma, \bot\Rightarrow\Delta$$

Logical rules:

$$\wedge R\frac{\Gamma\Rightarrow A, \Delta \quad \Gamma\Rightarrow B, \Delta}{\Gamma\Rightarrow A\wedge B, \Delta} \qquad \wedge L_1\frac{\Gamma, A\Rightarrow\Delta}{\Gamma, A\wedge B\Rightarrow\Delta} \qquad \wedge L_2\frac{\Gamma, B\Rightarrow\Delta}{\Gamma, A\wedge B\Rightarrow\Delta}$$

$$\vee R_1\frac{\Gamma\Rightarrow A, \Delta}{\Gamma\Rightarrow A\vee B, \Delta} \qquad \vee R_2\frac{\Gamma\Rightarrow B, \Delta}{\Gamma\Rightarrow A\vee B, \Delta} \qquad \vee L\frac{\Gamma, A\Rightarrow\Delta \quad \Gamma, B\Rightarrow\Delta}{\Gamma, A\vee B\Rightarrow\Delta}$$

$$\rightarrow R\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \qquad \rightarrow L\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

$$\forall R\frac{\Gamma \Rightarrow Aa, \Delta}{\Gamma \Rightarrow \forall xAx, \Delta} \qquad \forall L\frac{\Gamma, At \Rightarrow \Delta}{\Gamma, \forall xAx \Rightarrow \Delta} \qquad \exists R\frac{\Gamma \Rightarrow At, \Delta}{\Gamma \Rightarrow \exists xAx, \Delta} \qquad \exists L\frac{\Gamma, Aa \Rightarrow \Delta}{\Gamma, \exists xAx \Rightarrow \Delta}$$

Structural rules:

$$wL\frac{\Gamma \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \qquad wR\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow B, \Delta} \qquad cL\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad cR\frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$eL\frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \qquad eR\frac{\Sigma \Rightarrow \Gamma, A, B, \Delta}{\Sigma \Rightarrow \Gamma, B, A, \Delta}$$

$$\mathrm{Cut}\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Appendix B: Intuitionistic predicate logic IL

Axioms:

$$A \Rightarrow A$$
$$\Gamma, \bot \Rightarrow A$$

Logical rules:

$$\wedge R\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \qquad \wedge L_1\frac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \wedge L_2\frac{\Gamma, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$$

$$\vee R_1\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \vee R_2\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \qquad \vee L\frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

$$\rightarrow R\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad \rightarrow L\frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C}$$

$$\forall R\frac{\Gamma \Rightarrow Aa}{\Gamma \Rightarrow \forall xAx} \qquad \forall L\frac{\Gamma, At \Rightarrow B}{\Gamma, \forall xAx \Rightarrow B} \qquad \exists R\frac{\Gamma \Rightarrow At}{\Gamma \Rightarrow \exists xAx} \qquad \exists L\frac{\Gamma, Aa \Rightarrow B}{\Gamma, \exists xAx \Rightarrow B}$$

Structural rules:

$$wL\frac{\Gamma \Rightarrow A}{\Gamma, B \Rightarrow A} \qquad cL\frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad eL\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}$$

$$\mathrm{Cut}\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, A \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}$$

Appendix C: Intuitionistic predicate logic IL$^{>}$

Axioms:

$$A \Rightarrow A$$
$$\Gamma, \bot \Rightarrow \Delta$$

Logical rules:

$$\wedge R\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \qquad \wedge L_1\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \wedge L_2\frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$\vee R_1\frac{\Gamma\Rightarrow A,\Delta}{\Gamma\Rightarrow A\vee B,\Delta}\qquad \vee R_2\frac{\Gamma\Rightarrow B,\Delta}{\Gamma\Rightarrow A\vee B,\Delta}\qquad \vee L\frac{\Gamma,A\Rightarrow\Delta\quad \Gamma,B\Rightarrow\Delta}{\Gamma,A\vee B\Rightarrow\Delta}$$

$$\to R\frac{\Gamma,A\Rightarrow B}{\Gamma\Rightarrow A\to B}\qquad \to L\frac{\Gamma\Rightarrow A,\Delta\quad \Gamma,B\Rightarrow\Delta}{\Gamma,A\to B\Rightarrow\Delta}$$

$$\forall R\frac{\Gamma\Rightarrow Aa}{\Gamma\Rightarrow\forall xAx}\qquad \forall L\frac{\Gamma,At\Rightarrow\Delta}{\Gamma,\forall xAx\Rightarrow\Delta}$$

$$\exists R\frac{\Gamma\Rightarrow At,\Delta}{\Gamma\Rightarrow\exists xAx,\Delta}\qquad \exists L\frac{\Gamma,Aa\Rightarrow\Delta}{\Gamma,\exists xAx\Rightarrow\Delta}$$

Structural rules:

$$wL\frac{\Gamma\Rightarrow\Delta}{\Gamma,B\Rightarrow\Delta}\qquad wR\frac{\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow B,\Delta}\qquad cL\frac{\Gamma,A,A\Rightarrow\Delta}{\Gamma,A\Rightarrow\Delta}\qquad cR\frac{\Gamma\Rightarrow A,A,\Delta}{\Gamma\Rightarrow A,\Delta}$$

$$eL\frac{\Gamma,A,B,\Delta\Rightarrow\Sigma}{\Gamma,B,A,\Delta\Rightarrow\Sigma}\qquad eR\frac{\Sigma\Rightarrow\Gamma,A,B,\Delta}{\Sigma\Rightarrow\Gamma,B,A,\Delta}$$

$$\mathrm{Cut}\frac{\Gamma_1\Rightarrow\Delta_1,A\quad \Gamma_2,A\Rightarrow\Delta_2}{\Gamma_1,\Gamma_2\Rightarrow\Delta_1,\Delta_2}$$

Appendix D: Classical linear logic CLL

Axioms:

$$A\Rightarrow A$$

$$\Rightarrow\mathbf{1}\qquad \bot\Rightarrow$$

$$\Gamma,\mathbf{0}\Rightarrow\Delta\qquad \Gamma\Rightarrow\top,\Delta$$

Logical rules:

$$\mathbf{1}L\frac{\Gamma\Rightarrow\Delta}{\Gamma,\mathbf{1}\Rightarrow\Delta}\qquad \bot R\frac{\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\bot,\Delta}$$

$$\otimes L\frac{\Gamma,A,B\Rightarrow\Delta}{\Gamma,A\otimes B\Rightarrow\Delta}\qquad \otimes R\frac{\Gamma_1\Rightarrow A,\Delta_1\quad \Gamma_2\Rightarrow B,\Delta_2}{\Gamma_1,\Gamma_2\Rightarrow A\otimes B,\Delta_1,\Delta_2}$$

$$\&L_1\frac{\Gamma,A\Rightarrow\Delta}{\Gamma,A\,\&\,B\Rightarrow\Delta}\qquad \&L_2\frac{\Gamma,B\Rightarrow\Delta}{\Gamma,A\,\&\,B\Rightarrow\Delta}\qquad \&R\frac{\Gamma\Rightarrow A,\Delta\quad \Gamma\Rightarrow B,\Delta}{\Gamma\Rightarrow A\,\&\,B,\Delta}$$

$$\wp R\frac{\Gamma\Rightarrow A,B,\Delta}{\Gamma\Rightarrow A\wp B,\Delta}\qquad \wp L\frac{\Gamma_1,A\Rightarrow\Delta_1\quad \Gamma_2,B\Rightarrow\Delta_2}{\Gamma_1,\Gamma_2,A\wp B\Rightarrow\Delta_1,\Delta_2}$$

$$\oplus R_1\frac{\Gamma\Rightarrow A,\Delta}{\Gamma\Rightarrow A\oplus B,\Delta}\qquad \oplus R_2\frac{\Gamma\Rightarrow B,\Delta}{\Gamma\Rightarrow A\oplus B,\Delta}\qquad \oplus L\frac{\Gamma,A\Rightarrow\Delta\quad \Gamma,B\Rightarrow\Delta}{\Gamma,A\oplus B\Rightarrow\Delta}$$

$$\multimap R\frac{\Gamma,A\Rightarrow B,\Delta}{\Gamma\Rightarrow A\multimap B,\Delta}\qquad \multimap L\frac{\Gamma_1\Rightarrow A,\Delta_1\quad \Gamma_2,B\Rightarrow\Delta_2}{\Gamma_1,\Gamma_2,A\multimap B\Rightarrow\Delta_1,\Delta_2}$$

$$!L_1\frac{\Gamma\Rightarrow\Delta}{\Gamma,!A\Rightarrow\Delta}\qquad !L_2\frac{\Gamma,A\Rightarrow\Delta}{\Gamma,!A\Rightarrow\Delta}\qquad !R\frac{!\Gamma\Rightarrow C,?\Delta}{!\Gamma\Rightarrow !C,?\Delta}\qquad !c\frac{\Gamma,!A,!A\Rightarrow\Delta}{\Gamma,!A\Rightarrow\Delta}$$

$$?R_1\frac{\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow ?A,\Delta}\qquad ?R_2\frac{\Gamma\Rightarrow A,\Delta}{\Gamma\Rightarrow ?A,\Delta}\qquad ?L\frac{!\Gamma,C\Rightarrow ?\Delta}{!\Gamma,?C\Rightarrow ?\Delta}\qquad ?c\frac{\Gamma\Rightarrow ?A,?A,\Delta}{\Gamma\Rightarrow ?A,\Delta}$$

$$\forall R\frac{\Gamma\Rightarrow Aa,\Delta}{\Gamma\Rightarrow\forall x\,Ax,\Delta}\qquad \forall L\frac{\Gamma,At\Rightarrow\Delta}{\Gamma,\forall xAx\Rightarrow\Delta}\qquad \exists R\frac{\Gamma\Rightarrow At,\Delta}{\Gamma\Rightarrow\exists xAx,\Delta}\qquad \exists L\frac{\Gamma,Aa\Rightarrow\Delta}{\Gamma,\exists xAx\Rightarrow\Delta}$$

Structural rules:

$$\text{Cut}\,\frac{\Gamma_1 \Rightarrow A,\, \Delta_1 \quad \Gamma_2,\, A \Rightarrow \Delta_2}{\Gamma_1,\, \Gamma_2 \Rightarrow \Delta_1,\, \Delta_2}$$

$$eL\,\frac{\Gamma,\, A,\, B,\, \Delta \Rightarrow \Sigma}{\Gamma,\, B,\, A,\, \Delta \Rightarrow \Sigma} \qquad eR\,\frac{\Sigma \Rightarrow \Gamma,\, A,\, B,\, \Delta}{\Sigma \Rightarrow \Gamma,\, B,\, A,\, \Delta}$$

Appendix E: Intuitionistic linear logic ILL

Axioms:

$$A \Rightarrow A$$
$$\Rightarrow \mathbf{1}$$
$$\Gamma, \mathbf{0} \Rightarrow A \qquad \Gamma \Rightarrow \top$$

Logical rules:

$$\mathbf{1}L\,\frac{\Gamma \Rightarrow B}{\Gamma, \mathbf{1} \Rightarrow B}$$

$$\otimes L\,\frac{\Gamma,\, A,\, B \Rightarrow C}{\Gamma,\, A \otimes B \Rightarrow C} \qquad \otimes R\,\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1,\, \Gamma_2 \Rightarrow A \otimes B}$$

$$\&L_1\,\frac{\Gamma,\, A \Rightarrow C}{\Gamma,\, A \,\&\, B \Rightarrow C} \qquad \&L_2\,\frac{\Gamma,\, B \Rightarrow C}{\Gamma,\, A \,\&\, B \Rightarrow C} \qquad \&R\,\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B}$$

$$\oplus R_1\,\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \oplus B} \qquad \oplus R_2\,\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \oplus B} \qquad \oplus L\,\frac{\Gamma,\, A \Rightarrow C \quad \Gamma,\, B \Rightarrow C}{\Gamma,\, A \oplus B \Rightarrow C}$$

$$\multimap R\,\frac{\Gamma,\, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \qquad \multimap L\,\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2,\, B \Rightarrow C}{\Gamma_1,\, \Gamma_2,\, A \multimap B \Rightarrow C}$$

$$!L_1\,\frac{\Gamma \Rightarrow B}{\Gamma,\, !A \Rightarrow B} \qquad !L_2\,\frac{\Gamma,\, A \Rightarrow B}{\Gamma,\, !A \Rightarrow B} \qquad !R\,\frac{!\Gamma \Rightarrow C}{!\Gamma \Rightarrow\, !C} \qquad !c\,\frac{\Gamma,\, !A,\, !A \Rightarrow B}{\Gamma,\, !A \Rightarrow B}$$

$$\forall R\,\frac{\Gamma \Rightarrow Aa}{\Gamma \Rightarrow \forall x A x} \qquad \forall L\,\frac{\Gamma,\, At \Rightarrow B}{\Gamma,\, \forall x A x \Rightarrow B} \qquad \exists R\,\frac{\Gamma \Rightarrow At}{\Gamma \Rightarrow \exists x A x} \qquad \exists L\,\frac{\Gamma,\, Aa \Rightarrow B}{\Gamma,\, \exists x A x \Rightarrow B}$$

Structural rules:

$$\text{Cut}\,\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2,\, A \Rightarrow C}{\Gamma_1,\, \Gamma_2 \Rightarrow C} \qquad eL\,\frac{\Gamma,\, A,\, B,\, \Delta \Rightarrow C}{\Gamma,\, B,\, A,\, \Delta \Rightarrow C}$$

Appendix F: Intuitionistic linear logic ILL$^{>}$

Axioms:

$$A \Rightarrow A$$
$$\Rightarrow \mathbf{1}$$
$$\Gamma, \mathbf{0} \Rightarrow \Delta \qquad \Gamma \Rightarrow \top$$

Logical rules:

$$\mathbf{1}L\,\frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathbf{1} \Rightarrow \Delta}$$

$$\otimes L\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A\otimes B \Rightarrow \Delta} \qquad \otimes R\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A\otimes B, \Delta_1, \Delta_2}$$

$$\& L_1\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A\& B \Rightarrow \Delta} \qquad \& L_2\frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A\& B \Rightarrow \Delta} \qquad \& R\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A\& B, \Delta}$$

$$\oplus R_1\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A\oplus B, \Delta} \qquad \oplus R_2\frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A\oplus B, \Delta} \qquad \oplus L\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A\oplus B \Rightarrow \Delta}$$

$$\multimap R\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A\multimap B} \qquad \multimap L\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A\multimap B \Rightarrow \Delta_1, \Delta_2}$$

$$!L_1\frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad !L_2\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad !R\frac{!\Gamma \Rightarrow C}{!\Gamma \Rightarrow !C} \qquad !c\frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$

$$\forall R\frac{\Gamma \Rightarrow Aa, \Delta}{\Gamma \Rightarrow \forall xAx, \Delta} \qquad \forall L\frac{\Gamma, At \Rightarrow \Delta}{\Gamma, \forall xAx \Rightarrow \Delta} \qquad \exists R\frac{\Gamma \Rightarrow At, \Delta}{\Gamma \Rightarrow \exists xAx, \Delta} \qquad \exists L\frac{\Gamma, Aa \Rightarrow \Delta}{\Gamma, \exists xAx \Rightarrow \Delta}$$

Structural rules:

$$\mathrm{Cut}\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$eL\frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \qquad eR\frac{\Sigma \Rightarrow \Gamma, A, B, \Delta}{\Sigma \Rightarrow \Gamma, B, A, \Delta}$$

Appendix G: The fragment $\mathcal{F}$ of CLL

Axioms:

$$A \Rightarrow A$$
$$\Gamma, 0 \Rightarrow \Delta$$

Logical rules:

$$\oplus L\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A\oplus B \Rightarrow \Delta} \qquad \oplus R_1\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A\oplus B, \Delta} \qquad \oplus R_2\frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A\oplus B, \Delta}$$

$$\& L_1\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A\& B \Rightarrow \Delta} \qquad \& L_2\frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A\& B \Rightarrow \Delta} \qquad \& R\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A\& B, \Delta}$$

$$\multimap L\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A\multimap B \Rightarrow \Delta_1, \Delta_2} \qquad \multimap R\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A\multimap B, \Delta}$$

$$!L_1\frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad !L_2\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad !R\frac{!\Gamma \Rightarrow C}{!\Gamma \Rightarrow !C} \qquad !c\frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$

$$\forall L\frac{At, \Gamma \Rightarrow \Delta}{\forall xAx, \Gamma \Rightarrow \Delta} \qquad \forall R\frac{\Gamma \Rightarrow Aa, \Delta}{\Gamma \Rightarrow \forall xAx, \Delta} \qquad \exists L\frac{Aa, \Gamma \Rightarrow \Delta}{\exists xAx, \Gamma \Rightarrow \Delta} \qquad \exists R\frac{\Gamma \Rightarrow At, \Delta}{\Gamma \Rightarrow \exists xAx, \Delta}$$

Structural rules:

$$eL\frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \qquad eR\frac{\Sigma \Rightarrow \Gamma, A, B, \Delta}{\Sigma \Rightarrow \Gamma, B, A, \Delta}$$