Call-by-Name and Call-by-Value in Normal Modal Logic

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Abstract. This paper provides a call-by-name and a call-by-value calculus, both of which have a Curry-Howard correspondence to the minimal normal logic **K**. The calculi are extensions of the $\lambda\mu$-calculi, and their semantics are given by CPS transformations into a calculus corresponding to the intuitionistic fragment of **K**. The duality between call-by-name and call-by-value with modalities is investigated in our calculi.

1 Introduction

Modal logics have a long history since logics with strict implications, and are now widely accepted both theoretically and practically. Especially, studies of modal logics by Kripke semantics [18] are quite active and a large number of results exist, for example, [7] is a textbook about such studies. Since Kripke semantics concern only provability, equality on proofs is less studied on modal logics compared with traditional logics.

It is well-known that the intuitionistic propositional logic exactly corresponds to the simply typed λ-calculus: formulae as types and proofs as terms. Such a correspondence is called a Curry-Howard correspondence after Howard's work [15]. A Curry-Howard correspondence enables us to study equality on proofs computationally. Though the correspondence can be extended to higher-order and predicate logics as shown in [3], we investigate only propositional logics in this paper. The aim of this study is to give a proper calculus that have a Curry-Howard correspondence to the modal logic **K**. Through a Curry-Howard correspondence, any type system can be regarded as a logic by forgetting terms. In this sense, modal logics are contributing to practical studies for programming languages, *e.g.*, staged computations [8] and information flow analysis [23]. Since **K** is known as a minimal modal logic, this paper focuses on **K** rather than **S4**.

Before defining a calculus for **K**, we consider the intuitionistic fragment of **K**, which is called **IK** in this paper. In Section 2, the calculus for **IK** is defined as a refinement of Bellin *et al.*'s calculus [4] rather than Martini and Masini's[22]. Our calculus is sound and complete for the categorical semantics given in [4]. The study [19] about simply typed λ-calculus and cartesian closed categories is a typical study of categorical semantics. Categorical semantics of modal logics are studied by Bierman and de Paiva, and Bellin *et al.* in [6] and [4]. Their semantics are based on studies about semantics of linear logics (*e.g.*, [29] and [5]) since the exponential of the linear logic [12] is a kind of **S4** modality.

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A Curry-Howard correspondence between the classical propositional logic and the λ-calculus with continuations was provided in [13] by Griffin. Parigot has proposed the $\lambda\mu$-calculus as a calculus for the classical logic in [26]. Now, kinds of $\lambda\mu$-calculi exist and some of them are defined by CPS transformations. A CPS transformation was originally introduced in [11], and the relation between call-by-value and CPS semantics was first studied by Plotkin in [27]. De Groote defines a CPS transformation on a call-by-name $\lambda\mu$-calculus in [9], but in this paper, we adopt Selinger's CPS transformation [30], which is an extension of Hofmann and Streicher's [14]. In Section 3, we provide a call-by-name $\lambda\mu$-calculus with a box modality, which has a Curry-Howard correspondence to **K**, by the CPS semantics into the calculus for **IK** defined in Section 2. A call-by-value $\lambda\mu$-calculus is provided in [25] by Ong and Stewart. We define a call-by-value calculus for **K** also as an extension of Selinger's call-by-value $\lambda\mu$-calculus [30] via the CPS transformation in Section 4.

The duality between call-by-name and call-by-value is an important property of the classical logic. The duality on a programming language with first-class continuations was first formalized by Filinski in [10]. It has been formalized on the $\lambda\mu$-calculi in [30] by Selinger, and reformulated as sequent calculi in [34] by Wadler. In [16], the duality is developed with recursion by the author. In Section 5, we study such duality on the classical modal logic **K**.

In addition, we investigate the logic **S4** with the CPS semantics. It is shown in Section 6 that a diamond modality is a monad in call-by-name **S4**.

Notations

We introduce notations specific to this paper.

- The symbol "$\equiv$" denotes the α-equivalence.
- We may omit superscripted and subscripted types if they are trivial.
- A notation "$\overrightarrow{M}$" is used for a sequence of meta-variables "$M_1, \ldots, M_n$". Hence, an expression "$\overrightarrow{M}, \overrightarrow{N}$" stands for the concatenation of $\overrightarrow{M}$ and $\overrightarrow{N}$.
- For a unary operator $\Phi(-)$, we write "$\Phi(\overrightarrow{M})$" for "$\Phi(M_1), \ldots, \Phi(M_n)$".
- We write "$\overrightarrow{N}(\theta\overrightarrow{x}.\,M)$" for "$N_1(\theta x_1.\cdots N_n(\theta x_n.\,M)\cdots)$" and "$[\overrightarrow{a}](\theta\overrightarrow{x}.\,M)$" for "$[a_1](\theta x_1.\cdots[a_n](\theta x_n.\,M)\cdots)$", where θ is λ or μ.
- We write "$\neg\tau$" for "$\tau \to \bot$".

2 Calculus for Intuitionistic Normal Modal Logic

In this section, we study the intuitionistic modal logic **IK**. Intuitionism of a diamond modality is not trivial, for example, [33] gives an account of it, but this section focuses on the box fragment of **IK**. We call also this fragment itself **IK** in this paper. A diamond modality is investigated in a classical logic after the next section.

It is well-known that the λ-calculus with conjunctions and disjunctions exactly corresponds to the intuitionistic propositional logic. Therefore, we extend