Translations Between Logics

Tutorial

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The method of studying inter-relations between logical systems by the analysis of translations between them was originally introduced by Kolmogoroff, in 1925.

The first known 'translations' involving classical logic, intuitionistic logic and modal logic were presented by Kolmogoroff and Glivenko in 1929, Lewis and Langford in 1932, Gödel in two papers of 1933, and Gentzen in 1933. Kolmogoroff, Gentzen and one of Gödel's papers were developed mainly in order to show relative consistency of the classical logic with respect to the intuitionistic one.

However, in spite of these papers dealing with inter-relations between the studied systems, they are not interested in the meaning of the term translation between logics. Several terms are used by the authors, such as translations, interpretations, transformations among others. Since then, translations between logics have been used to different purposes.

Prawitz and Malmnäs (1968) survey these historical papers and this is the first paper in which a general definition for the concept of translation between logical systems is introduced.

Brown and Suszko (1973) construct a general framework of the theory of abstract logics, concerned with algebraic properties of abstract logics of a same similarity type. Though motivating, they are not interested in the study of inter-relations between general abstract logics and continuous functions are defined as generalizations of the familiar continuous functions between topological spaces.

Szczerba (1977) defines the concept of interpretation function between structures. The corresponding functions mapping formulas to formulas are

called translations, but we may say that the author is only concerned with translations between models.

Wójcicki (1988) and Epstein (1990) are the first works with a general systematic study on translations between logics, both studying inter-relations between propositional calculi in terms of translations.

D'Ottaviano (1973) studies variants of Tarskian closure operators characterized by interpretations. Hoppmann (1973) uses this characterization and claims that, from a logical point of view, continuous functions between closure structures correspond to functions that preserve deductions, in a remarkable coincidence with the underlying approach of Brown and Suszko (1973). Given two closure structures $\mathbf{K}_1 = \langle L_1, C_1 \rangle$ and $\mathbf{K}_2 = \langle L_2, C_2 \rangle$, an *interpretation* or a *translation* from $\mathbf{K}_1$ into $\mathbf{K}_2$ is a function $f : \mathcal{P}(L_1) \to \mathcal{P}(L_2)$ such that: (i) for $A, B \subseteq L_1, f(C_1(A)) \subseteq C_2(f(A))$; (ii) the inverse image of a closed set of L_2 is a closed set of L_1; and (iii) if $A \vdash_{C_1} B$ then $f(A) \vdash_{C_2} f(B)$.

This is apparently the first time in the literature when the term "translation between general logic systems" is used to mean a function preserving derivability.

Da Silva, D'Ottaviano and Sette (1999), motivated by such early treatments and explicitly interested in the study of inter-relations between logic systems in general, propose a general definition for the concept of translation between logics, in order to single out what seems to be in fact the essential feature of a logical translation: logics are characterized as pairs constituted by a set (ignoring the fact that in general a logic deals with formulas of a language) and a consequence operator, and translations between logics are defined as maps preserving consequence relations.

Definition: A *translation* from a logic $\mathbf{A} = < A, C_A >$ into a logic $\mathbf{B} = < B, C_B >$ is a map $t : A \to B$ such that

$$t(C_A(X)) \subseteq C_B(t(X))$$

for any $X \subseteq A$.

If $\mathbf{A}$ and $\mathbf{B}$ are logics, with A and B formal languages and $\vdash_{C_A}$ and $\vdash_{C_B}$ being the associated consequence relations, respectively, then t is a translation if, and only if, for $\Gamma \cup \{\alpha\} \subseteq \mathbf{Form}\ (A)$:

$$\Gamma \vdash_{C_A} \alpha \quad \text{implies} \quad t(\Gamma) \vdash_{C_B} t(\alpha).$$

In this Tutorial, we will begin by a historical survey of the use of translations for the study of inter-relations between logical systems, and will discuss and compare the different approaches to the use of the term 'translation'.

We will present an initial segment of a theory of translations and will also investigate some connections between translations involving logics and uniformly continuous functions between spaces of their theories. Several of the results we have obtained are relevant to the study of general properties of logic systems from the point of view of translations between them.

We will study an important subclass of translations that preserve and conserve consequence relations, the conservative translations, introduced and investigated in Feitosa (1997) and in Feitosa and D'Ottaviano (2001). We will prove that the class constituted by logics and conservative translations determines a co-complete subcategory of the bi-complete category whose objects are the logics and whose morphisms are the translations between them.

We will present some conservative translations involving classical logic, intuitionistic logics, modal logics, the many-valued logics of Lukasiewicz and Post and several known paraconsistent logics (see D'Ottaviano and Feitosa 1999, D'Ottaviano and Feitosa 2000, D'Ottaviano and Feitosa 2007).

Based on Scheer and D'Ottaviano (2006), we will also initiate the study of a theory of conservative translations involving cumulative non-monotonic logics.

By dealing with the Lindenbaum-Tarski algebraic structures associated to the logics, we will study the problem, several times mentioned in the literature, of the existence of conservative translations from intuitionistic logic and from Lukasiewicz infinite-valued logic into classical logic (see Cignoli, D'Ottaviano and Mundici 2000, D'Ottaviano and Feitosa 2006, Feitosa and D'Ottaviano 200–).

The category whose objects are topological spaces and whose morphisms are the continuous functions between them is a full sub-category of the bi-complete category of logics and translations. This is in line with our intuition that topological spaces can be seen as particular cases of logics.

Our notion of translation accommodates certain maps that seem to be intuitive examples of translations, such as the identity map from intuitionistic into classical logic and the forgetful map from modal logics into classical logic; such cases would be ruled out if the stricter notion of conservative translation were imposed. In this sense, the more abstract notion of translations given by da Silva, D'Ottaviano and Sette a genuine advance in the scope of relating logic systems, based upon which further unfoldings can be devised.

Translations in the sense of Prawitz and Malmnäs do not coincide either with conservative translations, or with translations in the sense of our definition. Translations in Wójcicki's sense are particular cases of conservative translations, being derivability preserving schematic translations in Prawitz and Malmnäs' sense. Epstein's translations are instances of conservative translations, and his grammatical translations are particular cases of Prawitz and Malmnäs' schematic translations with respect to derivability (and coincide with schematic conservative translations). None of them attempted a more general conception.

Other developments of the wider notion of translation sprung forth: Carnielli (1990) proposes a new approach to formal semantics for non-classical logics using translations, the so-called *possible-translations semantics*. Juliana Bueno-Soler, in her Master Dissertation at UNICAMP in 2004, also introduces the *possible-translations algebraic semantics*, in which translations play an essential role. Victor Fernandez, in his Doctoral Thesis at UNICAMP in 2005, uses translations in order to investigate combinations of logics, more particularly fibring of logics.

In the Tutorial, we will present the notion of *transfer* and the underlying model-theoretic approach to translations between logics as developed by Coniglio and Carnielli (2002).

We will also present the notion of *contextual translation*, as proposed by Coniglio (2005) and studied by Carnielli, Coniglio and D'Ottaviano (2007).

Finally, we will compare the concepts of conservative translations, transfers and contextual translations in the scope of our general definition of translation between logics.

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