THE JOURNAL OF SYMBOLIC LOGIC
Volume 13, Number 1, March 1948

SOME THEOREMS ABOUT THE SENTENTIAL CALCULI OF LEWIS AND HEYTING

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In this paper we shall prove theorems about some systems of sentential calculus, by making use of results we have established elsewhere regarding closure algebras[1] and Brouwerian albegras.[2] We shall be concerned mostly with the Lewis system[3] and the Heyting system.[4] Some of the results here are new (in particular, Theorems 2.4, 3.1, 3.9, 3.10, 4.5, and 4.6); others have been stated without proof in the literature (in particular, Theorems 2.1, 2.2, 4.4, 5.2, and 5.3).

The proofs to be given here will be found to be mostly very simple; generally speaking, they amount to drawing conclusions from the theorems established in McKinsey and Tarski [10] and [11]. We have thought it might be worth while, however, to publish these rather elementary consequences of our previous work—so as to make them readily available to those whose main interest lies in sentential calculus rather than in topology or algebra.

In referring to closure algebras and Brouwerian algebras, we shall use the notation used in McKinsey and Tarski [11]. When referring to theorems from McKinsey and Tarski [10] we shall put an asterisk after the number of the theorem; and we shall put two asterisks after the numbers of theorems from McKinsey and Tarski [11].

1. The Lewis system and closure algebra. The aim of this section is to establish certain connections between the Lewis system of sentential calculus and closure algebras.

By the Lewis system of sentential calculus we understand what has been referred to in the literature as the system S4. The construction of this system can be briefly described as follows.

The symbols of the system consist of variables, constants, and parentheses. There are infinitely many variables, and they are assumed to be arranged in an infinite sequence. Thus we can speak of the nth variable, or of the variable with index n; we denote it by "v_n". We put, in particular,

$$p = v_1, \quad q = v_2, \quad r = v_3, \quad s = v_4.$$

As regards constants, they are three in number: the negation sign, the conjunction sign, and the possibility sign. The expression formed from two given expressions α and β by putting a conjunction sign between them (and enclosing the whole in parentheses) is called the conjunction of α and β—in symbols, $\alpha \wedge \beta$. The expression formed from α by putting a negation sign, or a possibility sign, in front of it is called the negation, or the possibility, of α—in symbols, $\sim\alpha$, or $\Diamond\alpha$

Received May 27, 1947.

[1] See McKinsey and Tarski [10]. (The numbers in square brackets refer to the bibliography at the end of the paper.)

[2] See McKinsey and Tarski [11].

[3] See Lewis and Langford [5], expecially Chapter VI and Appendix II.

[4] See Heyting [4].

respectively. It should be noticed that the symbols "v_n", "p", "q", "r", $\cdots$, "$\wedge$", "$\sim$", "$\Diamond$", etc., occur throughout this paper as meta-logical symbols used to denote certain expressions of sentential calculus and certain operations on expressions; we never employ symbols occurring in the calculus itself. On the other hand, symbols referring to various kinds of algebras will always be used in a mathematical, not a metamathematical, sense.

Expressions formed from variables by applying finitely many times the operations of conjunction, negation, and possibility are called *formulas*. In other words, the class of formulas is the smallest class of expressions which contains all the variables and is closed under these operations. Thus, for instance, the following expressions are formulas:

$$p\,, \quad p \wedge q\,, \quad \sim(p \wedge q)\,, \quad \Diamond p \wedge \sim q\,.$$

By the *index* of a formula we mean the index of the variable of highest index appearing in it.

We introduce for convenience the following abbreviations:

$$\begin{aligned}
&\alpha \vee \beta && \text{for} \quad \sim(\sim\alpha \wedge \sim\beta)\,;\\
&\alpha \rightarrow \beta && \text{for} \quad \sim(\alpha \wedge \sim\beta)\,;\\
&\alpha \leftrightarrow \beta && \text{for} \quad (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)\,;\\
&\alpha \supset \beta && \text{for} \quad \sim\Diamond(\alpha \wedge \sim\beta)\,;\\
&\alpha \equiv \beta && \text{for} \quad (\alpha \supset \beta) \wedge (\beta \supset \alpha)\,.
\end{aligned}$$

These symbols are called, respectively, the *disjunction sign*, *sign of material implication*, *sign of material equivalence*, *sign of strict implication*, and *sign of strict equivalence*.

A formula is called an *axiom* if it is of one of the eleven following kinds (where α, β, γ, and δ are arbitrary formulas):

*L*1. $(\alpha \wedge \beta) \supset (\beta \wedge \alpha)$.
*L*2. $(\alpha \wedge \beta) \supset \alpha$.
*L*3. $\alpha \supset (\alpha \wedge \alpha)$.
*L*4. $[(\alpha \wedge \beta) \wedge \gamma] \supset [\alpha \wedge (\beta \wedge \gamma)]$.
*L*5. $[(\alpha \supset \beta) \wedge (\beta \supset \gamma)] \supset (\alpha \supset \gamma)$.
*L*6. $[\alpha \wedge (\alpha \supset \beta)] \supset \beta$.
*L*7. $\Diamond\Diamond\alpha \supset \Diamond\alpha$.
*L*8. $(\alpha \supset \beta) \supset (\alpha \rightarrow \beta)$.
*L*9. $\alpha \rightarrow [\beta \rightarrow (\alpha \wedge \beta)]$.
*L*10. $(\alpha \supset \beta) \supset (\sim\beta \supset \sim\alpha)$.
*L*11. $[(\alpha \supset \beta) \wedge (\gamma \supset \delta)] \supset [(\alpha \wedge \gamma) \supset (\beta \wedge \delta)]$.
*L*12. $(\alpha \supset \beta) \supset (\Diamond\alpha \supset \Diamond\beta)$.

By the result of *detachment* of a formula β from a formula α we understand the (uniquely determined) formula γ such that

$$\alpha = (\beta \rightarrow \gamma)\,.$$

A formula α is said to be *derivable* from a set Φ of formulas if α belongs to every set of formulas which includes Φ and is closed under the operation of detachment. A formula which is derivable from the set of axioms is called a *provable* formula (of the Lewis system).

THEOREM 1.1. The system of sentential calculus defined above is equivalent to the Lewis system S4—i.e., a formula is provable in the calculus defined above, if and only if it is provable in S4.

Proof. That every formula of our system is provable in S4, follows from Lewis's formulas $B1$, $B2$, $B3$, $B4$, $B6$, $B7$, $A8$, and $C10.1$ (on pages 493 and 497 of Lewis and Langford [5]) and formulas 14.29, 12.43, 19.68, and 14.1 (of Chapter VI of Lewis and Langford [5]).

To show that every formula of S4 is provable in our system, we notice first that, by $L8$, the rule of detachment for strict implication (see page 126 of Lewis and Langford [5]) holds in our system and that, by $L9$, Lewis's rule of adjunction holds in our system. From $L1$ and $L2$ we see that, if $\alpha \wedge \beta$ is provable in our system, then α and β are provable in the system. From $L10$, $L11$, and $L12$ we then see by induction that Lewis's first rule of substitution holds of our system. It is obvious that the second rule of substitution holds, since we have taken axiom schemata instead of single axioms. Thus all Lewis's rules hold of our system By $L1$, $L2$, $L3$, $L4$, $L5$, $L6$, and $L7$ we see that Lewis's axioms $B1$, $B2$, $B3$, $B4$, $B6$, $B7$, and $C10.1$ are provable in our system. $B5$ can be proved in our system by the proof given in McKinsey [6]. $B8$ can then be proved by the proof given on page 148 of Parry [12].

In discussing the Lewis system we shall use the familiar notion of a matrix. *Matrices* are quintuples $\mathfrak{M} = \langle A, D, \cdot, -, C \rangle$, where A and D are sets, $\cdot$ is a binary operation defined over A, and $-$ and C are unary operations defined over A; A is supposed to include D, and to be closed under the operations $\cdot$, $-$, and C. A is sometimes referred to as the set of *elements* of $\mathfrak{M}$, and D as the set of *designated* elements. The matrix $\mathfrak{M} = \langle A, D, \cdot, -, C \rangle$ is called *regular* if it satisfies the following condition: whenever x and $-(x \cdot -y)$ are elements of D, then y is an element of D.

We assume it to be known under what conditions a formula α is said to be satisfied by such a matrix $\mathfrak{M}$; a recursive definition of this notion presents no difficulty[5] (the operations $\cdot$, $-$, and C correspond respectively to the constants $\wedge$, $\sim$, and $\Diamond$.) A matrix $\mathfrak{M}$ of the kind considered is called a *Lewis matrix* if it satisfies every provable formula of the Lewis system; it is easily seen that a regular matrix is a Lewis matrix if it satisfies all the axioms of the Lewis system. A matrix is called a *characteristic* Lewis matrix, if it is also the case that, conversely, every formula satisfied by it is provable in the Lewis system.

Every quintuple $\mathfrak{M}$ of the kind considered above can clearly be regarded as a sort of algebraic system, in which A is the domain of the algebra, D is a distinguished set of elements of this domain, and $\cdot$, $-$, and C are the three fundamental operations of the algebra. We shall be particularly interested in systems in

[5] See Tarski [13], p. 106.

which A is a closure algebra under[6] $\cdot$, $-$, and C, and in which D consists of a single element d—in fact, of the unit element of the closure algebra: $D = \{d\}$, where $d = 1$. Such systems are involved in the following two theorems, which establish the fundamental relations between the Lewis system and closure algebras.

THEOREM 1.2. For $\mathfrak{M} = \langle A, \{d\}, \cdot, -, C\rangle$ to be a regular Lewis matrix it is necessary and sufficient that A be a closure algebra under $\cdot$, $-$, and C, and that d be the unit element of this closure algebra.

Proof. From the definition of closure algebra,[7] together with Theorems 3, 9, and 10 of McKinsey [8].

THEOREM 1.3. If A is a dissectable closure algebra under $\cdot$, $-$, and C (thus, in particular, if A is a closure algebra over Euclidean space of any number of dimensions), then $\mathfrak{M} = \langle A, \{1\}, \cdot, -, C\rangle$ is a regular characteristic matrix for the Lewis system.

Proof. By Theorem 11 of McKinsey [8] we see that every formula α which is not provable in the Lewis system fails to be satisfied by some normal Lewis matrix $\mathfrak{M} = \langle A, \mathrm{D}, \cdot, -, C\rangle$.[8] By Theorem 1 of McKinsey [8], we see that A is a Boolean algebra with respect to $\cdot$ and $-$. If 1 is the unit element of this Boolean algebra, and we set $\mathfrak{M}' = \langle A, \{1\}, \cdot, -, C\rangle$, then it is easily seen that α also fails for $\mathfrak{M}'$. Our theorem now follows from Theorem 1.2, and Theorem 5.9* (of McKinsey and Tarski [10]).

The results just stated can also be formulated without using the notion of a matrix. With this in view we correlate with every formula α of the Lewis system a closure-algebraic function[9] $f^{(\alpha)}$; in fact, with a formula of index n a closure-algebraic function of n arguments. This is done recursively in the following way:

(i) If $\alpha = v_n$ for some n, then $f^{(\alpha)}$ is the function determined by the equation

$$f^{(\alpha)}(x_1, \cdots, x_n) = x_n$$

(for all elements $x_1, \cdots, x_n$ of every closure algebra).[10]

(ii) If α is a formula of index m, β a formula of index n, and $r = \max(m, n)$, then $f^{(\alpha \wedge \beta)}$ is the function defined by the equation

$$f^{(\alpha \wedge \beta)}(x_1, \cdots, x_r) = f^{(\alpha)}(x_1, \cdots, x_m) \cdot f^{(\beta)}(x_1, \cdots, x_n) .$$

[6] In McKinsey and Tarski [10] a closure algebra was considered as constituted by a set, the unary operations of complementation and closure, and the two binary operations of sum and product. Since sum can be defined in terms of product and complement (by De Morgan's law), however, it is clear that it is also possible to consider a closure algebra as constituted by the set of elements, the two unary operations, and the single binary operation of product.

[7] See McKinsey and Tarski [10], Definition 1.1.

[8] A normal matrix (see McKinsey [8]) is one which satisfies the conditions: (i) if x is in D and $x \supset y$ is in D, then y is in D; (ii) if $x \equiv y$ is in D, then $x = y$.

[9] For a definition of closure-algebraic functions see §4 of McKinsey and Tarski [10].

[10] As in McKinsey and Tarski [10], when there is no danger of misunderstanding, we omit the subscript on symbols for functions to indicate the closure algebra to which the elements $x_1, \cdots, x_n$ belong.

(iii) If α is a formula of index n, then $f^{(\sim\alpha)}$ and $f^{(\Diamond\alpha)}$ are functions defined by the equations

$$f^{(\sim\alpha)}(x_1, \cdots, x_n) = -f^{(\alpha)}(x_1, \cdots, x_n),$$
$$f^{(\Diamond\alpha)}(x_1, \cdots, x_n) = Cf^{(\alpha)}(x_1, \cdots, x_n).$$

The results stated in Theorems 1.2 and 1.3 can now be put in the following form:

THEOREM 1.4. For every formula α of the Lewis calculus the following conditions are equivalent:

(i) α is provable in the Lewis system;

(ii) $f^{(\alpha)}$ is identically equal to 1 in every closure algebra.

Moreover, if $\mathfrak{A} = \langle A, \cdot, -, C\rangle$ is any dissectable closure algebra, then each of the above conditions is equivalent to the following:

(iii) $f^{(\alpha)}$ is identically equal to 1 in $\mathfrak{A}$.

2. Theorems about the Lewis system. The following theorem was taken by Gödel (in [2]) as a primitive rule in his axiomatization of the Lewis system; so far as we know, however, no proof of it has been published.

THEOREM 2.1. If α is provable in the Lewis system, then $\sim\Diamond\sim\alpha$ is provable in the Lewis system.

Proof. Since α is provable in the Lewis system, we see by Theorem 1.4 that $f^{(\alpha)}$ is identically equal to 1 in every closure algebra. From this we easily conclude that $-C-f^{(\alpha)}$, or $f^{(\sim\Diamond\sim\alpha)}$, is identically equal to 1 in every closure algebra from which it follows that $\sim\Diamond\sim\alpha$ is provable in the Lewis system.

Remark. Theorem 2.1 can also be proved directly, by an induction on the length of the proof of α in the Lewis system.

The following result was conjectured by Gödel in [2]:

THEOREM 2.2. If $\sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta$ is provable in the Lewis system, then either α or β is provable in the Lewis system.

Proof. Since $\sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta$ is provable in the Lewis system, we see by 1.4 that, setting $\gamma = \sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta$, $f^{(\gamma)}$ is identically equal to 1 in every closure algebra. Hence, by Definition 4.1*, $-C-f^{(\alpha)} + -C-f^{(\beta)}$ is identically equal to 1 in every closure algebra. Hence, using 4.12*, we conclude that either $f^{(\alpha)}$ is identically 1 or $f^{(\beta)}$ is identically 1; our theorem now follows from 1.4.

Remark. Theorem 2.2 can also be proved directly, without making use of 1.4 and 4.12*. In order to do this, it is convenient to consider the class $\mathfrak{A}$ of all ordered couples $\langle\alpha, \beta\rangle$ of formulas of the Lewis calculus. We define operations $-$, $\cdot$, and C on the elements of $\mathfrak{A}$ as follows:

(1) $$-\langle\alpha, \beta\rangle = \langle\sim\alpha, \sim\beta\rangle;$$

(2) $$\langle\alpha_1, \beta_1\rangle\cdot\langle\alpha_2\ \beta_2\rangle = \langle\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2\rangle;$$

if $\sim\alpha$ and $\sim\beta$ are both provable, then

(3) $$C\langle\alpha, \beta\rangle = \langle\Diamond\alpha, \beta\rangle;$$

if $\sim\alpha$ and $\sim\beta$ are not both provable, then

$$C\langle\alpha, \beta\rangle = \langle\Diamond\alpha, p \vee \sim p\rangle. \tag{4}$$

We then set $\mathfrak{M} = \langle\mathfrak{A}, \mathfrak{D}, \cdot, -, C\rangle$, where $\mathfrak{A}$, $\cdot$, $-$, and C are as defined above, and $\mathfrak{D}$ is the set of all those members $\langle\alpha, \beta\rangle$ of $\mathfrak{A}$ such that both α and β are provable in the Lewis system. It is then easily verified that $\mathfrak{M}$ is a regular Lewis matrix. Moreover it is seen that if each variable v_i in a formula α is replaced by $\langle v_i, v_i\rangle$, then there is some formula α' such that α assumes, when evaluated in $\mathfrak{M}$, the value $\langle\alpha, \alpha'\rangle$. Hence we see that, if neither α nor β is provable in the Lewis system,

$$\begin{aligned}
&-C-\alpha(\langle p_1, p_1\rangle, \cdots, \langle p_n, p_n\rangle) + -C-\beta(\langle p_1, p_1\rangle, \cdots, \langle p_n, p_n\rangle)\\
&= -C-\langle\alpha, \alpha'\rangle + -C-\langle\beta, \beta'\rangle = -C\langle\sim\alpha, \sim\alpha'\rangle + -C\langle\sim\beta, \sim\beta'\rangle\\
&= -\langle\Diamond\sim\alpha, p \vee \sim p\rangle + -\langle\Diamond\sim\beta, p \vee \sim p\rangle\\
&\qquad\qquad = \langle\sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta, \sim(p \vee \sim p) \vee \sim(p \vee \sim p)\rangle,
\end{aligned}$$

which is not in $\mathfrak{D}$. Thus $\sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta$ is not satisfied by $\mathfrak{M}$, and hence is not provable in the Lewis system.

DEFINITION 2.3. A system S of sentential calculus is called *n-reducible* if, for every formula α which is not provable in S, there exists a formula β, which results from α by substitution, contains at most n variables, and is not provable in S either. A system is called *reducible* if there exists an n such that it is n-reducible.

Remark. If there is a finite characteristic matrix, with n elements, for a system S, then it is easily seen that S is n-reducible. On the other hand, a system can be n-reducible even though there exists no finite characteristic matrix for it: for example, if by formulas we mean only those built up from sentential variables by means of the connective "$\rightarrow$", and if S consists of just those formulas which have the form

$$\alpha \rightarrow \alpha,$$

then S is 2-reducible, but has no finite characteristic matrix. Thus the following theorem is seen to be stronger than the result, established by Dugundji in [1], that there is no finite characteristic matrix for the Lewis system.

THEOREM 2.4. The Lewis system is not reducible.

Proof. Suppose, if possible, that, for some n, the Lewis system is n-reducible. Then it can be shown, by an argument similar to that explained in Theorem 4 of McKinsey [8], that there is a regular characteristic matrix for the Lewis system which has n generators and only one designated element: such a matrix can be constructed by considering first the class of all formulas of the Lewis system which involve at most the n variables $p_1, \ldots, p_n$, and then identifying formulas α and β when $\alpha \equiv \beta$ is provable in the Lewis system. From 1.4 we then conclude that there is a functionally free closure algebra with n generators. Since this contradicts 5.6*, our theorem follows.

3. Extensions of the Lewis system. A system S' is called an *extension* of a system S, if every provable formula of S is also provable in S'.

It is clear that a system S can be an extension of the Lewis system without being closed under detachment. Moreover, we have:

THEOREM 3.1. Let S be the smallest system which is closed under detachment and has as axioms, in addition to the axioms of the Lewis system, all formulas of the form

$$\Diamond(\Diamond\alpha \rightarrow \sim\Diamond\sim\alpha) .$$

Then S does not satisfy Theorem 2.1: i.e., there is a formula α which is provable in S, while $\sim\Diamond\sim\alpha$ is not provable in S.

Proof. Let $\mathfrak{A}$ be the class of all subsets of the set $\{2, 3, 4, 5\}$. Let $\mathfrak{D}$ be those members of $\mathfrak{A}$ to which 5 belongs. If X and Y are any members of $\mathfrak{A}$, let $X \cdot Y$ be the intersection of X and Y, and let $-X$ be the complement of X with respect to $\{2, 3, 4, 5\}$. Let

$$\begin{aligned} C\{2\} &= \{2, 5\}, \\ C\{3\} &= \{3, 4, 5\} , \\ C\{4\} &= \{3, 4, 5\} , \\ C\{5\} &= \{5\} ; \end{aligned}$$

if X contains more than one member, let CX be the union of all sets CY, where Y is a one-element subset of X. Let $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{D}, \cdot, -, C \rangle$.

It is easily verified that the matrix $\mathfrak{M}$ is regular, and satisfies every axiom of S—and hence also every provable formula of S. On the other hand, $\mathfrak{M}$ does not satisfy the formula $\sim\Diamond\sim\Diamond(\Diamond p \rightarrow \sim\Diamond\sim p)$; for this formula fails if we replace p by $\{4\}$.

Remark. It is clear, however, that every reasonable extension of $S4$ must be closed under substitution. Moreover, from Lemmas 2 and 4 of McKinsey [9], it is seen that the most important extensions of $S4$ are closed under detachment and satisfy 2.1. We are therefore led to the following definition.

DEFINITION 3.2. An extension S of the Lewis system is called *normal* if it is closed under substitution and detachment, and satisfies the condition: whenever α is provable in S, then $\sim\Diamond\sim\alpha$ is provable in S.

THEOREM 3.3. Every normal extension of the Lewis system is closed under Lewis's four rules.

Proof. Similar to the proof of 1.1.

THEOREM 3.4. Let S be an extension of the Lewis system formed by adding new axioms all of which are of the form $\sim\Diamond\alpha$, and suppose S is closed under the rules of substitution and detachment. Then S is a normal extension of the Lewis system.

Proof. It suffices to show that, whenever α is provable in S, then $\sim\Diamond\sim\alpha$ is provable in S. This can be done by an induction on the number of steps in the proof of α.

THEOREM 3.5. The system $S5$ is a normal extension of the Lewis system.

Proof. By 3.4, since $S5$ can be obtained from our system by adding as axioms all formulas of the form $\Diamond\alpha \supset \sim\Diamond\sim\Diamond\alpha$.

THEOREM 3.6. Let S be a normal extension of the Lewis system. Then there exists a matrix $\mathfrak{M} = \langle A, \{d\}, \cdot, -, C\rangle$ which is a characteristic matrix for S, and such that $\Gamma = \langle A, \cdot, -, C\rangle$ is a closure algebra. Γ can, moreover, be supposed to be a homomorphic image of a subalgebra of the closure algebra over Euclidean space.

Proof. By a construction similar to that given in Theorem 4 of McKinsey [8], we can show that there exists a regular characteristic matrix $\mathfrak{M} = \langle A, D, \cdot, -, C\rangle$ for S. From the way in which $\mathfrak{M}$ is constructed it is easily seen that: (1) A contains at most $\aleph_0$ elements; (2) if x is in D, then $-C-x$ is in D; (3) if $x \supset y$ and $y \supset x$ are in D, then $x = y$.

From (2) and (3) above, we easily conclude that D contains only one element. For let x and y be in D. By (2) we see that $-C-x$ is in D. Since

$$\sim\Diamond\sim p \rightarrow (q \supset p)$$

is provable in the Lewis system[11] (and hence in S) we conclude that $-C-x \rightarrow (y \supset x)$ is in D. Since $\mathfrak{M}$ is regular, we therefore conclude that $y \supset x$ is in D. Similarly $x \supset y$ is in D. Hence, from (3), we see that $x = y$, as was to be shown.

Since $\mathfrak{M}$ is *a fortiori* a Lewis matrix, we see from 1.2 that Γ is a closure algebra.

Since A contains at most a countable infinity of generators, the closure algebra Γ is a homomorphic image of the free closure algebra with $\aleph_0$ generators, and hence, by 5.17*, a homomorphic image of a subalgebra of the closure algebra over Euclidean space.

Remark. Theorem 3.6 would no longer be true if we omitted from the definition of a normal extension the condition that $\sim\Diamond\sim\alpha$ is provable in S whenever α is provable in S; for a characteristic matrix for an S not satisfying this condition would clearly have to have at least two designated elements.

THEOREM 3.7. A necessary and sufficient condition that a formula α be provable in the Lewis system $S5$, is that $f^{(\alpha)}$ be identically 1 in every closure algebra in which every closed element is also open (and hence, every open element is also closed).

Proof. By 3.6 and 3.5, in view of the fact that $S5$ could also be obtained from the Lewis system by adding as axioms all formulas of the form $\Diamond\alpha \equiv \sim\Diamond\sim\Diamond\alpha$.

In order to prove our next theorem it is convenient first to formulate a lemma regarding general algebra.

LEMMA 3.8. Let $\mathfrak{A}$ be an equationally definable class of algebras. Then in order that there exist a finite functionally free $\mathfrak{A}$-algebra it is necessary and sufficient that, for some integer n: (i) there exist a functionally free $\mathfrak{A}$-algebra with n generators; and (ii) there be only a finite number of $\mathfrak{A}$-algebraic functions of n variables.

[11] By 19.75 and 14.1 of Lewis and Langford [5].

Proof. The conditions are obviously necessary. To see that they are also sufficient, we notice that (i) implies that the free $\mathfrak{A}$-algebra with n generators is functionally free (since every $\mathfrak{A}$-algebra with n generators is a homomorphic image of this one), and that (ii) implies that the free $\mathfrak{A}$-algebra with n generators is finite.

THEOREM 3.9. The Lewis system $S5$ is not reducible.

Proof. In order to establish our theorem it suffices, by 3.7, to show that, if $\mathfrak{A}$ is the class of closure algebras in which every closed element is also open, then there exists no functionally free $\mathfrak{A}$-algebra with a finite number of generators. Since Dugundji has shown (in [1]) that there is no finite characteristic matrix for $S5$—from which it follows that there is no finite functionally free $\mathfrak{A}$-algebra—we see from Lemma 3.8 that it suffices to show that there are only a finite number of $\mathfrak{A}$-algebraic functions of n variables.

From the definition of $\mathfrak{A}$-algebras we can easily show that each of the following equations is identically satisfied in every $\mathfrak{A}$-algebra:

$$-C-Cx = Cx\ ;$$

$$C-Cx = -Cx\ ;$$

$$C-C-x = -C-x\ ;$$

$$C[x \cdot Cy] = Cx \cdot Cy\ ;$$

$$C[x \cdot -Cy] = Cx \cdot -Cy\ .$$

From these equations we see that every $\mathfrak{A}$-algebraic function of n variables can be expressed as a product of expressions of the form

$$E_1 + CE_2 + -CE_3 + \ldots + -CE_r\,,$$

where each of the expressions $E_1, \ldots, E_r$ represents a Boolean-algebraic function of n variables (i.e., involves only the operations $\cdot$ and $-$). Since there are only 2^{2^n} Boolean-algebraic functions of n variables, it follows that there are only a finite number of $\mathfrak{A}$-algebraic functions of n variables, as was to be shown.

Remark. Theorem 2.2 is not true of the system $S5$, for the formula

$$\sim\Diamond\sim\sim p \vee \sim\Diamond\sim\Diamond p$$

is provable in $S5$, while neither $\sim p$ nor $\Diamond p$ is provable in $S5$. It may be wondered, however, whether 2.2 is true of some other normal extension of the Lewis system. This question is answered affirmatively by the following theorem.

THEOREM 3.10. Let S be the smallest system which is closed under detachment and has as axioms, in addition to the axioms of the Lewis system, all formulas of the form

$$\sim\Diamond\sim\Diamond(\Diamond\alpha \supset \alpha)\ .$$

Then S is a proper normal extension of the Lewis system which satisfies the condition: if $\sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta$ is provable in S, then either α or β is provable in S.

Proof. That S is a normal extension of the Lewis system follows immediately from 3.4. To show that S is not identical with the Lewis system, it suffices to notice that the equation

$$-C-C-C[CX \cdot -X] = 1$$

is not identically satisfied by the closure algebra over the Euclidean line; for if X is the set of points with rational coordinates, then $-C-C-C[CX \cdot -X]$ is the empty set.

To show that S satisfies the given condition, we make the construction described in the proof of 3.9*, and then argue as in the proof of 4.12* and 2.2.

4. The Heyting system and Brouwerian algebra. We shall use the same variables, and the same names for them, in the Heyting system as in the Lewis system. There are four constants in the Heyting system: the negation sign, the conjunction sign, the disjunction sign, and the implication sign. The expression formed from an expression α by putting a negation sign in front of it is called the negation of α—in symbols, $\sim\alpha$. The expression formed from the two expressions α and β by putting a conjunction sign (or a disjunction sign, or an implication sign) between them, and enclosing the whole in parentheses, is called the conjunction (or disjunction, or implication, respectively) of α and β—in symbols, $\alpha \wedge \beta$ (or $\alpha \vee \beta$, or $\alpha \rightarrow \beta$, respectively).

The class of *formulas* is the smallest class of expressions which contains all the variables and is closed under the operations of forming negations, conjunctions, disjunctions, and implications. We define the *index* of a formula as in §1.

We write $\alpha \leftrightarrow \beta$ as an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

A formula is called an *axiom* of the Heyting calculus if it is of one of the eleven following kinds (where α, β, and γ are arbitrary formulas):

*H*1. $\alpha \rightarrow (\alpha \wedge \alpha)$.
*H*2. $(\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)$.
*H*3. $(\alpha \rightarrow \beta) \rightarrow [(\alpha \wedge \gamma) \rightarrow (\beta \wedge \gamma)]$.
*H*4. $[(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma)] \rightarrow (\alpha \rightarrow \gamma)$.
*H*5. $\beta \rightarrow (\alpha \rightarrow \beta)$.
*H*6. $[\alpha \wedge (\alpha \rightarrow \beta)] \rightarrow \beta$.
*H*7. $\alpha \rightarrow (\alpha \vee \beta)$.
*H*8. $(\alpha \vee \beta) \rightarrow (\beta \vee \alpha)$.
*H*9. $[(\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)] \rightarrow [(\alpha \vee \beta) \rightarrow \gamma]$.
*H*10. $\sim\alpha \rightarrow (\alpha \rightarrow \beta)$.
*H*11. $[(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \sim\beta)] \rightarrow \sim\alpha$.

We define *derivability* and *provability*, as in the Lewis system, in terms of the operation of detachment.

By a *matrix* for the Heyting calculus we mean a system $\mathfrak{M} = \langle A, D, \cdot, +, \div, \neg \rangle$, where A and D are sets, $\cdot$, $+$, and $\div$ are binary operations defined over A and $\neg$ is a unary operation defined over A; A is supposed to include D, and to be closed under the operations $\cdot$, $+$, $\div$, and $\neg$. The matrix $\mathfrak{M}$ is called *regular* if, whenever x and $x \div y$ are in D, then y is in D.

We assume it to be known under what circumstances a formula α is said to be satisfied by such a matrix $\mathfrak{M}$ (here the operations $\cdot$, $+$, $\div$, and $\neg$ correspond respectively to the constants $\wedge$, $\vee$, $\rightarrow$, and $\sim$). A matrix which satisfies every provable formula of the Heyting calculus is called a *Heyting matrix*; a *characteristic* Heyting matrix satisfies a formula if and only if it is provable in the Heyting calculus.

The proof of the following theorem presents no essential difficulties:[12]

THEOREM 4.1. The matrix $\mathfrak{M} = \langle A, \{d\}, \cdot, +, \div, \neg \rangle$ is a regular Heyting matrix if and only if A is a Brouwerian algebra with respect to $\cdot$, $+$, $\neg$, and the operation $\dot{-}$ defined by the equation $x \dot{-} y = y \div x$; in this Brouwerian algebra, is to be considered as sum, $+$ as product, and the element d as the zero-element.

THEOREM 4.2. If the Brouwerian algebra of Theorem 4.1 is dissectable,[13] then the matrix of Theorem 4.1 is a characteristic matrix for the Heyting calculus. Thus, in particular, we obtain a characteristic matrix $\mathfrak{M}$ for the Heyting calculus if we set $\mathfrak{M} = \langle \mathfrak{A}, \{\Lambda\}, \cup, \cap, \div, \neg \rangle$, where $\mathfrak{A}$ is the set of all closed subsets of Euclidean space, Λ is the empty set, $\cup$ and $\cap$ mean union and intersection, and $\div$ and $\neg$ are defined by the equations:

$$X \div Y = C(-X \cap Y) ;$$

$$\neg X = C - X .$$

Dually, we obtain a characteristic matrix for the Heyting calculus if we set $\mathfrak{M} = \langle \mathfrak{B}, \{E\}, \cap, \cup, \div, \neg \rangle$, where $\mathfrak{B}$ is the set of all open subsets of Euclidean space, E is the set of all points of Euclidean space, $\cap$ and $\cup$ mean intersection and union, and $\div$ and $\neg$ are defined by the equations:

$$X \div Y = I(-X \cup Y) ;$$

$$\neg X = I - X .$$

Proof. Similar to the proof of Theorem 1.3, but using 3.24** instead of 5.9*.

Just as, in §1, we correlated closure-algebraic functions with formulas of the Lewis calculus, so we can correlate with each formula α of the Heyting calculus a Brouwerian-algebraic function $f^{(\alpha)}$. If $\alpha = v_n$ for some n, then $f^{(\alpha)}$ is the function determined by the equation

$$f^{(\alpha)}(x_1, \ldots, x_n) = x_n$$

(for all elements $x_1, \ldots, x_n$ of every Brouwerian algebra). If α is a formula of

[12] See 1.1** of McKinsey and Tarski [11], for a definition of Brouwerian algebras. In order to prove the theorem it is convenient to make use of Theorem 1.4** of McKinsey and Tarski [11].

[13] This term is defined in 3.3**.

index m, then $f^{(\sim\alpha)}$ is the function defined by the equation:

$$f^{(\sim\alpha)}(x_1, \ldots, x_m) = \neg f(x_1, \ldots, x_m) .$$

If α is a formula of index m, β is a formula of index n, and $r = \max(m, n)$, then $f^{(\alpha \wedge \beta)}, f^{(\alpha \vee \beta)}$, and $f^{(\alpha \to \beta)}$ are the functions defined by the equations:

$$f^{(\alpha \wedge \beta)}(x_1, \ldots, x_r) = f^{(\alpha)}(x_1, \ldots, x_m) + f^{(\beta)}(x_1, \ldots, x_n) ;$$

$$f^{(\alpha \vee \beta)}(x_1, \ldots, x_r) = f^{(\alpha)}(x_1, \ldots, x_m) \cdot f^{(\beta)}(x_1, \ldots, x_n) ;$$

$$f^{(\alpha \to \beta)}(x_1, \ldots, x_r) = f^{(\beta)}(x_1, \ldots, x_m) \dot{-} f^{(\alpha)}(x_1, \ldots, x_n) .$$

By means of this correlation, the results stated in Theorems 4.1 and 4.2 can now be put in the following form:

THEOREM 4.3. For every formula α of the Heyting calculus the two following conditions are equivalent:

(i) α is provable in the Heyting calculus;

(ii) $f^{(\alpha)}$ vanishes identically in every Brouwerian algebra.

Moreover, if $\mathfrak{A} = \langle A, +, \cdot, \dot{-}, \neg \rangle$ is any dissectable Brouwerian algebra, then each of the above conditions is equivalent to the following:

(iii) $f^{(\alpha)}$ vanishes identically in $\mathfrak{A}$.

We come now to some theorems asserting internal properties of the Heyting calculus. Our first theorem was stated without proof by Gödel.[14]

THEOREM 4.4. If $\alpha \vee \beta$ is provable in the Heyting calculus, then either α is provable in the Heyting calculus, or β is provable in the Heyting calculus.

Proof. Similar to the proof of 2.2, but using 2.25** instead of 4.12*, and 4.3 instead of 1.4.

Our next theorem is equivalent to 2.27**, the proof of which is due to Gödel.

THEOREM 4.5. There exist in the Heyting calculus infinitely many non-equivalent formulas involving only one variable.

Proof. By 2.27** and 4.3.

Remark. An examination of the functions used in the proof of 2.27** reveals that we could actually state a stronger theorem: namely, that there are infinitely many non-equivalent formulas which involve only one variable and only the signs "$\to$", "$\vee$", and "$\sim$" (the sign "$\sim$" is involved because the operation $\neg$ is introduced implicitly when we write "$f(x, 1)$").

On the other hand, there are only six non-equivalent formulas involving only one variable and only the signs "$\sim$" and "$\to$":

$$p,\ \sim p,\ \sim\sim p,\ p \to p,\ \sim(p \to p),\ \sim\sim p \to p.$$

There are only five non-equivalent formulas involving only one variable and only the signs "$\wedge$" and "$\sim$":

[14] In Gödel [2].

$$p,\ \sim p,\ \sim\sim p,\ p \wedge \sim p,\ \sim(p \wedge \sim p).$$

There are only seven non-equivalent formulas involving only the signs "$\wedge$", "$\vee$", and "$\sim$":

$$p,\ \sim p,\ \sim\sim p,\ p \wedge \sim p,\ \sim(p \wedge \sim p),\ p \vee \sim p,\ \sim p \vee \sim\sim p.$$

From the last fact, together with 4.5, we can conclude that "$\rightarrow$" is not definable in terms of "$\wedge$", "$\vee$", and "$\sim$".[15]

The following theorem constitutes an improvement on the result established in Gödel [3].

THEOREM 4.6. The Heyting calculus is not reducible.

Proof. Similar to the proof of 2.4, but using 4.9** instead of 5.6*.

5. Relations between the Heyting calculus and the Lewis system. In this section we shall prove three theorems[16] which provide methods of "translating" the Heyting calculus into the Lewis system. In view of the decision method for the Lewis system given in McKinsey [8], each of these theorems provides a new decision method for the Heyting calculus.

THEOREM 5.1. Let T be a function defined over all formulas of the Heyting calculus, assuming as values formulas of the Lewis calculus, and satisfying the following conditions (where v_i is any sentential variable, and α and β are arbitrary formulas):

(i) $T(v_i) = \sim\Diamond\sim v_i$;
(ii) $T(\alpha \vee \beta) = T(\alpha) \vee T(\beta)$;
(iii) $T(\alpha \wedge \beta) = T(\alpha) \wedge T(\beta)$;
(iv) $T(\alpha \rightarrow \beta) = T(\alpha) \supset T(\beta)$;
(v) $T(\sim\alpha) = \sim\Diamond T(\alpha)$.

Then, for any formula α of the Heyting calculus, α is provable in the Heyting calculus if and only if $T(\alpha)$ is provable in the Lewis system.

Proof. Let α be any formula of the Heyting calculus; suppose that α is of index n. By Theorem 1.4 we see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(T(\alpha))}(x_1, \ldots, x_n) = 1 \tag{1}$$

is true for all elements $x_1, \ldots, x_n$ of every closure algebra. By condition (i) of the hypothesis of our theorem, it is then seen that $T(\alpha)$ is provable in the Lewis system if and only if (1) is true for all open elements of every closure algebra. By means of conditions (ii)-(v) of the hypothesis of our theorem, the principle of duality for closure algebras, and 1.14** and 1.15**, we then see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(\alpha)}(x_1, \ldots, x_n) = 0$$

[15] For a proof that no one of the symbols "$\sim$", "$\vee$", "$\wedge$", and "$\rightarrow$" is definable in terms of the others, see McKinsey [7] or Wajsberg [14].

[16] Theorems 5.2 and 5.3 were conjectured by Gödel in [2].

is true for all elements of every Brouwerian algebra. Our theorem now follows by Theorem 4.3.

THEOREM 5.2. Let T' be a function defined over all formulas of the Heyting calculus, assuming as values formulas of the Lewis system, and satisfying the following conditions (for every sentential variable v_i, and for arbitrary formulas α and β):

(i) $T'(v_i) = v_i$;

(ii) $T'(\alpha \vee \beta) = \sim\Diamond\sim T'(\alpha) \vee \sim\Diamond\sim T'(\beta)$;

(iii) $T'(\alpha \wedge \beta) = T'(\alpha) \wedge T'(\beta)$;

(iv) $T'(\alpha \rightarrow \beta) = \sim\Diamond\sim T'(\alpha) \rightarrow \sim\Diamond\sim T'(\beta)$;

(v) $T'(\sim\alpha) = \Diamond\sim T'(\alpha)$.

Then, for any formula α of the Heyting calculus, α is provable in the Heyting calculus if and only if $T'(\alpha)$ is provable in the Lewis system.

Proof. It is easily shown, by an induction on the number of symbols occurring in α, that, for α any formula of the Heyting calculus, the formula

$$\sim\Diamond\sim T'(\alpha) \equiv T(\alpha)$$

(where T is the function introduced in 5.1) is provable in the Lewis system. Hence, by 5.1 and 2.1, the theorem.

THEOREM 5.3. Let T'' be a function defined over all formulas of the Heyting calculus, assuming as values formulas of the Lewis system, and satisfying the following conditions (for v_i any sentential variable, and α and β arbitrary formulas):

(i) $T''(v_i) = v_i$;

(ii) $T''(\alpha \vee \beta) = \sim\Diamond\sim T''(\alpha) \vee \sim\Diamond\sim T''(\beta)$;

(iii) $T''(\alpha \wedge \beta) = \sim\Diamond\sim T''(\alpha) \wedge \sim\Diamond\sim T''(\beta)$;

(iv) $T''(\alpha \rightarrow \beta) = \sim\Diamond\sim T''(\alpha) \rightarrow \sim\Diamond\sim T''(\beta)$;

(v) $T''(\sim\alpha) = \sim\Diamond\sim\Diamond\sim T''(\alpha)$.

Then, for any formula α of the Heyting calculus, α is provable in the Heyting calculus if and only if $T''(\alpha)$ is provable in the Lewis system.

Proof. Similar to the proof of 5.2.

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