Example: Finding the Optimal Combination of Goods, to Maximize Utility, Subject to a Budget Constraint*

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Abstract: In this example we find the optimal combination of two goods to maximize a representative consumer's utility (satisfaction), subject to their budget constraint.

The Problem:

Suppose the representative consumer is given a choice to purchase 2 goods, C_1 and C_2. The representative consumer's utility function can be written as $U(C_1, C_2)$. In this case, the consumer values both goods the same. The consumer has a budget of $M = \$500$. The price of good 1 is equal to \$5 and the price of good 2 is equal to \$10. Therefore, the consumer's budget constraint can be written as $M = P_1C_1 + P_2C_2$.

1. Choose a quantity (C_1, C_2) to
2. Maximize Utility $U(C_1, C_2)$
3. Subject to $M = P_1C_1 + P_2C_2$

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Solution:

To find the optimal combination of goods we use the Lagrangian function. The Lagrangian function states that:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

Our application of the Lagrangian function states that:

$$L(C_1, C_2, \lambda) = U(C_1, C_2) + \lambda(M - P_1 C_1 - P_2 C_2)$$

Here, we re-write the budget constraint as $M - P_1 C_1 - P_2 C_2 = 0$, for simplicity. We use the variable λ (Lagrange Multiplier) which allows us to find the maximum of this multivariable function.

For our example, the Lagrangian function is:

$$L(C_1, C_2, \lambda) = U(C_1, C_2) + \lambda(500 - 5C_1 - 10C_2)$$

Next, we distribute λ:

$$L(C_1, C_2, \lambda) = U(C_1, C_2) + \lambda 500 - \lambda 5C_1 - \lambda 10C_2$$

From here we can take the partial derivatives of the function L with respect to $C_1, C_2,$ $and\ \lambda$ denoted by ∂L. First, we will take the partial derivative of function L with respect to C_1:

$$\frac{\partial L}{\partial C_1} = C_1 C_2 + \lambda 500 - \lambda 5C_1 - \lambda 10C_2$$

$$= C_1^0 C_2 - \lambda 5C_1^0$$

$$= C_2 - \lambda 5$$

Since $\lambda 500$ and $\lambda 10C_2$ do not have C_1, they are constants. Therefore, using the constant rule, we can eliminate them. From there we use the power rule to find the rest of the derivative.

Next, we will take the partial derivative of the function L with respect to C_2:

$$\frac{\partial L}{\partial C_2} = C_1 C_2 + \lambda 500 - \lambda 5C_1 - \lambda 10C_2$$

$$= C_1 C_2^0 - \lambda 10C_2^0$$

$$= C_1 - \lambda 10$$

Here we apply the same principles as above, but with respect to C_2. Since $\lambda 500\ and\ \lambda 5C_1$ do not have C_2, they are constants and can be eliminated using to constant rule. From there we use the power rule to find the rest of the derivative.

Next, we take the partial derivative of the function L with respect to λ:

$$\frac{\partial L}{\partial \lambda} = C_1C_2 + \lambda 500 - \lambda 5C_1 - \lambda 10C_2$$

$$= \lambda^0 500 - \lambda^0 5C_1 - \lambda^0 10C_2$$

$$= 500 - 5C_1 - 10C_2$$

Since C_1C_2 does not have λ it is a constant and can be eliminated using the constant rule. From there we use the power rule to find the rest of the derivative.

Now we have 3 expressions.

1. $C_2 - \lambda 5$
2. $C_1 - \lambda 10$
3. $500 - 5C_1 - 10C_2$

Next, we find the ratio of $C_1 and\ C_2$. To start, we must set the Lagrange Multiplier equal to each expression.

$$\lambda = \frac{C_2}{5}$$

$$\lambda = \frac{C_1}{10}$$

Next, since both functions are set equal to the Lagrange Multiplier, we can set them equal to each other and solve for the ratio.

$$\frac{C_2}{5} = \frac{C_1}{10}$$

$$\rightarrow 5C_1 = 10C_2$$

$$\rightarrow C_1 = 2C_2$$

Here we can simply cross multiply the two fractions and divide both sides by 5.

Now that we know the ratio of C_1 to C_2, we can set our partial derivative of function L with respect to λ to zero and plug our ratio to find the value of C_2:

$$500 - 5C_1 - 10C_2 = 0$$

$$\rightarrow 500 - 5(2C_2) - 10C_2 = 0$$

$$\rightarrow 500 - 10C_2 - 10C_2 = 0$$

$$\rightarrow 500 - 20C_2 = 0$$

$$\rightarrow 500 = 20C_2$$

$$\rightarrow 25 = C_2$$

Finally, since we know the value of C_2, we can find the value of C_1 by plugging in our value of C_2 into our ratio:

$$C_1 = 2C_2$$

$$C_1 = 2(25)$$

$$C_1 = 50$$

So, the representative consumer's optimal combination of goods, that maximizes their utility, subject to their budget, is 50 units of C_1, and 25 units of C_2.