Calculus II

Notes from TAU Course with Additional Information
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1 Functional Analysis

1.1 Sequence of Functions

Definition 1.1.1 (Supremum)**.** *For a set $A \subset \mathbb{R}$, if A is bounded from above, the supremum of A is the lowest upper bound (denoted* $\sup A$*). Otherwise* $\sup A = \infty$*. That is, $M = \sup(A)$ iff $\forall\, a \in A\,,\, a \leq M$.*

Definition 1.1.2 (Sequence of Functions)**.** *A sequence of function is a family $\{f_n\}_{n\in\mathbb{N}}$ where $\forall\, n \in \mathbb{N}\,,\, f_n : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$. Observe, the interval $\mathcal{I}$ is the same domain for all f_n in the sequence.*

Definition 1.1.3 (Pointwise Convergence)**.** *For a sequence $\{f_n\}_{n\in\mathbb{N}}$, we say f_n converges pointwise to f (denoted $f_n \to f$) if, $\forall\, x_0 \in \mathcal{I}\,,\, f_n(x_0) \to f(x_0)$. That is, if f is defined explicitly $\forall\, x_0 \in \mathcal{I}\,,\, f(x_0) := \lim\limits_{n\to\infty} f_n(x_0)$.*

Remark 1.1.4. *The pointwise limit is unique, since $\lim\limits_{n\to\infty} f_n(x_0)$ is unique.*

Remark 1.1.5. *That pointwise limit of continuous functions can be discontinuous. For illustration, take $f_n : [0, 1] \to \mathbb{R}$ where $f_n(x) = x^n$. Then, $f_n(x) \to f(x)$ where:*

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

which is then discontinuous.

Definition 1.1.6 (Uniform Convergence)**.** *For a sequence $\{f_n\}_{n\in\mathbb{N}}$, we say f_n converges pointwise to f (denoted $f_n \xrightarrow{u} f$) if,*

$$\forall\, \epsilon > 0\,,\, \exists\, N \in \mathbb{N} : \forall\, n \geq N\,,\, \forall\, x \in \mathcal{I}\,,\, |f(x) - f_n(x)| < \epsilon$$

Lemma 1.1.7. *(UC $\Rightarrow$ PC) If $f_n \xrightarrow{u} f$, then $f_n \to f$.*

Proof. If $f_n \xrightarrow{u} f$, then $\forall\, x \in \mathcal{I}\,,\, f(x) = \lim\limits_{n\to\infty} f_n(x)$, by definition. $\square$

Definition 1.1.8 (Vector Space of Functions)**.** $\{f \mid f : \mathcal{I} \to \mathbb{R}\}$ *is a vector space over $\mathbb{R}$ with pointwise addition and scalar multiplication: $(f + g)(x) = f(x) + g(x)$ and $(\alpha \cdot f) = \alpha \cdot f(x)$*

Definition 1.1.9 (Uniform Norm)**.** *We define the following norm for functions $f : \mathcal{I} \to \mathbb{R}$:*

$$\|f\|_\infty = \sup_{x\in\mathcal{I}} |f(x)|$$

which we can check is a norm. Also, f is bounded iff $\|f\|_\infty < \infty$.

Remark 1.1.10. *The idea of using $\|\cdot\|_\infty$ is to bound independent of x, since $\|f-g\|_\infty$ only depends on f and g. We can substitute: $\|f-g\|_\infty \leq \epsilon \Leftrightarrow \forall\, x \in \mathcal{I}\,,\, |f(x) - g(x)| \leq \epsilon$ (cf. 1.1.1).*

Remark 1.1.11 (Banach Algebra)**.** $\forall\, f, g : \mathcal{I} \to \mathbb{R}\,,\, \|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$, *where $\cdot$ is pointwise multiplication.*

Lemma 1.1.12. *$f_n \xrightarrow{u} f$ iff $\|f - f_n\|_\infty \to 0$*

Proof. We prove each direction:

($\Rightarrow$) $\forall\, \epsilon > 0\,,\, \exists\, N \in \mathbb{N} : \forall\, n \geq N\,,\, \forall\, x \in \mathcal{I}\,,\, |f(x) - f_n(x)| < \epsilon/2$ (cf. 1.1.3). Taking the supremum on $x \in \mathcal{I}$, $\forall\, n \geq N\,,\, \|f - f_n\|_\infty \leq \epsilon/2 < \epsilon$. That is, $\|f - f_n\|_\infty \to 0$ by definition.

($\Leftarrow$) $\|f - f_n\|_\infty \to 0 \Leftrightarrow \forall\, \epsilon > 0\,,\, \exists\, n \in \mathbb{N} : \forall\, n \geq N\,,\, \|f - f_n\|_\infty < \epsilon$. Then, $\forall\, n \geq N\,,\, \forall\, x \in \mathcal{I}\,,\, |f(x) - f_n(x)| < \epsilon$ (cf. 1.1.1, 1.1.10).

Hence, $\|\cdot\|_\infty$ is the norm that defines uniform continuity. $\square$

Lemma 1.1.13. *If $f_n \to f$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly iff $f_n \xrightarrow{u} f$.*

Proof. We prove each direction:

($\Rightarrow$) If $f_n \xrightarrow{u} g$, by 1.1.7 $f_n \to g$ and by 1.1.4, $g = f$.

($\Leftarrow$) If $f_n \xrightarrow{u} f$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly.

Hence, if $f_n \to f$, $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly iff $\lim\limits_{n\to\infty} \|f - f_n\|_\infty = 0$. $\square$

Remark 1.1.14. *If we change our domain, we way have UC. Going back to the example of $f_n(x) = x^n$, we get $\{f_n\}_{n\in\mathbb{N}} \xrightarrow{u} f \equiv 0$ in $\mathcal{I} = [0, t]$ for $t < 1$ since $\|f - f_n\|_\infty = \sup\limits_{x\in\mathcal{I}} |x|^n = t^n \to 0$.*

Lemma 1.1.15 (Bounded Limit)**.** *If $f_n \xrightarrow{u} f$ and $\forall\, n \in \mathbb{N}\,,\, f_n$ is bounded, then f is bounded.*

Proof. By definiton of uniform limit (cf. 1.1.6) $\exists\, N \in \mathbb{N} : \|f - f_N\|_\infty < 1$. By the triangle inequality: $\|f\|_\infty \leq \|f - f_N\|_\infty + \|f_N\|_\infty < 1 + \|f_N\|_\infty < \infty$ $\square$

Theorem 1.1.16 (Uniform Limit)**.** *Every uniformly convergent sequence of continuous, the limit is continuous.*

Proof. Let $f_n \xrightarrow{u} f$. For any $\epsilon > 0$, let $N \in \mathbb{N}$ s.t. $\|f - f_N\|_\infty < \epsilon/3$, that is, $\forall\, x \in \mathcal{I}\,,\ |f(x) - f_N(x)| < \epsilon/3$. Since f_N is continuous, $\forall\, a \in \mathcal{I}\,,\ \exists\, \delta > 0 :$ $\forall\, x \in (a - \delta, a + \delta) \subseteq \mathcal{I}\,,\ |f_N(x) - f_N(a)| < \epsilon/3$. Putting all the terms together, and using triangle inequality:

$$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| < \epsilon$$

Hence, $\forall\, a \in \mathcal{I}\,,\ f$ is continuous at a. Therefore, f is a continuous on $\mathcal{I}$. $\square$

Remark 1.1.17. *Defining the set of:*

- *Bounded Functions on $\mathcal{I}$, $B(\mathcal{I})$*
- *Continuous Functions on $\mathcal{I}$, $C(\mathcal{I})$*

Then, 1.1.16 and 1.1.15 imply $B(\mathcal{I})$ and $C(\mathcal{I})$ are closed under limits.

Theorem 1.1.18 ($B(\mathcal{I})$ and $C(\mathcal{I})$ are complete)**.** *If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{I}$, that is,*

$$\forall\, \epsilon > 0\,,\ \exists\, N \in \mathbb{N} : \forall\, m, n > N\,,\ \|f_m - f_n\|_\infty < \epsilon$$

then, $\exists\, f : \mathcal{I} \to \mathbb{R} : f_n \xrightarrow{u} f$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. As in 1.1.10,

$$\forall\, \epsilon > 0\,,\ \exists\, N \in \mathbb{N} : \forall\, m, n > N\,,\ \forall\, x \in \mathcal{I}\,,\ |f_m(x) - f_n(x)| < \epsilon$$

Therefore, for each $x \in \mathcal{I}$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Hence, since $\mathbb{R}$ is complete, each sequence converges to some $f(x)$. We define the pointwise limit $f(x) := \lim\limits_{n\to\infty} f_n(x)$, so it converges pointwise, which is necessary. Lastly, we need to prove $f_n \xrightarrow{u} f$. By the continuity of absolute value, we have $|f(x) - f_n(x)| = \lim\limits_{m\to\infty} |f_m(x) - f_n(x)|$. Since $\forall\, \epsilon > 0\,,\ \exists\, N \in \mathbb{N}$: $\forall\, m, n > N\,,\ \forall\, x \in \mathcal{I}\,,\ |f_m(x) - f_n(x)| < \epsilon/2$, we may take $m \to \infty$:

$$\forall\, \epsilon > 0\,,\ \exists\, N \in \mathbb{N} : \forall\, n > N\,,\ \forall\, x \in \mathcal{I}\,,\ |f(x) - f_n(x)| \leq \epsilon/2$$

So that, $\|f - f_n\|_\infty \leq \epsilon/2 < \epsilon$ (cf. 1.1.1, 1.1.10), hence, it converges uniformly. $\square$

Theorem 1.1.19 (Convergence of Integral)**.** *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of continuous functions in* $[a,b]$*. Suppose* $f_n \xrightarrow{u} f$ *in* $[a,b]$*. then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx$$

which is defined, cf. 1.1.16.

Proof. Calculating: $\left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| = \left|\int_a^b \Big[f(x) - f_n(x)\Big]\,dx\right| \leq \int_a^b \big|f(x) - f_n(x)\big|\,dx \leq \|f - f_n\|_\infty \cdot (b-a) \to 0$ □

Remark 1.1.20. *Uniform limit in 1.1.19 is necessary. For example, take* $f_n : [0,1] \to \mathbb{R}$ *where:* $f_n(x) = \begin{cases} n & \textit{if } 0 < x < \dfrac{1}{n} \\ 0 & \textit{otherwise} \end{cases}$*. We get:* $f_n \to f \equiv 0$*, but* $\int_0^1 f_n(x)\,dx = 1 \not\to 0$*.*

Theorem 1.1.21 (UC of Derivative)**.** *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of differentiable functions in* $\mathcal{I}$*. Suppose* $f_n \to f$ *(pointwise) and* $f_n' \xrightarrow{u} g$ *in* $\mathcal{I}$*. then* f *is differentiable and* $f' = g$*.*

Proof. By FTC II (Newton-Leibnitz), $f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt$ for some $a \in \mathcal{I}$, taking limit of both sides, for a fixed $x \in \mathcal{I}$, we get:

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(a) + \lim_{n\to\infty}\int_a^x f_n(t)\,dt = f(a) + \int_a^x g(t)\,dt$$

the last equality by 1.1.19. Hence, by FTC I, $f' = g$. □

1.2 Series of Functions

Definition 1.2.1. *Let $f_n : \mathcal{I} \to \mathbb{R}$, we define:*

1. *$\sum_{n=1}^{\infty} f_n$ converges pointwise iff $\{\sum_{k=1}^{n} f_k\}_{n\in\mathbb{N}}$ converges pointwise, i.e. $\forall\, x_0 \in \mathcal{I}\,,\ \sum_{n=1}^{\infty} f_n(x_0)$ converges (cf. 1.1.3).*

2. *$\sum_{n=1}^{\infty} f_n$ converges uniformly iff $\{\sum_{k=1}^{n} f_k\}_{n\in\mathbb{N}}$ converges uniformly (cf. 1.1.6).*

Lemma 1.2.2. *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly in $\mathcal{I}$ iff it converges pointwise and $\lim\limits_{n\to\infty} \sup\limits_{x\in\mathcal{I}} |\sum_{k=n}^{\infty} f_k(x)| = 0$*

Proof. Let $S_n = \sum_{k=1}^{n} f_k$, the partial sums. By definition (cf. 1.2.1), $\sum_{n=1}^{\infty} f_n$ converges uniformly iff $\{S_n\}_{n\in\mathbb{N}}$ converges uniformly. It converges uniformly to S, if it converges pointwise to S and $\|S - S_n\|_\infty \to 0$ (cf. 1.1.12,1.1.7). Then, $S_n \to S : x \mapsto \sum_{k=1}^{\infty} f_k(x)$. It is N&S $\|S - S_n\|_\infty \to 0$, that is, $\lim\limits_{n\to\infty} \|S - S_n\|_\infty = \lim\limits_{n\to\infty} \sup\limits_{x\in\mathcal{I}} \left|\sum_{k=n+1}^{\infty} f_k(x)\right| = 0$. □

Theorem 1.2.3 (AbsC ⇒ UC of Series)**.** *If $\sum_{n=1}^{\infty} \|f_n\|_\infty$ converges (absolutely), then $\sum_{n=1}^{\infty} f_n$ converges uniformly.*

Proof. Let $S_n = \sum_{k=1}^{n} f_k$. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} \|f_n\|_\infty$ converges, $\exists\, N \in \mathbb{N} : \forall\, m > n \geq N\,,\ \sum_{k=n+1}^{m} \|f_k\|_\infty < \epsilon$. Then, we get directly by triangle inequality: $\forall\, m > n \geq N\,,\ \forall\, x \in \mathcal{I}\,,$

$$|S_m(x) - S_n(x)| = \left|\sum_{k=n+1}^{m} f_k(x)\right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} \|f_k\|_\infty < \epsilon$$

Hence, $\sum_{n=1}^{\infty} f_n$ converges uniformly by Cauchy (cf. 1.1.18). □

Corollary 1.2.4 (Weierstrass M-test)**.** *Let $f_n : \mathcal{I} \to \mathbb{R}$ be a sequence of functions. Suppose there is a (non-negative) sequence $\{M_n\}_{n\in\mathbb{N}}$ such that:*

1. *$\forall\, n \in \mathbb{N}\,,\ \forall\, x \in \mathcal{I}\,,\ |f_n(x)| \leq M_n$, that is, $\forall\, n \in \mathbb{N}\,,\ \|f_n\|_\infty \leq M_n$*

2. *$\sum_{n=1}^{\infty} M_n$ converges (absolutely).*

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Proof. By comparision test of $\{M_n\}_{n\in\mathbb{N}}$ with $\{\|f_n\|_\infty\}_{n\in\mathbb{N}}$, if $\sum_{n=1}^{\infty} M_n$ converges, $\sum_{n=1}^{\infty} \|f_n\|_\infty$ converges. By 1.2.3, $\sum_{n=1}^{\infty} f_n$ converges uniformly. □

Lemma 1.2.5. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions in $[a,b]$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly, then*

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

Proof. By linearity of the integral, $\int_a^b \sum_{k=1}^{n} f_k(x)\,dx = \sum_{k=1}^{n} \int_a^b f_k(x)\,dx$. Taking the limit of both sides, it follows from 1.1.19 and the definition (cf. 1.2.1). □

Lemma 1.2.6. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable functions in $[a,b]$. If $\sum_{n=1}^{\infty} f_n$ converges pointwise and $\sum_{n=1}^{\infty} f_n'$ converges uniformly, then*

$$\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

Proof. By linearity of the derivative, $(\sum_{k=1}^{n} f_k)' = \sum_{k=1}^{n} f_k'$. Taking the limit of both sides, it follows from 1.1.21 and the definition (cf. 1.2.1). □

1.3 Power Series

Definition 1.3.1 (Power Series)**.** *Given a sequence of real numbers* $\{a_n\}_{n\in\mathbb{N}}$ *and* $a \in \mathbb{R}$*, its power series is the series of functions* $f_n(x) = a_n(x-a)^n$ *for* $n \in \mathbb{N}_0$*. That is, the power series is:*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

where the left hand side converges uniformly on some interval $\mathcal{I}$*. Of course, it converges pointwise at* $x = a$*.*

Lemma 1.3.2. *If* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges at* $x = x_0$*, then, it converges uniformly in* $(a-r, a+r)$ *for any* $r < |x_0 - a|$*.*

Proof. Let $\mathcal{I} = (a-r, a+r)$. We calculate: $\|f_n\|_\infty = \sup\limits_{x\in\mathcal{I}} |a_n(x-a)^n| = |a_n|r^n$. Since $\sum_{n=0}^{\infty} a_n(x_0-a)^n$ converges, $\{a_n(x_0-a)^n\}_{n\in\mathbb{N}}$ is bounded (by M). Hence, $\|f_n\|_\infty \leq M\big(\frac{r}{|x_0-a|}\big)^n$. Since $\frac{r}{|x_0-a|} < 1$, it follows the series of f_n converges uniformly by Weierstrass M-test (cf. 1.2.4). □

Corollary 1.3.3. *If* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges at* x_0*, the pointwise limit (which exists by 1.3.2 and 1.1.7 taking* $|x-a| < r < |x_0-a|$*) is continuous at* $(a - |x_0-a|, |x_0-a|)$*.*

Definition 1.3.4 (Radius of Convergence)**.** *R is a radius of convergence of* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *iff, for any given* $x \in \mathbb{R}$

$\forall\, x \in (a-R, a+R)\,,\ \sum_{n=0}^{\infty} a_n(x-a)^n$ *converges*
$\forall\, x \notin [a-R, a+R]\,,\ \sum_{n=0}^{\infty} a_n(x-a)^n$ *diverges*

Lemma 1.3.5 (Cauchy Hadamard Formula)**.** *Given a sequence of real numbers* $\{a_n\}_{n\in\mathbb{N}}$*, the radius of convergence (cf. 1.3.4) satisfies:*

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

if $\limsup \sqrt[n]{|a_n|} = 0$*, then* $R = \infty$
if $\limsup \sqrt[n]{|a_n|} = \infty$*, then* $R = 0$

Proof. It is a direct result of Cauchy's Criteria (Root Test), we get the formula: $|x-a| \cdot \dfrac{1}{R} < 1$. The second proposition is the contrapositive of the divergence criteria. □

Remark 1.3.6. *The radius of convergence only shows pointwise convergence. Moreover, we have to check the endpoints* $x = a \pm R$ *separately.*

Corollary 1.3.7. *By 1.3.2, for any integral* $\mathcal{I} \subsetneq (a - R, a + R)$*, the power series* $\sum_{n=0}^{\infty} a_n(x - a)^n$ *converges uniformly in* $\mathcal{I}$*.*

Remark 1.3.8. *In general, nothing can be said about uniform convergence on* $(a - R, a + R)$*.*

Lemma 1.3.9. *Differentiation and Integration term-by-term (cf. 1.1.19 and 1.1.21) is valid for power series on the radius of convergence.*

Proof. □

Corollary 1.3.10 (Taylor Series)**.** *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *with a positive radius of convergence, then* f *is infinetly differentiable in* $(a - R, a + R)$ *and* $\forall\, n \in \mathbb{N}_0\,,\ a_n = \dfrac{f^{(n)}(a)}{n!}$

Remark 1.3.11 (Analytic Functions)**.** *Let* T_n *be the* n*-th Taylor Polynomial of* f*. It is not necessarily true that* $T_n \xrightarrow{u} f$*. If it is true, we say* $f \in C^{\omega}$*.*

2 Multivariable Analysis

2.1 Multivariable Geometry

Definition 2.1.1 ($\mathbb{R}^n$ as a vector space)**.** *Let* $\mathbb{R}^n = \{(a_1, \cdots, a_n) \mid a_1, \cdots, a_n \in \mathbb{R}\}$. *We have the following operations:*

Addition: $(a_1, \cdots, a_n) + (b_1, \cdots, b_n) = (a_1 + b_1, \cdots, a_n + b_n)$
Scalar multiplication: $\lambda \cdot (a_1, \cdots, a_n) = (\lambda \cdot a_1, \cdots, \lambda \cdot a_n)$
Norm: $\|(a_1, \cdots, a_n)\| = \sqrt{\sum_{i=1}^n a_i^2}$
Scalar product: $(a_1, \cdots, a_n) \bullet (b_1, \cdots, b_n) = \sum_{i=1}^n a_i \cdot b_i$

With those operations, $\mathbb{R}^n$ *is an Euclidean Space (cf. Linear Algebra).*

Lemma 2.1.2 (Angles)**.** *We measure the angle between two vector as* $\theta := \arccos\left(\frac{a \bullet b}{\|a\| \cdot \|b\|}\right)$.

Corollary 2.1.3 (Perpendicularity)**.** $a \perp b \Leftrightarrow a \bullet b = (0, 0, 0)$

Definition 2.1.4 (Vector Product)**.** *In* $\mathbb{R}^3$*, we define the following operation* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ *as:*

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 \cdot b_3 - a_3 \cdot b_2, a_3 \cdot b_1 - a_1 \cdot b_3, a_1 \cdot b_2 - a_2 \cdot b_1)$$

further, we can use the short hand using determinants, by formally expanding Laplace's formula (cf. Linear Algebra) on the first row:

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $e_1 = (1, 0, 0)$*,* $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$*, the standard basis.*

Lemma 2.1.5. *The cross product obeys:*

Antisymmetry: $\forall\, a, b \in \mathbb{R}^3\,,\ a \times b = -b \times a$
Linearity: $\forall\, \alpha, \beta \in \mathbb{R}\,,\ \forall\, a, b, c \in \mathbb{R}^3\,,\ (\alpha \cdot a + \beta \cdot b) \times c = \alpha \cdot a \times c + \beta \cdot b \times c$ *and* $c \times (\alpha \cdot a + \beta \cdot b) = \alpha \cdot c \times a + \beta \cdot c \times b$
Perpendicularity: $a \times b \perp a, b$.

Proof. Antisymmetry and linearity follow directly from the definition with determinants. For perpendicularity, we only need to check $a \bullet (a \times b) = 0$ and $b \bullet (a \times b) = 0$, by explicit definition (cf. 2.1.4). $\square$

Corollary 2.1.6. *$a \times b = (0,0,0) \Leftrightarrow a, b$ are linearly dependent.*

Definition 2.1.7 (Right Handed)**.** *A basis (f_1, f_2, f_3) of $\mathbb{R}^3$ is right handed iff $f_1 \times f_2 = f_3$. The standard basis (e_1, e_2, e_3) is right handed (direct calculatation with 2.1.4).*

Definition 2.1.8 (Lines and Planes)**.** *We define the following geometrical objects: For $a, b, c \in \mathbb{R}^3$*

Line: $\{a + t(b-a) \mid t \in \mathbb{R}\} = a + \mathrm{Span}(b-a)$

Segment: $[A, B] = \{a + t(b-a) \mid t \in [0,1]\}$

Ray: $\{a + t(b-a) \mid t \in [0, \infty)\}$

Hyperplane: $\{a + t(b-a) + s(c-a) \mid t, s \in \mathbb{R}\} = a + \mathrm{Span}(b-a, c-a)$

Lemma 2.1.9. *For a plane equation $ax + by + cz = d$, we can convert into $A + \mathrm{Span}(B-A, C-A)$.*

If $a, b, c \neq 0$, then $A = (d/a, 0, 0), B = (0, d/b, 0), C = (0, 0, d/c)$.

If any of those are zero, change the corresponding vector entry to 1.