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EXACT SOLUTION OF THE ISOTROPIC HEISENBERG CHAIN WITH ARBITRARY SPINS: Thermodynamics of the model

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An exactly solvable generalization of the Heisenberg spin chain for arbitrary spins is found. The corresponding ground-state energy is calculated. The thermodynamics is studied on the basis of the exact solution. Magnetic susceptibility and specific heat are calculated (when $H \ll 1$ and $T \ll 1$) for the antiferromagnetic case.

1. Introduction

The progress of the last decade in studying two-dimensional exactly solvable models of quantum field theory and lattice statistics was evoked to some extent by using the Yang-Baxter equations. These equations were first discovered by Yang [1]; they appeared in the problem of non-relativistic $1+1$ dimensional particles with δ-function interaction, as the self-consistency condition for Bethe's ansatz. Analogous relations were derived by Baxter [2], who investigated the eight-vertex lattice model. These relations guarantee the commutativity of transfer-matrices with different values of the anisotropy parameter λ. The Yang–Baxter equations are also the central part of the theory of the relativistic purely elastic (factorized) S-matrix in $1+1$ dimensions. These equations bound the elements of the two-particle S-matrix; they represent the conditions necessary for the factorization of the multiparticle S-matrix into two-particle ones. These relations are to be an effect of the infinite set of conservation laws in the models. The Yang–Baxter equations in $1+1$ relativistic scattering theory and two-dimensional lattice statistics have the form (see fig. 1)

$$R^{k_1,k_2}_{i_1,i_2}(\lambda)R^{j_1,k_3}_{k_1,i_3}(\lambda+u)R^{j_2,j_3}_{k_2,k_3}(u)=R^{k_2,k_3}_{i_2,i_3}(u)R^{k_1,j_3}_{i_1,k_3}(\lambda+u)R^{j_1,j_2}_{k_1,k_2}(\lambda), \tag{1}$$

where $R^{j_1,j_2}_{i_1,i_2}(\lambda)$ is the two-particle S-matrix in the scattering theory case; indices i_1, i_2 (j_1, j_2) run over $1, 2, \ldots, r$ and denote the kinds of initial (final) particles. λ is the difference of the rapidity of colliding particles. In the case of lattice statistics $R^{j_1,j_2}_{i_1,i_2}(\lambda)$ is the vertex weight, the indices designate the states of the vertex bonds and λ is the anisotropy parameter associated with the angle at the vertex.

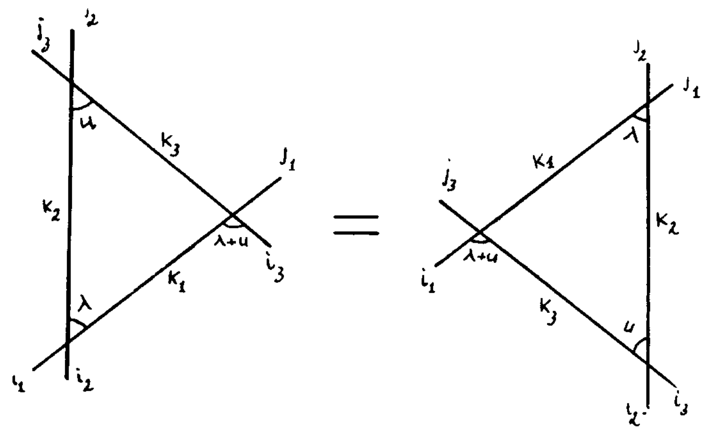

Fig. 1.

In this paper we construct generalized Heisenberg magnets with arbitrary spins using the Yang–Baxter equations (1). It is well known that the isotropic Heisenberg chain [3]

$$H_{1/2} = -\frac{1}{\eta} \sum_{n=1}^{N} \boldsymbol{\sigma}_n \boldsymbol{\sigma}_{n+1} \tag{2}$$

was the first model which was solved exactly 50 years ago using Bethe's ansatz by Bethe [4]. In (2) $\boldsymbol{\sigma}_n = (\sigma_n^x, \sigma_n^y, \sigma_n^z)$ are Pauli matrices at the nth site of the chain and $\boldsymbol{\sigma}_{N+1} = \boldsymbol{\sigma}_1$. Thermodynamic properties and classification of excitations have been investigated in [5–12]. We show that the generalization of the hamiltonian (2) for arbitrary spins s has the form

$$H_s = \sum_{n=1}^{N} Q_{2s}(\boldsymbol{s}_n \boldsymbol{s}_{n+1}), \tag{3}$$

where $Q_{2s}(x_n)$ is a polynomial of degree $2s$ of the SU(2) invariant quantities $x_n = \boldsymbol{S}_n \boldsymbol{S}_{n+1}$

$$Q_{2s}(x) = -\frac{1}{\eta} \sum_{j=1}^{2s} \left(\sum_{K=1}^{j} \frac{1}{k} \right) \prod_{\substack{l=0 \\ l \neq j}}^{2s} \frac{x - x_l}{x_j - x_l}. \tag{4}$$

In (3) and (4) $s_n = (s_n^x, s_n^y, s_n^z)$ are operators of arbitrary spin s at the nth site and $x_l = \frac{1}{2}[l(l+1) - 2s(s+1)]$. The diagonalization of the hamiltonian (3) is based on the algebraic formulation of Bethe's method, i.e. the quantum inverse scattering method developed by Faddeev, Skylanin and Takhtajan [13]. The paper is arranged as follows. In sect. 2 a certain vertex model is defined on the lattice, the transfer-matrix of which leads to the hamiltonian (3). In sect. 3 we diagonalize hamiltonian (3) using the quantum inverse scattering method. In sect. 4 equilibrium thermodynamics is investigated. Note that we consider only the antiferromagnetic case $\eta > 0$ (we put $\eta = 1$)*.

2. Construction of the vertex weights

Let us consider a two-dimensional homogeneous $M \times N$ lattice with localized spins s attributed to each bond of the lattice. Cyclic boundary conditions are imposed. The interaction takes place only between spins located on neighbouring bonds and is described by the vertex weight matrix $R^{j_1,j_2}_{i_1,i_2}(\lambda)$. It is useful to define the linear operator $R^{12}(\lambda)$ acting in the tensor product of two spaces $V_1 \otimes V_2$. It acts on the basic vectors $l_{i_1} \otimes l_{i_2}$ of the space $V_1 \otimes V_2$ as follows:

$$R^{12}(\lambda)(l_{i_1} \otimes l_{i_2}) = R^{k_1,k_2}_{i_1,i_2}(\lambda)(l_{k_1} \otimes l_{k_2}) \,. \tag{5}$$

Operators $R^{12}(\lambda)$, $R^{13}(\lambda)$ and $R^{23}(\lambda)$ act on the vectors from spaces 3, 2 and 1 as unit operators. Taking this into account we can write the Yang–Baxter equation (1) in the compact form

$$R^{12}(\lambda)R^{13}(\lambda+u)R^{23}(u) = R^{23}(u)R^{13}(\lambda+u)R^{12}(\lambda) \,. \tag{6}$$

We shall use both (1) and (6) as Yang–Baxter equations. We need the SU(2) invariant solution of eq. (6). Therefore we impose the following condition on the R-matrix:

$$O_1^{-1} \otimes O_2^{-1})R^{12}(\lambda)(O_1 \otimes O_2) = R^{12}(\lambda) \,, \tag{7}$$

where the matrix O belongs to the SU(2) group or its higher representation. Indices 1, 2 number spaces in which matrix O acts. The well-known solution of eq. (6) [15] (see fig. 2)

$${}_{\sigma}R^{12}(\lambda) = \frac{\eta - 2\lambda}{2\eta} I^1 \otimes I^2 + \tfrac{1}{2}\boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_2 \tag{8}$$

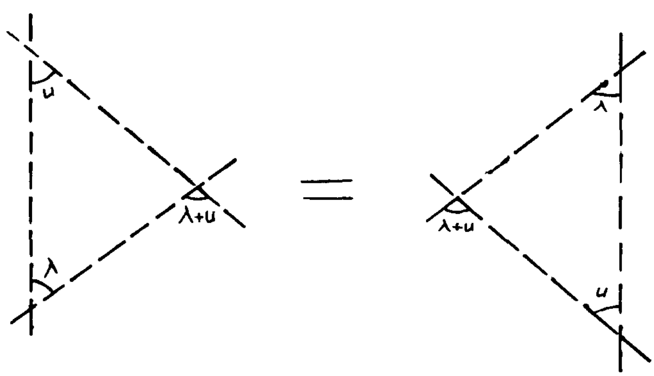

Fig. 2.

* When this paper was in preparation the author was informed about the work of Takhtajan [14] in which the ground-state energy and the S-matrix of the hamiltonian (3) have been obtained.

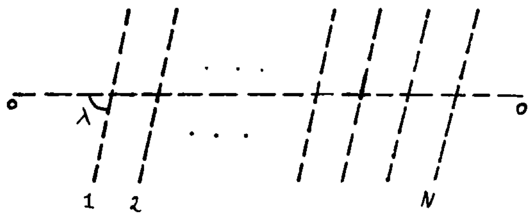

Fig. 3.

corresponds to the case when $s=\frac{1}{2}$ and spaces 1 and 2 in (7) are two-dimensional. Here η is an arbitrary parameter and I is unit operator. The vertex model with vertex weight (8) leads to hamiltonian (2) according to the formula

$$H_{1/2}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln T(\lambda)|_{\lambda=0}\,, \tag{9}$$

where $T(\lambda)$ is a transfer-matrix of the vertex model (see fig. 3 for $J(\lambda)$)

$$T(\lambda)=\mathrm{tr}_0\,J(\lambda)=\mathrm{tr}_0\,R^{0N}(\lambda)\cdots R^{02}(\lambda)R^{01}(\lambda)\,. \tag{10}$$

The product and trace in eq. (10) are taken in the auxiliary space denoted by index 0. By direct substitution it is easy to verify that

$${}_{\sigma s}R^{12}(\lambda)=\frac{\eta-2\lambda}{2\eta}I^1\otimes I^2+\boldsymbol{\sigma}_1\otimes\boldsymbol{S}_2 \tag{11}$$

is the solution of the Yang–Baxter equation in the following form [16] (see fig. 4):

$${}_{\sigma s}R^{12}(\lambda)\,{}_{\sigma}R^{13}(\lambda+u)\,{}_{\sigma s}R^{23}(u)={}_{\sigma s}R^{23}(u)\,{}_{\sigma}R^{13}(\lambda+u)\,{}_{\sigma s}R^{12}(\lambda)\,. \tag{12}$$

In (11) $\boldsymbol{s}=(s^x,s^y,s^z)$ is an operator of arbitrary spin s, I^1 and I^2 are unit operator in the 2-dimensional and $2s+1$ dimensional spaces accordingly. The generalization of solution (8) for the case when the matrix in (7) belongs to an arbitrary representation of the group SU(2) and spaces 1, 2, 3 in (6) have the dimensions $2s+1$, has

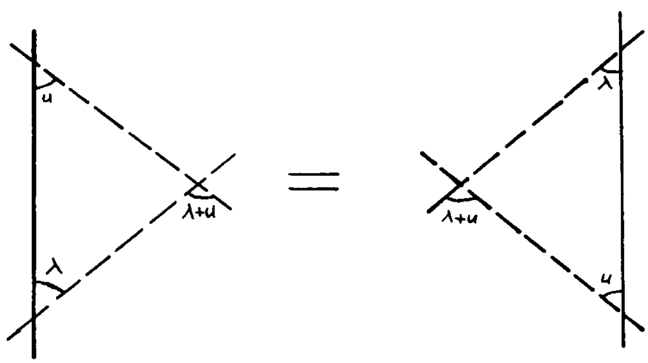

Fig. 4.

been found in [17, 18]. It has the following form:

$$ {}_{s}R^{12}(\lambda) = -\sum_{j=0}^{2s} \prod_{k=1}^{j} \frac{\lambda - k\eta}{\lambda + k\eta} P^{j} . \tag{13}$$

Here P^j is a projector acting in the space which is a tensor product of two spin s spaces. The projector fixes the state with total spin j, i.e. if $|l\rangle$ is a state with total spin l, then

$$P^{j}|l\rangle = \delta_{j,l}|l\rangle . \tag{14}$$

It is evident that P^j can be presented in the form of the polynomial of degree $2s$ of $x = s_1 \otimes s_2$,

$$P^{j} = \prod_{\substack{l=0 \\ l \neq j}}^{2s} \frac{x - x_l}{x_j - x_l} . \tag{15}$$

The transfer matrix $T_s(\lambda)$ constructed with the use of ${}_sR(\lambda)$ leads to the hamiltonian (3) according to relation (9). In the case when spin $s = 1$, (3) has the simple form [19]

$$H_1 = \tfrac{1}{4} \sum_{n=1}^{N} s_n s_{n+1} - (s_n s_{n+1})^2 . \tag{16}$$

In (16) we have taken $\eta = 1$. The ${}_sR(\lambda)$ matrix is the solution not only of the eq. (6), but also the next Yang–Baxter equation [20] (see fig. 5)

$$ {}_{\sigma s}R^{12}(\lambda)\, {}_{s}R^{13}(\lambda + u)\, {}_{\sigma s}R^{23}(u) = {}_{\sigma s}R^{23}(u)\, {}_{s}R^{13}(\lambda + u)\, {}_{\sigma s}R^{12}(\lambda) . \tag{17}$$

As we have noticed in the introduction, in relativistic scattering theory the Yang–Baxter equations are the conditions of factorization of the multiparticle S-matrix into two-particle ones. In our case when there is the invariance under the group SU(2) the factorization theory of scattering has a simple interpretation. Because of SU(2) invariance each act of collision conserves the set of spins (which

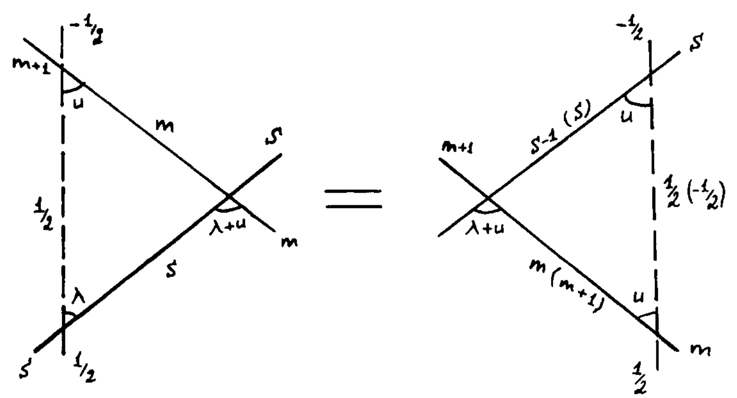

Fig. 5.

corresponds to the continuity of both lines (-$\stackrel{s=1/2}{---}$- and $\stackrel{s}{—}$ in figs. 2, 4, 5) and the sum of z-projections in $|\text{in}\rangle$ and $\langle\text{out}|$ states. Therefore, the matrix elements of R-matrices (8), (11) and (13) satisfy the following condition:

$$R^{s_1\bar{m}_1, s_2\bar{m}_2}_{s_1 m_1, s_2 m_2}(\lambda) \neq 0\,, \qquad m_1 + m_2 = \bar{m}_1 + \bar{m}_2 \tag{18}$$

where s_1, s_2 are spins of colliding particles and m_1, $m_2(\bar{m}_1, \bar{m}_2)$ are their z-projections in initial (final) states.

3. The eigenvalues and eigenfunctions of H_s and $T_s(\lambda)$

Following Faddeev and others [13] we introduce the monodrom matrix

$$T_s(\lambda) = \mathrm{tr}_0\, J^s(\lambda) = \mathrm{tr}_0\, {}_sR^{0N}(\lambda) \cdots {}_sR^{02}(\lambda)\,{}_sR^{01}(\lambda)\,. \tag{19}$$

The monodrom matrix $J^s(\lambda)$ in auxiliary space is the $(2s+1) \times (2s+1)$ operator matrix. In order to diagonalize $T_s(\lambda)$ the auxiliary transfer matrix $T_\sigma(\lambda)$ is introduced. $T_\sigma(\lambda)$ is constructed with the use of vertex weight ${}_{\sigma s}R(\lambda)$:

$$T_\sigma(\lambda) = \mathrm{tr}_0\, J^\sigma(\lambda) = \mathrm{tr}_0\, {}_{\sigma s}R^{0N}(\lambda) \cdots {}_{\sigma s}R^{02}(\lambda)\,{}_{\sigma s}R^{01}(\lambda)\,. \tag{20}$$

In eq. (20) $J^\sigma(\lambda)$ is the operator matrix 2×2 and ${}_{\sigma s}R(\lambda)$ is presented in the form

$${}_{\sigma s}R^{0n}(\lambda) = \begin{pmatrix} \dfrac{1-2\lambda}{2} + s_n^z & s_n^- \\ s_n^+ & \dfrac{1-2\lambda}{2} - s_n^z \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{21}$$

where $s_n^\pm = s_n^x \pm i s_n^y$. Transfer matrices $T_\sigma(\lambda)$ and $T_s(\lambda)$ commute. In fact eq. (17) can be written in the following form:

$${}_{\sigma s}R(\lambda - u)(J^\sigma(\lambda) \otimes J^s(u)) = (J^s(u) \otimes J^\sigma(\lambda))\,{}_{\sigma s}R(\lambda - u) \tag{22}$$

(see fig. 6, 7 and 8). Then multiplying eq. (22) from the left by ${}_{\sigma s}R^{-1}(\lambda)$ and taking the trace we have

$$[T_\sigma(\lambda),\, T_s(u)] = 0\,. \tag{23}$$

At these points where ${}_{\sigma s}R^{-1}(\lambda)$ doesn't exist, commutativity follows from the analytical continuation principle. Note, that in order to obtain the right-hand side

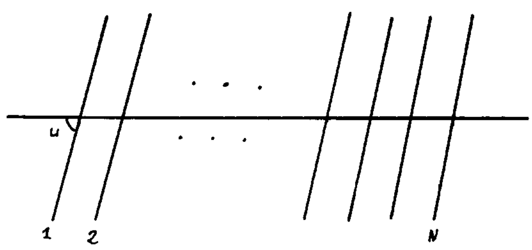

Fig. 6.

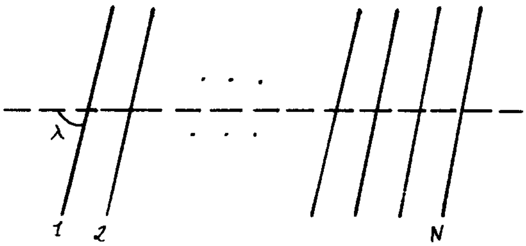

Fig. 7.

of fig. 8 from the left-hand side it is necessary to pull the vertical straight lines across the right vertex with the use of eq. (17). Since eq. (23) is satisfied we can simultaneously diagonalize $T_\sigma(\lambda)$ and $T_s(\lambda)$. First we find the eigenfunctions of $T_\sigma(\lambda)$ and then show that they are eigenfunctions for $T_s(\lambda)$ also. This diagonalization method of $T_s(\lambda)$ has been proposed by Fateev [21]. Diagonalization of $T_\sigma(\lambda)$ was performed in [16] in connection with the Kondo problem. Here we shall describe this diagonalization. From eq. (12) we have $[T_\sigma(\lambda), T_\sigma(\mu)] = 0$. This means that

$$_\sigma R(\lambda-\mu)(J^\sigma(\mu)\otimes J^\sigma(\lambda)) = (J^\sigma(\mu)\otimes J^\sigma(\lambda))_\sigma R(\lambda-\mu)\,. \tag{24}$$

Eq. (24) gives us the commutators between the elements of $J^\sigma(\lambda)$:

$$J^\sigma(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{25}$$

The matrix $_\sigma R(\lambda)$ in eq. (24) has the form

$$_\sigma R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b(\lambda) = \frac{1}{1-\lambda}, \quad c(\lambda) = -\frac{\lambda}{1-\lambda}. \tag{26}$$

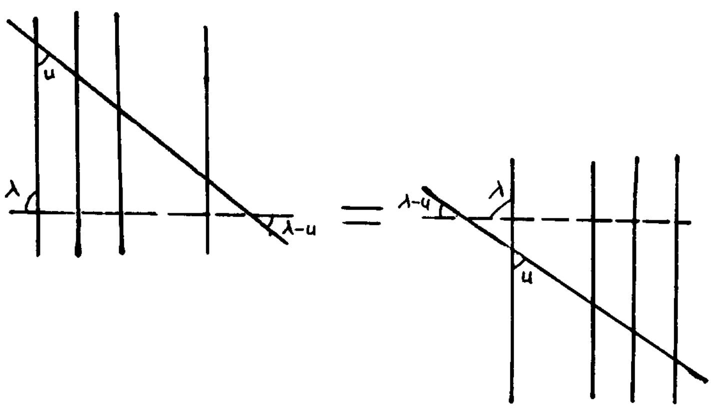

Fig. 8.

We write down the essential commutators from (24):

$$
\begin{aligned}
&[A(\lambda), A(\mu)]=0\,, \qquad [D(\lambda), D(\mu)]=0\,, \qquad [B(\lambda), B(\mu)]=0\,,\\
&[C(\lambda), C(\mu)]=0\,,\\
&\quad B(\lambda)A(\mu)=b(\lambda-\mu)B(\mu)A(\lambda)+c(\lambda-\mu)A(\mu)B(\lambda)\,,\\
&\quad B(\mu)D(\lambda)=b(\lambda-\mu)B(\lambda)D(\mu)+c(\lambda-\mu)D(\lambda)B(\mu)\,.
\end{aligned}
\tag{27}
$$

Let us consider the vector

$$
|\Phi_N\rangle=|\varphi_1\rangle\otimes|\varphi_2\rangle\otimes\cdots\otimes|\varphi_N\rangle\,, \tag{28}
$$

$$
|\varphi\rangle=\begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix} 2s\,.
$$

It follows from (21) that

$$
\begin{aligned}
&\alpha|\varphi\rangle=\left(\frac{1-2\lambda}{2}+s\right)|\varphi\rangle\,,\\
&\delta|\varphi\rangle=\left(\frac{1-2\lambda}{2}-s\right)|\varphi\rangle\,,\\
&\gamma|\varphi\rangle=0\,.
\end{aligned}
\tag{29}
$$

Thus we have the next relations for the elements of the monodromy matrix:

$$
\begin{aligned}
&A(\lambda)|\Phi_N\rangle=\left(\frac{1-2\lambda}{2}+s\right)^N|\Phi_N\rangle\,,\\
&D(\lambda)|\Phi_N\rangle=\left(\frac{1-2\lambda}{2}-s\right)^N|\Phi_N\rangle\,,\\
&C(\lambda)|\Phi_N\rangle=0\,.
\end{aligned}
\tag{30}
$$

From eqs. (27), (28) and (30) it follows that the vector

$$
\Psi(\lambda_1,\lambda_2\cdots\lambda_M)=\prod_{l=1}^{M}B(\lambda_l)|\Phi_N\rangle \tag{31}
$$

is an eigenfunction for the $T_\sigma(\lambda)=A(\lambda)+D(\lambda)$ with the eigenvalues

$$
\begin{aligned}
\Lambda^\sigma(\lambda,\lambda_1,\lambda_2\cdots\lambda_M)=&\left(\frac{1-2\lambda}{2}+s\right)^N\prod_{l=1}^{M}\frac{1}{c(\lambda_l-\lambda)}\\
&+\left(\frac{1-2\lambda}{2}-s\right)^N\prod_{l=1}^{M}\frac{1}{c(\lambda-\lambda_l)}\,,
\end{aligned}
\tag{32}
$$

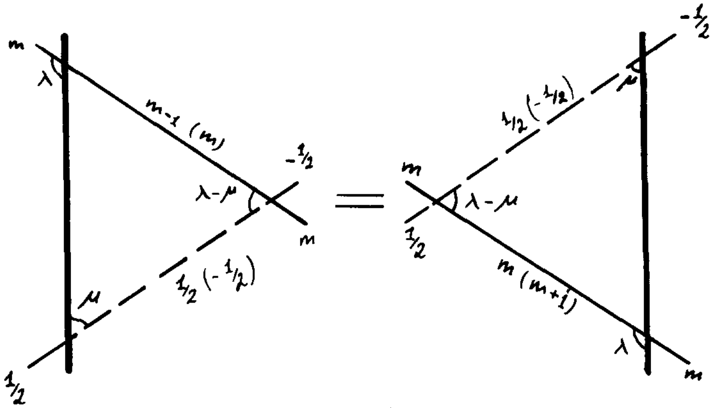

Fig. 9.

if the numbers λ_l satisfy the following system of equations:

$$\left(\frac{\lambda_j - s - \frac{1}{2}}{\lambda_j + s - \frac{1}{2}}\right)^N = \prod_{\substack{l=1 \\ l \neq j}}^{M} \frac{c(\lambda_l - \lambda_j)}{c(\lambda_j - \lambda_l)}. \tag{33}$$

To find the eigenvalues of $T_s(\lambda)$ it is necessary to act by $T_s(\lambda) = \sum_{m=-s}^{s} J^s_{m,m}(\lambda)$ on Bethe's vector (31). For this we must know the commutators between $J^s_{m,m}(\lambda)$ and $B(\mu)$ and their action on the vector $|\Phi_N\rangle$. From eqs. (22) we calculate the commutators between $J^s_{m,m}(\lambda)$ and $B(\mu)$. From eqs. (22) we write-down two equations according to two possible choices of indices in eqs. (22), as shown figs. 9 and 10:

$$\begin{aligned} &A(\mu) J^s_{m,m-1}(\lambda) R^{-1/2,m}_{1/2,m-1}(\lambda-\mu) + B(\mu) J^s_{m,m}(\lambda) R^{-1/2,m}_{-1/2,m}(\lambda-\mu) \\ &\quad = J^s_{m,m}(\lambda) B(\mu) R^{1/2,m}_{1/2,m}(\lambda-\mu) + J^s_{m+1,m}(\lambda) D(\mu) R^{-1/2,m+1}_{1/2,m}(\lambda-\mu), \end{aligned} \tag{34}$$

$$\begin{aligned} &A(\mu) J^s_{m,m-1}(\lambda) R^{1/2,m-1}_{1/2,m-1}(\lambda-\mu) + B(\mu) J^s_{m,m}(\lambda) R^{1/2,m-1}_{-1/2,m}(\lambda-\mu) \\ &\quad = J^s_{m,m-1}(\lambda) A(\mu) R^{1/2,m}_{1/2,m}(\lambda-\mu) + J^s_{m+1,m-1}(\lambda) C(\mu) R^{-1/2,m+1}_{1/2,m}(\lambda-\mu), \end{aligned} \tag{35}$$

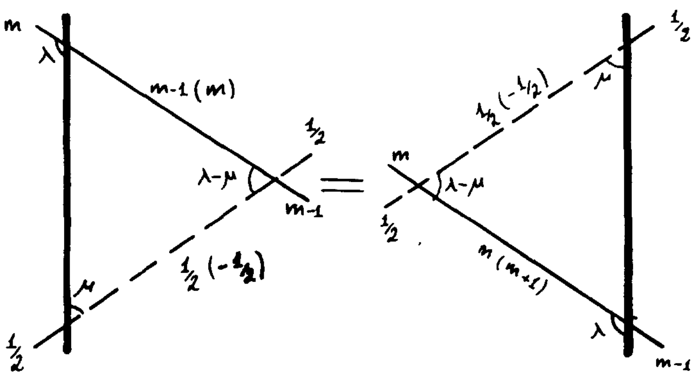

Fig. 10.

In these figures solid vertical lines correspond to the quantum indices of $J^s(\lambda)$ and $J^\sigma(\lambda)$, which are omitted in eqs. (34), (35). The full lines and dotted lines correspond to the auxiliary spaces $J^s(\lambda)$ and $J^\sigma(\lambda)$ respectively. Substituting the matrix elements of $R(\lambda-\mu)={}_{\sigma s}R(\lambda-\mu)$ from (11) into (34) and (35) and including the term $A(\mu)J^s_{m,m-1}(\lambda)$ from eqs. (34), (35) we obtain

$$\begin{aligned} J^s_{m,m}(\lambda)B(\mu) = C_m(\lambda-\mu)B(\mu)J^s_{m,m}(\lambda) + C^1_m(\lambda-\mu) \\ \times J^s_{m,m-1}(\lambda)A(\mu) + C^2_m(\lambda-\mu)J^s_{m+1,m}(\lambda)D(\mu) \\ + C^3_m(\lambda-\mu)J^s_{m+1,m-1}(\lambda)C(\mu)\,, \end{aligned} \tag{36}$$

where

$$\begin{aligned} C_m(\lambda) &= (\lambda-s-\tfrac{1}{2})(\lambda+s-\tfrac{1}{2})/(\lambda-m-\tfrac{1}{2})(\lambda-m+\tfrac{1}{2})\,, \\ C^1_m(\lambda) &= -[(s+m)(s-m+1)]^{1/2}/(\lambda-m+\tfrac{1}{2})\,, \\ C^2_m(\lambda) &= [(s-m)(s+m+1)]^{1/2}/(\lambda-m-\tfrac{1}{2})\,, \\ C^3_m(\lambda) &= (s^2-m^2)^{1/2}[(s+1)^2-m^2]^{1/2}/(\lambda-m-\tfrac{1}{2})(\lambda-m+\tfrac{1}{2})\,. \end{aligned} \tag{37}$$

Now we act by matrix elements in the auxiliary space of the ${}_sR(\lambda)$ matrix on the state (28) $|\varphi\rangle = |s,s\rangle = |s\rangle$:

$$\langle s,m'|{}_sR(\lambda)|s,m\rangle|s,s\rangle = \langle m'|{}_sR(\lambda)|m\rangle|s\rangle\,. \tag{38}$$

Then in (38) we express the state $|m\rangle|s\rangle$ as a sum of states with the total spin l:

$$|m\rangle|s\rangle = \sum_{l=0}^{2s}\langle l, m+s|m,s\rangle|l,m+s\rangle\,. \tag{39}$$

Let us act by ${}_sR(\lambda)$ on the states $|l,m+s\rangle$, taking into account eq. (14). After that we expand these states in initial set of states and obtain

$$\begin{aligned} \langle m'|{}_sR(\lambda)|m\rangle|s\rangle = -\sum_{j=0}^{2s}\prod_{k=1}^{j}\frac{\lambda-k}{\lambda+k}\langle j,m+s|m,s\rangle \\ \times\langle m', m_2|j,m+s\rangle|m_2\rangle\,, \end{aligned} \tag{40}$$

where $m_2 = m+s-m'$. Taking into account that $m_s \leq s$ we obtain that $\langle m'|{}_sR(\lambda)|m\rangle|s\rangle = 0$ for $m'<m$. Elements which have $m'>m$ are creation operators. The state $|s\rangle$ is the eigenfunctions for the diagonal elements

$$\langle m|{}_sR(\lambda)|m\rangle|s\rangle = -\sum_{j=0}^{2s}\prod_{k=1}^{j}\frac{\lambda-k}{\lambda+k}\left(\langle j,m+s|m,s\rangle\right)^2|s\rangle\,, \tag{41}$$

and at $m=s$ we have

$$\langle s|\langle s|{}_sR(\lambda)|s\rangle|s\rangle = -\prod_{k=1}^{2s}\frac{\lambda-k}{\lambda+k} = R^{s,s}_{s,s}(\lambda)\,. \tag{42}$$

Eigenvalues (41) are complicated and can not be expressed in a simple form. Nevertheless, they can be obtained from eqs. (12) with the appropriate choice of indices as shown in fig. 5. From fig. 5 we obtain the following relation:

$$R^{s,m}_{s,m}(\lambda) = \prod_{k=m+1}^{s} \frac{\lambda - k + s}{\lambda - k - s} R^{s,s}_{s,s}(\lambda) . \tag{43}$$

Taking into account (42) we have

$$\alpha_m(\lambda) = R^{s,m}_{s,m}(\lambda) = -\prod_{k=m+1}^{s} \frac{\lambda - k + s}{\lambda - k - s} \prod_{k'=1}^{2s} \frac{\lambda - k'}{\lambda + k'} . \tag{44}$$

Acting by transfer matrix $T_s(\lambda) = \sum_{m=-s}^{s} J^{s}_{m,m}(\lambda)$ on the vector (31) and taking into account (36), (37), (40) and (44) we obtain

$$T_s(\lambda)\Psi(\lambda_1, \lambda_2 \cdots \lambda_M) = \Lambda^s(\lambda, \lambda_1, \lambda_2 \cdots \lambda_M)\Psi(\lambda_1, \lambda_2 \cdots \lambda_M) + \text{“unwanted” terms} \tag{45}$$

where Λ^s are eigenvalues of $T^s_M(\lambda)$:

$$\Lambda^s(\lambda, \lambda_1, \lambda_2 \cdots \lambda_M) = \sum_{m=-s}^{s} (\alpha_m(\lambda))^N \prod_{l=1}^{M} c_m(\lambda - \lambda_l) . \tag{46}$$

We have obtained (46) taking into account only the first term in (36). Of course, it would be correct to keep watch over all of the “unwanted” terms and to find the equations for $\{\lambda_j\}$ from the condition of their equality to zero. This procedure would give us eqs. (33). There is the following argument for this statement. From (46) and (37) we see that the $\Lambda^s(\lambda, \lambda_1, \cdots \lambda_M)$ formally have poles at the points $\lambda = \lambda_j + n + \frac{1}{2}$; $-s \leq n \leq s$. Residues in the poles must be equal to zero since Λ^s does not know about them. It is easy to verify that this condition coincides with (33). We obtain the eigenvalues of hamiltonian (3) from (9) and (46):

$$E_s = -\sum_{j=1}^{M} \frac{s}{\lambda_j^2 + s^2} . \tag{47}$$

Only the term with $m = s$ from the sum (46) contributes to (47). The total spin z-projection is an important characteristic:

$$s^a = \sum_{n=1}^{N} s^a_n . \tag{48}$$

It is evident that the s^a commute with H_s. Let us calculate the eigenvalues of the vector (31). For this we use the commutator $[s^a, B(\lambda)]$. We have

$$[s^a, J^\sigma(\lambda)] = \sum_{n=1}^{N} {}_{\sigma s}R^{0N}(\lambda) \cdots [s^a, {}_{\sigma s}R^{0n}(\lambda)] \cdots {}_{\sigma s}R^{01}(\lambda) . \tag{49}$$

From (11) we obtain that

$$[s^a_n, {}_{\sigma s}R^{0n}(\lambda)] = -\tfrac{1}{2}[\sigma^a_0, {}_{\sigma s}R^{0n}(\lambda)]. \tag{50}$$

Eq. (50) gives us

$$[s^a, J^\sigma(\lambda)] = -\tfrac{1}{2}[\sigma^a_0, J^\sigma(\lambda)]. \tag{51}$$

We find the commutator $[s^z, B(\lambda)]$ from eq. (51):

$$[s^z, B(\lambda)] = -B(\lambda). \tag{52}$$

The eigenvalues of s^z are obtained by acting s^z on Bethe's vector (31). Taking into account eq. (52) we have

$$s^z = NS - M. \tag{53}$$

4. Thermodynamics of the model

The investigation of thermodynamic properties of the model is based on eqs. (33) and (47). Technically it is the same as in the case of $s=\frac{1}{2}$ [5, 6, 10] except that the ground state is constructed from the strings.

After the substitution $\lambda_j \to i\lambda_j + \frac{1}{2}$ eqs. (33) take the form

$$\left(\frac{\lambda_j - is}{\lambda_j + is}\right)^N = \prod_{\substack{l=1\\ l\neq j}}^{M} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i}. \tag{54}$$

It is well known [5, 15] that the general solution of eq. (54) in the thermodynamic limit lies in the complex plane λ and forms strings of the length n:

$$\lambda_j \equiv \lambda_j^{n,\alpha} = \lambda_j^n + i(n+1-2\alpha), \qquad \alpha = 1, 2, \cdots, n. \tag{55}$$

Here λ_j^n are the centres of the string and they are real numbers. Integers n are connected with M by the following relation:

$$\sum_n n\xi_n = M,$$

where ξ_n is a number of strings of the length n. Substituting (55) into (54) and taking a product of these equations we obtain

$$N\theta_{m,2s}(\lambda_j^m) = 2\pi J_j^m + \sum_{k=1}^{\infty} \sum_{\iota=1}^{\xi_k} \Xi_{m,k}(\lambda_j^m - \lambda_\iota^k), \tag{56}$$

where

$$\theta_n(\lambda) = 2\,\mathrm{arctg}\,\frac{\lambda}{n},$$

$$\theta_{m,p}(\lambda) = \sum_{l=1}^{\min(m,p)} \theta_{m+1+p-2l}(\lambda), \tag{57}$$

$$\Xi_{m,k}(\lambda)=\begin{cases}\theta_{|m-k|}(\lambda)+2\theta_{|m-k|+2}(\lambda)\cdots 2\theta_{m+k-2}(\lambda)+\theta_{m+k}(\lambda)\,, & m\neq k\,,\\ 2\theta_2(\lambda)+2\theta_1(\lambda)\cdots 2\theta_{2m-2}(\lambda)+\theta_{2m}(\lambda)\,, & m=k\,.\end{cases}\tag{58}$$

There is a useful relation between (57) and (58):

$$\theta_{m,p}(\lambda)=\hat{p}(\Xi_{m,p}(\lambda)+\pi\ \mathrm{sgn}\ \lambda\delta_{m,p})\,,\tag{59}$$

where

$$\hat{p}g(\lambda)=\int_{-\infty}^{\infty}p(\lambda-\lambda')g(\lambda')\,\mathrm{d}\lambda'$$

$$p(\lambda)=\left(4\ \mathrm{ch}\frac{\pi}{2}\lambda\right)^{-1}.$$

The numbers $2J_j^m$ arise due to taking a logarithm. Each set of numbers $\{2J_j^m\}$ uniquely determines the set of numbers $\{\lambda_j^m\}$ according to eq. (56). Numbers $2J_j^m$ form a finite lattice. The corresponding solution $\{\lambda_j^m\}$ of eq. (56) is called "particles". Numbers $\{\tilde{\lambda}_j^m\}$ satisfy the equation

$$2\pi\tilde{J}_j^m=N\theta_{m,2s}(\tilde{\lambda}_j^m)-\sum_{k=1}^{\infty}\sum_{i=1}^{\xi_k}\Xi_{m,k}(\tilde{\lambda}_j^m-\lambda_i^k)\,,\tag{60}$$

where $2\tilde{J}_j^m$ numbers are omitted on the lattice J_j^m. The corresponding solutions $\tilde{\lambda}_j^m$ are called "holes". In the thermodynamic limit our system is described in the language of partition functions of the particles $\rho_m(\lambda)$ and holes $\tilde{\rho}_m(\lambda)$ which are defined as follows:

$$\begin{aligned}\rho_m(\lambda)&=\lim_{N\to\infty}\frac{1}{N(\lambda_{j+1}^m-\lambda_j^m)}\,,\\ \tilde{\rho}_m(\lambda)&=\lim_{N\to\infty}\frac{1}{N(\tilde{\lambda}_{j+1}^m-\tilde{\lambda}_j^m)}\,.\end{aligned}\tag{61}$$

Let us designate the left-hand side of eq. (60) $Nh_m(\lambda)$. It is equal to $2\pi J_j^m$ at $\lambda=\lambda_j^m$ and $2\pi\tilde{J}_j^m$ at $\lambda=\tilde{\lambda}_j^m$. Then in the thermodynamic limit we have

$$h_m(\lambda)=\theta_{m,2s}(\lambda)-\sum_{k=1}^{\infty}\int\Xi_{m,k}(\lambda-\lambda')\rho_k(\lambda')\,\mathrm{d}\lambda'\,,\tag{62}$$

$$\frac{\mathrm{d}h_m(\lambda)}{\mathrm{d}\lambda}=2\pi(\rho_m(\lambda)+\tilde{\rho}_m(\lambda))\,.\tag{63}$$

Differentiating eq. (62) and taking into account (63) we obtain

$$\tilde{\rho}_m(\lambda)=-\sum_{n=1}^{\infty}\hat{A}_{m,n}(\rho_n(\lambda)-p(\lambda)\delta_{n,2s})\,,\tag{64}$$

where

$$A_{m,n}=\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\Xi_{m,n}(\lambda)+\delta(\lambda)\delta_{n,m}\,. \tag{65}$$

Energy (47) and s^z (53) in the language of distribution functions are of the form

$$E_{s/N}=-\tfrac{1}{2}\sum_{n=1}^{\infty}\int\theta'_{n,2s}(\lambda)\rho_n(\lambda)\,\mathrm{d}\lambda\,, \tag{66}$$

$$s^z/N=s-\sum_{n=1}^{\infty}n\int\rho_n(\lambda)\,\mathrm{d}\lambda\,. \tag{67}$$

For obtaining eq. (66) we have used the following formula:

$$\sum_{\alpha=1}^{n}\frac{s}{(\lambda_j^{n,\alpha})^2+s^2}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\theta_{n,2s}(\lambda_j^n)=\tfrac{1}{2}\theta'_{n,2s}(\lambda_j^n)\,.$$

Entropy of the system is a logarithm of state numbers. For given distribution functions $\rho_n(\lambda)$ and $\tilde{\rho}_n(\lambda)$ in a small interval from λ to $\lambda+\mathrm{d}\lambda$ there are $\rho_n(\lambda)N\,\mathrm{d}\lambda$ particles and $\tilde{\rho}_n(\lambda)N\,\mathrm{d}\lambda$ holes. Here we assume $N\,\mathrm{d}\lambda\gg1$. So entropy of our system is

$$S/N=\sum_{n=1}^{\infty}\int[(\rho_n+\tilde{\rho}_n)\ln(\rho_n+\tilde{\rho}_n)-\rho_n\ln\rho_n-\tilde{\rho}_n\ln\tilde{\rho}_n]\,\mathrm{d}\lambda\,. \tag{68}$$

In order to find the equilibrium distribution functions at the finite temperature T we must minimize the free energy

$$F=E_s-TS-HS^z\,. \tag{69}$$

Substituting eqs. (66)–(68) into (69), then minimizing with respect to the ρ_n, $\tilde{\rho}_n$ and taking into account (64) we obtain

$$\ln\left(1+\frac{\tilde{\rho}_n}{\rho_n}\right)=\frac{1}{T}(Hn-\tfrac{1}{2}\theta'_{n,2s})+\sum_{m=1}^{\infty}\hat{A}_{n,m}\ln\left(1+\frac{\rho_m}{\tilde{\rho}_m}\right), \tag{70}$$

or in the language of functions $\varepsilon_n=T\ln(\tilde{\rho}_n/\rho_n)$

$$\ln(1+\mathrm{e}^{\varepsilon_n/T})=\frac{1}{T}(Hn-\tfrac{1}{2}\theta'_{n,2s})+\sum_{m=1}^{\infty}\hat{A}_{n,m}\ln(1+\mathrm{e}^{-\varepsilon_m/T})\,. \tag{71}$$

Acting by the inverse matrix

$$\hat{A}^{-1}_{n,m}=\delta_{n,m}-\hat{p}(\delta_{n+1,m}+\delta_{n-1,m}) \tag{72}$$

on eq. (71) we find another form which will be used later:

$$\begin{aligned}\varepsilon_1&=-T\hat{p}\ln f(\varepsilon_2)\\ \varepsilon_n&=-T\hat{p}\ln f(\varepsilon_{n+1})f(\varepsilon_{n-1})-\pi p\delta_{n,2s}\,,\end{aligned} \tag{73}$$

where $f(\varepsilon_n)=(1+e^{\varepsilon_n/T})^{-1}$ is the Fermi function. From (71) it is also easy to find that

$$\lim_{n\to\infty}\frac{\varepsilon_n}{n}=H. \tag{74}$$

The free energy is expressed by means of the solution of eqs. (71), (73) in the following two forms:

$$\frac{1}{N}F(T,H)=-sH+T\sum_{n=1}^{\infty}\int A_{n,2s}(\lambda)\ln(1-f(\varepsilon_n))\,d\lambda, \tag{75}$$

$$\frac{1}{N}F(T,H)=-\tfrac{1}{2}\int p(\lambda)\theta'_{2s,2s}(\lambda)\,d\lambda+T\int p(\lambda)\ln f(\varepsilon_{2s}(\lambda))\,d\lambda. \tag{76}$$

We see from eqs. (71) and (73) that for $n\neq 2s$ the functions $\varepsilon_n(\lambda)\geqslant 0$, while $\varepsilon_{2s}(\lambda)$ can change sign at some points $\lambda=\pm b$. Let us consider the case $T=0$. Taking the limit $T\to 0$ in (71), we find

$$\varepsilon_n(\lambda)=Hn-\tfrac{1}{2}\theta'_{n,2s}(\lambda)-\hat{A}_{n,2s}\varepsilon^-_{2s}(\lambda), \qquad n\neq 2s, \tag{77}$$

$$\varepsilon^+_{2s}(\lambda)=H(2s)-\tfrac{1}{2}\theta'_{2s,2s}(\lambda)-\hat{A}_{2s,2s}\varepsilon^-_{2s}(\lambda), \tag{78}$$

where

$$\varepsilon^+_{2s}(\lambda)=\begin{cases}\varepsilon_{2s}(\lambda), & \varepsilon_{2s}>0\\ 0, & \varepsilon_{2s}<0\end{cases}; \qquad \varepsilon^-_{2s}(\lambda)=\begin{cases}\varepsilon_{2s}, & \varepsilon_{2s}<0,\\ 0, & \varepsilon_{2s}>0.\end{cases}$$

From eq. (78) it is easy to establish that $\varepsilon^+_{2s}(\lambda)=0$ when $H=0$, i.e. $\varepsilon_{2s}(\lambda)$ is negative. Taking into account the relation

$$\theta'_{2s,2s}(\lambda)=2\pi\hat{A}_{2s,2s}p(\lambda),$$

we find the solution of (78), (79):

$$\varepsilon_n(\lambda)=-\frac{\pi}{4}\frac{\delta_{n,2s}}{\operatorname{ch}\frac{1}{2}\pi\lambda}. \tag{79}$$

Quantities ε_n are the energies of a string with length n. As we see from (79) a string with length $2s$ is present in the ground state [14, 22]. Taking the limit $T\to 0$ in (76) we obtain the ground-state energy

$$E^0_s=F(0,0)=-N\sum_{k=1}^{2s}\beta(k), \tag{80}$$

where

$$\beta(k)=\int_0^1 x^{k-1}\frac{dx}{1+x}.$$

For integer s we have

$$E^0_s=-N\sum_{k=1}^{s}\frac{1}{2k-1},$$

and for half-integer s

$$E_s^0 = -N\left(\ln 2 + \sum_{k=1}^{s-1/2} \frac{1}{2k}\right).$$

Later we shall show that in the ground state $\tilde{\rho}_n = 0$, i.e. there are no holes and $\rho_n(\lambda) = P(\lambda)$. In this state the total spin is equal to

$$s_N^z = s - \sum_{n=1}^{\infty} n\rho_n(\lambda)\,\mathrm{d}\lambda = 0\,,$$

as it should be.

Let us consider the case when $0 < H \ll 1$ and $T = 0$ as before and introduce the following designation:

$$1 + \hat{k} = \hat{A}_{2s,2s}\,.$$

The resolvent $\hat{J}$ of eq. (78) for $H = T = 0$ is defined as

$$(1+\hat{J})(1+\hat{k}) = 1\,.$$

We can rewrite eq. (78) in the following form:

$$(1+\hat{k})\varepsilon_{2s}(\lambda) = H(2s) - \pi(1+\hat{k})p(\lambda) + \hat{k}(1-\hat{\tau})\varepsilon_{2s}(\lambda)\,, \tag{81}$$

where $\hat{\tau}$ is a projector. It fixes the interval $[-b, b]$ where ε_{2s} is negative. Acting from the left on eq. (81) by the operator $(1+\hat{J})$ we obtain

$$\varepsilon_{2s}(\lambda) = \tfrac{1}{2}H - \pi P(\lambda) - \hat{J}(1-\hat{\tau})\varepsilon_{2s}(\lambda)\,, \tag{82}$$

or

$$\varepsilon_{2s}(\lambda) = \tfrac{1}{2}H - \pi P(\lambda) - \int_b^{\infty} J(\lambda-\lambda')\varepsilon_{2s}(\lambda')\,\mathrm{d}\lambda' - \int_{-\infty}^{-b} J(\lambda-\lambda')\varepsilon_{2s}(\lambda')\,\mathrm{d}\lambda'\,. \tag{83}$$

Write $y(\lambda) = \varepsilon_{2s}(\lambda + b)$. Then

$$y(\lambda) = \tfrac{1}{2}H - P(\lambda+b) - \int_0^{\infty} J(\lambda-\lambda')y(\lambda')\,\mathrm{d}\lambda' - \int_0^{\infty} J(\lambda+\lambda'+2b)y(\lambda')\,\mathrm{d}\lambda'\,. \tag{84}$$

We assume that the magnetic field $H \ll 1$ which signifies $b \gg 1$. Then it is easy to show that the last term in eq. (84) is of the order $(1/b)(J(\lambda+\lambda'+2b) \sim 1/b)$. Thus we can solve the integral equation (84) by iteration:

$$y_1(\lambda) = \tfrac{1}{2}H - \pi P(\lambda+b) - \int_0^{\infty} J(\lambda-\lambda')y_1(\lambda')\,\mathrm{d}\lambda'\,, \tag{85}$$

$$y_2(\lambda) + \int_0^{\infty} J(\lambda-\lambda')y_2(\lambda')\,\mathrm{d}\lambda' = -\int_0^{\infty} J(\lambda+\lambda'+2b)y_1(\lambda')\,\mathrm{d}\lambda'\,. \tag{86}$$

Eqs. (85) and (86) are of the Winer–Hopt kind and can be solved analytically [23]. The solution of eq. (85) is of the form

$$y_1(\lambda)=\frac{1}{2\pi}\int \mathrm{e}^{-i\omega\lambda}\,y_1(\omega)\,\mathrm{d}\omega\,,$$

$$y_1(\omega)=\pi H\delta^+(\omega)G^-(0)G^+(\omega)-\tfrac{1}{2}\pi\sum_{n=0}^{\infty}\chi_n(\omega)\rho^{2n+1}(-1)^n\,, \tag{87}$$

where

$$\chi_n(\omega)=G^+(\omega)G^-(-\tfrac{1}{2}i\pi(2n+1))/(\tfrac{1}{2}\pi(2n+1)-i\omega)\,,$$

$$G^\pm(\omega)=\sqrt{\frac{\pi}{s}}\frac{\Gamma(\mp i\omega/\pi)f_\pm^{2s}(\omega)}{\Gamma(\mp i2s\omega/\pi)\Gamma(\frac{1}{2}\mp i\omega/\pi)}\,,$$

$$f_\pm(\omega)=\left(\mathrm{e}^{\pm i\pi/2}\frac{\omega}{\mathrm{e}\pi}\right)^{\pm i\omega/\pi}\,,$$

$$\rho=\mathrm{e}^{-\pi b/2}\,.$$

$G^\pm(\omega)$ are analytic in the upper and lower half-planes respectively and they factorize the function

$$1+k(\omega)=2\,\mathrm{cth}\,\omega\;\mathrm{sh}\,2s\omega\times\mathrm{e}^{-2s|\omega|}=G^+(\omega)G^-(\omega)\,.$$

Holding only the first term of the sum in (83) and using boundary conditions $y_1(0)=\varepsilon_{2s}(b)=0$ we obtain the following equation for b:

$$b=-\frac{2}{\pi}\ln\frac{HG^-(0)}{\pi G^-(-\frac{1}{2}i\pi)}\,. \tag{88}$$

The solution of eq. (86) is of the form

$$y_2(\omega)=-\frac{sH}{2b}G^-(0)G^+(\omega)\delta^+(\omega)\,. \tag{89}$$

We obtain the free energy (76) by substituting the solution $y=y_1+y_2$ into (76) when $T=0$:

$$\frac{1}{N}F(0,H)=F(0,0)-\frac{sH^2}{\pi^2}\left(1+\frac{s}{2\ln\rho}\right)\,. \tag{90}$$

For the magnetic susceptibility we have

$$\chi_{T=0}=\frac{2s}{\pi^2}\left(1+\frac{s}{2\ln\rho}\right)\,, \tag{91}$$

where

$$\rho=\frac{2H}{\pi^2}s\Gamma(s)(2\mathrm{e})^{-s}\,.$$

Let us now consider the case when $H=0$ and $T \ll 1$. In this case we can calculate the specific heat by the method described in [24]. Acting by the inverse matrix $\hat{A}_{kn}^{-1}$ on eq. (64), we obtain

$$\rho_n + \tilde{\rho}_n = \hat{p}(\tilde{\rho}_{n+1} + \tilde{\rho}_{n-1}) - \delta_{n,2s} p(\lambda) . \tag{92}$$

After the substitution $\lambda \to \lambda - (2/\pi) \ln(2T/\pi)$ in eqs. (73) and (92) and differentiation of eq. (73), when $T \ll 1$, we obtain the following relations:

$$\rho_n = \frac{2}{\pi^2} \frac{d\varepsilon_n}{d\lambda} f(\varepsilon_n) , \tag{93}$$

$$\tilde{\rho}_n = \frac{2}{\pi^2} \frac{d\varepsilon_n}{d\lambda} (1 - f(\varepsilon_n)) . \tag{94}$$

If $T \to 0$ in (94), taking into account (79) we obtain that $\tilde{\rho}_n^0 = 0$. Then from (64) it follows that

$$\rho_n(\lambda) = \delta_{n,2s} p(\lambda) . \tag{95}$$

Since by definition ρ_n and $\tilde{\rho}_n$ are positive functions, it is easy to see from eqs. (93), (94) that $d\varepsilon_n/d\lambda \geq 0$. Substituting (93) and (94) into (68) we find

$$S/N = -\frac{2T}{\pi^2} \sum_{n=1}^{\infty} \int_{\min \varphi_n}^{\max \varphi_n} [f(\varphi_n) \ln f(\varphi_n) + (1 - f(\varphi_n)) \ln(1 - f(\varphi_n))] \, d\varphi_n , \tag{96}$$

where

$$\varphi_n(\lambda) = \frac{1}{T} \varepsilon_n \left(\lambda - \frac{2}{\pi} \ln \frac{2T}{\pi} \right) .$$

We calculate the maxima and minima of φ_n in the following way. After the substitution $\lambda \to \lambda - (2/\pi) \ln(2T/\pi)$, eq. (73), (74) take the form (when $T \ll 1$)

$$\begin{gathered} \varphi_1 = -\hat{p} \ln f(\varphi_2) , \\ \varphi_n = -\hat{p} \ln [f(\varphi_{n+1}) f(\varphi_{n-1})] - \delta_{n,2s} e^{-\pi\lambda/2} , \\ \lim_{n \to \infty} \frac{\varphi_n}{n} = 0 . \end{gathered} \tag{97}$$

Write $\hat{\varphi}_n = \max \varphi_n = \varphi_n(\infty)$ and $\check{\varphi}_n = \min \varphi_n = \varphi_n(-\infty)$. At $\lambda \to \infty$ in (97) we obtain the equations for $\hat{\varphi}_n$:

$$\begin{gathered} \hat{\varphi}_1 = -\tfrac{1}{2} \ln f(\hat{\varphi}_2) , \\ \hat{\varphi}_n = -\tfrac{1}{2} \ln [f(\hat{\varphi}_{n+1}) f(\hat{\varphi}_{n-1})] , \\ \lim_{n \to \infty} \frac{\hat{\varphi}_n}{n} = 0 . \end{gathered} \tag{98}$$

The solutions of these equations has the form

$$\tilde{\varphi}_n = \ln\left[(n+1)^2 - 1\right]. \tag{99}$$

At $\lambda \to -\infty$ in (97) we have

$$\begin{aligned} \bar{\varphi}_1 &= -\tfrac{1}{2}\ln f(\bar{\varphi}_2)\,, \\ \bar{\varphi}_n &= -\tfrac{1}{2}\ln\left[f(\bar{\varphi}_{n+1})f(\bar{\varphi}_{n-1})\right], \\ \bar{\varphi}_{2s-1} &= -\tfrac{1}{2}\ln f(\bar{\varphi}_{2s-2})\,, \end{aligned} \tag{100}$$

and

$$\begin{aligned} \bar{\varphi}_{2s} &= -\infty\,, \\ \bar{\varphi}_n &= -\tfrac{1}{2}\ln\left[f(\bar{\varphi}_{n+1})f(\bar{\varphi}_{n-1})\right], \\ &\lim_{n\to\infty}\frac{\bar{\varphi}_n}{n} = 0\,. \end{aligned} \tag{101}$$

It is easy to verify that solutions of this equations are of the form

$$\bar{\varphi}_n = \ln\left[\sin^2\frac{\pi(n+1)}{2(s+1)}\sin^{-2}\frac{\pi}{2(s+1)} - 1\right], \qquad n < 2s\,, \tag{102}$$

$$\bar{\varphi}_n = \ln\left[(n-2s+1)^2 - 1\right], \qquad n \geqslant 2s\,. \tag{103}$$

Calculation of the entropy (96) with taking into account (99), (102) and (103) gives us

$$S/N = \tfrac{2}{3}T - \frac{2T}{\pi^2}\sum_{n=1}^{2s-1}\int_0^{1/x_n^2} g(x)\,\mathrm{d}x\,, \tag{105}$$

where

$$g(x) = \frac{1}{x}\ln(1-x) + \frac{1}{1-x}\ln x\,,$$

$$x_n^2 = \sin^2\frac{\pi(n+1)}{2(s+1)}\sin^{-2}\frac{\pi}{2(s+1)}\,.$$

We obtain the specific heat at $s = \frac{1}{2}$ and $s = 1$,

$$c_{1/2} = \tfrac{2}{3}T\,, \qquad c_1 = T\,.$$

In conclusion we give that argument that Bethe's vectors (31) are complete, i.e. their number is equal to $(2s+1)^N$. Note that in the case when $s = \frac{1}{2}$ and $s = 1$ there exists an exact proof of this statement [4, 14].

Substituting $\lambda \to \lambda - (2/\pi)\ln(2T/\pi)$ in (76), taking $T \to \infty$ and taking into account (99) we obtain

$$\frac{1}{N}F(T,0) = F(0,0) - T\ln(2s+1)\,,$$

We have for the entropy

$$S(T) = \ln (2s+1)^N .$$

This equation signifies that the number of states is equal to $(2s+1)^N$.

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References

[1] C.N. Yang, Phys. Rev. 168 (1968) 1920
[2] R.J. Baxter, Ann. of Phys. 70 (1972) 193
[3] W. Heisenberg, Z. Phys. 49 (1928) 619
[4] H. Bethe, Z. Phys. 71 (1931) 205
[5] C.N. Yang and C.P. Yang, Phys. Rev. 150 (1966) 321
[6] C.N. Yang and C.P. Yang, Phys. Rev. 150 (1966) 321
[7] L. Hulten, Arkiv für Matematik, Astronomi och Fysik 26A (1938) 1
[8] J. Des Cloizeaux and J. Pearson, Phys. Rev. 128 (1962) 2131
[9] R. Orbach, Phys. Rev. 112 (1958) 309
[10] M. Takahashi, Prog. Theor. Phys. 46 (1971) 401
[11] A.A. Ovchinikov, JETP (Sov. Phys.) 56 (1969) 1354
[12] L.D. Faddeev and L.A. Takhtajan, Phys. Lett. 85A (1981) 375
[13] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Theor. Mat. Fiz. 40 (1979) 194
[14] L.A. Takhtajan, Phys. Lett. 87A (1982) 479
[15] R.J. Baxter, Ann. of Phys. 70 (1972) 323
[16] V.A. Fateev and P.B. Wiegmann, Phys. Rev. Lett. 46 (1981) 1955
[17] V.A. Fateev, unpublished.
[18] P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. 5 (1981) 393
[19] A.B. Zamolodchikov and V.A. Fateev, Yad. Fis. 32 (1980) 581
[20] V.A. Fateev, unpublished
[21] V.A. Fateev, unpublished
[22] V.A. Fateev and P.B. Wiegmann, unpublished
[23] M.G. Krein, Usp. Mat. Nauk 13 (1958) 3
[24] V M. Filyov, A.M. Tsvelick and P.B. Wiegman, Phys. Lett. 81A (1981) 175