Quantifier Guided Aggregation Using OWA Operators

Ronald R. Yager
Machine Intelligence Institute; Iona College, New Rochelle, New York 10801

We consider multicriteria aggregation problems where, rather than requiring all the criteria be satisfied, we need only satisfy some portion of the criteria. The proportion of the critera required is specified in terms of a linguistic quantifier such as **most.** We use a fuzzy set representation of these linguistic quantifiers to obtain decision functions in the form of OWA aggregations. A methodology is suggested for including importances associated with the individual criteria. A procedure for determining the measure of "orness" directly from the quantifier is suggested. We introduce an extension of the OWA operators which involves the use of triangular norms. © 1996 John Wiley & Sons, Inc.

I. INTRODUCTION

Starting with the classic work of Bellman and Zadeh,[1] fuzzy set theory has been used as a tool to develop and model multicriteria decision problems. In this framework the criteria are represented as fuzzy subsets over the space of decision alternatives and fuzzy set operators are used to aggregate the individual criteria to form the overall decision function. As originally suggested by Bellman and Zadeh, the criteria are combined by the use of an intersection operation which implicitly implies a requirement that **all** the criteria be satisfied by a solution to the problem. As noted by Yager,[2,3] this condition may not always be the appropriate relationship between the criteria. For example, a decision maker may be satisfied if **most** of the criteria are satisfied. In this work we look at the issue of the formulation of these softer decision functions which we call **quantifier guided aggregations.** In Ref. 4 we suggested the use of the Ordered Weighted Averaging (OWA) operators as a tool to implement these kinds of aggregations. Here we develop this approach further by considering environments in which the individual criteria have importances associated with them. We also provide for an extension of the basic OWA aggregation which allows us to include triangular norm operations. A number of other related issues such as the measure of *orness* and determination of weights in OWA aggregation are discussed.

INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 11, 49–73 (1996)

CCC 0884-8173/96/010049-25

II. LINGUISTIC QUANTIFIERS

In natural language we find many examples of what Zadeh[5] called linguistic quantifiers. These objects are exemplified by terms such as *most, many, at least half, some,* and *few*. Classical logic uses only two of these terms; the existential quantifier, *there exists,* and the universal quantifier, *all,* in forming logical propositions. In Ref. 5 Zadeh suggested a formal representation of these linguistic quantifiers using fuzzy sets. Furthermore, Zadeh distinguished between two classes of linguistic quantifiers absolute and relative. Absolute quantifiers can be represented as a fuzzy subset of the *non-negative reals*. In particular, if we have an absolute quantifier, such as *about 10,* we can represent it as a fuzzy subset Q of the non-negative reals, R^+. In this representation for any $X \in R^+$ we use $Q(x)$ to indicate the degree to which x satisfies the concept conveyed by the linguistic quantifier. A relative or proportional linguistic quantifier indicates a proportional quantity such as *most, few,* or *about half.* Zadeh suggested that any relative quantifier can be expressed as a fuzzy subset Q of *the unit interval, I.* Again in this representation for any proportion $y \in I$, $Q(y)$ indicates the degree to which y satisfies the concept conveyed by the term Q.

In Ref. 6 Yager further distinguished three categories of these relative quantifiers. A fuzzy subset Q of the real line is called a **R**egular **I**ncreasing **M**onotone (RIM) quantifier if

1. $Q(0) = 0$; **2.** $Q(1) = 1$; **3.** $Q(x) \geq Q(y)$ if $x > y$.

Examples of this kind of quantifier are *all, most, many, at least* α.

A fuzzy subset Q of the real line is called a **R**egular **D**ecreasing **M**onotone (RDM) quantifier if

1. $Q(0) = 1$; **2.** $Q(1) = 0$; **3.** $Q(y) \geq Q(x)$ if $x > y$

Examples of these kinds of quantifier are *at most one, few, at most* α.

A fuzzy subset Q of the real line is called a **R**egular **U**ni**M**odal (RUM) quantifier if

1. $Q(0) = 0$
2. $Q(1) = 0$
3. there exists two values a and $b \in I$, where $a < b$, such that
   (i) For $y < a$, $Q(x) \leq Q(y)$ if $x < y$.
   (ii) For $x \in [a, b]$, $Q(x) = 1$.
   (iii) For $x > b$, $Q(x) \geq Q(y)$ if $x < y$.

An example of this class ia *about* α.

Some interesting relationships exist between these three classes of relative quantifiers. Noting that the antonym $\hat{F}$ of a fuzzy F [7] on the real line is defined as $\hat{F}(x) = F(1 - x)$, we see that if Q is a RIM quantifier, then its antonym is a RDM quantifier and vice versa. Examples of these antonym pairs are *few* and *many,* and *at least* α and *at most* α.

Furthermore, any RUM quantifier can be expressed as the intersection of a RIM and RDM quantifier. Assume Q is a unimodal quantifier such that $Q(x) = 1$ when $x \in [a, b]$. Let $Q_1(x)$ be a quantifier such that

(1) $Q_1(x) = Q(x)$ for $x \leq b$; (2) $Q_1(x) = 1$ for $x > b$.

It is easily seen that $Q_1(x)$ is monotone increasing quantifier. Let $Q_2(x)$ be a quantifier such that

(1) $Q_2(x) = 1$ for $x < a$; (2) $Q_2(x) = Q(x)$ for $x \geq a$.

This can be easily seen to be monotone decreasing quantifier. Furthermore, if we define

$$Q^+ = Q_1 \cap Q_2$$

where

$$Q^+(x) = T[Q_1(x), Q_2(x)]$$

and T is any t-norm [8], i.e., min or product, then $Q^+(x) = Q(x)$.

The quantifier **for all** is represented by the fuzzy subset Q_* where

$$Q_*(1) = 1 \text{ and } Q_*(x) = 0 \text{ for all } x \neq 1.$$

The quantifier **there exist,** not none, is defined as

$$Q^*(0) = 0 \text{ and } Q^*(x) = 1 \text{ for all } x \neq 0.$$

Both of these are examples of RIM quantifiers.

The antonym of *all* is the quantifier $\hat{Q}$ with $\hat{Q}(0) = 1$ and $\hat{Q}(x) = 0$ for all $x \neq 0$. It is semantically equivalent to the linguistic term *none*.

Consider the parameterized fuzzy subset defined on I such that

$$Q(r) = r^\alpha \qquad \alpha \geq 0$$

We can see that this formulation defines a family of RIM quantifiers. Three special cases of these family are worth noting:

1. For $\alpha = 1$ we get $Q(r) = r$. This is called the **unitor** quantifier
2. For $\alpha \to \infty$ we get Q_*, the universal quantifier.
3. For $\alpha \to 0$ we get Q^*, the existential quantifier.

Another parameterized class of fuzzy subsets, that provide a family of RIM quantifiers is

$$Q(r) = \frac{2}{1 + e^{-\lambda}\left(\frac{r}{1-r}\right)} - 1,$$

$\lambda \in [0, \infty]$. Again here as $\lambda \to \infty$ we get Q_* while $\lambda \to 0$ gives us Q^*. The following theorem shows that all RIM quantifiers are bound by Q^* and Q_*.

THEOREM. *Assume Q is any RIM quantifier than for all $x \in I$*

$$Q_*(x) \leq Q(x) \leq Q^*(x).$$

Proof. Since $0 \leq Q(x) \leq 1$ the result follows from the definitions of RIM, Q_* and Q^*.

III. QUANTIFIER GUIDED AGGREGATION

Assume we are faced with a decision problem in which we have a collection of n criteria of interest. We denote these criteria as $A_1, \ldots, A_n$. For any solution x we can evaluate the degree to which it satisfies any of the criteria, we shall denote this as $A_i(x) \in [0, 1]$. In this framework A_i can be viewed as a fuzzy subset over the set of alternatives. In order to determine the appropriateness of a particular alternative x as the solution to our problem, we must aggregate its scores to the individual criteria to find some overall single value to associate with the alternative. In order to obtain this overall evaluation, implement the aggregation, some information must be provided on the relationship between the criteria that are to be aggregated. In their classic work Bellman and Zadeh[1] suggested an approach to this problem which uses

$$\mathbf{Agg}(A_1(x), A_2(x), \ldots\ldots A_n(x)) = \mathrm{Min}_i[A_i(x)].$$

Essentially, this approach is assuming that we desire **all** the criteria be satisfied by an acceptable solution. One then selects, as the best solution, the alternative with the highest aggregated value.

In Ref. 2 Yager suggested a generalization of this approach which he called *quantifier guided aggregation*. In order to formally express this technique we must first recall the OWA aggregation operator [4, 9].

DEFINITION. *An aggregation operator F,*

$$F: I^n \rightarrow I$$

is called an ***Ordered Weighted Averaging*** *(OWA) operator of dimension n if it has associated with it a weighting vector W,*

$$W = \begin{bmatrix} \omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \omega_n \end{bmatrix},$$

such that **1.** $\omega_i \in [0, 1]$ *and* **2.** $\Sigma_{i=1}^n \omega_i = 1$ *and where*

$$F(a_1, \ldots\ldots, a_n) = \sum_{j=1}^{n} \omega_j b_j$$

in which b_j is the jth largest of the a_i.

An essential feature of this aggregation is the reordering operation, a nonlinear operator, that is used in the process. Thus in the OWA aggregation the weights are not associated with a particular argument but with the ordered position of the arguments.

In Ref. 4 Yager shows that OWA aggregation has the following properties:

(1) **Commutativity:** The indexing of the arguments is irrelevant
(2) **Monotonicity:** If $a_i \geq \hat{a}_i$ for all i then $F(a_1, \ldots\ldots, a_n) \geq F(\hat{a}_1, \ldots\ldots, \hat{a}_n)$.
(3) **Idempotency:** $F(a, \ldots a) = a$.
(4) **Bounded:** $\text{Max}_i[a_i] \geq F(a_1, \ldots\ldots, a_n) \geq \text{Min}_i[a_i]$

We note that these conditions imply that the OWA operator is a mean operator [10, 11].

The form of the aggregation is very strongly dependent upon the weighting vector used. In Ref. 12 Yager investigates various different families of OWA aggregation operators. A number of special cases of weighting vector are worth noting. The weighting vector W^* defined such that

$$\omega_1 = 1 \text{ and } \omega_j = 0 \text{ for all } j \neq 1$$

gives us the aggregation $F^*(a_1, \ldots, a_n) = \text{Max}_i[a_i]$. Thus W^* provides the largest possible aggregation.

The weighting vector W_* defined such that

$$\omega_n = 1 \text{ and } \omega_i = 0 \text{ for } i \neq n$$

gives us the aggregation $F_*(a_1, \ldots a_n) = \text{Min}_i[a_i]$. This weighting provides the smallest aggregation of the arguments.

The weighting vector W_a defined such that $\omega_i = \dfrac{1}{n}$ for all i gives us the simple average

$$F_A(a_1, \ldots a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

The weighting vector W^k defined such that $\omega_k = 1$ and $\omega_i = 0$ for $i \neq k$ gives us $F(a_1, \ldots a_n) = b_k$ where b_k is the kth largest of the a_i.

One other weighting worth noting, is one we shall call the olympic aggregate. In this case

$$\omega_1 = 0$$

$$\omega_n = 0$$

$$\omega_i = \frac{1}{n-2} \qquad \text{for } i \neq 1 \text{ or } n$$

Having introduced the OWA aggregation operator we are now in a position to describe the process of quantifier guided aggregation. Again consider that we have a collection of $A_1, \ldots, A_n$ of criteria. These criteria are represented as fuzzy subsets over the set of alternatives X. In the process of quantifier guided aggregation the decision maker provides a linguistic quantifier Q indicating the proportion of criteria he feels is necessary for a good solution. Essentially in this framework the decision maker is providing an agenda indicating the structure to be used to aggregate the individual criteria to get an overall decision function. The form of decision function implicit in this approach is

Q criteria are satisfied by a good solution.

The formal procedure used to evaluate this decision function is expressed in the following. The quantifier is used to generate an OWA weighting vector W of dimension n. This weighting vector is then used in an OWA aggregation to determine the overall evaluation for each alternative. For each alternative the argument of this OWA aggregation is the satisfaction of the alternative to each of the criteria, $A_i(x)$, $i = 1 \ldots n$. Thus the process used in quantifier guided aggregation is as follows

(1) Use Q to generate a set of OWA weights, $\omega_1, \ldots \omega_n$.
(2) For each alternative x in X calculate the overall evaluation

$$D(x) = F(A_1(x), A_2(x), \ldots A_n(x))$$

where F is an OWA aggregation using the weights found in 1.

The procedure used for generating the weights from the quantifier depends upon the type of quantifier provided. We shall here consider the case in which Q is a RIM quantifier. In this case the weights are generated as

$$\omega_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \qquad \text{for } i = 1 \ldots n$$

Because of the nondecreasing nature of Q it follows that $\omega_i > 0$. Furthermore, from the regularity of Q, $Q(1) = 1$ and $Q(0) = 0$, it follows that $\Sigma_i \omega_i = 1$. Thus we see that the weights generated are an acceptable class of OWA weights.

The use of a RIM quantifier to guide the aggregation essentially implies that the more criteria satisfied the better the solution. This condition seems to be one that is naturally desired in criteria aggregation. Thus most quantifier guided aggregation would seem to be based upon the use of these types of quantifiers. Not withstanding the above observation the technique of quantifier guided aggregation can be applied to other types of quantifiers. In Refs. 9 and 13 Yager describes the process used in the case in which we have RDM and RUM quantifiers. We shall not pursue this issue here and in the following assume all quantifiers are RIM.

IV. MEASURE OF ORNESS OF A QUANTIFIER

As we previous noted the class of RIM quantifiers are bound by the quantifiers "there exists," Q^*, and "for all," Q_*. Thus for any quantifier of this type $Q_*(r) \leq Q(r) \leq Q^*(r)$. As we have indicated Q^* leads to the weighting vector

$$W^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which results in the OWA aggregation $D(x) = \text{Max}_i[A_i(x)]$. This can be seen as an *oring* of the criteria. At the other extreme Q_* leads to the weighting vector

$$W_* = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

which results in the OWA aggregation $D(x) = \text{Min}_i[A_i(x)]$. This can be seen as an "anding" of the criteria. Thus we see that this family of quantifiers provide for aggregation of the satisfaction to the criteria lying between an "anding" and "oring."

In Ref. 4 Yager associated with any OWA aggregation a measure of its degree of *orness*. In particular, if we have a weighting vector W of dimension n then the measure of *orness* is defined as

$$orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\omega_i.$$

It is easy to show that this measure lies in the unit interval. Furthermore, it was shown in Ref. 4 that $orness(W^*) = 1$, $orness(W_{ave}) = 0.5$ and $orness(W_*) = 0$. We can easily extend this measure to the case in which the weights are generated by any quantifier. Given a linguistic quantifier Q if we generate the weights by $\omega_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)$, then we can associate with this quantifier a degree of *orness* as

$$orness(Q) = \frac{1}{n-1}\sum_{j=1}^{n}(n-j)\left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right)$$

Algebraic manipulation of the formula leads to the form

$$orness(Q) = \frac{1}{n-1}\sum_{j=1}^{n-1} Q\left(\frac{j}{n}\right)$$

Furthermore, if we let $n \to \infty$ then we can show that

$$orness(Q) = \int_0^1 Q(r)\,dr.$$

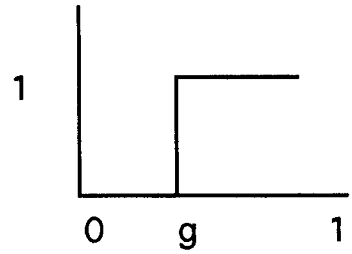

Figure 1. Quantifier.

Thus the nominal degree of *orness* associated with a RIM linguistic quantifier is equal to the area under the quantifier.

This standard definition for the measure of *orness* of quantifier provides a simple useful method of obtaining this measure. Consider, for example (see Fig. 1), the class of quantifiers defined by

$$Q(r) = 0 \qquad r \leq g$$

$$Q(r) = 1 \qquad r < g$$

In this case calculating the degree of *orness* as the area under the quantifier we get

$$orness(Q) = 1 - g.$$

We note in the special case when $g = 0$, we get the pure "or" with *orness* $(Q) = 1$ and when $g = 1$ we get the pure "and" with $orness(Q) = 0$.

If we consider the quantifier

$$Q(r) = r^{\alpha} \qquad \alpha \geq 0$$

then

$$orness(Q) = \int_0^1 r^{\alpha}\, dr = \frac{1}{\alpha + 1} r^{\alpha+1}|_0^1 = \frac{1}{\alpha + 1}$$

A number of interesting properties can be associated with this measure of *orness*. If Q_1 and Q_2 are two quantifiers such that $Q_1(x) \geq Q_2(x)$ for all x then $orness(Q_1) \geq orness(Q_2)$. In addition, since any regular quantifier is normal we see that $orness(Q) = 0$ iff $Q = Q_*$.

In Refs. 14 and 15 Yager associated with any fuzzy subset a measure called the degree of specificity. The specificity measures the degree to which the fuzzy subset consists of exactly one element. In Ref. 16 he describes a number of properties required of any measure of specificity. Using these properties it can be shown that for the class of RIM quantifiers the measure of *orness* is inversely related to the measure of specificity. Thus

$$S_p(Q) = 1 - orness(Q),$$

increasing the *orness* decreases the specificity. In particular the pure "or" quantifier, Q^*, has maximal *orness* and minimal specificity. On the other hand, the pure "and" quantifier, Q_*, has minimal *orness* and maximal specificity. In this way we can introduce an idea of specificity of aggregation with the "and" being the most specificity and the "or" the least specific.

In some applications of fuzzy subsets, particularly in the theory of approximate reasoning,[17–19] considerable use is made of the principle of minimal specificity.[19,20] This principle says that if Δ is some operation on fuzzy subsets which can be implemented in various different ways, then select the implementation that leads to a resulting fuzzy subset with minimal specificity. Under the imperative of this principle when no further requirements are made on a quantifier other than it be RIM, the preferred choice is the quantifier Q^* in that it leads to the minimal specificity.

V. ALTERNATIVE GENERATION OF WEIGHTS

In the framework of this article we are mainly focusing on the issue of developing aggregation structures in situations in which we have at our disposal some linguistic quantifier to guide the aggregation. In this framework, as we indicated, we use the information $\omega_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)$ to provide the weights associated with the quantifier. In other situations where a quantifier is not specified we may have to use different methods to generate the weights. One very useful approach to this problem was suggested by O'Hagan.[21] In this approach rather than starting with a quantifier Q we have at our disposal a measure of *orness*, α, associated with the aggregation process and we generate the weights by solving the following constraint optimization problem:

$$\textbf{Max: } -\sum_{i=1}^{n} \omega_i \ln \omega_i$$

such that

1. $\omega_i > 0$

2. $\sum_{i=1}^{n} \omega_i = 1$

3. $\frac{1}{n-1}\sum_{i=1}^{n} \omega_i(n-i) = \alpha$

In this approach constraints one and two assures us that the weights satisfy the OWA conditions. Constraint three assures us that the weights have an *orness* value of α. The objective function is a measure of entropy or dispersion associates with the weights O'Hagan calls the weights generated by the technique ME-OWA weights, indicating maximal entropy weights. In choosing this objective function we are essentially selecting the weights in a manner that makes maximal use of the information in the arguments. Consider a situation in which $\alpha = 0.5$. This degree of *orness* can be obtained in a number of different ways among these are: $w_1 = 0.5$ and $w_n = 0.5$; $\frac{w_{n+1}}{2} = 1$; and $w_i = \frac{1}{n}$ for all i. The case in which $w_i = \frac{1}{n}$ for all i, the one selected by the ME-OWA algorithm, is the one which most uniformly uses the information in arguments.

Another class of problems involving the generation of the weights are situations in which we have observations in which we have a collection of arguments and an associated aggregated value and we want to use these to generate the weights. In Ref. 22 Filev and Yager suggested an approach to solving this problem. In the following, we shall consider some alternatives to that approach.

Assume we have a observation consisting of a set of arguments, $A_1(x), \ldots, A_p(x)$ and an associated aggregated value $D(x) = d$. We can now consider the ordering of these scores to give us $b_1, \ldots, b_n$ along with d. In the spirit of the approach suggested by O'Hagan, we can consider the generation of the weights underlying this aggregation process by the following mathematical program problem:

$$\textbf{Max:} - \sum_{i=1}^{n} \omega_i \ln \omega_i$$

such that

1. $\omega_i > 0$
2. $\sum_{i=1}^{n} \omega_i = 1$
3. $\sum_{i=1}^{n} \omega_i b_i = d.$

In the above we replaced condition three by the requirement that the aggregation equals d.

In situations in which we have a collection of observations of the above type we can proceed as follows. Assume we have a collection of data each of the type (B_j, d_j) where B_j is an ordered vector $B_j = [b_{j1}, b_{j2}, \ldots b_{jn}]$ consisting of the arguments and d_j is its aggregated value. For each observation we can solve the preceding mathematical program problem to obtain a weighting vector W^j. We can apply some further procedure to obtain a weighting vector $\hat{W}$ that best matches this collection of vectors. For example, we could use a least squares fit.

Another approach could be to convert each of these W^j into its associated measure of *orness*, $orness(W^j) = \alpha_j$. We can then find the average of these degrees of *orness*. Assuming we have K samples then

$$\hat{\alpha} = \frac{1}{K}\sum_{j=1}^{K} \alpha_j$$

Using O'Hagan's original approach we can then use $\hat{\alpha}$ to generate an OWA weight $\hat{W}$ which can be considered as representative of the aggregation process generating the data.

As we noted above in many applications of fuzzy set theory an often used imperative for furnishing missing information is the principle of minimal specificity. In Refs. 19 and 20 Dubois and Prade discuss the use of this principle in considerable detail. Motivated by this principle we can suggest another procedure for generating weights from observations. Assume we have an observation $b_1, \ldots . b_n$ and d. The b_is are the ordered argument and the d the aggregated value. As we have already indicated, the measure of *orness* is inversely related to the measure of specificity. Using this relationship we can consider the following mathematical programming problem to generate the underlying weight

$$\text{Max:} \frac{1}{n-1}\sum_{i=1}^{n} w_i(n-i)$$

such that

1. $\omega_i > 0$
2. $\sum_{i=1}^{n} \omega_i = 1$
3. $\sum_{i=1}^{n} \omega_i b_i = d$.

In previous section we described a procedure for obtaining the OWA weights of dimension n from a given RIM quantifier, the inverse problem is that of determining a quantifier associated with a given weighting vector. Assume W is an OWA vector of dimension n we shall associate with this vector a quantifier Q. Furthermore, we can assign to this quantifier some values.

$$Q\left(\frac{i}{n}\right) = \sum_{j=1}^{i} w_j \qquad \text{for } i = 0, \ldots, n$$

We must now provide for the values of $Q(r)$ between these fixed points. One approach in the spirit of maximal entropy is to use a piecewise linear construction of Q. Thus in this case

$$Q(r) = \omega_i(nr - i) + Q\left(\frac{i}{n}\right) \qquad \text{for } \frac{i-1}{n} \leq r \leq \frac{i}{n}.$$

A second approach in the spirit of minimal specificity is to generate Q from the weights as

$$Q(0) = 0$$

$$Q(r) = Q\left(\frac{i}{n}\right) \qquad \text{for } \frac{i-1}{n} < r \le \frac{i}{n}$$

One important application of the generation of quantifiers from weighting vectors is in the extension of OWA operators. Assume we have an OWA operator of dimension n with weighting vector W. Consider now we are interested in extending this aggregation to the case in which our dimension of aggregation is m. In this case we can proceed as follows. Assume there exists some underlying quantifier Q. Generate the form of the quantifier by the preceding approach. For example

$$Q(r) = \omega_i(nr - i) + \sum_{j=1}^{i} \omega_i \qquad \text{for } \frac{i-1}{n} \le r \le \frac{i}{n}$$

We then can generate the weights associated with the aggregation of dimension m as

$$\hat{\omega}_i = Q\left(\frac{i}{m}\right) - Q\left(\frac{i-1}{m}\right).$$

VI. IMPORTANCE WEIGHTED QUANTIFIER GUIDED AGGREGATION

In the section we turn to the problem of quantifier guided aggregation in environments in which the criteria to be aggregated have importances associated with them. Related approaches can be found in Refs. 4 and 23. In this environment we shall again assume we have a set of n criteria expressed as fuzzy subsets over the space of alternative solutions X. We again denote these criteria as A_i, where $A_i(x)$ is the satisfaction of alternative x to ith criteria. In introducing quantifier guided aggregation we essentially considered as our overall decision function the statement *Q criteria are satisfied by x*. Where Q is some RIM linguistic quantifier. We now additionally assume that we can associate with each criteria a value V_i indicating the importance of that criteria. We shall consider the V_is to lie in the unit interval $V_i \in [0, 1]$, with the understanding that the larger the value the more important the criteria. We make no restrictions on the total value of importances, that is they need not sum to one.

Again considering Q to be some RIM quantifier we now assume the overall evaluation function to be *Q **important** criteria are satisfied by x*. In the following we describe the procedure to evaluate the overall satisfaction of alternative x. First, we note for a given alternative x we have a collection of n pairs $(V_i, A_i(x))$. The first step in this process is to order the $A_i(x)$s in descending order. Thus we let b_j be the jth largest of $A_i(x)$. Furthermore, we let u_j denote the importance associated with the criteria that has the jth largest satisfaction to x. Thus if $A_5(x)$ is the largest of the $A_i(x)$ then $b_1 = A_5(x)$ and $u_1 = V_5$. At this point we can consider our information regarding the alternative x to be a collection of n pairs (u_j, b_j) where the b_js are in descending ordering.

Our next step is to obtain the OWA weights associated with this aggregation. To obtain these weights we proceed as follows

$$\omega_j(x) = Q\left(\frac{\sum_{k=1}^{j} u_k}{T}\right) - Q\left(\frac{\sum_{k=1}^{j-1} u_k}{T}\right)$$

in the above $T = \sum_{K=1}^{n} u_k$, the total sum of importances. Having obtained the weights we can now calculate the evaluation associated with x_j, which we denote as $D(x)$:

$$D(x) = F_w(b_1, \ldots, b_n)$$

thus

$$D(x) = \sum_{j=1}^{n} b_j \omega_j(x)$$

We emphasize that the weights used in this aggregation will generally be different for each x. This is due to the fact that the ordering of the A_is will be different and in turn lead to different u_js. The following example illustrates the application of the above method.

EXAMPLE. Assume we have two alternatives x and y. We shall assume four criteria A_1, A_2, A_3, A_4. The importances associated with these criteria are: $V_1 = 1$, $V_2 = 0.6$, $V_3 = 0.5$, and $V_4 = 0.9$. Furthermore, the satisfaction of each of the criteria by the alternatives is given by the following:

$$A_1(x) = 0.7 \qquad A_2(x) = 1 \qquad A_3(x) = 0.5 \qquad A_4(x) = 0.6$$
$$A_1(y) = 0.6 \qquad A_2(y) = 0.3 \qquad A_3(y) = 0.9 \qquad A_4(y) = 1$$

We shall assume the quantifier guiding this aggregation to be "most" which is defined by $Q(r) = r^2$. We first consider the aggregation for x. In this case the ordering of the criteria satisfaction give us

	b_j	u_j
A_2	1	0.6
A_1	0.7	.1
A_4	0.6	0.9
A_3	0.5	0.5

We note $T = \sum_{i=1}^{4} u_j = 3$. Calculating the weights associated with x, which we denote $\omega_i(x)$, we get

$$\omega_1(x) = Q\left(\frac{0.6}{3}\right) - Q\left(\frac{0}{3}\right) = (0.2)^2 - 0 = 0.04$$

$$\omega_2(x) = Q\left(\frac{1.6}{3}\right) - Q\left(\frac{0.6}{3}\right) = .28 - .04 = .24$$

$$\omega_3(x) = Q\left(\frac{2.5}{3}\right) - Q\left(\frac{1.6}{3}\right) = .69 - .28 = .41$$

$$\omega_4(x) = \left(\frac{3}{3}\right) - Q\left(\frac{2.5}{3}\right) = 1 - .69 = .31$$

To obtain $D(x)$ we calculate

$$D(x) = \sum_{i=1}^{4} \omega_i(x)b_i = (.04)(1) + (.24)(.7) + (.41)(.6) + (.31)(.5) = 0.609$$

To calculate the evaluation for y we proceed as follows. In this case the ordering of the criteria satisfaction is

	b_j	u_j
A_4	1	0.9
A_3	0.9	0.5
A_1	0.6	1
A_2	0.3	0.6

The weights associated with the aggregation are:

$$\omega_1(y) = Q\left(\frac{.9}{3}\right) - Q\left(\frac{0}{3}\right) = .09 - 0 = .09$$

$$\omega_2(y) = Q\left(\frac{1.4}{3}\right) - Q\left(\frac{.9}{3}\right) = .22 - .09 = .13$$

$$\omega_3(y) = Q\left(\frac{2.4}{3}\right) - Q\left(\frac{1.4}{3}\right) = .64 - .22 = .42$$

$$\omega_4(y) = Q(1) - Q\left(\frac{2.4}{3}\right) 1 - .64 = .36$$

To obtain $D(y)$ we calculate

$$D(y) = \sum_{i=1}^{4} \omega_i(y)b_i = (.09)(1) + (.13)(.9) + (.42)(.6) + (.36)(.3) = 0.567$$

Hence in this example x is the preferred alternative.

It is important to observe as we previously noted the weights are different for the two aggregations.

In the calculation of the weights we make considerable use of summations of the form $\Sigma_{k=1}^{j} u_k$ we shall find it convenient to note these as S_j, thus

$$S_j = \sum_{k=1}^{j} u_k.$$

Using this notation we get

$$\omega_j = Q\left(\frac{S_j}{T}\right) - Q\left(\frac{S_j - 1}{T}\right)$$

where

$$T = S_n = \sum_{k=1}^{n} u_k$$

VII. CHARACTERISTICS OF IMPORTANCE WEIGHTED AGGREGATION

In using this approach for the inclusion of importances associated with the criteria, we should point out that any criteria having zero importance plays no role in the formulation of the overall evaluation function. We see this from the following. Assume that A_i is a criteria with importance V_i and assume that $A_i(x)$ is the jth largest of the satisfactions. In this case $b_j = A_i(x)$ and $u_j = V_i$. The weights ω_j associated with this component in the OWA aggregation is

$$\omega_j = Q\left(\frac{S_j}{T}\right) - Q\left(\frac{S_j - 1}{T}\right).$$

Since $S_j - S_j - 1 = u_j$ and in the case of zero importance $u_j = 0$, we get $S_j = S_{j-1}$ and hence $\omega_j = 0$. Thus a component with zero importance has zero weight and provides no contribution to the aggregation. As a matter of fact we can remove it from the whole process.

A second technical consideration we must investigate is the situation in which two criteria have the same satisfaction. Without loss of generality assume $A_1(x) = A_2(x) = \alpha$. Assume that in the ordering process these turn out to be the j and $j + 1$ biggest of the satisfactions. Thus we have $b_j = \alpha$ and $b_{j+1} = \alpha$. However, if these two criteria have different importances we have two different ways of assigning the u_j,

$$u_j = V_1 \text{ and } u_{j+1} = V_2$$

or

$$u_j = V_2 \text{ and } u_{j+1} = V_1.$$

In the following we shall show it does not make a difference which of these we use. Consider the first assignment. In this case

$$\omega_j = Q\left(\frac{S_j}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right) = Q\left(\frac{S_{j-1} + V_1}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right)$$

$$\omega_j + 1 = Q\left(\frac{S_{j+1}}{T}\right) - Q\left(\frac{S_j}{T}\right) = Q\left(\frac{S_{j-1} + V_1 + V_2}{T}\right) - Q\left(\frac{S_{j-1} + V_1}{T}\right)$$

In calculating the overall satisfaction the sum of contributions we get from these two components is

$$\omega_j b_j + \omega_{j+1} b_{j+1},$$

since $b_j = b_{j+1} = \alpha$ we get

$$\omega_j \alpha + \omega_{j+1}\alpha = (\omega_j + \omega_{j+1})\alpha.$$

However,

$$\omega_j + \omega_{j+1} = Q\left(\frac{S_{j-1} + V_1}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right) + Q\left(\frac{S_{j-1} + V_1 + V_2}{T}\right) - Q\left(\frac{S_{j-1} + V_1}{T}\right)$$

$$\omega_j + \omega_{j+1} = Q\left(\frac{S_{j-1} + V_1 + V_2}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right).$$

Thus we see that the choice does not make any difference.

We shall now look at some special cases of this importance weighted quantifier guided aggregation in order to get a better understanding of the process. We first consider the special case when all the criteria have the same importances. In this situation we have $V_i = \alpha$ for all i. In this case independent of the ordering of the $A_i(x)$, the resulting u_j, will all be equal to α, thus $u_j = \alpha$ for all j. Hence in this case

$$\omega_j(x) = Q\left(\frac{S_j}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right)$$

and hence

$$\omega_j(x) = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right).$$

Thus the ω_j are independent of x and are obtained in the same manner as those obtained when we did not include the importances. Thus the case in which we have equal importances for all the arguments results in the same structure as the case in which we do not consider importances at all. In this case the weights are the same for all x.

We now consider the special case when the linguistic quantifier Q is the unitor quantifier $Q(r) = r$. Again starting with the satisfaction to the criteria,

$A_i(x)$, and their importances V_i we order these satisfactions to give us b_j and u_j. We then calculate the associated weights,

$$\omega_j(x) = Q\left(\frac{\sum_{i=1}^{j} u_i}{T}\right) - Q\left(\frac{\sum_{i=1}^{j-1} u_i}{T}\right)$$

Since $Q(r) = r$ we get

$$\omega_j(x) = \frac{\sum_{i=1}^{j} u_i}{T} - \frac{\sum_{i=1}^{j-1} u_i}{T} = \frac{u_j}{T}$$

Calculating the OWA aggregation using these weights we obtain

$$D(x) = \sum_{j=1}^{n} \omega_j(x) b_j = \frac{1}{T} \sum_{j=1}^{n} u_j b_j$$

Recalling that u_j is that importance associated with the criteria that provided the value for b_j we see that in this case we get

$$D(x) = \frac{1}{T} \sum_{i=1}^{n} A_i(x) V_i,$$

the ordinary weighted average. Thus the ordinary weighted average, the sum of the product of the relative importance times the score, is a special case of the above method when the quantifier is the unitor quantifier.

We now consider the special case when the quantifier guiding the aggregation is defined as

$$Q(r) = 0 \qquad r < g$$
$$Q(r) = 1 \qquad g \leq r \leq 1$$

Semantically this quantifier corresponds to *at least g percent*. To find the OWA weights we again order the arguments and bring along their associated importances, this process gives us b_j and u_j. We then calculate the associated OWA weights as $\omega_j = Q\left(\frac{S_j}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right)$. From the form of the quantifier Q we see that

$$\omega_j = 0 \qquad \text{for all } j \text{ for which } \frac{S_j}{T} < g$$

$$\omega_j = 1 \qquad \text{for the first } j \text{ for which } \frac{S_j}{T} \geq g$$

$$\omega_j = 0 \qquad \text{for all other } j$$

We note that in this case our OWA weighting vector always consist of one value equal to one and all other values equal to zero. Since the aggregated value for alternative x is $D(x) = \sum_{j=1}^{n} \omega(x)_j b_j$ we see that $D(x) = b_{j^*}$ where j^* is the first value of j for which $\frac{S_j}{T} \geq g$.

The effective aggregation process in this case can be seen to be the following simple process. We order the criteria satisfactions and carry along their associated importances. This results in a table of the kind shown below.

Score	Importance	Proportion
b_1	u_1	S_1/T
b_2	u_2	S_2/T
b_3	u_3	S_3/T
b_n	u_n	S_n/T

We then select as our aggregated score the value b_{j*} for which the value S_{j*}/T exceeds or is equal to g for the first time. A number of special cases of this situation are worth noting. In this case when $g = 1$ we effectively have

$$Q(r) = 0 \qquad r < 1$$

$$Q(r) = 1 \qquad r = 1$$

This is a situation corresponding to the quantifier *for all*. In this case we see that our approach has as its evaluation the smallest criteria satisfaction that has nonzero importance. Thus in this case

$$D(x) = \underset{\text{all } i \text{ s.t. } V_i \neq 0}{\text{Min}} A_i(x).$$

We now consider the case for which

$$Q(r) = 1 \qquad \text{for all } r \neq 0$$

$$Q(0) = 0$$

This corresponds to the case in which $g = \varepsilon \to 0$. In this case we get

$$D(x) = \underset{\text{all } i \text{ s.t. } V_i \neq 0}{\text{Max}} A_i(x)$$

This in this case our value is the largest satisfaction that has any nonzero importance.

Another special case of the above is the situation in which

$$Q(r) = 0 \qquad r < 1/2$$

$$Q(r) = 1 \qquad r \geq 1/2$$

This case, which is a kind of **median** aggregation, has $g = 1/2$. Thus we select the ordered criteria b_{j*} for which S_{j*}/T equals or exceeds 0.5 for the first time. Thus we begin adding up the importances of the ordered satisfactions and stop as soon as the total equals or exceeds one half the total importance. The value of ordered satisfaction that occur at this point is our overall evaluation. In this situation we have provided a methodology for obtaining a **weighted median** aggregation.

In the earlier section we introduced a measure of *orness* associated with a given weighting vector W defined as

$$orness(w) = \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \omega_j(n-j)$$

Since in the situation in which we have importances associated with a quantifier the weighing vector is dependent upon the ordering of the objects the measure of *orness* is different for each alternative, then in this case

$$orness(W(x)) = \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \omega_j(x)(n-j)$$

Using the formulation

$$\omega_j(x) = Q\left(\frac{S_j(x)}{T}\right) - Q\left(\frac{S_{j-1}(x))}{T}\right)$$

where $S_j(x)$ is the sum of importances for the j highest scoring criteria under alternative x we get for the *orness* associated with the quantifier Q for the alternative x

$$orness(Q/x) = \frac{1}{n-1} \sum_{j=1}^{n} \left(Q\left(\frac{S_j(x))}{T}\right) - Q\left(\frac{S_{j-1}(x))}{T}\right)(n-j)\right)$$

Doing some simple algebraic manipulations we can show that

$$orness(Q/x) = \frac{1}{n-1} \sum_{j=1}^{n-1} Q\left(\frac{S_j(x)}{T}\right)$$

VIII. TRIANGULAR NORM TYPE OWA OPERATORS

In using quantifier guided aggregation technique we are essentially building a decision function by balancing two factors. The first factor is related to our quantifier Q. This factor stipulates that $Q\left(\frac{i}{n}\right)$ is the degree of satisfaction we attain if we satisfy any i of the criteria. With Q a RIM quantifier this is an increasing function of i. Furthermore, ω_i is the increase in satisfaction we get in going from satisfying $i - 1$ to i criteria. The second concern in the construction of a quantifier guided decision function is that we can find i criteria that are satisfied. We shall let $G(i)$ be the degree to which i criteria are satisfied. Combining these two factors we obtain an aggregation function of the form

$$D(x) = \sum_{i=1}^{n} \omega_i G(i).$$

In the above the value of the $G(i)$ term is obtained from the satisfaction of the individual criteria,

$$G(i) = f_i(A_1(x), \ldots \ldots A_n(x)).$$

Since $G(i)$ just stipulates we need find any i criteria that are satisfied we only need consider the i most satisfied criteria. If we assume that b_j is the jth largest of the criteria scores then

$$G(i) = F(b_1, \ldots \ldots b_i).$$

Since $G(i)$ requires that *all* the i criteria considered are satisfied the appropriate formulation for the construction of $G(i)$ is to use a t-norm aggregation of the i most satisfied criteria scores, $b_1, \ldots . b_i$. Thus

$$G(i) = T(b_1, \ldots . b_i)$$

where T is some t-norm operator. We recall that a t-norm is a mapping

$$T: [0, 1]\, x\, [0, 1] \rightarrow [0, 1]$$

s.t.

(1) $T(a, b) = T(b, a)$	**Commutativity**
(2) $T(a, b) \geq T(c, d)$ if $a \geq c$ and $b \geq d$	**Monotonicity**
(3) $T(a, T(b, c)) = T(a, T(b, c))$	**Associativity**
(4) $T(1, a) = a$	**Identity of One**

As extensively discussed in the literature the t-norm provides a general class of *and* aggregation operators.

Using these operators we can provide a general class of OWA operators.

DEFINITION. *An aggregation operator*

$$F_T: I^n \rightarrow I$$

is called a T type ordered weighted averaging operator of dimension n if it has associated with it a weighting vector W

$$W = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \cdot \\ \cdot \\ \omega_n \end{bmatrix}$$

such that

(1) $\omega_i \in [0, 1]$
(2) $\Sigma\, \omega_i = 1$

and where

$$F_T(a_1, \ldots \ldots a_n) = \sum_{j=1}^{n} \omega_j T(B_j)$$

where T is any t-norm and $B_j = (b_1, b_2, \ldots, b_j)$ *and where* b_j *is the jth largest of the* a_i. *We note* $T(B_1)$ *is defined as* b_1. *In the above we call* B_j *the top j dimension ordered bag of* $(a_1, \ldots, a_n)$.

A number of special cases of this operator are worth noting. Assume $T(a, b) = \text{Min}[a, b]$. In this case

$$T_{\text{Min}}(B_j) = \text{Min}[b_1, \ldots, b_j] = b_j$$

and thus

$$F_{\text{Min}}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \omega_j b_j$$

which is the ordinary OWA operator.

If $T(a, b) = a\, b$ then

$$T_{\Pi}(B_j) = \prod_{K=1}^{j} b_K$$

and

$$F_{\Pi}(\mathrm{a}_1, \ldots a_n) = \sum_{j=1}^{n} \omega_j \left(\prod_{K=1}^{j} b_K \right)$$

This product type OWA aggregation can be expressed as

$$F_{\Pi}(a_1, \ldots a_n) = b_1[\omega_1 + b_2(\omega_2 + b_3(\omega_3 + b_4(\ldots + b_n\omega_n)))]$$

Another special case is when we use the t-norm

$$T_L(a, b) = (a + b - 1) \vee 0.$$

In this case

$$T_L(B_j) = \left(\left(\sum_{K=1}^{j} b_K \right) - (j - 1) \right) \vee 0$$

For this operator we note that if for some m $T_L(B_m) = 0$, then $T_L(B_j) = 0$ for all $j \geq m$. Thus if m^* is the smallest integer for which

$$\sum_{K=1}^{m^*} b_j \leq m^* - 1$$

then

$$F_L(a_1, \ldots, a_n) = \sum_{K=1}^{m^*} \omega_j \left(\left(\sum_{K=1}^{j} b_j \right) - (j - 1) \right)$$

Furthermore, we note that

$$a + b - 1 = a - (1 - b) = a - \bar{b}$$

and hence we can express $T_L(B_j)$ as

$$T_L(B_j) = \left(b_1 = \sum_{k=2}^{j} \bar{b}_k\right) \vee 0.$$

In this case we see that $T(B_j) = 0$ for all j such that $\Sigma_{k=2}^{j} \bar{b}_k \geq b_1$. Thus if m^* is the smallest integer for which $\Sigma_{k=2}^{m^*} \bar{b}_k \geq b_1$, then

$$F_L(a_1, \ldots, a_n) = \sum_{j=1}^{m^*} \omega_j(b_1 - R_j)$$

where $R_j = \Sigma_{k=2}^{j} \bar{b}_k$.

It is well established in the literature[10] that Min is the largest of the t-norm. Thus for any t-norm T

$$\text{Min}[B_j] \geq T(B_j)$$

and hence any weighting vector W

$$F_{\text{Min}}(a_1, \ldots a_n) \geq F_T(a_1, \ldots a_n).$$

Thus the ordinary type OWA is the largest of the class of T type OWA operators.

If we consider the weighting vector W^* where $\omega_1 = 1$ and $\omega_i = 0$ for all other i, then for any T

$$F_T(a_1, \ldots a_n) = T(B_1) = b_1 = \text{Max}[a_i].$$

If we consider the weighting vector W_* where $\omega_n = 1$ and $\omega_i = 0$ for all other i, then for any T

$$F_T(a_1, \ldots a_n) = T(B_j) = T(a_1, \ldots a_n).$$

If we consider the weights vector W_A where $\omega_i = \dfrac{1}{n}$ for all i then for any T

$$F_T(a_1, \ldots a_n) = \frac{1}{n}\sum_{j=1}^{n} T(B_j).$$

The following theorem puts some bounds on T type OWA operators.

THEOREM. *Assume T is any t-norm then for any weighting vector W.*

$$\text{Max}_i[a_i] \geq F_T(a_1, \ldots a_n) \geq T(a_1, \ldots a_n).$$

Proof. Since for any t-norm T

$$T(x_1, x_2, \ldots x_n) \geq T(x_1, x_2 \ldots x_n, x_{n+1})$$

it follows that

$$T(B_j) \geq T(B_i) \qquad \text{for } j > j.$$

From this it follows that

$$F_T(a_1, \ldots a_n) \leq F^*_T(a_1, \ldots . a_n) \leq \text{Max}(a_1, \ldots a_n)$$

and

$$F_T(a_1, \ldots a_n) \geq F_{*T}(a_1, \ldots . a_n) \geq T(a_1, \ldots . a_1)$$

where F^*_T and F_*T use the weighting vectors W^* and W_*, respectively.

THEOREM. *Assume T is any t-norm then for any weighting vector W, $F_T(a_1, \ldots . a_n)$ is monotonic with respect to the a_is.*

This theorem follows directly from the monotonicity of the t-norm operators.

Inspired by the T type OWA operators we can consider a class of S type OWA operators. Let S be any t-conorm,[10] we recall that S satisfies the same properties as a t-norm accept instead of (4) it has $S(0, a) = a$. The prototypical example of a t-conorm is the Max operator. Assume $(a_1, \ldots, a_n)$ is a bag of arguments to be aggregated and again let b_j be the jth largest of these. We shall let $D_j = (b_j, b_{j+1}, \ldots, b_n)$. Thus D_j is the subbag of the $n + 1 - j$ smallest values to be aggregated. We now consider the aggregation.

$$F_S(a_1, \ldots, a_n) = \sum_{j=1}^{n} \omega_j S(D_j)$$

where $S(D_j)$ is a t-conorm aggregation of the elements in D_j. We shall call this S type OWA aggregation.

If we consider the special case when S is equal to the Max we note that

$$S(D_j) = \text{Max}[b_j, b_{j+1}, , b_n] = b_j$$

Thus for S = Max

$$F_S(a_1, \ldots, a_n) = \sum_{j=1}^{n} \omega_j b_j$$

which is the ordinary OWA operator. Recalling that Max is the smallest t-conorm[10] it follows that for any S

$$\text{Max}(B_j) \leq S(B_j)$$

and hence for any W and S

$$F_{\text{Max}}(a_1, \ldots, a_n) \leq F_S(a_1, \ldots, a_n).$$

The following theorem mirrors the one for T type OWA aggregation operators.

THEOREM. *Assume S is any t-conorm then for any weighting vector W*

$$\text{Min}_i[a_i] \leq F_S(a_1, \ldots, a_n) \leq S(a_1, \ldots, a_n).$$

Furthermore, it can be easily shown that these operators are monotonic.

IX. CONCLUSION

We have looked at the issue of quantifier guided multicriteria decision making. We have suggested that the OWA operators provide an appropriate tool for the construction of these types of decision functions. We have suggested a method for including importances associated with the different criteria to be aggregated.

References

1. R.E. Bellman and L.A. Zadeh, "Decision-making in a fuzzy environment," *Manage. Sci.*, **17**(4), 141–164 (1970).
2. R.R. Yager, "Quantifiers in the formulation of multiple objective decision functions," *Inf. Sci.*, **31**, 107–139 (1983).
3. R.R. Yager, "Aggregating criteria with quantifiers," in *Proceedings of the International Symposium on Methodologies for Intelligent Systems,* Knoxville, TN, pp. 183–189, 1986.
4. R.R. Yager, "On ordered weighted averaging aggregation operators in multi-criteria decision making," *IEEE Trans. Syst. Man Cybern.*, **18**, 183–190 (1988).
5. L.A. Zadeh, "A computational approach to fuzzy quantifiers in natural languages," *Comput. Math. Appl.*, **9**, 149–184 (1983).
6. R.R. Yager, "Connectives and quantifiers in fuzzy sets," *Fuzzy Sets Syst.*, **40**, 39–76 (1991).
7. L. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning: Part 1," *Inf. Sci.*, **8**, 199–249 (1975).
8. E.P. Klement, "Characterization of fuzzy measures constructed by means of triangular norms," *J. Math. Anal. Appl.*, **86**, 345–358 (1982).
9. R.R. Yager and D.P. Filev, *Essentials of Fuzzy Modeling and Control,* Wiley, New York, 1994.
10. D. Dubois and H. Prade, "A review of fuzzy sets aggregation connectives," *Inf. Sci.*, **36**, 85–121 (1985).
11. R.R. Yager, "On mean type aggregation," IEEE Transactions on Systems, Man and Cybernetics, to appear.
12. R.R. Yager, "Families of OWA operators," *Fuzzy Sets Syst.*, **59**, 125–148 (1993).
13. R.R. Yager, "Some extensions of constraint propagation of label sets," *Int. J. Approximate Reasoning,* **3**, 417–436 (1989).
14. R.R. Yager, "Entropy and specificity in a mathematical theory of evidence," *Int. J. General Syst.* **9**, 249–260 (1983).
15. R.R. Yager, "Measuring the quality of linguistic forecasts," *Int. J. Man Mach. Stud.*, **21**, 253–257 (1984).
16. R.R. Yager, "On the specificity of a possibility distribution," *Fuzzy Sets Syst.*, **50**, 279–292 (1992).
17. L.A. Zadeh, "A theory of approximate reasoning," in *Machine Intelligence,* Vol. 9, J. Hayes, D. Michie, and L. I. Mikulich, Eds., Halstead Press, New York, 1979, pp. 149–194.
18. R.R. Yager, "Deductive approximate reasoning systems," *IEEE Trans. Knowl. Data Eng.*, **3**, 399–414 (1991).
19. D. Dubois and H. Prade, "Fuzzy sets in approximate reasoning Part I: Inference with possibility distributions," *Fuzzy Sets Syst.*, **40**, 143–202 (1991).
20. D. Dubois and H. Prade, "The principle of minimum specificity as a basis for evidential reasoning," in *Uncertainty in Knowledge-Based Systems,* B. Bouchon and R. R. Yager, Eds., Springer-Verlag, Berlin, Germany, 1987, pp. 75–84.
21. M. O'Hagan, "Aggregating template or rule antecedents in real-time expert systems

with fuzzy set logic," in *Proceedings of the 22nd Annual IEEE Asilomar Conference on Signals, Systems and Computers,* Pacific Grove, CA, pp. 681–689, 1988.
22. D.P. Filev and R.R. Yager, "Learning OWA operator weights from data," in *Proceedings of the Third IEEE International Conference on Fuzzy Systems,* Orlando, FL, pp. 468–473, 1994.
23. R.R. Yager, "Fuzzy quotient operators for fuzzy relational data bases," in *Proceedings of the International Fuzzy Engineering Symposium,* Yokohama, Japan, pp. 289–296, 1991.