Weekly Exercises Fuzzy Logic

IMAT 1223

Week 1 – Submit Exercises 3, 5 and 7 in Learning Zone

Exercise 1. *For classical sets (also referred to as crisp sets in Fuzzy Set Theory) A and B, prove:*

(a) Commutative law: $A \cup B = B \cup A$; $A \cap B = B \cap A$.

(b) Associative law: $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$.

(c) Distributive law: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(d) De Morgan's law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$; $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Exercise 2. *Establish membership functions for*

(a) Real numbers approximately equal to 6.

(b) Integers far from 10.

(c) Integers very far from 10.

Exercise 3. *A variable is something that takes more than one value. If only one value is possible, then the variable is referred to as a constant variable (or singleton in Fuzzy Set Theory). A linguistic variable is a variable that takes linguistic values or terms, i.e, values the variable is described with are words or combinations of words, like fast, slow, very fast, etc. Describe the linguistic variable "age" referred to a person. Does your modelling of this variable apply to a building or work of art?*

Exercise 4. *Describe the variable "weight" and "height" using five descriptors or linguistic terms. Does your modelling apply to both male and female human beings? Words may mean different things not just when applied to different things but they may mean different things for different people.*

Exercise 5. *For fuzzy sets A and B with MFs:* $\mu_A(x) = \mu_B(x) = \begin{cases} 1 - |x|, & -1 < x < 1; \\ 0; & \textit{otherwise.} \end{cases}$.

(a) Sketch $\mu_{A \cap B}$ for minimum and product t-norms.

(b) Sketch $\mu_{A \cup B}$ for maximum and probabilistic sum (algebraic sum) t-conorms.

Do the same when the Mrs are: $\mu_A(x) = e^{-\frac{1}{2}(x-3)^2}$ *and* $\mu_B(x) = e^{-\frac{1}{2}(x-4)^2}$.

Exercise 6. *Prove that the minimum operator is the strongest t-norms while the maximum operator is the weakest t-conorm.*

Exercise 7. *Prove that $T_L = \max\{x + y - 1, 0\}$ $(\forall x, y \in [0, 1])$ is a t-norm, which is known as the Łukasiewicz t-norm. Determine the Łukasiewicz t-conorm, $S_L = \min\{x + y, 1\}$ using the duality property*

$$S_L(x, y) = 1 - T_L(1 - x, 1 - y).$$

Week 2 – Submit Exercises 8, 9 and 14 in Learning Zone

Definition 1. *A 'fuzzy' set A on the universe X is characterised by a function $\mu_A\colon X \to [0,1]$ that associates each element of the universe $x \in X$ a value in $\mu_A(x) \in [0,1]$ that represents the degree up to which such element verifies the property defining A.*

Note 1. *In some cases, the membership degree is interpreted as how similar the element x is to the property defining A or how well the property defining A applies to the element x.*

In mathematical notation, a fuzzy set set is represented as follows:

$$A = \{(x, \mu_A(x)) | x \in X \wedge \mu_A(x) \in [0,1]\}. \tag{1}$$

Note 2. *When the property defining A is precise and there is no room to interpretation on whether the property is verified or not verified, then $\mu_A(x)$ can only be 0 or 1 ($\mu_A(x) \in \{0,1\}$) and set A is a classical set, which in fuzzy set theory is referred to as a 'crisp' set. For crisp sets the membership function is called the characteristic function or indicator function of the set.*

The representation (1) is called the vertical representation of a fuzzy set. This is because there is horizontal representation of a fuzzy set using its alpha-cuts.

Definition 2. *The alpha-cut of a fuzzy set A with membership function μ_A, denoted A_α or A^α , is the crisp set*

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}$$

where $\alpha \in [0,1]$. The strong alpha cut, $A_{\alpha+}$ or $A^{\alpha+}$, is defined with strict inequality.

Theorem 1 (The Representation Theorem). *A fuzzy set A with membership function μ_A can be represented using its alpha-cuts as follows:*

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha. \tag{2}$$

In other words:

$$\forall x \in X : \qquad \mu_A(x) = \sup_{\alpha \in [0,1]} \alpha A_\alpha(x), \quad \textit{where} \quad \alpha A_\alpha(x) = \begin{cases} \alpha & x \in A_\alpha \\ 0 & \textit{otherwise.} \end{cases} \tag{3}$$

The representation of a fuzzy set A with its strong alpha-cuts is the same as (2) *replacing A_α with $A_{\alpha+}$.*

The set of all distinct numbers $\alpha \in [0,1]$ that are employed as membership grades of the elements of X in A is called the level set of A, and denoted by $L(A)$. For discrete (finite) fuzzy sets, $L(A)$ is also discrete (finite) and the supremum operator $\sup$ in (3) becomes the maximum operator.

Note 3. *The representation of a fuzzy set with its alpha-cuts is important to derive properties of fuzzy sets, or to prove results that apply for fuzzy sets. For example, if we have the membership function of a positive fuzzy number, the we can get the membership of its corresponding negative fuzzy number using the extension principle and the representation of the fuzzy number with its alpha cuts. In practice, the vertical representation is used to visualise a fuzzy set and interpret it.*

Exercise 8. *Assume minimum and maximum operators for the intersection and union of fuzzy sets. Answer the following:*

1. *Given any two fuzzy sets A and B, prove that the following properties hold:*
   a) *If $\alpha \geq \beta$, then $A_\alpha \subseteq A_\beta$*
   b) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$
   c) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$
2. *Given the discrete fuzzy set $A = \{(0.2, x_1), (0.4, x_2), (0.6, x_3), (0.8, x_4), (1, x_5)\}$, obtain $L(A)$, and provide all the distinct alpha-cuts of A.*
3. *Show that* (3) *is verified fro the discrete set A given in item 2.*

Definition 3 (The Extension Principle). *Let $X_1 \times X_2 \times \cdots \times X_n$ be a universal product set and $f\colon X_1 \times X_2 \times \cdots \times X_n \to Y$ be a function that maps elements of $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$ to element $y = f(x_1, x_2, \cdots, x_n)$ of the universal set Y. Let A_i be a fuzzy set on X_i with membership function μ_{A_i} $(i = 1, 2, \cdots, n)$. Let us denote*

$$f^{-1}(y) = \{(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n | y = f(x_1, x_2, \cdots, x_n)\}.$$

Then, $B = f(A_1, A_2, \cdots, A_n)$ is the fuzzy set on Y with membership function

$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1, x_2, \cdots, x_n) \in f^{-1}(y)} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)\} & \textit{if } f^{-1}(y)) \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$$

Exercise 9. *Prove using the extension principle, to extend function $f(x) = -x$, that the opposite of a fuzzy number N on $\mathbb{R}$, $-N$ has membership function $\mu_{-N}(x) = \mu_N(-x)$, where μ_N is the membership function of N.*

A fuzzy relation represents a degree of presence or absence of association, interaction, or interconnectedness between the elements of two or more (fuzzy) sets. Some examples of binary fuzzy relations are:

- x is much larger than y;
- y is very close to z;
- a is more profitable than b.

Formally, given two crisp sets X and Y, a fuzzy relation between the elements of set X and the elements of set Y is defined as a fuzzy set on the Cartesian product space $X \times Y$, $F(X \times Y)$, where tuples (x, y) may have varying degrees of membership $\mu_F(x, y)$ within the relation:

$$F(X \times Y) = \{((x, y), \mu_F(x, y)) \,|(x, y) \in X \times Y \wedge \mu_F(x, y) \in [0, 1]\}$$

Because fuzzy relations are fuzzy sets in a Cartesian product space, set theoretic and algebraic operations can be defined for them using the operators for fuzzy union (t-norms like maximum), intersection (t-norms like minimum or product) and complement.

Exercise 10. *Let $X = \{2, 12\}$ and $Y = \{1, 7, 13\}$. Find the membership functions of the fuzzy relations*
"x is close to y" and "x is smaller than y;" and "x is close to y" or "x is smaller than y."
Assume that the MFs for the relations 'close to' and 'smaller than' are

$$\mu_c(x,y) = \begin{array}{c} \\ 2 \\ 12 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 7 & 13 \end{array} \\ \begin{pmatrix} 0.9 & 0.4 & 0.1 \\ 0.1 & 0.4 & 0.9 \end{pmatrix} \end{array}; \qquad \mu_s(x,y) = \begin{array}{c} \\ 2 \\ 12 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 7 & 13 \end{array} \\ \begin{pmatrix} 0 & 0.6 & 1 \\ 0 & 0 & 0.3 \end{pmatrix} \end{array}$$

The composition of two fuzzy relations $R(X \times Y)$ and $S(Y \times Z)$ is a fuzzy relation $S \circ R(X \times Z)$ with membership function

$$\mu_{S \circ R}(x, z) = \sup_{y \in Y} \mu_R(x, y) \star \mu_S(y, z)$$

where $\star$ is a t-norm (usually minimum or product). As noted before, when X, Y, Z are discrete, the supremum operator becomes the maximum.

Exercise 11. *Let X and Y be the fuzzy sets of exercise 10, and $Z = \{4, 8\}$. Let the fuzzy relation on $Y \times Z$ 'much bigger than' be characterized by the membership function*

$$\mu_c(x,y) = \begin{array}{c} \\ 1 \\ 7 \\ 13 \end{array}\begin{array}{c} \begin{array}{cc} 4 & 8 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0.6 & 0 \\ 1 & 0.7 \end{pmatrix} \end{array}.$$

Compute the membership function of the statement "x is close to y and y is much bigger than z" using composition of fuzzy relations with both minimum and product as the $\star$ operation.

When $X = Y$ and X is a fuzzy set, we can define the fuzzy relation $R(X \times X)$ with membership function $\mu_R(x, x) = \mu_R(x)$. In this case, the composition $S \circ R(X \times Z)$ will have membership function

$$\mu_{S \circ R}(z) = \sup_{x \in X} \mu_R(x) \star \mu_S(x, z),$$

i.e. will be a fuzzy set in Z. Thus, a fuzzy set on X becomes a fuzzy set on Z when there is a (fuzzy) relation between the elements of X and the elements of Y. This is known as Zadeh's compositional rule of inference.

Exercise 12. *Consider the relation 'close to' between set x and Y of exercise 10. Let fuzzy set 'small' on X be defined with membership function*

$$\mu_s(x) = \begin{array}{c} \begin{array}{cc} 4 & 8 \end{array} \\ \begin{pmatrix} 0.9 & 0.1 \end{pmatrix} \end{array}.$$

Compute the membership function of the statement "x is small and x is close to y" using both minimum and product as the $\star$ operation.

One of the major components of a fuzzy system is 'Rules,' i.e. logical implications in the forms of IF–THEN statements:

IF x is A, THEN y is B; where $x \in X$ and $y \in Y$.

where A and/or B are fuzzy sets on X and Y, respectively. An IF–THEN rule is a special kind of relation between A and B, and its membership function is denoted $\mu_{A \to B}(x, y)$.

A tautology is a proposition formed by combining other propositions which is true regardless of the truth or falsehood of the propositions combined. The most important tautology involving implication is

$$p \to q \quad \longleftrightarrow \quad \sim [p \wedge (\sim q)]$$

This tautology can also be expressed as

$$p \to q \quad \longleftrightarrow \quad (\sim p) \vee q$$

The importance of these tautologies is that they allow to express the membership function for $p \to q$ in terms of the membership functions of propositions p and q and the operators for minimum/product for $\wedge$, maximum for $\vee$ and complement for $\sim$.

The first tautology leads to the following implication membership function:

$$\mu_{p \to q}(x, y) = 1 - \min\{\mu_p(x), 1 - \mu_q(y)\},$$

and the second tautology to Kleene-Dienes implication membership function

$$\mu_{p \to q}^{KD}(x, y) = \max\{1 - \mu_p(x), \mu_q(y)\}.$$

These are not the only ones that give agreement with classical propositional logic for $p \to q$. Two others are:

Reichenbach : $\mu_{p \to q}^{R}(x, y) = 1 - \mu_p(x)\,[1 - \mu_q(y)]$

Lukasiewicz : $\mu_{p \to q}^{L}(x, y) = \min\{1, 1 - \mu_p(x) + \mu_q(y)\}$

Exercise 13. *For the IF–THEN rules below*

IF distance is big, THEN acceleration is medium

Model 'big' as the triangular fuzzy number with parameters $(50, 75, 100)$ *and 'medium' as the fuzzy number with parameters* $(2, 5, 8)$. *Give the expression of the membership degree of the fuzzy rule using the four different implication membership functions given above.*

In engineering applications an implication is activated only when the antecedent is true, so that we can imply the consequent is also true. This is known as *Modus Ponens:*

Premise: x is A

Implication: IF x is A THEN y is B

Consequence: y is B

Thus, in this case , we only need to consider an approach to obtain the truth value of a proposition $p \to q$ for the case that the antecedent truth value is positive. The two proposed approaches to compute the truth value of an implication for engineering applications are

Mamdani : $\mu_{p \to q}(x, y) = \min\{\mu_p(x), \mu_q(y)\}$

Larsen : $\mu_{p \to q}(x, y) = \mu_p(x)\mu_q(y)$

In crisp logic a rule will be activated if the premise is exactly the same as the antecedent of the rule, and the result of such rule firing is the rule's actual consequent. In fuzzy logic, a rule is fired so long as the degree of similarity between the premise and the antecedent of the rule is non-zero, and the result of such rule firing is a consequent that has a non-zero degree of similarity to the rule's consequent. Thus, in fuzzy logic, Modus Ponens in extended to *Generalised Modus Ponens:*

$$\begin{aligned} \text{Premise:} \quad & x \text{ is } A^* \\ \text{Implication:} \quad & \text{IF } x \text{ is } A \text{ THEN } y \text{ is } B \\ \text{Consequence:} \quad & y \text{ is } B^* \end{aligned}$$

Using Zadeh's compositional rule of inference, we have

$$\forall y \in Y : \quad \mu_{B^*}(y) = \sup_{x \in X} \mu_{A^*}(x) \star \mu_{A \to B}(x, y). \tag{4}$$

When the value observed for X is x', then A^* can be considered the fuzzy singleton (a classical set with one element) with membership

$$\mu_{A^*}(x) = \begin{cases} 1 & if x = x' \\ 0 & if x \neq x. \end{cases}$$

For this observation, Generalised Modus Ponens (4) results in

$$\forall y \in Y : \quad \mu_{B^*}(y) = \sup_{x \in X} \mu_{A^*}(x) \star \mu_{A \to B}(x, y) = \max\{\mu_{A \to B}(x', y), 0\} = \mu_{A \to B}(x', y). \tag{5}$$

Considering minimum implication, this becomes

$$\forall y \in Y : \quad \mu_{B^*}(y) = \min\{\mu_A(x')\mu_B(y)\}.$$

Considering product implication, it becomes

$$\forall y \in Y : \quad \mu_{B^*}(y) = \mu_A(x')\mu_B(y).$$

Thus, a crisp input is converted by a fuzzy rule into a fuzzy set.

Exercise 14. *For the fuzzy rule of exercise 13 compute the fuzzy outputs when the inputs are* $x' = 50$, $x' = 60$, $x' = 75$, $x' = 80$ *and* $x' = 100$, *respectively.*