Type-Reduction of General Type-2 Fuzzy Sets: The Type-1 OWA Approach

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For general type-2 fuzzy sets, the defuzzification process is very complex and the exhaustive direct method of implementing type-reduction is computationally expensive and turns out to be impractical. This has inevitably hindered the development of type-2 fuzzy inferencing systems in real-world applications. The present situation will not be expected to change, unless an efficient and fast method of deffuzzifying general type-2 fuzzy sets emerges. Type-1 ordered weighted averaging (OWA) operators have been proposed to aggregate expert uncertain knowledge expressed by type-1 fuzzy sets in decision making. In particular, the recently developed *alpha-level approach* to type-1 OWA operations has proven to be an effective tool for aggregating uncertain information with uncertain weights in real-time applications because its complexity is of linear order. In this paper, we prove that the mathematical representation of the type-reduced set (TRS) of a general type-2 fuzzy set is equivalent to that of a special case of type-1 OWA operator. This relationship opens up a new way of performing type reduction of general type-2 fuzzy sets, allowing the use of the *alpha-level approach* to type-1 OWA operations to compute the TRS of a general type-2 fuzzy set. As a result, a fast and efficient method of computing the centroid of general type-2 fuzzy sets is realized. The experimental results presented here illustrate the effectiveness of this method in conducting type reduction of different general type-2 fuzzy sets.

1. INTRODUCTION

Type-2 fuzzy sets initially proposed by Zadeh in 1975[1] offer the advantage of modeling higher level uncertainty in human decision-making process than using type-1 fuzzy sets. In a type-2 fuzzy inference system (FIS), type-2 fuzzy sets are used in the antecedent and/or consequent parts of all or some of its fuzzy rules. Type-2 FISs have gained successful applications in various areas where uncertainties occur, such as in diagnostic medicine[2,3] and in intelligent signal processing.[4,5]

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INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 28, 505–522 (2013)

View this article online at wileyonlinelibrary.com. • DOI 10.1002/int.21588

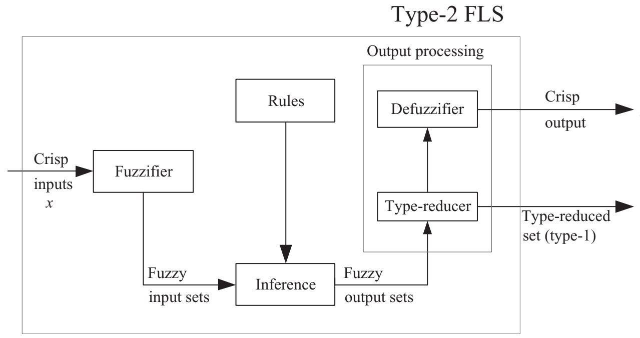


Figure 1. Type-2 FIS.[16]

Generally speaking, there are five stages in any FIS: *fuzzification*, *antecedent computation*, *implication*, *aggregation,* and *defuzzification*. The defuzzification process becomes necessary and important because as Zadeh[6] pointed out, fuzzy sets might need to be defuzzified in those situation in which a person is presented with a fuzzy statement but its implementation or execution is to be done via the use of a single real value. The defuzzification of a type-1 fuzzy set does not present any challenges from a mathematical point of view; however, this is not true in the case of a type-2 fuzzy set. The defuzzification of a type-2 fuzzy set consists of two steps (see Figure 1):[7]

(a) *Type-reduction of type-2 fuzzy set*: The procedure by which a type-2 fuzzy set is converted to a type-1 fuzzy set, known as the *type-reduced set (TRS)*; and
(b) *Defuzzification of type-1 fuzzy set* : The TRS is defuzzified to give a crisp number, known as the *centroid* of the type-2 fuzzy set.

The computation of the TRS is a very challenging step in type-2 FIS modeling. The consequence is that most researchers concentrate exclusively on the development of theoretical results and practical applications of interval type-2 fuzzy sets.[4,5,8–13] The defuzzification of an interval type-2 fuzzy set has been greatly simplified in recent years with the development of novel, accurate, and fast interval methods such as the *Greenfield–Chiclana collapsing defuzzifier*[10] and the *enhance iterative algorithm with stop condition*[11,14] variant of the *Karnik–Mendel Iterative procedure*.[15]

For general type-2 fuzzy sets, the direct method of implementing type reduction is computationally expensive and inefficient, because it involves identifying the centroids of an extraordinarily large number of type-1 fuzzy sets, called embedded type-2 fuzzy sets.[7,16] This has inevitably hindered the development of general type-2 FISs in real-world applications. The present situation will not be expected to change, unless an efficient and fast method to deffuzzify general type-2 fuzzy sets emerges.

The idea of developing such a method of defuzzifying general type-2 fuzzy sets turns out to be possible while we investigate a seemingly different research problem—*aggregation of uncertain information* based on type-1 OWA operator.[17] Type-1 OWA operators provide us with a new technique for directly aggregating uncertain information modeled by type-1 fuzzy sets via OWA mechanism in soft decision making and data mining.

It is known that aggregation is a necessary step in many applications, in particular the multiexpert decision making, multicriteria decision making.[18–20] type-1 OWA operators can be used to aggregate expert knowledge expressed by type-1 fuzzy sets in decision making and they also have the potential of merging fuzzy sets in fuzzy modeling to improve model interpret ability and transparency.[21–23] However, the direct approach to performing type-1 OWA operation involves high computational load,[17] which inevitably curtailed further applications of type-1 OWA operator to real-world decision making. To overcome this issue, a new approach to type-1 OWA operations, called the *alpha-level approach*, has been developed based on the α-cuts of fuzzy sets.[24] This approach benefits from the so-called *representation theorem of type-1 OWA operators*.[24] This *representation theorem* states that a type-1 OWA operator can be decomposed into a series of its α-level type-1 OWA operators. The *alpha-level approach* has proven to be an effective tool for performing type-1 OWA operations. Indeed, the complexity of this *alpha-level approach* is of linear order, so it can be used in real-time soft decision making, database integration, and information fusion that involve aggregation of uncertain information.

The aggregation of crisp information via an OWA operator[25] and the defuzzification of a type-1 fuzzy set[6] have up to now being treated as different and unconnected problems in fuzzy set theory research. A similar situation applies to the aggregation of uncertain information via a type-1 OWA operator[17,24] and the defuzzification of a type-2 fuzzy set.[7] However, a close inspection of their mathematical representation suggests that the centroid of a type-1 fuzzy set and the TRS of a type-2 fuzzy set could be seen as a special case of an OWA operator and type-1 OWA operator, respectively.

Mathematically, the centroid of a type-1 fuzzy set can be seen as the output of an OWA operator applied to a set of crisp values. This means that in practice, the computation process of the TRS of a type-2 fuzzy sets could be carried out by applying its equivalent OWA computation process. The TRS of a type-2 fuzzy set and the type-1 OWA operator were both developed via the application of Zadeh's extension principle.[1] We hypothesize that a result connecting the mathematical representation of the TRS of a type-2 fuzzy set and the representation of a type-1 OWA operator can be proved. Indeed, in this paper, we will prove that the TRS of a type-2 fuzzy set, as defined in Ref. 7 is equivalent to that of a type-1 OWA operator, as defined in Ref. 17. We extend this equivalent mathematical representation to the TRS of interval type-2 fuzzy sets and the α-level type-1 OWA operators.

In summary, the main contribution of this paper is that the link between the type reduction of general type-2 fuzzy sets and type-1 OWA aggregation is established. As a result, a fast and efficient method for computing the TRS of a general type-2 fuzzy set emerges via the *alpha-level approach* to type-1 OWA aggregation.

2. TYPE-2 FUZZY SETS AND TYPE-REDUCED SETS

Let X be a universe of discourse. A fuzzy set A on X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$, and is expressed as follows:

$$A = \{(x, \mu_A(x)) | \mu_A(x) \in [0, 1]\ \forall x \in X\}. \tag{1}$$

Note that

(1) The membership grades of A are crisp numbers. This type of fuzzy set is also referred to as a type-1 fuzzy set. In the following, we will use the notation $U = [0, 1]$.
(2) A crisp number a can also be represented as a type-1 fuzzy set $\hat{a}$ with the following membership function: $\mu_{\hat{a}}(x) = 1$ if $x = a$; and $\mu_{\hat{a}}(x) = 0$ if $x \neq a$. This special type-1 fuzzy set $\hat{a}$ is called the *singleton fuzzy set*.

The *representation theorem* of type-1 fuzzy sets provides an alternative and convenient way to define type-1 fuzzy sets via their corresponding family of crisp α-level sets. The α-level set of a type-1 fuzzy set A is defined as

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\} \tag{2}$$

The set of crisp sets $\{A_\alpha | 0 < \alpha \leq 1\}$ is said to be a representation of the type-1 fuzzy set A. Indeed, the type-1 fuzzy set A can be represented as

$$A = \bigcup_{0<\alpha\leq 1} \alpha A_\alpha \tag{3}$$

with membership function

$$\mu_A(x) = \bigvee_{\alpha : x \in A_\alpha} \alpha \tag{4}$$

This is the so-called "horizontal" representation of a type-1 fuzzy set.

The definition of the *centroid* of a type-1 fuzzy set A in X, also referred to as the *center of gravity* or *center of mass*, requires the universe of discourse to be a subset of the set of real numbers. Therefore, from now on we will assume that the domain of the type-1 fuzzy set is of such type.

The centroid for a continuum universe of discourse X is defined as

$$C_A = \frac{\displaystyle\int_X x \cdot \mu_A(x) dx}{\displaystyle\int_X \mu_A(x) dx}, \tag{5}$$

while when the domain X is discretized into n points is

$$C_A = \frac{\sum_{i=1}^{n} x_i \cdot \mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)} \tag{6}$$

Note that in this discrete form of the centroid of a type-1 fuzzy set it is true that $x_1 < x_2 < \cdots < x_n$.

2.1. Definition of a Type-2 Fuzzy Set

A type-2 fuzzy set $\tilde{A}$ on X is a fuzzy set whose membership grades are themselves fuzzy, i.e., $\mu_{\tilde{A}}(x)$ is a type-1 fuzzy set on U for all x, i.e.,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | \mu_{\tilde{A}}(x) \in \tilde{P}(U)\ \forall x \in X\}. \tag{7}$$

where $\tilde{P}(U)$ is the set of fuzzy sets on U.

This implies that for all $x \in X$ there exists a subset of U, J_x, such that $\mu_{\tilde{A}}(x)\colon J_x \to U$. Applying (1), we have

$$\mu_A(x) = \{(u, \mu_{\tilde{A}}(x)(u)) | \mu_{\tilde{A}}(x)(u) \in U \forall u \in J_x \subseteq U\}. \tag{8}$$

where X is called the primary domain of J_x the primary membership of x while U is known as the secondary domain and $\mu_{\tilde{A}}(x)$ is called the secondary membership of x.

Putting (7) and (8) together, we have

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u))) | \mu_{\tilde{A}}(x)(u) \in U, \forall x \in X, \forall u \in J_x \subseteq U\}. \tag{9}$$

Geometrically, a type-2 fuzzy set may be viewed as a surface in space represented by (x, u, z) coordinates. The *footprint of uncertainty (FOU)* of a type-2 fuzzy set is the projection of the set onto the $x - u$ plane. Figure 2 shows the FOU of a general type-2 fuzzy set with Gaussian primary membership function and triangular secondary membership functions. Note that in this example, both primary and secondary domains are the unit interval; however, in general, the primary domain although numeric in nature may be different to the secondary domain (see Section 5). The *lower (upper) membership function* of a type-2 fuzzy set is the type-1 membership function associated with the lower (upper) bound of the FOU.

Interval Type-2 Fuzzy Set. An interval type-2 fuzzy set is a type-2 fuzzy set with constant secondary membership function 1, i.e., $\mu_{\tilde{A}}(x)(u) = 1, \forall u \in J_x$. The intersection of a plane parallel to the $x - u$ plane at a height $\alpha \in U$ with a general

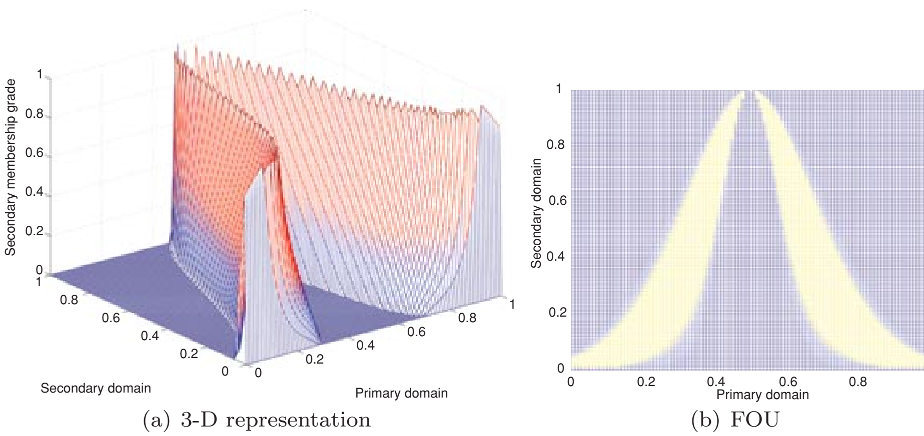


Figure 2. Type-2 fuzzy set: Gaussian primary MF and triangular secondary MFs.[16]

type-2 fuzzy sets produces an interval type-2 fuzzy set. This is known as a horizontal slice of a type-2 fuzzy set or α-plane.[26] In particular, the FOU of a general type-2 fuzzy set is obtained using the strong α-plane with $\alpha = 0$.

2.2. Type Reduced Set of a Type-2 Fuzzy Set

An mentioned before, for type-2 fuzzy sets the defuzzification process has two steps. First, through a procedure known as *type reduction*, a type-1 set is derived. This set is known as the *type-reduced set (TRS)*. Defuzzifying the type-1 TRS is relatively straightforward, and this is the second step of type-2 defuzzification.

Type reduction is dependent on the concept of an *embedded type-2 set*. An embedded type-2 set (or "set" for short) is a special kind of type-2 fuzzy set. It relates to the type-2 fuzzy set in which it is embedded in this way: For every primary domain value, x, there is a unique secondary domain value, u, plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)$.

Definition 1(Embedded Set). *Let $\tilde{A}$ be a type-2 fuzzy set in X. For discrete universes of discourse X and U, an embedded type-2 set $\tilde{A}_e$ of $\tilde{A}$ is defined as the following type-2 fuzzy set:*

$$\tilde{A}_e = \{(x_i, (u_i, \mu_{\tilde{A}}(x_i)(u_i)))|\ \forall i \in \{1, \ldots, n\} :\ x_i \in X\ \ u_i \in J_{x_i} \subseteq U\}. \tag{10}$$

where $\tilde{A}_e$ contains exactly one element from $J_{x_1}, J_{x_2}, \ldots, J_{x_N}$, namely $u_1, u_2, \ldots, u_N$, each with its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_1)$, $\mu_{\tilde{A}}(x_2)(u_2), \ldots, \mu_{\tilde{A}}(x_N)(u_N)$.

The TRS is defined via the application of Zadeh's extension principle, and only after the primary domain X has been discretized.

DEFINITION 2. *The TRS associated with a type-2 fuzzy sets $\tilde{A}$ with domain X discretized into n points is*

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{n} x_i \cdot u_i}{\sum_{i=1}^{n} u_i}, \mu_{\tilde{A}}(x_1)(u_1) * \cdots * \mu_{\tilde{A}}(x_n)(u_n) \right) \times \middle| \forall i \in \{1, \ldots, n\} : \ x_i \in X \ \ u_i \in J_{x_i} \subseteq U \right\}. \tag{11}$$

Note that the TRS is a type-1 fuzzy set in U. Again in this case, we have $x_1 < x_2 < \cdots < x_n$. The type reduction stage requires the application of a t-norm $(*)$ to the secondary membership grades. Because the product t-norm does not produce meaningful results for type-2 fuzzy sets with general secondary membership functions, the minimum t-norm $(\wedge)$ is used.[7]

TRS of an Interval Type-2 Fuzzy Sets. In the case of $\tilde{A}$ being an interval type-2 fuzzy set, i.e., $\mu_{\tilde{A}}(x)(u) = 1 \ \forall x, u$, we have that the TRS is the crisp set

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{n} x_i \cdot u_i}{\sum_{i=1}^{n} u_i}, 1 \right) \middle| \ \forall i \in \{1, \ldots, n\} : \ x_i \in X \ \ u_i \in J_{x_i} \subseteq U \right\}. \tag{12}$$

2.3. Type-Reduction Algorithm

The TRS is a type-1 fuzzy set in U, and its computation in practice requires the secondary domain U to be discretized as well. Algorithm 1, adapted from Mendel,[7] is used to compute the TRS of a type-2 fuzzy sets. This stratagem has become known as the *exhaustive method*, as every embedded set is processed.[10,16]

Algorithm 1: Type-reduction of a discretized type-2 fuzzy set to a type-1 fuzzy set.

Input: a discretized generalized type-2 fuzzy set
Output: a discrete type-1 fuzzy set
forall the *embedded sets* **do**
  find the minimum secondary membership grade (z);
  calculate the primary domain value (x) of the type-1 centroid of the type-2 embedded set;
  pair the secondary grade (z) with the primary domain value (x) to give set of ordered pairs (x, z) {some values of x may correspond to more than one value of z};
end
forall the *primary domain* (x) *values* **do**
  select the maximum secondary grade {make each x correspond to a unique secondary domain value};
end

The exhaustive method direct implementation is slow and inefficient, because of the extraordinarily large number of embedded sets into which the type-2 fuzzy set is decomposed.[7,16] This has inevitably hindered the development of type-2 FISs for real applications, a situation that will not be expected to change, unless an efficient and fast method to deffuzzify general type-2 fuzzy sets is developed. Indeed, as it was mentioned before, a consequence of this being that most researchers concentrate exclusively on the development of theoretical results and practical applications for interval type-2 fuzzy sets.

3. TYPE-1 OWA OPERATORS

In 1988, Yager introduced an aggregation technique based on the OWA scheme.[25]

DEFINITION 3. *An OWA operator of dimension n is a mapping $\phi : \mathbb{R}^n \to \mathbb{R}$, which has an associated set of weights $W = (w_1, \ldots, w_n)^T$ to it, so that $w_i \in [0,\ 1]$, $\sum_{i=1}^{n} w_i = 1$,*

$$\phi(a) = \phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)} \tag{13}$$

and

$$\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $a_{\sigma(i)}$ is the ith highest element in the set $\{a_1, \ldots, a_n\}$.

Generally speaking, the OWA operator based aggregation process consists of three steps: (i) the first step is the reordering the input arguments in a descending order. In this way, a particular element to aggregate is not associated with a particular weight, but rather a weight is associated with a particular ordered position of an aggregated object; (ii) the second step is to determine the weights for the operator in a proper way; (iii) finally, the OWA weights are used to aggregate the reordered arguments.

3.1. Definition of Type-1 OWA Operators Based on the Extension Principle

Unlike Yager's OWA operator that aggregates crisp values, the type-1 OWA operator is able to aggregate type-1 fuzzy sets with uncertain weights, with these uncertain weights being also modeled as type-1 fuzzy sets. As a generalization of Yager's OWA operator, and based on Zadeh's extension principle,[1] a type-1 OWA operator is defined as follows:[17]

DEFINITION 4. *Given n linguistic weights $\left\{W^i\right\}_{i=1}^{n}$ in the form of type-1 fuzzy sets defined on the domain of discourse U, a type-1 OWA operator is a mapping, $\mathbf{\Phi}$,*

$$\begin{aligned} \mathbf{\Phi} : \tilde{P}(\mathbb{R}) \times \ldots \times \tilde{P}(\mathbb{R}) &\longrightarrow \tilde{P}(\mathbb{R}) \\ (A^1, \ldots, A^n) &\mapsto Y \end{aligned}$$

such that

$$\mu_Y(y) = \sup_{\substack{\sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U,\, a_i \in X}} \left(\mu_{W^1}(w_1) \wedge \cdots \wedge \mu_{W^n}(w_n) \wedge \mu_{A^1}(a_1) \wedge \cdots \wedge \mu_{A^n}(a_n)\right) \tag{14}$$

where

$$\bar{w}_i = \frac{w_i}{\sum_{i=1}^{n} w_i},$$

and

$$\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)},\ \forall i = 1, \ldots, n-1$, *i.e.,* $a_{\sigma(i)}$ *is the* i*th highest element in the set* $\{a_1, \ldots, a_n\}$.

A direct approach to performing type-1 OWA operation was suggested in Ref.17. However, this approach is computationally expensive, which inevitably curtails further applications of the type-1 OWA operator to real-world decision making. A fast approach to type-1 OWA operations has been developed based on the α-cuts of fuzzy sets.[24]

3.2. Definition of Type-1 OWA Operators Based on the α-Cuts of Fuzzy Sets

DEFINITION 5. *Given the n linguistic weights* $\left\{W^i\right\}_{i=1}^{n}$ *in the form of type-1 fuzzy sets defined on the domain of discourse* U, *then for each* $\alpha \in U$, *an* α*-level type-1 OWA operator with* α*-cuts of weight sets* $\left\{W_\alpha^i\right\}_{i=1}^{n}$ *to aggregate the* α*-cuts of type-1 fuzzy sets* $\left\{A^i\right\}_{i=1}^{n}$ *is given as*

$$\Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right) = \left\{ \frac{\sum_{i=1}^{n} w_i a_{\sigma(i)}}{\sum_{i=1}^{n} w_i} \middle| \forall i \in \{1, \ldots, n\} : w_i \in W_\alpha^i \wedge a_i \in A_\alpha^i \right\} \tag{15}$$

where $W_\alpha^i = \{w | \mu_{W_i}(w) \geq \alpha\}$, $A_\alpha^i = \{x | \mu_{A_i}(x) \geq \alpha\}$, *and* $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *is a permutation function such that* $a_{\sigma(i)} \geq a_{\sigma(i+1)},\ \forall\, i = 1, \ldots, n-1$, *i.e.,* $a_{\sigma(i)}$ *is the* i*th largest element in the set* $\{a_1, \ldots, a_n\}$.

According to the representation theorem of type-1 fuzzy sets, the α-level sets $\Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right)$ obtained via Definition 5. can be used to construct the following type-1 fuzzy set on $\mathbb{R}$

$$\Phi\left(A^1, \ldots, A^n | W^1, \ldots, W^n\right) = \bigcup_{0<\alpha\leq 1} \alpha \Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right) \tag{16}$$

with membership function

$$\mu_\Phi(x) = \bigvee_{\alpha : x \in \Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_\alpha} \alpha \tag{17}$$

3.3. Representation Theorem of Type-1 OWA Operators

The two apparently different aggregation results in (14) and (16) obtained according to Zadeh's extension principle and the α-level of type-1 fuzzy sets, respectively, are equivalent as proved in Ref. 24:

THEOREM 1. *Given the n linguistic weights $\left\{W^i\right\}_{i=1}^n$ in the form of type-1 fuzzy sets defined on the domain of discourse U, and the type-1 fuzzy sets $A^1, \ldots, A^n$, then we have that*

$$Y\left(A^1, \ldots, A^n | W^1, \ldots, W^n\right) = \Phi\left(A^1, \ldots, A^n | W^1, \ldots, W^n\right)$$

where $Y\left(A^1, \ldots, A^n | W^1, \ldots, W^n\right)$ is the aggregation result defined in (14) and $\Phi\left(A^1, \ldots, A^n | W^1, \ldots, W^n\right)$ is the result defined in (16).

Theorem 1. is called the *representation theorem of type-1 OWA operators*. Therefore, an effective and practical way of carrying out type-1 OWA operations is to decompose the type-1 OWA aggregation into the α-level type-1 OWA operations and then reconstruct it via the above representation theorem. This α-level approach has been proved to be much faster than the direct approach,[24] so it can be used in real time decision making and data mining applications.

3.4. Alpha-Level Type-1 OWA Operators of Fuzzy Numbers

When the linguistic weights and the aggregated sets are fuzzy number, the alpha-level type-1 OWA operator produces closed intervals:[24]

THEOREM 2. *Let $\left\{W^i\right\}_{i=1}^n$ be fuzzy numbers on U and $\left\{A^i\right\}_{i=1}^n$ be fuzzy numbers on $\mathbb{R}$. Then for each $\alpha \in U$, $\Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)$ is a closed interval.*

Based on this result, the computation of the type-1 OWA output according to, (16), G, reduces to compute the left end-points and right end-points of the intervals $\Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)$:

$$\Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_- \text{ and } \Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_+,$$

where $A^i_\alpha = [A^i_{\alpha-}, A^i_{\alpha+}]$, $W^i_\alpha = [W^i_{\alpha-}, W^i_{\alpha+}]$.

For the left end-points, we have

$$\Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right)_- = \min_{\substack{W_{\alpha-}^i \leq w_i \leq W_{\alpha+}^i \\ A_{\alpha-}^i \leq a_i \leq A_{\alpha+}^i}} \sum_{i=1}^{n} w_i a_{\sigma(i)} \Big/ \sum_{i=1}^{n} w_i \tag{18}$$

while for the right end-points, we have

$$\Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right)_+ = \max_{\substack{W_{\alpha-}^i \leq w_i \leq W_{\alpha+}^i \\ A_{\alpha-}^i \leq a_i \leq A_{\alpha+}^i}} \sum_{i=1}^{n} w_i a_{\sigma(i)} \Big/ \sum_{i=1}^{n} w_i \tag{19}$$

It can be seen that (18) and (19) are programming problems. Solutions to these problems, so that the type-1 OWA aggregation operation can be performed efficiently, are available from Ref. 24.

3.5. Alpha-Level Approach to Type-1 OWA Aggregation Algorithm

Given n linguistic weights $\left\{W^i\right\}_{i=1}^n$, the procedure to aggregate $\left\{A^i\right\}_{i=1}^n$ by a type-1 OWA operator via the α-level aggregation scheme is given in Algorithm 2.[24]

Algorithm 2: Procedure of the *Alpha-Level Approach* to type-1 OWA operation.

Input: a set of linguistic weights and a set of type-1 fuzzy sets to aggregate
Output: a type-1 fuzzy set
Step 1 To set up the α- level resolution in [0, 1];
Step 2 For each $\alpha \in [0, 1]$;
 Step 2.1. **To calculate** $\rho_{\alpha+}^{i_0^*}$;
 1 Let $i_0 = 1$;
 2 **If** $\rho_{\alpha+}^{i_0} \geq A_{\alpha+}^{\sigma(i_0)}$ **then** $\rho_{\alpha+}^{i_0}$ is the solution, stop; **else** go to *2.1-3*;
 3 $i_0 \leftarrow i_0 + 1$, go to *2.1-2*;
 Step 2.2. **To calculate** $\rho_{\alpha-}^{i_0^*}$;
 1 Let $i_0 = 1$;
 2 **If** $\rho_{\alpha-}^{i_0} \geq A_{\alpha-}^{\sigma(i_0)}$ **then** $\rho_{\alpha-}^{i_0}$ is the solution, stop; **else** go to *2.2-3*;
 3 $i_0 \leftarrow i_0 + 1$, go to step *2.2-2*;
Step 3 To construct the aggregation resulting fuzzy set Φ based on all the available intervals $\left[\rho_{\alpha-}^{i_0^*},\ \rho_{\alpha+}^{i_0^*}\right]$:

$$\mu_\Phi(x) = \bigvee_{\alpha : x \in \left[\rho_{\alpha-}^{i_0^*},\ \rho_{\alpha+}^{i_0^*}\right]} \alpha$$

In this approach, the α values are required to cover all the available membership grades $\{\mu_{W^i}(w_i)\}$ and $\{\mu_{A^i}(a_i)\}$, and $\rho_{\alpha-}^{i_0}$ and $\rho_{\alpha+}^{i_0}$ are defined as

$$\rho_{\alpha-}^{i_0} \triangleq \frac{\sum_{i=1}^{i_0-1} W_{\alpha-}^{i} A_{\alpha-}^{\sigma(i)} + \sum_{i=i_0}^{n} W_{\alpha+}^{i} A_{\alpha-}^{\sigma(i)}}{J_{i_0}} \tag{20}$$

where

$$J_{i_0} \triangleq \sum_{i=1}^{i_0-1} W_{\alpha-}^{i} + \sum_{i=i_0}^{n} W_{\alpha+}^{i} \tag{21}$$

and

$$\rho_{\alpha+}^{i_0} \triangleq \frac{\sum_{i=1}^{i_0-1} W_{\alpha+}^{i} A_{\alpha+}^{\sigma(i)} + \sum_{i=i_0}^{n} W_{\alpha-}^{i} A_{\alpha+}^{\sigma(i)}}{H_{i_0}} \tag{22}$$

where

$$H_{i_0} \triangleq \sum_{i=1}^{i_0-1} W_{\alpha+}^{i} + \sum_{i=i_0}^{n} W_{\alpha-}^{i} \tag{23}$$

4. TYPE-1 OWA APPROACH TO TYPE REDUCTION OF GENERAL TYPE-2 FUZZY SETS

The following theorem establishes the relationship between the TRS of a type-2 fuzzy set and the type-1 OWA operator.

THEOREM 3. *Given a general type-2 fuzzy set $\tilde{A}$, with domain X discretized in a set of n points such that $x_1 < x_2 < \cdots < x_n$, the TRS of $\tilde{A}$ is*

$$C_{\tilde{A}} = \Phi\left(\hat{x_1}, \ldots, \hat{x_n} | W^1, \ldots, W^n\right)$$

where Φ is a type-1 OWA operator with set of uncertain weights defined by $\tilde{A}$s secondary membership functions as $W^1 = \mu_{\tilde{A}}(x_n), \ldots, W^n = \mu_{\tilde{A}}(x_1)$ to aggregate the singleton type-1 fuzzy sets $\hat{x_1}, \ldots, \hat{x_n}$.

Proof. We note that the type-1 fuzzy set derived after the application of a type-1 OWA operator can be rewritten as follows:

$$Y = \left\{ \left(\frac{\sum_{i=1}^{n} w_i a_{\sigma(i)}}{\sum_{i=1}^{n} w_i},\ \mu_{W^1}(w_1) \wedge \cdots \wedge \mu_{W^n}(w_n) \wedge \mu_{A^1}(a_1) \wedge \cdots \wedge \mu_{A^n}(a_n) \right) \right| \\ \times \forall i \in \{1, \ldots, n\} :\ w_i \in S(W^i),\ a_i \in S(A^i) \right\}. \tag{24}$$

where $S(W^i)$ and $S(A^1)$ are the support sets of W^i and A^i, respectively, for all $i = 1, \ldots, n$, i.e.

$$S(W^i) = \{w \in U | \mu_{W^i}(w) > 0\};$$

and

$$S(A^i) = \{a \in X | \mu_{A^i}(a) > 0\}.$$

Because $x_1 < x_2 < \cdots < x_n$, it is clear that expression (24) with $A^1 = \hat{x_1}, \ldots, A^n = \hat{x_n}$, and $W^1 = \mu_{\tilde{A}}(x_n), \ldots, W^n = \mu_{\tilde{A}}(x_1)$ reduces to

$$Y = \left\{ \left(\frac{\sum_{k=1}^{n} w_i x_{\sigma(i)}}{\sum_{i=1}^{n} w_i}, \ \mu_{W^1}(w_1) \wedge \cdots \wedge \mu_{W^n}(w_n) \right) \right.$$
$$\left. \times \middle| \forall i \in \{1, \ldots, n\} : \ x_i \in X, w_i \in J_{x_{n-i+1}} \right\}. \tag{25}$$

Considering $x_1 < x_2 < \cdots < x_n$, i.e., $x_{\sigma(i)} = x_{n-i+1}$, and using the notation $w_i = u_{n-i+1}$, then the (25) can be written as

$$Y = \left\{ \left(\frac{\sum_{k=1}^{n} x_i u_i}{\sum_{i=1}^{n} u_i}, \ \mu_{\tilde{A}}(x_1)(u_1) \wedge \cdots \ \wedge \mu_{\tilde{A}}(x_n)(u_n) \right) \right.$$
$$\left. \times \middle| \forall i \in \{1, \ldots, n\} : \ x_i \in X, u_i \in J_{x_i} \right\}. \tag{26}$$

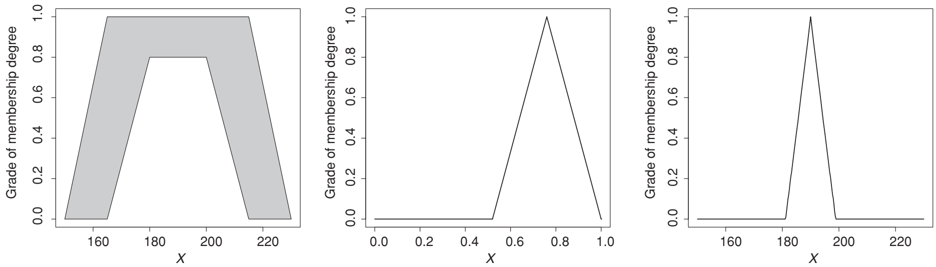
This expression just coincides with the TRS associated with a type-2 fuzzy sets $\tilde{A}$ with domain X discretized into n points as per the expression (11) given in Definition 2. ■

In the case of an interval type-2 fuzzy set $\tilde{A}$ with domain X discretized into n points x_i, the primary membership of x_i, J_{x_i}, is a closed interval and therefore $C_{\tilde{A}}$ is also closed. On the other hand, when the inputs of an α-level type-1 OWA operator $A_\alpha^1, \ldots, A_\alpha^n$ reduce to singleton points, the aggregation result (15) reduces to (27). Therefore, in this case, both mathematical representations (15) and (27) are equivalent. We have the following corollary:

COROLLARY 1. *Given an interval type-2 fuzzy set $\tilde{A}$, with domain X discretized in a set of n points such that $x_1 < x_2 < \cdots < x_n$, the TRS of $\tilde{A}$ is*

$$C_{\tilde{A}} = \Phi_1 \left(x_1, \ldots, x_n | J_{x_n}, \ldots, J_{x_1} \right)$$

where Φ_1 is a special type-1 OWA operator with weights $J_{x_n}, \ldots, J_{x_1}$ to aggregate the crisp points $x_1, \ldots, x_n$.



(a) FOU (b) SMF at $x = 174.75$ (c) TRS

Figure 3. Case study 1: (a) FOU; (b) example of secondary MF; and (c) TRS.

In this way, the TRS of an interval type-2 fuzzy set reduces to

$$C_{\tilde{A}} = \left\{ \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \middle| \forall i \in \{1, \ldots, n\} : w_i \in J_{x_{n-i+1}} \right\} \tag{27}$$

In other words, the TRS of an interval type-2 fuzzy set can be computed via the application of the α-level type-1 OWA operator as per Definition 5. where the value of α is set to be 1. So a special *alpha-level approach* with only considering $\alpha = 1$ is used to compute the centroid of an interval type-2 fuzzy set, and more generally to aggregate conventional intervals values with intervals weights.

5. EXPERIMENTAL RESULTS

In this section, we provide some case studies to show the application of the type-1 OWA operators to computing the TRS of different type-2 fuzzy sets, and ultimately their centroid as per expression (6).

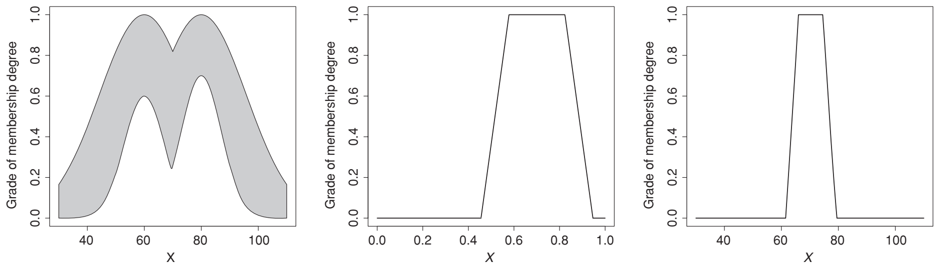
5.1. Case Study 1: General Type-2 Fuzzy Set with Trapezoidal FOU and Triangular Secondary Membership Functions

In this case study, a general type-2 fuzzy set is defined with trapezoidal FOU and triangular secondary membership functions as per Figure 3a. More specifically, the upper membership function of the FOU, $u(x)$, is a trapezoidal function

$$u(x) = \begin{cases} \frac{(u_1 - x)\cdot e}{u_1 - u_2}, & u_1 \leq x \leq u_2 \\ e, & u_2 \leq x \leq u_3 \\ \frac{(u_4 - x)\cdot e}{u_4 - u_3}, & u_3 \leq x \leq u_4 \\ 0, & \text{otherwise.} \end{cases}$$

with apexes $(u_1,\ u_2,\ u_3, u_4)$ chosen as

$$(u_1,\ u_2,\ u_3, u_4) = (150,\ 165,\ 215,\ 230)$$



(a) FOU (b) SMF at $x = 54.75$ (c) TRS

Figure 4. Case study 2: (a) FOU; (b) example of secondary MF; and (c) TRS.

and the maximum value of $u(x)$ set to be $e = 1$. The lower membership function of the FOU, $l(x)$, is set as a trapezoidal functions with apexes

$$(l_1,\ l_2,\ l_3, l_4) = (165,\ 180,\ 200,\ 215)$$

and the maximum value of $l(x)$ set to be $e = 0.8$. For any $x \in X$, the associated secondary membership function is defined as a triangular function with apexes $(tr_1,\ tr_2,\ tr_3)$ defined as follows:

$$tr_1 = l(x)$$
$$tr_2 = [l(x) + u(x)]/2$$
$$tr_3 = u(x)$$

Figure 3b shows an example of such secondary MF at $x = 174.75$. Note that this type-2 fuzzy set is symmetrical with respect to the value $x = 190$, which is its centroid. Then application of the *alpha-level approach* to type-1 OWA aggregation to calculate the TRS of this general type-2 fuzzy set results in the following type-1 fuzzy set depicted in Figure 3c. This TRS is symmetrical with respect to the value $x = 190$, which is its centroid and coincides with the observation made above for the type-2 fuzzy set from which it was derived.

5.2. Case Study 2: General Type-2 Fuzzy Set with Mixed Gaussian FOU and Trapezoidal Secondary Membership Functions

In this case study, the FOU of a general type-2 fuzzy set is shown in Figure 4a. The upper membership function of the FOU, $u(x)$, is defined as $u(x) = \max\{u_1(x), u_2(x)\}$ with $u_1(x)$ and $u_2(x)$ being the following Gaussian membership functions:

$$u_1(x) = \exp\left(-\frac{(x-60)^2}{500}\right)$$

$$u_2(x) = \exp\left(-\frac{(x-80)^2}{500}\right)$$

The lower membership function, $l(x)$, is defined similarly $l(x) = \max\{l_1(x), l_2(x)\}$ using the following two Gaussian functions:

$$l_1(x) = 0.6 \exp\left(-\frac{(x-60)^2}{100}\right)$$

$$l_2(x) = 0.7 \exp\left(-\frac{(x-80)^2}{100}\right)$$

For any $x \in X$, the associated secondary membership function is defined as a trapezoidal function with following apexes $(tr_1,\ tr_2,\ tr_3,\ tr_4)$:

$$\begin{aligned} tr_1 &= l(x) \\ tr_2 &= (l(x) + mid(x))/2 \\ tr_3 &= (mid(x) + u(x))/2 \\ tr_4 &= u(x) \end{aligned}$$

where $\mathrm{mid}(x) = (l(x) + u(x))/2$. Figure 4b shows an example of such secondary MF. The TRS of this general type-2 fuzzy set obtained by applying the *Alpha-Level Approach* to type-1 OWA aggregation is illustrated in Figure 4c.

5.3. Case Study 3: Interval Type-2 Fuzzy Set with Gaussian FOU

In here, we consider an interval type-2 fuzzy set with Gaussian upper and lower membership functions

$$u(x) = \exp\left(-\frac{(x-70)^2}{500}\right)$$

and

$$l(x) = 0.6 \exp\left(-\frac{(x-70)^2}{100}\right)$$

as illustrated in Figure 5.

Note that this interval type-2 fuzzy set is symmetrical with respect to the value $x = 70$, which is its centroid. We use the special the *alpha-level approach* to type-1 OWA aggregation with only $\alpha = 1$ to generate its TRS. In this case, the TRS will be a closed interval with the centroid of this fuzzy set as midpoint. For any $x \in X$, the intervals $[l(x),\ u(x)]$ are used as weights in the type-1 OWA aggregation to generate the TRS of this interval type-2 fuzzy set, which is $TRS = [61, 79]$ with 70 as its midpoint.

6. CONCLUSIONS

In this paper, we have shown that the apparent disparate problems consisting of the computation of the TRS of a type-2 fuzzy set and the type-1 OWA aggregation

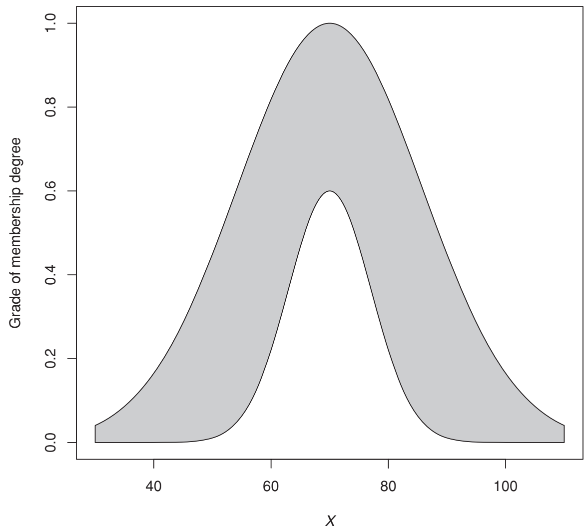


Figure 5. Case study 3: Interval type-2 fuzzy set.

of type-1 fuzzy sets are closely related. In essence, both problems are aggregation problems. Based on the *type-1 OWA representation theorem*, we have proved that the TRS of a type-2 fuzzy sets is a special case of a type-1 OWA operator. In particular, the centroid of an interval type-2 fuzzy sets is a particular case of an α-level type-1 OWA operator.

The main contribution of this paper is the realization of a fast and efficient method to compute the centroid of a general type-2 fuzzy set via the type-1 OWA operator. This could inspire an increase use of general type-2 fuzzy sets in real-world applications.

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