Fuzzy Logic

Mathematics for Fuzzy Logic

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Definition

A set is a collection of things that can be distinguished from one another as individuals and that share some property.
Each individual in this collection is called a member, or element, of the set.

Notation

a is an element of (or belongs to) the set A: $a \in A$
a does not belong to A: $a \notin A$

A set can be described in different ways:

LIST METHOD: Enumerate the elements that belong to the set (finite sets)

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

RULE METHOD Describe the set analytically by stating conditions for membership

$$A = \{x|P(x)\}$$

"the set C is composed of elements x, such that (every) x has the property P"

$$A = \{x|0 \leq x \leq 1\}$$

Universal Set Set that consists of all the individuals that are of interest in the context of a particular application

Empty Set Set which contains no elements and is denoted by the symbol $\emptyset$

$A \subseteq B \Leftrightarrow$ every element of A is also an element of B

$$A \subseteq B \Leftrightarrow [\forall x \in A \Rightarrow x \in B]$$

$$A = B \Leftrightarrow [A \subseteq B \subseteq A]$$

$$A \subset B \Leftrightarrow [A \subseteq B \wedge A \neq B]$$

The set which consists of all possible subsets of a given set X

$$\mathcal{P}(X) = \{A | A \subseteq X\}$$

The complement of a set A, denoted by $\overline{A}$ is the set of all the elements in the universal set X which are not in A

$$\overline{A} = \{x \in X | x \notin A\}$$

$$\overline{\overline{A}} = A$$

$$\overline{X} = \emptyset$$

$$\overline{\emptyset} = X$$

The union of set A and set B is the set containing all the elements belonging to A or B, or to both

$$A \cup B = \{x | x \in A \; or \; x \in B\}$$

$$A \cup X = X$$

$$\text{If } A \subseteq B \Longrightarrow A \cup B = B$$

$$A \cup A = A$$

$$A \cup \overline{A} = X$$

The intersection of set A and set B is the set containing all the elements belonging to both sets A and B simultaneously

$$A \cap B = \{x | x \in A \ and \ x \in B\}$$

Two sets are A and B are disjoints if they have no element in common, i.e. $A \cap B = \emptyset$

$$A \cap X = A$$

$$\text{If } A \subseteq B \Longrightarrow A \cap B = A$$

$$A \cap A = A$$

$$A \cap \overline{A} = \emptyset$$

The difference of sets A and B is a set that consists of all the elements which belong to A, but which do not belong to B

$$A - B = \{x|x \in A \; and \; x \notin B\}$$

$$A - B \neq B - A$$

$$X - A = \overline{A}$$

$$A \subseteq B \Longrightarrow A - B = \emptyset$$

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

A set $A = \{x|P(x)\}$ can be represented using the following characteristic function:

$$\delta_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

1 indicates membership (property $P(x)$ is true) and 0 non-membership (property $P(x)$ is false).

This method is particularly important in FL since, contrary to the other two methods, it can be generalised via the concept of membership function to fuzzy sets.

Let X be the universal set:

$$A \subseteq B \Leftrightarrow \delta_A(x) \leq \delta_B(x) \; \forall x \in X$$

$$A \cap B = \{x | \delta_A(x) = 1 \wedge \delta_B(x) = 1\}$$

In terms of the characteristic functions

$$\delta_{A \cap B}(x) = \min\{\delta_A(x), \delta_B(x)\}$$

$$A \cup B = \{x|\delta_A(x) = 1 \vee \delta_B(x) = 1\}$$

In terms of the characteristic functions

$$\delta_{A\cup B}(x) = \max\{\delta_A(x), \delta_B(x)\}$$

$$\delta_{\overline{A}}(x) = 1 - \delta_A(x) \ \forall x \in X$$

- Intersection

$$\delta_{A\cap B}(x) = \delta_A(x)\cdot\delta_B(x) \text{ product}$$

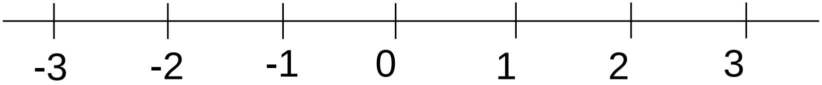
$$\delta_{A\cap B}(x) = \max\{0, \delta_A(x)+\delta_B(x)-1\}$$

- Union

$$\delta_{A\cup B}(x) = \delta_A(x)+\delta_B(x)-\delta_A(x)\cdot\delta_B(x)$$

$$\delta_{A\cup B}(x) = \min\{1, \delta_A(x)+\delta_B(x)\}$$

Often represented by an x-axis with each point on the axis corresponding to a real number in $\mathbb{R}$



The set of all points between given points a and b on the real line $(a \leq b)$ is called an interval

- Closed $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
- Open $(a, b) = \{x \in \mathbb{R} | a < x < b\}$ (or $]a, b[$)
- Half-open
  $(a, b] = \{x \in \mathbb{R} | a < x \leq b\} (or$]a,b])
  $[a, b) = \{x \in \mathbb{R} | a \leq x < b\} (or$[a,b[)

The cartesian product of two real numbers is the two-dimensional Euclidean space, usually called a plane

Can be represented by the so-called Cartesian Coordinates or $x - y$ axes

Given A and B two sets, the cartesian product $A \times B$ is the set of *ordered* pairs composed from elements $a \in A$ and $b \in B$

In general $(a, b) \neq (b, a)$

When $A = B$ notation used is: $A \times A = A^2$

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0} \cdot (x - x_0)$$

$$(x_0, 0), (x_1, 1) \Rightarrow y = \frac{x - x_0}{x_1 - x_0}$$

$$(x_0, 1), (x_1, 0) \Rightarrow y = \frac{x - x_1}{x_0 - x_1}$$

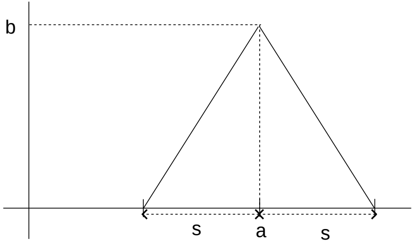
Represent

$$A(x) = \begin{cases} x-5, & 5 \leq x \leq 6; \\ 7-x, & 6 < x \leq 7; \\ 0, & \text{otherwise.} \end{cases}$$

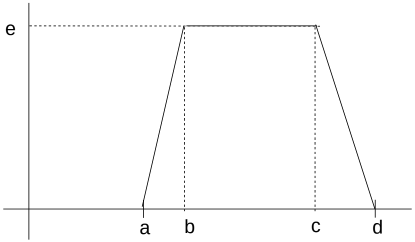
Evaluate

- $A(3)$
- $A(5)$
- $A(5.5)$
- $A(6)$
- $A(6.5)$
- $A(7)$
- $A(9)$

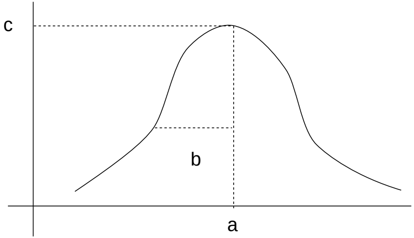
$$A(x) = \begin{cases} b\left(1 - \dfrac{|x-a|}{s}\right), & a-s \le x \le a+s; \\ 0, & \text{otherwise.} \end{cases}$$



$$A(x) = \begin{cases} \dfrac{(a-x)\cdot e}{a-b}, & a \leq x \leq b; \\ e, & b \leq x \leq c; \\ \dfrac{(d-x)\cdot e}{d-c}, & c \leq x \leq d; \\ 0, & \text{otherwise.} \end{cases}$$



$$A(x) = c \cdot e^{-\frac{(x-a)^2}{2b^2}}$$



Plot the following two functions

$$A(x) = \begin{cases} 1 - \dfrac{|x-3|}{2}, & 1 \leq x \leq 5, \\ 0, & \text{otherwise.} \end{cases}$$

$$B(x) = \begin{cases} 1 - \dfrac{|x-4|}{2}, & 2 \leq x \leq 6; \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the analytic expression of

- $\overline{A}(x) = 1 - A(x)$
- $\overline{B}(x) = 1 - B(x)$
- $\min\{A(x), B(x)\}$
- $\max\{A(x), B(x)\}$

$$[a_1, b_1] \star [a_2, b_2] = \{x_1 \star x_2 | a_1 \leq x_1 \leq b_1 \wedge a_2 \leq x_2 \leq b_2\};$$

$$\star \in \{+\ ,-\ ,\cdot\ ,/\};$$

and the expression $[a_1, b_1]/[a_2, b_2]$ is not defined if $0 \in [a_2, b_2]$.

Since

$$\star\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$$

is a continuous function defined in a compact set, then it takes its minimum and maximum values as well as all the values in between.

So, it is

$$[a_1, b_1] \star [a_2, b_2] = [\min B, \max B]$$

$$B = \{a_1 \star a_2, a_1 \star b_2, b_1 \star a_2, b_1 \star b_2\}.$$

If $a_1 = b_1$ and $a_2 = b_2$, then $\min B = \max B$ and interval arithmetic becomes number arithmetic.

$$[2,4]+[-3,-1]=[-1,3]$$

$$[2,4]-[-3,-1]=[3,7]$$

$$[2,4]\cdot[-3,-1]=[-12,-2]$$

$$[2,4]/[-3,-1]=[-4,-2/3]$$