Fuzzy Sets and Logic

Basics of Fuzzy Logic

MSc IS/ISR

Overview

- This lecture introduces the basics of fuzzy logic, with a review of classical logic and its operations, logical implication, and certain classical inference mechanisms such as Tautologies or equivalences.
- The operations of disjunction, conjunction and negation as well as classical implication and equivalences are reviewed.
- Operations on propositions are shown to be isomorphic with operations on sets; hence an algebra of propositions is developed using the algebra of operations covered in the last lesson.
- Fuzzy Logic is then shown to be an extension of classical logic when partial truth values between true and not true are included to extend bi-valued logic.

Objectives

- To know the main five classical operations with propositions:
  - disjunction, conjunction, negation, implication and equivalence.
- To learn that a proposition can be modelled as a set.
- To know how to derive the truth value of the conjunction, disjunction, implication and equivalence of two propositions from the truth values associated of the individual propositions.
- To obtain the membership function associated to the conjunction, disjunction, implication and equivalence of two propositions described using vague concepts modelled via fuzzy sets.

Fuzzy Logic

- The notion fuzzy logic has three different meanings.
  - In most cases, the term fuzzy logic refers to fuzzy logic in the broader sense, including all *applications and theories where fuzzy sets or concepts are involved*.
  - The second and narrower meaning of the term fuzzy logic focuses on the field of *approximative reasoning where fuzzy sets are used and propagated within an inference mechanism* as it is for instance common in expert systems/fuzzy control.
  - Finally, fuzzy logic in the narrow sense, which is the topic of this section, considers *fuzzy logic as an extension of classical logic* – from two valued logic to multi-valued logic – and is devoted *to* issues connected to logical calculi and the associated deduction mechanisms.

From Classical Logic to Fuzzy Logic

- We cannot provide a complete introduction to fuzzy logic as a multi-valued logic.
- In this session, we will introduce those notions of fuzzy logic which are necessary or useful to understand fuzzy controllers, the following topic in this module.
- We mainly need the concepts of fuzzy logic to introduce the set theoretical operations for fuzzy sets.
    - The basis for operations like union, intersection and complement are the logical connectives disjunction, conjunction and negation, respectively.
    - Therefore, we briefly repeat some fundamental concepts from classical logic in order to generalize them to the field of fuzzy logic.

Propositions and Truth Values

- Classical propositional logic deals with the formal handling of statements (propositions) to which one of the two truth values 1 (for true) or 0 (for false) can be assigned.
- We represent these propositions by Greek letters φ , ψ etc.
- Typical propositions, for which the formal symbols φ_1 and φ_2 may stand are

  φ_1 : Four is an even number.

  φ_2 : $2+5 = 9$.
- The truth value which is assigned to a proposition φ is denoted by $[[\varphi]]$.
- For the above propositions we obtain $[[\varphi_1]] = 1$ and $[[\varphi_2]] = 0$.

Combined Propositions

- If the truth values of single propositions are known, we can determine the truth values of combined propositions using
  - truth tables that define the interpretation of the corresponding logical connectives.
- The most important logical connectives are
  - the logical AND $\land$ (conjunction),
  - the logic OR $\lor$ (disjunction),
  - the negation NOT $\neg$, and
  - the IMPLICATION $\rightarrow$.

Truth Values of Combined Propositions

- The conjunction $\varphi \land \psi$ of two propositions φ and ψ is true, if and only if both φ and ψ are true.
- The disjunction $\varphi \lor \psi$ of φ and ψ obtains the truth value 1 (true), if and only if at least one of the two propositions is true.
- The implication $\varphi \rightarrow \psi$ is only false, if the antecedent φ is true and the consequent ψ is false.
- The negation $\neg \varphi$ of the proposition φ is false, if and only if φ is true.

Truth Tables

Conjunction

$[[\phi]]$	$[[\psi]]$	$[[\phi \wedge \psi]]$
1	1	1
1	0	0
0	1	0
0	0	0

Truth Tables

Disjunction

[[ϕ]]	[[ψ]]	[[$\phi \vee \psi$]]
1	1	1
1	0	1
0	1	1
0	0	0

Truth Tables

Implication

$[[\phi]]$	$[[\psi]]$	$[[\phi \to \psi]]$
1	1	1
1	0	0
0	1	1
0	0	1

Truth Tables
Negation

$[[\phi]]$	$[[\neg\phi]]$
1	0
0	1

Example

This definition implies that the propositions

Four is an even number AND 2+ 5 = 9.

Four is an even number IMPLICATION 2 + 5 = 9.

are false, whereas the propositions

Four is an even number OR 2+ 5 = 9.

NOT 2+ 5 = 9.

are true.

Formally expressed, this means that we have

$[[\varphi 1 \wedge \varphi 2]] = 0$, $[[\varphi 1 \rightarrow \varphi 2]] = 0$, $[[\varphi 1 \vee \varphi 2]] = 1$ and $[[\neg \varphi 2]] = 1$

Fuzzy Propositions

- The assumption that a statement is either true or false is suitable for mathematical issues.
- For many expressions formulated in natural language such a strict separation between true and false statements would be unrealistic and would lead to counterintuitive consequences.
- A driver will not calculate the distance he will need for stopping his car abruptly on a wet road by using another friction constant in some mathematical formula to calculate this distance.
- He will consider the rule: the wetter the road, the longer the distance needed for breaking.
- In order to model this human information processing in a more appropriate way, we use gradual truth values for statements.
    - This means that a statement can not only be true (truth value 1) or false (truth value 0) but also more or less true expressed by a value between 0 and 1.

From Fuzzy Sets to Fuzzy Statements

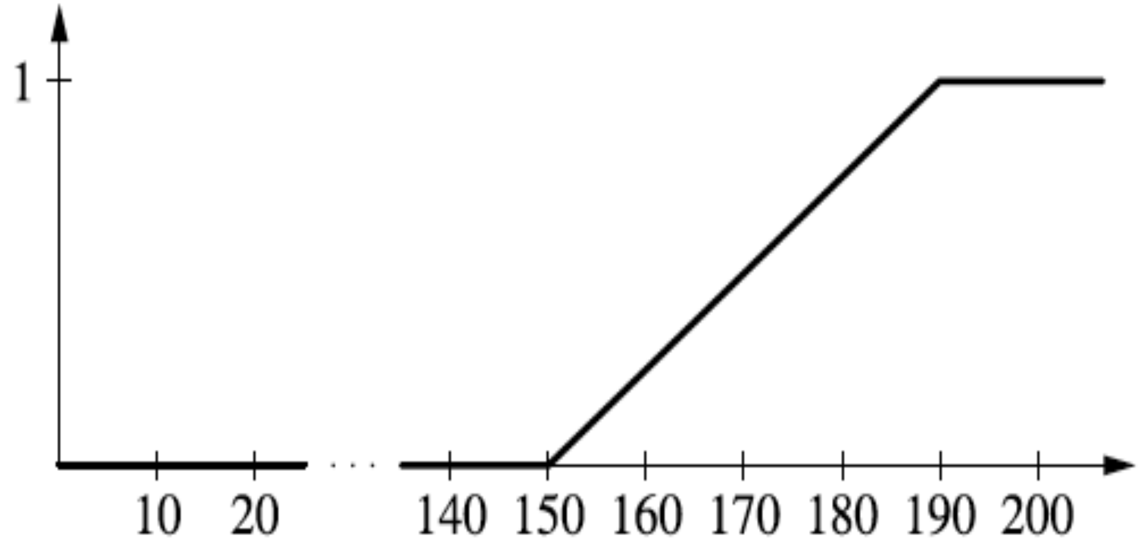
- A fuzzy set models a property that elements of the universe of discourse can have more or less.
  - For example, let us consider the fuzzy set of high velocities

That means the degree of membership of a specific velocity v to the fuzzy set of high velocities corresponds to the “truth value” which is assigned to the statement “v is a high velocity”. In this sense, a fuzzy set determines the corresponding truth values for a set of statements—in our example for all statements we obtain, when we consider in a concrete velocity value for v . In order to understand how to operate with fuzzy sets, it is first of all useful to consider classical crisp propositions.

From Fuzzy Sets to Fuzzy Statements

A fuzzy set models a property that elements of the universe of discourse can have more or less.

- For example, let us consider the fuzzy set of high velocities (right hand side)
- That means the degree of membership of a specific velocity `v’ to the fuzzy set of `high velocities’ corresponds to the “truth value” which is assigned to the statement “v is a high velocity”.
- In this sense, a fuzzy set determines the corresponding truth values for a set of statements—in our example for all statements we obtain its truth value when we consider in a concrete velocity value for v .
- In order to understand how to operate with fuzzy sets, it is first of all useful to consider classical crisp propositions.



Dealing with combined fuzzy statements

- Dealing with combined propositions like

  “160 km/h is a high velocity AND the stopping distance is about 110 m”

requires the extension of the truth tables of the classical logical connectives for conjunction, disjunction, implication or negation.

- For conjunction, disjunction and implication this truth function assigns to each combination of two truth values (the truth value assigned to φ and ψ) one truth value (the truth value of the conjunction, disjunction of φ and ψ or the implication $\varphi \rightarrow \psi$).
- The truth function assigned to the negation has only one truth value as argument.

Truth Functions

- If we denote the truth function by w_* associated with the classical logical connective $* \in \{\wedge, \vee, \rightarrow, \neg\}$, then w_* is a binary or unary function.

$$w_\wedge,\ w_\vee,\ w_\rightarrow : \{0,\ 1\}^2 \rightarrow \{0,\ 1\}, \text{ while } w_\neg : \{0,\ 1\} \rightarrow \{0,\ 1\}$$

- For fuzzy propositions, where the unit interval $[0,\ 1]$ replaces the binary set $\{0,\ 1\}$ as set of possible truth values, we have to assign truth functions to the logic connectives accordingly.

Truth Functions (2)

- These truth functions have to be defined on the unit square or the unit interval.

$$w_\wedge, w_\vee, w_\rightarrow : [0,\ 1]^2 \rightarrow [0,\ 1], \text{ while } w_\neg : [0,\ 1] \rightarrow [0,\ 1]$$

- A minimum requirement we demand of these functions is that, limited to the values 0 and 1, they should provide the same values as the corresponding truth function associated with the connectives of classical logic.
- This requirement says that a combination of fuzzy propositions which are actually crisp (non-fuzzy), because their truth values are 0 or 1 were, coincide with the usual combination of classical crisp propositions.

Truth Functions (3)

- The most frequently used truth functions for conjunction and disjunction in fuzzy logic are the minimum or maximum.

$$w_{\wedge}(\alpha,\beta) = \min\{\alpha,\beta\} \text{ , } w_{\vee}(\alpha,\beta) = \max\{\alpha,\beta\}$$

- Normally the negation is defined by $w_{\neg}(\alpha) = 1 - \alpha$.
- The implication is often understood in the sense of the Łukasiewicz implication

$$w_{\rightarrow}(\alpha,\beta) = \min\{1-\alpha+\beta, 1\}$$

or the Gödel implication

$$w_{\rightarrow}(\alpha,\beta) = \begin{cases} 1 & if\ \alpha \leq \beta \\ \beta & otherwise. \end{cases}$$

Truth Functions (4)

In Fuzzy Logic Control, however, the main implication membership functions used are:

Mamdani implication: $w_{\rightarrow}(\alpha, \beta) = \min\{\alpha, \beta\}$

Larsen implication: $w_{\rightarrow}(\alpha, \beta) = \alpha \cdot \beta$

and not the the Łukasiewicz implication and the Gödel implication obtained extending propositional logic.

Truth Functions (4)

- In the framework of fuzzy logic, we should always choose a t-norm as the truth function for conjunction.
- The minimum can be considered as a special t-norm, since it is the only idempotent t-norm which means that only the minimum satisfies the property

$$t\,(\alpha,\alpha) = \alpha \quad \text{for all } \alpha \in [\,0,\ 1]\,.$$

- Only the idempotence of a t-norm can guarantee that the truth values of the proposition φ and $\varphi \wedge \varphi$ coincide, which at first sight seems to be a canonical requirement, letting the minimum seem to be the only reasonable choice for the truth functions for the conjunction in the context of fuzzy logic.
- However, the following example shows that the idempotency property is not always desirable.

Example of conjunction with t-norm

A buyer has to decide between the houses A and B . The houses are very similar in most aspects. So, the buyer makes the decision considering the criteria good price and good location. After careful consideration the following “truth values” are assigned to the decisive aspects:

	statement	truth value $[\![\varphi_i]\!]$
φ_1	The price of house *A* is good.	0.9
φ_2	The location of house *A* is good.	0.6
φ_3	The price of house *B* is good.	0.6
φ_4	The location of house *B* is good.	0.6

Example (cont.)

- He chooses house $x \in \{A,B\}$ for which the proposition

  "The price of house x is good AND The location of house x is good"

yields the greater truth value.

- This means that the buyer will choose house A if

$$[[\varphi 1 \wedge \varphi 2]] > [[\varphi 3 \wedge \varphi 4]],$$

and house B otherwise.

- When we determine the truth value of the conjunction by the minimum, we would obtain the value 0.6 for both of the houses and thus the houses would be regarded as equally good.
- But this is counterintuitive because house A has definitely a better price than house B and the locations are equally good.
- However, when we choose a non-idempotent t-norm, for example, the algebraic product or the Łukasiewicz t-norm, as truth function for the conjunction, we will always favour house A .

Truth Functions (5)

- Equally, in the framework of fuzzy logic, we should always choose a t-conorm as the truth function for disjunction.
- Like the minimum is the only idempotent t-conorm which means that only the minimum satisfies the property

$$s(\alpha,\alpha) = \alpha \quad \text{for all } \alpha \in [0, 1] .$$

- Applying the disjunction in the sense of the maximum only the proposition with the greatest truth value determines the truth value of the disjunction of the propositions.
- We can avoid this disadvantage, if we give up idempotency.

t-norms and Implications

- In addition to the connection between t-norms and t-conform (duality), we can also find connections between t-norms and implications.

- A continuous t-norm t induces the residuated implication $\vec{t}$ by the formula

$$\vec{t}(\alpha,\beta) = \sup\{\gamma \in [0,1] | t(\alpha,\gamma) \leq \beta\}.$$

- Thus, by residuation we obtain
  - the Łukasiewicz implication from the Łukasiewicz t-norm, and
  - the Gödel implication from the minimum t-norm.

Biimplication or Equivalence

- In classical logic the biimplication or equivalence is defined as

$$[[\varphi \leftrightarrow \psi]] = [[(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]]$$

- If $\alpha = [[\varphi]]$ and $\beta = [[\psi]]$ then using the residuated implication $\vec{t}$ we would have that the formula for the biimplication would be

$$\overleftrightarrow{t}(\alpha, \beta) = \vec{t}(\max\{\alpha, \beta\}, \min\{\alpha, \beta\})$$
$$= t\,(\vec{t}(\alpha, \beta), \vec{t}(\beta, \alpha))$$
$$= \min\{\vec{t}(\alpha, \beta), \vec{t}(\beta, \alpha)\}.$$

Quantifiers

- Besides the logical operators like conjunction, disjunction, implication or negation in (fuzzy) logic, there also exist the quantifiers ∀ (all) and ∃ (exists).
- The universal quantifier ∀ and the existential quantifier ∃ are closely related to the conjunction and the disjunction, respectively.
- If the universal set X is finite, e.g. $X = \{x_1, \ldots, x_n\}$, then

$$(\forall x \in X)(P(x)) \text{ is equivalent to } [P(x_1) \wedge \ldots \wedge P(x_n)]$$

The truth value of the statement $(\forall x \in X)(P(x))$ would be:

$$[[(\forall x \in X)(P(x))]] = \inf\{[[P(x)]];\ x \in X\}$$

- Other t-norms than the minimum are normally not used for the universal quantifier, because the non-idempotent property leads easily to the truth value zero in the case of an infinite universe of discourse.

Quantifiers (2)

The same consideration about the existential quantifier

$$(\exists\ x \in X)(P\ (x)) \text{ is equivalent to } [P(x_1) \vee \ldots \vee P(x_n)]$$

leads to

$$[[(\exists\ x \in X)(P\ (x))]] = \sup\{[[P(x)]];\ x \in X\}$$

Example

Consider the predicate P(x) with the interpretation "x is a high velocity". Let the truth value [[P(x)]] be given by the fuzzy set of the high velocities from Figure shown before

$$[[P(x)]] = \mu_{hv}(x)$$

We have for instance

$$[[P(150)]] = 0,\ [[P(170)]] = 0.5 \text{ and } [[P(190)]] = 1.$$

Thus, the statement

$(\forall x \in [170, 200])(P(x))$ = ("All velocities between 170 km/h and 200 km/h are high velocities")

has the truth value

$$[[\forall x \in [170, 200] : P(x)]] = \inf\{[[P(x)]];\ x \in [170, 200]\} = \inf\{\mu_{hv}(x) \mid x \in [170, 200]\} = 0.5.$$

Analogously, we obtain

$$[[\exists x \in [100, 180] : P(x)]] = 0.75.$$

Summary

- In this session about fuzzy logic we have discussed
  - various ways of combining fuzzy propositions.
  - An essential assumption we have used is that of truth functionality.
  - This means that the truth value of the combination of several propositions depends only on the truth values of the propositions, but not on the individual propositions.
- If we understand the conjunction in its classical sense, a conjunctive combination of a proposition with itself should be equivalent to itself, which is not satisfied for nonidempotent t-norms.
- Another possibility is to understand the conjunction as a list of pro and con arguments for a thesis or as a proof.
- In any case, the repeated use of the same (fuzzy) argument within a proof might result in a loss of credibility and thus idempotency is not desirable, even for a conjunction of a proposition with itself.
- Fortunately, for fuzzy control these consideration are of minor importance, because in this application area fuzzy logic is used in a more restricted context, where we do not have to worry about combining the same proposition with itself.