Fuzzy Logic

Operations on Fuzzy Sets

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Definition (Fuzzy Set)

Let X be a universal set defined in a specific problem, with a generic element denoted by x. A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_A(x))|x \in X\}$$

where $\mu_A \colon X \to [0,1]$ is called the membership function of A and $\mu_A(x)$ represents the degree of membership of the element x in A

Interpretation

The nearer $\mu_A(x)$ to unity, the higher the grade of membership of x in A

Corollary

When the range of the membership function reduces to $\{0,1\}$ we get the ordinary or classical set.

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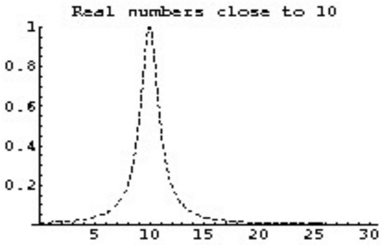
(a) $X = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

$$\begin{aligned} A =& \{(5, 0), (6, 0.2), (7, 0.4), (8, 0.6), (9, 0.8), (10, 1), \\ & (11, 0.8), (12, 0.6), (13, 0.4), (14, 0.2), (15, 0)\} \end{aligned}$$

(b) $X = \mathbb{R}$

$$A = \{(x, \mu_A(x) | x \in \mathbb{R}\}$$

$$\mu_A(x) = \frac{1}{1 + (x - 10)^2}$$



Exercise

Model the following expressions as fuzzy sets:

a) *Real numbers considerably larger than 10*

b) *Large integers*

c) *Very small positive numbers*

A fuzzy set is *empty*

if and only if

its membership function is identically zero on X

Two fuzzy sets A and B are *equal*

if and only if

$$\mu_A(x) = \mu_B(x)\ \forall x \in X$$

A fuzzy set A is a *subset of* the fuzzy set B

if and only if

$$\mu_A(x) \leq \mu_B(x) \ \forall x \in X$$

In symbols:

$$A \subseteq B \Leftrightarrow \mu_A \leq \mu_B$$

- Defined via the membership function
  - It is the crucial component of a fuzzy set
- Obvious extensions of the corresponding definitions for ordinary sets

The intersection of two sets A and B is the largest set $A \cap B$ contained in both A and B

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$

$$\wedge \equiv logical\ AND$$

A	B	$A \cap B$
0	0	0
0	1	0
1	0	0
1	1	1

AND

- What membership function?
- Preservation of the above truth table

Zadeh's proposal:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

- What membership function?
- Preservation of the above truth table

Zadeh's proposal:

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1. Since

$$\min\{\mu_A(x), \mu_B(x)\} \leq \mu_A(x) \quad \text{and} \quad \min\{\mu_A(x), \mu_B(x)\} \leq \mu_B(x)$$

We conclude from the previous definition of containment that a fuzzy set with membership function $\min\{\mu_A(x), \mu_B(x)\}$ is contained in both A and B

2. Let C be a fuzzy set contained in both A and B then

$$\mu_C(x) \leq \mu_A(x) \quad \text{and} \quad \mu_C(x) \leq \mu_B(x),$$

and hence

$$\mu_C(x) \leq \min\{\mu_A(x), \mu_B(x)\},$$

which means that C is a subset of the fuzzy set with membership function

$$\min\{\mu_A(x), \mu_B(x)\}.$$

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The union of two sets A and B is the smallest set $A \cup B$ containing both A and B

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$

$$\vee \equiv logical\ OR$$

A	B	$A \cup B$
0	0	0
0	1	1
1	0	1
1	1	1

OR

Zadeh's proposal:

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

The complement of a set A is the set $\overline{A}$ of elements that do not belong to A

A	$\overline{A}$
0	1
1	0

NOT

Zadeh's proposal:

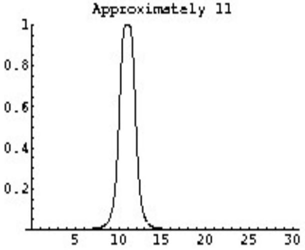
$$\mu_{\overline{A}}(x) = 1 - \mu_A(x) \ \forall x \in X$$

Exercise

Prove that Zadeh's proposal extends the classical definitions of intersection, union and complement

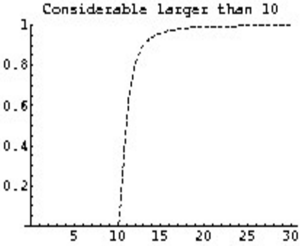
$A =$ "x approximately 11"

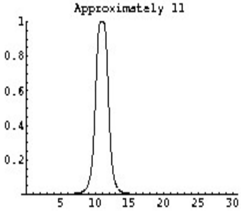
$$\mu_A(x) = \frac{1}{1+(x-11)^4}$$

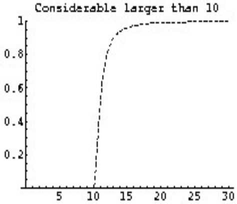


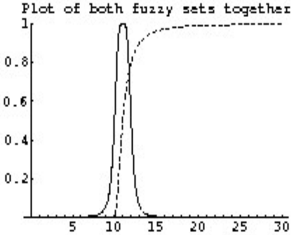
$B =$ "x considerable larger than 10"

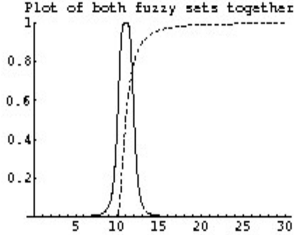
$$\mu_B(x) = \begin{cases} 0 & x \leq 10 \\ 1 - \frac{1}{1+(x-10)^2} & x > 10 \end{cases}$$




Approximately 11
1
0.8
0.6
0.4
0.2
5 10 15 20 25 30


Considerable larger than 10
1
0.8
0.6
0.4
0.2
5 10 15 20 25 30


Plot of both fuzzy sets together
1
0.8
0.6
0.4
0.2
5 10 15 20 25 30


Plot of both fuzzy sets together
1
0.8
0.6
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5
10
15
20
25
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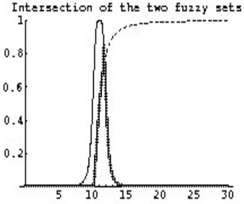
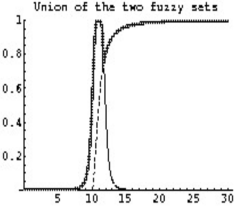

Intersection of the two fuzzy sets
1
0.8
0.6
0.4
0.2
5
10
15
20
25
30


Union of the two fuzzy sets
1
0.8
0.6
0.4
0.2
5
10
15
20
25
30

Exercise

Determine the unions, intersections and complements of the fuzzy sets you modelled before using Zadeh's proposal

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

- Interpreting
  - Intersection as "logical and"
  - Union as "logical or"
  - "The element x belongs to the set A" can be accepted as more or less true
- They proved that under "reasonable" conditions:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

- Zadeh's triplet is not the only possible way to extend classic set theory consistently
- Most fuzzy logic applications make use these operators
- In some sense, they are becoming the "classical" operators for operations on fuzzy sets
- In general:
  - Intersection: Triangular norms (t-norms)
  - Union: Triangular conorms (t-conorms)

$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x) \text{ [product]}$$

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \text{ [probabilistic sum]}$$

$$\mu_{\overline{A}} = 1 - \mu_A(x)$$

are also consistent with the classical set theory

Definition

A triangular norm (briefly t-norm) is a binary operation T on the unit interval [0, 1] which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T\colon [0,1] \times [0,1] \to [0,1]$ such that for all $x, y, z \in [0,1]$:

1. $T(x,1) = x \ \forall x$ (neutral element)
2. $T(x,y) = T(y,x) \ \forall x, y$ (commutativity)
3. $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)
4. $T(x, T(y,z)) = T(T(x,y), z)$ (associativity)

1. $T(x,1) = x \ \forall x$ (neutral element)

   This property imposes the correct generalization to classical sets

2. $T(x,y) = T(y,x) \ \forall x,y$ (commutativity)

   This property indicates that the operator is indifferent to the order of the fuzzy sets to be combined

3. $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)

   This property implies that a decrease in the membership values in A or B cannot produce an increase in the membership value in $A \cap B$

4. $T(x,T(y,z)) = T(T(x,y),z)$ (associativity)

   This property allows to take the intersection of any number of sets by recursively applying a t-norm operator

- The boundary condition and the monotonicity conditions were given in the above definition in their minimal form
- Together with commutativity it follows that, for all $x \in [0,1]$, each t-norm satisfies

$$T(0,x) = T(x,0) = 0,\ T(1,x) = x$$

(all t-norms coincide on boundary of unit square $[0,1]^2$)

$$T(x_1, y_1) \leq T(x_2, y_2) \text{ whenever } x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

(joint monotonicity in both components)

EXERCISE

Prove the truth of the above conditions.

Name of operator	Expression
Minimum	$T_M(x,y) = \min\{x,y\}$
Product	$T_P(x,y) = x \cdot y$
Łukasiewicz t-norm	$T_L(x,y) = \max\{x+y-1,0\}$
Drastic product	$T_D(x,y) = \begin{cases} 0 & (x,y) \in [0,1[^2 \\ \min\{x,y\} & \text{otherwise} \end{cases}$

Definition

If, for two t-norms T_1 and T_2, we have

$$T_1(x, y) \leq T_2(x, y) \ \forall (x, y) \in [0, 1]^2,$$

then we say that T_1 is *weaker* than T_2 or, equivalently, that T_2 is *stronger* than T_1, and we write in this case $T_1 \leq T_2$.

We shall write

$$T_1 < T_2 \text{ if } T_1 \leq T_2 \quad \text{and} \quad T_1 \neq T_2,$$

i.e., if $T_1 \leq T_2$ and if $T1(x_0, y_0) < T_2(x_0, y_0)$ for some $(x_0, y_0) \in [0, 1]^2$.

Result 1

The drastic product T_D is the weakest t-norm, and the minimum T_M is the strongest t-norm

$$T_D(x, y) \leq T \leq T_M(x, y) \;\; \forall\, x, y \in [0, 1]$$

Result 2

Between the four basic t-norms we have these strict inequalities:

$$T_D(x, y) < T_L(x, y) < T_P(x, y) < T_M(x, y) \;\; \forall\, x, y \in [0, 1]$$

DEFINITION

A triangular conorm (t-conorm) is a binary operation S on the unit interval [0, 1] which is commutative, associative, monotone and has 0 as neutral element, i.e., it is a function $S\colon [0,1] \times [0,1] \to [0,1]$ such that for all $x, y, z \in [0,1]$:

1. $S(x,0) = x \ \forall x$ (neutral element)
2. $S(x,y) = S(y,x) \ \forall x, y$ (commutativity)
3. $S(x,y) \leq S(x,z)$ if $y \leq z$ (monotonicity)
4. $S(x, S(y,z)) = S(S(x,y), z)$ (associativity)

Name of operator	Expression
Maximum	$S_M(x,y) = \max\{x,y\}$
Probabilistic sum	$S_P(x,y) = x + y - x \cdot y$
Łukasiewicz t-conorm	$S_L(x,y) = \min\{x+y, 1\}$
Drastic sum	$S_D(x,y) = \begin{cases} 1 & (x,y) \in]0,1]^2 \\ \max\{x,y\} & \text{otherwise} \end{cases}$

- Any t-conorm S can be generated from a t-norm T, and viceversa

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

- When this is the case (T, S) is said to be a dual pair of t-norm and t-conorm

$$(T_M, S_M),\ (T_P, S_P),\ (T_L, S_L),\ (T_D, S_D)$$

are dual pairs of t-norms and t-conorms

Exercise

Prove that $T_H(x,y) = \dfrac{x \cdot y}{x + y - x \cdot y}$ *is a t-norm and give the expression of its dual t-conorm* S_H.

Exercise

Prove that if $(T_1, S_1),\ (T_2, S_2)$ *are two dual pairs of t-norms and t-conorms with* $T_1 \leq T_2$ *then* $S_1 \geq S_2$.

- All the operators mentioned so far include the case of dual logic as special case
- Why are there unique definitions for intersection and union in dual logic and traditional set theory and so many suggested definitions in fuzzy set theory?
  - The answer is simply that many operators perform in exactly the same way if the degrees of membership are restricted to the values 0 and 1
- Are the only ways to "combine" or aggregate fuzzy sets the intersection or the union, or are there other possibilities of aggregation?
  - There are other ways of combining fuzzy sets with "and" and "or" being limiting cases

- Other general operators in the sense that they do not distinguish between the intersection and the union of fuzzy sets
- Aggregation of fuzzy sets are based on aggregation procedures frequently used in utility theory
- They realise the idea of trade-offs between conflicting goals when compensation is allowed
- The resulting trade-offs lie between the most optimistic lower bound and the most pessimistic upper bound: minimum and the maximum degree of membership of the aggregated sets
- They are called averaging operators
- Adequate for human aggregation procedures in decision environments – we will cover these later in the module

- Fuzzy Sets Operations are extensions of Classical Set Operations
- Many possibilities
- Justification ranges from:
  - Intuitive argumentation (Zadeh)
  - Empitical justification (Zimmermann and Zysno)
  - Axiomatic (Bellman and Giertz, Hamacher)
- Most fuzzy logic applications use Zadeh's $\min,\ \max$
- Averaging Operator suitable in decision making context

Additional Readings in Blackboard

D. Dubois, H. Prade
Fuzzy Set and Systems: Theory and Applications
New York: Academic Press, 1980

E.P. Klement, R. Mesiar, E. Pap
Triangular norms. Position paper I: Basic analytical and algebraic properties
Fuzzy Sets and Systems, 143 (1):5 – 26, 2004

L. A. Zadeh
Fuzzy Sets
Information and Control, 8:338 - 353, 1965

R.R. Yager
On Ordered Weighted Averaging Aggregation Operators in Multicriteria Decision Making
IEEE TSMC, 18:183 - 190, 1998