Important Theorem and Principle

Representation Theorem and Extension Principle

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- The Representation Theorem of Fuzzy Sets
- The Extension Principle of Fuzzy Sets

1. The Representation Theorem of Fuzzy Sets
   - Alpha cuts
   - The Representation Theorem
2. The Extension Principle of Fuzzy Sets
3. Application to continuous convex membership functions

- We have already studied the definition of a fuzzy set and the main operations on fuzzy sets.
- Up to know, we have presented concepts and results on fuzzy sets and fuzzy logic based on what is known as the vertical representation of a fuzzy set.
- An alternative representation of a fuzzy set is based on the decomposition of a fuzzy set in crisp sets using a horizontal approach.
    - "The Representation Theorem of a Fuzzy Sets."
    - It is useful in solving mathematical equations/problems that involve fuzzy concepts: these are first decomposed in their collection of crisp sets to allow solving crisp versions of the fuzzy equation/problem of interest; the fuzzy solution is then obtained by reconstructing it with the crisp solutions using the representation theorem of a fuzzy sets.
- Up to know we have only introduced the following operations on fuzzy sets: intersection, union and the complement of fuzzy sets.
- To do operations like "about 2" plus "about 3", we need to apply what is known as
    - "The Extension Principle of Fuzzy Sets."

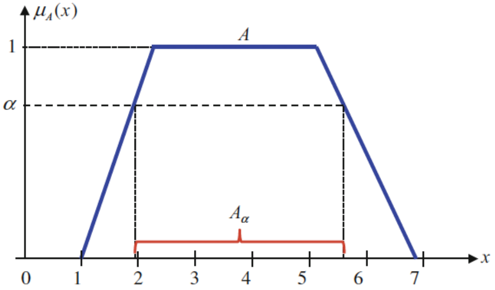
1 The Representation Theorem of Fuzzy Sets

- Alpha cuts
- The Representation Theorem

2 The Extension Principle of Fuzzy Sets

3 Application to continuous convex membership functions

Let A be a fuzzy set with membership function μ_A depicted in blue



- The interval in the x-axis shown in red is called the alpha cut of fuzzy set A: A_α
- Notice that the membership function is normal and convex.
- As we will see next, the definition of alpha cut does not require these properties.
- However, when dealing with fuzzy numbers they are verified.

Definition

The alpha cut of a fuzzy set A with membership function μ_A, denoted A_α or A^α , is the crisp set

$$A_\alpha = \{x|\mu_A(x) \geq \alpha\}$$

where $\alpha \in [0, 1]$. The strong alpha cut, $A_{\alpha+}$ or $A^{\alpha+}$, is defined with strict inequality.

Definition

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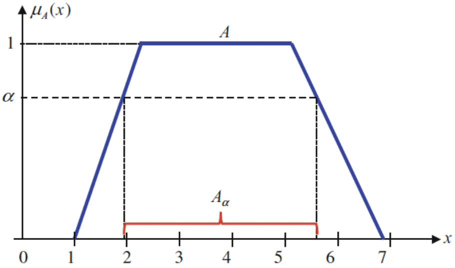


The level set of A

$$L(A) = \{\alpha \in [0,1] | \exists x \in X : \mu_A(x) = \alpha\}$$

is finite and we can list all alpha cuts of the fuzzy set.

EXAMPLE

Let $A = \{(7, 0.3), (8, 0.6), (9, 0.9), (10, 1), (11, 0.9), (12, 0.6), (13, 0.3)\}$ with $X = \mathbb{Z}$.

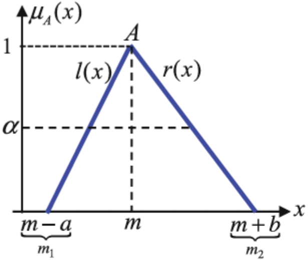
1. Level set: $L(A) = \{0, 0.3, 0.6, 0.9, 1\}$
2. Alpha cuts:

$$\begin{aligned} A_0 &= X \\ A_{0.3} &= \{7, 8, 9, 10, 11, 12, 13\} \\ A_{0.6} &= \{8, 9, 10, 11, 12\} \\ A_{0.9} &= \{9, 10, 11\} \\ A_1 &= \{10\} \end{aligned}$$

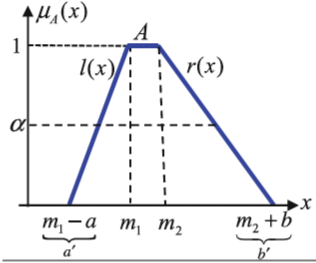
- How to obtain the alpha cut when the membership function is given with our plotting it?
- We are assuming the membership function is continuous
- Not easy unless the membership function is convex.
- Another reason for liking convex membership functions to non-convex ones!
- The process to get the alpha cuts of a (normal) convex fuzzy set is the following:
  - Solve (in x) the equation: $\mu_A(x) = \alpha$.
  - Will result in at most 2 solutions: $a_\alpha \leq b_\alpha$.
  - In general the alpha cut is the closed interval: $A_\alpha = [a_\alpha, b_\alpha]$
  - When only one solution, the alpha cut can be a set with just one element (e.g. triangular membership functions with $\alpha = 1$) or a closed interval with the other end point being the infimum or maximum element of the support of set A (e.g. first/last labels of a linguistic variable with shoulder membership functions).

Obtaining the alpha cuts of non discrete fuzzy sets

It is easy to see how to obtain the alpha cut when the membership function is plotted



$$\begin{aligned} A_\alpha &= [a_\alpha, b_\alpha] \\ &= [m - a(1-\alpha), m + b(1-\alpha)] \\ &= [m_1 + (m - m_1)\alpha, m_2 - (m_2 - m)\alpha] \end{aligned}$$



$$\begin{aligned} A_\alpha &= [a_\alpha, b_\alpha] \\ &= [m_1 - a(1-\alpha), m_2 + b(1-\alpha)] \\ &= [a' + (m_1 - a')\alpha, b' - (b' - m_2)\alpha] \end{aligned}$$

1. If $\alpha \geq \beta$, then $A_\alpha \subseteq A_\beta$
2. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$
3. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$
4. $A \subseteq B \iff A_\alpha \subseteq B_\alpha \quad \forall\alpha$
5. $A = B \iff A_\alpha = B_\alpha \quad \forall\alpha$

- The principal role of alpha cuts cuts in fuzzy set theory is their capability to represent fuzzy sets.
- A fuzzy set can uniquely be represented by the family of all its alpha cuts (the representation theorem).
- This representation allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.
- In each extension, a given classical (crisp) property or operation is required to be valid for each crisp set involved in the representation.
- Such extended properties or operations are called either cutworthy.

Let A be a fuzzy set on X with membership function μ_A:

$$A = \{(x, \mu_A(x)) \,|x \in X\}.$$

Let us denote the characteristic function of the crisp alpha cut set A_α as follows:

$$A_\alpha(x) = \begin{cases} 1 & x \in A_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

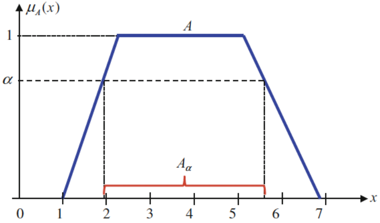
The set

$$\alpha A_\alpha = \{(x, \alpha A_\alpha(x)) \,|x \in X\}$$

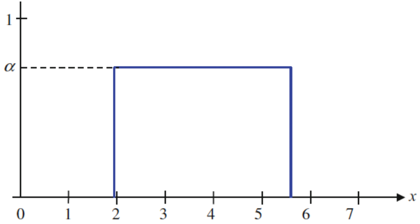
is a fuzzy set on X with membership function

$$\alpha A_\alpha(x) = \begin{cases} \alpha & x \in A_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

This function raises the alpha cut A_α off the x-axis to the height (level) α.



A_α



αA_α

Theorem (Representation Theorem)

Fuzzy set A on X can be represented as

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha.$$

In other words,

$$\forall x \in X : \qquad \mu_A(x) = \sup_{\alpha \in [0,1]} \alpha A_\alpha(x).$$

Proof.

Case $\mu_A(x) = 0$. In the case $x \notin A_\alpha$ for $\alpha > 0$, which means that $\alpha A_\alpha(x) = 0$ for $\alpha > 0$. Since $\alpha A_\alpha(x) = 0$ when $\alpha = 0$, we have that

$$\alpha A_\alpha(x) = 0 \quad \forall \alpha \in [0,1] \implies \sup_{\alpha\in[0,1]} \alpha A_\alpha(x) = \sup_{\alpha\in[0,1]} 0 = 0 = \mu_A(x).$$

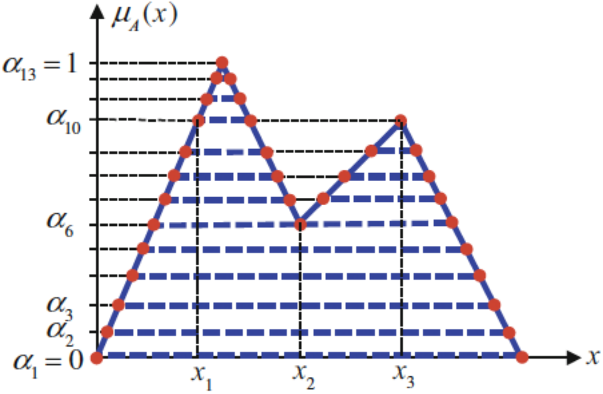
Case $\mu_A(x) > 0$. Denoting $a = \mu_A(x)$, then it is

$$\sup_{\alpha\in[0,1]} \alpha A_\alpha(x) = \max\left\{ \sup_{\alpha\in[0,a]} \alpha A_\alpha(x), \sup_{\alpha\in(a,1]} \alpha A_\alpha(x) \right\}.$$

1. $\alpha \in (a,1] \implies a < \alpha \implies x \notin A_\alpha \implies A_\alpha(x) = 0 \implies \alpha A_\alpha(x) = 0.$
2. $\alpha \in [0,a] \implies a \geq \alpha \implies x \in A_\alpha \implies A_\alpha(x) = 1 \implies \alpha A_\alpha(x) = \alpha.$

$$\sup_{\alpha\in[0,1]} \alpha A_\alpha(x) = \max_{\alpha\in[0,a]} \{\alpha, 0\} = a = \mu_A(x).$$

□


$\mu_A(x)$
$\alpha_{13} = 1$
α_{10}
α_6
α_3
α_2
$\alpha_1 = 0$
x_1
x_2
x_3
x

Theorem (Representation Theorem 2)

Fuzzy set A on X can be represented as

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_{\alpha+}.$$

In other words,

$$\forall x \in X : \qquad \mu_A(x) = \sup_{\alpha \in [0,1]} \alpha A_{\alpha+}(x).$$

- We say that a crisp function $f\colon X \to Y$ is fuzzified when it is extended to act on fuzzy sets defined on X and Y.
- The sets of fuzzy sets on X is usually denoted as $\mathcal{F}(X)$.
- The fuzzified function, for which the same symbol f is usually used, has the form $f\colon \mathcal{F}(X) \to \mathcal{F}(Y)$
- A principle for fuzzifying crisp functions is called the extension principle.
- This is applied to extend for example operation or properties from real numbers to the case of fuzzy real numbers such as 'about 2' $+$ 'about 3', 'small'^{2} absolute value of 'about -0.5', for example.

Example

Let us assume that ‘small’ refers to natural numbers and that is described by the following discrete fuzzy set

$$small = \{(5, 0.1), (4, 0.2), (3, 0.5); (2, 0.9), (1, 1)\}.$$

- We aim to extend the square function applied to natural numbers to the case of this fuzzy set of small natural numbers.
- We first apply the square function to the elements of the support set of ‘small’ to obtain the support set of ‘small’^{2}
$$\{25, 16, 9, 4, 1\}.$$
- 25 is obtained from 5, so its membership degree in ‘small’^{2} will be equal to the membership degree of 5 in ‘small’.
- Similarly, we obtain: $`small'^2 = \{(25, 0.1), (16, 0.2), (9, 0.5), (4, 0.9), (1, 1)\}.$

Exercise

Fuzzify the absolute value function $f(x) = |x|$ to your own discrete version of the fuzzy set 'about -0.5' with universe $[-1.5, 2.5]$.

Example

Let us assume the fuzzy sets ‘about 2’ and ‘about 3’ are both discrete fuzzy sets

$$\begin{aligned} \text{`}\textit{about } 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ \text{`}\textit{about } 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$\begin{aligned} `about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ `about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

- Since both are fuzzy sets, we expect that heir sum 'about 2' + 'about 3' will also be a fuzzy set.
- A fuzzy set is characterised by its membership function, so we need to find the membership function of the set 'about 2' + 'about 3'.
  1. First, we find the support set of the fuzzy set, i.e. the crisp set of elements that will have a positive membership degree in 'about 2' + 'about 3'.
  2. Second, we find their actual degree of membership.

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$\begin{aligned} \text{`}about\ 2' &= \{(0, 0.1), (1; 0.5), (2, 1), (3, 0.5), (4, 0.1)\} \\ \text{`}about\ 3' &= \{(1, 0.1), (2; 0.5), (3, 1), (4, 0.5), (5, 0.1)\} \end{aligned}$$

To find the support set of 'about 2' + 'about 3' we use the support sets of each fuzzy set:

		about 3:				
		1	2	3	4	5
about 2:	0					
	1					
	2					
	3					
	4					

Example

Let us assume the fuzzy sets ‘about 2’ and ‘about 3’ are both discrete fuzzy sets

$$\begin{aligned} \text{‘}about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ \text{‘}about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

Apply the operation on these support elements to get the support set of ‘about 2’ + ‘about 3’:

		about 3:					
		1	2	3	4	5	
about 2:	0	1	2	3	4	5	
	1	2	3	4	5	6	
	2	3	4	5	6	7	about 2 + about 3
	3	4	5	6	7	8	
	4	5	6	7	8	9	

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$\begin{aligned} \text{`}about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ \text{`}about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

To get the membership degree of each element in the support set of 'about 2' + 'about 3', we list all possible combinations that give them:

- $\mu_{\text{'about 2' + 'about 3'}}(1)$ — Membership degree of 1 in 'about 2' + 'about 3'
  - 1 in 'about 2' + 'about 3' is obtained with 0 in 'about 2' **and** 1 in 'about 3';
  - and → conjunction → **minimum** (to a t-norm) applies

$$\mu_{\text{'about 2' + 'about 3'}}(1) = \min\{\mu_{\text{'about 2'}}(0), \mu_{\text{'about 3'}}(1)\} = \min\{0.1, 0.1\} = 0.1$$

EXAMPLE

Let us assume the fuzzy sets ‘about 2’ and ‘about 3’ are both discrete fuzzy sets

$$\begin{aligned} \text{‘}about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ \text{‘}about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

- μ ‘about 2’ + ‘about 3’(2) — Membership degree of 2 in ‘about 2’ + ‘about 3’
  - 2 in ‘about 2’ + ‘about 3’ is obtained with 0 in ‘about 2’ **and** 2 in ‘about 3’ **or** 1 in ‘about 2’ **and** 1 in ‘about 3’
  - and $\rightarrow$ conjunction $\rightarrow$ **minimum** (to a t-norm) applies; or $\rightarrow$ disjunction $\rightarrow$ **maximum** (to a t-conorm) applies;

  μ ‘about 2’ + ‘about 3’$(2) = \max\{\min\{0.1, 0.5\}, \min\{0.5, 0.1\}\} = \max\{0.1, 0.1\} = 0.1$

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$
\begin{aligned}
\text{`}about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\
\text{`}about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\}
\end{aligned}
$$

- $\mu_{\text{'about 2' + 'about 3'}}(3)$ — Membership degree of 3 in 'about 2' + 'about 3'
  - 3 is obtained as 0+3 or 1+2 or 2+1

$$\mu_{\text{'about 2' + 'about 3'}}(3) = \max\{\min\{0.1,1\},\min\{0.5,0.5\},\min\{1,0.1\}\} = 0.5$$

- $\mu_{\text{'about 2' + 'about 3'}}(4)$ — Membership degree of 4 in 'about 2' + 'about 3'
  - 4 is obtained as 0+4 or 1+3 or 2+2 or 3 +1

$$\mu_{\text{'about 2' + 'about 3'}}(4) = \max\{\min\{0.1,0.5\},\min\{0.5,1\},\min\{1,0.5\},\min\{0.5,0.1\}\} = 0.5$$

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$
\begin{aligned}
`about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\
`about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\}
\end{aligned}
$$

- $\mu_{\text{'about 2' + 'about 3'}}(5)$ — Membership degree of 5 in 'about 2' + 'about 3'
  - 5 is obtained as 0+5 or 1+4 or 2+3 or 3 +2 or 4+1

$$
\begin{aligned}
\mu_{\text{'about 2' + 'about 3'}}(5) &= \max\{\min\{0.1,0.1\},\min\{0.5,0.5\},\min\{1,1\},\min\{0.5,0.5\},\min\{0.1,0.1\}\} \\
&= 1
\end{aligned}
$$

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$\begin{aligned} \text{`}about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ \text{`}about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

- $\mu_{\text{'about 2' + 'about 3'}}(6)$ — Membership degree of 6 in 'about 2' + 'about 3'
  - 6 is obtained as 1+5 or 2+4 or 3+3 or 4 +2

$$\mu_{\text{'about 2' + 'about 3'}}(6) = \max\{\min\{0.5,0.1\},\min\{1,0.5\},\min\{0.5,1\},\min\{0.1,0.5\}\} = 0.5$$

- $\mu_{\text{'about 2' + 'about 3'}}(7)$ — Membership degree of 7 in 'about 2' + 'about 3'
  - 7 is obtained as 2+5 or 3+4 or 4+3

$$\mu_{\text{'about 2' + 'about 3'}}(7) = \max\{\min\{1,0.1\},\min\{0.5,0.5\},\min\{0.1,1\}\} = 0.5$$

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$\begin{aligned} `about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ `about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

- μ 'about 2' + 'about 3'(8) — Membership degree of 8 in 'about 2' + 'about 3'
  - 8 is obtained as 3+5 or 4+4 o

  μ 'about 2' + 'about 3'$(8) = \max\{\min\{0.5, 0.1\}, \min\{0.1, 0.5\}\} = 0.1$

- μ 'about 2' + 'about 3'(9) — Membership degree of 9 in 'about 2' + 'about 3'
  - 9 is obtained as 4+5

  μ 'about 2' + 'about 3'$(9) = \min\{0.1, 0.1\} = 0.1$

Example

Let us assume the fuzzy sets 'about 2' and 'about 3' are both discrete fuzzy sets

$$\begin{aligned} `about\ 2' &= \{(0, 0.1), (1; 0.5), (2, 1), (3, 0.5), (4, 0.1)\} \\ `about\ 3' &= \{(1, 0.1), (2; 0.5), (3, 1), (4, 0.5), (5, 0.1)\} \end{aligned}$$

$$`about\ 2' + `about\ 3' = \{(1, 0.1), (2, 0.1), (3, 0.5), (4, 0.5), (5, 1), (6, 0.5), (7, 0.5), (8, 0.1), (9, 0.1)\}$$

Example

Let us assume the fuzzy sets ‘about 2’ and ‘about 3’ are both discrete fuzzy sets

$$\begin{aligned} `about\ 2' &= \{(0,0.1),(1;0.5),(2,1),(3,0.5),(4,0.1)\} \\ `about\ 3' &= \{(1,0.1),(2;0.5),(3,1),(4,0.5),(5,0.1)\} \end{aligned}$$

Note

We will come back to this example when we have seen the continuous case and notice that if the discrete fuzzy sets ‘about 2’ and ‘about 3’ are obtained from their continuous convex fuzzy representations, then this application of the extension principle is not the correct one since the resultant ‘about 5’ membership function is not of the same convex type and it is not possible to reconstruct its continuous convex membership function.

DEFINITION

Let X be a universal set and $f\colon X \to Y$ be a function that maps the element $x \in X$ to element $y = f(x)$ of the universal set Y. Let A be a fuzzy set on X with membership function μ_A. Let us denote

$$f^{-1}(y) = \{x \in X | y = f(x)\}.$$

Then, $B = f(A)$ is the fuzzy set on Y with membership function

$$\mu_B(y) = \begin{cases} \sup\limits_{y=f(x)} \mu_A(x) & \text{if } f^{-1}(y)) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Definition

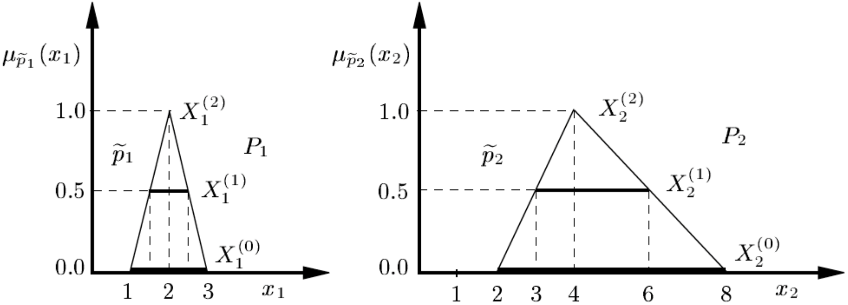
Let $X_1 \times X_2 \times \cdots \times X_n$ be a universal product set and $f\colon X_1 \times X_2 \times \cdots \times X_n \to Y$ be a function that maps elements of $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$ to element $y = f(x_1, x_2, \cdots, x_n)$ of the universal set Y. Let A_i be a fuzzy set on X_i with membership function μ_{A_i} $(i = 1, 2, \cdots, n)$. Let us denote

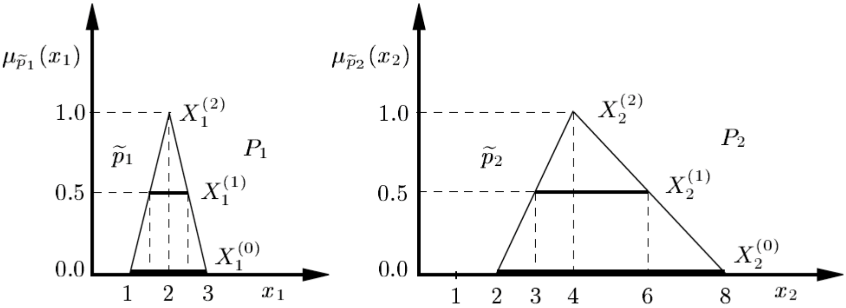
$$f^{-1}(y) = \{(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n | y = f(x_1, x_2, \cdots, x_n)\}.$$

Then, $B = f(A_1, A_2, \cdots, A_n)$ is the fuzzy set on Y with membership function

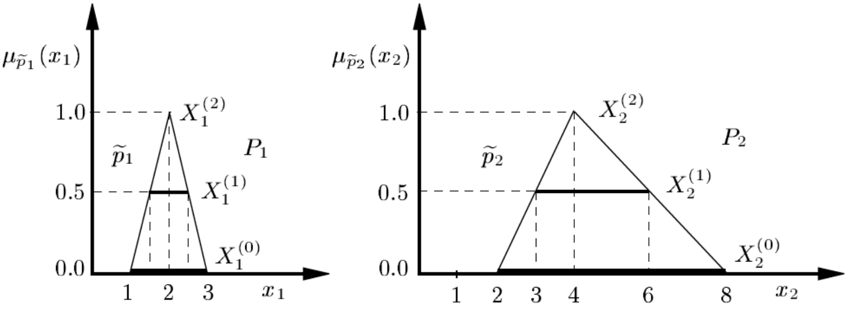
$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1, x_2, \cdots, x_n) \in f^{-1}(y)} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)\} & \text{if } f^{-1}(y)) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Since there are infinite combinations of product of two support elements from each fuzzy set that results in the same support element of the product fuzzy set, the process most appropriate here is to use the representation theorem using their alpha cuts.





Here only 3 alpha values are depicted: $\alpha_0 = 0; \alpha_1 = 0.5; \alpha_2 = 1$ the raised alpha cuts off the x-axis are represented as $X_i^{(j)}$ with $i \in \{1, 2\}$ used to represent each of the fuzzy sets $\tilde{p}_1$ and $\tilde{p}_2$ and $j \in \{0, 1, 2\}$ to represent each of the three alpha levels (for $\alpha = 0$, the strong cuts are represented.)

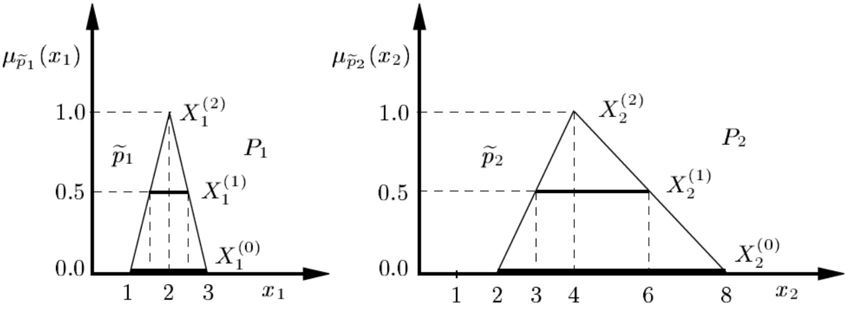


The membership function of fuzzy number $\tilde{p}_1$ and $\tilde{p}_1$ are:

$$\mu_{\tilde{p}_1} = \begin{cases} x-1 & \text{if } 1 < x \leq 2 \\ 3-x & \text{if } 2 < x \leq 3 \\ 0 & \text{otherwise.} \end{cases}; \qquad \mu_{\tilde{p}_2} = \begin{cases} \dfrac{x-2}{2} & \text{if } 2 < x \leq 4 \\ \dfrac{8-x}{4} & \text{if } 4 < x \leq 8 \\ 0 & \text{otherwise.} \end{cases}$$

Representation Theorem is needed

Suppose we want to multiply two fuzzy numbers $\tilde{p}_1$ and $\tilde{p}_2$



$\tilde{p}_1 : P_1 = \{(1,0),(1.5,0.5),(2,1),(2.5,0.5),(3,0)\}$
$X_1^{(0)} = [1,3]$
$X_1^{(1)} = [1.5,2.5]$
$X_1^{(3)} = [2,2]$

$\tilde{p}_2 : P_2 = \{(2,0),(3,0.5),(4,1),(6,0.5),(8,0)\}$
$X_2^{(0)} = [2,8]$
$X_2^{(1)} = [3,6]$
$X_2^{(3)} = [4,4]$

Theorem

Let $f: X \to Y$ be an arbitrary crisp function. Then, for any $A \in \mathcal{F}(X)$ and all $\alpha \in [0,1]$ the following property of f fuzzified by the extension principle holds:

$$f(A)_{\alpha+} = f(A_{\alpha+}).$$

Proof.

For all $y \in Y$,

$$\begin{aligned}
Y \in f(A)_{\alpha+} &\iff \mu_{f(A)}(y) > \alpha \\
&\iff \sup_{x|y=f(x)} \mu_A(x) > \alpha \\
&\iff \exists x_0 \in X: \quad y = f(x_0) \wedge \mu_A(x_0) > \alpha \\
&\iff \exists x_0 \in X: \quad y = f(x_0) \wedge x_0 \in A_{\alpha+} \\
&\iff y \in f(A_{\alpha+}).
\end{aligned}$$

□

This property together with the representation theorem(s) imply that

- $f(A) = \bigcup_{\alpha \in [0,1]} \alpha f(A)_{\alpha+} = \bigcup_{\alpha \in [0,1]} \alpha f(A)_{\alpha}$.
- Recall the example 'about 2" + 'about 3'!
- To apply a crisp function to a fuzzy set, we apply it to the (strong) alpha cuts to get the (strong) alpha cuts of the resultant fuzzy set, which can be reconstructed using the representation theorem.
  - This property tells us that when working with discretised versions of continuous fuzzy sets, the fuzzified function by the extension principle must be applied to the strict alpha cuts.
  - If the membership functions are convex like fuzzy numbers, then the strict alpha cuts are open intervals, where the algebraic operation we are intended to fuzzify is to be applied.
- In practice, we apply the algebraic operation to the alpha cuts (closed intervals) and the resultant will be (if the function is continuous) a closed intervals (alpha cuts), which can be used to reconstruct the continuous fuzzy number output.

In the previous example:

$$\tilde{p}_1 : P_1 = \{(1,0),(1.5,0.5),(2,1),(2.5,0.5),(3,0)\} \quad \tilde{p}_2 : P_2 = \{(2,0),(3,0.5),(4,1),(6,0.5),(8,0)\}$$

$$X_1^{(0)} = [1,3] \qquad X_2^{(0)} = [2,8]$$

$$X_1^{(1)} = [1.5,2.5] \qquad X_2^{(1)} = [3,6]$$

$$X_1^{(3)} = [2,2] \qquad X_2^{(3)} = [4,4]$$

The three alpha cuts for the fuzzy output $\tilde{q} = \tilde{p}_1 \cdot \tilde{p}_2$ are:

$$\tilde{q}_1 : \tilde{Q}^* = \{(2,0),(4.5,0.5),(8,1),(15,0.5),(24,0)\}$$

$$Z_1^{(0)} = [2,24]$$

$$Z_1^{(1)} = [4.5,15]$$

$$Z_1^{(3)} = [8,8]$$

Applying the representation theorem we reconstruct the fuzzy output (with more alpha cuts).