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The Stratic Defuzzifier for discretised general type-2 fuzzy sets

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ABSTRACT

Stratification is a feature of the type-reduced set of the general type-2 fuzzy set, from which a new technique for general type-2 defuzzification, Stratic Defuzzification, may be derived. Existing defuzzification strategies are summarised. The stratified structure is described, after which the Stratic Defuzzifier is presented and contrasted experimentally for accuracy and efficiency with both the Exhaustive Method of Defuzzification (to benchmark accuracy) and the α-Planes/Karnik–Mendel Iterative Procedure strategy, employing 5, 11, 21, 51 and 101 α-planes. The Stratic Defuzzifier is shown to be much faster than the Exhaustive Defuzzifier. In fact the Stratic Defuzzifier and the α-Planes/Karnik–Mendel Iterative Procedure Method are comparably speedy; the speed of execution correlates with the number of planes participating in the defuzzification process. The accuracy of the Stratic Defuzzifier is shown to be excellent. It is demonstrated to be more accurate than the α-Planes/Karnik–Mendel Iterative Procedure Method in four of six test cases, regardless of the number of α-planes employed. In one test case, it is less accurate than the α-Planes/Karnik–Mendel Iterative Procedure Method, regardless of the number of α-planes employed. In the remaining test case, the α-Planes/Karnik–Mendel Iterative Procedure Method with 11 α-Planes gives the most accurate result, with the Stratic Defuzzifier coming second.

1. Introduction

A concept initially proposed in Zadeh's pioneering 1975 paper [48], Type-2 Fuzzy Sets (T2FSs) extend Type-1 Fuzzy Sets (T1FSs) in such a way that their membership grades are themselves T1FSs as opposed to crisp numbers. T2FSs are particularly suited to inferencing under conditions of high uncertainty [35,23,15]. They exist in two forms, the interval, whose secondary membership grades are uniformly 1, and the general, with secondary membership grades in [0, 1]. Interval type-2 Fuzzy Inferencing Systems (FISs) are computationally simpler than their general counterparts [35]; for them varied applications have been developed [11,5,38,1,7,8,39,42,3,9,46]. As yet, owing to its colossal computational complexity, relatively few general type-2 fuzzy logic applications have been developed [30,12,29,6,13,40,4,41,10], though this number is growing. In recent years algorithms have been devised that dramatically reduce the general type-2 FIS's computational complexity [21,25,31,49,20].

In this paper we present an innovative computational complexity reducing algorithm based on *stratification*, an intrinsic feature of the type-reduced set of a general T2FS. This new technique resembles Liu's α-Planes Method [31] in so far as the

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general type-2 fuzzy set is decomposed into horizontal planes; the difference resides in the number of planes involved and their positions on the z-axis. For the established α-Planes Method these are parameters chosen by the developer, whereas with Stratic Defuzzification they are a naturally occurring by-product of stratification. Moreover, it is more closely aligned with the data than is the α-Planes Method; for this reason, it is reasonable to expect that it gives more accurate results. Thus the two motivations for presenting this new technique are its intuitive appeal and the expectation of its engendering highly accurate results. This strategy may be applied in a wide variety of contexts, but it is particularly suited to control applications for which accuracy and speed are essential.

The Mamdani Type-2 FIS (Fig. 1) takes crisp numerical inputs and processes them through three stages: fuzzification, inferencing, and lastly defuzzification, which is the focus of this paper. Via defuzzification, the *aggregated set*, a T2FS generated by the second stage of inferencing, is transformed into a single crisp value. Defuzzification of a discretised T1FS is a straightforward, one-stage operation; for a discretised T2FS defuzzification is more complex, comprising two stages [34]:

1. Type-reduction, transforming the T2FS into a T1FS termed the *Type-Reduced Set (TRS)*;
2. Defuzzification of the TRS.

Hence, as a by-product of defuzzification, the TRS is an intermediary linking the originating T2FS to the defuzzified output.

The next section sets out the definitions and assumptions pertaining to the remainder of the paper. Section 3 concerns type-2 defuzzification techniques, and in Section 4 the Stratic Defuzzifier is presented. Experiments evaluating this new algorithm are reported on in Section 5. The paper concludes with Section 6.

2. Preliminaries

A subset A of a larger set X, normally referred to as the universe of discourse, can be defined using the characteristic or indicator function that takes the value 1 (full membership) on A and 0 elsewhere. A subset A is a collection of elements with a common property, and it is the fulfilment of this property that determines whether an element of X is a member of the subset A (indicator function output 1) or not (indicator function output 0). In this paper, it is assumed that the universe of discourse is the continuous closed unit interval, i.e. $X \equiv U = [0, 1]$. Partial membership of elements of X in a subset A is the characteristic of the 'fuzzy' subset concept. For simplicity, it is habitual to write 'set A on X' rather than 'subset A on X'; this convention is adopted in this paper. Depending on the precision to which membership degrees are presented, there are two main sorts of fuzzy sets in the literature:

Type-1 fuzzy set (T1FS).
These are characterised by precise numeric membership functions $\mu_A : X \to U$, i.e. given an element $x \in X$ its membership grade of $A, \mu_A(x)$, can be precisely quantified by a number in U. The concept of membership function is more general than the concept of indicator function, which means that a subset on X is a special kind of T1FS. Formally, a T1FS A on X is defined as follows [47]:

$$A = \left\{ \left(x, \mu_A(x)\right) \mid \mu_A(x) \in U \quad \forall x \in X \right\}. \tag{1}$$

Type-2 fuzzy set (T2FS).
When (at least one of) the values $\mu_{\tilde{A}}(x)$ cannot be precisely measured with one number in U but a set on U or, in general, a type-1 fuzzy set on U is needed, we say that the set $\tilde{A}$ on X is a T2FS. Formally, a T2FS $\tilde{A}$ on X is defined as follows [2,37]

$$\tilde{A} = \left\{ \left(x, u, \mu_{A^\sim}(x, u)\right) \mid \quad x \in X; \quad u \in U; \ \mu_{A^\sim}(x, u) \in U \right\}, \tag{2}$$

where $\mu_{\tilde{A}} : X \times U \to U$ is the membership function of $\tilde{A}$.

The set $J_x = \left\{ (x, u) | \mu_{\tilde{A}}(x, u) > 0 \right\}$ is known as the primary membership of x, while the values $\mu_{\tilde{A}}(x, u)$ $(\forall u)$ are known as the secondary membership grades of x. When secondary membership grades are all identical and constant on their respective primary memberships then the T2FS is called an *interval* T2FS (IT2FS). In particular, a normal IT2FS's secondary membership grades are all identically 1 and it is defined as follows:

$$\tilde{A} = \{(x, u, 1)) \mid x \in X; \ u \in U\}. \tag{3}$$

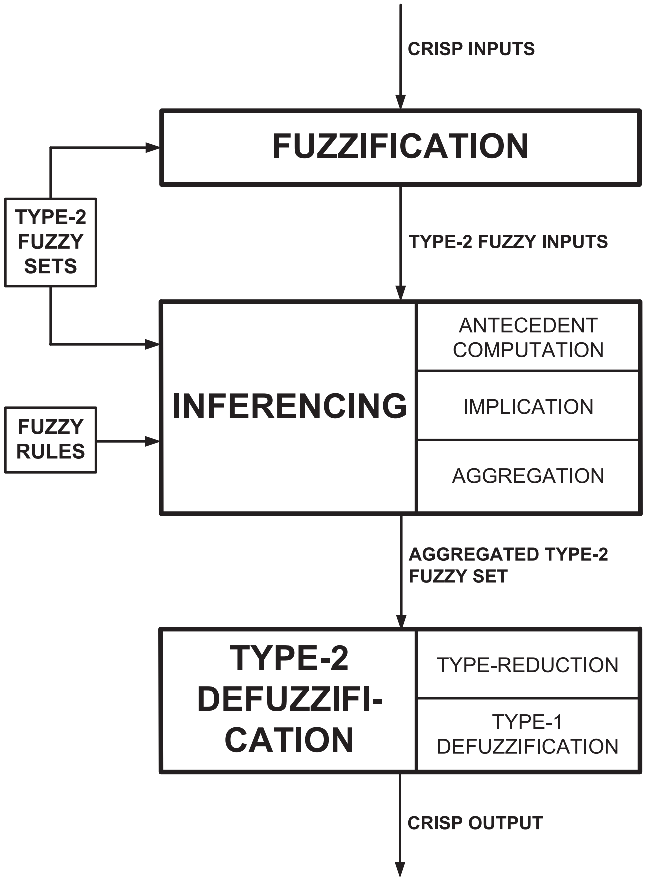


Fig. 1. The Mamdani Type-2 FIS [18].

Because $X \equiv [0,1]$, a T2FS is a surface within the unit cube represented by (x,u,z) co-ordinates such that $x \in U,\ u \in J_x \subseteq U,$ and $z = \mu_{\widetilde{A}}(x,u) \in U$. The intersection of a vertical plane through the x-axis parallel to the $u - z$ plane or a horizontal plane through the z-axis, parallel to the $x - u$ plane of a T2FS surface produces a vertical or a horizontal *slice*, respectively. In mathematical notation, the T2FS's $\widetilde{A}$ vertical slice at x will be the set $\left\{\left(x,u,\mu_{\widetilde{A}}(x,u)\right)|\ u \in U; \mu_{\widetilde{A}}(x,u) \in U\right\}$, while its horizontal slice at z will be the interval IT2FS $\tilde{A}_z$ with membership function

$$\mu_{\tilde{A}_z}(x,u) = \begin{cases} 0, & \mu_{\widetilde{A}}(x,u) < z \\ z, & \mu_{\widetilde{A}}(x,u) \geqslant z \end{cases}. \tag{4}$$

Both the primary and secondary domains of a T2FSs need to be discretised for computer processing. The degree of discretisation is the separation of the corresponding slices (vertical or horizontal) of the T2FS.

Throughout this paper, we are assuming the following [20]:

1. Secondary membership functions are convex, i.e. $\forall x \in X$

$$\mu_{\widetilde{A}}(x, \lambda u_1 + (1-\lambda)u_2) \geqslant \lambda\mu_{\widetilde{A}}(x,u_1) + (1-\lambda)\mu_{\widetilde{A}}(x,u_2) \quad \forall (x,u_1), (x,u_2) \in J_x; \quad \lambda \in [0,1]. \tag{5}$$

Thus, when the first variable (x) is fixed, $\mu_{\widetilde{A}}(x,u)$ is continuous on the second variable (u) and $J_x = \left[\underline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{A}}(x)\right] \subseteq U$, where $\underline{\mu}_{\widetilde{A}}(x) = \inf\{u|\ (x,u) \in J_x\}$ and $\overline{\mu}_{\widetilde{A}}(x) = \sup\{u|\ (x,u) \in J_x\}$are respectively called the lower and upper membership functions of T2FS $\widetilde{A}$. The closed region whose boundaries are the lower and the upper membership functions of $\widetilde{A}$ is known as its Footprint Of Uncertainty (FOU).

2. The *grid method of discretisation* for general T2FSs [14,25] is employed. In this straightforward procedure the $x-u$ plane is divided evenly into a rectangular grid in accordance with the degrees of discretisation of the x and u-axes (Fig. 2). The surface of the fuzzy set, consisting of the secondary membership grades corresponding to each grid point (x,u) of the FOU, may be represented by a matrix of the secondary grades, for which the x and u co-ordinates are implied by the secondary grade's position within the matrix.
3. The centroid method of defuzzification for T1FSs is used [28, Page 336]: Let A be a non-empty type-1 fuzzy set that has been discretised into n vertical slices (at $x_1, x_2, \ldots, x_n$). X_A, the centroid of A, is calculated by this formula:

$$X_A = \frac{\sum_{i=1}^{i=n}\mu_A(x_i)x_i}{\sum_{i=1}^{i=n}\mu_A(x_i)} = \frac{\sum_{i=1}^{i=n}u_i x_i}{\sum_{i=1}^{i=n}u_i}. \tag{6}$$

4. The minimum t-norm is employed.

3. Type-reduction of the T2FS

This section is concerned with defuzzification of the T2FS via type-reduction. Existing strategies for the general T2FS are presented.

3.1. The Wavy-Slice Representation Theorem

An *embedded T2FS (embedded set)* or *wavy-slice* [35] (Fig. 3) is a special kind of T2FS. The process to construct an embedded set of a given T2FS is: For every primary domain value only one of the possible secondary domain values is selected plus the corresponding associated secondary membership grade [19]. Formally, given a T2FS $\widetilde{A}$ with membership function $\mu_{\widetilde{A}} : X \times U \to U$, for discrete sets $X_d = \{x_1, x_2, \ldots, x_N\} \subseteq X$ and $U_d = \{u_1, u_2, \ldots, u_M\} \subseteq U$, an embedded T2FS $\widetilde{A}_e$ of $\widetilde{A}$ is defined as the following T2FS

$$\widetilde{A}_e = \left\{ (x_i, u_{x_i}, \mu_{A^\sim}(x_i, u_{x_i})) |\ u_{x_i} \in J_{x_i} \subseteq U_d \quad \forall i = 1, \ldots, N \right\}. \tag{7}$$

In 2002 Mendel and John proved that a T2FS can be defined as the union of its embedded T2FSs [35], i.e. $\widetilde{A} = \bigcup_{j=1}^{n}\widetilde{A}_e^j$,where $\widetilde{A}_e^j = \left\{ \left(x_i, u_i^j, \mu_{\widetilde{A}}\left(x_i, u_i^j\right)\right) | u_i^j \in J_{x_i}\ \forall i = 1, \ldots, N \right\}$ and $n = \prod_{i=1}^{N} \# J_{x_i}$. This result is known as the Wavy-Slice Representation Theorem.

3.2. Exhaustive Defuzzification

Type-2 defuzzification techniques are derived from and incorporate type-1 defuzzification strategies. The algorithm based on the Wavy-Slice Representation Theorem, known as *Exhaustive Defuzzification*, processes every embedded set in turn and, consequently, it is absolutely precise[1] [35]. Unfortunately, its enormous computational complexity means that it is a highly inefficient method. The first stage, which overwhelmingly bears the brunt of the computational cost, is the type-reduction of the T2FS into its TRS [17,19]:

$$C_{A^\sim} = \left\{ \left(\frac{\sum_{i=1}^{N} x_i \cdot u_{k_i}}{\sum_{i=1}^{N} u_{k_i}}, \mu_{A^\sim}(x_1, u_{k_1}) * \ldots * \mu_{A^\sim}(x_N, u_{k_N}) \right) | \forall (u_{k_1}, u_{k_2}, \ldots, u_{k_N}) \in J_{x_1} \times J_{x_2} \times \ldots \times J_{x_N} \right\}, \tag{8}$$

where $*$ is a t-norm.

3.3. Efficient alternatives to exhaustive defuzzification

Since the early years of the millennium a number of techniques have been devised that drastically reduce the computational cost of general type-2 defuzzification [33,25,49,31,20]. Greenfield and Chiclana, in [17], describe the evaluation of three algorithms relative to Exhaustive Defuzzification, the Sampling Defuzzifier, Vertical Slice Centroid Type-Reduction, and the α-Planes Method. The latest addition to the options available for type-2 defuzzification is the General Greenfield–Chiclana Collapsing Defuzzifier [20].

3.3.1. The Sampling Defuzzifier

This technique [25] is a highly efficient, cut-down version of the Exhaustive Defuzzifier. The principle is to reduce the computational cost of type-reduction by randomly selecting and processing a sample of embedded T2FSs (SubSection 3.1).

[1] Discretisation unavoidably brings with it an element of approximation. However the Exhaustive Method does not introduce further inaccuracies.

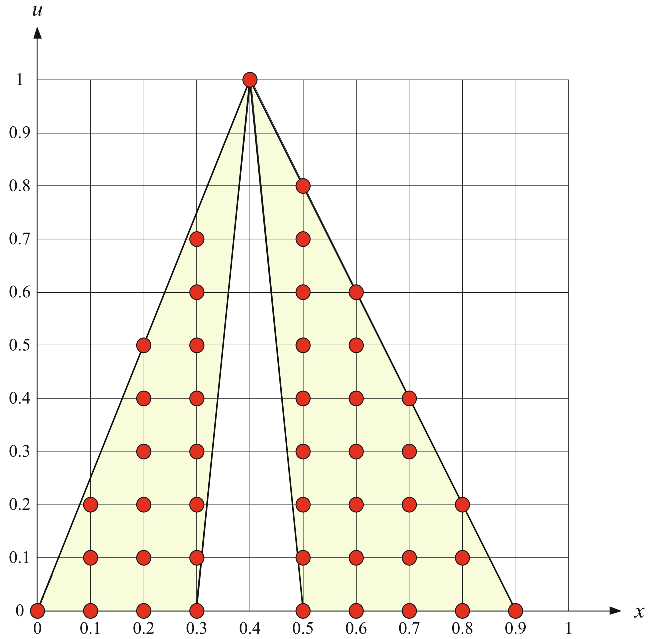


Fig. 2. The Grid Method of Discretisation, in which each red spot indicates a point of intersection that falls within the FOU.

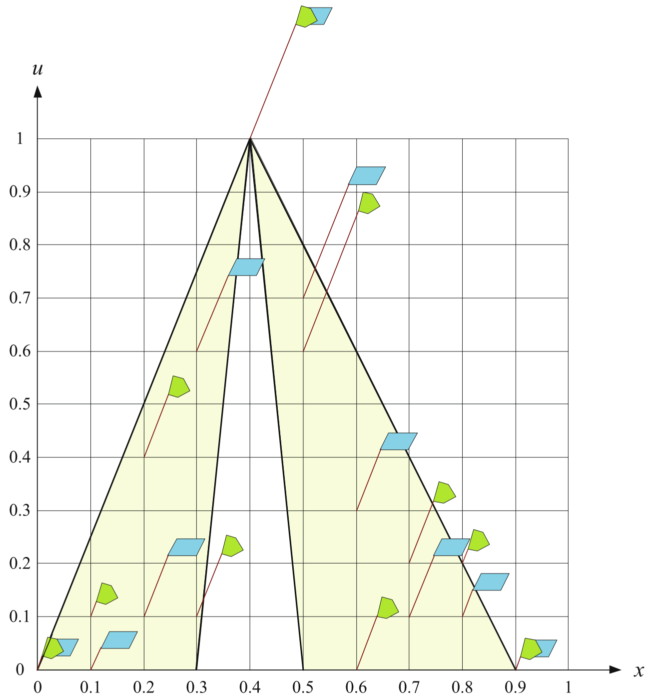


Fig. 3. Two embedded T2FSs, represented by different flag styles. The height of the flag indicates the secondary membership grade. For both the primary and secondary domains the degree of discretisation is 0.1. The shaded region indicates the FOU.

A full description of this strategy may be found in [25]. This approach is non-deterministic; the sample size is the choice of the FIS developer.

3.3.2. Vertical Slice Centroid Type-Reduction

Proposed by Lucas et al. in 2007 [33], this approach is extremely straightforward and intuitive. This technique works by cutting the T2FS into vertical slices. As each slice is a T1FS, it is easily defuzzified. The slice's domain value is paired with its defuzzified value, so constructing a T1FS which is readily defuzzified, giving the defuzzified value of the originating T2FS.

3.3.3. The α-Plane Representation

In 2008 Liu [31,36] propounded the α-Planes Representation[2], originally conceived as a mechanism to generalise the *Karnik–Mendel Iterative Procedure*. This technique works by decomposing the general T2FS into a collection of evenly spaced α-planes. These are horizontal slices similar to interval T2FSs. Through repeated invocation of an interval defuzzification technique such as the KMIP, the general T2FS is type-reduced. The recent increase in the number of applications employing general type-2 fuzzy logic [12,29,6,13,40,41] may be largely attributed to the α-Planes Method.

3.3.4. The General Greenfield–Chiclana Collapsing Defuzzifier

A recent addition to the catalogue of type-2 defuzzification options, the General Greenfield–Chiclana Collapsing Defuzzifier (GGCCD) converts a general T2FS into a T1FS that is an approximation to the *Representative Embedded Set (RES)*, for which the defuzzified value is defined to equate to that of the originating T2FS. This T1FS set is termed the *Generalised Representative Embedded Set Approximation (GRESA)*; as a T1FS the GRESA is easily defuzzified. Consequently the collapsing algorithm diminishes the computational cost of general type-2 defuzzification. The GGCCD is an iterative approach, full details of which may be found at [20].

3.4. Interval defuzzification via the Karnik–Mendel Iterative Procedure

The *Karnik–Mendel Iterative Procedure (KMIP)* [26] is a widely adopted approach to type-reduction for an interval T2FS. Type-reduction via the KMIP results in an interval which may be regarded as a T1FS. The KMIP is a highly efficient method for finding the interval endpoints. The defuzzified value is deemed to be situated at the interval's midpoint, and is an approximation, as in general the distribution of the TRS tuples over the interval is not symmetrical.

Since 2001, when the KMIP was first published, various enhancements have been proposed [44,32] that produce the same output whilst employing differing search strategies. In 2011 the optimum version of the KMIP in terms of efficiency was shown to be the *Enhanced Iterative Algorithm with Stop Condition (EIASC)* Section III [45]. Since EIASC is a highly efficient version of the KMIP, it is employed in the experiments described in Section 5.

4. Stratic Defuzzification

In the previous section existing options for the defuzzification of the type-2 fuzzy were discussed. Now we turn to the theme of this paper, the Stratic Defuzzifier.

4.1. The stratified structure of the TRS

The Stratic Method of Defuzzification exploits an inherent characteristic of the TRS, namely *stratification*. The stratified structure of the TRS was first described by Greenfield and John in 2008 [22]. The TRS membership function of a discretised general T2FS is essentially a set of tuples. Figs. 4–6 show typical TRSs obtained from randomly generated embedded set samples of sizes 50, 500, and 5000, drawn from the same T2FS. The tuples are represented by dots which visibly align themselves into layers or *strata*. That these strata manifest themselves is attributable to the fact that type-reduction involves the selection of the minimum secondary membership grade for each embedded set. Repeatedly the same minima reoccur, each time associated with a different domain value. In this way a stratum is formed (see Fig. 7).

A stratum may be formally defined thus:

Definition 1 (*Stratum [22]*). "Let T be the TRS of a discretised general type-2 fuzzy set. A *stratum*, S_ω, is a subset[3] of T for which every element has the same membership grade.

$$S_\omega = \{(x, \mu_T(x)) \in T | \mu_T(x) = \omega, \text{ for some } \omega \in [0,1]\}." \tag{9}$$

4.2. The TRS Shell

The Stratic Defuzzifier is an innovation based on the novel concept of the *TRS Shell*.

Definition 2 (*TRS Shell (TRSS)*). The T1FS

[2] Independently of Liu, Wagner and Hagras proposed the notion of *zSlices*, which are equivalent conceptually to α-planes [43].

[3] Here the widely understood *crisp* subset is intended. In [27, Page 19] the *fuzzy* definition of subset is given.

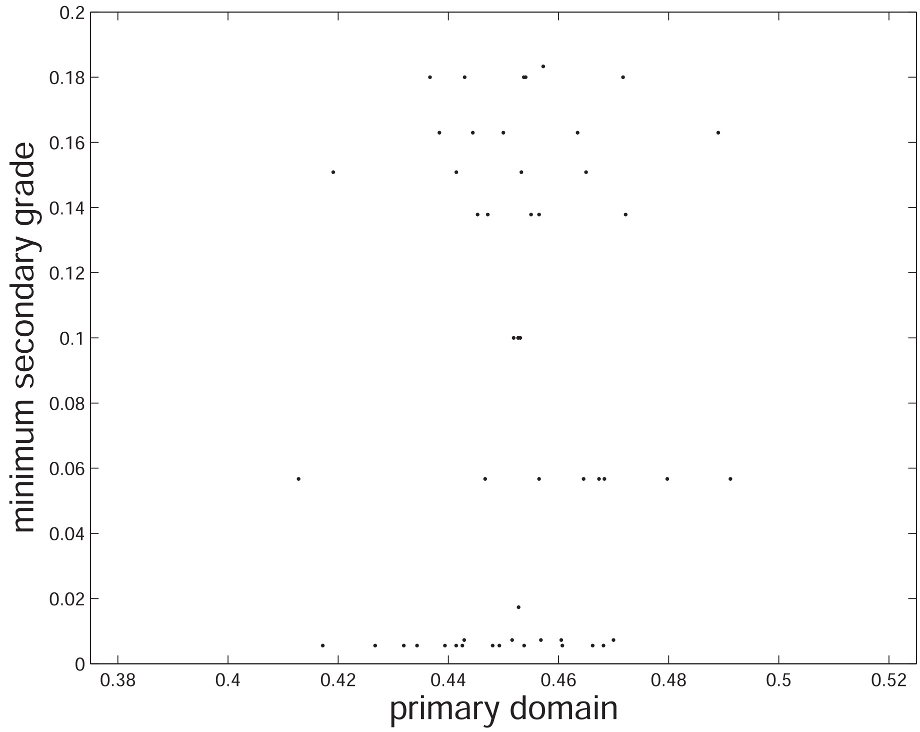


Fig. 4. Strata formed by sampling 50 TRS tuples from a discretised general T2FS.

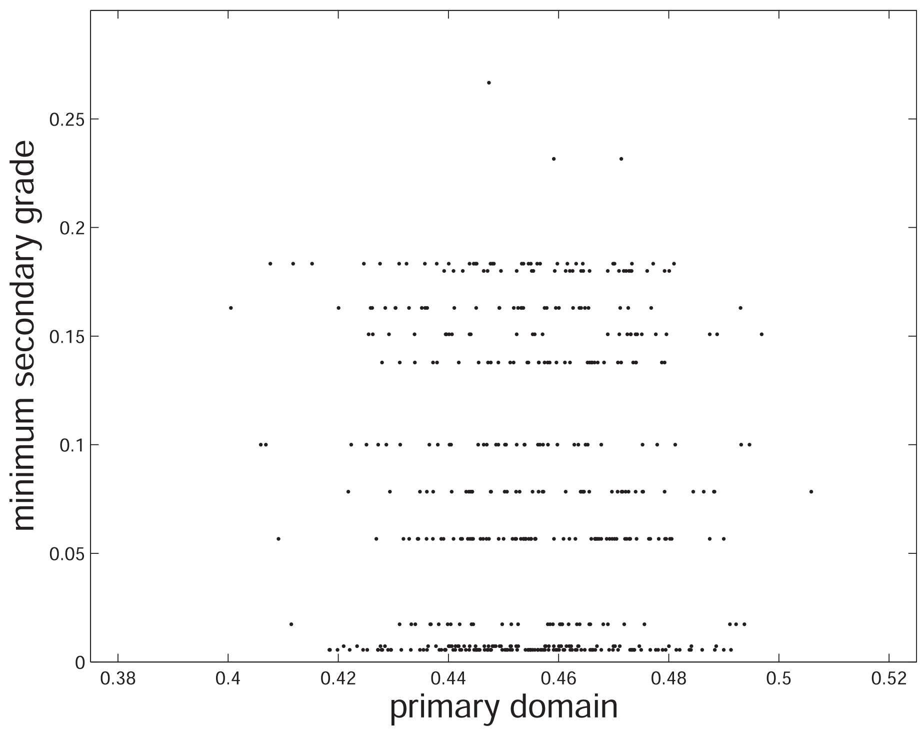


Fig. 5. Strata formed by sampling 500 TRS tuples from the same discretised general T2FS shown in Fig. 4.

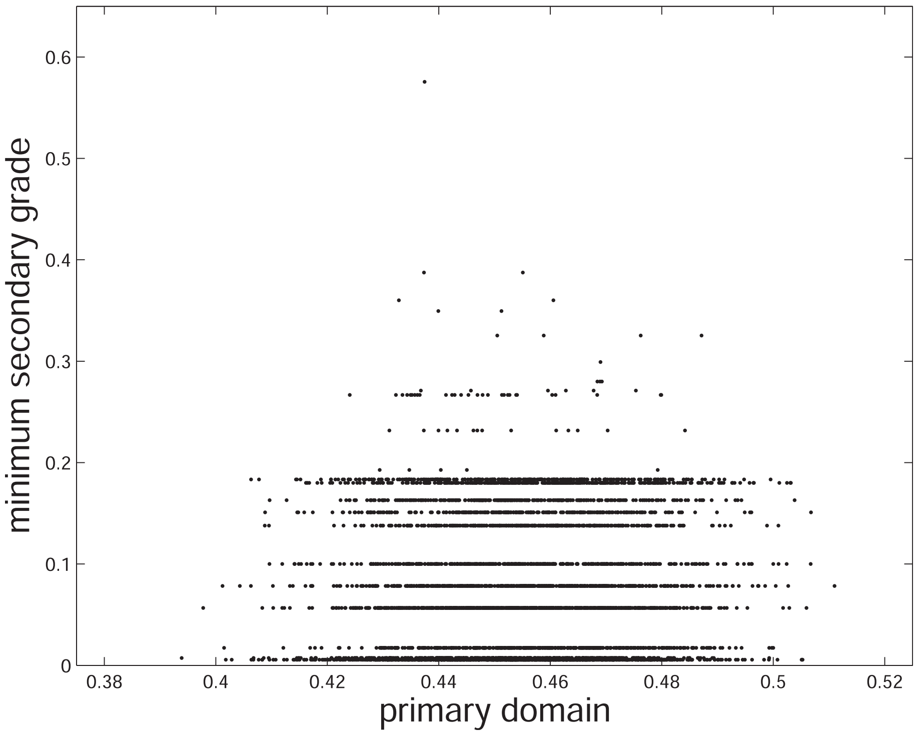
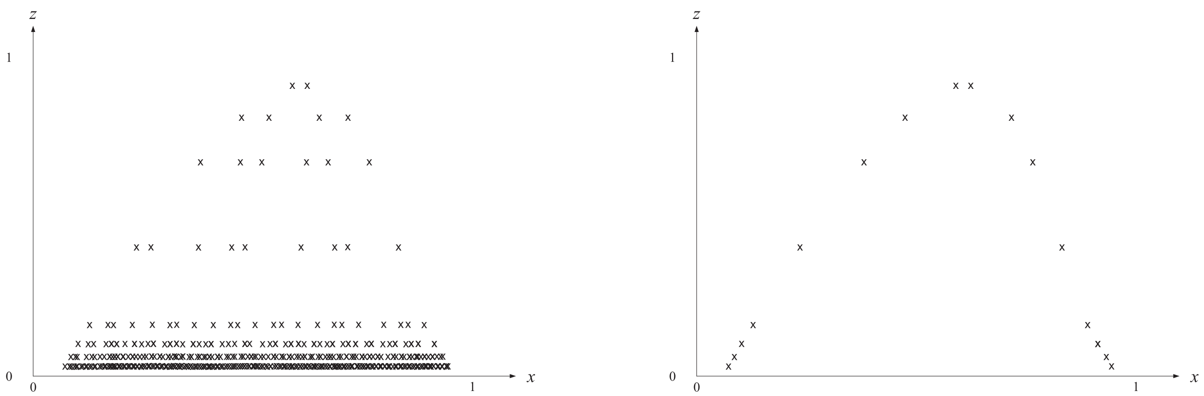


Fig. 6. Strata formed by sampling 5000 TRS tuples from the same discretised general T2FS shown in Fig. 4.



(a) A TRS, with the strata clearly visible.

(b) The TRSS corresponding to the TRS depicted in Figure 7(a)

Fig. 7. By eliminating all the tuples that lie between the end points of the strata, the TRS is transformed into the TRSS.

$$shell_T = \bigcup_{\omega \in [0,1]} shell_\omega, \tag{10}$$

where $shell_\omega = \{(x_m, \mu_T(x_m)), (x_M, \mu_T(x_M))\}; x_m = \min\{x|(x, \mu_T(x)) \in S_\omega\}$ and $x_M = \max\{x|(x, \mu_T(x)) \in S_\omega\}$, consists solely of the TRS tuples that lie at the end points of the strata, and is termed the *shell* of the TRS T of a discretised general T2FS.

The only totally accurate way of constructing the TRS is via Exhaustive Defuzzification. The TRSS is a T1FS approximating to the TRS. The idea of Stratic Defuzzification is to use the TRSS as a substitute for the TRS. This has the advantage that the TRSS is generated by an algorithm whose computational complexity is minimal relative to that of the Exhaustive Method. By defuzzifying the TRSS, a crisp number that approximates to the defuzzified value of the originating T2FS is obtained.

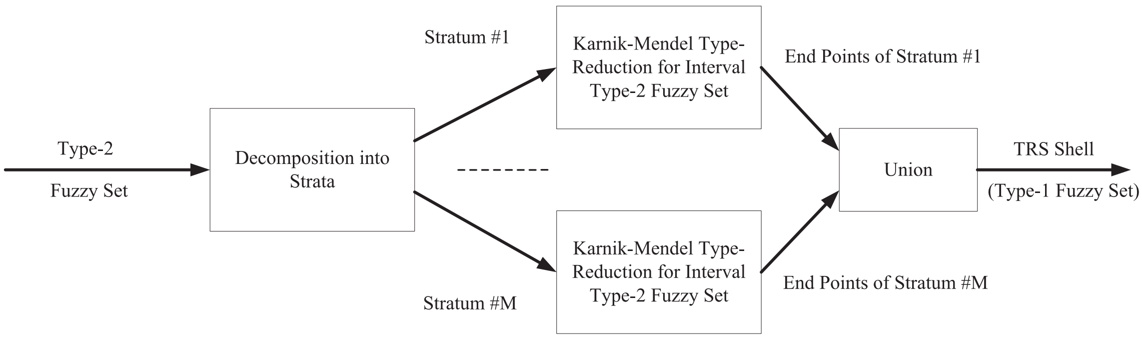


Fig. 8. Principle stages of the Stratic Defuzzifier.

The TRSS is formed of all the strata end points; the Stratic Type-Reduction strategy (Algorithm 1, Fig. 8) consists of determining the heights at which the strata lie, then locating their end points by applying the KMIP to each one in turn. Stratic Defuzzification, as with Exhaustive Defuzzification (and indeed all the efficient type-2 defuzzification alternatives listed in SubSection 3.3), is a two-stage procedure of type-reduction (to the TRSS), followed by type-1 defuzzification of the TRSS. The originality of this strategy lies in its exploitation of the T2FS's naturally occurring characteristic of stratification. From the T2FS's stratified structure the end points and height of each stratum may be extracted and amalgamated to form the TRSS.

The interval KMIP (and its variants such as EIASC) may be seen as special cases of Stratic Defuzzification, in which only one stratum is involved. In this case the TRSS consists of the left and right end points of that stratum. The reason that the KMIP and the Stratic Method of Defuzzification are both approximate techniques is that those tuples within the TRS but not on the TRSS are disregarded.

Algorithm 1: Stratic Defuzzification.

Input: a discretised general T2FS
Output: defuzzified value of the T2FS

```
initialise the TRSS to consist of 0 tuples {The TRSS is initially empty.} ;
determine the heights (secondary membership grades) of all the strata ;
forall the strata do
    apply EIASC to find the stratum's end points ;
    pair the each stratum end point (x) with the stratum height (z) to give
    two tuples (x,z) ;
    store both tuples in the TRSS ;
end
defuzzify the TRSS {The TRSS is a T1FS.} ;
```

5. Experimental evaluation of the Stratic Defuzzifier

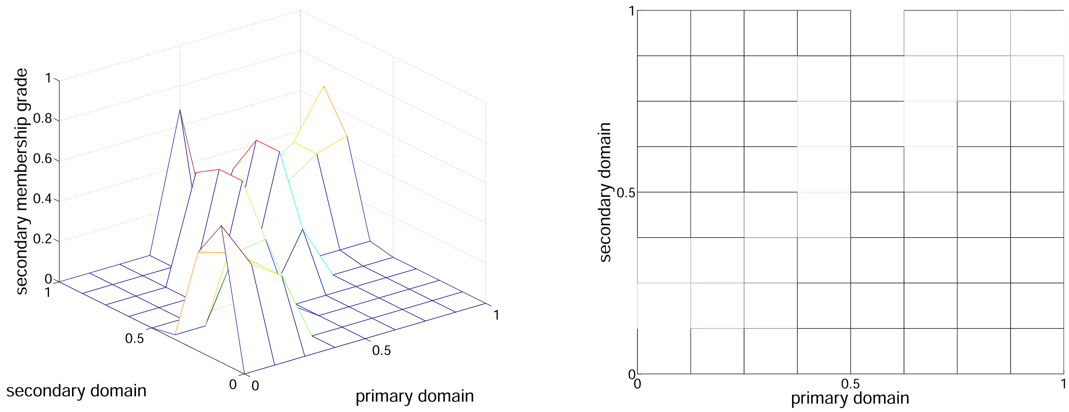
To be acceptable as a type-2 defuzzification technique, the Stratic Defuzzifier has to be demonstrated to possess good accuracy and efficiency. The experiments described below assess the algorithm in both respects.

Six discretised general T2FSs were generated (Figs. 9–14), to be used as test sets. These aggregated sets are products of the inferencing stage of a general type-2 FIS prototype (Fig. 1), coded[4] in Matlab™. They possess diverse properties; Table 1 summarises their salient features. For each inference a sufficiently coarse degree of discretisation was selected to enable Exhaustive Defuzzification to complete within a reasonable time, so that the benchmark defuzzified value could be determined. Three rule sets were used [17]. The FIS was invoked twice for each rule set, with differing input parameters.

The defuzzification algorithms were coded in Matlab™R2014a and tested on a PC with an Intel(R) Core™i5-4570 CPU and 8.00 GB RAM, running at a clock speed 3.20 GHz, using the MS Windows 10 Education operating system. Each test program process was run with priority higher than that of the operating system, going some way to eliminating timing errors arising from unrelated operating system processes.

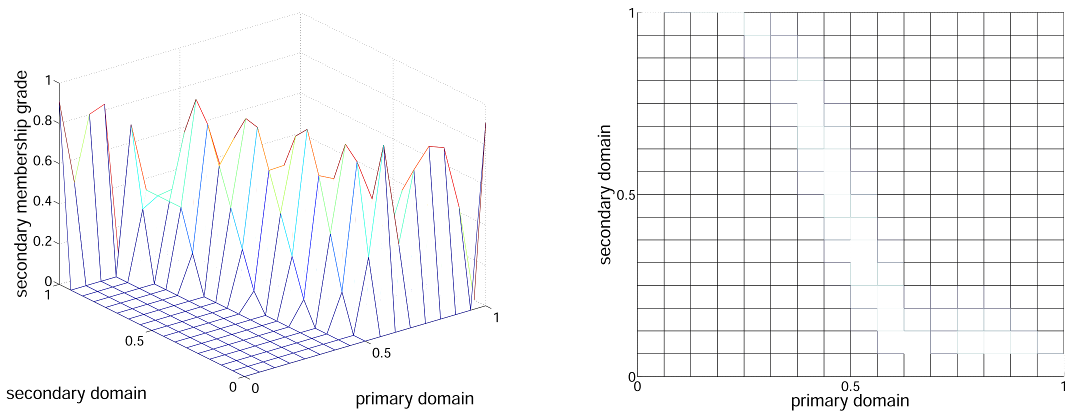
For the α-Planes/KMIP Method, it is necessary for the developer to choose the number of α-planes. In contrast, neither the Exhaustive Defuzzifier nor the Stratic Defuzzifier require this extra parameter.

[4] The code for the creation and defuzzification of the test sets is located at http://www.tech.dmu.ac.uk/ sarahg/.



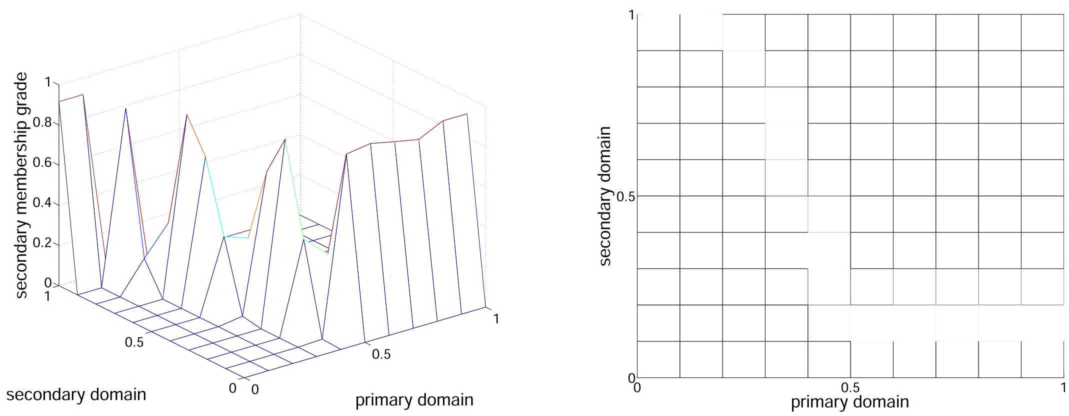
(a) 3-D representation. (b) FOU.

Fig. 9. Heater0.125 — a general type-2 fuzzy test set generated by the Heater FIS; both the domain and co-domain degrees of discretisation are 0.125.



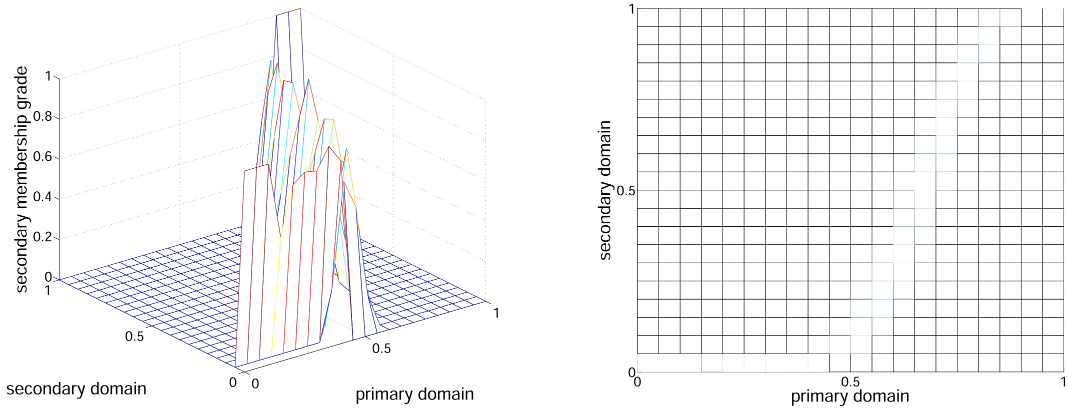
(a) 3-D representation. (b) FOU.

Fig. 10. Heater0.0625 — a general type-2 fuzzy test set generated by the Heater FIS; both the domain and co-domain degrees of discretisation are 0.0625.



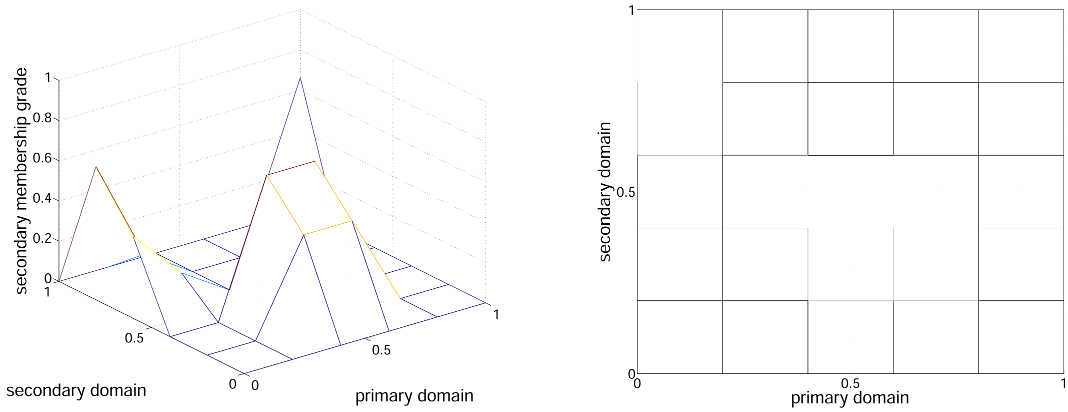
(a) 3-D representation. (b) FOU.

Fig. 11. Powder0.1 — a general type-2 fuzzy test set generated by the Powder FIS; both the domain and co-domain degrees of discretisation are 0.1.



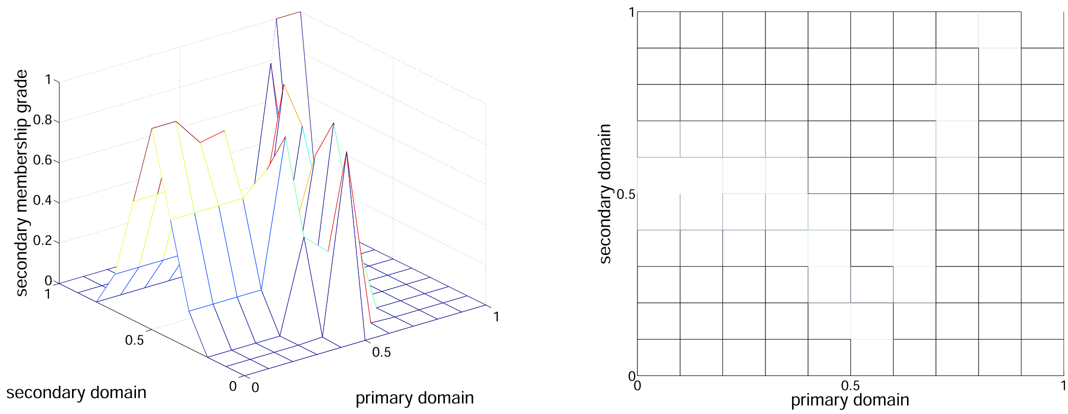
(a) 3-D representation. (b) FOU.

Fig. 12. Powder0.05 — a general type-2 fuzzy test set generated by the Powder FIS; both the domain and co-domain degrees of discretisation are 0.05.



(a) 3-D representation. (b) FOU.

Fig. 13. Shopping0.2 — a general type-2 fuzzy test set generated by the Shopping FIS; both the domain and co-domain degrees of discretisation are 0.2.



(a) 3-D representation. (b) FOU.

Fig. 14. Shopping0.1 — a general type-2 fuzzy test set generated by the Shopping FIS; both the domain and co-domain degrees of discretisation are 0.1.

Table 1
Salient properties of the test sets.

Test set	Normal FOU?	Normal Sec. MF?	Narrow FOU?	No. of Emb. Sets	No. of Strata
Heater0.125	✓	×	×	14580	4
Heater0.0625	✓	×	✓	13778100	17
Powder0.1	✓	×	✓	24300	13
Powder0.05	✓	✓	✓	3840000	35
Shopping0.2	×	×	×	16	1
Shopping0.1	✓	✓	×	312500	13

For tests on the Stratic Defuzzifier and α-Planes/EIASC Method, in order to allow for unmitigated, irrelevant operating system processes, 100 timings were taken; the means and standard deviations were calculated. This meant that each test set was defuzzified.

- once exhaustively;
- 100 times using the Stratic Defuzzifier;
- 100 times through the α-Planes/EIASC Method using 5 evenly spaced α-planes;
- 100 times through the α-Planes/EIASC Method using 11 evenly spaced α-planes;
- 100 times through the α-Planes/EIASC Method using 21 evenly spaced α-planes;
- 100 times through the α-Planes/EIASC Method using 51 evenly spaced α-planes;
- 100 times through the α-Planes/EIASC Method using 101 evenly spaced α-planes.

For each test set, the number of embedded sets and the number of strata are included in Table 1.

5.1. Results and discussion

Tables 2–7 show the results in relation to accuracy; Tables 8–13 show the results in relation to speed of execution.

5.2. Accuracy

5.2.1. The relationship between the α-planes method and exhaustive defuzzification

Are the α-Planes Method and Exhaustive Defuzzification equivalent? Theorem 2 of Liu's 2008 paper on the α-Planes Method states,

> "*For minimum t-norm operation, centroid type-reduction for a type-2 fuzzy set $\widetilde{A}$ is the union of the centroids of its associated type-2 fuzzy sets $\widetilde{A}(\alpha)$, with $\alpha \in [0,1]$, i.e.,*

$$Y_c = \int_{\alpha\in[0,1]} \text{Centroid}\left(\widetilde{A}(\alpha)\right) = \int_{\alpha\in[0,1]} \alpha/\text{domain}\left(\text{Centroid}\left(\widetilde{A}(\alpha)\right)\right).\text{''}$$

However the proof presented [31, Page 2235] does not take Exhaustive Defuzzification as its starting point; no mention is made of embedded sets, let alone the determination of the minimum secondary grade of each embedded set[5]. Consequently it has not been shown that the α-Planes Method is equivalent to Exhaustive Defuzzification; these two methods cannot be expected to give identical results. Since the Exhaustive Defuzzifier provides the benchmark for accuracy, it follows that the α-Planes Method is approximate.

Comparative experiments bear this out [17, Page 15]. These indicate an issue with the α-Planes Method in that as the number of α-planes increases, the α-Planes Method results fail to converge to the value obtained by General Exhaustive Defuzzification, but actually converge to another value. Moreover even the α-Planes/Interval Exhaustive results do not converge to this value. This discrepancy reveals an issue with the α-planes method itself, which has been previously discussed in [24,16].

For the α-Planes Method, since the concept of the *truncation grade* [19] is absent, slicing is performed in the region above the truncation grade [19]. Doing this impairs accuracy and wastes time.

Rather than regarding Stratic Defuzzification as a new technique which resembles the α-Planes Method, the α-Planes Method is best viewed as an approximation to Stratic Defuzzification, which is itself an approximation to Exhaustive defuzzification.

[5] Moreover, the highly pertinent concept of *truncation* [19] is not touched upon.

Table 2
Defuzzified values for the test set Heater0.125 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC. Exhaustive Defuzzification provides the standard for accuracy, from which the error is calculated and hence the ranking derived.

Test set: Heater0.125	Defuzzified value	DV error	Accuracy ranking
Exhaustive Defuzzification	0.6313618377	N/A	
Stratic Defuzzification: 4 Planes	0.6344757121	0.0031138744	1
α-Planes/EIASC: 11 Planes	0.6203643676	−0.0109974701	2
α-Planes/EIASC: 21 Planes	0.6178091252	−0.0135527125	3
α-Planes/EIASC: 51 Planes	0.6151293494	−0.0162324883	4
α-Planes/EIASC: 101 Planes	0.6148483161	−0.0165135216	5
α-Planes/EIASC: 5 Planes	0.6015861938	−0.0297756439	6

Table 3
Defuzzified values for the test set Heater0.0625 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC. Exhaustive Defuzzification provides the standard for accuracy, from which the error is calculated and hence the ranking derived.

Test set: Heater0.0625	Defuzzified value	DV error	Accuracy ranking
Exhaustive Defuzzification	0.2621587894	N/A	
Stratic Defuzzification: 17 Planes	0.2556804862	−0.0064783032	1
α-Planes/EIASC: 11 Planes	0.2784258103	0.0162670209	2
α-Planes/EIASC: 21 Planes	0.2832087215	0.0210499321	3
α-Planes/EIASC: 5 Planes	0.2834726301	0.0213138407	4
α-Planes/EIASC: 51 Planes	0.2837306962	0.0215719068	5
α-Planes/EIASC: 101 Planes	0.2849751274	0.0228163380	6

Table 4
Defuzzified values for the test set Powder0.1 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC. Exhaustive Defuzzification provides the standard for accuracy, from which the error is calculated and hence the ranking derived.

Test set: Powder0.1	Defuzzified value	DV error	Accuracy ranking
Exhaustive Defuzzification	0.2806983775	N/A	
α-Planes/EIASC: 11 Planes	0.2856771537	0.0049787762	1
α-Planes/EIASC: 21 Planes	0.2899246043	0.0092262268	2
α-Planes/EIASC: 101 Planes	0.2905238149	0.0098254374	3
α-Planes/EIASC: 51 Planes	0.2924872462	0.0117888687	4
α-Planes/EIASC: 5 Planes	0.2985191723	0.0178207948	5
Stratic Defuzzification: 17 Planes	0.2508410831	−0.0298572944	6

Table 5
Defuzzified values for the test set Powder0.05 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC. Exhaustive Defuzzification provides the standard for accuracy, from which the error is calculated and hence the ranking derived.

Test set: Powder0.05	Defuzzified value	DV error	Accuracy ranking
Exhaustive Defuzzification	0.8180632180	N/A	
Stratic Defuzzification: 35 Planes	0.8210022735	0.0029390555	1
α-Planes/EIASC: 5 Planes	0.8133958352	−0.0046673828	2
α-Planes/EIASC: 11 Planes	0.8031191474	−0.0149440706	3
α-Planes/EIASC: 21 Planes	0.8002826730	−0.0177805450	4
α-Planes/EIASC: 51 Planes	0.7990107273	−0.0190524907	5
α-Planes/EIASC: 101 Planes	0.7986380906	−0.0194251274	6

Table 6
Defuzzified values for the test set Shopping0.2 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC. Exhaustive Defuzzification provides the standard for accuracy, from which the error is calculated and hence the ranking derived.

Test set: Shopping0.2	Defuzzified value	DV error	Accuracy ranking
Exhaustive Defuzzification	0.5481044441	N/A	
Stratic Defuzzification: 1 Plane	0.5488888889	0.0007844448	1
α-Planes/EIASC: 11 Planes	0.5382327803	−0.0098716638	2
α-Planes/EIASC: 21 Planes	0.5365971981	−0.0115072460	3
α-Planes/EIASC: 51 Planes	0.5363332587	−0.0117711854	4
α-Planes/EIASC: 101 Planes	0.5362490859	−0.0118553582	5
α-Planes/EIASC: 5 Planes	0.5359967320	−0.0121077121	6

Table 7
Defuzzified values for the test set Shopping0.1 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC. Exhaustive Defuzzification provides the standard for accuracy, from which the error is calculated and hence the ranking derived.

Test set: Shopping0.1	Defuzzified value	DV error	Accuracy ranking
Exhaustive Defuzzification	0.5954109472	N/A	
α-Planes/EIASC: 11 Planes	0.5949939045	−0.0004170427	1
Stratic Defuzzification: 13 Planes	0.5967388964	0.0013279492	2
α-Planes/EIASC: 21 Planes	0.5934031254	−0.0020078218	3
α-Planes/EIASC: 51 Planes	0.5924808745	−0.0029300727	4
α-Planes/EIASC: 101 Planes	0.5924190421	−0.0029919051	5
α-Planes/EIASC: 5 Planes	0.6021700303	0.0067590831	6

Table 8
Timings for the test set Heater0.125 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC.

Test Set: Heater0.125	No. of reps.	Mean defuzz. time	SD of defuzz. times	Time per plane
Exhaustive Defuzzification	1	1.0126530	N/A	N/A
α-Planes/EIASC: 101 Planes	100	0.0038446	0.0000329	0.0000381
α-Planes/EIASC: 51 Planes	100	0.0019563	0.0000260	0.0000384
α-Planes/EIASC: 11 Planes	100	0.0004315	0.0000209	0.0000392
α-Planes/EIASC: 21 Planes	100	0.0008262	0.0000318	0.0000393
α-Planes/EIASC: 5 Planes	100	0.0003198	0.0011503	0.0000640
Stratic Defuzz.: 4 Planes	100	0.0004614	0.0012418	0.0001154

Table 9
Timings for the test set Heater0.0625 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC.

Test Set: Heater0.0625	No. of reps.	Mean defuzz. time	SD of defuzz. times	Time per plane
Exhaustive Defuzzification	1	1013.0866	N/A	N/A
α-Planes/EIASC: 5 Planes	100	0.0004208	0.0000923	0.0000842
α-Planes/EIASC: 101 Planes	100	0.0086291	0.0000559	0.0000854
α-Planes/EIASC: 51 Planes	100	0.0043839	0.0000351	0.0000860
α-Planes/EIASC: 21 Planes	100	0.0018436	0.0000305	0.0000878
Stratic Defuzz.: 17 Planes	100	0.0015460	0.0001042	0.0000909
α-Planes/EIASC: 11 Planes	100	0.0010071	0.0000244	0.0000916

Table 10
Timings for the test set Powder0.1 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC.

Test set: Powder0.1	No. of reps.	Mean defuzz. time	SD of defuzz. times	Time per plane
Exhaustive Defuzzification	1	1.7176950	N/A	N/A
α-Planes/EIASC: 101 Planes	100	0.0061603	0.0000508	0.0000610
α-Planes/EIASC: 51 Planes	100	0.0031180	0.0000268	0.0000611
α-Planes/EIASC: 21 Planes	100	0.0013062	0.0000197	0.0000622
α-Planes/EIASC: 5 Planes	100	0.0003181	0.0000871	0.0000636
α-Planes/EIASC: 11 Planes	100	0.0007065	0.0000191	0.0000642
Stratic Defuzz.: 17 Planes	100	0.0010072	0.0001116	0.0000775

Table 11
Timings for the test set Powder0.05 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC.

Test set: Powder0.05	No. of reps.	Mean defuzz. time	SD of defuzz. times	Time per plane
Exhaustive Defuzzification	1	305.0580000	N/A	N/A
Stratic Defuzz.: 35 Planes	100	0.0037333	0.0001257	0.0001067
α-Planes/EIASC: 101 Planes	100	0.0112556	0.0000826	0.0001114
α-Planes/EIASC: 51 Planes	100	0.0057244	0.0000740	0.0001122
α-Planes/EIASC: 21 Planes	100	0.0024003	0.0000365	0.0001143
α-Planes/EIASC: 11 Planes	100	0.0012847	0.0000221	0.0001168
α-Planes/EIASC: 5 Planes	100	0.0006142	0.0000927	0.0001228

Table 12
Timings for the test set Shopping0.2 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC.

Test set: Shopping0.2	No. of reps.	Mean defuzz. time	SD of defuzz. times	Time per plane
Exhaustive Defuzzification	1	0.0013220	N/A	N/A
α-Planes/EIASC: 101 Planes	100	0.0032082	0.0000314	0.0000318
α-Planes/EIASC: 51 Planes	100	0.0016533	0.0000326	0.0000324
α-Planes/EIASC: 21 Planes	100	0.0007042	0.0000269	0.0000335
α-Planes/EIASC: 11 Planes	100	0.0003745	0.0000249	0.0000340
α-Planes/EIASC: 5 Planes	100	0.0002108	0.0001115	0.0000422
Stratic Defuzz.: 1 Plane	100	0.0001745	0.0000858	0.0001745

Table 13
Timings for the test set Shopping0.1 under Exhaustive Defuzzification, Stratic Defuzzification and α-Planes/EIASC.

Test set: Shopping0.1	No. of reps.	Mean defuzz. time	SD of defuzz. times	Time per plane
Exhaustive Defuzzification	1	22.3481430	N/A	N/A
α-Planes/EIASC: 101 Planes	100	0.0068777	0.0000549	0.0000681
α-Planes/EIASC: 51 Planes	100	0.0034977	0.0000265	0.0000686
α-Planes/EIASC: 21 Planes	100	0.0014698	0.0000215	0.0000700
α-Planes/EIASC: 11 Planes	100	0.0007865	0.0000191	0.0000715
Stratic Defuzz.: 13 Planes	100	0.0009551	0.0000896	0.0000735
α-Planes/EIASC: 5 Planes	100	0.0004042	0.0000826	0.0000808

5.3. Speed

The experimental results relating to speed of execution are shown in Tables 8–13. It is clear that both the Stratic Defuzzifier and the α-Planes/EIASC Method execute many times faster than the Exhaustive Defuzzifier.

For each test set:

1. For the α-Planes/EIASC Method, the defuzzification time divided by the number of α-planes is fairly constant;
2. The defuzzification time per stratum is similar to, though usually slightly greater than, the defuzzification times per α-plane.

Thus the speed of execution correlates with the number of planes (whether strata or α-planes) involved in the defuzzification process.

6. Conclusions

This paper reports on the Stratic Method of Defuzzification for discretised, general T2FSs. This novel strategy makes use of the naturally occurring feature of stratification evident within the TRS.

Strata and α-planes are indistinguishable in terms of properties; the difference between a stratum and an α-plane is solely a matter of how each is generated. It is intuitively appealing to defuzzify a general T2FS by slicing it horizontally to produce α-planes or strata. However the Stratic Defuzzifier is a more elegant solution to general type-2 defuzzification than the α-Planes Method as it is a self-contained technique; it does not require the introduction of arbitrary, user defined parameters.

Using the standard of Exhaustive Defuzzification, experiments show the Stratic Defuzzifier to have excellent accuracy as well as outstanding speed. Compared to the α-Planes/EIASC Method, the Stratic Defuzzifier has greater accuracy in four out of six test cases. As regards efficiency, the Stratic Defuzzifier and the α-Planes/EIASC Method are comparable. Excepting test sets with low numbers of strata, there is a correlation between the defuzzification time per stratum and the defuzzification time per α-plane. (See Table 14).

6.1. Future research

- It would be useful to employ rigourous statistical tests to contrast the performance of the Stratic Defuzzifier with other options for type-2 defuzzification (besides the α-Planes Method). In 2013 Greenfield and Chiclana evaluated and contrasted general type-2 defuzzification techniques [17], but it is appropriate to update this research as since then other strategies have been developed, among them the GGCCD [20].
- Experiments contrasting the performance of the Stratic Defuzzifier with the α-Planes Method using the same number of planes for each strategy may give more insight into their differing accuracies[6].
- An evaluation of the impact of using other interval defuzzification techniques instead of EIASC would be valuable.

[6] The Stratic Defuzzifier would determine the number of α-planes to be used.

Table 14
Accuracy ranking for the Stratic Defuzzification relative to 5 instances of the α-Planes/EIASC Method employing different numbers of α-planes.

Test set	Table number	Ranking of sStratic Defuzzifier
Heater0.125	2	1
Heater0.0625	3	1
Powder0.1	4	6
Powder0.05	5	1
Shopping0.2	6	1
Shopping0.1	7	2

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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