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FUZZY DECISION PROCEDURES WITH BINARY RELATIONS

Towards A Unified Theory

by

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To

Dennis and Irina

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Foreword

In decision theory there are basically two approaches to the modeling of individual choice: one is based on an absolute representation of preferences leading to a numerical expression of preference intensity. This is utility theory. Another approach is based on binary relations that encode pairwise preference. While the former has mainly blossomed in the Anglo-Saxon academic world, the latter is mostly advocated in continental Europe, including Russia. The advantage of the utility theory approach is that it integrates uncertainty about the state of nature, that may affect the consequences of decision. Then, the problems of choice and ranking from the knowledge of preferences become trivial once the utility function is known. In the case of the relational approach, the model does not explicitly accounts for uncertainty, hence it looks less sophisticated. On the other hand it is more descriptive than normative in the first stand because it takes the pairwise preference pattern expressed by the decision-maker as it is and tries to make the best out of it. Especially the preference relation is not supposed to have any property. The main problem with the utility theory approach is the gap between what decision-makers are and can express, and what the theory would like them to be and to be capable of expressing. With the relational approach this gap does not exist, but the main difficulty is now to build up convincing choice rules and ranking rules that may help the decision process.

This book is devoted to a systematic investigation of choice and ranking rules for both fuzzy and non-fuzzy binary preference relations, a topic to which Russian scholars have contributed a lot. At the representation level, the main benefit of the use of fuzzy relations is to account for shades of preference rather than uncertainty about states of nature. Most of this book may look like a paradox to utility theory tenants because the algebraic framework of binary preference relations, fully explored here, is cast in a continuous setting. Yet, the "tour de force" of the author is to show that the study of fuzzy relations can shed new light on classical choice rules in the non-fuzzy case by laying bare reasons why some choice rules are better than others, and by suggesting that the fuzzy setting can enrich the potential rankings of alternatives stemming from standard binary relations. The study of multiple-valued decision rules also gives the opportunity for a thorough and original study of basic issues in fuzzy set theory such as fuzzy set inclusions and eigen-fuzzy sets on which many results are presented in the following pages. This book will certainly be considered as a worthwhile contribution to the mathematics of decision science and of fuzzy sets. It is also a window that opens on the research world of Eastern Europe in these fields. May it help starting again the dialogue between Eastern and Western mathematics of decision, a dialogue that had been cut off for so-many years.

Didier DUBOIS Henri PRADE

Preface

This book is aimed at building new bridges between Fuzzy Set Theory, and Crisp Set Theory. It often occurs in fuzzy studies that generalized concepts, originated with crisp prototypes, keep aloof from crisp problems and never return to their crisp "motherland". In our opinion, the backward influence of a generalized theory upon its source should (and can) be much more fertilizing. Dealing with vast material and possessing highly developed technique, fuzzy concepts can be used to achieve a more profound understanding of the nature of crisp problems. In many cases, they can suggest new solutions of crisp problems, though sometimes these solutions may seem somewhat paradoxical. Furthermore, fuzzy ideas can be instrumental in building new crisp constructions.

Therefore, the interaction between fuzzy and crisp theories can be conceived of as a two-sided road

$$\text{CRISP} \quad \underset{\text{explanation, new constructions}}{\overset{\text{generalization}}{\rightleftarrows}} \quad \text{FUZZY}$$

The author thanks in advance all the readers who will make efforts to follow this general idea everywhere in the book, irrespectively of a specific subject, as well as of technical details.

In the present research, the above "super-task" is developed within a conventional branch of decision theory, namely, in decision-making with binary relations.

Aggregation of preferences, choice and ranking of alternatives with respect to binary relations represent an acknowledged tool both in theory and in practice of decision-making. As in many other fields of applied mathematics, recent developments in this area can be characterized as erosion of classical boundaries between the descriptive ("empiric") and the normative ("scientific") approaches. Increasing variety of aggregation models in multicriteria optimization, in voting theory, and in other domains of decision-making considerably broadens the class of binary preference relations and imposes rejection of the conventional transitivity, antisymmetry, acyclicity, and completeness properties of the relations. In addition, "valued" relations, reflecting ingenious structure of preferences, are extensively used in many studies. Under these conditions, classical choice rules, mainly based on a concept of a non-dominated alternative, very often fail. Therefore, it is not surprising that the number of papers devoted to "non-classical" choice and ranking rules grows very rapidly, bringing forth a new significant problem of comparative study of the rules, as well as of the resulting choices and rankings. In our opinion, a conventional apparatus of choice theory lacks "distinguishing capacity" for the adequate solution of the latter problem. On the one hand, the list of choice functions, simultaneously satisfying several "rational choice axioms", proves to be very restricted; in addition, this list only includes the functions which fail in an extended domain of preference relations. On the other hand, many of non-classical choice rules can satisfy certain partial rationality conditions, but, again, it puts considerable a priori restrictions on the class of preference relations.

Wide dissemination of applied decision support systems (*DSS'*) introduces additional topicality in the subject and adds many new aspects to the above problems. Incontestable prevalence of empiric, descriptive component of these systems forwards a new and significant problem of efficiency of decision rules. A concept of efficiency is multidimensional. For instance, a designer and/or a user of a *DSS* can be interested in the probability of success when applying decision rules implemented in the system to diverse classes of preference relations. Otherwise, he can ask either about "definiteness" of the resulting choice or about resoluteness of ranking. Algorithmic efficiency of implementations of decision rules cannot be omitted either.

In our opinion, this list of problems claims for new approaches in decision-making based on binary relations. In the present research, a general approach to constructing and comparative analysis of choice and ranking rules with ordinary and valued binary relations is proposed. The purpose of our study is to discover a convincing answer to the single question:

"What are the *adequate* decision rules ensuring choice and ranking of alternatives on the basis of an arbitrary binary preference relation (be that an ordinary or a valued one), with no a priori restrictions put on the properties of a relation?"

Once posed, this question gives rise to a number of related questions, among which the most essential one is perhaps:

"What is the correct understanding of the term 'adequate'?"

In the traditional choice theory, the answer is immediate:

"Adequate choice functions must satisfy the axioms of rational choice".

This viewpoint seems indisputable in the sense that decision rules are supposed to fall under one or another rationality condition(s). In this book it is the appropriately formulated mathematical model of a notion of rationality that constitutes the central component of generalized decision rules. However, for the above-stated reasons we deviate from a method of investigation of decision rules based on "rational choice axioms" and propose an alternative method of quality estimation of these rules.

In brief, our approach to decision-making with binary relations includes new formalism for decision rules, supplemented with a new understanding of consistency of these rules. We start the development of this alternative method with systematization of choice rules. This study is based on the observation that "rationality concepts" underlying a considerable number of classical and modern choice rules possess a rather regular algebraic structure. First, each of these rules can be represented as a combination of three primary notions of rationality. Next, the primary notions themselves can be expressed by similar "dichotomous" operational formulas based on a composition law. A composition law, in its turn, is understood as a way of action of a binary relation on subsets of alternatives. Investigation of invariant properties of composition laws interpreted as homomorphisms of semilattices results in a family of "algebraically sound" laws consisting of eight members. Two conjugated laws in this family take fundamental place in the setting of choice/ranking rules. Synthesis of choice rules based on dichotomous formulas and combinations of primary notions leads to a pair of families (lattices) of dichotomous rationality concepts, each containing eighteen concepts. The first family is based on a Boolean composition and contains a considerable number of well-known choice rules such as the Core, and the Stable Core, von Neumann - Morgenstern Solution, internal and external stability, *GOCHA* rule, etc. On the opposite, the filling of another family, which is based on a non-traditional composition law, was only recently begun by T.Schwartz (*GETCHA* rule). Decision rules based on dichotomous rationality concepts represent the main subject of the subsequent study.

Two generalizations underlie our approach to quality estimating, and to a comparative study of decision rules. One is the admission of multifold choice, another is expanding the scale of preferences. A "multifold choice assumption" means that we consider all subsets of alternatives satisfying a given rationality concept as having equal rights to be chosen. This understanding predetermines the refusal of aggregation, as well as of "supplementing of definition" of the rational subsets of alternatives. In traditional decision theory, these two methods are often used and are aimed at choosing the unique final subset; for this purpose, aggregation in the form of union, intersection, maximum/minimum of rational subsets with respect to inclusion is very common. However, remaining in the bounds of Boolean 'yes/no' ('chosen/rejected') scale of preferences, we risk a considerable increase of uncertainty with multifold choice (this is, probably, the intrinsic reason why the theory of multivalent choice functions develops rather slowly). Completely new perspectives in this field are related to drawing infinite and continuous preference scale (say, a unit interval).

In a broad outline, principal advantages of enriching the scale of preferences are as follows. In the conventional choice theory, research is concentrated on discrete collections of subsets of alternatives, and the whole picture of the choice process is, to a certain extent, static. With multi-valued preferences, and especially with continuous ones, this picture acquires more vivid and dynamic features. It turns out that significant "events" occur not only in the choice space, but also inside the very scale of preferences. The most principal event is the transformation of the logic of the choice, which changes from a binary mode chosen/rejected to a ternary mode chosen/uncertain/rejected. In this way, uncertainty is organically absorbed by classical choice rules. Furthermore, diverse combinations of the three above values constitute a conceptual system of estimates of quality of choice rules. It is of special interest that, starting with dichotomous formulations of choice rules and with an infinite scale of preferences, we invariably arrive to a three-term evaluation of the results of choice. Within the above system of estimates, a notion of contensiveness of decision rules acquires a natural meaning; there also appear new tools for comparative study of the rules.

In this research, construction and investigation of decision rules with the enhanced scale of preferences is based on fuzzy set theory. Immersion of binary relations and of decision rules in fuzzy environment provides a sufficient level of generalization (thus, all results can automatically be applied to valued preference relations) and involves a highly developed fuzzy technique.

The key concept of this book is the notion of Fuzzy Decision Procedure with Binary Relations. Such a procedure is interpreted as a kind of "decision machine" processing, on the basis of certain fuzzy rationality concept, an input binary relation contained in some preference domain, into an output multifold fuzzy choice. The latter contains all the "most rational" fuzzy subsets of the set of original alternatives taken from a fixed ranking domain (environment). The term "ranking domain" is due to the conception of fuzzy subsets of alternatives as of a priori test rankings. A fuzzy rationality concept is considered as a fuzzy binary relation between preference domain, and environment. Informally, it estimates, with a given binary preference relation, "degree of rationality" of test rankings when applying a procedure to decision-making with this relation. This understanding of Fuzzy Decision Procedure as an aggregate

(original alternatives, preference domain, environment,
rationality concept)

makes it an efficient tool both for solving particular problems (choice and ranking with specific binary relations) and for general considerations (contensiveness and efficiency of decision procedures in diverse preference and ranking domains).

The central concept of contensiveness of a fuzzy decision procedure is derived from the same concept regarding a multifold fuzzy choice. Contensiveness of a multifold fuzzy choice with respect to a specialization of a decision procedure associated with a given binary relation is understood as certain coherence of the corresponding collection of "most rational" fuzzy rankings. A coherence, in its turn, is considered as an ability of this collection to induce a non-trivial crisp choice or ranking, which is usually supplied with numeric (generally, interval) quality estimates. So far, both choice, and ranking with fuzzy binary relations are treated in a unified framework, differing only in concrete "contensiveness formulas". As to the entire decision procedure, a rather weak condition of contensiveness is accepted: it is only demonstration of contensivenes for at least one binary relation in a preference domain that is required. However, even this weak requirement represents an insurmountable obstacle for a considerable number of conventional decision rules.

The above principal quality estimate is supplemented with a specific index of efficiency in preference domain, which is interpreted in a probabilistic manner as a relative measure of a subdomain of a preference domain, consisting of "contensive" fuzzy preference relations.

A method of comparative study of fuzzy decision procedures based on contensiveness and efficiency criteria is later applied to investigation of parametric variety of families of Fuzzy Dichotomous Decision Procedures representing fuzzy versions of the above dichotomous rationality concepts. In fuzzy environment, these concepts gain an additional structural parameter. Except for the composition law, they depend on a model of fuzzy inclusion which is considered as a fuzzy binary relation between fuzzy subsets of alternatives.

In this connection, a separate chapter of the present book is devoted to fuzzy inclusions. It should be noticed that the problem of constructing adequate models of fuzzy inclusions is a fundamental problem of fuzzy set theory which is far beyond the scope of fuzzy decision-making. We propose simple axiomatics motivated by a simultaneous examining of common requirements to fuzzy inclusions, and to fuzzy implications. On the basis of representation theorem for fuzzy inclusions satisfying the above axiomatics, a geometric approach is developed, and a comprehensive study of algebraic and topological properties of fuzzy inclusions is undertaken. A surprising result of this research is that none of the fuzzy inclusions can possess well-defined algebraic properties together with continuity. For this reason, we select two distinguished representatives for the subsequent study, each representing one class of the properties. One is a conventional L.Zadeh's inclusion, satisfying all possible algebraic requirements, but discontinuous. Another is Kleene - Dienes inclusion associated with a maximal t-norm 'min', representing a continuous and, in a certain sense, the only linear model of fuzzy inclusion. Besides being characteristic of the entire family of fuzzy inclusions, these two representatives reflect two different applications of fuzzy decision procedures. Kleene - Dienes inclusion is responsible for choice structures, whereas L.Zadeh' inclusion gives rise to fuzzy ranking structures.

Finally, for each of the eighteen different dichotomous rationality concepts, we consider four fuzzy versions (2 composition laws × 2 fuzzy inclusions). In addition, each of the 72 resulting fuzzy rationality concepts is studied in two environments. A universal environment includes all fuzzy test rankings; in a restricted environment with "prohibited trivial choice", the smallest (empty) subset as well as the greatest (the whole set of alternatives) fuzzy subset are excluded.

The results of investigation of fuzzy decision procedures are rather unexpected; in our opinion, these results motivate an essential revision of traditional ideas of choice and ranking on the basis of binary relations. They also dictate considerable changes in the acknowledged fuzzy decision models.

The first result states that in universal environment the overwhelming majority of decision procedures proves to be "identically incontensive" in any preference domain. In other words, rationality concepts underlying these procedures induce a consistent fuzzy multifold choice with none of the fuzzy or crisp binary relations. We emphasize that, within a two-valued scale of preferences, this effect can in no way be recognized: indicators of crisp subsets represent too poor a collection of "test rankings".

A simple explanation of unsatisfactory behavior of a considerable number of decision procedures can be achieved in terms of the above-mentioned three-valued choice logic. Truth values in this logic are simulated by the corresponding subsets of preferences within a unit interval:

"chosen" - $\{1\}$, "uncertain" - $[0,1]$, "rejected" - $\{0\}$.

With incontensive decision procedures, multifold fuzzy choice is represented either as a collection of "uncertain dichotomies" of the type uncertain/rejected or as a collection of triangulations of the type chosen/uncertain/rejected. In both cases, multifold fuzzy choice gives no reason to positively prefer any of the crisp subsets of alternatives.

Paradoxically, the list of rationality concepts giving rise to incontensive procedures includes the notion of "non-dominated alternative" in its crisp interpretation and in any of the conventional fuzzy ones. In other words, a fundamental idea of graphodominant choice function (the choice of the Core of a binary relation), when considered from a general viewpoint, proves to be inconsistent. In the book, a possible explanation of this controversy with choice theory is proposed.

On the whole, incontensive procedures with a valued (fuzzy) binary relation "hint" at the possibility of ordinary choice with respect to their crisp prototypes, provided that the latter are applied to extremal cuts of a valued relation (1-cut, strict 0-cut). However, this "extremism" proves to be unproductive because of eroding of crisp choice by uncertain preferences.

Only two rationality concepts result in decision procedures which prove to be contensive under the most general conditions. One is von Neumann - Morgenstern Solution basing on classical composition law, another is Stable Core with the same composition law. Fuzzy versions of these procedures possess a number of remarkable features which are mainly opposite to the properties of incontensive decision rules. First of all, both procedures can be used both for choice and for ranking purpose. Switching from a choice mode to a ranking mode is achieved merely by changing the model of fuzzy inclusion: choice is associated with Kleene - Dienes inclusion, whereas ranking is due to L.Zadeh inclusion. Next, the choice with contensive procedures, in contrast to the incontensive ones, is connected with a more balanced median cut (strict 1/2-cut) of a valued relation, which is nothing but the nearest crisp relation in Hamming metric. Furthermore, contensive procedures do not "erode" crisp choice. On the contrary, fuzzy von Neumann - Morgenstern Solution sharpens crisp choice with respect to its crisp prototype. Moreover, from a probabilistic viewpoint induced crisp choice with respect to a fuzzy version is almost always unifold (it should be mentioned that crisp von Neumann - Morgenstern Solution with a random ordinary graph does not possess this property!). Finally, in the whole variety of fuzzy dichotomous decision procedures, only the two contensive procedures preserve a classical choice logic chosen/rejected, supplemented with a symmetric interval estimate of preferences of the chosen subset.

The study of ranking facilities of fuzzy dichotomous decision procedures based on L.Zadeh inclusion is encumbered by a very complicated structure of the corresponding multifold fuzzy choice. To cope with this problem, we propose special technique of "$\varkappa$-mappings" enabling one to determine the induced crisp quasi-ordering, and the interval numeric estimates of the preferences of the classes of this quasi-ordering in an implicit way, escaping the straightforward construction of multifold fuzzy choice. Roughly speaking, ranking properties of a valued relation depend on its C-spectral properties, and on its singular body, which are determined by two attractors of a unity orbit with respect to the above-mentioned $\varkappa$-mapping. In the book classification of possible cases is suggested on the basis of the size of the singular body.

Application of the above results to valued versions of quasi-orderings demonstrates considerable difference of decision-making with "weighted" relations as compared to the classical case. Thus, an ordinary quasi-ordering possesses unique von Neumann - Morgenstern Solution which coincides with the Core, and hence represents the Stable Core. With valued quasi-ordering, von Neumann - Morgenstern Solution is also unique and coincides with the Stable Core, but must not necessarily be the Core. For this reason, consistent ranking of alternatives on the basis of valued binary preference relation considerably differs, even in the case of order relations, from the conventional ranking using a concept of "fuzzy non-dominated alternative".

Since ordinary relations represent a particular case of valued relations, all the above results of comparative study of fuzzy choice rules remain valid in the classical field. In other words, we have new reasons (derived from a generalized theory) to confirm the principal statement that, in universal environment, only the von Neumann - Morgenstern Solution, and the Stable Core are worthy of note. Some other decision rules (GOCHA, GETCHA, etc.) can also be consistent under the condition of eliminating trivial choice from the environment. However, the structure of the choice with crisp binary relations must also be revised: usual dichotomy must be changed for a triangulation with an "uncertain member", and what's more, any part of this new member can be either included in or excluded from the overall choice.

However, fuzzy decision procedures can be used not only for the purposes of explanation and adjustment of known decision rules; they can be used for creating new decision-making tools. An example of this kind is the graphtheoretical concept of dipole decomposition derived from a fuzzy von Neumann - Morgenstern Solution and representing a far going "ranking generalization" of the crisp von Neumann - Morgenstern Solution.

As was mentioned above, in the context of applied decision support systems we face an important problem of efficiency of decision rules. In this book, three aspects of efficiency are considered, namely, efficiency in preference domain (probability of a contensive choice on the basis of a random relation), efficiency in ranking domain (relative measure of a multifold fuzzy choice), and efficiency of the induced crisp choice (mean percent of rejected alternatives). The results of this research based on methods of random graphs narrow the limits of the class of well-defined decision rules up to a single element. Actually, in the initial list of 72 fuzzy decision procedures, only one procedure, namely, a fuzzy version of von Neumann - Morgenstern Solution basing on classical composition law and on Kleene - Dienes inclusion, proves to be asymptotically efficient in all aspects. Additional simulation tour in low dimensions assures high efficiency of this procedure.

Three principal conclusions for applied *DSS'* can be derived from the study of efficiency. First, implementing a *DSS* with a wide variety of aggregation models resulting in a corresponding variety of types of binary preference relation one cannot count on successful ranking of alternatives; conversely, a sufficiently sharp choice can be surely expected, though the very chosen subset must not necessarily be unique (multifold choice). Second, involving valued relations in a decision process, one can considerably improve the quality of choice. Third, the list of decision rules for the purpose of analysis of arbitrary preference relations (be that ordinary or valued relations) can be restricted to four representatives. These distinguished rules are fuzzy von Neumann - Morgenstern Solution, and fuzzy Stable Core, each basing on classical composition law, and on two alternative models of fuzzy inclusion. In addition, the use of three out of four rules in this list is well founded only in the case when a *DSS* possesses special means for regular processing of quasi-orderings, or at least of acyclic relations.

It cannot be overlooked that the significant place of analogs of von Neumann - Morgenstern Solution for the analyses of weighted preference relations was foreseen by B.Roy ('ELECTRE' method). It also should be noticed that the present research motivates a simplified version of this method: instead of "scanning" all level cuts, as implemented in 'ELECTRE', it suffices, from a theoretical viewpoint, to use only median cut.

Another area of applications of fuzzy decision procedures in applied *DSS'* is due to new perspectives in constructing "soft" estimates of concordance between *a priori* and *a posteriori* preferences of a decision-maker. These perspectives give life to new ideas in the development of a "reciprocal user interface", and of a "retrospective portrait" of a decision-maker. In the book, the problem of simultaneous estimate of competence and of resoluteness of a decision-maker during the process of solving of multicriteria decision problems is briefly considered and exemplified. A practical example involves fuzzy majority method of preference aggregation in multipurpose decision problems with non-equally significant criteria. An axiomatic basis of this method is proposed.

Internal logic of the research required the study of diverse concepts which are auxiliary in the main context of this book, but can be useful in other branches of fuzzy set theory. Except for the above-mentioned fuzzy inclusions, we mark out a one more "by-product", namely, the study of mainsprings and of algebraic essence of general cut technique. When combined with methods of graph theory, and with some topological details, this technique proved to be a powerful and straightforward tool in the research of fuzzy relational systems, in particular, in solving fuzzy relational equations and inequalities of diverse types. This is demonstrated with a complete algorithmically sound description of an acknowledged family of eigen fuzzy subsets, and of two concerned but less known families of invariant, and of antiinvariant fuzzy subsets. In the book, we confine ourselves to conventional fuzzy subsets; nevertheless, all results remain in force for a wide class of L-fuzzy subsets.

Thematically, this book is subdivided into five parts. In Chapters 1-5, motivations, main definitions, systematization of choice rules with binary relations, and formalism of fuzzy decision procedures are presented. A separate Chapter 6 is devoted to the theory of fuzzy inclusions. Ties between this chapter, and the remaining material are explained in Section 6.6. In Chapters 7-13, we bring together main results of the research of fuzzy dichotomous decision procedures (structure of multifold fuzzy choice, and of induced crisp choice or ranking, contensiveness and efficiency considerations). Chapter 10 includes a special study of cut technique together with applications to invariant, antiinvariant, and eigen fuzzy subsets. This chapter, as well as Chapter 6, goes beyond the boundaries of decision-making. In Chapter 14, the influence of the above results upon a traditional decision theory is observed. In Chapter 15, we present additional descriptive considerations and applications to multicriteria decision problems.

All Definitions, assertions (Lemmas, Propositions, Theorems, and Corollaries), Examples, and Notes are enumerated consequently in each chapter (Definition 3.1, Theorem 9.2, Note 9.1...). In many cases, assertions are subdivided into parts, which are indicated by small latin numerals in brackets: (i), (ii),..., (vi)... A typical example of a reference is "Theorem 8.1 (ii)".

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Chapter 1

Introduction

Two main approaches in modern Decision Theory are the *descriptive* approach - the "art" of decision-making, and the *prescriptive* approach - the "science" of decision-making. A scientific background of descriptive decision-making is, first of all, psychology, and then social and management sciences, whereas prescriptive branch is essentially based on mathematical considerations.

In applied Decision Support Systems, descriptive approach is prevailing. But perhaps, for a "prescriptive" scientist, such *DSS* would show like something raw and non-motivated. Conversely, a "prescriptively sound" *DSS* would seem rather rigid and non-realistic for a common user.

Reuniting of these two branches seems to be the most challenging and promising field both in decision science, and in applications. However, there exist serious collisions preventing this union. Two problems of this kind are the problem of *preference domain*, and the problem of *choice rules* (see Figure 1.1).

It is common knowledge that the majority of choice rules can be modeled as choice with binary preference relations. It is also known in prescriptive theory that "rational choice axioms" force binary preference relations to possess one or another transitivity property. Perhaps, the weakest of these properties, namely, *pseudo-transitivity* $R \circ I_R \subseteq R$, was recently formulated by D.Danilov and A.Sotskov [1] (in fuzzy set theory, conditions of this kind were studied by S.Ovchinnikov [1,7], and by V.Kuzmin [1]). However, in descriptive theory there exist numerous

The Problem of "Preference Domain"

DESCRIPTIVE APPROACH – *Intransitivity* of preferences		PRESCRIPTIVE APPROACH – *Transitivity* of preferences
(Tversky [1])		pseudo-transitivity
Voting theory (majority relations)	⇒ × ⇐	$R \circ I_R \subseteq R$
		(Danilov, Sotskov, [1])
Empiric aggregation models in multicriteria decision-making		

The Problem of "Choice rules"

Rational choice axioms		Available choice rules
Heritage		Pareto (*H*+*I*+*C*)
Independence	⟶	Core (*H*+*C*)
Concordance		Joint extremal (*H*+*I*)

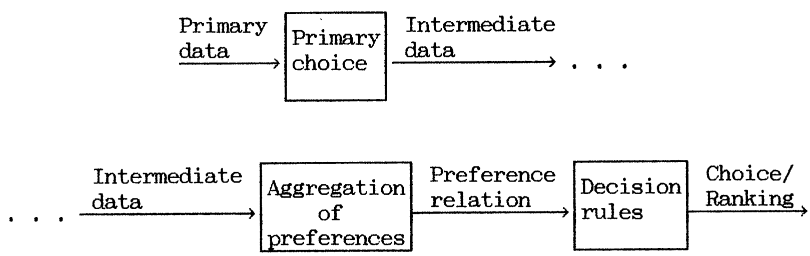
(Berezovski, Borzenko and Kempner [1])

Fig. 1.1. Two problems in Decision Theory

motivations of the converse effect - non-transitivity of preferences (we mark out special research by A.Tversky [1], and great diversity of empiric aggregation rules in voting theory, and in multicriteria decision-making, described by D.Dubois [3], J.Fodor and M.Roubens [1], B.Roy [1], and many others, and resulting in a non-transitive aggregated relation. Let us consider, for example, a collection of aggregation rules concerning multicriteria decision problems and using the notion of "significant criteria", and let us assume that all local criteria are orderings. One can formulate various aggregating notions - say, x is preferred to y iff it is preferred for at least one significant criterion, for all, for a sufficient number of such criteria. In the whole collection, only one rule - pessimistic aggregation "for all criteria irrespective of their significance" - is transitivity preserving (see Kitainik, Orlovski, and Roubens [1], and Chapter 15 for details).

Another problem concerns choice rules themselves. Formally, these rules are expected to satisfy the above axioms of rational choice. There exist several dozens of axioms of this kind (see Aizerman [1], Arrow [1], Chernoff [1], Fishburn [1,2], Sen [1], Schwartz [1]); in a survey by Z.Lezina [1], systematization of these axioms is proposed; rationality properties of choice functions in fuzzy environment versions are studied by Barrett, Pattaniak and Salles [1], Blin [1], Bouyssou [1], Choleva [1], Fodor and Roubens [1], Fung and Fu [1], Montero [1], Ovchinnikov and Roubens [1], Roubens [1], and many others. Let us examine a conventional triple of axioms: *Heritage*, *Independence*, and *Concordance*. It is known (see, e.g., Berezovski, Borzenko and Kempner [1]) that the only choice functions satisfying combinations of these axioms are the *Pareto choice* (*H+I+C*), the *Core* (non-dominated alternatives of a digraph) (*H+C*), and the so called *joint extremal choice* (*H+I*). For many reasons, this list is too restrictive.

A more practical consideration aiming at the development of applied Decision Support Systems, adds new problems to this list. A multistage information processing in a *DSS* can be roughly represented by the following diagram



We suppose that all "obviously non-fitting" alternatives (say, those Pareto dominated) are sifted through the Primary Choice module. So, we are interested in the two bottom blocks. The Decision Rules block can be considered as a kind of "decision-making machine", processing the "stream" of preference relations, resulting from preference aggregation models, into the final choices or rankings of alternatives. What is the efficiency of this machine? How often is it successful?

The answer depends on both the parameters of the input stream of preference relations and the contents of the "decision box". In our opinion, the diversity and complexity of current models of preference aggregation enables one to assume the random nature of the stream of input relations. So, as a model example, let us suppose that the input stream can be represented as a sample of independent random relations $G_{n,q}$ (n - dimension, q - probability for any ordered pair of alternatives to represent a preference; for more details, see (Erdos and Spencer [1]). Furthermore, let us suppose that a "decision box" contains a conventional graphodominant choice, that is, selects all non-dominated alternatives (the Core) of a relation. Under these conditions, a natural efficiency estimate is its "reliability", that is, probability of success. Clearly, a "successful application" of a choice rule presupposes at least the non-emptiness of the resulting choice. At the same time, an easy calculation shows that the efficiency of a graphodominant choice in this example makes $\approx nq^n$, thus exponentially decreasing when the number of alternatives grows (even with 10 alternatives, and $q=0.5$, the efficiency is smaller than 0.1). This means that the choice rule almost always fails. So, the *efficiency problem* is also worthy of note.

Numerous efforts in solving these problems gave rise to several trends in modern Decision Theory:

A. Relaxing transitivity property and generalizing order relations (see, e.g. Doignon, Monjardet, Roubens, and Vincke [1]; Ovchinnikov, Roubens and Fodor [2], Fodor and Roubens [2], Ovchinnikov [3], Roubens and Vincke [1]);

B. Developing "non-classical" choice rules; this stream also includes two main approaches:

B1. Using sophisticated aggregation schemes in order to reduce the problems to the conventional case of transitive preference relations (see Fishburn [1], Miller [1], Richelson [1]);

B2. "Inventing" choice rules themselves (the Core, von Neumann - Morgenstern solution, the Stable Core, *GOCHA*, and *GETCHA* rules, recently introduced by T.Schwartz [1], and so on; see a survey by V.Volskiy [1] for numerous examples of choice rules with binary relations).

The co-existence of a great variety of "non-classical" choice rules with binary relations sets a one more *problem of comparative study* of these rules. The above argumentation demonstrates the shortage of traditional prescriptive tools for such study. On the one hand, a sound approach of the Choice Theory, based on combinations of rational choice axioms, proves to be too restrictive (see Figure 1.1). On the other hand, relaxation of rationality requirements also leads to a disappointing result: any "modern" choice rule satisfies, under some additional conditions, one or another axiom of rational choice (see Schwartz [1] for extended discussion). So, there is a need for non-traditional approaches in the field.

In this volume, a unified approach is proposed, consolidating the mentioned trends in Decision Theory and considering the listed problems within a general framework of *Fuzzy Decision Procedures with Binary Relations*.

But why "fuzzy" decision procedures? Because, surprisingly, new ideas and tools come into sight in fuzzy set theory when it is considered, roughly speaking, as involving extended scale of preferences (multi-valued and continuous). It looks as if some conventional results in crisp decision theory were casual just because they were originally based on an incomplete two-valued scale of preferences... On the whole, "completion effects" are common in many branches of mathematics. A generally accepted example can be found in real and complex analysis: the values of *real* integrals depend on singularities somewhere on the *complex* plane...

Our approach consists of the following principal parts:

- introducing new *Formalism for Fuzzy Decision Procedures;*
- *comparative study* of crisp and fuzzy choice rules with binary relations using *contensiveness* and *efficiency* concepts;
- *adjusting* crisp choice rules, and conventional fuzzy decision models;
- *producing* new crisp and fuzzy *choice rules*.

A *Formalism for Fuzzy Decision Procedures* includes, in its turn:

- general consideration of *fuzzy decision procedure*, and of *multifold fuzzy choice*;
- developing *contensiveness* and *efficiency criteria for multifold fuzzy choice*;
- *systematization of choice rules with binary relations* based on invariant algebraic description, and introducing families of *fuzzy dichotomous decision procedures*;
- the study of *composition laws*, and of *fuzzy inclusions* as of structural parameters of fuzzy dichotomous decision procedures.

On the basis of new formalism, the following principal results for specific decision procedures are obtained:

considerable reduction of the number of well-defined choice rules;

discovering "proper" structures of choice and ranking with binary relations (be that crisp or fuzzy representatives) which were "hidden" in the crisp case under the incompleteness of scale of preferences;

recognizing efficient choice rules that can be used with most general preference relations, thus constituting a sound basis for applied Decision Support Systems.

A one more trait of the approach is that it does not distinguish between *choice* and *ranking*: both are considered within a common framework. To process from one to another, one needs just to select appropriate parameters of decision procedures.

Based on the above notions, new ideas in *aggregating preferences in multipurpose decision-making*, and in *co-ordination of preferences* in interactive decision support systems can be developed. In particular, this includes new opportunities for "measuring" psychological characteristics of a decision-maker (such as are, for instance, *competence*, and *resoluteness*).

The concepts under consideration must not be confined to decision theory. Thus, the notion of efficiency can be applied to fuzzy cluster analysis as well. Fuzzy inclusions represent one of the basic concepts in fuzzy set theory. Alternative models of composition laws, as well as invariant, and eigen fuzzy subsets have diverse applications, say, in fuzzy dynamic systems. Extended α-cut method provides new facilities for the research of a wide class of relational systems, etc.

Chapter 2

Common Notations

The following basic notations are used all over the text.

$I=[0,1]$ - unit interval;

X - finite set, containing n ordinary initial alternatives (the support);

$\tilde{\mathcal{P}}(X)$ - the set of all fuzzy subsets (f.s.) of X;

$\mathcal{P}(X)$ - the set of all crisp (ordinary) subsets of X.

Crisp objects are generally denoted by *capital* letters, fuzzy objects - by *small* letters.

$\mu_a(x)$ - membership function for $a\in\tilde{\mathcal{P}}(X)$;

$\chi_Z(x)$ - characteristic function for $Z\in\mathcal{P}(X)$.

Marked elements in $\tilde{\mathcal{P}}(X)$ are constants $\alpha\cdot 1$ with $\mu_{\alpha\cdot 1}(x)\equiv\alpha$; in particular, $0=0\cdot 1=\chi_\emptyset$, $1=1\cdot 1=\chi_X$, $1/2=1/2\cdot 1$.

$\tilde{\mathcal{P}}(X^2)$ - the set of all fuzzy binary relations (FR's) on X;

$\tilde{\mathcal{P}}_0(X^2)$ - the set of all *antireflexive* FR's - $\mu_R(x,x)\equiv 0$;

$\tilde{\mathcal{P}}_1(X^2)$ - the set of all *reflexive* FR's - $\mu_R(x,x)\equiv 1$;

$\tilde{\mathcal{P}}^{(2)}(X)$ - the set of all *fuzzy level two subsets* of X.

Depending on the context, symbol $\vee$ is used as lattice and ordinary max, sup, and also denotes a union of f.s., $\wedge$ is used to denote min, inf and intersection of f.s. In particular, $\overset{\circ}{\vee}a = \bigvee_{y\in Y} \mu_a(y)$ stands for height (upper bound), $\overset{\circ}{\wedge}a = \bigwedge_{y\in Y} \mu_a(y)$ stands for lower bound of arbitrary f.s. $a\in\tilde{\mathcal{P}}(X)$.

All variants of α-cuts are used in the text (conventional a_α, strict $a_{>\alpha}$, inverse $a_{\le\alpha}$ and $a_{<\alpha}$). Notations a_α, and A_α mean the same thing. By $A_{=\alpha}=a_{=\alpha}=\{y\in X|\mu_a(y)=\alpha\}$ we denote *level set* of a.

$\tilde{\mathcal{P}}(X)$ is considered as poset with respect to L.Zadeh' inclusion $\subseteq$; all intervals $[a,b]$ in $\tilde{\mathcal{P}}(X)$ are due to this order.

Interior of a subset Z of topological space is denoted by $\overset{\circ}{Z}$.

If $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$ is a crisp subset of the set of all f.s.' of X, that is, an ordinary collection of f.s.', then $\overset{\sqcup}{\mathfrak{X}}$ will denote the set of all supplements of elements of $\mathfrak{X}$, $\overset{\sqcup}{\mathfrak{X}}=\{\bar{a}|a\in\mathfrak{X}\}$.

$\odot$ - general symbol of a *composition law* between fuzzy binary relation on X and fuzzy subset of X,

$$\odot:\tilde{\mathcal{P}}(X^2)\times\tilde{\mathcal{P}}(X)\longrightarrow\tilde{\mathcal{P}}(X),\ (R,a)\longrightarrow R\odot a;$$

the two main representatives of a family of composition laws are the Boolean ($\vee$-$\wedge$) composition $\circ$, and the dual law $\overline{|\circ}$;

We introduce a new binary operation $|\pi|$ between a f.s. and an interval f.s., $|\pi|:\tilde{\mathcal{P}}(X)\times\tilde{\mathcal{P}}^2(X)\longrightarrow\tilde{\mathcal{P}}(X)$: $a|\pi|[b,c]$ is defined as projection in Hamming metric (ρ_H) of a fuzzy subset a on an interval $[b,c]$,

$$a|\pi|[b,c]=\arg\bigwedge_{f\in[b,c]}\rho_H(a,f);$$

more explicitly,

$$\mu_{a|\pi|[b,c]}(x)=\begin{cases}\mu_b(x), & \mu_a(x)<\mu_b(x)\\ \mu_a(x), & \mu_b(x)\le\mu_a(x)\le\mu_c(x)\\ \mu_c(x), & \mu_a(x)>\mu_c(x)\end{cases}$$

$inc\in\tilde{\mathcal{P}}(\tilde{\mathcal{P}}^2(X))$ - fuzzy *inclusion* between f.s.' of X;

P - fuzzy *decision procedure*;

$\mathfrak{R}$ - *preference domain*, an ordinary subset of the set $\tilde{\mathcal{P}}(X^2)$ of all fuzzy binary relations on X;

$\mathcal{E}$ - *ranking domain*, or *environment*, an ordinary subset of the set $\tilde{\mathcal{P}}(X)$ of all f.s.' of X;

p - *rationality concept* associated with a fuzzy decision procedure P, that is, a fuzzy relation between $\mathcal{R}$, and $\mathcal{E}$;

$(p,R,\mathcal{E})$ - *specialization* of a fuzzy decision procedure with a given preference relation R;

$\mu^*(p,R,\mathcal{E})$ - *maximum value* (supremum) of a specialization;

$\mathcal{D}(p,R,\mathcal{E})$ - *multifold fuzzy choice*;

$\Delta_1(\odot)$, $\Delta_2(\odot)$, $\Delta_3(\odot)$ - *basic crisp* $\odot$*-dichotomies*;

$\Delta_1(\odot,\mathrm{inc})$, $\Delta_2(\odot,\mathrm{inc})$, $\Delta_3(\odot,\mathrm{inc})$ - *basic fuzzy* $(\odot,inc)$*-dichotomies*;

Δ_{ij}, Δ_{ijk} - brief notation for conjunctions ("mini-terms") $\Delta_i\wedge\Delta_j$, $\Delta_i\wedge\Delta_j\wedge\Delta_k$;

$\mathfrak{P}(\odot,\mathrm{inc})$ - a family (lattice) of *fuzzy dichotomous decision procedures* associated with a given composition law $\odot$ and fuzzy inclusion inc.

$\square$ - symbol of *superposition of abstract mappings*, $(f\square g)(x)=f(g(x))$.

Given a crisp/fuzzy binary relation R on a support X, several families of crisp/fuzzy subsets will be often used in the subsequent study:

the set of all *invariant* subsets of R

$$\mathfrak{Inv}(R)=\{a\in\tilde{\mathcal{P}}(X)\,|\,R\circ a\subseteq a\};\quad \mathfrak{Inv}_0(R)=\mathfrak{Inv}(R)\cap\mathcal{P}(X);$$

the set of all *antiinvariant* subsets of R

$$\mathfrak{Ainv}(R)=\{a\in\tilde{\mathcal{P}}(X)\,|\,a\subseteq R\circ a\};\quad \mathfrak{Ainv}_0(R)=\mathfrak{Ainv}(R)\cap\mathcal{P}(X);$$

the set of all *eigen* subsets of R

$$\mathfrak{Eig}(R)=\mathfrak{Inv}(R)\cap\mathfrak{Ainv}(R)=\{a\in\tilde{\mathcal{P}}(X)\,|\,R\circ a=a\};\quad \mathfrak{Eig}_0(R)=\mathfrak{Eig}(R)\cap\mathcal{P}(X);$$

the set of all *internally stable* subsets of R

$$\mathfrak{I}(R)=\{a\in\tilde{\mathcal{P}}(X)\,|\,R\circ a\subseteq\bar{a})\};\quad \mathfrak{I}_0(R)=\mathfrak{I}(R)\cap\mathcal{P}(X);$$

the set of all *externally stable* subsets of R

$$\mathfrak{E}(R)=\{a\in\tilde{\mathcal{P}}(X)\,|\,\bar{a}\subseteq R\circ a\};\quad \mathfrak{E}_0(R)=\mathfrak{E}(R)\cap\mathcal{P}(X);$$

With Q being a *crisp* binary relation, we denote by $\mathfrak{I}^*(Q)$ (resp., by $\mathfrak{E}_*(Q)$) the set of all maximal (resp., minimal) with respect to set inclusion *crisp* subsets of $\mathfrak{I}_0(Q)$ (resp., $\mathfrak{E}_0(Q)$).

Given a fuzzy binary relation $R\in\tilde{\mathcal{P}}(X^2)$, we will denote by $\mathfrak{R}$, and M respectively the set of all crisp von Neumann - Morgenstern solutions, and the set of all non-dominated alternatives (the Core) of a strict median-cut of R, that is, of crisp relation $R_{>1/2}$.

Other notations are introduced when needed.

In addition to the above common notations, the following abbreviations are used in the text:

MCDP - Multicriteria Decision Problem;

DSS - Decision Support System;

FR - fuzzy binary relation;

FDP - fuzzy decision procedure;

FDDP - fuzzy dichotomous decision procedure;

MFC - multifold fuzzy choice;

FI - fuzzy inclusion.

Chapter 3

Systematization of Choice Rules with Binary Relations

Let us suppose that R is a binary preference relation on a finite set X of initial crisp alternatives; for x, y in X, xRy means "x is preferred to y". For the above-stated reasons, no special *a priori* properties of R (transitivity, antisymmetry, weak completeness, etc.) are required. The problem is to determine, on the basis of R, ranking of X or at least the "subset of best alternatives".

3.1. RATIONALITY CONCEPT. MULTIFOLD CHOICE

As was mentioned in Chapter 1, in classical crisp decision theory, binary preference relations are supposed to possess one or another version of transitivity. Even more, they are often required to be either antisymmetric (for all $x \neq y$, $xRy \Rightarrow \neg(yRx)$) or weakly complete (for all $x \neq y$, either xRy or yRx holds). With such relations, there always exist non-dominated alternatives (x is non-dominated iff, for all $y \in X$, $\neg(yRx)$), or at least "non-dominated collections" of alternatives (say, "winning cycles" in voting theory, see Aizerman [1], Aizerman and Malishevski [1], Danilov [1]). For this reason, the graphodominant choice, that is, the rule prescribing the selection of all non-dominated alternatives, is in general use. However, modern approaches to aggregation of preferences are very diverse (see the above notes in Chapter 1, and examples in Chapter 15); in many cases, these aggregation schemes leave no hope for a "well-defined" preference relation. In response to this challenge, a variety of "non-classical" choice rules with binary relations was developed (other trends in modern decision-making with preference relations were outlined in Chapter 1). Some of these new rules are aimed at "converting" the "bad" relation to the conventional class. We dwell on several examples of this type.

Example 3.1. 1) According to Fishburn rule (Fishburn [1]), the resultant choice with a relation R contains all non-dominated alternatives of a modified relation $R_{\mathcal{F}}$ defined as

$$xR_{\mathcal{F}}y \Leftrightarrow ((\forall\ z\in X)(zRx\Rightarrow zRy)\&(\exists\ z\in X)((zRy)\&\neg(zRx)))$$

In other words, x is preferred to y with respect to $R_{\mathcal{F}}$ iff the subset $R^{-1}(\{x\})$ containing all alternatives that are "R-better" than x, is strictly smaller than $R^{-1}(\{y\})$. Clearly, $R_{\mathcal{F}}$ is a semi-ordering (transitive and antisymmetric relation).

2) Miller rule (Miller [1]) is dual to the previous one: the overall choice includes all non-dominated alternatives of $R_{\mathcal{M}}$, where

$$xR_{\mathcal{M}}y \Leftrightarrow ((\forall\ z\in X)(yRz\Rightarrow xRz)\&(\exists\ z\in X)((xRz)\&\neg(yRz)))$$
$$\Leftrightarrow R(\{y\})\subset\subset R(\{x\})$$

It is easy to prove that $R_{\mathcal{M}}=((R^{-1})_{\mathcal{F}})^{-1}$.

3) In the same way, Richelson rule (Richelson [1]) is based on the intersection $R_{\mathcal{R}}=R_{\mathcal{F}}\cap R_{\mathcal{M}}$ (see Volskiy [1] for more details) [1].

4) In fuzzy decision-making, this approach is in common use. Let us consider, for example, nine *crisp* choice functions with *FR*'s proposed by C.R.Barrett, P.K.Pattaniak and M.Salles [1] (see also Nurmi [1], Orlovski [1], Switalski [1], and many other papers). Actually, each of these functions is the Core of an appropriate crisp relation. Thus,

$$\text{max-MF}(X,R)=\{x\in X\,|\,(\forall\ y\in X)(\bigvee_{z\neq x}\mu_R(x,z)\geq\bigvee_{z\neq y}\mu_R(y,z))\},$$

is the core of R_{MF}, $xR_{MF}y \Leftrightarrow \bigvee_{z\neq x}\mu_R(x,z)>\bigvee_{z\neq y}\mu_R(y,z)$ (see Bouyssou [1] for another characterization of the above choice functions) ■

For the reasons that will be stated later in this section, we are more

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[1] Significant generalization of these considerations to fuzzy environment was recently undertaken by J.Fodor [2].

interested in choice rules differing from the graphodominant choice. The process of choice with the majority of such rules can be decomposed into two steps: determining "rational subsets" of alternatives, and constructing the final choice. At the first step, a *rationality concept* is introduced; it can be interpreted as a predicate on a power set of X; truth function of such predicate defines a collection of "rational subsets".

We give seven examples of thus defined rationality concepts. In Section 3.4, a more regular approach to their synthesis will be developed.

Example 3.2. Crisp rationality concepts ($Y \subseteq X$ is rational iff ...):

1) *GOCHA* concept (Schwartz [1]) -

$$(\forall\ x \in Y)(\forall\ y \in \overline{Y})(\neg(yRx)).$$

2) Internal stability (von Neumann and Morgenstern [1], Berge [1]) or independence (Swami and Thulasiraman [1]) -

$$(\forall\ x,y \in Y)(\neg(yRx)).$$

3) External stability (von Neumann and Morgenstern [1], Berge [1]) or domination (Swami and Thulasiraman [1]) -

$$(\forall\ y \in \overline{Y})(\exists\ x \in Y)(xRy).$$

4) Non-dominated alternatives (Aizerman [1]) -

$$(\forall\ y \in X)(\neg(yRx)).$$

The set of all non-dominated alternatives is called the Core of a binary relation.

5) von Neumann - Norgenstern solution (von Neumann and Morgenstern [1]) or kernel (Berge [1]) - both the Internal, and the External stability.

6) Stable Core (von Neumann and Morgenstern [1], Muto [1]) - a von Neumann - Norgenstern solution consisting of non-dominated alternatives.

7) *GETCHA* concept (Schwartz [1]) -

$$(\forall\ x \in Y)(\forall\ y \in \overline{Y})(xRy)\ \blacksquare$$

Three illustrations to rationality concepts in the above list are the key ones (see Figure 3.2, and Table 3.1 in Section 3.2). With respect to the first one (*GOCHA* rationality concept), a subset is rational iff none of the alternatives contained in this subset is worse than any "outside" alternative. Next is *internal stability* (*independence*), requiring that no pair of alternatives within a rational subset is tied by preferences; the last one is *external stability* (*domination*): any of the outside alternatives is worse than at least one among the rational alternatives.

The main problem caused by the first step of non-classical rules is the excessive number of rational subsets. This effect is due to the fact that, in contrast with the notion of non-dominated alternative, which can be formulated for each alternative separately, all the remaining rationality conditions deal with the entire collections of alternatives. Thus, maximal number of internally stable subsets (as well as of von Neumann - Morgenstern solutions) of a binary relation is achieved with the so called Moon - Moser graphs (Moon, Moser [1]) and makes $\approx 3^{n/3}$ (see also Chapter 12). Therefore, the result of the first step of a choice rule can be very ambiguous.

Therefore, the purpose of the second step of choice rules is to eliminate this non-uniqueness. What are the ideas of such reduction? One is to restrict preference domain to those classes of binary relations which guarantee the unique choice.

Example 3.3. It is known (see, e.g., Berge [1]) that von Neumann - Morgenstern Solution is unique for an acyclic relation, whereas *GETCHA* rule provides unique choice with any antisymmetric relation (Schwartz [1]; for necessary and sufficient conditions of uniqueness of *GETCHA* choice, see Corollaries 3.1, 3.2 at the end of this chapter) ∎

Another approach is to transform the original relation, in order to reduce the problem to the latter case (see the above Example 3.1). A one more method is to "aggregate" rational subsets, so that the final [unique] choice is considered as their union, intersection, maximum/minimum with respect to inclusion, and so forth.

Example 3.4. 1) The resulting choice with *GOCHA* choice rule, as defined in (Schwartz [1]), is the union of minimal with respect to inclusion non-empty rational subsets (see Theorem 3.2 (i) for the exhaustive description of these subsets).

2) The graphodominant choice is defined as a Core of a binary relation, that is, the set of all non-dominated alternatives, which can be considered as the only maximal with respect to inclusion rational subset with respect to a "non-dominatedness" rationality concept.

3) The resulting *GETCHA* choice is defined as a minimal rational subset (a *GETCHA*-rational subset is necessarily non-empty). To guarantee the uniqueness of *GETCHA* choice, the class of binary relation should be restricted (see Example 3.3) ∎

So far, the first, and the second method of "sharpening" of crisp rational choice is operating in the "preference domain" (the space of binary relations), whereas the latter method transforms the "choice domain" itself (the power set of a support). In our opinion, none of these approaches is adequate enough: the first one narrows the class of preference relations, the two remaining methods are arbitrary, thus distorting either the original preferences or the resulting choice. In this issue, we adhere to an alternative idea that the non-unique, *multifold choice* is a legal structure. Say, the choice, which is represented as an ensemble of two subsets, each containing three alternatives, is nowise worse than a single subset containing six alternatives. A straightforward next step is that the multifold choice should be considered *in toto*. Therefore, the subject of the subsequent study is moved from determining the *unique final choice* to introducing *quality estimates* of the *multifold choice*. We point out that the idea of multifold choice attracts growing attention in modern decision theory (see, e.g., recent research by $\boxed{\text{O. Bondareva}}$ [1,2] of rational choice axioms for "multivalent" choice). In our opinion, fuzzy environment provides much more favorable field for such considerations in comparison with crisp environment, thanks to the fact that multifold fuzzy choice is both the continual, and the continuous one.

3.2. BASIC DICHOTOMIES: INVARIANT DESCRIPTION

We precede a more detailed study of choice rules by a conventional geometric interpretation of binary preference relations. It is common knowledge that preferences are often considered as darts (directed edges) of a directed graph (digraph, see Tutte [1]), and that graph technique is of common use in decision theory.

Example 3.5. Let $X=\{x_1,\ldots,x_6\}$; let us consider a binary relation

$$R = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

The corresponding digraph is shown on Figure 3.1. Fourteen darts of this digraph correspond to the fourteen pairwise preferences between alternatives (1's in the matrix R) ■

Let us consider the resulting choice with this relation according to the above-mentioned choice rules.

Example 3.6. With the relation R, defined in Example 3.5, the final choice according to the choice rules described in Examples 3.1, 3.2, 3.4 is as follows (we omit details; for *GOCHA*, and *GETCHA* rules, see Example 3.11):

GOCHA rule - $\{x_1,x_2,x_3\}$;
the Core - $\{x_1\}$;
von Neumann - Morgenstern Solutions - $\{x_1,x_2\}$, $\{x_1,x_3\}$;

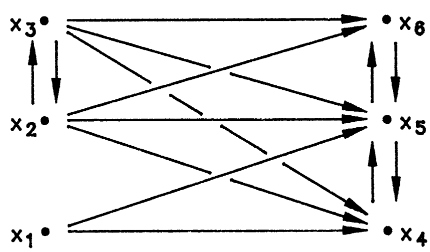


Fig.3.1. Digraph of the relation R

Stable Core - $\emptyset$;
GETCHA rule - $\{x_1,x_2,x_3,x_4,x_6\}$;
Fishburn rule - $\{x_1\}$;
Miller rule - $\{x_2,x_3\}$;
Richelson rule - $\{x_1,x_2,x_3\}$.

So, the only alternative "missed" in these choices is x_5, and none of the alternatives is included in all choices ■

From now on, a digraph of a binary preference relation will be identified with the relation itself and will be denoted by the same letter.

Three basic rationality concepts introduced in the previous section are illustrated in the graph terms on Figure 3.2.

Two observations give rise to the invariant study of choice rules with binary relations: one is that all the three basic rationality concepts can be written in a similar algebraic form; another is that the mentioned concepts generate a regular structure of choice rules which contains many of the conventional rules, and a number of new rules.

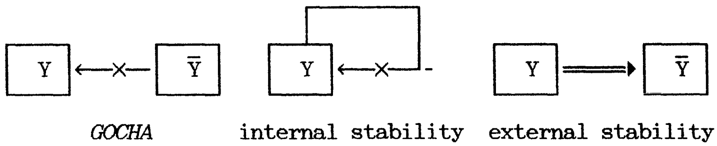


Fig.3.2. Basic rationality concepts

To obtain the announced algebraic description, one should consider a binary preference relation as an operator on the power set of a support. A specific form of effect of a relation upon subsets of X will be called a *composition law* (see Dubois, Prade [11]). A conventional example of this kind is *Boolean composition* $\circ$ defined as

$$R \circ Y = \{ y \in X \mid (\exists\ x \in Y)(xRy) \}$$

Using the matrix of a relation R, the result of Boolean composition $R \circ Y$ can be calculated as Boolean product of a characteristic function of a subset Y, which is represented in the form of row vector, upon the columns of a matrix R (in other words, the matrix is a right-hand factor).

Note 3.1. Formally, the above explanation motivates the notation $Y \circ R$ rather than $R \circ Y$. However, we preserve the latter version as a more "functional" notation, since R in this term is considered as an operator on $\mathcal{P}(X)$ ■

Example 3.7. Given the relation R, considered in Example 3.5, and a subset $Y = \{x_1, x_4\}$, the result $R \circ Y$ can be expressed as

$$R \circ Y = (1\ 0\ 0\ 1\ 0\ 0) \circ \begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} = (0\ 0\ 0\ 1\ 1\ 0),$$

which is nothing but the characteristic function of a subset $\{x_4, x_5\}$ ■

In the graph terms, the result of Boolean composition can be obtained as the set of all end vertices of darts with their initial vertices in Y.

Example 3.8. Under the conditions of Example 3.7, $R \circ Y$ is depicted on Figure 3.3 (only the needed fragment of a digraph R is shown) ■

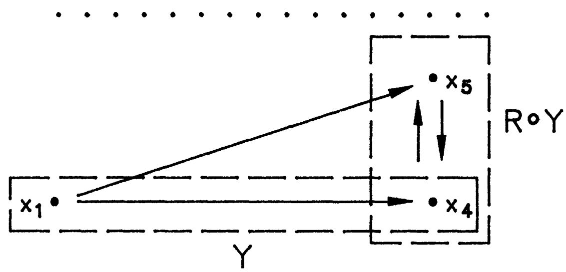


Fig. 3.3. Boolean composition in graph terms

To establish the correspondence between composition laws, on the one hand, and choice rules, on the other hand, we introduce the principal notion of *basic dichotomies*.

Definition 3.1. *Basic crisp dichotomies* associated with a composition law $\odot$ are the three crisp predicates $\Delta_1(\odot)$, $\Delta_2(\odot)$, $\Delta_3(\odot)$ on $\mathcal{P}(X)$, depending on a binary relation in the following way: a subset $Y \subseteq X$ belongs to the truth set of such predicate iff

$$\Delta_1(\odot)(R) - R \circ \overline{Y} \subseteq \overline{Y}; \quad \Delta_2(\odot)(R) - R \circ Y \subseteq \overline{Y}; \quad \Delta_3(\odot)(R) - \overline{Y} \subseteq R \circ Y$$

(a dichotomy $\Delta_2(\odot)$ is defined only for antireflexive relations: $(\forall\, x \in X)(\neg(xRx)))$ ∎

Table 3.1

Basic crisp dichotomies based on Boolean composition law

($Y \subseteq X$ is rational iff <Concept>)

Notation	Δ_1	Δ_2	Δ_3
Name	*GOCHA*	Internal stability	External stability
Concept: verbal expression	No alternative in $\overline{Y}$ is better than any alternative in Y	No alternative in Y is better than any alternative in Y	Any alternative in $\overline{Y}$ is worse than at least one alternative in Y
formula	$R \circ \overline{Y} \subseteq \overline{Y}$	$R \circ Y \subseteq \overline{Y}$	$\overline{Y} \subseteq R \circ Y$

Proposition 3.1. Truth sets of $\Delta_1(\circ)$, $\Delta_2(\circ)$, $\Delta_3(\circ)$ coincide with the sets of all rational subsets of respectively *GOCHA*, internal stability, and external stability rationality concepts ■

Proof. $\Delta_1(\circ)(R)(Y)$ is satisfied iff $R\circ\overline{Y}\subseteq\overline{Y}$. Using the definition of Boolean composition, and routine set-theoretical facts, we come to the following chain of equivalent assertions:

$$R\circ\overline{Y}\subseteq\overline{Y} \Leftrightarrow \{y\in X\,|\,(\exists\ x\in\overline{Y})(xRy)\} \subseteq \overline{Y} \Leftrightarrow \{y\in X\,|\,(\exists\ x\in\overline{Y})(xRy)\}\cap Y=\emptyset$$

$$\Leftrightarrow(\forall\ y\in Y)(\neg(\exists\ x\in\overline{Y})(xRy)) \Leftrightarrow (\forall\ x\in\overline{Y})(\forall\ y\in Y)(\neg(xRy)).$$

The latter predicate is exactly the definition of *GOCHA* rationality concept (see Example 3.2). The proof for $\Delta_2(\circ)$, $\Delta_3(\circ)$ is similar ■

However, rationality concepts based on Boolean composition do not exhaust the universe of concepts. Thus, *GETCHA* rule can not be directly represented in the above form. But, as a matter of fact, it turns to be nothing but a version of external stability derived from an alternative composition law. In order to expand the area of invariant descriptions of choice rules, additional study of *composition laws* is undertaken.

3.3. COMPOSITION LAWS

To set a background for invariant consideration of composition laws, we propose a purely algebraic characterization of Boolean composition $\circ$ on the basis of a well-known fact that $\circ$ preserves the union of subsets, $R\circ(Y\cup Z)=(R\circ Y)\cup(R\circ Z)$ (see, e.g., Dubois, Prade [1]). The latter equality means that, with each binary relation R, the mapping $b_R:\mathcal{P}(X)\longrightarrow\mathcal{P}(X)$, $b_R(Y)=R\circ Y$, is an *endomorphism* of an upper semilattice $\mathcal{P}_U$ of subsets of X.

Proposition 3.2. The correspondence $b:R\longrightarrow b_R$ induces an isomorphism between the set $\mathcal{P}(X^2)$ of all binary relations on X and the set $\mathrm{Hom}(\mathcal{P}_U,\mathcal{P}_U)$ of all endomorphisms of $\mathcal{P}_U$ ■

Proof. Let $R\neq Q$ be two binary relations on X. In such case, there exist $x,y\in X$, satisfying either $(xRy)\&(\neg(xQy))$ or $(xQy)\&(\neg(xRy))$; let us suppose, for definiteness, that the first of these propositions is true.

According to the definition of Boolean composition, this is equivalent to $(y\in b_R(\{x\}))\&(y\notin b_Q(\{x\}))$. It follows that $b_R(\{x\})\neq b_Q(\{x\})$, and we conclude that $b_R\neq b_Q$. Hence, b is a monomorphism.

Conversely, given an endomorphism of $\mathcal{P}_\cup$, $e\in\mathrm{Hom}(\mathcal{P}_\cup,\mathcal{P}_\cup)$, we define a binary relation R_e using the formula $xR_ey \Leftrightarrow y\in e(\{x\})$. Then, with each subset $Y\in\mathcal{P}(X)$, $e(Y)=\bigcup_{x\in Y} e(\{x\})$; therefore,

$$e(Y)=\{y\in X\,|\,(\exists\ x\in Y)(y \in e(\{x\}))\}=\{y\in X\,|\,(\exists\ x\in Y)(xR_ey)\}=R_e\circ Y;$$

this is equivalent to the equality $e=b_{R_e}$, so that b is epimorphism ■

But why confine oneself to endomorphisms of $\mathcal{P}_\cup$? The dual meet semilattice $\mathcal{P}_\cap$, that is, the power set of X supplied with intersection as semilattice operation, is nowise worse. Therefore, for the sake of completeness, let us consider all combinations of homo-/endo-morphisms between/inside the two semilattices $\mathcal{P}_\cup$, and $\mathcal{P}_\cap$.

A regular method for constructing such homomorphisms in the spirit of Proposition 3.2 is based on an observation that the involution $\nu:Y\longrightarrow\overline{Y}$ belongs to both families $\mathrm{Hom}(\mathcal{P}_\cup,\mathcal{P}_\cap)$, and $\mathrm{Hom}(\mathcal{P}_\cap,\mathcal{P}_\cup)$. Let e be a homomorphism from $\mathcal{P}_\cup$ into $\mathcal{P}_\cap$. It is clear that $\nu\Box e\in\mathrm{Hom}(\mathcal{P}_\cup,\mathcal{P}_\cup)$ (we recall that $\Box$ denotes superposition of mappings, see Chapter 2). In virtue of Proposition 3.2, there exists a binary relation R satisfying $(\nu\Box e)(Y)=\overline{e(Y)}=R\circ Y$ for all $Y\in\mathcal{P}_\cup$. Hence, a general formula of a homomorphism from $\mathcal{P}_\cup$ into $\mathcal{P}_\cap$ is $e(Y)=\overline{R\circ Y}$. Alternatively, for an endomorphism $e\in\mathrm{Hom}(\mathcal{P}_\cap,\mathcal{P}_\cap)$, $\nu\Box e\Box\nu\in\mathrm{Hom}(\mathcal{P}_\cup,\mathcal{P}_\cup)$, so that, for an appropriate R, $e(Y)=\overline{R\circ\overline{Y}}$, and so on.

This method of synthesis of composition laws yields eight representatives; Table 3.2 contains explicit formulas for the resultant laws. In this table, mnemonic notation for "non-classical" composition laws is used: the vertical line | to the left (resp., to the right) of the sign of Boolean composition $\circ$ indicates that the Boolean composition should be modified by taking the supplement of a relation (resp., the supplement of a subset): $R|\circ a=\overline{R}\circ a$, $R\circ|a=R\circ\overline{a}$; the horizontal line $^{-}$ over the sign $\circ$ means the negation of the whole expression: $R\bar{\circ}A=\overline{R\circ A}$. Of course, there are exactly eight combinations of the three additional symbols "left line", "right line", top line". The formulas themselves can easily be derived from these notations.

Open problem. Though the "left-right-top-line method" is exhausted by the presented formulas, Table 3.2 is only half-filled. What is the *natural* way of filling empty cells in the table? In particular, can there be invented a "bottom line" of the square surrounding the symbol $\circ$, in order to complete the above mnemonic notations? ∎

Table 3.2

Composition laws as semilattice homomorphisms

Binary relations on X	Subsets of a support X			
	$Hom(\mathcal{P}_{\cup},\mathcal{P}_{\cup})$	$Hom(\mathcal{P}_{\cup},\mathcal{P}_{\cap})$	$Hom(\mathcal{P}_{\cap},\mathcal{P}_{\cup})$	$Hom(\mathcal{P}_{\cap},\mathcal{P}_{\cap})$
$Hom(\mathcal{P}^2_{\cup},\mathcal{P}^2_{\cup})$	$\circ$ Boolean composition		$\circ\vert$ $R\circ\vert A=R\circ\bar{A}$	
$Hom(\mathcal{P}^2_{\cup},\mathcal{P}^2_{\cap})$		$\bar{\circ}$ $R\bar{\circ}A=\overline{R\circ A}$		$\circ\rceil$ $R\circ\rceil A=\overline{R\circ\bar{A}}$
$Hom(\mathcal{P}^2_{\cap},\mathcal{P}^2_{\cup})$	$\vert\circ$ $R\vert\circ A=\bar{R}\circ A$		$\vert\circ\vert$ $R\vert\circ\vert A=\bar{R}\circ\bar{A}$	
$Hom(\mathcal{P}^2_{\cap},\mathcal{P}^2_{\cap})$		$\lceil\circ$ $R\lceil\circ A=\overline{\bar{R}\circ A}$		$\lceil\circ\rceil$ $R\lceil\circ\rceil A=\overline{\bar{R}\circ\bar{A}}$

Unfortunately, a more careful examining of the "capacity" of thus obtained composition laws to provide a sound basis for developing sensitive choice rules essentially decreases the list of non-classical laws. We give an example of such "testing".

Example 3.9. With a non-classical composition law $\circ\rceil$, $R\circ\rceil A=\overline{R\circ\bar{A}}$, let us consider a basic dichotomy $\Delta_2(\circ\rceil)$. According to this rationality concept,

$$A\in\mathcal{P}(X) \text{ is rational iff } R\circ\rceil A\subseteq\bar{A}.$$

Direct calculation yields the following chain of inclusions: $R\circ\rceil A\subseteq\bar{A} \Leftrightarrow \overline{R\circ\bar{A}}\subseteq\bar{A} \Leftrightarrow R\circ\bar{A}\supseteq A$. The latter expression results in the following "rationality predicate" for $\Delta_2(\circ\rceil)$

$$\bar{A} \text{ is } \Delta_3(\circ)\text{-rational iff } \bar{A} \text{ is externally stable.}$$

In other words,

"A subset is rational iff each of the alternatives included in this subset is worse than at least one of the outside alternatives".

Thus defined "rationality concept" can scarcely be considered as a logically consistent one ■

Exhaustive logical testing, applied to non-classical composition laws, leaves only one new representative $\lceil\!\circ$ (see Kitainik [7]). This law was implicitly introduced by T.Schwartz [1], and was independently studied by K.Izumi, H.Tanaka, K.Asai [1]. New set-theoretical interpretations and results concerning this law were recently obtained by D.Dubois and H.Prade [6]. The law $\lceil\!\circ$ is dual to Boolean composition in the same meaning as "arbitrariness" quantor is dual to the "existence" quantor: instead of

$$R\circ Y=\{y\in X \mid (\underline{\exists\ x\in Y})(xRy)\},$$

the dual law implies the formula

$$R\lceil\!\circ Y=\{y\in X \mid (\underline{\forall\ x\in Y})(xRy)\},$$

thus introducing a concept of "strong domination".

Example 3.10. For a binary relation from Example 3.5, and a subset of alternatives $Y=\{x_1,x_4\}$ (see Example 3.7), the difference between the two composition laws is demonstrated on Figure 3.4.

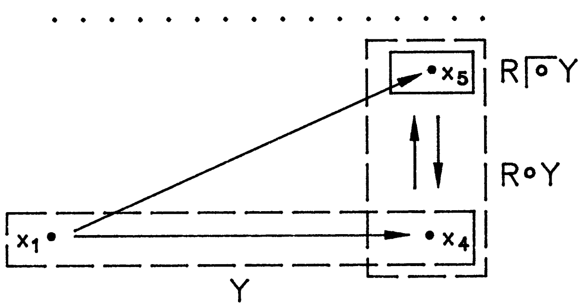


Fig. 3.4. The difference between Boolean composition $\circ$, and the dual composition law $\lceil\!\circ$

Proposition 3.3. Truth set of a predicate $\Delta_3(\lceil\!\circ)$ coincides with the sets of all rational subsets of *GETCHA* rationality concept ■

Proof. By definition of basic dichotomies, $\Delta_3(\lceil\!\circ)(R)(Y)$ is satisfied iff $\bar{Y}\subseteq R\lceil\!\circ Y$. The latter statement can be written as $(\forall\ y\in\bar{Y})(\forall\ x\in Y)(xRy))$ which is exactly the original formula of a *GETCHA*-rational subset ■

3.4. SYNTHESIS OF RATIONALITY CONCEPTS

A regular structure of rationality concepts comes into being when analyzing in more details the sample of these concepts presented in Example 3.2. We already know that four in the seven concepts in the list, namely, with numbers 1 to 3, and number 7, represent basic dichotomies. What are the remaining concepts? They are nothing but Boolean polynomials depending on basic dichotomies:

the Core - $\Delta_1(\circ)\wedge\Delta_2(\circ)$;
von Neumann - Morgenstern Solution - $\Delta_2(\circ)\wedge\Delta_3(\circ)$;
the Stable Core - $\Delta_1(\circ)\wedge\Delta_2(\circ)\wedge\Delta_3(\circ)$.

A natural generalization of this fact is the following "closure" idea. Let us consider basic dichotomies as independent variables ("atoms"), and the universe of choice rules - as the collection of all non-decreasing Boolean polynomials depending on these variables. In other words, we propose to extend the list of rationality concepts to all "positive" combinations of basic dichotomies. We recall that a non-decreasing Boolean polynomial can be represented in a disjunctive form, with its conjunctions (mini-terms) containing none of the negations of variables. For example, this list contains the following specific rationality concept $\Delta_1(\circ)\wedge\Delta_2(\circ)\vee\Delta_3(\circ)$: a subset of alternatives is rational iff it is either the Core or an arbitrary externally stable subset. On the contrary, a "non-positive" assertion $\Delta_1(\circ)\wedge\Delta_2(\circ)\vee\neg(\Delta_3(\circ))$ (either the Core or *not* an externally stable subset) seems to be unfitting the idea of rationality, and therefore it is not included in the family.

Definition 3.2. A *crisp dichotomous rationality concept* based on a composition law $\odot$ is a non-decreasing Boolean polynomial depending on the basic dichotomies $\Delta_1(\odot)$, $\Delta_2(\odot)$, $\Delta_3(\odot)$ ■

We denote by $\mathfrak{P}(\odot)$ the family of thus defined rationality concepts associated with a composition law $\odot$:

$$\mathfrak{P}(\odot) = \{\text{non-decreasing Boolean polynomials depending on } \Delta_1(\odot),\Delta_2(\odot),\Delta_3(\odot)\}$$

Each family of rationality concepts determines a free distributive lattice with three generators (denoted as D_{18}, see Birkhoff [11]), containing 18 rationality concepts. On Figure 3.5, Hasse diagram of $\mathfrak{P}(\odot)$ is presented.

As stated above, the family $\mathfrak{P}(\circ)$ of rationality concepts based on the Boolean composition law is rather densely occupied by known choice rules. As far as the author knows, only one in the seven mini-terms, namely, $\Delta_1(\circ)\wedge\Delta_3(\circ)$, was somehow "missed" in crisp decision-making, though the corresponding rationality concept, "separating" a rational subset from its supplement, is rather sensitive (on Figure 3.5, known rationality concepts are indicated).

As to the family $\mathfrak{P}(\ulcorner\circ)$ of rationality concepts based on the alternative composition law $\ulcorner\circ$ (see Figure 3.6), its study was only recently begun by T.Schwartz [1] who introduced *GETCHA* concept (dual external stability) in crisp decision theory. Nevertheless, there are many interesting rules in the family, and perhaps, in the near future some of them will appear in the papers on crisp choice theory. For the moment, we confine ourselves to brief examples, and avoid the detailed study of new rationality concepts for the following principal reason. Proper explanation of the behavior, and of adequate choice on the basis of these decision rules, as well as of their acknowledged neighbors in the families $\mathfrak{P}(\odot)$, will be discovered in fuzzy environment. The results of Chapters 8 to 14 are essentially different from what can be carried out of the immediate crisp considerations.

So, we dwell only on dual basic dichotomies $\Delta_1(\ulcorner\circ)$, and $\Delta_2(\ulcorner\circ)$ (for more examples, see Kitainik [7]).

"Dual *GOCHA*" concept $\Delta_1(\ulcorner\circ)$ represents the following understanding of a rational subset: $Y\in\mathcal{P}(X)$ is rational iff each of the "inside" alternatives is not worse than at least one of the "outside" alternatives; formally: $(\forall\ x\in Y)(\exists\ y\in\bar{Y})(\neg(yRx))$. In its turn, dual internal stability $\Delta_2(\ulcorner\circ)$ requires that none of the alternatives in Y is worse than all the rest alternatives in Y: $(\forall\ x\in Y)(\exists\ y\in Y)(\neg(yRx))$. We see that both the $\Delta_1(\ulcorner\circ)$, and the $\Delta_2(\ulcorner\circ)$ put rather weak restrictions on the rational subsets, in contrast with $\Delta_3(\ulcorner\circ)$ which is very rigid. (Theorem 3.1 (ii) gives an explanation of this effect).

To demonstrate the power of formal technique, developed in this chapter, let us improve known results regarding non-classical choice rules. First, we establish significant correspondences between basic dichotomies of the two families $\mathfrak{P}(\circ)$, and $\mathfrak{P}(\ulcorner\circ)$ induced by the duality of composition laws.

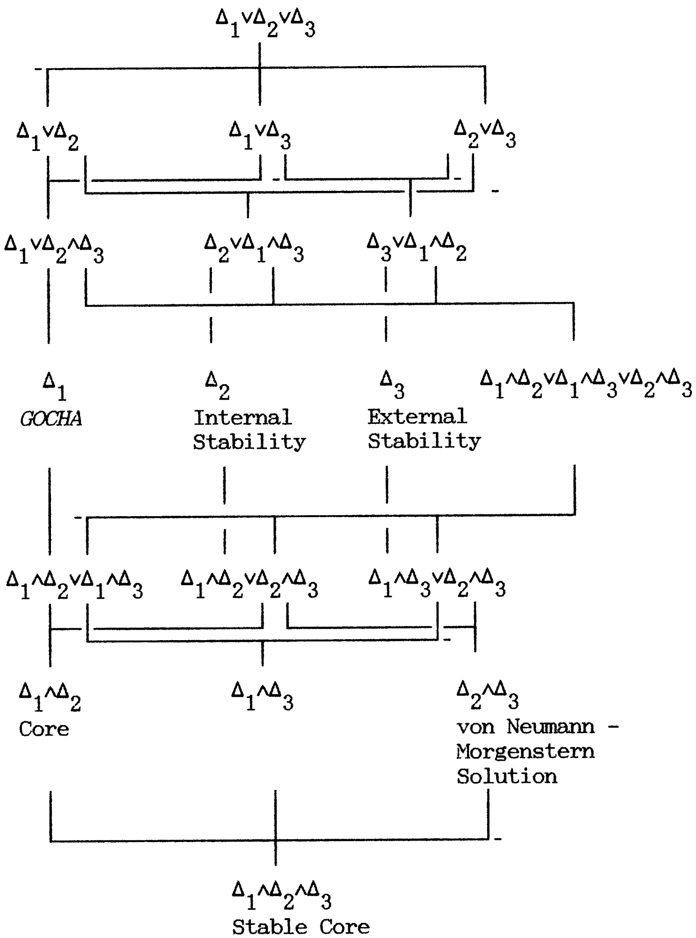


Fig. 3.5. Hasse diagram of a lattice $\mathfrak{B}(\odot)$ of dichotomous rationality concepts

$\Delta_1 \vee \Delta_2 \vee \Delta_3$

. . . .

Δ_1(?)	Δ_2 (?)	Δ_3 *GETCHA*
$\Delta_1 \wedge \Delta_2$ (?)	$\Delta_1 \wedge \Delta_3$ (?)	$\Delta_2 \wedge \Delta_3$ (?)

$\Delta_1 \wedge \Delta_2 \wedge \Delta_3$ (?)

Fig. 3.6. Fragment of a family $\mathfrak{P}(\overline{\lceil\circ})$ of rationality concepts based on alternative composition law (new choice rules are marked by (?))

Theorem 3.1 (Kitainik [7]). The following interrelations between basic dichotomies of families of crisp rationality concepts $\mathfrak{P}(\circ)$, and $\mathfrak{P}(\overline{\lceil\circ})$ take place:

(i) Equalities. For each binary relation $R\in\mathcal{P}(X^2)$, and a subset $Y\in\mathcal{P}(X)$:

$$\Delta_1(\circ)(R)(Y)=\Delta_3(\overline{\lceil\circ})(\bar{R})(\bar{Y})$$

$$\Delta_3(\circ)(R)(Y)=\Delta_1(\overline{\lceil\circ})(\bar{R})(\bar{Y})$$

(ii) Inequalities

$$\Delta_1(\circ)(R)(Y)\le\Delta_1(\overline{\lceil\circ})(R)(Y)$$

$$\Delta_2(\circ)(R)(Y)\le\Delta_2(\overline{\lceil\circ})(R)(Y)$$

$$\Delta_3(\circ)(R)(Y)\ge\Delta_3(\overline{\lceil\circ})(R)(Y) \blacksquare$$

Proof. (i) $\Delta_1(\circ)(R)(Y)=1 \Leftrightarrow R\circ\bar{Y}\subseteq\bar{Y} \Leftrightarrow \overline{R\circ\bar{Y}}\supseteq Y \Leftrightarrow \overline{\bar{R}\circ\bar{Y}}\supseteq Y \Leftrightarrow \bar{R}\overline{\lceil\circ}\bar{Y}\supseteq Y$
$\Leftrightarrow \Delta_3(\circ)(R)(Y)=1.$

The proof of the second equality is similar.

(ii) We notice that the inclusion $R\overline{\lceil\circ}Y\subseteq R\circ Y$ obviously holds for each non-empty Y. Hence,

$$\Delta_1(\circ)(R)(Y)=1 \Leftrightarrow R\circ\bar{Y}\subseteq\bar{Y} \Rightarrow R\ \overline{\lceil\circ}\ \bar{Y}\subseteq\bar{Y} \Rightarrow \Delta_1(\overline{\lceil\circ})(R)(Y)=1,$$

so that $\Delta_1(\circ)(R)(Y)\le\Delta_1(\overline{\lceil\circ})(R)(Y)$. The two remaining inequalities can be verified analogously $\blacksquare$

In that way, implementation of the composition law $\overline{\lceil\circ}$ leads to diverse dualities between the families of dichotomous rationality concepts. On the one hand, we consider reciprocal rationality concepts as members of the same name, belonging to alternative families $\mathfrak{P}(\odot)$, $\mathfrak{P}(\overline{|\circ})$. On the other hand, Theorem 3.1 establishes a "double supplement plus transposition" duality between Δ_1, and Δ_3. As to the internal stability Δ_2, dual concept $\Delta_2(\overline{|\circ})$ corresponds to a notion of *antiinvariant subset* (Kitainik and Krystev [1], see also Chapter 10):

$$Y \text{ is rational with respect to } \Delta_2(\overline{|\circ}) \Leftrightarrow R\overline{|\circ}Y \subseteq \bar{Y} \Leftrightarrow \overline{\bar{R}\circ Y \subseteq \bar{Y}} \Leftrightarrow \bar{R}\circ Y \supseteq Y,$$

so that $Y \in \mathfrak{Ainv}(\bar{R})$ (see Chapter 2 for definitions) is a subset of alternatives with each element being $\bar{R}$-dominated by at least one alternative, which is also included in Y. It should be noticed that in case when $\bar{R}$ is antireflexive, an antiinvariant subset must include more than one element. In this way, dichotomous rationality concepts introduce a great variety of dichotomies (and also of "triangulations" - see Chapter 8) based on a regular "compositional behavior" of crisp/fuzzy subsets.

Based on Theorem 3.1, one can derive a complete algorithmic description of *GOCHA*, and of *GETCHA* choice (see Example 3.4 for definitions), and establish necessary and sufficient condition of the uniqueness of *GETCHA* choice. We recall that a *bicomponent* of a digraph R on a set X is a maximal with respect to inclusion strongly complete subgraph of a transitive closure R_t. The collection $\hat{X}$ of all bicomponents of R forms a partition of X and induces a factor-graph $\hat{R}$ on $\hat{X}$ which is called *condensation* of R and is always acyclic. Canonical projection $\pi: R \longrightarrow \hat{R}$ maps each $x \in X$ into the very bicomponent $\hat{x}$, containing x; a dart $(x,y) \in R$ (xRy) is transferred into a dart $(\hat{x},\hat{y})$ (see, e.g., Swami, Thulasiraman [1]). In these terms, *GOCHA* choice turns to be nothing but *the Core of condensation*, thus generalizing the conventional notion of "winning cycle" (Aizerman [1], Aizerman and Malishevski [1]). Next, the "cross-link" between $\mathfrak{P}(\circ)$, and $\mathfrak{P}(\overline{|\circ})$, stated in Theorem 3.1 (i), enables one to apply this result to *GETCHA* rule.

Theorem 3.2 (Kitainik [7]). (i) *GOCHA* choice coincides with the subset $\pi^{-1}((\Delta_1 \wedge \Delta_2)(\circ)(\hat{R})(\hat{X}))$, that is, with the union of all non-dominated bicomponents of R.

(ii) *GETCHA* choice (generally, not unique) coincides with the set $\{\pi^{-1}(\overline{\hat{R}^{-1}\circ\hat{X}})\}$ of all non-dominating bicomponents of $\hat{R}$ ■

Proof. First, we prove that "R-invariantness" $R\circ\bar{Y} \subseteq \bar{Y}$ can be transferred to a condensation $\hat{R}$. Let us suppose that $\hat{x}$ is a bicomponent of R, and let $\hat{x}\cap\overline{Y}\neq\emptyset$. If $|\hat{x}|=1$, then $\hat{x}\subseteq\overline{Y}$; otherwise, for each $k\geq 1$, $R^k\circ(\hat{x}\cap\overline{Y})\subseteq R^k\circ(\overline{Y})\subseteq\overline{Y}$. It follows that $R_{\tau}\circ(\hat{x}\cap\overline{Y})\subseteq\overline{Y}$. On the other hand, we can easily derive from the definition of a bicomponent $\hat{x}$ that $(\forall\ x\in\hat{x})(R_{\tau}\circ\{x\}\supseteq\hat{x})$. Hence, $\hat{x}\subseteq R_{\tau}\circ(\hat{x}\cap\overline{Y})\subseteq\overline{Y}$. In other words, any R-invariant subset $\overline{Y}$ is a union of entire bicomponents. Conversely, if $\bar{\hat{Y}}$ is a subset of vertexes of condensation, satisfying the inclusion $\hat{R}\circ\bar{\hat{Y}}\subseteq\bar{\hat{Y}}$, then the faithfulness of the co-image of this inclusion, $R\circ(\pi^{-1}(\bar{\hat{Y}}))\subseteq\pi^{-1}(\bar{\hat{Y}})$, is easily implied by the definition of condensation. It follows that *GOCHA* choice with R on X is a co-image (with respect to canonical mapping) of *GOCHA* choice with $\hat{R}$ on $\hat{X}$. Since $\hat{R}$ is acyclic, its minimal with respect to inclusion *GOCHA*-rational subsets are exactly its non-dominated vertexes; the union of these subsets is the Core of $\hat{R}$. Hence, the resulting *GOCHA* choice on X is the union of all non-dominated bicomponents.

(ii) In virtue of Theorem 3.1 (i), *GETCHA*-rational subsets of a binary relation R are exhausted by the collection of supplements of *GOCHA*-rational subsets of $\overline{R}$. Owing to this duality, the proof of the desired equivalence repeats the previous one, with the only difference that the transfer of the former construction to condensation involves *non-dominating* (instead of *non-dominated*, as in (i)) subsets of alternatives ∎

An easy consequence of this theorem is a necessary and sufficient condition of uniqueness of *GETCHA* choice.

Corollary 3.1 (Kitainik [7]). *GETCHA* choice with a binary relation R is unique iff the condensation of a supplementary relation $\overline{R}$ has only one non-dominating vertex ∎

In Corollary 3.2, and in Example 3.11, we show that the latter condition of uniqueness really improves the result, formerly obtained by T.Schwartz [1].

Corollary 3.2. The class of binary relations, satisfying the condition of Corollary 3.1, contains the class of antisymmetric relations ∎

Proof. Obviously, (R is antisymmetric) $\Leftrightarrow$ ($\overline{R}$ is weakly complete). It easily follows that the condensation of $\overline{R}$ is also weakly complete.

Clearly, the only class of acyclic and weakly complete binary relations is the class of linear orderings. In such case, the smallest element of the ordering induced by $\hat{\bar{R}}$ is the only non-dominating vertex of a condensation. So, any antisymmetric binary relation satisfies the condition of Corollary 3.1, thus providing the uniqueness of *GETCHA* choice ∎

Example 3.11. With a binary relation R from Example 3.5, the bicomponents are as follows: $\hat{x}_1=\{x_1\}$, $\hat{x}_2=\{x_2,x_3\}$, $\hat{x}_3=\{x_4,x_5,x_6\}$; the condensation of R can be represented as

$$\hat{R} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

The corresponding digraph looks like this:

$$\begin{tikzcd} \hat{x}_1 \bullet \arrow[d] & \\ \hat{x}_3 \bullet & \bullet\, \hat{x}_2 \arrow[l] \end{tikzcd}$$

The Core of this digraph is $\{\hat{x}_1,\hat{x}_2\}$. According to Theorem 3.2 (i), *GOCHA* choice results in the union of the corresponding bicomponents, that is, in the subset $\{x_1,x_2,x_3\}$. As to the binary relation $\bar{R}$, its bicomponents are the two subsets $\hat{x}_1=\{x_1,x_2,x_3,x_4,x_6\}$, $\hat{x}_2=\{x_5\}$, and the condensation of $\bar{R}$ is represented by a matrix

$$\hat{\bar{R}} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

with the corresponding digraph being

$$\begin{tikzcd} \hat{x}_1 \bullet & \bullet\, \hat{x}_2 \arrow[l] \end{tikzcd}$$

It follows that $\hat{\bar{R}}$ has the only non-dominating vertex $\hat{x}_1$. Hence, *GETCHA* choice with R is the subset $\{x_1,x_2,x_3,x_4,x_6\}$ (see Theorem 3.2 (ii)). It should be noticed that the original relation R is neither antisymmetric (3 pairs of alternatives are connected in both directions) nor weakly complete (6 pairs of alternatives are not connected). Hence, the class of binary relations providing the unique *GETCHA* choice strictly contains the class of antisymmetric relations ∎

More consequences of Theorem 3.2 concerning the duality between *GOCHA* and *GETCHA* crisp choice rules can be found in (Schwartz [1], Kitainik [7]). Thus, with tournaments $\bar{R}=R^{-1}$, these two choices are identical. With antisymmetric relations $R^{-1}\subseteq\bar{R}$, *GOCHA* choice is included in *GETCHA* choice, whereas weakly complete relations provide the converse inclusion *GETCHA*$\subseteq$*GOCHA*.

Although the developed technique is a powerful feature enabling one to establish significant links between rationality concepts, and between the choice rules, we still have no idea of how to compare the results of these choices. Each of the rationality concepts has its advantages, as well as disadvantages. Say, the Core guarantees a relatively small choice; it also has a sound basis in classical choice theory (see Chapter 1). However, within a general preference domain, containing not only acyclic relations, this rule too often fails.

To work out a basis for comparative study of arbitrary rationality concepts, we go over to fuzzy versions.

Chapter 4

Fuzzy Decision Procedures

We get down to developing a more unified approach to decision-making with binary relations in fuzzy environment, keeping in mind the Problem of Preference Domain, the Problem of Choice Rules, the Problem of Efficiency, and the necessity of comparative study of choice rules (see Chapter 1). When imbedding crisp choice rules into fuzzy environment, one gains the possibility of using the enhanced scale of preferences, thus producing "soft" decision-making tools and supplementing discrete considerations with the continuity idea.

The structure of rationality concepts introduced in the previous chapter makes it easy to directly formulate fuzzy versions (for an alternative understanding of fuzzy rationality see, e.g., a paper by J.Montero [1]). However, a more adequate approach is to extend the very definition of a rationality concept in order to make it suitable for the purpose of general estimates of validity of rationality concepts (contensiveness, efficiency) as well as for concrete analyses of FR's (structure of multifold fuzzy choice, and of induced crisp choice or ranking). More precisely, we develop in more details the idea of interpretation of a decision rule as of a "decision-making machine", processing an input preference relation into a resulting choice/ranking (see Chapter 1).

In Chapters 1, 3, the influence of an input class of binary preference relations on properties and efficiency of choice rules was demonstrated.

Another noticeable factor that would be taken into consideration is an *a priori* collection of "admissible" subsets of a support, among which the resulting choice must be done. In classical choice theory, the term *environment* is sometimes used to indicate such collection (Danilov [1], Danilov and Sotskov [1]). The reasons for restricting choice domain may be very diverse. Two common examples are the requirement of a non-empty choice, which is modeled by an environment with empty subset eliminated, and limitation of the number of elements in the resulting choice (see, e.g., Aizerman and Malishevski [1], Danilov and Sotskov [1]). Motivated by these considerations, we formulate the key concept of a *Fuzzy Decision Procedure with Fuzzy Binary Relations.*

Definition 4.1 (Kitainik [10]). A *Fuzzy Decision Procedure with Fuzzy Binary Relations (FDP)* is an aggregate $P=\{X,\mathcal{R},\mathcal{E},\hat{p}\}$. The terms of this aggregate have the following meaning (see Figure 4.1):

X is initial finite set of crisp alternatives;

$\mathcal{R}\subseteq\tilde{\mathcal{P}}(X^2)$ is *preference domain* - a collection of fuzzy binary relations on X; any of these relations can be considered as an input to decision procedure;

$\mathcal{E}\subseteq\tilde{\mathcal{P}}(X)$ is *environment*, or *ranking domain* - a collection of admissible fuzzy subsets of X, considered as fuzzy "trial" rankings;

$\hat{p}\in\tilde{\mathcal{P}}(\mathcal{R}\times\mathcal{E})$ is *fuzzy rationality concept* - a fuzzy binary relation between $\mathcal{R}$, and $\mathcal{E}$ ■

4.1. FUZZY RATIONALITY CONCEPT

More canonically, a fuzzy rationality concept can be considered as a kind of fuzzy mapping from $\mathcal{R}$ into the set of all fuzzy subsets of $\mathcal{E}$,

$$p:\mathcal{R}\longrightarrow\tilde{\mathcal{P}}(\mathcal{E})\subseteq\tilde{\mathcal{P}}^{(2)}(X), \quad \mu_{p(R)}(a) \triangleq \mu_{\hat{p}}(R,a),$$

When applied to a specific *FR* R, contained in the preference domain, fuzzy rationality concept can be identified with a fuzzy level two subset p(R) of a support X. Semantically, with each *FR* $R\in\mathcal{R}$, and a trial ranking $a\in\mathcal{E}$, membership function of p(R), that is, $\mu_{p(R)}(a)$ is interpreted as

> degree of rationality of a trial ranking a, when applying procedure P to decision-making with FR R.

$P = \{X, \mathcal{R}, \mathcal{E}, \hat{p}\}$

support

preference domain
fuzzy binary relations

environment (ranking domain)
admissible fuzzy rankings of X

fuzzy rationality concept $\hat{p}$
fuzzy binary relation between $\mathcal{R}$ and $\mathcal{E}$.

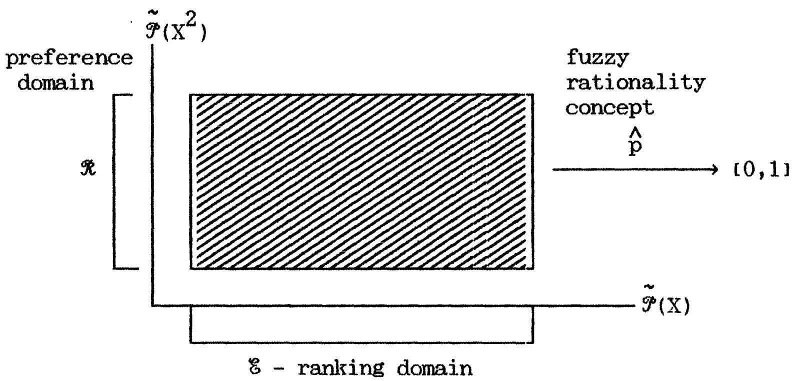


Fig. 4.1. Fuzzy Decision Procedures with Fuzzy Binary Relations

In other words, $\mu_{p(R)}(a)$ measures the degree to which a specific *a priori* ranking satisfies the notion of rationality, established by a concept p, and specified for an *FR* R, with due regard for an environment $\mathcal{E}$. In this connection, a triple $(p,R,\mathcal{E})$ will be called in the sequel a *specialization* of an *FDP* P.

4.2. MULTIFOLD FUZZY CHOICE

In crisp case, degree of rationality is measured in a two-valued yes-no scale: any subset of alternatives can be either rational or not; the collection of all rational subsets represents a preliminary basis of choice (see Section 3.1). In fuzzy case, the picture is slightly different. Naturally, we are interested in the collection of "most rational" rankings; in terms of Bellman - Zadeh principle (Bellman, Zadeh [1]), these rankings form a *maximal decision* $\mathcal{D}(p,R,\mathcal{E})$. From now on, this crisp subset of the set of all f.s. of an environment will be called a *Multifold Fuzzy Choice* (*MFC*).

Definition 4.2. A *Multifold Fuzzy Choice* (*MFC*) is as an ordinary subset of environment $\mathscr{E}$, containing all fuzzy subsets that are best fitting a given specialization:

$$\mathscr{D}(p,R,\mathscr{E})=\mu^{-1}_{p(R)}(\mu^{*}(p,R,\mathscr{E}))\cap\mathscr{E},$$

where

$$\mu^{*}(p,R,\mathscr{E})=\bigvee_{a\in\mathscr{E}}\mu_{p(R)}(a)$$

is the greatest possible value that can be attained by a specialization $(p,R,\mathscr{E})$ (see Figure 4.2) ■

So far, *MFC* is considered as the main output from a *FDP* resulting from the analysis of a specific fuzzy binary preference relation.

Note 4.1. In general, not only "maximal", but also "high enough" degree of rationality is of interest. In fact, α-cuts of decision procedures are studied below as well as maximal decisions themselves. However, the very maximal decision proves to be sufficiently large to give answers to all questions connected with consistency, and efficiency of a procedure (see Chapters 7-10) ■

4.3. FAMILIES OF FUZZY DICHOTOMOUS DECISION PROCEDURES

To work out explicit formulas for fuzzy versions of basic dichotomies, we point out that, given a "De Morgan triple" $(\vee,\wedge,\bar{\ })$ of fuzzy connectives (union, intersection, supplement) - see, e.g., Fodor [2] for formal definition of De Morgan triple, one can build both the $\vee$-$\wedge$ composition $\circ$, and the dual composition law $\overline{\circ}$ (the latter is introduced exactly as in crisp case, $\mu_{R\overline{\circ}a}(x)=\mu_{\overline{\bar{R}\circ a}}(x)$). In order to construct basic dichotomies in fuzzy domain, we need also to extend the model of inclusion (see Chapter 6 for the detailed study of fuzzy inclusions). With all these structural parameters being defined, we arrive to the following expressions for basic fuzzy dichotomies, and the definition of a lattice of *fuzzy dichotomous rationality concepts*.

Definition 4.3. (i) *Basic fuzzy dichotomies* $\Delta_1(\odot,\text{inc})$, $\Delta_2(\odot,\text{inc})$, $\Delta_3(\odot,\text{inc})$ are the three fuzzy rationality concepts:

$$\mu_{\Delta_1(\odot,\mathrm{inc})(R)}(a) = \mu_{\mathrm{inc}}(R\odot\bar{a},\bar{a})$$
$$\mu_{\Delta_2(\odot,\mathrm{inc})(R)}(a) = \mu_{\mathrm{inc}}(R\odot a,\bar{a})$$
$$\mu_{\Delta_3(\odot,\mathrm{inc})(R)}(a) = \mu_{\mathrm{inc}}(\bar{a},R\odot a)$$

(ii) The lattice $\mathfrak{P}(\odot,\mathrm{inc})$ of *fuzzy dichotomous rationality concepts* is a family

$$\mathfrak{P}(\odot,\mathrm{inc})=\{\text{non-decreasing } \vee\text{-}\wedge \text{ polynomials depending on } \Delta_1,\Delta_2,\Delta_3\}$$

(iii) a *Fuzzy Dichotomous Decision Procedure* (*FDDP*) is an *FDP* with its fuzzy rationality concept belonging to $\mathfrak{P}(\odot,\mathrm{inc})$ ∎

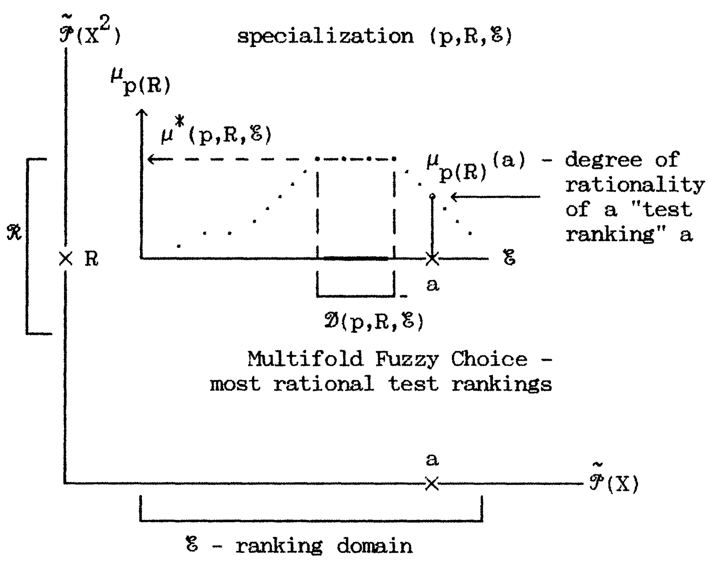


Fig. 4.2. Fuzzy rationality concept, specialization, and Multifold Fuzzy Choice

Note 4.2. (i) In the above definition, $\odot$, and inc are respectively one of the composition laws $\circ$, $\ulcorner\!\circ$, and a fuzzy inclusion.

(ii) Basic dichotomy $\Delta_2(\circ,\mathrm{inc})$ can be used only in *FDP*'s with preference domain included in the set of all antireflexive *FR*'s ($\mu_R(x,x)\equiv 0$) ∎

For brevity, in the subsequent reasoning we use the terms "$(\odot,\mathrm{inc})$-*FDDP*", "$\odot$-*FDDP*", "inc-*FDDP*".

Similarly to the crisp case, the duality between composition laws $\circ$, and $\lceil\circ$ induces a "cross-link" between basic dichotomies of the two families $\mathfrak{P}(\circ,\mathrm{inc})$, and $\mathfrak{P}(\lceil\circ,\mathrm{inc})$.

Proposition 4.1. With any fuzzy inclusion inc, satisfying the *contraposition property*

$$(\forall\ a,b\in\tilde{\mathcal{P}}(X))(\mu_{\mathrm{inc}}(a,b)=\mu_{\mathrm{inc}}(\bar{b},\bar{a})),$$

the following formulas are in force:

$$\mu_{\Delta_1(\circ,\mathrm{inc})(R)}(a)=\mu_{\Delta_3(\lceil\circ,\mathrm{inc})(\bar{R})}(\bar{a})$$
$$\mu_{\Delta_3(\circ,\mathrm{inc})(R)}(a)=\mu_{\Delta_1(\lceil\circ,\mathrm{inc})(\bar{R})}(\bar{a})\ \blacksquare$$

Proof. $\mu_{\Delta_1(\circ,\mathrm{inc})(R)}(a)=\mu_{\mathrm{inc}}(R\circ\bar{a},\bar{a})$; with respect to contraposition property of inc, this can be rewritten as $\mu_{\mathrm{inc}}(a,\overline{R\circ\bar{a}})=\mu_{\mathrm{inc}}(a,R\lceil\circ a)$, which is nothing but $\mu_{\Delta_3(\lceil\circ,\mathrm{inc})(\bar{R})}(\bar{a})$. The proof of the second equality is similar $\blacksquare$

Corollary 4.1. If an environment $\mathcal{E}$ is closed with respect to supplement, $\overset{\sqcup}{\mathcal{E}}=\mathcal{E}$, then *MFC*'s with the cross-linked basic dichotomies can be represented as

$$\mathfrak{D}(\Delta_1(\circ,\mathrm{inc}),R,\mathcal{E})=\overset{\sqcup}{\mathfrak{D}(\Delta_3(\lceil\circ,\mathrm{inc}),\bar{R},\mathcal{E})}$$

$$\mathfrak{D}(\Delta_3(\circ,\mathrm{inc}),R,\mathcal{E})=\overset{\sqcup}{\mathfrak{D}(\Delta_1(\lceil\circ,\mathrm{inc}),\bar{R},\mathcal{E})}\ \blacksquare$$

Proof. Obvious $\blacksquare$

Note 4.3. Inequalities between basic dichotomies of the two families $\mathfrak{P}_\Delta(\circ,\mathrm{inc})$, and $\mathfrak{P}_\Delta(\lceil\circ,\mathrm{inc})$ stated in Theorem 3.1 (ii) must not be true in fuzzy environment $\blacksquare$

Up to this moment, fuzzy versions of *FDDP*'s were nothing but straightforward generalizations of the corresponding crisp notions. Principal difference with classical choice theory appears at the next step. As was mentioned in Chapter 3, the existing approaches to determining the final choice on the basis of "rational subsets" (eliminating non-uniqueness) do not seem to be well-defined. For this reason, let us consider *MFC* $\mathfrak{D}(p,R,\mathcal{E})$ without any restrictions and modifications, attempting to work out new *contensiveness criteria* for its quality estimate.

Chapter 5

Contensiveness Criteria

5.1. MOTIVATIONS AND POSTULATES FOR MULTIFOLD FUZZY CHOICE

First of all, let us establish *contensiveness criteria* for *MFC* associated with a specialization $(p,R,\mathfrak{E})$. Contensiveness criteria for fuzzy decision procedures, and for fuzzy binary relations will then be derived from these original notions.

Our approach to quality estimate of *MFC* is based on two simple postulates (Kitainik [5]):

PT_1. [Egalitarianism, Minimax]. All elements (fuzzy subsets) contained in *MFC* are equal in rights. Therefore, the overall quality of the choice should be estimated according to the potentially worst element.

PT_2. [Reduction to crisp case]. The quality of *MFC* should be measured in accordance with its ability to induce confident crisp choice ∎

So, the notion of contensiveness, when applied to a *specialization* $(p,R,\mathfrak{E})$ of a *FDP*, is interpreted as certain "coherence" of the corresponding *MFC* $\mathfrak{D}(p,R,\mathfrak{E})$, resulting in a well-defined crisp choice.

To work out adequate explicit forms of coherence of *MFC*, we recall that in Decision Theory there exist two polar judgments on what can be admitted as a satisfactory result of decision-making. According to the first interpretation, the result should be represented as a *collection of "good" alternatives* which can be accepted on a final or intermediate step of decision-making. This approach is most closely related to choice theory (Aizerman, Malishevski [1]). With a more rigid approach, only *ranking* of alternatives is a well-defined ultimate end of decision-making (this viewpoint is more characteristic of utility theory, see Fishburn [2]).

Note 5.1. There exist other interpretations of the result of decision-making; thus, A.Belkin and M.Levin [1] adhere to an idea that, in extremely complicated decision problems, with hundreds of alternatives, constructing of a more or less structured preference relation can itself be viewed as a sufficiently good result ■

In case of fuzzy decision procedures, and of multifold fuzzy choice, the above considerations lead to the following specific forms of contensiveness criteria. *Dichotomous contensiveness* estimates the "ability" of *MFC* to produce crisp choice, based on the whole body of fuzzy subsets contained in *MFC*. Naturally, thus induced crisp choice is, in its turn, also multifold. A more restrictive criterion of *ranking contensiveness* is understood as a simultaneous "coherent" ranking of a support using all "trial rankings" in *MFC*. We emphasize that the approach under consideration treats the choice, and the ranking concepts in a general framework of estimating the quality of *MFC*.

In the next two sections, these two forms of contensiveness criteria are introduced on a formal basis.

5.2. DICHOTOMOUSNESS AND δ-CONTENSIVENESS OF MULTIFOLD FUZZY CHOICE, PROCEDURES, AND RELATIONS

Let us discuss more precisely the above postulates of the contensiveness of *MFC*. It is clear that the worst possible case in the process of analysis of a specific *FR* R using a *FDP* P occurs when the corresponding *MFC* $\mathfrak{D}(p,R,\mathfrak{S})$ contains at least one constant $\alpha\cdot 1$. Indeed, the presence of such constant in *MFC* makes the original crisp alternatives indistinguishable with respect to this specific "constant trial ranking", which means, in virtue of PT_1, incontensiveness of the *MFC* itself.

Furthermore, if one considers the preference scale I, in the spirit of theory of measurement, as a continuous "scale of intervals" (Pfanzagl

[1]), that is, the scale in which the notion of distance between preference values is meaningful, then the above "incontensiveness effect" is caused by any "constant-like sequence", that is, the sequence of f.s.' belonging to *MFC* and converging to a constant $\alpha\cdot 1$.

The above argumentation motivates the following quantitative expression of contensiveness. Let us define *dichotomousness* of a f.s. $a\in\tilde{\mathcal{P}}(X)$ as a "maximal gap" in the graph of a membership function (Kitainik [2]):

$$\delta(a)=\bigvee_{\substack{\alpha,\beta\in[\mathring{\wedge}a,\mathring{\vee}a]\\ a_\alpha=a_\beta}}(\beta-\alpha)$$

Semantically, $\delta(a)$ measures the greatest possible "polarization" of preferences, resulting from the crisp choice, based on the trial ranking associated with a f.s. This polarization is achieved through the choice of an α-cut a_α with α belonging to arbitrary "$\delta(a)$-gap"; clearly, thus defined choice does not depend on a specific value α inside a maximal gap.

In the spirit of the Minimax Postulate PT_1, dichotomousness of an ensemble $\mathfrak{E}\subseteq\tilde{\mathcal{P}}(X)$ should be defined as

$$\delta(\mathfrak{E}) = \bigwedge_{a\in\mathfrak{E}} \delta(a)$$

Definition 5.1 (Kitainik [2,5]). (i) A specialization $(p,R,\mathfrak{E})$ is called *δ-contensive* (*dichotomously contensive, DC*) iff

$$\delta(p,R,\mathfrak{E})=\delta(\mathfrak{D}(p,R,\mathfrak{E}))>0$$

Otherwise, it is called *δ-trivial* (*dichotomously trivial, DT*).

(ii) *Induced crisp choice* with a *dichotomously contensive* specialization $(p,R,\mathfrak{E})$ is the collection of all α-cuts a_α of f.s.' included in $\mathfrak{D}(p,R,\mathfrak{E})$ and satisfying the inequality:

$$\mathring{\wedge}a\big|_{a_\alpha}-\mathring{\vee}a\big|_{\overline{a_\alpha}}\geq\delta(p,R,\mathfrak{E})$$

(iii) Let $\mathfrak{P}$ be a class of fuzzy decision procedures, possessing the same preference domain $\mathfrak{R}$, and the environment $\mathfrak{E}$. A procedure $P\in\mathfrak{P}$ (resp, an *FR* $R\in\mathfrak{R}$) is called δ-contensive (DC) in a domain $\mathfrak{R}$ (resp, in a class $\mathfrak{P}$) in an environment $\mathfrak{E}$ iff there exists a δ-contensive specialization $(p,R,\mathfrak{E})\in\mathfrak{P}\times\mathfrak{R}\times\{\mathfrak{E}\}$, containing this procedure (resp, *FR*). Otherwise, a procedure (resp, an *FR*) is called δ-trivial (DT) in the corresponding domain (resp, class), and environment ∎

The idea of dichotomous contensiveness is illustrated on Figure 5.1.

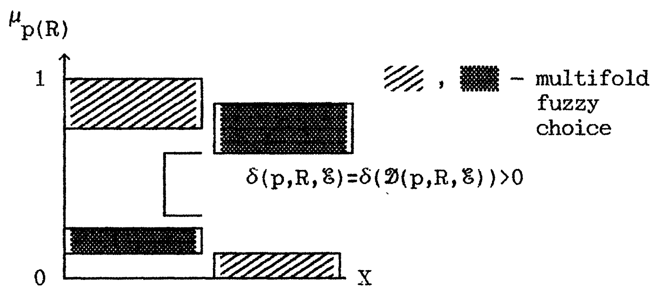


Fig. 5.1. Dichotomous contensiveness of *MFC*

So, in case of dichotomous contensiveness every f.s. in *MFC* stands for itself, producing its own "partial" choice due to maximal polarization of preferences, with the only restriction that the levels of resolution for these partial choices exceed, or are equal to the dichotomousness of the whole *MFC*; the induced multifold crisp choice is just the collection of all partial choices.

On the face of it, this collection of dichotomies is too easy. But, as a matter of fact, in case of dichotomously contensive decision procedures, the induced crisp choice proves to be pretty regular: in some meaning, it is almost always unique (see Chapter 8).

It may also seem that the introduced definition of the induced crisp choice does not fit the Maximal Decision Principle: why not choose maximal decisions for all f.s.' included in *MFC*? A somewhat surprising is that the two approaches, namely, the Maximal Decision Principle, and the Induced Crisp Choice with respect to maximal polarization of preferences, do coincide in the DC case (see Chapter 8).

Another remark concerns definitions of dichotomous contensiveness of a specific *FDP* in a class of *FR*'s or, dually, of a given *FR* in a class of *FDP*'s. In fact, both definitions are nothing but "∃-convolutions" of the primary definition of contensiveness of a specialization. Isn't this convolution too weak? More precisely, why consider a procedure as a contensive one according to just a "demonstration of contensiveness" for one or another fuzzy binary relation in a preference domain? Why not define contensiveness on the basis of a "∀-convolution"? The proper answer to these questions will be given in Chapters 8-12. Two principal motivations are as follows:

1) even the weak "existence"-criterion sets an insuperable obstacle for the great majority of conventional choice rules in the families $\mathfrak{P}(\odot,\mathrm{inc})$ (they turn to be incontensive in *any* preference domain);
2) in order to measure of the body of contensive specializations with respect to a given *FDP* another estimate, namely, the *efficiency* criterion is used in this book (see Chapter 12).

5.3. RANKING ALTERNATIVES USING MULTIFOLD FUZZY CHOICE

5.3.1. Connected Spectrum, Coherence, and Interval Ranking

In contrast with the above notion of dichotomous contensiveness based on the collection of partial choices according to all elements of *MFC*, in this section we introduce a more rigid concept of coherence of *MFC*. This type of coherence provides a possibility of crisp ranking of a support with fuzzy interval estimate of preferences of the members of a ranking.

From the very beginning, the problem of constructing of a "coherent" ranking seems to have an easy solution. Since every f.s. $a\in\mathfrak{D}(p,R,\mathfrak{E})$ is a "trial ranking", thus producing a "partial" crisp semi-ordering O_a on X, $xO_ay \Leftrightarrow \mu_a(x)>\mu_a(y)$, a natural idea of the "global" ranking consistent with *MFC* is associated with the strongest semi-ordering on X, which is in agreement with all O_a's. In lattice terms, this overall ranking O can be calculated as

$$O=\wedge([U,1]\cap\mathcal{O}),$$

where U is the exact upper bound of all O_a's in the lattice $\mathcal{P}(X^2)$ of all crisp relations on X, 1 - the unit relation (upper universal bound of $\mathcal{P}(X^2)$), $\mathcal{O}$ - the set of all semi-orderings on X. It should be noticed that, U itself must not belong to $\mathcal{O}$ (the union of semi-orderings can well be neither transitive nor antisymmetric). Unfortunately, this construction fails for the reasons that have been already discussed at the beginning of the previous section: the existence of constants (even less, of constant-like sequences, "equalizing" any two classes of O) in *MFC* depreciates the information contained in the resulting ordering.

Therefore, we suggest (see Kitainik [5,9,10]) a more rigid ranking criterion requiring that the whole *MFC* should be contained in some interval fuzzy ranking with distinct bounds of intervals. Let us suppose that $[a,b]$ is any interval in $\tilde{\mathcal{P}}(X)$. The union of intervals and/or points $\bigcup_{x\in X}[\mu_a(x),\mu_b(x)]\subseteq I$ is denoted by $\lambda_{[a,b]}$. *Connected spectrum* (C-*spectrum*) of $[a,b]$

$$\Lambda_{[a,b]}=\{J^i=[\alpha^i,\beta^i]\}$$

is defined as the set of all connected components of λ. Let $\xi=\{X^i\}\in\Pi_X$ be any element of the lattice Π_X of all crisp partitions of X (Birkhoff [1]; maximal element in Π_X is denoted by 1_Π, number of members of a partition - by $|\xi|$, order in Π_X - by $\prec$). For any interval $[a,b]\subseteq\tilde{\mathcal{P}}(X)$, an *induced partition* $\xi_{[a,b]}$, determined by $\Lambda_{[a,b]}$, can be naturally introduced:

$$\xi_{[a,b]}=\{X^i\},\ X^i=\{x\in X\,|\,[\mu_a(x),\mu_b(x)]\subseteq J^i\}$$

Note 5.2. From the point of view of three-valued logic, components J^i can be classified into the following three classes, determining rough structure of preferences:

dominating components ($J^i\subseteq]1/2,1]$),
neutral components ($1/2\in J^i$), and
dominated components ($J^i\subseteq[0,1/2[$) ■

In that way, an interval $[a,b]$ in the lattice $\tilde{\mathcal{P}}(X)$ of all f.s.' of X gives rise to an interval partition

$$\Lambda_{[a,b]}/\xi_{[a,b]}=\sum J^i/X^i$$

To translate this construction to any ordinary set of f.s.' of X, $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$, let us consider the minimal interval $[1_{\mathfrak{X}},t_{\mathfrak{X}}]\subseteq\tilde{\mathcal{P}}(X)$ containing $\mathfrak{X}$. Membership functions of the bounds $1_{\mathfrak{X}}$, $t_{\mathfrak{X}}$ of this interval can easily be calculated: $\mu_{1_{\mathfrak{X}}}(x)=\bigwedge_{a\in\mathfrak{X}}\mu_a(x)$; $\mu_{t_{\mathfrak{X}}}(x)=\bigvee_{a\in\mathfrak{X}}\mu_a(x)$.

In these terms, C-*spectrum* of $\mathfrak{X}$, and *induced* by $\mathfrak{X}$ *partition* are defined respectively as $\Lambda_{\mathfrak{X}}=\Lambda_{[1_{\mathfrak{X}},t_{\mathfrak{X}}]}$; $\xi_{\mathfrak{X}}=\xi_{[1_{\mathfrak{X}},t_{\mathfrak{X}}]}$.

In order to characterize induced partition in the terms of *coherence*, let us define, for any crisp subset $Z\subseteq X$, an interval $\mathfrak{X}(Z)$ in $\tilde{\mathcal{P}}(X)$:

$$\mathfrak{X}(Z)=[\bigwedge_{z\in Z}\mu_{1_{\mathfrak{X}}}(z),\ \bigvee_{z\in Z}\mu_{t_{\mathfrak{X}}}(z)]$$

A pair $(\mathfrak{X},\xi)$, with $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$ being an ordinary set of f.s.' of X, $\xi\in\Pi_X$ being a partition of X, is called *coherent* iff

$$(\forall\ i,j,\ 1\le i<j\le|\xi|)(\mathfrak{X}(X^i)\cap\mathfrak{X}(X^j)=\emptyset)$$

In other words, the explicit formulation of coherence requires that all f.s.' in $\mathfrak{X}$ should uniformly "distinguish" all members of partition.

Let us denote $\mathcal{C}(\mathfrak{X})=\{\xi\in\Pi_X \mid (\mathfrak{X},\xi)$ is coherent$\}$. Basic properties of thus defined coherence are established in the following three statements.

Proposition 5.1. With any $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$, $\mathcal{C}(\mathfrak{X})=\mathcal{C}([1_{\mathfrak{X}},t_{\mathfrak{X}}])$ ∎

Proof. Obvious ($1_{\mathfrak{X}}=1_{[1_{\mathfrak{X}},t_{\mathfrak{X}}]}$, $t_{\mathfrak{X}}=t_{[1_{\mathfrak{X}},t_{\mathfrak{X}}]}$) ∎

Proposition 5.2. The mapping $\mathcal{P}(\tilde{\mathcal{P}}(X)\setminus\{\emptyset\})\longrightarrow\mathcal{P}(\Pi_X)$, $\mathfrak{X}\longrightarrow\mathcal{C}(\mathfrak{X})$ is antitone with respect to set inclusion ∎

Proof. $\mathfrak{X}\subseteq\mathfrak{E}\Rightarrow(\forall\ Z\subseteq X)(\mathfrak{X}(Z)\subseteq\mathfrak{E}(Z))\Rightarrow(\forall\ \xi=\{X^i\}\in\mathcal{C}(\mathfrak{E}))(\forall\ i\neq j)(\mathfrak{X}(X^i)\cap\mathfrak{X}(X^j)=\emptyset)$
$\Rightarrow(\forall\ \xi)(\xi\in\mathcal{C}(\mathfrak{E})\Rightarrow\xi\in\mathcal{C}(\mathfrak{X}))$ ∎

A significant observation is that the structure of coherence can be completely discovered using only the induced partition $\xi_{\mathfrak{X}}$.

Proposition 5.3 (Kitainik [5]). With any $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$, $\mathcal{C}(\mathfrak{X})$ coincides with an interval $[\xi_{\mathfrak{X}},1_{\Pi}]$ in Π_X ∎

Proof is sub-divided into three statements:

(a) $\mathcal{C}(\mathfrak{X})$ is dual ideal in Π_X;

(b) $\xi_{\mathfrak{X}}$ belongs to $\mathcal{C}(\mathfrak{X})$;

(c) $\xi_{\mathfrak{X}}$ is the smallest element in $\mathcal{C}(\mathfrak{X})$.

(a) is obvious.

(b) Let $i\neq m$ be two integers; by definition,

$$(\forall\ x\in X^i,\ y\in X^m)(\mathfrak{X}(\{x\})\cap\mathfrak{X}(\{y\})\subseteq J^i\cap J^m=\emptyset\ ).$$

Now, let us suppose that the two intervals, $\mathfrak{X}(X^i_{\mathfrak{X}})$, and $\mathfrak{X}(X^m_{\mathfrak{X}})$, are not disjoint, $\mathfrak{X}(X^i_{\mathfrak{X}})\cap\mathfrak{X}(X^m_{\mathfrak{X}})\neq\emptyset$, and let β be any common point of these intervals. In such case, there exist sequences

$$\{x^i_q\}\subseteq X^i_{\mathfrak{X}},\ \{x^m_q\}\subseteq X^m_{\mathfrak{X}},\quad \{a_q\},\{b_q\}\subseteq\mathfrak{X}:$$

$$\mu_{a_q}(x^i_q)\xrightarrow[q\to\infty]{}\beta,\ \mu_{b_q}(x^m_q)\xrightarrow[q\to\infty]{}\beta.$$

Since both $X^i_{\mathfrak{X}}$, and $X^m_{\mathfrak{X}}$ are finite, there also exist $x^i\in X^i_{\mathfrak{X}}$, $x^m\in X^m_{\mathfrak{X}}$, and subsequences $\{a'_q\},\{b'_q\}\subseteq\mathfrak{X}$:

$$\mu_{a'_q}(x^i) \xrightarrow[q\to\infty]{} \beta, \qquad \mu_{b'_q}(x^m) \xrightarrow[q\to\infty]{} \beta;$$

this means, however, that $\beta\in\mathfrak{X}(\{x^i\})\cap\mathfrak{X}(\{x^m\})\neq\emptyset$; the contradiction proves that $\mathfrak{X}(\{X^i_{\mathfrak{X}}\})\cap\mathfrak{X}(\{X^m_{\mathfrak{X}}\})=\emptyset$, and $\xi_{\mathfrak{X}}\in\mathfrak{C}(\mathfrak{X})$.

(c) Let $\xi=\{X^i\}\in\mathfrak{C}(\mathfrak{X})$; it suffices to prove that

$$(\forall\ i,j)(X^i\cap X^j_{\mathfrak{X}}\neq\emptyset \Rightarrow X^j_{\mathfrak{X}}\subseteq X^i).$$

Let us suppose that $x\in X^i\cap X^j_{\mathfrak{X}}; y\in X^j_{\mathfrak{X}}\backslash X^i$, and let us consider the chain $\{x=x^0,x^1,x^2,\ldots,x^{m-1},y=x^m\}$, satisfying the property

$$[\mu_{1_{\mathfrak{X}}}(x^i),\mu_{t_{\mathfrak{X}}}(x^i)]\cap[\mu_{1_{\mathfrak{X}}}(x^{i+1}),\mu_{t_{\mathfrak{X}}}(x^{i+1})]\neq\emptyset$$

(the existence of such chain is guaranteed by definition of $\xi_{\mathfrak{X}}$). For some $i=0,\ldots,m-1$, we have both $x^i\in X^i$, and $x^{i+1}\notin X^i$; let us assume that $x^{i+1}\in X^q$; according to construction of the chain, $\mathfrak{X}(\{x^i\})\cap\mathfrak{X}(\{x^{i+1}\})\neq\emptyset$, so that $\mathfrak{X}(X^i)\cap\mathfrak{X}(X^q)\neq\emptyset$, in contradiction with the assumption $\xi\in\mathfrak{C}(\mathfrak{X})$; hence, $X^j_{\mathfrak{X}}\subseteq X^i$ ■

5.3.2. ρ-Contensiveness

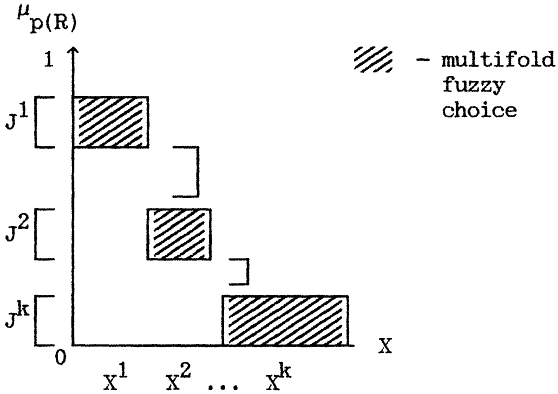
To introduce the notion of ranking contensiveness, nothing else is left but to apply the above construction to *MFC*. For brevity, in Definition 5.2 $\mathfrak{D}$ stands for $\mathfrak{D}(p,R,\mathfrak{E})$.

Definition 5.2 (Kitainik [5]). (i) A specialization $(p,R,\mathfrak{E})$ is called *ρ-contensive* (*ranking contensive, RC*) iff the interval $\mathfrak{C}(\mathfrak{D})$ in the lattice Π_X is non-trivial, that is, iff $\mathfrak{D}$ induces a non-trivial partition $\xi^o=\xi_{\mathfrak{D}}\neq 1_{\Pi}$, which is called *induced crisp ranking*. Otherwise, a specialization is called *ranking trivial* (*RT*).

(ii) Let $\mathfrak{P}$ be a class of fuzzy decision procedures, possessing the same preference domain $\mathfrak{R}$, and environment $\mathfrak{E}$. A procedure $P\in\mathfrak{P}$ (resp., an *FR* $R\in\mathfrak{R}$) is called *ρ-contensive* (RC) in a domain $\mathfrak{R}$ (resp., in a class $\mathfrak{P}$) in an environment $\mathfrak{E}$ iff there exists *ρ-contensive* specialization $(p,R,\mathfrak{E})\in\mathfrak{P}\times\mathfrak{R}\times\{\mathfrak{E}\}$, containing this procedure (resp., *FR*). Otherwise, a procedure (resp, an *FR*) is called *ρ-trivial* (RT) in the corresponding domain (resp., class) and environment ■

From now on, the set of disjoint intervals $\Lambda^{\circ}=\Lambda_{\mathfrak{D}}$ willbe called *connected spectrum* (*C-spectrum*) of a specialization $(p,R,\mathfrak{C})$. A partition $\xi^{\circ}=\xi_{\mathfrak{D}}$ will be referred to as *canonical partition* (*C-partition*), associated with a specialization $(p,R,\mathfrak{C})$. A pair $(\Lambda^{\circ},\xi^{\circ})$ is treated as *interval fuzzy ranking* $\Sigma J^i/X^i$ induced by a specialization, and called *canonical ranking* (*C-ranking*).

The idea of ranking contensiveness is illustrated on Figure 5.2.



$\xi^{\circ}=\{X^1,\dots,X^k\}$ - canonical partition

$\Lambda^{\circ}=\{J^1,\dots,J^k\}$ - connected spectrum

Fig. 5.2. Ranking contensiveness of *MFC*

Chapter 6

Fuzzy Inclusions

The problem of constructing adequate fuzzy inclusions goes far out of the bounds of decision-making. Fuzzy inclusion represents one of the basic tools in Fuzzy Set Theory. An intuitive motivation of the necessity of a comprehensive study of this topic is a need for "soft" models of inclusion between fuzzy subsets (and, maybe, even between crisp subsets), satisfying the continuity requirements. Historically, the first fuzzy inclusion $\subseteq$ was proposed by L.Zadeh [1] as a straightforward extension of ordinary inclusion:

$$a \subseteq b \Leftrightarrow (\forall\ x \in X)(\mu_a(x) \leq \mu_b(x))$$

A common reason for criticism with regard to this model is the "lack of softness". Let us consider, for example, the following three fuzzy subsets of a support $X=\{x_1,x_2,x_3\}$: $a=0.5/x_1+0.51/x_2+0.001/x_3$; $b=1/x_1+0.5/x_2+0.5/x_3$; $c=1/x_1+1/x_2+0/x_3=\chi_{\{x_1,x_2\}}$. On an intuitive level, it seems clear that a is "much more included" in b than in c. Nevertheless, no pair in $\{a,b,c\}$ is $\subseteq$-comparable. In a more formal manner, we are not satisfied by the fact that $\subseteq$ is a *crisp* binary relation between *fuzzy* subsets; of course, any crisp binary relation on $\tilde{\mathcal{P}}(X)$ is invariably discontinuous as a mapping from $\tilde{\mathcal{P}}(X)\times\tilde{\mathcal{P}}(X)$ into a unit interval I.

In addition to "naive" motivations of this type, the problem of fuzzy inclusions is connected with one of the most challenging problems in fuzzy science, namely, with the problem of simultaneous setting of Fuzzy Logic, and of Fuzzy Set Theory. In the first place, the latter problem includes investigating of the correspondence between logical connectives, on the one hand, and set operations, on the other hand. This type of problems requires a more solid argumentation, based on an axiomatic study.

With this understanding, one can note that, semantically, fuzzy inclusions are close to fuzzy implications (in a paper by E.Ruspini [1], an example of successful application of this idea to *AI* problems is demonstrated). For this reason, we begin the research of fuzzy inclusions with the comparative study of axiomatic bases underlying fuzzy implication as a connective in Fuzzy Logic, and fuzzy inclusion as a "connective" in Fuzzy Set theory (Section 6.1). This study provides with a more precise motivation of axiomatics of fuzzy inclusions, which is presented in Section 6.2 together with primary properties of fuzzy inclusions satisfying this axiomatics. In Section 6.3, a Representation Theorem is proved. This result introduces a powerful tool for the subsequent research of a family of fuzzy inclusions on the basis of the behavior of the corresponding "representing functions". In Section 6.4, algebraic properties of fuzzy inclusions (reflexivity, antisymmetry, transitivity) are studied in the terms of representations. A somewhat surprising is that well-defined algebraic properties of a fuzzy inclusion turn to be incompatible with its "softness", that is, with continuity. In particular, the classical fuzzy inclusion $\subseteq$ proves to be the "most algebraic" among all other inclusions; therefore, its discontinuity is not a chance property, but the result of the above incompatibility. In Section 6.5, general structure of binary operations on a family of fuzzy inclusions is established. The choice of distinguished fuzzy inclusions for the main purpose of this book (construction of decision procedures with binary relations) is motivated in Section 6.6, devoted to characterization of marked elements of the collection of fuzzy inclusions. In Section 6.7, new families of transitive fuzzy inclusions are constructed, and a considerable number of known models is studied from the viewpoint of the proposed axiomatics.

6.1. MOTIVATIONS. FUZZY INCLUSION AND FUZZY IMPLICATION

Fuzzy implications are studied in numerous papers. In general, three principal approaches can be found in thess studies:

- "pure" axiomatic study (Baldwin, Pilsworth [1]);
- introducing and theoretical research of diverse models motivated by classical and fuzzy logic (Bandler, Kohout [1]; Di Nola, Ventre [1]; Fodor [3,5]; Ovchinnikov, Roubens [1,2]; Yager [1]);
- building and verifying empiric models (Kiszka, Kochanska, Sliwinska [1]).

In a "t-fuzzy logic", derived from a t-norm t, two types of fuzzy implication are used most frequently: residuated implication (pseudo-complement), and Kleene-Dienes implication (see, e.g., papers by D.Dubois and H.Prade [2,4]; S.Gottwald [1], S.Gottwald and W.Pedrycz [1]).

In contrast with fuzzy implication, the concept of fuzzy inclusion is studied rather sporadically. In (Dubois,Prade [1]), classification of models of "ε-inclusions" was proposed. In (Fodor [1], Wygralak [1]), special classes of fuzzy inclusions were studied.

Let us compare axiomatic bases, that is, general "external" requirements to fuzzy inclusion, and to fuzzy implication. We recall that a t-norm is defined (Schweizer, Sclar [1]) as a non-decreasing function t on I×I with its values in I. So far, t determines a structure of an associative and commutative monoid with zero and unit elements 0, and 1; formally: for all $\alpha,\beta,\gamma\in I$, $t(\alpha,t(\beta,\gamma))=t(t(\alpha,\beta),\gamma)$; $t(\alpha,\beta)=t(\beta,\alpha)$; $t(\alpha,0)=0$; $t(\alpha,1)=1$. Set $I_0=]0,1[$, and let t be a t-norm. Following S.Gottwald [1], we denote the associated conjunction (resp., intersection), that is, t itself, by $\wedge_t$ ($\wedge$ stands for $\wedge_t$ with t=min); residuated implication is denoted by $\xrightarrow[t]{}$. For brevity, α^k is used instead of $\alpha\wedge_t\alpha\wedge_t\ldots\wedge_t\alpha$ (with k α's). Let $\mathcal{R}_k(t)$, $\mathcal{I}(t)$ be respectively the sets of all "k-nilpotents" ($\alpha^k=0$), and "idempotents" ($\alpha^2=1$) of the corresponding monoid. In this section, arbitrary implication, inclusion, and negation (supplement) are denoted respectively by $\rightarrow$, $\subseteq$ [1], and $\neg$. The latter is considered as an antitone automorphism (not necessarily involution) of I.

We start axiomatic considerations with postulating elementary

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[1] Starting with Section 6.2, fuzzy inclusion will be denoted by inc.

properties of fuzzy implications/inclusions. Let us denote by $\mathfrak{Z}\subseteq\tilde{\mathcal{F}}(I^2)$ a sublattice of all fuzzy binary relations $\longrightarrow$ on I satisfying two basic conditions:

M (Monotonicity). $\alpha\longrightarrow\beta$ is antitone/monotone with respect to α/β.

H (Heritage). Restriction of $\longrightarrow$ on $\{0,1\}$ is a conventional crisp implication.

Let $\underset{\mathbb{O}}{\longrightarrow}$, and $\underset{\mathbb{I}}{\longrightarrow}$ be the smallest, and the greatest elements of $\mathfrak{Z}$:

$$\alpha\underset{\mathbb{O}}{\longrightarrow}\beta=\begin{cases}1, & \alpha=0\vee\beta=1\\ 0, & \text{otherwise}\end{cases}\qquad \alpha\underset{\mathbb{I}}{\longrightarrow}\beta=\begin{cases}0, & (\alpha,\beta)=(1,0)\\ 1, & \text{otherwise}\end{cases}$$

Next, let $\mathfrak{Z}(P)$ be the truth set of a predicate P on $\mathfrak{Z}$.

Axiomatic approaches to fuzzy implication, and inclusion can be interpreted as constructing these operations on the basis of their links with "primary" connectives. Thus, the Modus Ponens MP_t $(\alpha\underset{t}{\wedge}(\alpha\longrightarrow\beta))\longrightarrow\beta$, the Modus Tollens MT_t $((\alpha\longrightarrow\beta)\underset{t}{\wedge}(\neg\beta))\longrightarrow\neg\alpha$, the Contraposition $C_\neg$ $(\alpha\longrightarrow\beta)\longleftrightarrow(\neg\beta\longrightarrow\neg\alpha)$, when treated as axiomatic requirements, links $\longrightarrow$, on the one hand, with $\underset{t}{\wedge}$, and $\neg$, on the other hand. In (Baldwin, Pilsworth [1]), families $\mathfrak{Z}(MP_t\&MT_t)$, and $\mathfrak{Z}(MP_t\&C_\neg)$ of implications were studied, with t, and $\neg$ being respectively the min-norm $\wedge$, and the classical negation $\neg\alpha=1-\alpha$. In what follows, we reject MT_t, which is implied by $MP_t\&C_\neg$. Turning to inclusions, we see that C_t has a one-to-one analog in ordinary set theory: $A\subseteq B \Longleftrightarrow \neg B\subseteq\neg A$. However, both the *MP*, and the *MT* do not seem to have distinct set-theoretical interpretations. Instead, the link between inclusion and intersection in ordinary set theory is given by the formula $A\subseteq(B\cap C) \Longleftrightarrow (A\subseteq B)\&(A\subseteq C)$; we refer to (Giles [1]) for motivations. In a "t-fuzzy case", this formula is naturally transformed into the equivalence $\alpha\subseteq(\beta\underset{t}{\wedge}\gamma) \Longleftrightarrow (\alpha\subseteq\beta)\underset{t}{\wedge}(\alpha\subseteq\gamma)$ which is nothing but Distributivity D_t of inclusion (Distributivity can also be considered as inference property: $\alpha\longrightarrow(\beta\underset{t}{\wedge}\gamma) \Longleftrightarrow (\alpha\longrightarrow\beta)\underset{t}{\wedge}(\alpha\longrightarrow\gamma)$).

Is it possible to combine the above-mentioned models? In other words, do there exist *unified fuzzy "implications-inclusions"* possessing well-defined inference properties MP_t, $C_\neg$, and satisfying the set-theoretical requirement D_t? In addition, can the combined models be continuous, or at least semi-continuous?

The answer essentially depends on the specific properties of t-norms. Below in this section, we use conventional fuzzy versions of Modus Ponens, Contraposition, and Distributivity properties, changing meta-implication, and meta-equivalence respectively for ordinary inequality, and equality:

MP_t: $(\alpha \underset{t}{\wedge} (\alpha \longrightarrow \beta)) \leq \beta$

$C_\neg$: $(\alpha \longrightarrow \beta) = ((\neg\beta) \longrightarrow (\neg\alpha))$

D_t: $\alpha \longrightarrow (\beta \underset{t}{\wedge} \gamma) = ((\alpha \longrightarrow \beta) \underset{t}{\wedge} (\alpha \longrightarrow \gamma))$

This means that, though the properties concern *fuzzy* objects, and connectives, they are formulated as *crisp* assertions.

Proposition 6.1 (basic considerations).

(i) Residuated implication $\underset{t}{\rightarrow}$ is the *exact upper bound* of $\mathfrak{Z}(MP_t)$ in a lattice $\mathfrak{Z}$. A necessary and sufficient condition under which $\underset{t}{\rightarrow}$ is the *greatest element* of $\mathfrak{Z}(MP_t)$ is lower continuity of $t(\alpha,\beta)$ with respect to β in each point $(\alpha, \alpha \underset{t}{\rightarrow} \beta)$.

(ii) With any $\longrightarrow \in \mathfrak{Z}$, $\alpha \in I$, $0 \longrightarrow \alpha = \alpha \longrightarrow 1 = 1$.

(iii) With any $\longrightarrow \in \mathfrak{Z}(D_t)$, $\mathcal{I}(t)$ is a left $\longrightarrow$-ideal, that is,

$$(\forall\ \alpha \in I,\ \beta \in \mathcal{I}(t))(\alpha \longrightarrow \beta \in \mathcal{I}(t)).$$

(iv) With any $\longrightarrow \in \mathfrak{Z}(D_t \& C_\neg)$, $\alpha \in I$, both $\alpha \longrightarrow 0$, and $1 \longrightarrow \alpha$ belong to $\mathcal{I}(t)$ ■

Proof. (i) By definition, $\alpha \underset{t}{\rightarrow} \beta = \sup\{\gamma \mid \alpha \underset{t}{\wedge} \gamma \leq \beta\}$. Obviously, $\underset{t}{\rightarrow} \in \mathfrak{Z}$. Next, the expression for MP_t shows that, for each $\longrightarrow \in \mathfrak{Z}(MP_t)$, $\alpha \longrightarrow \beta$ belongs to the above set of γ's, and hence, it cannot exceed $\alpha \underset{t}{\rightarrow} \beta$. Therefore, $\underset{t}{\rightarrow}$ is an upper bound of $\mathfrak{Z}(MP_t)$. To show that it is the exact upper bound, let us define, with an arbitrarily small $\varepsilon > 0$, an implication $\underset{\varepsilon}{\rightarrow}$:

$$\alpha \underset{\varepsilon}{\rightarrow} \beta = \begin{cases} 1, & \alpha \leq \beta \\ \alpha \underset{t}{\rightarrow} \beta \underset{1}{\wedge} \bar{\varepsilon}, & \text{otherwise} \end{cases}$$

with 1 being Lukasiewicz' norm. Clearly, $\underset{\varepsilon}{\rightarrow} \in \mathfrak{Z}$; also, since $\alpha \underset{\varepsilon}{\rightarrow} \beta$ is strictly smaller than $\alpha \underset{t}{\rightarrow} \beta$ if $\alpha \underset{t}{\rightarrow} \beta \neq 0$, $\underset{\varepsilon}{\rightarrow}$ satisfies MP_t. In addition, the distance between $\underset{t}{\rightarrow}$ and $\underset{\varepsilon}{\rightarrow}$ is sufficiently small: $|\alpha \underset{t}{\rightarrow} \beta - \alpha \underset{\varepsilon}{\rightarrow} \beta| \leq \varepsilon$ for all α, β. So, in any neighborhood of $\underset{t}{\rightarrow}$, there exists an implication from $\mathfrak{Z}(MP_t)$; it follows that $\underset{t}{\rightarrow}$ is an exact upper bound of $\mathfrak{Z}(MP_t)$. Furthermore, the additional continuity condition is equivalent to the requirement that, for any α, β, $\alpha \xrightarrow[t]{} \beta$ belongs to the set $\{\gamma | \alpha \underset{t}{\wedge} \gamma \leq \beta\}$; in its turn, this is equivalent to the assertion $\xrightarrow[t]{} \in MP_t$; in such case, $\xrightarrow[t]{}$ is the greatest element of $\mathfrak{Z}(MP_t)$.

(ii) Follows from the Heritage ($0 \longrightarrow 0 = 1 \longrightarrow 1 = 1$) and the Monotonicity ($0 \longrightarrow \alpha \geq 0 \longrightarrow 0$, $\alpha \longrightarrow 1 \geq 1 \longrightarrow 1$) conditions.

(iii) Let us suppose that $\beta \in \mathcal{I}(t)$, that is, $\beta^2 = \beta$. According to D_t, $\alpha \longrightarrow \beta = \alpha \longrightarrow \beta^2 = (\alpha \longrightarrow \beta)^2$, so that $\alpha \longrightarrow \beta \in \mathcal{I}(t)$.

(iv) In virtue of D_t, $(\alpha \longrightarrow 0)^2 = \alpha \longrightarrow (0^2) = \alpha \longrightarrow 0$; hence, $\alpha \longrightarrow 0 \in \mathcal{I}(t)$. Next, due to $C_\neg$, $1 \longrightarrow \alpha = \neg\alpha \longrightarrow 0 \in \mathcal{I}(t)$ ∎

Proposition 6.1 (i) causes a natural question: can residuated implications themselves possess other properties in the list, except for the MP_t? So, let us study the following dual problem: describe all those t-norms t for which residuated implications satisfy Modus Ponens, Contraposition, Distributivity axioms, and their combinations.

Let P be a "t-dependent" predicate, and P_t - its specialization for a t-norm t. Set $\mathfrak{R}(P) = \{\text{t-norms} | \xrightarrow[t]{} \in \mathfrak{Z}(P_t)\}$. Let $\mathfrak{C}$ be the $(\bigcup, \bigcap)$-lattice of all "upper closed" subsets of I, containing $\{0,1\}$ ($J \in \mathfrak{C}$ iff $\{0,1\} \subseteq J$, and, for any $K \subseteq J$, $\sup(K) \in J$; in particular, all such J's are complete $\{\vee, \wedge\}$-sublattices of I, but not vice versa). Let us denote by $\mathfrak{C}_1$ the sublattice of $\mathfrak{C}$, containing all closed subsets of I. With $J \in \mathfrak{C}$, $\alpha \in I$, we denote $\alpha_* = \sup([0,\alpha] \bigcap J)$, $\alpha^* = \inf\{]\alpha,1] \bigcap J)$; clearly, if $\alpha \in J$, then $\alpha_* = \alpha$.

Theorem 6.1 (t-norms with distributive residuated implications).

(i) $\mathfrak{R}(D)$ is a $(\vee, \wedge)$-sublatice of the lattice of all fuzzy subsets of I^2. The mapping $\iota : t \longrightarrow \mathcal{I}(t)$ establishes an isomorphism between $\mathfrak{R}(D)$ and $\mathfrak{C}$. The co-image $t = \iota^{-1}(J) \in \mathfrak{R}(D)$ of a subset $J \in \mathfrak{C}$, and the corresponding residuated implication can be calculated using the following formulas

$$\alpha^2 \equiv \alpha_*;$$

$$\alpha \underset{t}{\wedge} \beta = \begin{cases} \alpha \wedge \beta, & \alpha \vee \beta = 1 \\ (\alpha \wedge \beta)^2, & \text{otherwise} \end{cases}$$

$$\alpha \xrightarrow[t]{} \beta = \begin{cases} 1, & \alpha \leq \beta \\ \beta^*, & \text{otherwise} \end{cases}$$

(ii) (Modus Ponens plus Distributivity). $\mathfrak{R}(MP\&D) = \{\wedge\}$.

(iii) (Contraposition plus Distributivity). $\mathfrak{R}(\mathcal{C}\&\mathcal{D})=\{\mathbb{O}\}$.

(iv) (all properties). $\mathfrak{R}(\mathcal{MF}\&\mathcal{C}\&\mathcal{D})$ is empty.

(v) (Distributivity plus continuity). The only lower semi-continuous $t\in\mathfrak{R}(\mathcal{D})$ is the min-norm $\wedge$. The subset of all upper semi-continuous t-norms, belonging to $\mathfrak{R}(\mathcal{D})$, is exactly $\iota^{-1}(\mathbb{C}_1)$ ■

Proof. (i) Set $J=\mathcal{I}(t)$. We will prove that t is uniquely determined by J, and that any $J\in\mathbb{C}$ determines a t-norm $\iota^{-1}(J)\in\mathfrak{R}(\mathcal{D})$.

Since residuated implication is reflexive, $\alpha\underset{t}{\to}\alpha\equiv1$, we obtain from $\mathcal{D}_t$ that $\alpha\underset{t}{\to}\alpha^k=(\alpha\underset{t}{\to}\alpha)^k\equiv1$; it follows from the definition of residuated implication that $(\forall\ k)(\forall\ \gamma\in[0,1[)(\alpha\underset{t}{\wedge}\gamma\le\alpha^k)$. Using monotonicity of t, with $\gamma\in[\alpha,1[$, k=3, we come to $\alpha^2\le\alpha\underset{t}{\wedge}\gamma\le\alpha^3$; hence, $\alpha^2=\alpha^3$, and $(\forall\ \gamma\in[\alpha,1[)(\alpha\underset{t}{\wedge}\gamma=\alpha^2)$. It follows that:

(1) $(\forall\ k>1)(\alpha^k=\alpha^2)$;

(2) $\alpha^2\in\mathcal{I}(t)$;

(3) t can be reconstructed from the function α^2:

$$\alpha\wedge\beta=(\alpha\underset{t}{\wedge}\beta)\underset{t}{\wedge}(\alpha\vee\beta)=(\alpha\wedge\beta)^2.$$

Let us suppose that $\alpha^2<\alpha$; then, for each $\beta\in[\alpha^2,\alpha]$, $\beta^2\in[\alpha^2,\alpha^4]=\{\alpha^2\}$, so that $\alpha^2\big|_{[\alpha^2,\alpha]}$=const. Therefore, for each Archimedean element α, $\alpha^2=(\alpha^2)^2$ is the closest to the bottom non-Archimedean element. In other words, $\alpha^2=\alpha_*$.

Now, let us calculate $\alpha\underset{t}{\to}\beta$. Clearly, for all residuated t-norms, $\alpha\le\beta$ implies $\alpha\underset{t}{\to}\beta=1$, and $\alpha>\beta$ implies $\alpha\underset{t}{\to}\beta\ge\beta$. Hence, using (3), we obtain

$$\alpha\underset{t}{\to}\beta=\sup\{\gamma\,|\,\alpha\underset{t}{\wedge}\gamma\le\beta\}=\sup\{\gamma\,|\,(\alpha\underset{t}{\wedge}\gamma)^2\le\beta\}=\sup\{\gamma\in[\beta,1]\,|\,\gamma^2\le\beta\}$$

If $\gamma\in[\beta,\beta^*[$, then, $\gamma\ge\gamma^2\ge\beta^2$, and $\gamma^2\in\mathcal{I}(t)$, so that $\gamma^2\le\beta$, in which case, as is already proved, $\gamma^2=\beta^2\le\beta$. It follows that $\alpha\underset{t}{\to}\beta\ge\beta^*$. In case when $\gamma>\beta^*$, there exist two possibilities. If $\beta=\beta^*$, then $\beta=\inf\{]\beta,1]\cap\mathcal{I}(t)\}$; it follows that some $\eta\in]\beta,\gamma]$ also belongs to $\mathcal{I}(t)$, so that $\gamma^2\ge\eta^2=\eta>\beta^*=\beta$. If $\beta<\beta^*$, then $[\beta^*,\gamma]\cap\mathcal{I}(t)\ne\emptyset$, so that η can be selected in the latter intersection, and again, $\gamma^2\ge\eta^2=\eta\ge\beta^*>\beta$. So, no γ in $]\beta^*,1]$ satisfies the condition $\gamma^2\le\beta$;

hence, $\alpha \xrightarrow[t]{} \beta \le \beta^*$, which, together with the already proved $\alpha \xrightarrow[t]{} \beta \ge \beta^*$, yields $\alpha \xrightarrow[t]{} \beta = \beta^*$.

Since $J = \mathcal{J}(t)$ is obviously upper closed for any t-norm, we proved that ι is a monomorphism from $\mathfrak{R}(D)$ into $\mathbb{C}$, and that $\iota^{-1}(J)$ can be reconstructed using the given expressions. Now, let us prove that ι is an epimorphism. Let $J \in \mathbb{C}$; we define a function $\iota'(J): I^2 \longrightarrow I$,

$$\iota'(J)(\alpha,\beta) = \begin{cases} \alpha\wedge\beta, & \alpha\vee\beta = 1 \\ (\alpha\wedge\beta)_*, & \text{otherwise} \end{cases}$$

The only thing needed to prove is that $\iota'(J)$ is a t-norm, and that $\mathcal{J}(\iota'(J)) = J$. Commutativity, and boundary properties are straightforward (say, $\iota'(J)(0,\alpha) = (\alpha\wedge 0)_* = 0_* = 0$, since $0 \in J$). Associativity is implied by an obvious equality $((\alpha\wedge\beta)_*\wedge\gamma)_* = (\alpha\wedge(\beta\wedge\gamma)_*)_* = \alpha_*\wedge\beta_*\wedge\gamma_*$.
Since for $\alpha \notin J$, $\alpha_* \ne \alpha$, the equality $\mathcal{J}(\iota'(J)) = J$ is also proved.

(ii) If t possesses at least one Archimedean element γ, select $\alpha,\beta \in]\gamma^2, \gamma[$, $\alpha > \beta$. Then $\alpha \xrightarrow[t]{} \beta = \beta^* > \gamma > \alpha > \beta$, and $\alpha \wedge_t \alpha \xrightarrow[t]{} \beta = \alpha > \beta$, so that $\xrightarrow[t]{}$ does not satisfy MP_t. The min-norm $\wedge$ obviously satisfies $MP_\wedge$ $(\alpha\wedge\alpha \xrightarrow[\wedge]{} \beta = \alpha\wedge\beta \le \beta)$.

(iii) In virtue of (i), $(\forall\ \alpha>\beta)(\alpha \xrightarrow[t]{} \beta = \beta^*)$, so that $C_\neg$ can be written in the form $(\forall\ \alpha>\beta)(\beta^* = (\neg\alpha)^*)$, which is equivalent to $\beta^* \equiv 0^* = 1$, that is, $t = \mathbb{O}$.

(iv) $\mathfrak{R}(MP\&C\&D) = \mathfrak{R}(MP\&D) \cap \mathfrak{R}(C\&D) = \{\wedge\} \cap \{\mathbb{O}\} = \emptyset$.

(v) Clearly, $\wedge$ is continuous, and hence, lower semi-continuous. For each $t \in \mathfrak{R}(D_t) \setminus \{\wedge\}$, there exists at least one Archimedean element $\alpha^2 < \alpha$. It follows from (i) that $t|_{[(\alpha,\alpha),(\alpha,1)[} \equiv \alpha^2 < t(\alpha,1) = \alpha$, so that t cannot be lower semi-continuous.

Next, each $t \in \mathfrak{R}(D_t)$ is obviously upper semi-continuous in each point (α,β) with $\alpha\wedge\beta \in \mathcal{J}(t)$, and continuous in each (α,β) with $\alpha\wedge\beta \in]\gamma^2, \gamma[$ for an arbitrary $\gamma \notin \mathcal{J}(t)$. So, the only points under question are the limits of decreasing sequences from $\mathcal{J}(t)$. If $\{\alpha_n\}$ is the sequence of this type, and $\alpha = \lim \alpha_n \notin \mathcal{J}(t)$, then $\alpha^2 < \alpha = \lim \alpha_n = \lim \alpha_n^2$, so that t, together with the function α^2, is discontinuous in (α,α) (and in each (α,β), $\beta \ge \alpha$). On the opposite, if $\alpha = \lim \alpha_n \in \mathcal{J}(t)$, then $\alpha^2 = \alpha = \lim \alpha_n = \lim \alpha_n^2$, and α^2 is upper semi-continuous in the point α, which, together with the assertion (3) of (i), implies upper semi-continuity in all (α,β) with $\beta \ge \alpha$. So, the necessary and sufficient condition of upper semi-continuity of t is the lower closeness of $\mathcal{T}(t)$, that is, ordinary closeness, since $\mathcal{T}(t)$ is always upper closed ∎

Theorem 6.1 demonstrates the exclusive role of min-norm in the setting of fuzzy implications, and inclusions. In fact, this theorem provides new characterizations of the min-norm:

(1) it is the only continuous (even more - the only lower semi-continuous) t-norm possessing distributive residuated implication;

(2) it is the only t-norm (irrespective of continuity), for which residuated implication, in addition to distributivity, satisfies the Modus Ponens principle.

For an arbitrary, not necessarily residuated implication/inclusion, a surprising contrast between Lukasiewicz' norm, and the min-norm can be found.

Proposition 6.2 (Contraposition and Distributivity of implications for Archimedean t-norms).
For any $\longrightarrow\in\mathfrak{I}(C_{\neg}\&D_{t})$, with an upper semi-continuous Archimedean t-norm t ($\mathcal{T}(t)=\{0,1\}$), the following assertions hold:

(i) either $(\forall\alpha\in I)((\alpha\longrightarrow 0=0)\&(1\longrightarrow\alpha=0))$ or $(\forall\ \alpha\in I,\ \beta\neq 0)(\alpha\longrightarrow\beta=1)$.

(ii) If t has zero divisors ($\mathcal{R}_2(t)\neq\{0\}$) then $(\forall\alpha\in I)((\alpha\longrightarrow 0=0)\&(1\longrightarrow\alpha=0))$, and $\mathcal{R}_2(t)$ is left $\longrightarrow$-ideal.

(iii) $\longrightarrow$ can not be continuous ∎

Proof. (i) Let $\alpha_*=\inf\{\alpha^k\}$. Let us prove that $\alpha_*=0$ for all $\alpha\in I_0$. Indeed, let us assume that $\alpha>0$, and $\alpha_*>0$. Since t is Archimedean, $\alpha_*^2<\alpha_*$. On the other hand, upper semi-continuity of t implies $\alpha_*^2=\inf\{\alpha^{2k}\}=\inf\{\alpha^k\}=\alpha_*$ - a contradiction. In virtue of Proposition 6.1 (iv), $(\forall\ \alpha\in I)(\alpha\longrightarrow 0\in\mathcal{T}(t)=\{0,1\})$. Let us suppose that, for some α, $\alpha\longrightarrow 0=1$. Owing to $C_{\neg}$, this implies $1\longrightarrow\neg\alpha=1$, and, applying D_t, we come to $(\forall\ k)((1\longrightarrow(\neg\alpha)^k)=(1\longrightarrow\neg\alpha)^k=1)$. Since $\inf(\{\alpha^k\})=0$, Monotonicity implies that $(\forall\ \beta\neq 0)(1\longrightarrow\beta=1)$, and $(\forall\ \alpha\in I)(\forall\ \beta\neq 0)$ $(1\longrightarrow\beta=1)$.

(ii) The proof of the first assertion repeats (i), with the only difference that the existence of non-trivial zero divisors implies $\bigcup\mathcal{R}_k(t)=I_0$. Hence, $\alpha\longrightarrow 0=1\Rightarrow 1\longrightarrow\neg\alpha=1\Rightarrow 1\longrightarrow(\neg\alpha)^k\equiv 1$; taking k with $(\neg\alpha)^k=0$, we arrive to $1\longrightarrow 0=1$, which is a contradiction with the Heritage condition $(1\longrightarrow 0=0)$. Next, $\beta\in\mathcal{R}_2(t)\Leftrightarrow\beta^2=0\Rightarrow(\forall\ \alpha\in I)((\alpha\longrightarrow\beta)^2=\alpha\longrightarrow\beta^2=\alpha\longrightarrow 0=0\Rightarrow\alpha\longrightarrow\beta\in\mathcal{R}_2(t)$.

(iii) If $\rightarrow$ is continuous, then $((0\rightarrow 0=1)\&(1\rightarrow 0=0))$ implies that a subset $\{\alpha\rightarrow 0|\alpha\in I\}$ covers $[0,1]$, which is a contradiction with the condition $\alpha\rightarrow 0\in\mathcal{I}(t)=\{0,1\}$ ■

So far, the existence of implications, satisfying both the Contraposition, and the Distributivity properties is incompatible with the continuity of an Archimedean t-norm.

Theorem 6.2. For each upper semi-continuous Archimedean t-norm t with zero divisors, which is, in addition, lower semi-continuous in the point $(1,1)\in I^2$, $\mathfrak{I}(C_{\neg}\&D_t)=\{\underset{\mathbb{O}}{\rightarrow}\}$ ■

Proof. Let us suppose that, for some $\alpha\neq 0$, $\beta\neq 1$, $\alpha\rightarrow\beta>0$. Since t is lower semi-continuous in the point $(1,1)$, $\sup(\alpha^2)=1$. Let us select γ with $\gamma^2\geq\beta$; in virtue of D_t, $(\alpha\rightarrow\gamma)^2=\alpha\rightarrow\gamma^2\geq\alpha\rightarrow\beta>0$, so that $\alpha\rightarrow\gamma\notin\mathcal{R}_2(t)$, which is in contradiction with Proposition 6.2 (ii). So, $\alpha\rightarrow\beta\equiv 0$ for all $\alpha\neq 0$, $\beta\neq 1$. Once more referring to Proposition 6.2 (ii), we find $(\forall\alpha\in I)((\alpha\rightarrow 0=0)\&(1\rightarrow\alpha=0))$, so that $\rightarrow=\underset{\mathbb{O}}{\rightarrow}$ ■

So, no implication based on a t-norm of this class, including Lukasiewicz' norm [2], can satisfy both the Distributivity and the Contraposition conditions. In the remaining sections of this chapter, we will show that the picture is completely different with the min-norm, which gives rise to a continual family of fuzzy inclusions satisfying all the above-listed properties.

The results of this section show that, if one wishes to preserve the Contraposition and the Distributivity properties, classical De Morgan triple $(\vee,\wedge,\bar{\ })$ can constitute a good basis for an axiomatic study of fuzzy inclusions. In what follows, inclusions of this type will be studied in details. Even more, we expand the domain of fuzzy inclusions from the set I of "truth values" to the set of all fuzzy subsets of a given support X. This requires for an additional *Heritage* axiom, as will be shown in the next section.

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[2] Continuous t-norms of this class are equivalent to Lukasiewicz' norm - see, e.g. (Ovchinnikov and Roubens [1]).

6.2. AXIOMATICS

From now on, we use the notation inc for fuzzy inclusion, which is considered as a fuzzy binary relation on the set of all fuzzy subsets of X, $inc \in \mathcal{P}(\tilde{\mathcal{P}}(X) \times \tilde{\mathcal{P}}(X))$. In Table 6.1, four axioms of fuzzy inclusion are listed. All of them, except for the *symmetry axiom* A_3, were motivated in the previous section. As to the symmetry, it is related to the extension of fuzzy inclusion from the original set I of truth/preference values on a set of all f.s. of X. This axiom reflects the natural requirement that the degree of inclusion of fuzzy subsets a, and b does not depend on the permutation of the points of the support X thus being determined by the collection of the pairs of values $(\mu_a(x), \mu_b(x))$ for all $x \in X$.

Definition 6.1. A *Fuzzy inclusion* (*FI*) on a set X is defined as a fuzzy binary relation inc on the set of all fuzzy subsets of X, $inc \in \tilde{\mathcal{P}}(\tilde{\mathcal{P}}(X) \times \tilde{\mathcal{P}}(X))$, satisfying the four axioms listed in Table 6.1. The set of all fuzzy inclusions is denoted by $\mathcal{I}nc$.

Table 6.1

Axioms for fuzzy inclusions (Kitainik [1,2])

Crisp prototype $\subseteq$	Fuzzy version inc	Motivation
$A \subseteq B \Leftrightarrow \bar{B} \subseteq \bar{A}$	A_1. $\mu_{inc}(a,b) = \mu_{inc}(\bar{b},\bar{a})$	contraposition
$A \subseteq B \cap C \Leftrightarrow$ $\Leftrightarrow (A \subseteq B) \& (A \subseteq C)$	A_2. $\mu_{inc}(a, b \wedge c) =$ $\mu_{inc}(a,b) \wedge \mu_{inc}(a,c)$	distributivity
$A \subseteq B \Leftrightarrow \sigma A \subseteq \sigma B$	A_3. $\mu_{inc}(a,b) = \mu_{inc}(\sigma a, \sigma b)$	symmetry
	A_4. $inc\|_{\mathcal{P}^2(X)} = \subseteq$	heritage

In this table: A,B,C are crisp subsets of X;
a,b,c are fuzzy subsets of X;
σ is an automorphism of X.

All results presented in the remaining part of this Chapter were obtained in (Kitainik [1,2,3]).

6.3. REPRESENTATION THEOREM

First, let us establish elementary properties of *FI*s.

Proposition 6.3. (Basic properties of fuzzy inclusions).

(i) $\mu_{inc}(a\vee b,c)=\mu_{inc}(a,c)\wedge\mu_{inc}(b,c)$

(ii) $\mu_{inc}(a,b)$ is $\subseteq$-antitone with respect to a and $\subseteq$-monotone with respect to b.

(iii) $(\forall\alpha,\beta\in I)(\forall x,y\in X)(x\neq y \Rightarrow \mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{y\}}})=1)$.

(iv) $\mu_{inc}(a,b)=\bigwedge_{x\in X}\mu_{inc}(\mu_a(x)\chi_{\{x\}},\overline{\overline{\mu_b(x)}\chi_{\{x\}}})$

(v) $\mathcal{I}nc$ is closed with respect to α-cuts:

$$(\forall\alpha\in I)(\forall inc\in\mathcal{I}nc)(inc_{\alpha},inc_{>\alpha}\in\mathcal{I}nc)$$

Proof. (i) In virtue of A_1,

$$\mu_{inc}(a\vee b,c)=\mu_{inc}(\bar{c},\overline{a\vee b})=\mu_{inc}(\bar{c},\bar{a}\wedge\bar{b})\,.$$

Applying A_2, we come to $\mu_{inc}(a\vee b,c)=\mu_{inc}(\bar{c},\bar{a})\wedge\mu_{inc}(\bar{c},\bar{b})$; using A_1 once more, we obtain $\mu_{inc}(a\vee b,c)=\mu_{inc}(a,c)\wedge\mu_{inc}(b,c)$.

(ii) Let us suppose that a_1, $a_2\in\tilde{\mathcal{P}}(X)$, $a_1\subseteq a_2$; then $a_1\vee a_2=a_2$, and, owing to (i), for each $b\in\tilde{\mathcal{P}}(X)$, we have

$$\mu_{inc}(a_1\vee a_2,b)=\mu_{inc}(a_2,b)=\mu_{inc}(a_1,b)\wedge\mu_{inc}(a_2,b)\le\mu_{inc}(a_1,b),$$

so that inc is antitone with respect to the first argument. Monotonicity with respect to the second argument can be proved in the same way.

(iii)
$$\mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{y\}}})\ge\mu_{inc}(\chi_{\{x\}},\overline{\beta\chi_{\{y\}}})=\mu_{inc}(\chi_{\{x\}},\chi_{\overline{\{y\}}}\vee\bar{\beta}\chi_{\{y\}})$$
$$\ge\mu_{inc}(\chi_{\{x\}},\chi_{\overline{\{y\}}})=1;$$

in this chain, both inequalities are implied by (ii), since $\alpha\chi_{\{x\}}\subseteq\chi_{\{x\}}$, and $\chi_{\overline{\{y\}}}\subseteq\chi_{\overline{\{y\}}}\vee\bar{\beta}\chi_{\{y\}}$; the second equality follows from the condition of the assertion, and from A_4, because $x\neq y$ implies $\{x\}\subseteq\overline{\{y\}}$.

(v) Let us write arguments a, and b, in the form $a=\bigvee_{x\in X}\mu_a(x)\chi_{\{x\}}$, $b=\bigwedge_{y\in X}\overline{\overline{\mu_b(y)}\chi_{\{y\}}}$ (the first equality is obvious, the second equality is implied by the first one by changing b for $\bar{\bar{b}}$). Consequently applying (i), and A_2, we arrive to $\mu_{inc}(a,b)=\bigwedge_{x,y\in X}\mu_{inc}(\mu_a(x)\chi_{\{x\}},\overline{\mu_b(y)\chi_{\{y\}}})$. According to (iii), any of the right-hand side members of this equality with $x\neq y$ can be changed for 1; eliminating these members under the symbol $\wedge$, we come to

$$\mu_{inc}(a,b)=\bigwedge_{x\in X}\mu_{inc}(\mu_a(x)\chi_{\{x\}},\overline{\mu_b(x)\chi_{\{x\}}})$$

(vi) Obvious: all the three equalities A_1-A_3 are preserved by α-cuts ∎

We wish to underline the fact that, being introduced through an external description, that is, on a purely axiomatic basis, *FI*'s *automatically* fall under a conventional logical formula:

(a is included in b) $\Leftrightarrow$ $(\forall\, x\in X)(\mu_a(x)$ is included in $\mu_b(x))$;

furthermore, the arbitrariness quantor $(\forall\, x\in X)$ in a right-hand side predicate, is invariably modeled as $\bigwedge_{x\in X}$.

The main tool in the study of *FI*'s is the following *representation theorem*.

Theorem 6.3. $\mathcal{I}nc$ is isomorphic to the following set Φ of functions, defined on the triangle $T=\{\alpha,\beta\in I^2\,|\,\beta\leq\alpha\}$

$$\Phi=\{\varphi:T\longrightarrow I\,|\,\varphi \text{ is non-increasing with respect to both arguments; } \varphi(0,0)=\varphi(1,0)=1;\ \varphi(1,1)=0\}$$

A canonical bijection $\nu:\mathcal{I}nc:\longrightarrow\Phi$ is defined by the following formulas:

$$\varphi_{inc}(\alpha,\beta)=\mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}});$$

$$\mu_{inc_\varphi}(a,b)=\bigwedge_{x\in X}\varphi(\mu_a(x)\vee\overline{\mu_b(x)},\mu_a(x)\wedge\overline{\mu_b(x)})\ ∎$$

where the notation $\varphi_{inc}=\nu(inc)$, $inc_\varphi=\nu^{-1}(\varphi)$ will be used in the remaining of this chapter.

Proof. For arbitrary $inc\in\mathcal{I}nc$, $\alpha,\beta\in I$, $x\in X$, let us define

$$\varphi_{inc}(\alpha,\beta)=\mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}}).$$

We will prove that the correspondence $\nu:inc\longrightarrow\varphi$ determines a bijection between $\mathcal{I}nc$, and Φ.

(1) The correctness of definition of φ_{inc} is immediately implied by a symmetry axiom A_3.

(2) Let us prove that $\varphi_{inc}\in\Phi$. Anti-monotonicity of φ_{inc} with respect to the first argument is implied by Proposition 6.3 (ii), with respect to the second argument - by Proposition 6.3 (ii), and by A_1 ($\varphi_{inc}(\alpha,\beta)=\mu_{inc}(\beta\chi_{\{x\}},\overline{\alpha\chi_{\{x\}}})$). Boundary values of φ_{inc} in the angles of T can be calculated according to A_4: $\varphi_{inc}(0,0)=\mu_{inc}(0,1)=1$, etc.

(3) Injectivity. In virtue of Proposition 6.3 (v),

$$\mu_{inc}(a,b)=\bigwedge_{x\in X}\mu_{inc}(\mu_a(x)\chi_{\{x\}},\overline{\mu_b(x)\chi_{\{x\}}})$$

Hence, a fuzzy inclusion inc is uniquely determined by the set $\{\mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}})\}_{\alpha,\beta\in I}$. If $\beta>\alpha$, then $(\beta,\alpha)\in T$, and, owing to A_1, $\mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}})=\mu_{inc}(\beta\chi_{\{x\}},\overline{\alpha\chi_{\{x\}}})$. Therefore, inc is also determined by the set $\{\mu_{inc}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}})\}_{\alpha,\beta\in T}$.

(4) Surjectivity. Let $\varphi\in\Phi$. We define a fuzzy binary relation $inc_\varphi\in\tilde{\mathcal{P}}(\tilde{\mathcal{P}}(X)\times\tilde{\mathcal{P}}(X))$ by the formula

$$\mu_{inc_\varphi}(a,b)=\bigwedge_{x\in X}\varphi(\mu_a(x)\vee\overline{\mu_b(x)},\mu_a(x)\wedge\overline{\mu_b(x)})$$

In what follows, we prove that $inc_\varphi\in\mathcal{I}nc$.

(A) Axioms A_1, and A_3 for inc_φ are obviously satisfied (transposition of members under the symbol $\bigwedge_{x\in X}$, and under the symbols $\vee$, $\wedge$ inside the brackets).

(B) Verification of the axiom A_4. Let $a,b\in\mathcal{P}(X)$, $\mu_a=\chi_A$, $\mu_b=\chi_B$. If $A\subseteq B$, then $\chi_A\wedge\chi_{\bar{B}}=0$; therefore, all members under the symbol $\bigwedge_{x\in X}$ in the formula are equal either to $\varphi(0,0)=1$ or to $\varphi(1,0)=1$; it follows that $\mu_{inc_\varphi}(\chi_A,\chi_B)=1$. Conversely, let us suppose that $y\in A\setminus B=A\cap\bar{B}$; in such case, $\chi_A(y)=\overline{\chi_B(y)}=1$; hence, the corresponding member in the formula, together with $\mu_{inc_\varphi}(\chi_A,\chi_B)$, is equal to $\varphi(1,1)=0$.

(C) Verification of the axiom A_2 is more difficult. Set

$$\psi_1(x)=\varphi(\mu_a(x)\vee\mu_b(x)\vee\overline{\mu_c(x)},\mu_a(x)\vee\mu_b(x)\wedge\overline{\mu_c(x)});$$

$$\psi_2(x)=\varphi(\mu_a(x)\vee\overline{\mu_c(x)},\mu_a(x)\wedge\overline{\mu_c(x)});$$

$$\psi_3(x)=\varphi(\mu_b(x)\vee\overline{\mu_c(x)},\mu_b(x)\wedge\overline{\mu_c(x)}).$$

Let us prove that $\psi_1(x)\equiv\psi_2(x)\wedge\psi_3(x)$. This will immediately yield the validity of A_2, since

$$\mu_{inc_\varphi}(a\vee b,c)=\bigwedge_{x\in X}\psi_1(x),\ \text{and}$$

$$\mu_{inc_\varphi}(a,c)\wedge\mu_{inc_\varphi}(b,c)=(\bigwedge_{x\in X}\psi_2(x))\wedge(\bigwedge_{x\in X}\psi_3(x))$$

$$=\bigwedge_{x\in X}(\psi_2(x)\wedge\psi_3(x)).$$

Most easily, the equality $\psi_1=\psi_2\wedge\psi_3$ can be tested using the "table of values" for $\alpha=\mu_a(x)$, $\beta=\mu_b(x)$, $\gamma=\overline{\mu_c(x)}$. Thus, with $\alpha\ge\beta\ge\gamma$, $\psi_1(x)=\varphi(\alpha,\gamma)$; $\psi_2(x)=\varphi(\alpha,\gamma)$; $\psi_3(x)=\varphi(\beta,\gamma)$; since φ is non-increasing with respect to the first argument, $\varphi(\alpha,\gamma)\le\varphi(\beta,\gamma)$; it follows that $\psi_2(x)\wedge\psi_3(x)=\varphi(\alpha,\gamma)\wedge\varphi(\beta,\gamma)=\varphi(\alpha,\gamma)=\psi_1(x)$. With $\alpha\ge\gamma\ge\beta$, we have: $\psi_1(x)=\varphi(\alpha,\gamma)$; $\psi_2(x)=\varphi(\alpha,\gamma)$; $\psi_3(x)=\varphi(\gamma,\beta)$; since both $\alpha\ge\gamma$, and $\beta\ge\gamma$ are satisfied, non-increasing of φ with respect to both arguments implies $\varphi(\alpha,\gamma)\le\varphi(\gamma,\beta)$; the remaining equalities are the same as in the previous case. The other four possibilities of inequalities between α, β, and γ can be studied in the same way.

So, all the four axioms for inc_φ are fulfilled; hence, $inc_\varphi\in\mathcal{I}nc$.

(5) Reciprocality. Let $(\alpha,\beta)\in T$. In such case,

$$\varphi_{inc_\varphi}(\alpha,\beta)=\mu_{inc_\varphi}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}})=\bigwedge_{y\in X}\varphi(\alpha\chi_{\{x\}}(y)\vee\overline{\beta\chi_{\{x\}}}(y),\alpha\chi_{\{x\}}(y)\wedge\overline{\beta\chi_{\{x\}}}(y))$$

$$=(\bigwedge_{y\in\overline{\{x\}}}\varphi(0,0))\wedge\varphi(\alpha,\beta)=\varphi(\alpha,\beta).$$

Therefore, $\varphi_{inc_\varphi}=\varphi$, so that $\nu(inc_\varphi)=\varphi$, and the mapping $\varphi\longrightarrow inc_\varphi$ is nothing but the inverse to ν, that is, ν^{-1} ∎

We mark out two significant, though obvious consequences from this theorem.

Corollary 6.1. Φ is a universal representation: it does not depend on the cardinality $n=|X|$ ∎

Corollary 6.2. A *FI* inc is continuous (as a mapping $\tilde{\mathcal{P}}^2(X)\longrightarrow I$) iff φ_{inc} is a continuous function on T.

6.4. PROPERTIES OF FUZZY INCLUSIONS

Representation Theorem motivates a "geometric" style in the study of the properties of an *FI* inc according to the properties of the corresponding function φ_{inc}. Most of these properties prove to be determined by the behavior of φ_{inc} on the following subtriangles of the *representation domain* T (see Figure 6.1):

$$T_1'=\{\alpha,\beta\in I^2 \mid \beta\le\alpha\le\bar{\alpha}\le\bar{\beta}\};\quad T_2'=\{\alpha,\beta\in I^2 \mid \beta\le\bar{\alpha}<\alpha\le\bar{\beta}\};$$

$$T_1''=\{\alpha,\beta\in I^2 \mid \bar{\alpha}<\beta<\bar{\beta}\le\alpha\};\quad T_2''=\{\alpha,\beta\in I^2 \mid \bar{\alpha}\le\bar{\beta}\le\beta\le\alpha\};$$

$$T'=T_1'\cup T_2';\quad T''=T_1''\cup T_2'';\quad t'=T'\cap\hat{T}''=\{\alpha,\beta\in I^2 \mid \alpha=\bar{\beta}\}.$$

A useful auxiliary technique in a geometric study of *FI*'s is based on the concept of *antipolyndroms* (Kaufmann [1]). With $\alpha,\beta,\gamma,\ldots\in I$, let us denote by $\alpha\beta\gamma\ldots$ the chain of inequalities $\alpha\le\beta\le\gamma\ldots$; strict inequality will be distinguished by putting a dot between the corresponding variables (e.g., $\alpha.\beta\bar{\beta}.\bar{\alpha}$ is the same as $\alpha<\beta\le\bar{\beta}<\bar{\alpha}$); equality is denoted by putting a sign '=' ($\alpha=\bar{\beta}.\gamma.\bar{\gamma}.\beta=\bar{\alpha}$). The chain of inequalities is called an *antipolyndrom* iff it contains all supplements $\bar{\alpha},\bar{\beta},\bar{\gamma},\ldots$; otherwise, it is called an *antipolyndromial interval*. For each antipolyndrom (or antipolyndromial interval) ω depending on the k variables, a truth set $T(\omega)\subseteq I^k$ is defined as a subset of all values satisfying the corresponding chain of inequalities. For a set Ω of antipolyndroms (and/or antipolyndromial intervals), set $T(\Omega)=\bigcup_{\omega\in\Omega} T(\omega)$. In these terms, the above partition of T can be written as

$$T=T(\beta\alpha);\quad T_1'=T(\beta\alpha\bar{\alpha}\bar{\beta});\quad T_2'=T(\beta\bar{\alpha}.\alpha\bar{\beta});\quad T_1''=T(\bar{\alpha}.\beta.\bar{\beta}.\alpha);\quad T_2''=T(\bar{\alpha}\bar{\beta}\beta\alpha);$$

$$T'=T(\beta\alpha\bar{\beta})=T(\beta\bar{\alpha}\bar{\beta});\quad T''=T(\bar{\alpha}.\beta\alpha)=T(\bar{\alpha}\bar{\beta}.\alpha);\quad t'=T(\alpha=\bar{\beta})=T(\bar{\alpha}=\beta).$$

In this section, we study conditions under which an *FI* satisfies common algebraic properties of binary relations: reflexivity, antisymmetry, transitivity. A surprising result of this study is that "well-defined" algebraic properties often turn to be incompatible with continuity of *FI*'s (see Section 6.1).

6.4.1. Reflexivity, Antisymmetry

We recall that a binary relation R on a set X with its values in a poset $\mathfrak{D}$ with universal bounds $\mathbb{O}$, $\mathbb{I}$ is called:

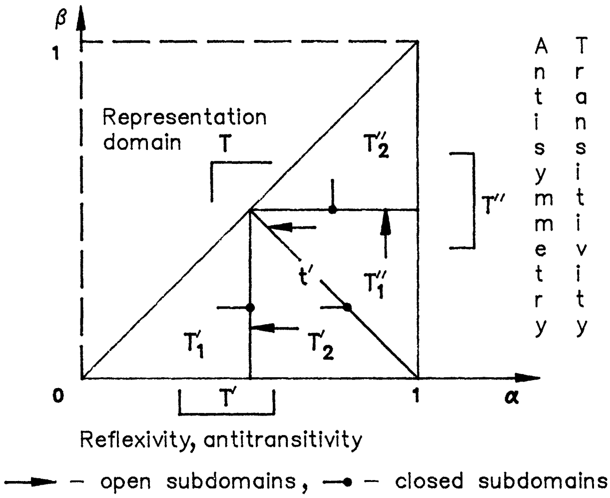


Fig. 6.1. Representation domain, and its subdomains

reflexive iff $R(x,x)\equiv\mathbb{I}$;

antireflexive iff $R(x,x)\equiv\mathbb{O}$;

weakly reflexive iff $(\forall\ x,y\in X)(R(x,y)\le R(x,x))$;

perfectly antisymmetric (*antisymmetric in the sense of L.Zadeh*) iff

$$(\forall\ x\ne y\in X)(R(x,y)>\mathbb{O} \Rightarrow R(y,x)=\mathbb{O});$$

antisymmetric in the sense of A. Kaufmann iff

$$(\forall\ x,y\in X)((R(x,y)=R(y,x)>\mathbb{O}) \Rightarrow y=x).$$

First, let us study *reflexivity* and *antireflexivity* of *FI*'s.

Theorem 6.4. (i) The following properties of a *FI* $inc\in\mathcal{I}nc$ are equivalent:

(1) inc is reflexive;

(2) $\varphi_{inc}|_{T'}\equiv 1$;

(3) inc is weakly reflexive;

(4) inc is stronger than $\subseteq$ ($a\subseteq b \Rightarrow \mu_{inc}(a,b)=1$)

(ii) $inc\in\mathcal{I}nc$ is antireflexive iff $\varphi_{inc}|_{\hat{T}''}\equiv 0$.

Proof. (i) (1) $\Rightarrow$ (2). For any $\beta\in I$, set $b=\beta\cdot 1$. Using definitions of reflexivity, and of φ_{inc}, we come to $\mu_{inc}(b,b)=\varphi_{inc}(\beta\vee\bar\beta,\beta\wedge\bar\beta)=1$. It follows that $\varphi_{inc}|_{t'}\equiv 1$. With each $(\alpha,\beta)\in T'$, an antipolyndromial interval $\beta\alpha\bar\beta$ is in force; therefore, $(\alpha,\beta)\le(\alpha,\bar\beta)$. Finally, monotonicity of φ_{inc} implies $\varphi_{inc}(\alpha,\beta)\ge\varphi_{inc}(\beta,\bar\beta)=1$.

(2) $\Rightarrow$ (1). Using the definition of ν^{-1} (see Theorem 6.3), we arrive to the following inequalities:

$$(\forall\ a\in\tilde{\mathcal{P}}(X))(\mu_{inc}(a,a)=\bigwedge_{x\in X}\varphi_{inc}(\mu_a(x)\vee\overline{\mu_a(x)},\mu_a(x)\wedge\overline{\mu_a(x)})$$

$$\geq\bigwedge_{\alpha\in I}\varphi_{inc}(\alpha\vee\bar{\alpha},\alpha\wedge\bar{\alpha})=\bigwedge_{(\alpha,\beta)\in T'}\varphi_{inc}(\alpha,\beta)=1).$$

(1) $\Rightarrow$ (3). Follows from definitions.

(3) $\Rightarrow$ (1). Owing to Proposition 6.3 (ii), μ_{inc} is antitone with respect to the second argument; hence, $\mu_{inc}(a,a)\geq\mu_{inc}(a,1)\geq\mu_{inc}(1,1)=1$ (the latter equality is due to axiom A_4).

(2) $\Leftrightarrow$ (4). Follows from the equality $\nu(\subseteq)=\varphi_{\subseteq}=\chi_{T'}$.

(ii) Let us suppose that inc is antireflexive. Arguing as in the proof of (i), (1) $\Rightarrow$ (2), we come to $\varphi_{inc}(\beta\vee\bar{\beta},\beta\wedge\bar{\beta})=0$. Dually to (i), (1) $\Rightarrow$ (2), with each $(\alpha,\beta)\in T''$, $(\alpha,\beta)\geq(\alpha,\bar{\alpha})$; monotonicity of φ_{inc} implies $\varphi_{inc}(\alpha,\beta)\leq\varphi_{inc}(\alpha,\bar{\alpha})=0$. The converse assertion is proved dually to (i), (2) $\Rightarrow$ (1) ■

In that way, diverse definitions of reflexivity known in the theory of binary relations determine the same family $\mathcal{I}nc_r$ of reflexive fuzzy inclusions. The smallest element in $\mathcal{I}nc_r$ is L.Zadeh inclusion $\subseteq$.

We begin the study of antisymmetry with a simple assertion on a "conjugation of joint spectrum". Let us define *joint spectrum* of a pair of fuzzy subsets $a,b\in\tilde{\mathcal{P}}(X)$ as a discrete subset $\mathcal{S}(a,b)$ of a triangle T

$$\mathcal{S}(a,b)=\{\mu_a(x)\vee\overline{\mu_b(x)},\mu_a(x)\wedge\overline{\mu_b(x)}\}\subseteq T$$

Proposition 6.4. $\mathcal{S}(b,a)=\tau(\mathcal{S}(a,b))$, where τ is conjugation, $\tau:T\longrightarrow T$, $\tau((\alpha,\beta))=(\bar{\beta},\bar{\alpha})$ ■

Proof. $\mathcal{S}(b,a)=\{\mu_b(x)\vee\overline{\mu_a(x)},\mu_b(x)\wedge\overline{\mu_a(x)}\}$

$$=\{\overline{\mu_a(x)\wedge\overline{\mu_b(x)}},\overline{\mu_a(x)\vee\overline{\mu_b(x)}}\}=\{\bar{\beta},\bar{\alpha}\}_{(\alpha,\beta)\in\mathcal{S}(a,b)}=\tau(\mathcal{S}(a,b))\ ■$$

In other words, the spectrum (from now on, we omit the word "joint") of (b,a) can be obtained from the spectrum of (a,b) by means of reflection in the "diagonal" t'.

In the next theorem, characterization of diverse antisymmetry properties of fuzzy inclusions is presented.

Theorem 6.5. (i) A *FI* inc is perfectly antisymmetric iff $\varphi_{inc}\big|_{T''}\equiv 0$.

(ii) For n=1, inc is antisymmetric in the sense of A.Kaufmann iff φ_{inc} is strictly τ-antisymmetric, that is, $(\forall K=(\alpha,\beta)\in T'')(\varphi(\tau K)>\varphi(K))$.

(iii) For $n\geq 2$, perfect antisymmetry of *FI* is equivalent to antisymmetry in the sense of A.Kaufmann ∎

Proof. (i) First, let us suppose that $\varphi_{inc}\big|_{T''}\equiv 0$. If, for some, $a,b\in\tilde{\mathcal{P}}(X)$, $a\neq b$, $\mu_{inc}(a,b)=\wedge\varphi_{inc}\big|_{\mathcal{S}(a,b)}>0$, then $\mathcal{S}(a,b)\subset T'$. In virtue of Proposition 6.4, $\mathcal{S}(b,a)\cap T''\neq\emptyset$ ($a\neq b$!); it follows that $\mu_{inc}(b,a)=\wedge\varphi_{inc}\big|_{\mathcal{S}(b,a)}=0$.

Conversely, if $(\alpha,\beta)\in T''$, and $\varphi_{inc}(\alpha,\beta)\geq 0$, set $a=\alpha\cdot 1$, $b=\bar{\beta}\cdot 1$; in such case, $a\neq b$, and $\mathcal{S}(a,b)=\{(\alpha,\beta)\}\geq\mathcal{S}(b,a)=\{(\bar{\beta},\bar{\alpha})\}$. It follows that $0<\mu_{inc}(a,b)\leq\mu_{inc}(b,a)$, so that inc does not possess perfect antisymmetry.

(ii) Directly follows from Proposition 6.4, because, with n=1, $\mathcal{S}(a,b)$ is reduced to a single point in T.

(iii) Let us suppose that $(\alpha,\beta)\in T''$, and $\varphi_{inc}(\alpha,\beta)>0$. Under this assumption, let X be the union of disjoint subsets U, and V, and let $a=\alpha/U+\bar{\beta}/V$; $b=\bar{\beta}/U+\alpha/V$ be two f.s.'; clearly, $\mathcal{S}(a,b)=\mathcal{S}(b,a)=\{(\alpha,\beta),(\bar{\beta},\bar{\alpha})\}$, and

$$\mu_{inc}(a,b)=\mu_{inc}(b,a)=\varphi_{inc}(\alpha,\beta)\wedge\varphi_{inc}(\bar{\beta},\bar{\alpha})=\varphi_{inc}(\alpha,\beta)>0,$$

so that inc is not antisymmetric in the sense of A.Kaufmann ∎

Let us denote by $\mathcal{I}nc_{pa}$ the set of all perfectly antisymmetric *FI*'s. As was already noticed, L.Zadeh inclusion $\subseteq$ represents the smallest element in the set $\mathcal{I}nc_{r}$ of all reflexive (and weakly reflexive) *FI*'s. In the next statement, we formulate, in the spirit of Theorem 6.4, several equivalent characteristic antisymmetry conditions for *FI*'s, and distinguish L.Zadeh inclusion, which proves to occupy dual position with respect to the families $\mathcal{I}nc_{r}$, $\mathcal{I}nc_{pa}$.

Corollary 6.3. With $n\geq 2$, the following properties of *FI* are equivalent:

(1) inc is perfectly antisymmetric;

(2) inc is antisymmetric (due to Kaufmann);

(3) inc is weaker than $\subseteq$ ($\mu_{inc}(a,b)>0 \Rightarrow a\subseteq b$) ∎

Proof. (1) $\Leftrightarrow$ (2). The same as in Theorem 6.5 (iii).

(1) $\Leftrightarrow$ (3). Follows from Theorem 6.5 (i) ($\varphi_{inc}\big|_{T''}\equiv 0$), and from the formula for $\varphi_{\subseteq}$ ($\varphi_{\subseteq}=\chi_{T''}$) ∎

Corollary 6.4. With $n\geq 2$, $\leq$ is the only *FI* possessing both the reflexivity, and the antisymmetry properties (be that perfect antisymmetry or antisymmetry in the sense of A.Kaufmann) ■

Proof. Direct consequence of Theorem 6.4 (i) and of Corollary 6.3 ■

The above results indicate the first difficulties in the synthesis of *FI*'s, possessing the well-defined algebraic properties together with the continuity.

Corollary 6.5. (i) Reflexive, antisymmetric in the sense of A.Kaufmann and continuous *FI*'s exist only with n=1.

(ii) A perfectly antisymetric *FI* can not be continuous ■

Proof. (i) With n=1, any continuous function $\varphi\in\Phi$, which is greater than $\varphi_{\leq}$ and strictly decreasing on T″, gives rise to a reflexive, antisymmetric in the sense of A.Kaufmann and continuous FI $\nu^{-1}(\varphi)$.

With n>1, any *FI*, which is antisymmetric in the sense of A.Kaufmann, is also perfectly antisymmetric (Theorem 6.5 (iii)) and hence coincides with $\leq$ (Corollary 6.4) thus being discontinuous.

(ii) Perfect antisymmetry of inc implies $\varphi_{\mathrm{inc}}\big|_{T''}\equiv 0$ (Theorem 6.5 (i)). Since, with any $\varphi\in\Phi$, $\varphi(1,0)=1$, φ is discontinuous in the point (1,0) ■

In the theory of fuzzy binary relations, and in fuzzy decision-making, a common "source" of antisymmetric relations is provided by the following "antisymmetrization procedure" $\alpha:\tilde{\mathcal{P}}(Y^2)\longrightarrow\tilde{\mathcal{P}}(Y^2)$, resulting in a "strict relation" (see, e.g., Kuzmin [1], Ovchinnikov and Roubens [1,2]):

$$\mu_{\alpha(R)}(y,z)=\begin{cases}0, & \mu_R(y,z)\leq\mu_R(z,y)\\ \mu_R(y,z)\ (\text{another version}:1), & \mu_R(y,z)>\mu_R(z,y)\end{cases}$$

Obviously, $\alpha(R)$ is perfectly antisymmetric. Can this canonical construction be useful in case of *FI*'s? With $n\geq 2$, the answer is negative. More exactly, the following statement is in force.

Theorem 6.6. (i) For n=1, $\alpha(\mathrm{inc})\in\mathcal{I}nc$ with any *FI* inc.

(ii) For $n\geq 2$, $\alpha(\mathrm{inc})\in\mathcal{I}nc$ iff $\alpha(\mathrm{inc})=\mathrm{inc}$ ■

Proof. Set $\mathrm{inc}_\alpha=\alpha(\mathrm{inc})$; for brevity, in this proof φ stands for φ_{inc}.

(i) Verification of axioms A_1, A_3, A_4 for inc_α is easy. To check A_2, let us consider two cases.

(1) $\mu_{inc_\alpha}(a\vee b,c)>0$. In this case,

$$\mu_{inc}(a\vee b,c)=\mu_{inc}(a,c)\wedge\mu_{inc}(b,c)>\mu_{inc}(c,a\vee b)\geq\mu_{inc}(c,a)\vee\mu_{inc}(c,b)$$

(here, equality is due to A_2, the first inequality follows from definition of α, the second inequality is implied by Proposition 6.3 (ii)). Hence, both inequalities

$$\mu_{inc}(a,c)\geq\mu_{inc}(a\vee b,c)>\mu_{inc}(c,a), \text{ and}$$

$$\mu_{inc}(b,c)\geq\mu_{inc}(a\vee b,c)>\mu_{inc}(c,b)$$

are satisfied; therefore,

$$\mu_{inc_\alpha}(a\vee b,c)=\mu_{inc}(a\vee b,c)=\mu_{inc}(a,c)\wedge\mu_{inc}(b,c)=\mu_{inc_\alpha}(a,c)\wedge\mu_{inc_\alpha}(b,c).$$

We emphasize that the proof of this fragment does not depend on n.

(2) $\mu_{inc_\alpha}(a\vee b,c)=0$. Under this assumption, we use the fact that with n=1, $\tilde{\mathcal{P}}(X)$ is a linear poset. Let us suppose, for definiteness, that $a\subseteq b$. Then $\mu_{inc}(c,a\vee b)=\mu_{inc}(c,b)\geq\mu_{inc}(a\vee b,c)=\mu_{inc}(b,c)$. It follows that $\mu_{inc_\alpha}(b,c)=0$, and hence, $0=\mu_{inc_\alpha}(a\vee b,c)=\mu_{inc_\alpha}(a,c)\wedge\mu_{inc_\alpha}(b,c)$.

(ii) A one-sided assertion ($inc\in\mathcal{I}nc_{pa}\Rightarrow inc_\alpha=inc$) is obvious. Now, let us suppose that $inc\notin\mathcal{I}nc_{pa}$. According to Theorem 6.5 (i), $\varphi(\alpha,\beta)>0$ for some $(\alpha,\beta)\in T''$. Since $n\geq2$, X can be represented as $U\cup V$, $U\cap V=\emptyset$. Set $a=\alpha/U+0/V$, $b=\bar{\beta}/U+1/V$. For these fuzzy subsets, $\mu_{inc}(a,b)=\varphi(\alpha,\beta)>\mu_{inc}(b,a)=0$; in other words, $\mu_{inc_\alpha}(a,b)>0$. However, since $T''=T(\bar{\alpha}.\beta\alpha)=T(\overline{\alpha\beta}.\alpha)$, it is easy to verify that neither of the inclusions $a\subseteq b$, $b\subseteq a$ takes place ($b|_U\subseteq a|_U$, $a|_U\subseteq b|_U$). This means, in its turn, that $inc_\alpha\in\mathcal{I}nc_{pa}$, and $\mu_{inc_\alpha}(a,b)>\mu_{\subseteq}(a,b)$, which is in contradiction with the already proved assertion that $\subseteq$ is the greatest element in $\mathcal{I}nc_{pa}$ (Corollary 6.3). Hence, $inc_p\notin\mathcal{I}nc_{pa}$ ∎

Theorem 6.6 shows that, with $n\geq2$, "antisymmetrization procedure" adds no antisymmetric *FI*'s, since application of this procedure to an *FI* either leaves it unchanged or brings it out of $\mathcal{I}nc$. However, there exists a regular alternative construction of antisymmetric *FI*'s: $p:\tilde{\mathcal{P}}(Y^2)\longrightarrow\tilde{\mathcal{P}}(Y^2)$, $p(inc)=inc\wedge\subseteq$ (the fact that $p(inc)\in\mathcal{I}nc$ will be established in the Theorem 6.8 (ii)). Obviously, $\varphi_{p(inc)}=\nu(p(inc))=\varphi_{inc}\wedge\chi_{T'}$; hence,

$p(inc)\in\mathcal{I}nc_{pa}$. It follows that $p(inc)$ is the *greatest perfectly antisymmetric FI contained in* inc.

6.4.2. Transitivity

We recall that a binary relation R on a set X with its values in a complete lattice is called:

transitive, iff $R^2\subseteq R$, that is, $(\forall\ x,y\in X)(\bigvee_{z\in X}\mu_R(x,z)\wedge\mu_R(z,y)\le\mu_R(x,y))$;

antitransitive, iff $R\subseteq R^2$, that is, $(\forall\ x,y\in X)(\mu_R(x,y)\le\bigvee_{z\in X}\mu_R(x,z)\wedge\mu_R(z,y))$.

In (Dubois and Prade [1]), transitivity of the so called ε-inclusions was studied. We also mark out that fuzzy implications satisfying the axiomatics of J.Baldwin, and B.Pilsworth [1] proved to be antitransitive. In this section, we concentrate mainly on the direct property, that is, on transitivity of *FI*'s. We consider only conventional $\vee$-$\wedge$ transitivity of *FI*'s, and the set of all transitive *FI*'s is denoted by $\mathcal{I}nc_t$.

A complete description of *transitivity* of *FI*'s is rather complicated. It requires the study of the behavior of representations φ_{inc} on the subtriangles T'_1, T'_2, T''_1, T''_2. It turns out that the transitivity condition puts no restrictions on the behavior of φ_{inc} on the subtriangle T' (except for the monotonicity on the whole T). On the contrary, most rigid constraints are put on the restriction $\varphi_{inc}|_{T''_2}$, whereas most difficult for the research is the behavior of a "transitive representation" on the subtriangle T''_1.

First of all, let us express transitivity in terms of representations.

Lemma 6.1. A *FI* inc is transitive iff it is "transitive with constants":

$$inc\in\mathcal{I}nc_t \Leftrightarrow (\forall\ \alpha,\beta,\gamma\in I)(\mu_{inc}(\alpha\cdot 1)\ge\mu_{inc}(\beta\cdot 1)\wedge\mu_{inc}(\gamma\cdot 1)) \blacksquare$$

Proof. The necessity of this condition is obvious; the sufficiency is implied by the formula for ν^{-1} (see Theorem 6.3) $\blacksquare$

Corollary 6.6. A *FI* inc is transitive iff

$$(\forall\ \alpha,\beta,\gamma\in I)(\varphi_{inc}(\alpha\vee\bar{\gamma},\alpha\wedge\bar{\gamma})\ge\varphi_{inc}(\alpha\vee\bar{\beta},\alpha\wedge\bar{\beta})\wedge\varphi_{inc}(\beta\vee\bar{\gamma},\beta\wedge\bar{\gamma})) \blacksquare$$

Proof. Straightforward consequence of Lemma 6.1 $\blacksquare$

So far, Corollary 6.6 gives rise to a well-tried tool of the research of transitivity in Φ instead of that in $\mathcal{I}nc$. Let us denote by $\Phi_t=\nu(\mathcal{I}nc_t)$ the set of all representations $\varphi\in\Phi$, satisfying the condition of Corollary 6.6.

To introduce a more geometric and concise style of the study, let us associate with any three numbers $\alpha,\beta,\gamma\in I$ the following three points in the triangle T: $K=(\alpha\vee\bar{\gamma},\alpha\wedge\bar{\gamma})$; $L=(\beta\vee\bar{\gamma},\beta\wedge\bar{\gamma})$; $M=(\alpha\vee\bar{\beta},\alpha\wedge\bar{\beta})$. We notice that, if at least one the pairs (K,L) or (K,M) is Pareto ordered ($K\le L$ or $K\le M$ is true), then monotonicity of φ implies $\varphi(K)\ge\varphi(L)\wedge\varphi(M)$, and hence, the condition of Corollary 6.6 is satisfied, so that the corresponding triple (K,L,M) does not put any restriction on transitivity. Therefore, we are interested only in a specific collection of "configurations" with $\neg(K\le L)\&\neg(K\le M)$. On Figure 6.2, the process, and the result of constructing configurations (K,L,M) is depicted. In the rest of this section, $\bigvee$, and $\bigwedge$ stand respectively for sup, and inf in a lattice T with the Pareto order $\le$.

Lemma 6.2 (Configurations (K,L,M)).

(i) With $K\in T'$, either $K\le L$ or $K\le M$ is satisfied.

(ii) With $K\in T''_1$, at least one of the statements (1)-(3) is satisfied:

(1) $K\le L$; (2) $K\le M$; (3) $(K>L)\&(K>M)\&(K=L\bigvee M)\&(L\bigwedge M\in t')$.

(iii) With $K\in T''_2$, at least one of the statements (1)-(4) is satisfied:

(1) $K\le L$; (2) $K\le M$; (3) $(K>L)\&(K>M)\&(K=L\bigvee M)\&(L\bigwedge M\in t')$

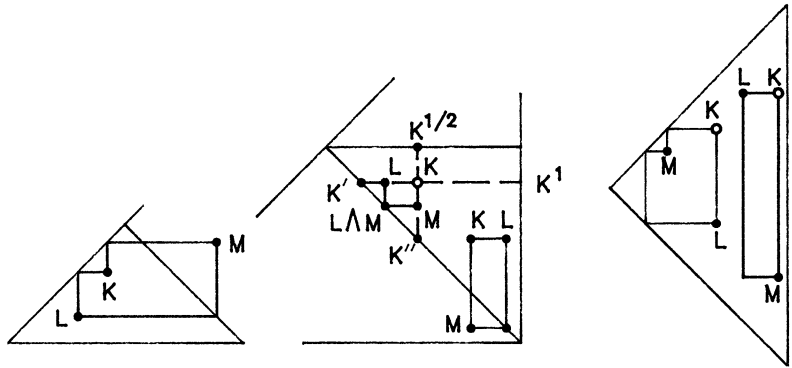
(4) $(L\bigvee M\le K)\&(L\bigwedge M\in T''_1)$ ∎

Proof. (i) Let us consider two cases: $K=(\bar{\gamma},\alpha)$, and $K=(\alpha,\bar{\gamma})$. In the first case, since $K=(\bar{\gamma},\alpha)\in T'$, $\alpha\overline{\gamma\alpha}$ is also true. Next, if $\beta\alpha$ holds then the resulting antipolyndrom containing α,β,γ is $\beta\alpha\bar{\gamma}\gamma\overline{\alpha\beta}$; it follows that $M=(\bar{\beta},\alpha)$, and $K\le M$. In case when $\alpha\beta$ is true, and in case $K=(\alpha,\bar{\gamma})$, the proof is similar.

(ii) Only the case (3) is of interest. Let us consider the same two variants as in (i).

1) With $K=(\bar{\gamma},\alpha)\in T''_1$, we have, by definition of $T''_1=T(\bar{\alpha}.\beta.\bar{\beta}\alpha)$, $\gamma\alpha\overline{\alpha\gamma}$. Since both K>L, and K>M are satisfied, it follows that $L=(\bar{\gamma},\beta)$, and $\beta.\alpha$ is true. Next, $\gamma\alpha\overline{\alpha\gamma}$ together with $\beta.\alpha$ implies $M=(\bar{\beta},\alpha)$. Combining the expressions for K,L,M, we obtain $L\bigvee M=(\bar{\gamma},\alpha)=K$, $L\bigwedge M=(\bar{\beta},\beta)\in t'$.

2) $K=(\alpha,\bar{\gamma})\in T''_1$ means $\bar{\alpha}.\bar{\gamma}.\gamma\alpha$. The conditions K>L, and K>M lead to $M=(\alpha,\bar{\beta})$ and $\bar{\beta}.\bar{\gamma}$; hence, $\bar{\alpha}\beta\bar{\gamma}\gamma\bar{\beta}\alpha$ is fulfilled, so that $L=(\beta,\bar{\gamma})$, $L\bigvee M=(\alpha,\bar{\gamma})=K$, $L\bigwedge M=(\bar{\beta},\beta)\in t'$.



a) $K \in T'$ b) $K \in T''_1$ c) $K \in T''_2$

o — restrictions on the behavior of representations

Fig. 6.2. Configurations (K,L,M)

(iii) With $K=(\bar\gamma,\alpha)\in T''_2$, we have, by definition of $T''_2=T''_2=T(\overline{\alpha\beta}\beta\alpha)$, that $\gamma\bar\alpha\alpha\bar\gamma$ is the case. As above, $K>L\&K>M$ implies $(L=(\bar\gamma,\beta)\&\beta.\alpha)$; when combining these two antipolyndroms, $\gamma\bar\alpha\alpha\bar\gamma$, and $\beta.\alpha$, we come to the following three alternatives:

1) $\gamma\beta\bar\alpha\alpha\overline{\beta\gamma}$; under this condition, $L\in T''_1$, $M=(\bar\beta,\alpha)\in T''_2$, $L\vee M=K$, $L\wedge M\in t'$, which is nothing but the case (3) in the formulation of lemma;

2) $\gamma\bar\alpha\beta\bar\beta\alpha\bar\gamma$; under this condition, $L\in T''_1$, $M=(\alpha,\bar\beta)\in T''_2$, $L\vee M=(\bar\gamma,\bar\beta)<K$, $L\wedge M=(\alpha,\beta)\in T''_1$, thus representing the case (4) (except for the special case $\gamma\bar\alpha=\beta.\bar\beta=\alpha\gamma$ yielding $L\vee M=K$, $L\wedge M\in t'$);

3) $\gamma\overline{\alpha\beta}\beta\alpha\bar\gamma$; under this condition, $L\in T''_2$, $M=(\alpha,\bar\beta)\in T''_1$, so that $M<L<K$, and $(L\wedge M)=M<(L\vee M)=L<K$ - again, the case (4) ■

This lemma demonstrates that the behavior of "transitive representations" in a subtriangle T'_1 is not limited by a transitivity condition. On the contrary, a very rigid restriction is put upon $\varphi|_{T''_2}$.

Lemma 6.3. For any $\varphi\in\Phi$, $\varphi|_{T''_2}$ does not depend on β, so that any level subset of $\varphi|_{T''_2}$ can be represented as a union of intervals $[(\alpha,1/2),(\alpha,\alpha)]$ ■

Proof. Let us select any $\alpha>1/2$ and define $\gamma=\bar\alpha$, $\beta=1/2$; in this case, the configuration (K,L,M) is as follows: $K=(\alpha\vee\bar\gamma,\alpha\wedge\bar\gamma)=(\alpha,\alpha)$, $L=M=(\alpha,1/2)<K$.

Combining the transitivity condition $\varphi(K)\geq\varphi(L)\wedge\varphi(M)$, (see Lemma 6.1), with the monotonicity of representation $\varphi(K)\leq\varphi(L)=\varphi(M)$ (Theorem 6.3), we arrive to conclusion that $\varphi(K)=\varphi(L)=\varphi(M)$. Therefore, φ is constant on a "vertical" interval $[(\alpha,1/2),(\alpha,\alpha)]$ ∎

As to the subtriangle T_1'', the description of $\varphi|_{T_1''}$ is very tedious. We obtain this description in the terms of *connected components* $\mathcal{L}_\gamma$ of *level subsets* $\varphi^{-1}(\gamma)\cap T_1''$; the latter components prove to have a relatively simple geometric shape. In the sequel, we denote by $\hat{T}_1''$ the closure of T_1'' in T. The points $K,L,L^*\in T_1''$ are called *conjugate* points iff $L\wedge L^*\in t'$, $L\wedge L^*=K$.

In virtue of Lemmas 6.1, 6.2, transitivity condition can be easily reformulated in the terms of conjugated points:

with any K,L,L^*, and an arbitrary $\varphi\in\Phi_t$, either $\varphi|_{[L,K]}$ or $\varphi|_{[L^*,K]}$ is constant.

With $K=(\alpha,\beta)\in T_1''$, let $K'=(\bar{\beta},\beta)$, $K''=(\alpha,\bar{\alpha})$, $K^{1/2}=(\alpha,1/2)$, $K^1=(\beta,1)$ be the "α-projections", and the "β-projections" of K on the sides of the triangle T_1''; we denote by $\mathcal{M}_K'=]K',K[$, $\mathcal{M}_K''=]K'',K[$ two legs of the right triangle $K'KK''$ (the vertex K itself is not included!).

To continue the research, we need several obvious facts concerning the semilattice structure, and the "ideals" of the triangle T_1'':

T_1'' is a $\vee$-subsemilattice of T;

$\hat{T}_1''$ is a $\vee$-subsemilattice of T with an upper universal bound $\mathbb{I}=(1,1/2)$;

an ideal $]\mathbb{O},K]=\{M\in T_1''\,|\,M\leq K\}$ is geometrically a (non-closed) subtriangle with vertexes K,K',K'';

a dual ideal $[K,\mathbb{I}]=\{M\in T''\,|\,K\leq M\}$ is geometrically a (non-closed) quadrangle with its vertexes being $K,K^1,\mathbb{I},K^{1/2}$.

The schedule of the subsequent study of the behavior of the restriction $\varphi|_{T_1''}$ of a "transitive representation" φ consists of two steps:

1) determination of the shape of $\mathcal{L}_\gamma$'s;
2) discovering the structure of "factorization" of a semilattice T_1'' with respect to the equivalence relation

"$K\approx L$ iff K, and L belong to the same $\mathcal{L}_\gamma$".

When examining the first step, we consider separately *one-dimensional* and *two-dimensional* connected components of level subsets of φ. Each of the components proves to be a subsemilattice of T_1'', thus possessing a single *top point* $K_\gamma=\bigvee\mathcal{L}_\gamma$. More explicitly, one-dimensional components turn to be either intervals $\mathcal{M}_K'$, $\mathcal{M}_K''$ or "angles" $\mathcal{M}_K'\cup\mathcal{M}_K''\cup\{K\}$, with an appropriate $K=K_\gamma$, whereas two-dimensional components have the form of a "comb" $]0,K]\setminus\bigcup_m]0,K_m]$, with finite or countable number of "teeth" (in particular, it can be a single triangle $]0,K]$).

At the next step, we discover that the quotient structure $T_1''/(\approx)$ can be efficiently studied using the above top points of connected components. These points can also be subdivided into two classes. Top points of the first class represent one-dimensional $\mathcal{L}_\gamma$'s and form a finite or a countable family of connected monotone curves $\mathcal{K}_j$ inside T_1''. The union of one-dimensional components adjacent to a given $\mathcal{K}_j$ is geometrically a solid (2-dimensional) angle; solid angles are pairwise disjoint. Another class consists of isolated points which prove to be top points of two-dimensional $\mathcal{L}_\gamma$'s. In general, all "level two" connected components of the set $\mathcal{K}=\{$top points of $\mathcal{L}_\gamma$'s$\}$ form a directed set Ω with a specific "tree-like" structure.

In what follows, we give a formal explication of the above description, omitting routine and complicated details in most of the proofs.

Lemma 6.4. Any $\mathcal{L}_\gamma$ is $\bigvee$-semilattice ■

Proof. Let us suppose that $L,M\in\mathcal{L}_\gamma$, $K=L\bigvee M$. The only non-trivial case is $K\notin\{L,M\}$; for definiteness, let $L\in]K',K[$, $M\in]K'',K[$. Then we consider two points in T_1'': $L_0=\bigvee(\mathcal{L}_\gamma\bigcap]K',K[)\le L$; $M_0=\bigvee(\mathcal{L}_\gamma\bigcap]K'',K[)\le L$. We prove that $L_0=M_0=K$; in such case, Corollary 6.6 immediately implies $\varphi(K)=\gamma$, and $K\in\mathcal{L}_\gamma$.

Let us suppose that $L_0\ne K$; then $\varphi(K)<\gamma$. With a sufficiently small $\varepsilon>0$, select $L_1\in]K',L_0[$ with $|L_1-L_0|<\varepsilon$ (since L_1, and L_0 belong to an interval $]K',K[$, $|L_1-L_0|$ is nothing but the distance between these points). It easily follows from the definition of L_0 that, for each $K_1\in]L_0,K]$, $\varphi(L_1)\ge\gamma>\varphi(K_1)$. Let us consider the triple of conjugated points K_1, L_1, L_1^*. Since $\varphi\big|_{[L_1,K_1]}\ne\text{const}$, the above reformulation of transitivity condition in the terms of conjugated points requires that $\varphi(K_1)=\varphi(L_1^*)<\gamma$. Hence, $[L_1^*,K_1]\bigcap\mathcal{L}_\gamma=\emptyset$. According to the choice of L_1 ($|L_1-L_0|<\varepsilon$), $|L_1^*-L_0''|$ is also smaller than ε.

Varying ε, and K in the assertion $([L_1^*,K_1]\cap\mathcal{L}_\gamma=\emptyset)\&(|L_1^*-L_0''|<\varepsilon)$, we conclude that $]L_0'',K]\cap\mathcal{L}_\gamma=\emptyset$; in addition, the above stated inequality $\varphi(K)<\gamma$ implies that $[K,\mathbb{I}[\cap\mathcal{L}_\gamma=\emptyset$. It follows that the two subsets, $\mathcal{L}_\gamma$, and $\mathcal{W}=]L_0'',K]\cup[K,\mathbb{I}[$ are disjoint. However, the subset $\mathcal{W}$ cuts T_1'' into two non-intersecting subsets, and the two original points $L\in\mathcal{L}_\gamma$, and $M\in\mathcal{L}_\gamma$ belong to different linear connected components of a subset $T_1''\setminus\mathcal{W}$. This is in contradiction with linear connectivity of $\mathcal{L}_\gamma$. It follows that the original assumption $L_0\neq K$ is wrong; therefore, $L_0=K$. The equality $M_0=K$ can be proved in the same way ∎

As was mentioned above, components $\mathcal{L}_\gamma$ can be sub-divided into two classes. *Two-dimensional* components (denoted by $\mathcal{L}_\gamma^{(2)}$) are defined as components with a non-empty interior, $\overset{\circ}{\mathcal{L}}{}_\gamma^{(2)}\neq\emptyset$; all the remaining components are called *one-dimensional* ($\mathcal{L}_\gamma^{(1)}$).

Since each $\mathcal{L}_\gamma$ is a $\bigvee$-subsemilattice of T_1'', it is quite natural to think of its upper bound $K_\gamma=\vee\mathcal{L}_\gamma$, which is called *top point* of $\mathcal{L}_\gamma$.

Corollary 6.7. With any $\mathcal{L}_\gamma$, top point K_γ belongs to the closure of $\mathcal{L}_\gamma$ ∎

Proof. Directly follows from Lemma 6.4 ∎

The collection of top points of all $\mathcal{L}_\gamma$'s is denoted by $\mathcal{K}$, $\mathcal{K}=\{K_\gamma\}\subset T_1''$. Let $\mathfrak{L}$ be the set of all connected components of $\mathcal{K}$, $\mathfrak{L}=\{\mathcal{K}_j\}$. Next, with each $\mathcal{K}_j$, a *bottom point* $K_j^\wedge$, and a "level two" *top point* $K_j^\vee$ is associated: $K_j^\wedge=\bigwedge\mathcal{K}_j$, $K_j^\vee=\bigvee\mathcal{K}_j$. Basic properties of $\mathcal{K}$, $\mathcal{K}_j$, and $\mathfrak{L}$ are presented in the following two statements.

Lemma 6.5. (i) For any $K_\gamma,K_\delta\in\mathcal{K}$, either (1) or (2) is fulfilled:

(1) $K_\gamma\wedge K_\delta\in T'$;

(2) K_γ,K_δ are $\leq$-comparable, that is, $K_\gamma\bigvee K_\delta\in\{K_\gamma,K_\delta\}$.

(ii) $\overset{\circ}{\mathcal{K}}=\emptyset$.

(iii) Any $\mathcal{K}_j$ is a continuous monotone curve in $\hat{T}_1''$, joining the points $K_j^\wedge$, and $K_j^\vee$.

(iv) With $j\neq l$, the statement (i) is fulfilled for all $K_\gamma\in\mathcal{K}_j$, $K_\delta\in\mathcal{K}_l$ simultaneously ∎

Proof. (i) We prove that $\neg(1) \Rightarrow (2)$. Let us suppose that $K_\gamma \wedge K_\delta \in T''_1$; then we set $N=K_\gamma \vee K_\delta$. According to Lemma 6.1, $\varphi(N)$ is equal either to $\varphi(K_\gamma)$ or to $\varphi(K_\delta)$. In such case, Corollary 6.6 shows that N belongs either to $\hat{\mathcal{L}}_\gamma$ or to $\hat{\mathcal{L}}_\delta$. Next, the definition of top point implies that N is either K_γ or K_δ.

(ii) Follows from (i), because, in any open subset of a triangle T''_1, there exist two points K_γ, K_δ satisfying neither (1) nor (2).

(iii) Let K_γ, $K_\delta \in \mathcal{K}_j$ be two different top points in a connected component $\mathcal{K}_j$, and let π be the path in $\mathcal{K}_j$ joining these points (we identify the mapping $\pi: I \longrightarrow \mathcal{K}_j$, $\pi(0)=K_\gamma$, $\pi(1)=K_\delta$ with its image $\pi(I) \subseteq \mathcal{K}_j \subset T''_1$). Let us prove that π is a monotone curve in T''_1. Indeed, let us assume that π is not monotone. In such case, for an arbitrarily small $\varepsilon>0$, there exist two $\leq$-incomparable points K_1, $K_2 \in \pi$ with $|K_1-K_2| \leq \varepsilon$. Since π is closed, the distance between π, and t′ is positive, $\rho(\pi,t')>0$; let us select $\leq$-incomparable points K_1, K_2 with $|K_1-K_2|<\rho(\pi,t')/2$. It follows that $K_1 \wedge K_2 \in T''_1$, which is in contradiction with (i) (K_1, and K_2 are incomparable, and $K_1 \wedge K_2 \notin T'$). Therefore, π is monotone. Making K_γ, K_δ to converge to $\overset{\wedge}{K_j}$, $\overset{\vee}{K_j}$, we reconstruct the whole $\mathcal{K}_j$ (we mark out that $\overset{\wedge}{K_j}$, and $\overset{\vee}{K_j}$ must not be included in $\mathcal{K}_j$).

(iv) Similarly to the method of verification of (i), we prove that $\neg(1) \Rightarrow (2)$. Let $K_\gamma \in \mathcal{K}_j$, $K_\delta \in \mathcal{K}_1$, $K_\gamma \wedge K_\delta \in T''_1$. Under this assumption, K_γ and K_δ are $\leq$-comparable (see (i)); let us suppose, for definiteness, that $K_\gamma \leq K_\delta$. Then we prove that $\overset{\vee}{K_j} \leq \overset{\wedge}{K_1}$. If we assume the contrary, then we can derive from (i), (iii) that $\overset{\vee}{K_j} \wedge \overset{\wedge}{K_1} \in T'$; in such case, $K_\gamma \wedge \overset{\wedge}{K_1} \leq \overset{\vee}{K_j} \wedge \overset{\wedge}{K_1} \in T'$. Next, since $\overset{\vee}{K_j} \leq \overset{\wedge}{K_1}$ is invalid, the intersection $\mathcal{K}_1 \cap [\overset{\vee}{K_j}, \mathbb{I}]$ is non-empty. Set $K_\eta = \wedge(\mathcal{K}_1 \cap [\overset{\vee}{K_j}, \mathbb{I}])$. Then $K_\gamma \leq \overset{\vee}{K_j} \leq K_\eta$, and hence, $K_\gamma \wedge K_\eta = K_\gamma \in T''_1$. Now, let $\pi: I \longrightarrow \mathcal{K}_1$, $\pi(0)=\overset{\wedge}{K_1}$, $\pi(1)=K_\eta$ be a path from $\overset{\wedge}{K_1}$ to K_η. In virtue of continuity of π and of operation $\wedge$, with α sufficiently close to 1, we come to a conclusion that both conditions $\pi(\alpha) \leq K_\eta$ and $\pi(\alpha) \wedge K_\gamma \in T''_1$ are satisfied, which is in contradiction with (i) ■

Lemma 6.5 (iv) shows that the partial order $\vdash\!\!<$ on $\mathfrak{L}$:

$$\mathcal{K}_j \vdash\!\!< \mathcal{K}_l \Leftrightarrow (\exists\ K_\gamma \in \mathcal{K}_j,\ K_\delta \in \mathcal{K}_l)(K_\gamma \le K_\delta),$$

is well-defined, since the definition does not depend on a specific choice of K_γ, K_δ.

With respect to this order, $\mathfrak{L}$ forms rather regular structure.

Corollary 6.8. (i) $\mathfrak{L}$ is a directed set, that is, for any two elements $\mathcal{K}_i, \mathcal{K}_j \in \mathfrak{L}$, either (1) or (2) is satisfied:

$$(1)\ \mathcal{K}_i \vdash\!\!< \mathcal{K}_j; \qquad (2)\ (\exists \mathcal{K}_m \in \mathfrak{L})((\mathcal{K}_i \vdash\!\!< \mathcal{K}_m) \& (\mathcal{K}_j \vdash\!\!< \mathcal{K}_m));$$

moreover, any two elements of $\mathfrak{L}$ have an exact upper bound in $\mathfrak{L}$.

(ii) Any interval $[\mathcal{K}_i, \mathcal{K}_j]$ in $\mathfrak{L}$ is completely ordered ■

Proof. (i) If $\mathcal{K}_i, \mathcal{K}_j \in \mathfrak{L}$ are $\vdash\!\!<$-incomparable, set $L = \check{K}_i \vee \check{K}_j$, and let $\mathcal{L}_\gamma$, and K_m be, respectively, the "primary" component, containing L, and the "level two" component, containing top point of $\mathcal{L}_\gamma$. Obviously, both $\mathcal{K}_i \vdash\!\!< \mathcal{K}_m$, and $\mathcal{K}_j \vdash\!\!< \mathcal{K}_m$ are satisfied. It can be easily proved using Lemma 6.5 (iv) that K_m is in fact the exact upper bound of $\mathcal{K}_i$, and $\mathcal{K}_j$.

(ii) Directly follows from Lemma 6.5 (iv) ■

So, $\mathfrak{L}$ is a tree-like semilattice. Let us clarify the shape of $\mathcal{L}_\gamma$'s with their top points belonging to a non-single-point component $\mathcal{K}_j$.

Lemma 6.6. Let $K = K_\gamma$ be the top point of a component $\mathcal{L}_\gamma$; suppose that K is included in the subset $\mathcal{K}_j \setminus \{\hat{K}_j, \check{K}_j\}$. The following assertions are true:

(i) If K is the point of strict lower monotonicity of $\mathcal{K}_j$, that is, $\mathcal{M}'_K \cap \mathcal{K}_j = \mathcal{M}''_K \cap \mathcal{K}_j = \emptyset$, then $\varphi|_{\mathcal{M}'_K} = \text{const}$, $\varphi|_{\mathcal{M}''_K} = \text{const}$, and the shape of $\mathcal{L}_\gamma$ is due to the following cases:

(A) $\mathcal{L}_\gamma = \mathcal{M}'_K \cup \{K\}$; in such case, $\varphi|_{\mathcal{M}''_K} > \gamma$;

(B) $\mathcal{L}_\gamma = \mathcal{M}''_K \cup \{K\}$; in such case, $\varphi|_{\mathcal{M}'_K} > \gamma$;

(C) $\mathcal{L}_\gamma = \mathcal{M}'_K \cup \mathcal{M}''_K \cup \{K\}$; in such case, $\varphi|_{\mathcal{M}'_K} = \varphi|_{\mathcal{M}''_K} = \gamma$;

moreover, if φ is lower semi-continuous in the point $K \in T''_1$, then (C) is in force.

(ii) If $\mathcal{M}'_K\cap\mathcal{X}_j\neq\emptyset$, then $\varphi|_{\mathcal{M}''_K}$=const. Set $K'_0=\wedge(\mathcal{M}'_K\cap\mathcal{X}_j)$, $K^{0\prime}=\vee(]K',K^1]\cap\mathcal{X}_j)$; then $K'_0\leq K\leq K^{0\prime}$. Furthermore, the shape of $\mathcal{L}_\gamma$ satisfies one of following possibilities:

(A) $\mathcal{L}_\gamma=\mathcal{M}'_K\cup\{K\}$; in such case, $\varphi|_{\mathcal{M}''_K}>\gamma$;

(B) $\mathcal{L}_\gamma=\mathcal{M}'_K\cup\mathcal{M}''_K\cup\{K\}$;

(B) $\mathcal{L}_\gamma=\mathcal{M}''_K\cup\{K\}$.

In addition, (A) and (B) are mutually exclusive, and each of these possibilities takes place in at most one point of the interval $[K'_0,K^{0\prime}]$. All these statements remain true when changing $'$ for $''$, and vice versa ∎

Proof. Lemma 6.5 (i) implies $\mathcal{M}'_K\cap\mathcal{X}=\mathcal{M}''_K\cap\mathcal{X}=\emptyset$. Let L_1 be any point in a subset $\mathcal{M}'_K\cap\mathcal{L}_\delta$. In such case, $L_1=L_1\wedge K_\delta\leq K\cap K_\delta\in T''_1$. Owing to Lemma 6.5 (i), K_δ and K are $\leq$-comparable; hence, $K_\delta\leq K$ (the assumption $K_\delta>K$ is in contradiction with monotonicity of φ). It follows that $K_\delta\in[L_1,K]$, and hence, in virtue of strict monotonicity of $\mathcal{X}_j$ in the point K, $K_\delta=K$. According to Lemma 6.4, $[L_1,K]\subset\mathcal{L}_\delta$; since L_1 is any point of $\mathcal{M}'_K\cap\mathcal{L}_\delta$, it follows that $\varphi|_{\mathcal{M}'_K}$=const. The proof for $\mathcal{M}''_K$ is similar. Note that the point K is adjacent only with one-dimensional components $\mathcal{L}_\gamma$; otherwise, φ must be constant in certain neighborhood $U(K)\cap]0,K]$, which is in contradiction with continuity and strict monotonicity of $\mathcal{X}_j$ in the point K. Hence, all possibilities for the adjacent $\mathcal{L}_\gamma$'s are exhausted by (A)-(C). Finally, if φ is lower semi-continuous in K, then $\varphi|_{\mathcal{M}'_K}=\varphi|_{\mathcal{M}''_K}=\limsup_{L\to K}\varphi(L)=\varphi(K)$, which is nothing but the variant (C).

(ii) Repeats the proof of (i) with insignificant modifications ∎

It follows from our consideration of "one-dimensional" $\mathcal{X}_j$'s, that each strictly monotone component $\mathcal{X}_j$ defines a solid angle

$$\mathcal{A}_j=]0,\check{K}_j]\setminus]0,\hat{K}_j],$$

and that all level sets of $\varphi|_{\mathcal{A}_j}$ are either "horizontal" intervals $\mathcal{M}'_K$ or "vertical" intervals $\mathcal{M}'_K$ or, maybe, their union - the angle $\mathcal{M}_K=\mathcal{M}'_K\cup\mathcal{M}''_K\cup\{K\}$. In case when connected components of level sets of φ are just intervals, exactly one of these components contains the point K itself. With non-strictly monotone component $\mathcal{K}_j$, the picture is more complicated.

This is the most general representation of the structure, and the disposition of one-dimensional connected components of level sets of φ, $\mathcal{L}_\gamma^{(1)}$. However, solid angles $\mathcal{A}_j$ can totally cover T_1'' only in case of uniqueness of $\mathcal{K}_j$, that is, when $\Omega=\{\mathcal{K}_1\}$, $K_1^\wedge \in t'$, $K_1^\vee = \mathbb{I}$, so that $\mathcal{K}_1$ cuts T_1'' into two disjoint parts. But in general, the supplement $T_1'' \setminus \bigcup \mathcal{A}_j$ contains numerous "gaps". Just these gaps give room to both the one-point "level two" components $\mathcal{K}_j=\{K_1^\wedge\}=\{K_1^\vee\}$ and the two-dimensional "primary" components $\mathcal{L}_\gamma$.

In order to describe in details the shape of two-dimensional components, we introduce several additional definitions. With $\mathcal{K}_j \in \Omega$, the two intervals $\mathcal{M}'_{K_j}\wedge$, $\mathcal{M}''_{K_j}\wedge$ (resp., $\mathcal{M}'_{K_j}\vee$, $\mathcal{M}''_{K_j}\vee$) are called *lower intervals of* $\mathcal{K}_j$ (resp., *upper intervals of* $\mathcal{K}_j$). The union of these intervals is called *lower angle* (resp., *upper angle*) of $\mathcal{K}_j$. A component $\mathcal{K}_j$ is called *lower/upper limiting* (resp., *lower/upper isolated*) iff there exists (resp., iff there does not exist) an increasing/decreasing sequence $\{K_m^\vee\}/\{K_m^\wedge\}$ converging in T_1'' to $K_j^\wedge/K_j^\vee$. The set of all lower isolated components $\mathcal{K}_j$ is denoted by Ω_{iso}. A *free part* of an upper/lower interval or of an upper/lower angle is defined as their maximal connected subset, which is free of top points (except for "their own" top point $K_j^\wedge/K_j^\vee$). If a free part coincides with the corresponding upper/lower interval or an angle, we speak of *free interval*, and of *free angle*. The bounds of free part are denoted by $\underline{Q}'_j$, $\underline{Q}''_j$, $\overline{Q}'_j$, $\overline{Q}''_j$. Next, we call two components $\mathcal{K}_1$, $\mathcal{K}_m$ *semi-conjugate* iff $K_1^\vee \wedge K_m^\vee \in t'$; three components $\mathcal{K}_j$, $\mathcal{K}_1$, $\mathcal{K}_m$ are called *conjugate* iff $\mathcal{K}_1$, $\mathcal{K}_m$ are *semi-conjugate* and, in addition, the equality $K_1^\vee \vee K_m^\vee = K_j^\wedge$ takes place. Let us denote by $\mathcal{O}(\mathcal{K}_j)$ the set of all $\mathcal{K}_m \in \Omega$ that are *covered* by $\mathcal{K}_j$ in a $\vee$-semilattice Ω. Finally, with $K_j^\wedge$ being a bottom point of a component $\mathcal{K}_j$, we denote by Ξ_j the "comb"

$$\Xi_j =]\Phi, K_j^{\wedge}] \setminus \bigcup_{\mathcal{K}_m \vdash\!< \mathcal{K}_j}]\Phi, K_m^{\vee}]$$

With the latter definitions, complete description of the behavior of a "transitive representation" $\varphi \in \Phi_t$ in the "gaps" between the solid angles $\mathcal{A}_j$ is as follows.

Lemma 6.7. Set $\gamma = \varphi(K_j^{\wedge})$.

(i) If $\mathcal{K}_j$ is lower limiting, then at least one of lower intervals, or a lower angle of $\mathcal{K}_j$ is free and makes one (or two) one-dimensional connected component(s) of level sets of φ; K is a top point of this component (these components). In addition, if both intervals are free and constitute two different components $\mathcal{L}_\gamma^{(1)}$, $\mathcal{L}_\delta^{(1)}$, then $\delta > \gamma$, and $K_j^{\wedge} \in \mathcal{L}_\gamma^{(1)}$.

(ii) If $\mathcal{K}_j$ is lower isolated, then Ξ_j is a non-empty open subset of T_1''; in addition, $\gamma \le \delta = \varphi|_{\Xi_j} = \text{const}$. More precisely, the corresponding two-dimensional component $\mathcal{L}_\delta^{(2)}$ can have the following shape:

(1) If at least one of the two lower intervals of $\mathcal{K}_j$ is not free, then $\gamma = \delta$, and $\mathcal{L}_\delta^{(2)}$ includes, except for the open comb Ξ_j, the whole free part of $\mathcal{K}_j$;

(2) If there exist $\mathcal{K}_l$, and $\mathcal{K}_m$, making, together with $\mathcal{K}_j$, a conjugate triple of components, then $\mathcal{L}_\delta^{(2)}$ includes, in addition to what is written in (1), one or two among the intervals $](\underline{Q}_j')'', \underline{Q}_j']$, $](\underline{Q}_j'')', \underline{Q}_j'']$.

(3) If two lower intervals of $\mathcal{K}_j$ are free, then one of the following three assertions takes place:

(A) $\mathcal{L}_\delta^{(2)}$ does not intersect lower angle of $\mathcal{K}_j$; in this case, $\delta > \gamma$;

(B) $\mathcal{L}_\delta^{(2)}$ contains one of the lower intervals of $\mathcal{K}_j$, and $\delta > \gamma$;

(C) $\mathcal{L}_\delta^{(2)}$ contains lower angle of $\mathcal{K}_j$; in this case, $\delta = \gamma$;

(4) If $\mathcal{K}_l \in \mathcal{O}(\mathcal{K}_j)$ is a single-point component, then one of the following two possibilities may occur:

(A) In case when $\mathcal{K}_1$ is either lower limiting or it is lower isolated, but its upper angle (since $\mathcal{K}_1$is one-point, it is simultaneously a lower angle), is not free, $\mathcal{L}_\delta^{(2)}$ is disjoint with the last angle;

(B) In case when $\mathcal{K}_1$ is lower isolated, and its upper angle is free, $\mathcal{L}_\delta^{(2)}$ contains, either one of the upper intervals of $\mathcal{K}_1$ (together with the point $\overset{\vee}{K}_1$) or the whole upper interval of $\mathcal{K}_1$, together with this point;

(5) If $\mathcal{K}_1 \in \mathcal{O}(\mathcal{K}_j)$ is a non-single-point (that is, a one-dimensional) component, then one of the following two possibilities may occur:

(A) In case when upper angle of $\mathcal{K}_1$ is free, $\mathcal{L}_\delta^{(2)}$ is the same as in (4)(B);

(B) In case when upper angle of $\mathcal{K}_1$ is not free, $\mathcal{L}_\delta^{(2)}$ contains a free part of upper interval of $\mathcal{K}_1$ (if exists) together with the point $\overset{\vee}{K}_1$.

(iii) With any two-dimensional connected component of level sets of φ, at least one of the sub-items of (ii) takes place ∎

Proof. (i) Let us suppose that both lower intervals of $\mathcal{K}_j$ are not free. In such case, it can be derived from Lemma 6.5 (i) that there exists an open neighborhood of a bottom point $\hat{K}_j$ of a component $\mathcal{K}_j$, which does not contain top points. It follows that $\mathcal{K}_j$ is lower isolated, which is in contradiction with the condition of (i). All the remaining assertions are obvious.

(ii) By definition, Ξ_j is a non-empty open subset of T_1'', containing none of the top points. Hence, any $\mathcal{L}_\eta$ which is intersecting with Ξ_j has the top point $\hat{K}_j$. Corollary 6.7 and monotonicity of φ imply that $\gamma \le \varphi|_{\Xi_j}$=const.

All the remaining statements are implied by Lemma 6.1, used for appropriate triples of conjugate points. For example,

(4) (A). A single-point and lower limiting $\mathcal{K}_1$ represents the top point of both its lower (=upper) intervals, or of its lower (=upper) angle, so that $\overset{\vee}{K}_1 \in \mathcal{L}_\eta^{(1)}$ for some η. If $\mathcal{L}_\eta^{(1)}$ is the whole upper angle of $\mathcal{K}_1$, then $\eta>\delta$ by definition of the top point; otherwise, if $\mathcal{L}_\eta^{(1)}$ is one of the upper intervals, then $\eta>\delta$, and, in addition, in any point L of the other

interval, $\varphi(L)>\delta$. It follows that $\mathcal{L}_\delta^{(1)}$ and the upper interval of $\mathcal{K}_1$ are disjoint.

(iii) It is sufficient to consider the top point K_δ of the $\mathcal{L}_\delta^{(2)}$, and the "level two" component $\mathcal{K}_j \ni K_\delta$. Since $\mathcal{L}_\delta^{(2)}$ is two-dimensional, $\mathcal{K}_j$ is lower isolated, and the proof is reduced to (ii) ■

Gathering all the above results, we arrive to the following *transitivity criterion for fuzzy inclusions.*

Theorem 6.7. A representation $\varphi \in \Phi_t$ of a transitive *FI* is completely defined by the following objects (1)-(6):

(1) A $\vee$-semilattice $(\mathfrak{L}, \vdash\!\!<)$, satisfying conditions of Corollary 6.8.

(2) An isotone mapping of $\mathfrak{L}$ into the partially ordered family of pairs of points $(\{(K_j^\wedge, K_j^\vee)\}, -\!\!\ll)$ in T_1'':

$$K_j^\wedge \in \hat{T}_1''; \quad K_j^\vee \in \hat{T}_1'' \setminus t'; \quad ((K_i^\wedge, K_i^\vee) -\!\!\ll (K_j^\wedge, K_j^\vee) \Leftrightarrow (K_i^\vee \le K_j^\wedge));$$

in addition, for each two $-\!\!\ll$-incomparable pairs $(K_i^\wedge, K_i^\vee)$, $(K_j^\wedge, K_j^\vee)$, the condition $K_i^\vee \wedge K_j^\vee \in T'$ is satisfied (see Lemma 6.5 (iv)).

(3) A family of continuous monotone curves $\mathcal{K}_j$ in T_1'', starting in $K_j^\wedge$ and ending in $K_j^\vee$ (when $K_j^\wedge = K_j^\vee$, $\mathcal{K}_j$ is single-point) - see Lemma 6.5 (iii).

(4) A family of finite or countable subsets $\mathcal{N}_j \subset \mathcal{K}_j$, containing at most one point on each interval of non-strict monotonicity of $\mathcal{K}_j$ (see Lemma 6.6 (ii)).

(5) A family of non-increasing functions $\varphi_j^-, \varphi_j^+ : \mathcal{K}_j \longrightarrow I$ satisfying conditions of Lemma 6.6 (ii) ($\varphi_j^-(K) = \varphi|_{\mathcal{M}_K'}$, $\varphi_j^+(K) = \varphi|_{\mathcal{M}_K''}$ are the values of φ "to the left" and "to the bottom" of the component $\mathcal{K}_j$ within the solid angles $\mathcal{A}_j$).

(6) A strictly antitone mapping $\mathfrak{L}_{iso} \longrightarrow I$, consistent with mapping (2) and satisfying, together with functions from (5), the conditions of Lemma 6.7 (values of φ_{inc} on two-dimensional components $\mathcal{L}_\gamma^{(2)}$).

Given all objects from (1)-(6), a representation $\varphi \in \Phi_t$ is evaluated as follows:

$\varphi|_{T''_1}$ - with respect to Lemmas 6.6, 6.7;

$\varphi|_{T''_2}$ - with respect to Lemma 6.3;

$\varphi|_{T'}$ - can be chosen arbitrarily, with the only requirement of preserving monotonicity of φ on T.

Finally, a *FI* inc is reconstructed, due to Theorem 6.3, as $\nu^{-1}(\varphi)$ ■

Proof. The necessity of all conditions was established during our research. Let us prove the sufficiency. Given a $\varphi\in\Phi$ satisfying (1)-(6), the conditions of Lemma 6.1 for all conjugate points $K,L,M\in T''_1$ are satisfied:

with $K\in\mathcal{A}_j$ - obviously;

with $K\in\mathcal{L}^{(2)}_\gamma$, and $L,M\notin\mathcal{X}$ - obviously;

with $K\in\mathcal{L}^{(2)}_\gamma$, $L=\check{K}_1,M=\check{K}_m$- owing to Lemma 6.7 (ii).

With $K\in T''_2$, in virtue of Lemmas 6.2, 6.3, the condition of Lemma 6.1 must be verified only when $K\in[(1/2,1/2),(1,1/2)]$, which is the same with the previous check. With $K\in T'$, in virtue of Lemma 6.2 (i), the check is obvious ■

A more or less general example of the disposition of level sets of φ on T''_1 is depicted on Figure 6.3.

So far, in spite of rather exotic description, the family of transitive *FI*'s is rather "populous". Thus, representations of transitive *FI*'s contain all functions $\varphi\in\Phi$ satisfying one of the following conditions:

1) $\varphi|_{T''}=\varphi(\alpha)$;

2) $\varphi|_{T''}=\varphi(\beta)$ ($\varphi(\beta)\equiv 0$, when $\beta\geq 1/2$)

(see Section 6.7 for more details). There also exist *continuous* and transitive *FI*'s - say, all *FI*'s presented in the latter example, with an appropriate choice of $\varphi(\beta)$, and with continuous extension to T'. Nevertheless, a representation of a transitive *FI* can not be *analytic* function on T. Indeed, with any continuous $\varphi\in\Phi_t$, there either exists a two-dimensional $\mathcal{L}^{(2)}_\gamma$ or $\varphi|_{T''_2}\equiv 0$. This makes analytic function to be constant on T, in contradiction with boundary values $1=\varphi(0,0)\neq\varphi(1,1)=0$.

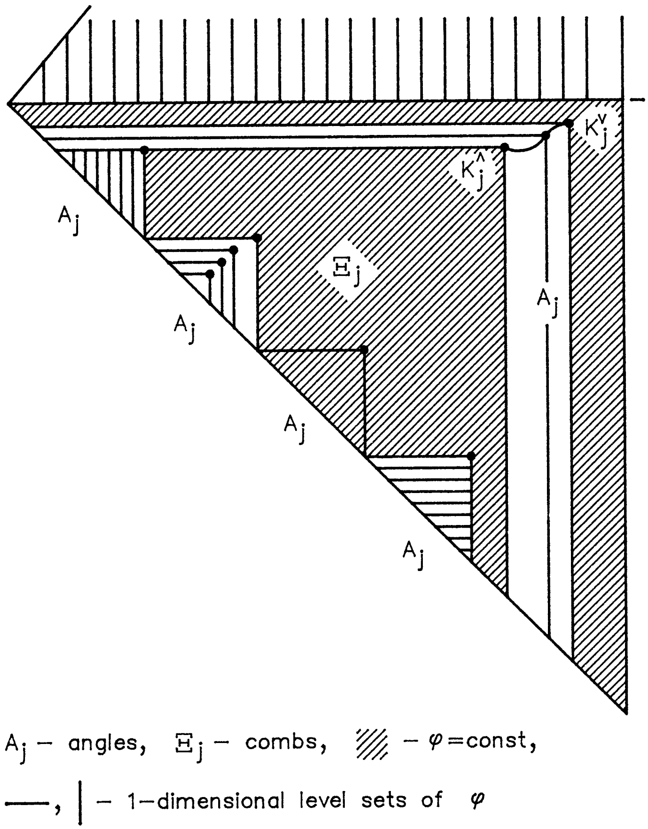


Fig. 6.3. Representation of a transitive *FI* on T_1'', T_2''

Antitransitivity of *FI*'s can be studied in a "dual" manner (all constructions are translated, with insignificant changes, from the transitive case through reflection in the diagonal t'). In fact, antitransitive *FI*'s form a smaller class $\mathcal{Inc}_{at}$, in the sense that the canonical mapping $\hat{\tau}:\Phi\longrightarrow\Phi$,

$$(\hat{\tau}(\varphi))(\alpha,\beta)=\begin{cases}\overline{\varphi(\tau(\alpha,\beta))}, & (\alpha,\beta)\in T\setminus[(0,0),(1,0)] \\ 1, & (\alpha,\beta)\in[(0,0),(1,0)]\end{cases}$$

causes imbedding of $\Phi_{at}=\nu(\mathcal{Inc}_{at})$ into Φ_t, whereas the inverse mapping $\hat{\tau}^{-1}$, being defined on a subset $\Phi_t^0=\{\varphi\in\Phi_t \ |\varphi|_{[(1,0),(1,1)]}\equiv 0\}\subset\Phi_t$, maps Φ_t^0 in a subset $\hat{\tau}^{-1}(\Phi_t^0)$ strictly containing Φ_{at}.

The study of compatibility/incompatibility of algebraic properties of *FI*'s with their continuity (see Corollary 6.5) can now be expanded. In the following two corollaries, several "positive", and "negative" interrelations between algebraic properties, and continuity of *FI*'s are established.

Corollary 6.9. (i) With $n\geq 2$, any antisymmetric *FI* is transitive.

(ii) Any perfectly antisymmetric *FI* is transitive ∎

Proof. We begin with (ii). According to Theorem 6.5. (i), $\varphi_{inc}|_{T''}\equiv 0$, so that φ_{inc} obviously satisfies the conditions of Theorem 6.7 ($\varphi_{inc}|_{T''_1}$ is constant, $\varphi_{inc}|_{T''_1}$ has the only two-dimensional component $\mathcal{L}^{(2)}_\gamma=T''_1$, $\gamma=0$). Hence, $\varphi_{inc}\in\Phi_t$, and $inc\in\mathcal{I}nc_t$.

(i) In virtue of Corollary 6.3, with $n\geq 2$, an antisymmetric (in any meaning) *FI* is perfectly antisymmetric; hence, owing to (ii), inc is transitive ∎

Corollary 6.10. No *FI* can possess:

(1) reflexivity+transitivity+continuity;

(2) antisymmetry+antitransitivity+continuity ∎

Proof. (1) If an *FI* inc is reflexive, transitive and continuous, then, in virtue of reflexivity, $\varphi_{inc}|_{T'}\equiv 1$ (see Theorem 6.4. (i)); in particular, $\varphi_{inc}|_{t'}\equiv 1$. Using transitivity, let us consider any of the components $\mathcal{L}_\gamma$ with $K_\gamma=\mathbb{I}=(1,1/2)$. On the one hand, Lemma 6.3 results in the equalities $\varphi_{inc}(\mathbb{I})=\varphi_{inc}(1,1)=0$. Since $\varphi_{inc}|_{\mathcal{L}_\gamma}=\text{const}$, and $\mathbb{I}\in\hat{\mathcal{L}}_\gamma$ (Corollary 6.7) continuity of φ_{inc} implies the equality $\varphi_{inc}|_{\mathcal{L}_\gamma}\equiv 0$. On the other hand, any connected component $\mathcal{L}_\gamma$ has at least one limit point in a subset t' (see Lemmas 6.6, 6.7); arguing in the same way, we come to $\varphi_{inc}|_{\mathcal{L}_\gamma}\equiv 1$. This contradiction proves the non-existence of *FI*'s satisfying the above properties.

(2) The proof is dual to (1) ∎

In other words, *FI*'s cannot simultaneously possess well-defined algebraic and topological properties. For this reason, the criticism about classical fuzzy inclusion $\subseteq$ seems to be non-motivated: a *transitive* and *reflexive* *FI*, satisfying the above axiomatics, is necessarily *discontinuous*.

6.5. BINARY OPERATIONS WITH FUZZY INCLUSIONS

In this section, we study the structure of binary operations on $\mathcal{I}nc$, and on the set of representations Φ. Let $\ominus$ be any binary operation on I, extended to $\tilde{\mathcal{P}}(Y)$ for an arbitrary set Y by the "pointwise" formula

$$\mu_{a\ominus b}=\mu_a\ominus\mu_b, \tag{6-0}$$

with $a,b\in\tilde{\mathcal{P}}(Y)$ being any f.s.' of Y. The problem is to define necessary and sufficient conditions on $\ominus$, resulting in the closeness of $\mathcal{I}nc$, and Φ with respect to thus induced operation. In other words, given a pair $inc_1,inc_2\in\mathcal{I}nc$ $(\varphi_1,\varphi_2\in\Phi)$, we require that $inc_1\ominus inc_2$ must also be in $\mathcal{I}nc$ ($\varphi_1\ominus\varphi_2$ must belong to Φ).

Theorem 6.8. (i) The mapping $\nu:\mathcal{I}nc\longrightarrow\Phi$ (see Theorem 6.2) is a "local homomorphism" in the following meaning: suppose that $\ominus$ satisfies (6-0), and that inc_1, inc_2, $inc_1\ominus inc_2\in\mathcal{I}nc$. In such case,

$$\nu(inc_1\ominus inc_2)=\nu(inc_1)\ominus\nu(inc_2).$$

(ii) The mapping $\nu^{-1}:\Phi\longrightarrow\mathcal{I}nc$ is $\wedge$-homomorphism ■

Proof. Directly follows from the explicit expressions for ν, ν^{-1} (see Theorem 6.3) ■

Theorem 6.9. (i) With $n\geq 2$, $\mathcal{I}nc$ is closed with respect to $\ominus$ iff $\ominus$ is a *quasi-conjunction*: $\alpha\ominus\beta=f_\ominus(\alpha\wedge\beta)$, with $f_\ominus:I:\longrightarrow I$ being a non-decreasing function with respect to both arguments, satisfying boundary conditions $f_\ominus(0)=0$, $f_\ominus(1)=1$.

(ii) Φ is closed with respect to $\ominus$ (and also $\mathcal{I}nc$ is closed with respect to $\ominus$ for $n=1$) iff $\ominus$ is a non-decreasing function with respect to both arguments, $0\ominus 0=0, 1\ominus 1=1$ ■

Proof. (i) To prove closeness of $\mathcal{I}nc$ with respect to a binary operation $\ominus$, we need to check axioms A_1-A_4 for $inc_1\ominus inc_2$. We draw an attention to the fact that $inc_1 \ominus inc_2$ satisfies Contraposition Axiom A_1, and Symmetry Axiom A_3 for *any* binary operation. As to A_4, the only limitation imposed by this axiom concerns boundary values, $0\ominus 0=0$, $1\ominus 1=1$, thus making the latter condition necessary. So, the only non-trivial work is to check Distributivity axiom A_2.

Let us suppose that $\mathcal{I}nc$ is closed with respect to $\ominus$, $a,b,c\in\tilde{\mathcal{P}}(X)$ are f.s.' of X, and $inc_1, inc_2\in\mathcal{I}nc$ are *FI*'s. An explicit formula for Distributivity Axiom A_2 is

$$\mu_{inc_1\ominus inc_2}(a\vee b,c)=\mu_{inc_1\ominus inc_2}(a,c)\wedge\mu_{inc_1\ominus inc_2}(b,c).$$

Writing out definition of $\ominus$ according to (6-0), we come to

$$\mu_{inc_1}(a\vee b,c)\ominus\mu_{inc_2}(a\vee b,c)$$
$$=(\mu_{inc_1}(a,c)\ominus\mu_{inc_2}(a,c))\wedge(\mu_{inc_1}(b,c)\ominus\mu_{inc_2}(b,c)).$$

Next, using A_2 for inc_1, inc_2 in the left-hand side of the latter equality, we transform it into

$$(\mu_{inc_1}(a,c)\wedge\mu_{inc_1}(b,c))\ominus(\mu_{inc_2}(a,c)\wedge\mu_{inc_2}(b,c))$$
$$=(\mu_{inc_1}(a,c)\ominus\mu_{inc_2}(a,c))\wedge(\mu_{inc_1}(b,c)\ominus\mu_{inc_2}(b,c)) \tag{6-1}$$

For brevity, let us introduce the notations $\alpha_1=\mu_{inc_1}(a,c)$; $\alpha_2=\mu_{inc_1}(b,c)$; $\beta_1=\mu_{inc_2}(a,c)$; $\beta_2=\mu_{inc_2}(b,c)$. With these notations, (6-1) is nothing but the equality

$$(\alpha_1\wedge\alpha_2)\ominus(\beta_1\wedge\beta_2)=(\alpha_1\ominus\beta_1)\wedge(\alpha_2\ominus\beta_2) \tag{6-2}$$

First of all, (6-2) obviously implies non-decreasing of $\ominus$ with respect to both arguments. Furthermore, if $n\geq 2$, then, given a number $\alpha\in[0,1[$, we can find $a,b,c\in\tilde{\mathcal{P}}(X)$, inc_1, $inc_2\in\mathcal{I}nc$, providing the following specific combination of values of α_i, β_i: $\alpha_1=\beta_2=\alpha$, $\alpha_2=\beta_1=1$. Applying (6-2) to this combination, we come to $(\alpha\wedge 1)\ominus(1\wedge\alpha)=(\alpha\ominus 1)\wedge(1\ominus\alpha)$, that is, $\alpha\ominus\alpha=\alpha\ominus 1$. It follows that $(\forall\ \beta\geq\alpha)(\alpha\ominus\beta=\alpha\ominus\alpha)$; hence, $\alpha\ominus\beta\equiv(\alpha\wedge\beta)\ominus(\alpha\wedge\beta)$, and finally, $f_\ominus(\alpha)=\alpha\ominus\alpha$. (we mark out an obvious tie between this result and Theorem 6.1).

Conversely, with $\alpha\ominus\beta=f_\ominus(\alpha\wedge\beta)$, the validity of axiom A_2 for $inc_1\ominus inc_2$ follows from the fact that (6-2) is satisfied for all α_i's, β_i's.

(ii) Closeness of Φ with respect to a monotone $\ominus$ with $0\ominus0=0$, $1\ominus1=1$, is obvious (both the antimonotonicity, and the boundary values of representations survive). On the contrary, if $\ominus$ is non-monotone, then there can be easily found $\varphi_1,\varphi_2\in\Phi$ with a non-antitone $\varphi_1\ominus\varphi_2\notin\Phi$; hence, Φ is not closed under the influence of $\ominus$. Finally, if $\ominus$ has other boundary values, then the values of $\varphi_1\ominus\varphi_2$ in the angles of T are distorted, and again, $\varphi_1\ominus\varphi_2\notin\Phi$.

With n=1, closeness of $\mathcal{I}nc$ with respect to $\ominus$ follows from the fact that $\tilde{\mathcal{P}}(X)$ is completely ordered. Hence, in the formulas (6-1), (6-2), $\alpha_1<\alpha_2$ implies $a\geq b$, and consequently, $\beta_1\leq\beta_2$; conversely, $\beta_1<\beta_2$ leads to $\alpha_1\leq\alpha_2$. Under these conditions, (6-2) is equivalent to monotonicity of $\ominus$, so that, A_2 for $inc_1\ominus inc_2$ is true ■

6.6. CHARACTERISTIC FUZZY INCLUSIONS (POLYNOMIAL AND PIECEWISE-POLYNOMIAL MODELS)

In this section, we investigate distinguished representatives and subfamilies of *FI*'s. Apart from other reasons, this study motivates the choice of *FI*'s to be used in fuzzy decision procedures. Following the ideas of D.Dubois and H.Prade [1], we propose characterization of both the inclusion inc itself, and of the associated "ε-inclusions", that is, of ε-cuts inc_ε, and of strict ε-cuts $\dot{inc}_\varepsilon$ of the original inclusions (according to Proposition 6.3 (v), inc_ε, and $\dot{inc}_\varepsilon$ are also *FI*'s). Taking into consideration the influence of the book (Dubois and Prade [1]) upon conventional fuzzy notations, we confine ourselves to the original notation of *FI*'s proposed in this book. Correspondingly, we keep unchanged the notations for fuzzy inclusions/implications borrowed from other papers.

Theorem 6.10 (characterization of L.Zadeh' inclusion $\subseteq$).

(i) L.Zadeh' inclusion $\subseteq$ represents:

(1) the smallest *reflexive FI*;

(2) the greatest *perfectly antisymmetric FI*;

(3) the only *FI* possessing

reflexivity+perfect antisymmetry+transitivity.

(ii) Representation of $\subseteq$ is the characteristic function of the subtriangle T′: $\varphi_{\subseteq}=\chi_{T'}$ ■

Proof. (i) All (1)-(3) statements were already proved in Sections 6.4.1, 6.4.2 (Theorem 6.4, Corollary 6.3, Corollary 6.9).

(ii) Directly follows from (i), Theorem 6.4, and Corollary 6.3 ∎

Therefore, L.Zadeh' inclusion is the only *FI* in $\mathcal{I}nc$ satisfying all classical ideas of inclusion. We already know from Sections 6.4.1, 6.4.2 that the "ability" of a continuous *FI* to fulfill "well-defined" algebraic properties is very limited. However, continuity of an inclusion is a very natural and significant requirement in fuzzy environment. For this reason, we consider another class of algebraic properties, namely, the polynomial behavior of representations. In Theorems 6.11, 6.12, 6.14, we mark out several continuous *FI*'s possessing best possible properties of this class.

We recall that, given a De Morgan triple $(\vee,\wedge,\bar{\ })$, Kleene - Dienes implication $\xrightarrow[K]{}$ on the set of truth values I is defined by means of the formula (see, e.g., Dubois and Prade [2,4]):

$$\alpha \xrightarrow[K]{} \beta = \bar{\alpha} \vee \beta = \overline{\alpha \wedge \bar{\beta}} \tag{6-3}$$

(strictly speaking, this expression requires that the negation $\bar{\ }$ should be the *strong* negation, that is, the idempotent mapping of a unit interval into itself).

Most propagated examples of Kleene - Dienes implications are associated with a classical fuzzy negation $\bar{\alpha}=1-\alpha$, and with disjunction based on one of the three t-norms, namely:

the greatest t-norm $\wedge$ gives rise to a commonly used disjunction $\vee$;

the Lukasiewicz' t-norm 1 (for the above-mentioned reason of preserving of notations, we denote the corresponding disjunction, which is also known as *bold union*, by $\dot{\vee}$, $\alpha\dot{\vee}\beta=1\wedge(\alpha+\beta)$);

the probabilistic t-norm $\cdot$ induces the disjunction $\hat{+}$,

$\alpha\hat{+}\beta=\overline{\bar{\alpha}\cdot\bar{\beta}}=\alpha+\beta-\alpha\cdot\beta$ (*probabilistic sum*).

In case of fuzzy inclusions, and of "multidimensional" fuzzy subsets $(n>1)$, it seems natural to redefine the corresponding inclusion between f.s.', in the spirit of Representation Theorem, and of Proposition 6.3 (iv) using the formula

$$\mu_{inc_K}(a,b)= \bigwedge_{x\in X} (\mu_a(x) \xrightarrow[K]{} \mu_b(x))=\mathring{\lambda}(a \xrightarrow[K]{} b) \tag{6-4}$$

with $\bigwedge_{x\in X}$ being based on ordinary $\wedge$ (see Section 6.1). In the sequel, we denote the corresponding *FI*'s for $\wedge$, 1, and $\cdot$ respectively by I_5, I_4, and $inc_{\hat{\dagger}}$ (the first, and the second of these notations are used by D.Dubois and H.Prade [1]).

One can pose a question whether these Kleene - Dienes inclusions are really *FI*'s. The answer proves to be positive not only for the above representatives, but for an arbitrary t-norm.

Lemma 6.8. With a negation $\bar{\alpha}=1-\alpha$, and an arbitrary t-norm t, Kleene - Dienes inclusion inc_K based on a disjunction $\underset{t}{\vee}$ belongs to $\mathcal{I}nc$ ■

Proof. A_1 immediately follows from (6-3), idempotentness of a negation, and commutativity of a t-norm: $\overline{\alpha\underset{t}{\wedge}\bar{\beta}}=\overline{\bar{\beta}\underset{t}{\wedge}\bar{\bar{\alpha}}}$.

A_2 can be derived from the monotonicity of t, De Morgan properties of a conventional triple $(\vee,\wedge,\bar{\ })$, and the fact that any t-norm is distributive with respect to $\wedge$, and $\vee$:

$$(\alpha\vee\beta)\underset{K}{\rightarrow}\gamma=\overline{(\alpha\vee\beta)\underset{t}{\wedge}\bar{\gamma}}=\overline{(\alpha\underset{t}{\wedge}\bar{\gamma})\vee(\beta\underset{t}{\wedge}\bar{\gamma})}=\overline{(\alpha\underset{t}{\wedge}\bar{\gamma})}\wedge\overline{(\beta\underset{t}{\wedge}\bar{\gamma})}=(\alpha\underset{K}{\rightarrow}\gamma)\wedge(\beta\underset{K}{\rightarrow}\gamma)$$

In this chain of equalities, the first, and the last ones are fulfilled by definition of $\underset{K}{\rightarrow}$, the second equality is due to distributivity of $\underset{t}{\wedge}$ with respect to $\vee$, the third one represents to De Morgan duality. Clearly, the external convolution $\underset{x\in X}{\wedge}$ also commutes with $(\alpha\underset{K}{\rightarrow}\gamma)\wedge(\beta\underset{K}{\rightarrow}\gamma))$.

A_3 is easily implied by (6-4).

A_4 is a straightforward consequence of boundary properties of a t-norm. So, all axioms are fulfilled, and $inc_K\in\mathcal{I}nc$ ■

Theorem 6.11 (characterization of I_5).

Kleene - Dienes inclusion I_5 associated with a t-norm $\wedge$ is:

(i) the only inclusion in $\mathcal{I}nc$, possessing *linear representation* $\varphi_{I_5}(\alpha,\beta)=\bar{\beta}$.

(ii) the only inclusion in $\mathcal{I}nc$, *representative in polynomial form* $\mu_{inc}(a,b)=\overset{o}{\wedge}p(a,b)$, with p being a $\vee$-$\wedge$-polynomial form, depending on fuzzy variables a, b, $\bar{a}$, $\bar{b}$ ■

Proof. (i) The uniqueness of linear representation in Φ is obvious in virtue of the existence of three fixed values of any representation in the angles of T. It is also clear that the mentioned linear function in Φ is exactly $\varphi(\alpha,\beta)=\bar{\beta}$. Let us prove that $\bar{\beta}$ is exactly $\varphi_{I_5}=\nu(I_5)$; indeed,

$$\varphi_{I_5}(\alpha,\beta)=\mu_{I_5}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}})=\mathring{\lambda}(\overline{\alpha\chi_{\{x\}}}\vee\overline{\beta\chi_{\{x\}}})=\mathring{\lambda}(\bar{\alpha}\chi_{\{x\}}\vee\bar{\beta}\chi_{\{x\}}\vee\chi_{\overline{\{x\}}})=1\wedge(\bar{\alpha}\vee\bar{\beta})=\bar{\beta}$$

(ii) Let us suppose that $\mathring{\lambda}p(a,b)\in\mathcal{I}nc$. Let us write out a disjunctive normal form of p. According to A_4, $\mathring{\lambda}p|_{\mathcal{P}(X)\times\mathcal{P}(X)}$ must be a conventional crisp inclusion, $\mathring{\lambda}p(\chi_A,\chi_B)=\mathring{\lambda}\overline{\chi_A}\vee\chi_B$. Using the properties of non-fuzzy Boolean functions, we arrive to a conclusion that any conjunction, contained in a disjunctive form of p, except for $\bar{a}$, and b, must be trivial in crisp sense. It follows that any conjunction has to contain either $a\wedge\bar{a}$ or $b\wedge\bar{b}$. Obviously, any conjunction of this type is absorbed either by $\bar{a}$ or by b; therefore, the reduced form of p is $\bar{a}\vee b$ ■

Theorem 6.12 (characterization of I_4)

Kleene - Dienes inclusion I_4 associated with Lukasiewicz' norm l is the only inclusion in $\mathcal{I}nc$, simultaneously possessing the following three properties: (1) *reflexivity*; (2) *continuity*; (3) linear representation on T'' $\varphi_{I_4}(\alpha,\beta)=1\wedge(2-(\alpha+\beta))=\bar{\alpha}\dot{\vee}\bar{\beta}$ ■

Proof. The uniqueness of representation satisfying (1)-(3) can be established in the same manner as in Theorem 6.11. Next,

$$\varphi_{I_4}(\alpha,\beta)=\mu_{I_4}(\alpha\chi_{\{x\}},\overline{\beta\chi_{\{x\}}})=\mathring{\lambda}(\overline{\alpha\chi_{\{x\}}}\dot{\vee}\overline{\beta\chi_{\{x\}}})=\mathring{\lambda}(\bar{\alpha}\chi_{\{x\}}\dot{\vee}\bar{\beta}\chi_{\{x\}}\dot{\vee}\chi_{\overline{\{x\}}})=1\wedge(\bar{\alpha}\dot{\vee}\bar{\beta})=\bar{\alpha}\dot{\vee}\bar{\beta}$$ ■

Theorem 6.13 (characterization of ε-inclusions $I_{5\varepsilon}$, $I_{4\varepsilon}$, $\dot{I}_{5\varepsilon}$, $\dot{I}_{4\varepsilon}$, associated with I_5, I_4).

(i) Reflexivity.

(1) $I_{5\varepsilon}$ ($\dot{I}_{5\varepsilon}$) are reflexive for all $\varepsilon\le 1/2$ ($\varepsilon<1/2$);

(2) $I_{4\varepsilon}$ ($\dot{I}_{4\varepsilon}$) are reflexive for all ε.

(ii) Antisymmetry.

None of conventional and strict ε-inclusions for I_5, I_4 possesses antisymmetry, except for $I_{5\varepsilon}$, $I_{4\varepsilon}$ with $\varepsilon=1$.

(iii) Transitivity.

(1) $I_{5\varepsilon}$ ($\dot{I}_{5\varepsilon}$) are transitive for all $\varepsilon>1/2$ ($\varepsilon\geq 1/2$);

(2) No one of ε-inclusions for I_4 is transitive, except for $I_{4\varepsilon}$ with $\varepsilon=1$, which is $\leq$;

(iv) Antitransitivity. All ε-inclusions for I_5, I_4 are antitransitive ∎

Proof. Routine combination of Theorems 6.11, 6.12 with the results of Section 6.4 ∎

Note 6.1. 1) Theorem 6.13 generalizes well-known results, concerning "weak inclusions" $—\!\!<_\alpha=(I_5)_\alpha$ (see Dubois and Prade [1]) with $\alpha\neq 1/2$.

2) It may seem surprising that a special weak inclusion $—\!\!<$ (see Dubois and Prade [1]) is reflexive as well as transitive, whereas none of ε-inclusions for I_5, I_4 possesses these properties simultaneously. An explanation in the terms of representations is easy. In fact, $—\!\!<$ is obtained by means of "singular transformation" of $\dot{I}_{5\varepsilon}$ for $\varepsilon=1/2$: $\varphi_{—<}$ differs from the representation $\varphi_{5,1/2}=\nu(\dot{I}_{5,1/2})$ in the single point $(1/2,1/2)\in T$: $\varphi_{—<}(1/2,1/2)=1$, $\varphi_{5,1/2}(1/2,1/2)=0$; this transformation turns an irreflexive but transitive *FI* $\dot{I}_{5\varepsilon}$ into a reflexive *FI* $—\!\!<$ ∎

We go over to the second order polynomial models of *FI*'s.

Theorem 6.14 (characterization of *FI*'s with polynomial representations of degree 2). (i) Any polynomial representations of degree 2 of an *FI* can be written in the form

$$\varphi(\alpha,\beta)=\bar{\beta}\hat{+}(t\bar{\alpha}+u\bar{\beta}), \tag{6-5}$$

where $u\in[-1,1]$; $t\in[0,\bar{u}]$, and $\hat{+}$ is probabilistic sum.

(ii) Let $\mathcal{O}$ be the following set of operations in I: $\mathcal{O}=\{\vee,\wedge,\hat{+},\cdot,\bar{\ }\}$. There exist exactly three polynomials in the family described in (i), corresponding to "$\mathcal{O}$-algebraic" *FI*'s, that is, to *FI*'s possessing a membership function

$$\mu_{inc}(a,b)=\mathring{\lambda}q(a,b), \tag{6-6}$$

with q being any element of minimal algebra with listed operations, generated by a, b. These polynomials in Φ, and the corresponding distinguished elements of $\mathcal{I}nc$ are presented in the following table. In addition, representation of $inc_{\hat{+}}$ is the only irreducible polynomial model of degree 2 ∎

representation	*FI*	coefficients in (6-5)
$\varphi_{I_5^2}=\bar{\beta}^2=(1-\beta)^2$	$I_5^2=I_5\cdot I_5$	t=0, u=-1
$\varphi_{I_5^{\hat{+}}}=\bar{\beta}\hat{+}\bar{\beta}=\overline{\beta^2}=1-\beta^2$	$I_5^{\hat{+}}=I_5\hat{+}I_5$	t=0, u=1
$\varphi_{inc\hat{+}}(\alpha,\beta)=\bar{\beta}\hat{+}\bar{\beta}=\overline{\alpha\beta}=1-\alpha\beta$	$inc\hat{+}$	t=1, u=0

Proof. (i) Let us write out the explicit form of representation:

$$\varphi(\alpha,\beta)=\varphi_{inc}(\alpha,\beta)=q_0+q_1\alpha+q_2\beta+q_3\alpha^2+q_4\beta^2+q_5\alpha\beta. \qquad (6\text{-}7)$$

Using the definition of Φ, we come to the following system (6-8) with three equations, expressing the values in the vertices of T, and with two inequalities responsible for anti-monotonicity of φ (non-positiveness of $\frac{\partial\Phi}{\partial\alpha}$, $\frac{\partial\Phi}{\partial\beta}$):

values in the vertices of T	conditions
$\varphi(0,0)=1$	$q_0=1$
$\varphi(1,0)=1$	$q_0+q_1+q_3=1$
$\varphi(1,1)=0$	$q_0+q_1+q_2+q_3+q_4+q_5=0$
anti-monotonicity	conditions
$\frac{\partial\Phi}{\partial\alpha}\le 0$	$q_1+2q_3\alpha+q_5\beta\le 0$
$\frac{\partial\Phi}{\partial\beta}\le 0$	$q_2+2q_4\beta+q_5\alpha\le 0$ (6-8)

From the first three conditions, we derive that

$$q_0=1; \quad q_3=-q_1; \quad q_2+q_4+q_5=-1; \qquad (6\text{-}9)$$

In order to transform anti-monotonicity conditions into α,β-independent form, we mark out that a necessary and sufficient condition of non-positiveness of a linear function anywhere on an affine figure T is its non-positiveness in the vertices of T. Therefore, two "parametric" inequalities in ((6-8) are equivalent to six non-parametric inequalities (6-10). Solving (6-9), (6-10) (we omit routine calculations), we reduce φ to the form (6-11), with additional restrictions (6-12) on the values of the coefficients.

values in the vertices of T	conditions
$\frac{\partial\Phi}{\partial\alpha}(0,0)\le 0$	$q_1\le 0$
$\frac{\partial\Phi}{\partial\alpha}(1,0)\le 0$	$q_1+2q_3\le 0$
$\frac{\partial\Phi}{\partial\alpha}(1,1)\le 0$	$q_1+2q_3+q_5\le 0$
$\frac{\partial\Phi}{\partial\beta}(0,0)\le 0$	$q_2\le 0$
$\frac{\partial\Phi}{\partial\beta}(1,0)\le 0$	$q_2+q_5\le 0$
$\frac{\partial\Phi}{\partial\beta}(1,1)\le 0$	$q_2+2q_4+q_5\le 0$ (6-10)

$$\varphi(\alpha,\beta)=1-(1+q_4+q_5)\beta+q_4\beta^2+q_5\alpha\beta, \tag{6-11}$$

$$q_4\in[-1,1], \qquad q_5\in[-(1+q_4),0] \tag{6-12}$$

By setting $u=-q_4\in[-1,1]$, $t=-q_5\in[0,\bar{u}]$, (6-12) is transformed into

$$\varphi(\alpha,\beta)=1-(1-t-u)\beta-u\beta^2-t\alpha\beta; \tag{6-13}$$

Direct calculation confirms that (6-13) coincides with (6-5).

(ii) In virtue of Lemma 6.8, $inc_{\hat{+}}\in\mathcal{I}nc$. Belonging of I_5^2, $I_5^{\hat{+}}$, and to $\mathcal{I}nc$ is established similarly to the proof of Lemma 6.8. In fact, the only non-routine calculation the proof of A_2, which is based on the use of the distributivity of the t-norm $\cdot$, and of the t-conorm $\hat{+}$ with respect to $\wedge$. Say, with $inc=I_5^2$:

$$\mu_{inc}(a\vee b,c)=(\mu_{I_5}(a\vee b,c))^2=(\mu_{I_5}(a,c)\wedge\mu_{I_5}(b,c))^2=(\mu_{I_5}(a,c))^2\wedge(\mu_{I_5}(b,c))^2$$
$$=\mu_{inc}(a,c)\wedge\mu_{inc}(b,c).$$

Next, if all three inclusions are really *FI*'s, then the explicit form of representations for I_5^2, $I_5^{\hat{+}}$ is implied by Theorem 6.8. (i) (ν is local homomorphism). With $inc_{\hat{+}}$, the calculation of φ_{inc} is similar to the case of I_5, I_4 (see Theorems 6.11, 6.12).

A less trivial job is to verify the absence of other $\mathcal{O}$-algebraic *FI*'s of degree 2. An outline of the proof is as follows. Using distributivity of $\cdot$, $\hat{+}$ with respect to $\wedge$, $\vee$, together with common properties of Boolean functions, one can reduce any $\mathcal{O}$-algebraic function q(a,b) depending on the variables a, b to the following canonical form:

$$q(a,b)=\bigvee_{J\in\mathcal{J}_q}\ \bigwedge_{j\in J} s_j(a,b), \tag{6-14}$$

where $\mathcal{J}_q$ is a set of sets of multi-indices;

J is a set of multi-indices;

j is a multi-index;

$s_j(a,b)$ is a $\{^-,\cdot,\hat{+}\}$-term in the alphabet $\{a, b\}$.

An *FI* itself is expressed as

$$\mu_{inc}(a,b)=\mathring{\lambda}q(a,b), \tag{6-15}$$

With these notations, a representation $\varphi=\nu(q)$ can be written as (see Theorem 6.3)

$$\varphi(\alpha,\beta)=\mu_{inc}(\alpha\cdot\chi_{\{x\}},\overline{\beta\cdot\chi_{\{x\}}})=\mathring{\lambda}\bigvee_{J\in\mathcal{J}_q}\ \bigwedge_{j\in J} s_j(\alpha\cdot\chi_{\{x\}},\overline{\beta\cdot\chi_{\{x\}}}) \tag{6-16}$$

Next,

$$s_j(\alpha\cdot\chi_{\{x\}},\overline{\beta\cdot\chi_{\{x\}}})(y)=\begin{cases} s_j(\alpha,\overline{\beta}), & y=x \\ s_j(0,1), & y\neq x\end{cases} \tag{6-17}$$

It follows that

$$\varphi(\alpha,\beta)=q(\alpha,\overline{\beta})\wedge q(0,1)=q(\alpha,\overline{\beta}). \tag{6-18}$$

Since $q(\alpha,\beta)$ is required to express a single polynomial of degree 2 on T, it is intuitively obvious that it must be reduced to a *single* term $s_j(a,b)$. The formal proof is as follows. Let $\mathcal{G}_0=\bigcup_{J\in\mathcal{J}_q} J$ be the set of all terms s_j, identified with the corresponding indices. We suppose that $j_1\neq j_2$ implies $s_{j_1}\neq s_{j_2}$. With each $j\in\mathcal{G}_0$, we define a subset $\mathcal{B}_j$ of a triangle T,

$$\mathcal{B}_j=\{(\alpha,\beta)\in T\,|\,q(\alpha,\beta)=s_j(\alpha,\beta)\}.$$

It should be noticed that, owing to continuity of q, and of all terms s_j, $\mathcal{B}_j$ is a *closed* subset of T. Let $\mathcal{G}=\{j\in\mathcal{G}_0\,|\,\mathcal{B}_j\neq\emptyset\}$, so that $\mathcal{G}$ includes "essential" $\{^-,\cdot,\hat{+}\}$-terms of a polynomial q. Obviously,

$$\bigcup_{j\in\mathcal{G}}\mathcal{B}_j=T. \tag{6-19}$$

Let us denote by $\mathcal{Q}\subseteq\mathcal{P}(\mathcal{G})$ the set of all minimal with respect to inclusion subsets of $\mathcal{G}$, satisfying (6-19). We shall prove that each $S\in\mathcal{Q}$ is single-point. Assuming the contrary, we can select two different indices $k,l\in S$. Since S is minimal, each of the two *open* subsets $U_k=T\setminus\bigcup_{j\in\mathcal{G}\setminus\{k\}}\mathcal{B}_j$,

$U_1=T\setminus\bigcup_{j\in\mathcal{G}\setminus\{1\}}\mathcal{B}_j$ is non-empty. Since S itself satisfies (6-19), both inclusions $U_k\subseteq\mathcal{B}_k$, and $U_1\subseteq\mathcal{B}_1$ are true. In other words, we derived that, with two open subsets $U_k,U_1\subset T$, $s_k|_{U_k}\equiv q|_{U_k}$, and $s_1|_{U_1}\equiv q|_{U_1}$, with s_k,s_1,q being polynomials. Of course, this implies that, for any analytic function, $s_k\equiv s_1$, thus leading to a contradiction with the condition $k\neq 1$. So, we proved that each S is a singleton, so that there exists a term s_j, representing q anywhere on T. Obviously, any $\{^{-},\cdot,\hat{+}\}$-term of degree 2 can be written either as $\zeta_1\cdot\zeta_2$ or as $\zeta_1\hat{+}\zeta_2$, with $\zeta_1,\zeta_2\in\{\alpha,\beta,\bar{\alpha},\bar{\beta}\}$. Boundary values of φ in the vertices of T put additional constraints on the choice of s_j. In the first case, $\zeta_1\cdot\zeta_2$, we arrive to a term $\varphi=s_j=\bar{\beta}\cdot\bar{\beta}=\varphi_{I_5^2}$. In the second case, $\zeta_1\hat{+}\zeta_2$, there exist two possibilities: $\varphi=s_j=\overline{\bar{\beta}\hat{+}\bar{\beta}}=\varphi_{I_5^{\hat{+}}}$, and $\varphi=s_j=\overline{\bar{\beta}\hat{+}\bar{\alpha}}=\varphi_{inc_{\hat{+}}}$ ■

Theorem 6.15 (characterization of ε-inclusions for the distinguished polynomial models of degree 2).

Representations of ε-inclusions for the *FI*'s I_5^2, $I_5^{\hat{+}}$, $inc_{\hat{+}}$ can be expressed as the characteristic functions of the following subsets $H(inc,\varepsilon)$ of the triangle T:

$$H(I_5^2,\varepsilon)=\{(\alpha,\beta)\in T\,|\,\beta\leq 1-\sqrt{\varepsilon}\}=A(I_5,\sqrt{\varepsilon});$$

$$H(I_5^{\hat{+}},\varepsilon)=\{(\alpha,\beta)\in T\,|\,\beta\leq\sqrt{1-\varepsilon}\}=A(I_5,1-\sqrt{1-\varepsilon});$$

$$H(inc_{\hat{+}},\varepsilon)=\{(\alpha,\beta)\in T\,|\,\alpha\beta\leq\varepsilon\}\ ■$$

Hence, reflexivity, transitivity and antitransitivity of ε-inclusions for I_5^2, $I_5^{\hat{+}}$ can easily be discovered as modifications of Theorem 6.13 (i)-(iii). As to the $inc_{\hat{+}}$, no ε-inclusion of this *FI* is transitive, whereas all ε-inclusions with $\varepsilon\leq 3/4$ are reflexive.

An example of piecewise-rational *FI* $I_{(7)}$

$$I_{(7)}(a,b)=1\wedge\left(\frac{\bar{a}\cdot b}{a\cdot\bar{b}}\right)\quad\left(\varphi_{(7)}(\alpha,\beta)=1\wedge\left(\frac{\bar{\alpha}\cdot\bar{\beta}}{\alpha\cdot\beta}\right)\right)$$

(here, n=1!) can be found in (Baldwin, Guild [1]). This *FI* is reflexive (and hence, antitransitive), antisymmetric due to Kaufmann when n=1, and intransitive; ε-inclusions for $I_{(7)}$ are also reflexive, but possess neither antisymmetry nor transitivity.

In addition to the above "polynomial" members of the family of *FI*'s, we dwell on a characteristic fuzzy inclusion associated, in the spirit of Section 6.1, with "logical" considerations. If we interpret *FI*'s as models of fuzzy implications, then the Contraposition, and the Distributivity properties are satisfied owing to A_1, and A_2. What about the Modus Ponens property? We already know from Theorem 6.1 that the Contraposition, the Distributivity, and the Modus Ponens properties are incompatible within a "t-fuzzy logic" associated with a continuous Archimedean t-norm. But the t-norm $\wedge$ proves to be much more tolerant. First, let us formulate *MP* for a multi-dimensional case. We say that a *FI* inc satisfies *MP* iff

$$(\forall\ a,b\in\tilde{\mathcal{P}}(X))(\forall\ x\in X)(\mu_a(x)\wedge\mu_{inc}(\mu_a(x),\mu_b(x))\le\mu_b(x)) \tag{6-20}$$

Let us denote by $\mathcal{I}nc_{MP}$ the set of all *FI*'s satisfying (6-20).

Theorem 6.16 (characterization of *FI*'s satisfying *MP*). $\mathcal{I}nc_{MP}=[\Phi,inc_{MP}]\cap\mathcal{I}nc$ (the right-hand side interval is an ideal in the lattice of all FR's on $\tilde{\mathcal{P}}(X)$, containing all FR's that are smaller or equal to inc_{MP}). The greatest *FI* in this family, inc_{MP}, has the representation $\varphi_{inc_{MP}}(\alpha,\beta)=\bar{\alpha}$, and the membership function

$$\mu_{inc_{MP}}(a,b)=\begin{cases} 1, & a\subseteq b \\ \bigwedge\limits_{x\in X}\overline{\mu_a(x)}\wedge\mu_b(x), & \text{otherwise}\end{cases}$$

inc_{MP} is strongly reflexive and transitive. In addition, with n=1, this *FI* is antisymmetric due to Kaufmann ■

Proof. Suppose that $inc\in\mathcal{I}nc_{MP}$; set $\varphi=\varphi_{inc}$; with $a,b\in\tilde{\mathcal{P}}(X)$, let us denote $u=\mu_a(x)$, $v=\mu_b(x)$ $\alpha=u\vee\bar{v}$; $\beta=u\wedge\bar{v}$. Using Theorem 6.3, we can transform (6-20) into the following functional inequality

$$(\forall\ u,v\in I)(u\wedge\varphi(u\vee\bar{v},u\wedge\bar{v})\le v), \tag{6-21}$$

supplemented with two equalities

$$\alpha=u\vee\bar{v};\quad \beta=u\wedge\bar{v}. \tag{6-22}$$

To solve the system (6-21),(6-22), we use once more the technique of antipolyndroms (see Section 6.4). Let us write out antipolyndroms for (u,v), and observe the corresponding antipolyndroms for (α,β), and the induced restrictions upon the behavior of φ. Of course, if $u \le v$, then (6-21) is automatically fulfilled; therefore, we need to consider only four antipolyndroms, namely, $v\bar{u}u\bar{v}$, $vu\overline{uv}$, $\bar{u}v\bar{v}u$, $\overline{uv}vu$.

antipolyndrom for (u,v)	α	β	$\bar{\alpha}$	$\bar{\beta}$	antipolyndrom for (α,β)	subtri-angle	condition on φ
$v\bar{u}u\bar{v}$	$\bar{v}$	u	v	$\bar{u}$	$\overline{\alpha\beta}\beta\alpha$	T''_2	$\varphi(\alpha,\beta)\le\bar{\alpha}$
$vu\overline{uv}$	$\bar{v}$	u	v	$\bar{u}$	$\bar{\alpha}\beta\bar{\beta}\alpha$	T''_1	$\varphi(\alpha,\beta)\le\bar{\alpha}$
$\bar{u}v\bar{v}u$	u	$\bar{v}$	$\bar{u}$	v	$\overline{\alpha\beta}\beta\alpha$	T''_2	$\varphi(\alpha,\beta)\le\bar{\beta}$
$\overline{uv}vu$	u	$\bar{v}$	$\bar{u}$	v	$\bar{\alpha}\beta\bar{\beta}\alpha$	T''_1	$\varphi(\alpha,\beta)\le\bar{\beta}$

Summarizing the contents of the resulting table, we come to a pair of conditions: $\varphi\big|_{T''}\le\bar{\alpha}$, $\varphi\big|_{T''}\le\bar{\beta}$. Since, with $(\alpha,\beta)\in T''$, $\bar{\alpha}\le\bar{\beta}$, this pair of inequalities is equal to a single condition $(\forall\ (\alpha,\beta)\in T'')(\varphi(\alpha,\beta)\le\bar{\alpha})$.

As to the behavior of representation on t', it is not restricted at all (except for the general condition of antimonotonicity and boundary values of φ). Clearly, the greatest representation satisfying these properties is defined by the formula $\varphi\big|_{T'}\equiv 1$; $\varphi\big|_{T''}\equiv\bar{\alpha}$. In virtue of Theorems 6.4, 6.5, 6.7, the corresponding *FI* is reflexive, transitive, and antisymmetric in the sense of Kaufmann when $n=1$. An explicit formula for inc_{MP} follows from Theorem 6.3 ∎

On Figure 6.4, the shape of level sets of the representations of the four characteristic *FI*'s $\subseteq$, I_5, I_4, and inc_{MP} is depicted.

Motivated by the results of this section, we choose the *FI*'s $\subseteq$, and I_5 as the basic ones for the construction of fuzzy dichotomous decision procedures. In fact, the main result on contensiveness of FDDP's obtained in Chapter 7, is valid for all *FI*'s; also, most of the results achieved on the basis of L.Zadeh' inclusion $\subseteq$ remain in force with any strongly reflexive *FI* (for instance, with I_4).

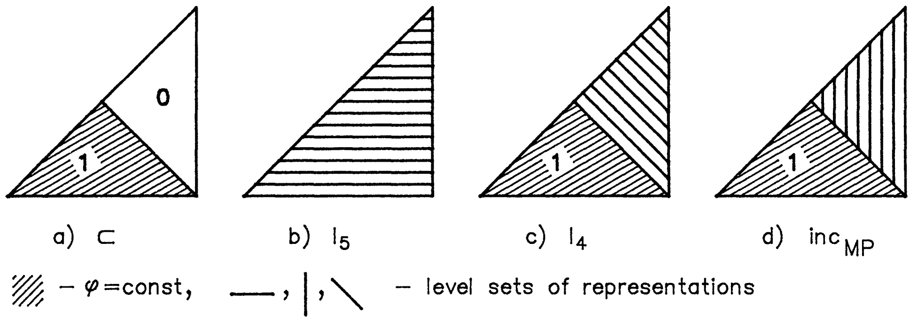


Fig. 6.4. Representations of distinguished *FI*'s

6.7. COMPARATIVE STUDY OF FUZZY INCLUSIONS

6.7.1. New Families of Transitive Inclusions

We introduce several new families of transitive *FI*'s in terms of representations $\varphi=\varphi_{\mathrm{inc}}$ (in what follows, only restrictions $\varphi_{\mathrm{inc}}|_{T''}$, or even $\varphi_{\mathrm{inc}}|_{T''_1}$ are given; the remaining values can be reconstructed with respect to Theorem 6.7).

1) $\varphi|_{T''}=\varphi(\alpha)$; with $\varphi(\alpha)\le\bar{\alpha}$, this family, in addition to transitivity, falls under the Modus Ponens property;

2) $\varphi|_{T''}=\varphi(\beta)$ ($\varphi(\beta)\equiv 0$, when $\beta\ge 1/2$); a parametric subfamily I^{γ} of this family with the representations $\varphi_5^{\gamma}(\alpha,\beta)=0\vee(\overline{\beta/\gamma})$ ($\gamma\le 1/2$) includes continuous and transitive *FI*s, which can be considered as modifications of I_5 in the sense that all of them possess piecewise-linear representations.

3) combinations of 1), and 2):

$$\varphi|_{T''}=\begin{cases}\varphi_1(\beta), & \alpha\le\alpha_0 \quad (\varphi_1|_{[1/2,\alpha_0]}\equiv\varphi_1(1/2))\\ \varphi_2(\alpha), & \alpha>\alpha_0\end{cases}$$

4) $$\begin{cases}\varphi|_{T''_2}\equiv 0,\\ \varphi(K)=\varphi_1((\pi\cap]K',K^1])\vee(\pi\cap]K'',K^{1/2}])) \text{ for all } K\in T''_1,\end{cases}$$

with π being the image of strictly monotone mapping $\pi:I\longrightarrow\hat{T}''_1$, $\pi(0)\in t'$, $\pi(1)=\mathbb{I}$ (in terms of Section 6.4.2, this is the case of one-dimensional $\mathcal{L}_\gamma$'s);

5) $\varphi\big|_{T_1''} = \sum_{m\in\Omega} c(m)\chi_{]0,\eta(m)]}$,

where Ω is a finite or countable tree (partial order - from leaves to the root),

$\eta:\Omega \longrightarrow T_1''$ is an isotone mapping of Ω into Pareto semilattice T_1'', satisfying two additional conditions:

(A) (m_1 and m_2 are incomparable in Ω) $\Rightarrow$ $(\eta(m_1)\wedge\eta(m_2)\in \overset{o}{T'})$,

(B) $\eta(m_0)=(1,1/2)$,

$c:\Omega \longrightarrow I$ is an antitone mapping (in terms of Section 6.4.2, this is the converse to the previous example of two-dimensional $\mathcal{L}_\gamma$'s);

6) the smallest transitive *FI* $inc_{t\wedge}$: $\varphi_{t\wedge}=\chi_{H_{t\wedge}}$, with $H_{t\wedge}$ being the interval $[(0,0),(1,0)]$; in fact, $inc_{t\wedge}$ is also the smallest element in $\mathcal{I}nc$;

7) the greatest transitive *FI* $inc_{t\vee}$: $\varphi_{t\vee}=\chi_{H_{t\vee}}$, $H_{t\vee}=T\setminus[(1,1/2),(1,1)]$.

Note 6.2. All *FI*s in the families 1, 3-7 are discontinuous.

6.7.2. Known Models Via Axiomatics

"Cardinality-based" inclusions (see Dubois and Prade [1])

$I_1(a,b)=|a\wedge b|/|a|$ - both A_1 and A_2 fail;

$I_2(a,b)=|\bar{a}\dot{\vee} b|$, and $I_3(a,b)=|\bar{a}\vee b|$ - A_1 is satisfied, A_2 fails;

(here, $|a|= \sum_{x\in X} \mu_a(x)/|X|$ is fuzzy cardinality).

"One-dimensional" inclusions (n=1; see Baldwin and Guild [1])

$I_{(3)}(a,b)=a\wedge b\wedge\bar{a}$ - both A_1 and A_2 fail;

(inference properties of $I_{(3)}$ were studied by Di Nola and Ventre [1]);

$I_{(4)}(a,b)=\begin{cases} 1, & a\le b \\ b, & a>b \end{cases}$, and $I_{(6)}(a,b)=1\wedge\left(\dfrac{b}{a}\right)$ - A_1 is satisfied, A_2 fails.

It should be noticed that $I_{(4)}$, and $I_{(6)}$ are residuated implications respectively for t-norm $\wedge$, and $\cdot$. In case of $I_{(4)}$, both the validity of A_2, and the failure of A_1 are implied by Theorem 6.1. The example of $I_{(6)}$ is a one more evidence of the "tolerance" of our axiomatics based on a t-norm $\wedge$: indeed, $I_{(6)}$ satisfies Distributivity axiom A_2; however, in virtue of Theorem 6.1, $I_{(6)}$ is *not* distributive with respect to its "own" t-norm $\cdot$!

In Table 6.2, a summary of the above comparative study of is presented (the *FI*'s I_5^2, $\hat{I}_5^{+}$, $inc_{\hat{+}}$, I_5^{γ}, $inc_{t\wedge}$, $inc_{t\vee}$ were introduced in this section).

Table 6.2

Main properties of fuzzy inclusions

Denotation	$\subseteq$	I_5	I_4	—<
Satisfying axioms:				
A_1	+	+	+	+
A_2	+	+	+	+
Properties:				
Characterization in $\mathcal{I}nc$ (! - "the only")	! reflexive+ antisymmetric+ transitive	!with linear represent-n ! ∨-∧-polynomially realisable	!reflexive+ continuous+ with linear represent-n on T″	-
Reflexivity	+	– ——— ε≤1/2	+ ——— +	+
Antisymmetry:				
perfect	+	– ——— –	– ——— –	–
due to Kaufmann	+	n=1 ——— –	n=1 ——— –	–
Transitivity	+	– ——— ε>1/2	– ——— –	+
Antitransitivity	+	+ ——— +	+ ——— +	+
Continuity	–	+	+	–
Comment	L.Zadeh' inclusion		Lukaciewicz' implication	Special weak inclusion

Table 6.2 (continued)

Denotation	I_5^2	$I_5^{\hat{+}}$	$\mathrm{inc}_{\hat{+}}$	I_7	inc_{MP}
Satisfying axioms:					
A_1	+	+	+	+	+
A_2	+	+	+	+	+
Properties:					
Characterization in $\mathcal{I}nc$ (! - "the only")	one of the two reducible polynomial degree 2 and σ-algebraic	same with I_5^2	! irreducible polynomial degree 2 and σ-algebraic	-	greatest satisfying Modus Ponens property
Reflexivity	$\frac{-}{\varepsilon\le 1/4}$	$\frac{-}{\varepsilon\le 3/4}$	$\frac{-}{\varepsilon\le 3/4}$	$\frac{+}{+}$	$\frac{+}{+}$
Antisymmetry:					
perfect	$\frac{-}{-}$	$\frac{-}{-}$	$\frac{-}{-}$	$\frac{-}{-}$	$\frac{-}{-}$
due to Kaufmann	$\frac{n=1}{-}$	$\frac{n=1}{-}$	$\frac{n=1}{-}$	$\frac{n=1}{-}$	$\frac{n=1}{-}$
Transitivity	$\frac{-}{\varepsilon>1/4}$	$\frac{-}{\varepsilon>3/4}$	$\frac{-}{-}$	$\frac{-}{-}$	$\frac{+}{+}$
Antitransitivity	$\frac{+}{+}$	$\frac{+}{+}$	$\frac{-}{-}$	$\frac{+}{+}$	$\frac{+}{+}$
Continuity	+	+	+	+	-
Comment					

Table 6.2 (continued)

Denotation	I_5^γ	$inc_{t\wedge}$	$inc_{t\vee}$	I_1	I_2
Satisfying axioms:					
A_1	+	+	+	-	+
A_2	+	+	+	-	-
Properties:					
Characterization in $\mathcal{I}nc$ (! - "the only")	-	!smallest transitive !smallest	! greatest transitive; ! greatest transitive+ reflexive	-	-
Reflexivity	$\frac{-}{-}$	-	+	+	+
Antisymmetry:					
perfect	$\frac{-}{-}$	+	-	-	-
due to Kaufmann	$\frac{n=1}{-}$	+	-	-	n=1
Transitivity	$\frac{+}{+}$	+	+	-	-
Antitransitivity	$\frac{+}{+}$	+	+	+	+
Continuity	+	-	-	+	+
Comment	Families of transitive and continuous FI's with piecewise linear representation				

Table 6.2 (continued)

Denotation	I_3	$I_{(3)}$	$I_{(4)}$	$I_{(6)}$
Satisfying axioms:				
A_1	+	-	-	-
A_2	-	-	+	+
Properties:				
Reflexivity	-	-	+	+
Antisymmetry:				
perfect	-	-	-	-
due to Kaufmann	n=1	-	+	+
Transitivity	-	-	+	-
Antitransitivity	+	+	+	+
Continuity	+	+	-	+

Numerator - properties of *FI*'s

Denominator - properties of ε-inclusions

! means "the only"

Chapter 7

Contensiveness of Fuzzy Dichotomous Decision Procedures in Universal Environment

We get over to a comprehensive study of fuzzy decision procedures with binary relations. We precede this study with a brief outline of the main results of the remaining of the book.. In what follows, we discover the structure of *MFC* with a collection of families of *FDDP*'s based on two composition laws $\circ$, and $\lceil\circ$, and on two fuzzy inclusions I_5, and $\subseteq$. In addition, we vary preference domain and consider *FDDP*'s in two environments: universal environment $\mathscr{E}=\tilde{\mathscr{P}}(X)$, and restricted environment with prohibited trivial choice $\mathscr{E}_0=\tilde{\mathscr{P}}(X)\setminus\{0,1\}$. The main purpose of this study is to determine the conditions under which one or another *FDDP/FR* is contensive. It turns out that the *FDDP*'s under consideration can be subdivided into three principal classes. Procedures included in the first class possess a very poor *MFC*; no specialization of these procedures can be contensive in any sensitive environment. These procedures are "unimproveable", so we do not pay much attention to a detailed description of *MFC*. The second class includes a very limited number of procedures that are contensive in universal environment. With this class, a broad study of the properties of *FR*'s satisfying the mentioned conditions of contensiveness is undertaken (in particular, investigation of fuzzy *contensive choice* requires completely different technique in comparison with fuzzy *contensive ranking*). The third class represents an intermediate set of procedures which are contensive in a restricted environment.

An important qualitative result is that the above characteristics of decision procedures are closely related to the transformation of *basic choice logic*. Starting with *dichotomous* formulations of rationality concepts, and with *continuous* preference scale, we arrive to a *three-valued* logic *chosen/uncertain/rejected*. With the above-mentioned class of "poor" *FDDP*'s, *MFC* induces either a one-term structure 'uncertain' or a two-term structure of the types 'uncertain/rejected' or 'chosen/uncertain'. For this reason, the induced crisp choice cannot be considered as a confident one. Conversely, the second class (well-defined *FDDP*'s) reproduces, with some modifications, the conventional crisp logic 'chosen/rejected'. The third class of procedures is characterized by a three-term shape of *MFC*.

As to the possibility of ranking alternatives on the basis of *FDDP*'s, it proves to be dependent on the characteristics of *connected spectrum* (see Chapter 5), and on the associated *regularity/singularity* properties of an *FR*.

It should be noticed that the above results motivate essential changes in a considerable body of conventional crisp and fuzzy decision rules. We dwell on the influence of our study upon conventional choice rules, upon decision-making in specific preference domains, including fuzzy preorderings, and reciprocal relations. New crisp ranking rule, namely, a *dipole decomposition*, is introduced. This rule represents a "ranking generalization" of von Neumann - Morgenstern Solutions for binary relations with a complicated structure of preferences.

Our considerations of contensiveness of *FDDP*'s are supplemented by a study of *efficiency*, which is understood in three diverse meanings: efficiency in preference domain, efficiency in ranking domain, and efficiency of the induced crisp choice. An overall result of the joint contensiveness/efficiency analysis indicates the *only* procedure possessing excellent properties, namely, the fuzzy version of von Neumann - Morgenstern Solution based on Boolean ($\vee$-$\wedge$) composition law, and on fuzzy Kleene - Dienes inclusion I_5.

For some *FDDP*'s under consideration, exhaustive description of *MFC* requires the study of three families of fuzzy subsets, namely, of *invariant*, of *antiinvariant*, and of *eigen* fuzzy subsets of a binary relation. Therefore, a complete algorithmically sound description of these families, and an additional study of their algebraic and topological properties is undertaken.

At the end of the book, we briefly discuss the accompanying *descriptive* decision problems. A kind of formalism of fuzzy majority approach is introduced for the purpose of aggregation of preferences in multipurpose decision-making. An approach to estimating concordance between *a priori* and *a posteriori* preferences of decision-maker, and to the subsequent study of his competence and resoluteness, is proposed.

In this small chapter, we present basic result on contensiveness of fuzzy dichotomous decision procedures, forming the families $\mathfrak{P}(\odot,\text{inc})$, in *universal environment* $\mathfrak{E}=\tilde{\mathcal{P}}(X)$ (correspondingly, the symbol $\mathfrak{E}$ in all notations is omitted - see Chapter 4).

Let us suppose that inc is arbitrary fuzzy inclusion, satisfying the axiomatics of Chapter 6; we consider two subfamilies of *FDDP*'s

$$\mathfrak{P}^{\text{DCU}}(\circ,\text{inc})=\{\Delta_{23},\ \Delta_{123}\},$$

$$\mathfrak{P}^{\text{DTU}}(\circ,\text{inc})=\mathfrak{P}(\circ,\text{inc})\backslash\mathfrak{P}^{\text{DCU}}(\circ,\text{inc}).$$

Theorem 7.1. (i) Any *FDDP* contained in $\mathfrak{P}^{\text{DTU}}(\circ,\text{inc})\cup\mathfrak{P}(\overline{\circ},\text{inc})$ is a *DT* procedure in any admissible preference domain.

(ii) Let $P\in\mathfrak{P}^{\text{DTU}}(\circ,\text{inc})\cup\mathfrak{P}(\overline{\circ},\text{inc})$. With any specialization (p,R), p(R) is a normal f.s. of $\tilde{\mathcal{P}}(X)$ ∎

Proof. We start with (ii). First, let us prove that

$$\mu_{\Delta_1(\circ,\text{inc})(R)}(0)=\mu_{\Delta_1(\circ,\text{inc})(R)}(1)=\mu_{\Delta_2(\circ,\text{inc})(R)}(0)=\mu_{\Delta_3(\circ,\text{inc})(R)}(1)=1.$$

Indeed, $\mu_{\Delta_1(\circ,\text{inc})(R)}(0)=\mu_{\text{inc}}(R\circ\overline{0},\overline{0})=\mu_{\text{inc}}(R\circ 1,1)=\mu_{\text{inc}}(0,\overline{R\circ 1})=1.$

The remaining equalities are proved in the same way. It follows that, for any "miniterm" $P=\Delta_{ijk}$ $(i,j,k\in\{\emptyset,1,2,3\})$, except for the Δ_{23}, and Δ_{123}, either 0 or 1 belongs to $\mathfrak{D}(p,R)$, and $\mu^*_{p(R)}=1$, so that P is normal and *DT*. For any other element $Q\in\mathfrak{P}(\circ,\text{inc})$, which is not a miniterm, let us consider its reduced form, that is, minimal disjunction of mini-terms, free of brackets. If at least one of the mini-terms, contained in the reduced form of Q, belongs to $\mathfrak{P}^{\text{DTU}}(\circ,\text{inc})$, then the above argumentation is valid, so that Q is normal and *DT*, thus proving (i) for $\mathfrak{P}^{DTU}(\circ,\mathrm{inc})$. Otherwise, Q is either Δ_{23} or Δ_{123} (disjunction of these *FDDPs* is clearly Δ_{23}). In the same way, it can be proved that

$$\mu_{\Delta_1(\lceil\circ,\mathrm{inc})(R)}(0)=\mu_{\Delta_2(\lceil\circ,\mathrm{inc})(R)}(0)=\mu_{\Delta_3(\lceil\circ,\mathrm{inc})(R)}(0)=\mu_{\Delta_3(\lceil\circ,\mathrm{inc})(R)}(1)=1.$$

Hence, for any *FDDP* $P\in\mathfrak{P}(\lceil\circ,\mathrm{inc})$, $0\in\mathfrak{D}(p,R)$, so that P is *DT*, and (i) is proved also for $\mathfrak{P}(\lceil\circ,\mathrm{inc})$ ■

Note 7.1. The proof of Theorem 7.1 shows that, as a matter of fact, only two axioms for fuzzy inclusions are required to ensure triviality of all mentioned procedures in universal environment, namely, the Contraposition axiom A_2, and the Heritage axiom A_4 (see Section 6.2). Therefore, Theorem 7.1 is true in a wider domain of *FDDP*'s ■

In that way, irrespective of the specific fuzzy inclusion, the majority of procedures in the families based on classical composition law, and all procedures based on the dual law, are "identically non-contensive", provided that the set of "trial rankings" is large enough.

Note 7.2. The proof of Theorem 7.1 may seem a somewhat tricky. What if we do not include 0, and 1 in ranking domain? A motivated answer to this question requires a more detailed study of *MFC*. For some of the procedures in both families $\mathfrak{P}(\circ,\mathrm{inc})$, and $\mathfrak{P}(\lceil\circ,\mathrm{inc})$ the reason of non-contensiveness is more essential than just belonging of trivial choices 0, 1 to *MFC*. For other procedures, these trivial choices turn to be "isolated" in *MFC*, and elimination of these two constants from ranking domain, that is, considering the restricted environment with "prohibited trivial choice", improves the quality of the choice (see Chapters 8, 11, 14 for a comprehensive research of this problem) ■

Note 7.3. It should be noticed that for *crisp* relations, constituting a specific preference domain, Theorem 7.1 also holds, thus motivating considerable reducing of the number of conventional choice rules (see Chapters 8, 14 for details) ■

An important consequence of Theorem 7.1 is a rather simple structure of *MFC* with all *DT* procedures. The only thing that should be studied is the form of *MFC* with specializations of basic dichotomies $\Delta_i(\odot,\widetilde{\mathrm{inc}})$. When these crisp subsets $\mathfrak{D}(\Delta_i(\odot,\mathrm{inc}),R)\subseteq\mathcal{P}(X)$ are determined, *MFC* with any specialization of a *FDDP* $P\in\mathfrak{P}^{DTU}(\circ,\mathrm{inc})\cup\mathfrak{P}(\ulcorner\circ,\mathrm{inc})$ can be derived quite formally, as an application of the corresponding $\vee$-$\wedge$ polynomial p, representing a rationality concept associated with P, to "partial" *MFC*'s, provided that $\vee$, and $\wedge$ are changed respectively for $\cup$, and $\cap$. More precisely, the following result holds.

Corollary 7.1. With any specialization $(p(\Delta_1,\Delta_2,\Delta_3),R)$ of a *FDDP* $P\in\mathfrak{P}^{DTU}(\circ,\mathrm{inc})\cup\mathfrak{P}(\ulcorner\circ,\mathrm{inc})$, the following equality holds:

$$\mathfrak{D}(p(\Delta_1,\Delta_2,\Delta_3),R)=p(\mathfrak{D}(\Delta_1,R),\mathfrak{D}(\Delta_2,R),\mathfrak{D}(\Delta_3,R)),$$

where the right-hand side polynomial is obtained from a rationality concept (that is, from a $\vee$-$\wedge$ polynomial) $p(\Delta_1,\Delta_2,\Delta_3)$ by changing operations $\underset{\sim}{\vee}$, $\wedge$ on fuzzy subsets of $\mathcal{F}(X)$ for operations $\cup$, $\cap$ on crisp subsets of $\mathcal{F}(X)$ ■

Proof. Let us denote by $\mathfrak{D}_q$ the truth function of an ordinary Boolean polynomial q. If we consider p as ordinary Boolean polynomial, depending on three Boolean variables b_1, b_2, b_3, then the equality

$$\mathfrak{D}_{p(b_1,b_2,b_3)}=p(\mathfrak{D}_{b_1},\mathfrak{D}_{b_2},\mathfrak{D}_{b_3}) \tag{7-1}$$

is a tautology. Since the $\vee$-$\wedge$ polynomial p, as well as any of Δ_i's, belongs to $\underset{\sim}{\mathfrak{P}}^{DTU}(\circ,\mathrm{inc})\cup\mathfrak{P}(\ulcorner\circ,\mathrm{inc})$, p(R), and all $\Delta_i(R)$, i=1,2,3, are normal f.s.' of $\mathcal{F}(X)$ (Theorem 7.1 (ii)), the following equivalencies are in force:

$$a\in\mathfrak{D}(p(\Delta_1,\Delta_2,\Delta_3),R) \Leftrightarrow \mu_{p(R)}(a)=1$$

$$a\in\mathfrak{D}(\Delta_i,R) \Leftrightarrow \mu_{\Delta_i(R)}(a)=1$$

If we assume $b_i=\mu_{\Delta_i(R)}(a)$, then the equivalence

$$a\in\mathfrak{D}(p(\Delta_1,\Delta_2,\Delta_3),R) \Leftrightarrow a\in p(\mathfrak{D}(\Delta_1,R),\mathfrak{D}(\Delta_2,R),\mathfrak{D}(\Delta_3,R))$$

is implied by (7-1) ■

In the next chapter, a generalization of this statement for α-cuts of *FDDP*'s will be obtained.

Chapter 8

Choice with Fuzzy Relations

In this chapter, we examine general structures of choice involved by fuzzy dichotomous decision procedures. According to the above considerations, these structures are derived from collections of "rational enough" (α-cuts), and of "extremely rational" (*MFC*) test rankings associated with each specialization of *FDDP* for a specific fuzzy binary relation. It turns out that the mentioned choice is associated with a collection of *FDDP*'s using Kleene - Dienes inclusion I_5. Based on the description of *MFC*'s with diverse procedures of this family, we discover intrinsic reasons for contensiveness/incontensiveness of *FDDP*'s, thus enriching the principal result of Chapter 7. In brief, the arguments *pro* and *contra* decision procedures are as follows:

a typical component of *MFC* with a *contensive* procedure proves to be the *contrasting* interval fuzzy subset with a non-zero dichotomousness; each of such components induces a conventional crisp dichotomy *chosen/rejected*, supplied with a confident preference estimate of a chosen crisp subset;

on the opposite, a characteristic component of *MFC* with an *incontensive* procedure includes a fuzzified *uncertainty domain*; the induced crisp choice is modeled by a dichotomy *uncertain/rejected* and does not reveal confident preferences, so that any crisp subset (as a candidate to crisp choice) is under question;

an important intermediate case is due to a *triangulation*, representing all values *chosen/uncertain/rejected* of the above "extended choice logic"; this kind of component of *MFC* results from a considerable number of *FDDP*'s based on Kleene - Dienes inclusion; with these procedures, the quality of choice can be improved in a restricted ranking domain.

8.1. BASIC TECHNIQUE. ELEMENTS OF MULTIFOLD FUZZY CHOICE

Basic technique for the extended study of the behavior of I_5-based FDDP's, and of the structure of the associated *MFC* includes two main components: algebraic properties of α-cut mappings, and calculations with fuzzy interval arithmetics. We summarize main properties of α-cut technique used in the subsequent analyses in the following lemma (see also Chapter 10).

Lemma 8.1. For any support Y, an arbitrary value $\alpha \in I$, and each f.s. $\tilde{a} \in \mathcal{P}(Y)$, the following statements hold:

(i) Both the conventional, and the strict α-cuts represent homomorphisms of a lattice $\tilde{\mathcal{P}}(Y)$ with operators in $\tilde{\mathcal{P}}(Y^2)$ onto the lattice $\mathcal{P}(Y)$ with operators in $\mathcal{P}(Y^2)$, provided that the action of $\tilde{\mathcal{P}}(Y^2)$ on $\tilde{\mathcal{P}}(Y)$ is a ∨-∧-composition ∘).

(ii) Let us consider the set $\{\geq, >\}$, supplied with an involution n, $n(\geq)=>$, $n(>)=\geq$. Then, with any $\vartheta \in \{\geq, >\}$,

$$(\bar{a})_{\vartheta\alpha} = \overline{a_{n(\vartheta)\bar{\alpha}}}\,.$$

(iii) Inequality $\overset{\circ}{\vee} a \leq \alpha$ is equivalent to equality $A_{>\alpha} = \emptyset$;

inequality $\overset{\circ}{\wedge} a \geq \alpha$ is equivalent to equality $A_{\alpha} = Y$.

(iv) "Cut mapping" is order representative:

$$a \subseteq b \Leftrightarrow (\forall\ \alpha)(A_{\alpha} \subseteq B_{\alpha})\ \blacksquare$$

Proof. (i) Well-known facts (see, e.g. Dubois and Prade [1]).

(ii) Let ϑ be >. Then

$$x \in \mathrm{supp}((\bar{a})_{>\alpha}) \Leftrightarrow \mu_{\bar{a}}(x) > \alpha \Leftrightarrow \mu_{a}(x) < \bar{\alpha} \Leftrightarrow \neg(\mu_{a}(x) \geq \bar{\alpha}) \Leftrightarrow x \in \overline{A_{\bar{\alpha}}}\,;$$

the proof when ϑ is '≥' is analogous.

(iii) Obvious.

(iv) Direct implication $a \subseteq b \Rightarrow (\forall\ \alpha)(A_{\alpha} \subseteq B_{\alpha})$ is evident. To prove the reverse implication, let us suppose that $a \subseteq b$ does not hold. In such case, there exists an $x\in X$ such that $\alpha=\mu_a(x)>\mu_b(x)$; therefore, both $x\in A_\alpha$, and $x\notin B_\alpha$ are satisfied, so that $A_\alpha\subseteq B_\alpha$ is not true ■

Corollary 8.1. With each specialization $(p(\Delta_1,\Delta_2,\Delta_3),R,\mathfrak{E})$ of a *FDDP* $P\in\mathfrak{P}(\odot,\mathrm{inc})$, and an arbitrary $\alpha\in I$, the following equality holds:

$$(p(R))_\alpha=p((\Delta_1)_\alpha,(\Delta_2)_\alpha,(\Delta_3)_\alpha)$$

Proof. Repeats the proof of Corollary 7.1, with due respect to Lemma 8.1 (i) and to the fact that p is a *monotone* polynomial, so that a "negation-based" cut formula from Lemma 8.1 (ii) can be avoided ■

Let $\odot$ be either the $\vee$-$\wedge$-composition law $\circ$ or the dual composition law $\ulcorner\circ$. An essential observation concerning the structure of *MFC* with $(\odot,I_5)$-based *FDDP*'s is that it consists of typical "elements", belonging to the following four classes of interval fuzzy subsets: *lower step*, *upper step*, *contrast*, and *triangulation* (see Figure 8.1). Formal definitions are as follows.

Definition 8.1. (i) Let $\alpha\in I$, $A\subseteq X$.

A *lower* (α,A)-*step* $\mathcal{L}(\alpha,A)$ is an interval f.s. of X:

$$\mathcal{L}(\alpha,A)=[0,\chi_A\vee\alpha\cdot\chi_{\bar{A}}]=[0,1]/A+[0,\alpha]/\bar{A}$$

An *upper* (α,A)-*step* $u(\alpha,A)$ is an interval f.s. of X:

$$u(\alpha,A)=[\alpha\cdot\chi_A,1]=[\alpha,1]/A+[0,1]/\bar{A}$$

An (α,A)-*contrast* $c(\alpha,A)$ is an interval f.s. of X:

$$c(\alpha,A)=[\alpha\cdot\chi_A,\ \chi_A\vee\bar{\alpha}\cdot\chi_{\bar{A}}]=[\alpha,1]/A+[0,\bar{\alpha}]/\bar{A}$$

Crisp subset A in the above formulas is called a *base* of the corresponding interval fuzzy subset (with $A=\emptyset$, any of the interval subsets is empty).

(ii) Let $\xi=\{A^1,A^2,A^3\}\in\Pi_X$ be any partition of X with $|\xi|\le3$ (one or two among A^i's may be empty).

A ξ-*triangulation* $t(\xi)$ is an interval f.s. of X:

$$t(\xi)=\{1\}/A^1+[0,1]/A^2+\{0\}/A^3$$

A crisp partition ξ is called a *base* of triangulation ■

Note 8.1. In the sequel, any of the interval fuzzy subsets introduced in the Definition 8.1, is also considered as a collection of "ordinary" fuzzy subsets, that is, as a crisp subset of $\tilde{\mathcal{P}}(X)$. In that way, expressions of the type $\bigcup\limits_{A}\mathcal{L}(\alpha,A)$, $\bigcup\limits_{\xi}\mathcal{t}(\xi)$, $\bigcup\limits_{A}(\mathcal{L}(\alpha,A)\cap\mathcal{u}(\alpha,A))$, etc., are understood as unions/intersections of the above crisp subsets, consisting of fuzzy subsets of X ■

Note 8.2. The class of all ξ-triangulations is rather wide. In particular, it contains the following subclasses:

crisp subsets ($A^2=\emptyset$);

lower (0,A)-steps ($A^1=\emptyset$);

upper (1,A)-steps ($A^3=\emptyset$);

the set of all f.s.' $\tilde{\mathcal{P}}(X)$, being treated as a single interval f.s. ($A^1=A^3=\emptyset$) ■

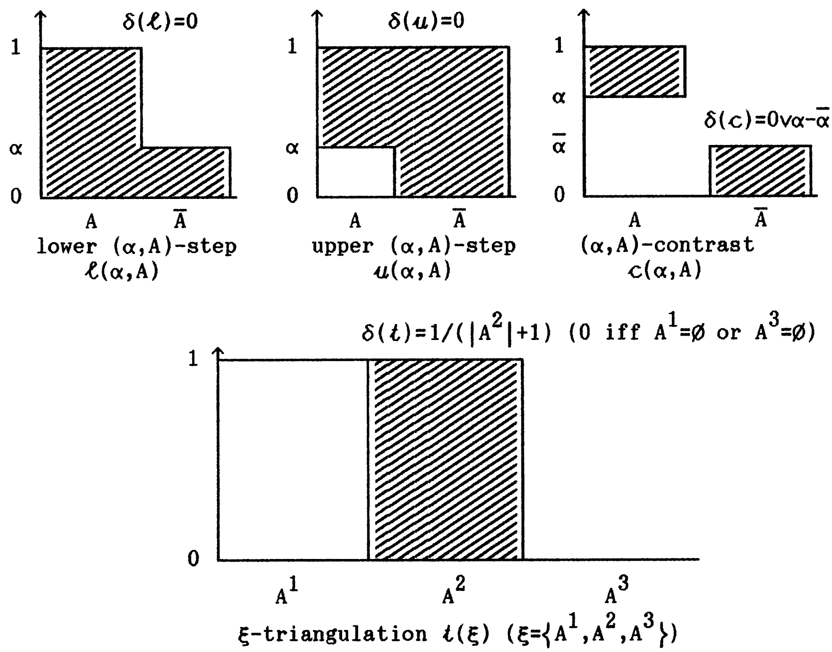


Fig. 8.1. Elements of multifold fuzzy choice (basic interval fuzzy subsets)

With the introduced elements of *MFC*, we are interested, in the spirit of Chapter 5, in the ability of these collections of f.s.' of X to produce confident crisp choice. It seems that lower, and upper steps "hint" on the possibility of the choice of the base. Say, one can find in $\mathcal{L}(\alpha,A)$, as well as in $u(\alpha,A)$, a characteristic function χ_A of a crisp subset A, which is the most determinant case, provided that the choice is considered as a maximal decision. However, the whole collection (with all f.s.' having equal rights to represent the induced crisp choice - see the Postulates for *MFC* in Chapter 5, and Definition 5.1 (i), (ii)) cannot be viewed as a well-defined *MFC*, because, despite for the characteristic function, this collection contains a "constant f.s." ($0\in\ell(\alpha,A)$, $1\in u(\alpha,A)$), and even a neighborhood of this constant in $\tilde{\mathcal{P}}(X)$. Hence, the dichotomousness of both the lower, and the upper step is $\delta(\ell)=\delta(u)=0$, and, as a matter of fact, both steps represent incontensive situation. The case of contrast is a somewhat different. Here, dichotomousness $\delta(c)=0\vee(\alpha-\bar{\alpha})$, thus being 0 for $\alpha\leq 1/2$ and $\alpha-\bar{\alpha}=2\alpha-1$ for $\alpha>1/2$; in the latter case, contrast represents a confident crisp choice of the base, with guaranteed interval preference estimate of a chosen subset $[\alpha,1]:[0,\bar{\alpha}]$, thus inducing a "resolution" (dichotomousness) $\alpha-\bar{\alpha}>0$. As to the triangulation, the situation is more complicated. With a one-term base ($|\xi|=1$), dichotomousness is always equal to 0. With a two-term base, dichotomousness is equal to 1 in case when $A^2=\emptyset$ (crisp subset!); otherwise, it is equal to 0. With a three-term base of a triangulation, its dichotomousness is greater than zero and depends on the number of points (alternatives) in the middle term A^2. More precisely, it is achieved with any "uniform" f.s. $u\in t(\xi)$, when membership values of $u|_{A^2}$ run through all values $\{i/|A^2|\}_{i=1,\dots,|A^2|}$; finally, in a three-term case $\delta(t)=\delta(u)=1/(|A^2|+1)$.

In the next statement, significant interrelations between the introduced interval fuzzy subsets are established.

Lemma 8.2. (i) For any $\beta>\alpha$, $A, B\subseteq X$,

$$\ell(\alpha,A)\cap u(\beta,B) = \begin{cases} \emptyset, & B\backslash A\neq\emptyset \\ [\beta,1]/B+[0,1]/(A\backslash B)+[0,\bar{\alpha}]/A, & B\backslash A=\emptyset \end{cases}$$

in particular, with any $\xi=\{A^1,A^2,A^3\}\in\Pi_X$,

$$t(\xi)=\ell(0,A^2\cup A^3)\cap u(1,A^1\cup A^2)$$

(ii) $\ell(\bar{\alpha},A)\cap u(\alpha,A)=c(\alpha,A)$.

(iii) $\ell(1/2,A)\cap u(1/2,B)=[0,1]/(A\backslash B)+(1/2)/(B\backslash A)+[0,1/2]/\overline{(A\cup B)}$.

(iv) Let $\xi=\{A^1,A^2,A^3\}\in\Pi_X$ be a partition of X with $|\xi|\leq 3$, and A be a subset of X, satisfying the property:

$$\text{either } A^1 \subseteq A \text{ or } A^3 \cap A = \emptyset.$$

Depending on which of these two cases takes place, let us define a partition ξ_A^m as

$$\text{either } \xi_A^m = \{A^1, A^2 \cap A, A^3 \cup A^2 \setminus A\} \quad \text{or } \xi_A^m = \{A^1 \cup A^2 \setminus A, A^2 \cap A, A^3\}$$

With these notations, intersection of a ξ-triangulation with lower $(0,A)$-step, and with upper $(1,A)$-step can be expressed as follows:

$$\ell(0,A) \cap t(\xi) = \begin{cases} \emptyset, & A^1 \setminus A \neq \emptyset \\ t(\xi_A^m), & \text{otherwise} \end{cases}$$

$$u(1,A) \cap t(\xi) = \begin{cases} \emptyset, & A^3 \cap A \neq \emptyset \\ t(\xi_A^m), & \text{otherwise} \end{cases}$$

(v) $\overbracket{\ell(\alpha,A)} = u(\bar{\alpha},\bar{A})$; $\overbracket{u(\alpha,A)} = \ell(\bar{\alpha},\bar{A})$; $\overbracket{c(\alpha,A)} = c(\alpha,A)$;

$\overbracket{t(\xi)} = t(\xi')$, where $\xi' = \{A^3, A^2, A^1\}$ is a "transponated" partition with respect to $\xi = \{A^1, A^2, A^3\}$.

(vi) For each $\alpha \in I$, the mapping $\ell_\alpha : \mathcal{P}(X) \longrightarrow \mathcal{P}(\tilde{\mathcal{P}}(X))$, $\ell_\alpha(A) = \ell(\alpha,A)$ is monotone with respect to inclusion of crisp subsets, whereas the mapping $u_\alpha : \mathcal{P}(X) \longrightarrow \mathcal{P}(\tilde{\mathcal{P}}(X))$, $u_\alpha(A) = u(\alpha,A)$ is antitone with respect to inclusion ∎

Proof. The statements (i)-(iv) can be easily derived from the following obvious "interval equality". Let us suppose that ι_1, and ι_2 are two fuzzy interval subsets, $\iota_1 = \sum J_i^1/A_i$, $\iota_2 = \sum J_i^2/B_i$, with $\{A_i\}, \{B_i\} \in \Pi_X$ being two partitions of X, $\{J_i^1\}$, $\{J_i^2\}$ being the corresponding collections of subintervals of I. In such case, an intersection $\iota_1 \cap \iota_2$ can be expressed as follows:

$$\iota_1 \cap \iota_2 = (J_i^1 \cap J_k^2)/(A_i \cap B_k), \text{ and } \iota_1 \cap \iota_2 = \emptyset \text{ iff } (\exists\ i,k)(J_i^1 \cap J_k^2 = \emptyset)$$

(v) From the definition of lower step, $\ell(\alpha,A) = [0, \chi_A \vee \alpha \cdot \chi_{\bar{A}}]$; since the mapping $n : \mathcal{P}(\tilde{\mathcal{P}}(X)) \longrightarrow \mathcal{P}(\tilde{\mathcal{P}}(X))$, $\mathfrak{X} \longrightarrow \overbracket{\mathfrak{X}}$ is obviously an automorphism, we come to

$$\overbracket{\ell(\alpha,A)} = [\overline{\chi_A \vee \alpha \cdot \chi_{\bar{A}}}, 1] = [\chi_{\bar{A}} \wedge \overline{\alpha \cdot \chi_{\bar{A}}}, 1] = [\chi_{\bar{A}} \wedge (\chi_A \vee \bar{\alpha} \cdot \chi_{\bar{A}}), 1] = [\alpha \cdot \chi_A, 1] = u(\bar{\alpha},\bar{A})$$

(geometrically, the graph of an upper step $u(\bar{\alpha},\bar{A})$ on Figure 8.1 is symmetric to the graph of a lower step $\mathcal{L}(\alpha,A)$ with respect to a horizontal line $y=1/2$). The remaining proofs are similar.

(vi) Obvious ■

After these preliminary calculations, let us investigate α-cuts of $(\circ,I_5)$-based *FDDPs*. With respect to Theorem 7.1, only two procedures (both based on the conventional composition law $\circ$) in the families $\mathfrak{P}(\circ,\mathrm{inc})$, $\mathfrak{P}(\ulcorner\circ,\mathrm{inc})$ of *FDDPs* can be considered as the candidates for contensiveness in universal environment, namely, fuzzy versions of von Neumann - Morgenstern solution $(\Delta_2\wedge\Delta_3)(\circ,\mathrm{inc})$, and of Fuzzy Stable Core $(\Delta_1\wedge\Delta_2\wedge\Delta_3)(\circ,\mathrm{inc})$. Nevertheless, we undertake the study of α-cuts of all basic dichotomies $\Delta_1,\Delta_2,\Delta_3$ (though all of them are incontensive), for three main reasons:

- to present more solid argumentation of their dichotomous triviality;
- to facilitate the research of contensive procedures;
- to discover new effects basing on fuzzy considerations and concerning crisp choice.

From now on, we use brief notations for mini-terms, depending on basic dichotomies $\Delta_1,\Delta_2,\Delta_3$ (see Chapter 2): Δ_{ij} stands for $\Delta_i\wedge\Delta_j$, Δ_{123} for $\Delta_1\wedge\Delta_2\wedge\Delta_3$, μ_i, μ_{ij} for $\mu_{\Delta_i(R)}$, $\mu_{\Delta_{ij}(R)}$, $(1\le i<j\le 3)$, μ_{123} for $\mu_{\Delta_{123}(R)}$. Using the formula from Chapter 6 for Kleene - Dienes inclusion I_5, $\mu_{I_5}(a,b)=\overset{\circ}{\wedge}\bar{a}\vee b$, let us write out extended expressions for basic $(\circ,I_5)$-based dichotomies:

$$\mu_1(a)=\mu_{I_5}(R\circ\bar{a},\bar{a})=\overset{\circ}{\wedge}(\overline{R\circ\bar{a}}\vee\bar{a})=\overset{\circ}{\vee}\overline{(R\circ\bar{a}\wedge a)} \tag{8-1}$$

$$\mu_2(a)=\mu_{I_5}(R\circ a,\bar{a})=\overset{\circ}{\wedge}(\overline{R\circ a}\vee\bar{a})=\overset{\circ}{\vee}\overline{(R\circ a\wedge a)} \tag{8-2}$$

$$\mu_3(a)=\mu_{I_5}(\bar{a},R\circ a)=\overset{\circ}{\wedge}(R\circ a\vee a) \tag{8-3}$$

The following result enables one to decrease the range of 'alphas' used in the research of *MFC*.

Lemma 8.3. For any $P\in\mathfrak{P}(\circ,I_5)$, $\mu_P(1/2)\ge 1/2$ ■

Proof. It is sufficient to prove the statement for basic dichotomies. For Δ_1, Δ_2, both expressions under the sign $\overset{\circ}{\vee}$ in (8-1), (8-2) are equal to

$(R\circ 1/2)\wedge 1/2\leq 1/2$; hence, the height $(\overset{\circ}{\vee})$ of the expression cannot exceed $1/2$, and the complement value $\overline{\overset{\circ}{\vee}}$ is not less than $1/2$. In case of Δ_3, it is clear that, for any $R\in\tilde{\mathcal{P}}(X^2)$, $R\circ 1/2\leq 1/2$, so that $(R\circ 1/2)\vee 1/2$ in (8-3) is exactly $1/2$, and $\mu_3(1/2)=1/2$ ∎

Corollary 8.2. For any $P\in\mathfrak{P}(\circ,I_5)$, $R\in\tilde{\mathcal{P}}(X^2)$, $\mu^*_{p(R)}\geq 1/2$ ∎

Proof. Directly follows from Lemma 8.3 ∎

So, in universal environment *MFC* for all $(\circ,I_5)$-based *FDDP*'s is related to α-cut of an *FR* with $\alpha\geq 1/2$. Even more, if $\mu^*_{p(R)}=1/2$, then the corresponding specialization is DT, since $\mathfrak{D}(p,R)$ contains a constant $1/2$.

In the sequel, we often refer to the following families of *crisp* subsets of a support X, depending on a *crisp* binary relation Q on X (see Chapter 2):

$\mathfrak{Inv}(Q)$ - the set of all Q-invariant subsets $(Q\circ A\subseteq A)$;

$\mathfrak{I}(Q)$ - the set of all internally stable subsets $(Q\circ A\subseteq\bar{A})$;

$\mathfrak{E}(Q)$ - the set of all externally stable subsets $(\bar{A}\subseteq Q\circ A)$;

$\mathfrak{I}^*(Q)$ - the set of all maximal subsets of $\mathfrak{I}(Q)$;

$\mathfrak{E}_*(Q)$ - the set of all minimal subsets of $\mathfrak{E}(Q)$;

$\mathfrak{K}$ - the set of all crisp von Neumann - Morgenstern solutions of a crisp relation $R_{>1/2}$, which is the strict 1/2-cut of a *FR* R;

M - the set of all non-dominated alternatives (the Core) of a crisp relation $R_{>1/2}$.

8.2. α-CUTS, AND MULTIFOLD FUZZY CHOICE WITH BASIC DICHOTOMIES

Using the above technical results, we can discover the structure of α-cuts of basic $(\circ,I_5)$-dichotomies. In particular, with $\alpha=\mu^*_{p(R)}$ we obtain the corresponding *MFC*.

Theorem 8.1. (α-cuts of basic $(\circ,I_5)$-dichotomies).

(i) For any $\alpha>1/2$, $(\Delta_1)_\alpha \subseteq \bigcup_{A\in\mathfrak{Inv}(R_{>\bar{\alpha}})} \mathcal{U}(\alpha,\bar{A})$

(ii) For any $\alpha>0$, $(\Delta_2)_\alpha = \bigcup_{A\in\mathfrak{I}^*(R_{>\bar{\alpha}})} \mathcal{L}(\bar{\alpha},A)$

(iii) For any $\alpha>0$, $\quad (\Delta_3)_\alpha = \bigcup_{A\in\mathfrak{C}_*(R_\alpha)} \mathcal{U}(\alpha,A)$ ∎

Proof. (i) $\mu_1(a)=\overline{\mathring{\vee}\,(R\circ\bar{a}\wedge a)}\geq\alpha \Leftrightarrow \mathring{\vee}(R\circ\bar{a}\wedge a)\leq\bar{\alpha}$. Owing to Lemma 8.1 (iii), the latter is equivalent to $(R\circ\bar{a}\wedge a)_{>\bar{\alpha}}=\emptyset$. Using Lemma 8.1 (i), we derive $(R_{>\bar{\alpha}}\circ(\bar{a})_{>\bar{\alpha}})\wedge(a)_{>\bar{\alpha}}=0$. With respect to Lemma 8.1 (ii), this is the same with $R_{>\bar{\alpha}}\circ\overline{A_\alpha}\cap A_{>\bar{\alpha}}=\emptyset$, that is, $R_{>\bar{\alpha}}\circ\overline{A_\alpha}\subseteq\overline{A_{>\bar{\alpha}}}$. Next, with $\alpha>1/2$, we have $A_{>\bar{\alpha}}\supseteq A_\alpha$, so that $\overline{A_{>\bar{\alpha}}}\subseteq\overline{A_\alpha}$. Hence, the already proved inclusion $R_{>\bar{\alpha}}\circ\overline{A_\alpha}\subseteq\overline{A_{>\bar{\alpha}}}$ implies $R_{>\bar{\alpha}}\circ\overline{A_\alpha}\subseteq\overline{A_\alpha}$, which is equivalent to $\overline{A_\alpha}\in\mathfrak{Jnd}(R_{>\bar{\alpha}})$. If we set $A=\overline{A_\alpha}\in\mathfrak{Jnd}(R_{>\bar{\alpha}})$, then, obviously, $a\in\mathcal{U}(\alpha,\bar{A})$; finally, $(\Delta_1)_\alpha\subseteq \bigcup_{A\in\mathfrak{Jnd}(R_{>\bar{\alpha}})} \mathcal{U}(\alpha,\bar{A})$

(ii) Following the same method as in (i), we obtain subsequently:

$$\mu_2(a)=\overline{\mathring{\vee}(R\circ a\wedge a)}\geq\alpha \Leftrightarrow \mathring{\vee}(R\circ a\wedge a)\leq\bar{\alpha} \Leftrightarrow (R\circ a\wedge a)_{>\bar{\alpha}}=\emptyset \Leftrightarrow R_{>\bar{\alpha}}\circ A_{>\bar{\alpha}}\cap A_{>\bar{\alpha}}=\emptyset$$

$$\Leftrightarrow A_{>\bar{\alpha}}\in\mathfrak{J}(R_{>\bar{\alpha}}) \Leftrightarrow (\Delta_2)_\alpha = \bigcup_{A\in\mathfrak{J}(R_{>\bar{\alpha}})} \mathcal{L}(\bar{\alpha},A) = \bigcup_{A\in\mathfrak{J}^*(R_{>\bar{\alpha}})} \mathcal{L}(\bar{\alpha},A)$$

(see Lemma 8.2 (vi)).

(iii) $\mu_3(a)=\mathring{\wedge}(R\circ a\vee a)\geq\alpha$; owing to Lemma 8.1 (iii), this is equivalent to $(R\circ a\vee a)_\alpha=X$. Similarly to (ii), we come to

$$R_\alpha\circ A_\alpha\cup A_\alpha=X \Leftrightarrow \overline{A_\alpha}\subseteq R_\alpha\circ A_\alpha \Leftrightarrow A_\alpha\in\mathfrak{C}(R_\alpha) \Leftrightarrow (\Delta_3)_\alpha = \bigcup_{A\in\mathfrak{C}(R_\alpha)} \mathcal{U}(\alpha,A)$$

$$\Leftrightarrow (\Delta_3)_\alpha = \bigcup_{A\in\mathfrak{C}_*(R_\alpha)} \mathcal{U}(\alpha,A) \quad ∎$$

A noticeable consequence of this theorem is the correspondence between fuzzy versions of rationality concepts, on the one hand, and their crisp prototypes, on the other hand. This correspondence can be informally expressed by the following "fuzzy statement" (italicized fragments represent "fuzzy terms"):

"To be *rational enough* with respect to a fuzzy basic dichotomy, a fuzzy test ranking should possess *high enough* membership values on some crisp subset, which is, in its turn, rational with respect to the corresponding crisp basic dichotomy, when the latter is applied to a *high enough* cut of the original fuzzy binary relation".

Corollary 8.3. (*MFC* with basic $(\circ, I_5)$-dichotomies).

(i) $\mathcal{D}(\Delta_1, R) = \bigcup_{A \in \mathfrak{I}nv(R_{>0})} \iota(\xi_A)$,

where $\xi_A = \{\bar{A}, A \backslash R_{>0} \circ A, R_{>0} \circ A\}$. In particular, $\mathcal{D}(\Delta_1, R)$ includes a lower step $\mathcal{L}(0, CND(R_{>0}))$, with $CND(R_{>0})$ being the Core of the strict 0-cut of an *FR*.

(ii) $\mathcal{D}(\Delta_2, R) = \bigcup_{A \in \mathfrak{I}^*(R_{>0})} \mathcal{L}(0, A)$.

$\mathcal{D}(\Delta_2, R)$ also includes $\mathcal{L}(0, CND(R_{>0}))$.

(iii) $\mathcal{D}(\Delta_3, R) = \bigcup_{A \in \mathfrak{C}_*(R_1)} u(1, A)$ ∎

Proof. (i) Using the proof of Theorem 8.1 (i), let us write down crisp characteristic equation for 1-cut of Δ_1: $b \in \mathcal{D}(\Delta_1, R) \Leftrightarrow R_{>0} \circ \overline{B_1} \subseteq \overline{B_{>0}}$. Comparing this inclusion with the obvious formula $\overline{B_{>0}} = B_{=0} \subseteq B_{<1} = \overline{B_1}$, we conclude that $\overline{B_1} = B_{<1} \in \mathfrak{I}nv(R_{>0})$, and that any $y \in B_{<1}$, which is $R_{>0}$-dominated by any $x \in B_{<1}$, belongs to $B_{=0}$; in other words, $\mu_b|_{R_{>0} \circ B_{<1}} \equiv 0$. As to the non-dominated alternatives of the induced relation $R_{>0}|_{B_{<1}}$ (these points form the subset $B_{<1} \backslash R_{>0} \circ B_{<1}$), membership value in these points can be anywhere in $[0,1[$. Denoting by A^1 an 1-cut of b, by A^2 the subset $B_{<1} \backslash R_{>0} \circ B_{<1}$, and by A^3 the subset $R_{>0} \circ B_{<1}$, we come to a partition ξ_A, with $A = B_{<1}$. The only thing remained to prove is that $\mu_a|_{A^2}$ can achieve the value 1. This assertion follows from the obvious fact that, for any $C \subseteq A^2$, a subset $D = A \backslash C$ is an invariant subset of $R_{>0}|_{A}$, so that $A^1 \cup C$ can be considered as the first term of a base ξ' of another triangulation, $\xi' = \{A^1 \cup C, D \backslash R_{>0} \circ D, R_{>0} \circ D\}$.

The second assertion is proved by assuming A=X; in this case, $A^1 = \bar{A} = \emptyset$, $A^2 = CND(R_{>0})$, $A^3 = \overline{CND(R_{>0})}$, so that $\xi = \{\emptyset, CND(R_{>0}), \overline{CND(R_{>0})}\}$, in which case $\iota(\xi) = \mathcal{L}(0, CND(R_{>0}))$.

(ii), and (iii) follows from Theorem 8.1 by assuming $\alpha = 1$. The second statement in (ii) is implied by an obvious fact that $CND(R_{>0})$ is an internally stable subset of $R_{>0}$ ■

The explicit form of *MFC* with Δ_2, Δ_3 shows that incontensiveness of these *FDDP*s has a more deep reason than just belonging of "trivial choice constants" 0, and 1 to *MFC* (see Theorem 7.1). In fact, these procedures "hint" at their crisp prototypes related to extremal cuts of the original *FR* ($R_{>0}$, R_1). However, no confident preference of the potential crisp choice is induced. Thus, with any $\varepsilon>0$, $A\in\mathfrak{I}(R_{>0})$, $\varepsilon\cdot\chi_A$ belongs to $\mathfrak{D}(\Delta_2,R)$ as well as χ_A itself. In the same way, with an arbitrarily small $\varepsilon>0$, and any crisp subset $B\in\mathfrak{C}(R_1)$, $1-\varepsilon\cdot\chi_B$ is included in $\mathfrak{D}(\Delta_3,R)$, together with χ_B itself. In other words, both the A, and the $\bar{B}$ represent "pure uncertainty domains" in the sense that membership functions of fuzzy trial rankings contained in *MFC* can take arbitrary values from 0 to 1 on these subsets.

A somewhat more surprising is that the same behavior is demonstrated by the most commonly used choice rule, namely, by the Core.

8.3. THE CORE IS UNFIT

In this section, we dwell on the most propagated choice rule in classical choice theory, namely, on the so called graphodominant choice rule (Berezovsky, Borzenko, and Kempner [1] , see also Chapters 1, 3), prescribing the choice of the Core, that is, of the subset of all non-dominated alternatives of a preference relation. Great majority of common crisp and fuzzy decision procedures refer to this rule (for crisp versions, see Examples 3.1, 3.3; more examples can be found, e.g., in a survey Volskiy [1]; fuzzy decision models based on diverse interpretations of this rule in fuzzy environment can be found in Orlovski [1,2], Ovchinnikov [2,4], Ovchinnikov and Roubens [1,2], etc., see also Example 3.1). We recall that the graphodominant choice function has sound axiomatic basis in choice theory (see Chapter 1): any crisp choice function satisfying the Heritage, and the Concordance axioms, can be represented as graphodominant choice with an appropriate binary relation (Berezovsky, Borzenko, and Kempner [1]). Moreover, the latter relation is transitive if, and only if, the original choice function satisfies also the Independence axiom. In this case, choice function can be modeled as Pareto choice in a linear space; the dimension of this space is exactly the width of a digraph associated with a binary relation.

In accordance with the idea of fuzzy rationality concept, we identify "fuzzy graphodominant choice" with a *FDDP* Δ_{12} (see Chapters 3, 4). We already know (Theorem 7.1) that this procedure is normal as a f.s. of $\tilde{\mathcal{P}}(X)$ and dichotomously trivial. An exhaustive description of *MFC* is presented in the following theorem.

Theorem 8.2. $\mathfrak{D}(\Delta_{12},R)=\mathcal{L}(0,\mathrm{CND}(R_{>0}))$ ∎

Proof. Owing to Corollary 7.1,

$$\mathfrak{D}(\Delta_{12},R)=\mathfrak{D}(\Delta_1,R)\cap\mathfrak{D}(\Delta_2,R). \tag{8-4}$$

With respect to additional statements of Corollary 8.3 (i), (ii), both the $\mathfrak{D}(\Delta_1,R)$, and the $\mathfrak{D}(\Delta_2,R)$ contain $\mathcal{L}(0,\mathrm{CND}(R_{>0}))$; hence, $\mathcal{L}(0,\mathrm{CND}(R_{>0}))$ is included in $\mathfrak{D}(\Delta_{12},R)$.

Substituting into (8-4) explicit formulas for the right-hand side terms from Corollary 8.3 (i), (ii) and using distributivity of set operations, we come to

$$\mathfrak{D}(\Delta_{12},R) = \bigcup_{\substack{A\in\mathfrak{I}nD(R_{>0})\\ B\in\mathfrak{I}^*(R_{>0})}} \ell(\xi_A)\cap\mathcal{L}(0,B), \tag{8-5}$$

where $\xi_A=\{\bar{A},A\backslash R_{>0}\circ A,R_{>0}\circ A\}$. In virtue of Lemma 8.2 (iv), (8-5) is equivalent to

$$\mathfrak{D}(\Delta_{12},R) = \bigcup_{\substack{A\in\mathfrak{I}nD(R_{>0})\\ B\in\mathfrak{I}^*(R_{>0})}} \ell((\xi_A)^{m}_{B}), \tag{8-6}$$

where $(\xi_A)^{m}_{B}=\{\bar{A},(A\backslash R_{>0}\circ A)\cap B,A\backslash B\}$, and $\bar{A}\subseteq B$. It follows that $\bar{B}=A\backslash B\in\mathfrak{I}nD(R_{>0})$, so that B is a maximal with respect to inclusion internally stable subset of a binary relation $R_{>0}$, possessing a $R_{>0}$-invariant supplement. Obviously, the unique subset satisfying these conditions is $B=\mathrm{CND}(R_{>0})$. Owing to the expression for $(\xi_A)^{m}_{B}$, with any $a\in\mathfrak{D}(\Delta_{12},R)$, $A_{>0}$ is included in $B=\mathrm{CND}(R_{>0})$. Hence, the reverse inclusion $\mathfrak{D}(\Delta_{12},R)\subseteq\mathcal{L}(0,\mathrm{CND}(R_{>0})$ is also in force ∎

Note 8.3. Theorem 8.2 generalizes a well-known result of S. Orlovski for transitive *FR*'s (Orlovski [1]); *MFC* associated with the concept of fuzzy non-dominated alternative "hints" at *crisply non-dominated alternatives* of

R. But the already observed uncertainty effect - "fuzzification" of the Core of $R_{>0}$ - leaves no hope for the confident preference of this subset over its supplement. Again, $\varepsilon\cdot\chi_A$ with an arbitrarily small ε belongs to $\mathfrak{D}(\Delta_{12},R)$ as well as χ_A itself, thus demonstrating dichotomous triviality of the procedure. It should be also noticed that in case when $CND(R_{>0})$ is empty (the only representative case with common non-transitive relations - see Chapter 1), *MFC* is reduced to 0 ■

We emphasize the result that the FND procedure $(\Delta_1\wedge\Delta_2)(\circ,I_5)$ is *identically incontensive* with *any* preference relation, be that *FR* or a crisp binary preference relation. In terms of Definition 5.1, this means that the corresponding *FDDP* is incontensive in *any* preference domain! This is a one more argument against the concept of "non-dominatedness", in addition to the classical criticism (see Chapter 1): not only the *potential emptiness* of the graphodominant choice, but also (which is even more significant) its *actual incontensiveness* is the proper reason to avoid this procedure.

Nevertheless, the unfitness of such an acknowledged choice rule looks as something paradoxical. Therefore, we suggest a one more explanation of this result.

As was mentioned in Chapter 1, binary relations used in classical choice theory commonly possess a kind of transitivity. Under this condition, the Core generally coincides with the Stable Core. In crisp environment, the result of the choice is completely determined by the very chosen subset; therefore, the answer to the question

"what is the difference between the Core, and the Stable Core of a transitive preference relation?"

is "no difference at all!". In fuzzy environment, the picture is unlike the crisp case: fuzzy rationality concept $(\Delta_1\wedge\Delta_2\wedge\Delta_3)(\circ,I_5)$ (the Stable Core) cardinally differs with $(\Delta_1\wedge\Delta_2)(\circ,I_5)$ (the Core). In the next section, we will observe not only *dichotomous*, but also *ranking contensiveness* of the $(\circ,I_5)$-based Stable Core in universal environment.

With provision to this reasoning, one can suppose that, in conventional choice theory, just the Stable Core is "unintentionally" used instead of the "ordinary" Core, because the difference between these two concepts is hidden under the incompleteness of a crisp preference domain. This leads to another significant conclusion that fuzzy decision procedures represent a more adequate and sensitive tool for the comparative study of choice rules.

Note 8.4. It should be noticed that an alternative version of the Core in fuzzy decision-making is associated with a single "ranking f.s." FND,

$$\mu_{FND(R)}(x)=1-\bigvee_{y\neq x}\mu_R(y,x);$$

which is a straightforward expression for the fuzzy statement:

"an alternative x belongs to FND iff there does not exist an y which is better than x with respect to R"

(see Orlovski [1,2], Ovchinnikov [2,4], Ovchinnikov and Roubens [1,2]). FND can be written in a more "invariant" form: $FND=\overline{R\circ 1}$. The reasons why this specific fuzzy ranking is also unfit will be forwarded in Chapter 9.

8.4. FUZZY VON NEUMANN - MORGENSTERN SOLUTION. FUZZY STABLE CORE

In this section, we study the remaining two procedures in the families $\mathfrak{P}(\odot,I_5)$, namely, the $(\circ,I_5)$-based von Neumann - Morgenstern Solution, and the $(\circ,I_5)$-based Stable Core. For the moment, we are not convinced that the above *FDDP*'s are contensive: we only know that the argumentation of Theorem 7.1, and of Sections 8.1, 8.2 is unfit. But, in fact, these procedures prove to be contensive, and the structure of *MFC*, as well as of the induced crisp choice, contrasts with the DT case.

Once more, we begin with the study of α-cuts of Δ_{23}, Δ_{123} for $\alpha>1/2$; in virtue of Lemma 8.3, only with $\alpha>1/2$ contensiveness of a specialization of any of these procedures can be expected.

First of all, let us prove two technical results.

Lemma 8.4. Let us suppose that

(1) $R_1\subseteq R_2$ are two crisp antireflexive binary relations on X;

(2) $A\in\mathfrak{Z}(R_2)$, $B\in\mathfrak{E}(R_1)$ are, respectively, an internally stable subsetof R_2, and an externally stable subset of R_1.

Under these assumptions, the inclusion $B\subseteq A$ is possible iff A=B is a crisp von Neumann - Morgenstern solution for any $R\in[R_1,R_2]\subseteq\mathcal{F}_0(X^2)$ ■

Proof. Let us denote C=A\B, so that $A=B\cup C$. Using conditions (1), (2), together with definitions of internal, and of external stability, we obtain the following chain of inclusions: $C\subseteq\bar{A}\cup C=\bar{B}\subseteq R_1\circ B\subseteq R_2\circ B\subseteq R_2\circ A\subseteq\bar{A}=\bar{B}\cap\bar{C}\subseteq\bar{C}$. Hence, $C=\emptyset$, $A=B$, and $\bar{A}=R_1\circ A=R_2\circ A$. Next, for any $R\in[R_1,R_2]$, we have $\bar{A}=R_1\circ A\subseteq R\circ A\subseteq R_2\circ A=\bar{A}$, so that $R\circ A=\bar{A}$ ∎

Assuming that $\alpha>1/2$ is a "rationality level", let us denote by $\mathfrak{K}_\alpha$ the set of all those crisp von Neumann - Morgenstern Solutions of a strict median cut $R_{>1/2}$ of the initial *FR* R, which satisfy the rationality concept $\Delta_{23}(R)$ (when the latter is considered as a fuzzy subset of $\tilde{\mathcal{P}}(X)$) to a degree not less than α: $\mathfrak{K}_\alpha=\mathfrak{K}\cap(\Delta_{23})_\alpha=\{K\in\mathfrak{K}\,|\,\mu_{23}(\chi_K)\geq\alpha\}$.

Lemma 8.5. For any $\alpha>1/2$, $\mathfrak{K}_\alpha=\mathfrak{I}(R_{>\bar{\alpha}})\cap\mathfrak{E}(R_\alpha)=\mathfrak{I}^*(R_{>\bar{\alpha}})\cap\mathfrak{E}_*(R_\alpha)$ ∎

Proof. From the definition of $\mathfrak{K}_\alpha$, and from Corollary 8.1, there easily follows the equivalence $K\in\mathfrak{K}_\alpha \Leftrightarrow (K\in\mathfrak{K})\&(\chi_K\in(\Delta_2)_\alpha)\&(\chi_K\in(\Delta_3)_\alpha)$. Owing to Theorem 8.1 (ii),(iii), this is equivalent to

$$(K\in\mathfrak{K})\&(\chi_K\in(\bigcup_{A\in\mathfrak{I}^*(R_{>\bar{\alpha}})}\mathcal{L}(\bar{\alpha},A))\cap(\bigcup_{A\in\mathfrak{E}_*(R_\alpha)}\mathcal{U}(\alpha,A))).$$

Clearly, the only crisp subsets belonging to both the lower, and the upper (α,A)-steps are the crisp subsets of A; hence, the latter condition is equivalent to $K\in\mathfrak{K}\cap\mathfrak{I}(R_{>\bar{\alpha}})\cap\mathfrak{E}(R_\alpha)$. Next, $\alpha>1/2$ implies $R_\alpha\subseteq R_{>1/2}\subseteq R_{>\bar{\alpha}}$; according to Lemma 8.4, $\mathfrak{I}(R_{>\bar{\alpha}})\cap\mathfrak{E}(R_\alpha)\subseteq\mathfrak{K}$, so that $\mathfrak{K}_\alpha=\mathfrak{K}\cap\mathfrak{I}(R_{>\bar{\alpha}})\cap\mathfrak{E}(R_\alpha)=\mathfrak{I}(R_{>\bar{\alpha}})\cap\mathfrak{E}(R_\alpha)$. In particular, any $K\in\mathfrak{K}_\alpha$ belongs to $\mathfrak{I}^*(R_{>1/2})\cap\mathfrak{I}(R_{>\bar{\alpha}})\subseteq\mathfrak{I}^*(R_{>\bar{\alpha}})$ as well as to $\mathfrak{E}_*(R_{>1/2})\cap\mathfrak{E}(R_\alpha)\subseteq\mathfrak{E}_*(R_\alpha)$, thus representing an element of $\mathfrak{I}^*(R_{>\bar{\alpha}})\cap\mathfrak{E}_*(R_\alpha)$; therefore, $\mathfrak{K}_\alpha\subseteq\mathfrak{I}^*(R_{>\bar{\alpha}})\cap\mathfrak{E}_*(R_\alpha)=\mathfrak{I}(R_{>\bar{\alpha}})\cap\mathfrak{E}(R_\alpha)=\mathfrak{K}_\alpha$ ∎

Combining these preliminary calculations with the results of Sections 8.1, 8.2, we are now able to completely describe α-cuts of Δ_{23} and Δ_{123}.

Theorem 8.3. For any $\alpha>1/2$,

(i) $(\Delta_{23})_\alpha=\bigcup_{K\in\mathfrak{K}_\alpha}c(\alpha,K)$

(ii) $(\Delta_{123})_\alpha\neq\emptyset$ iff both $\mathfrak{K}=\{M\}$ and $\mu_{123}(\chi_M)>1/2$ are satisfied. Under these conditions, $(\Delta_{123})_\alpha=c(\alpha,M)$ ∎

Proof. (i) Owing to Corollary 8.1, $(\Delta_{23})_\alpha=(\Delta_2)_\alpha\cap(\Delta_3)_\alpha$. Using Theorem 8.1 (ii), (iii), and distributivity of set operations, we arrive to the following equality:

$$(\Delta_{23})_\alpha=(\bigcup_{A\in\mathfrak{Z}^*(R_{>\bar\alpha})}\ell(\bar\alpha,A))\cap(\bigcup_{A\in\mathfrak{C}_*(R_\alpha)}u(\alpha,A))=\bigcup_{\substack{A\in\mathfrak{Z}^*(R_{>\bar\alpha})\\B\in\mathfrak{C}_*(R_\alpha)}}(\ell(\bar\alpha,A)\cap u(\alpha,B))$$

It follows from Lemma 8.2 (i), and from the inequality $\alpha>1/2>\bar\alpha$ that all intersections $\ell(\bar\alpha,A)\cap u(\alpha,B)$ with $B\setminus A\neq\emptyset$ are empty; hence, the latter equality can be rewritten as

$$(\Delta_{23})_\alpha=\bigcup_{\substack{A\in\mathfrak{Z}^*(R_{>\bar\alpha})\\B\in\mathfrak{C}_*(R_\alpha)\\B\subseteq A}}(\ell(\bar\alpha,A)\cap u(\alpha,B))$$

Since $R_{>\bar\alpha}\subseteq R_\alpha$, Lemma 8.4 implies that any inclusion $B\subseteq A$ in the latter formula is an equality, so that

$$(\Delta_{23})_\alpha=\bigcup_{A\in\mathfrak{Z}^*(R_{>\bar\alpha})\cap\mathfrak{C}_*(R_\alpha)}(\ell(\bar\alpha,A)\cap u(\alpha,A)),$$

which is, in virtue of Lemma 8.2 (ii), and of Lemma 8.5, exactly $\bigcup_{K\in\mathfrak{K}_\alpha}c(\alpha,K)$.

(ii) 1) Let us suppose that $\mathfrak{K}=\{M\}$, and $\mu_{123}(\chi_M)\geq\alpha$; then $\chi_M\in(\Delta_{123})_\alpha\neq\emptyset$. Since $\mu_{23}(\chi_M)\geq\mu_{123}(\chi_M)\geq\alpha>1/2$, it can be derived from (i) that $M\in\mathfrak{K}_\alpha$, $\mathfrak{K}_\alpha=\{M\}$, and $(\Delta_{23})_\alpha=c(\alpha,M)$; hence, $(\Delta_{123})_\alpha\subseteq c(\alpha,M)$. Next, the inequalities $\mu_1(\chi_M)\geq\mu_{123}(\chi_M)\geq\alpha$ imply (in virtue of Theorem 8.1 (i), and of the evident equality $(\chi_M)_\alpha=(\chi_M)_{>\bar\alpha}=M$) the inclusion $R_{>\alpha}\circ\overline{M}\subseteq\overline{M}$. For any $a\in c(\alpha,M)$, similar equality $A_{>\bar\alpha}=A_\alpha=M$ also holds; it follows that $R_{>\alpha}\circ\overline{A_\alpha}\subseteq\overline{A_{>\bar\alpha}}$; owing to the proof of Theorem 8.2 (i), this inclusion is equivalent to the inequality $\mu_1(a)\geq\alpha$, so that $a\in(\Delta_{123})_\alpha$, and $(\Delta_{123})_\alpha=c(\alpha,M)$.

2) Conversely, let $b\in(\Delta_{123})_\alpha\neq\emptyset$. According to (i), $(\Delta_{123})_\alpha\subseteq(\Delta_{23})_\alpha=\bigcup_{K\in\mathfrak{K}_\alpha}c(\alpha,K)$; hence, $B_\alpha=B_{>\bar\alpha}$ belongs to $\mathfrak{K}_\alpha$. Next, $\mu_1(b)\geq\mu_{123}(b)\geq\alpha$ also holds; in its turn, $\mu_1(b)\geq\alpha$ is equivalent to $R_{>\bar\alpha}\circ\overline{B_\alpha}\subseteq\overline{B_{>\bar\alpha}}=\overline{B_\alpha}$, which is the same with $\overline{B_\alpha}\in\mathfrak{Z}n\mathfrak{v}(R_{>\bar\alpha})\subseteq\mathfrak{Z}n\mathfrak{v}(R_{>\alpha})$. This means that B_α is the Stable Core (in particular, the unique crisp von Neumann - Morgenstern Solution) of $R_{>1/2}$,

so that $B_\alpha=M$, $\mathfrak{R}=\mathfrak{R}_\alpha=\{M\}$, and $(\Delta_{123})_\alpha\subseteq c(\alpha,M)$. For any $a\in c(\alpha,M)$, $A_\alpha=B_\alpha=A_{>\bar\alpha}=B_{>\bar\alpha}=M$, so that $R_{>\alpha}\circ\overline{A_\alpha}\subseteq\overline{A_{>\bar\alpha}}$, and hence, $\mu_1(a)\geq\alpha$. Taking into account that $\mu_{23}(a)\geq\alpha$ is also fulfilled, we come to $c(\alpha,M)\subseteq(\Delta_{123})_\alpha$. Together with the already proved in 1) converse inclusion, this yields $(\Delta_{123})_\alpha=c(\alpha,M)$ ■

The following Theorem 8.4 contains necessary and sufficient conditions for dichotomous contensiveness of a *FR* with respect to the two contensive procedures.

Theorem 8.4. Let R be a *FR*.

(i) Set $\mu^*_{23}=\mu^*(\Delta_{23},R)$, $\mathfrak{R}^*=\mathfrak{R}_{\mu^*_{23}}$. The following statements are equivalent:

(1) A specialization (Δ_{23},R) is *DC* in universal environment;

(2) $\mu^*_{23}>1/2$;

(3) $\emptyset\neq\mathfrak{R}^*\subseteq\mathfrak{I}(R_{1/2})$.

(ii) Set $\mu^*_{123}=\mu^*(\Delta_{123},R)$. The following statements are equivalent:

(1) A specialization (Δ_{123},R) is *DC* in universal environment;

(2) $\mu^*_{123}>1/2$;

(3) Crisp relation $R_{>1/2}$ possesses the Stable Core M, belonging to $\mathfrak{I}(R_{1/2})$ ■

Proof. (i) (1) ⇒ (2). Follows from Lemma 8.3 (for any $(\circ,I_5)$-based *FDDP* P, $\mu^*_{p(R)}\geq1/2$, and $\mu^*_{p(R)}=1/2=\mu_{p(R)}(1/2)$ corresponds to *DT* case).

(2) ⇒ (1). In virtue of Theorem 8.3 (i),

$$\mathfrak{D}(p,R)=(\Delta_{23})_{\mu^*_{23}}=\bigcup_{K\in\mathfrak{R}^*}c(\mu^*_{23},K).$$

Clearly,

$$\delta(c(\mu^*_{23},K))\underset{K\in\mathfrak{R}^*}{\equiv}\mu^*_{23}-\overline{\mu^*_{23}}=2\mu^*_{23}-1>0,$$

so that (Δ_{23},R) is *DC*.

(2) ⇒ (3). Non-emptiness of $\mathfrak{R}^*$ is guaranteed by Theorem 8.3 (i); the inclusion $\mathfrak{R}^* \subseteq \mathfrak{I}(R_{>\mu^*_{23}}) \subseteq \mathfrak{I}(R_{1/2})$ follows from Lemma 8.5.

(3) ⇒ (2). Let us suppose that $K \in \mathfrak{R}^*$. Then

$$\mu_2(\chi_K) = \overline{\overset{\circ}{\vee}\ (R \circ \chi_K \wedge \chi_K)} = \overline{\overset{\circ}{\underset{K\times K}{\vee}}\ R};$$

next, $K \in \mathfrak{I}(R_{1/2})$ implies the inequality $\overset{\circ}{\underset{K\times K}{\vee}} R < 1/2$; hence, $\mu_2(\chi_K) > 1/2$. Since $K \in \mathfrak{E}(R_{>1/2})$, $\mu_3(\chi_K)$ is also strictly greater than 1/2. Finally,

$$\mu^*_{23} \geq \mu_{23}(\chi_K) = \mu_2(\chi_K) \wedge \mu_3(\chi_K) > 1/2.$$

(ii) The proof repeats (i), with due respect to Theorem 8.3 (ii), and to the equality $\mu_1(\chi_M) = \overline{\overset{\circ}{\vee}\ (R \circ \overline{\chi_M} \wedge \chi_M)} = \overline{\overset{\circ}{\underset{\overline{M}\times M}{\vee}}\ R}$ ■

So far, the final structure of *MFC* in dichotomously contensive case is as follows:

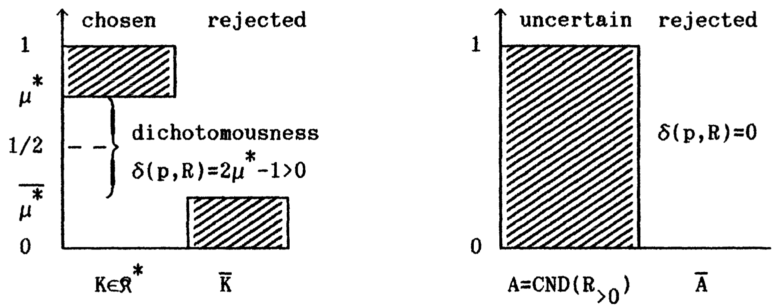
$$\mathfrak{D}(\Delta_{23}, R) = \bigcup_{K \in \mathfrak{R}^*} c(\mu^*_{23}, K) \qquad (\mu^*_{23} > 1/2, \quad \delta(\Delta_{23}, R) = 2\mu^*_{23} - 1 > 0)$$

$$\mathfrak{D}(\Delta_{123}, R) = c(\mu^*_{123}, M) \qquad (\mu^*_{123} > 1/2, \ \delta(\Delta_{123}, R) = 2\mu^*_{123} - 1 > 0)$$

In case of Δ_{23}, induced crisp choice is also multifold: any of the "best fitting" ($K \in \mathfrak{R}^*$) crisp von Neumann - Morgenstern solutions of the nearest crisp relation $R_{>1/2}$ can be chosen with the same preference $[\mu^*_{23}, 1]/K : [0, \overline{\mu^*_{23}}]/\overline{K}$. In our opinion, the specialization (Δ_{23}, R) does not contain more information enabling further selection from $\mathfrak{R}^*$. As to the Δ_{123}, induced crisp choice contains the only subset M with similar preference estimate $[\mu^*_{123}, 1]/M : [0, \overline{\mu^*_{123}}]/\overline{M}$.

On Figure 8.2, the difference between typical representatives of contensive (fuzzy von Neumann - Morgenstern solution $\Delta_{23}(\circ, I_5)$) and trivial (Fuzzy Core $\Delta_{12}(\circ, I_5)$) *FDDP*'s is demonstrated.

We stress on the following characteristics of contensive $(\circ, I_5)$-*FDDP*'s. First, *MFC* with respect to both contensive procedures is rather "balanced": it is based on the *nearest* to an original *FR* (in Hamming metric, see Kaufmann [1]) crisp relation $R_{>1/2}$, in contrast with *DT*



a) Fuzzy von Neumann-Morgenstern solution (FNMS) $(\Delta_2\wedge\Delta_3)(\circ,I_5)$

b) Fuzzy Core $(\Delta_1\wedge\Delta_2)(\circ,I_5)$

Fig. 8.2. Comparison of contensive and trivial multifold fuzzy choice

procedures, referring to *extremal* (0-, 1-) cuts. Another significant and surprising effect is that a contensive *fuzzy* decision procedure, instead of *fuzzifying* the choice, makes it, in some meaning, *sharper* than it was in crisp case. Indeed, a fuzzy $(\circ,I_5)$-version of von Neumann - Morgenstern Solution joins ranking and choice properties (see Figure 8.3). An algorithm for constructing *MFC* can be viewed as consisting of two steps. At the first step, the set $\mathfrak{K}$ of all crisp von Neumann - Morgenstern Solutions of a median cut $R_{>1/2}$ is ordered with respect to the membership function of $\mu_{23}(\chi_K)$ $(K\in\mathfrak{K})$. It occurs that, in order to guarantee "α-rationality" $\mu_{23}(a)\geq\alpha\geq1/2$, a trial ranking $a\in\tilde{\mathcal{P}}(X)$ should be included in an (α,K)-contrast, with its base K belonging to the corresponding class $\mathfrak{K}_\alpha$ of the above ordering. At the second step, the highest class of the mentioned ordering, namely, $\mathfrak{K}^*$, corresponding to a maximum possible degree of rationality μ^*_{23}, is included in the overall induced crisp choice, and the whole *MFC* is nothing but the union of the "highest contrasts". Though the "confidence estimate" for the induced crisp choice is fuzzified and makes not 1:0, but $[\mu^*_{23},1]:[0,\overline{\mu^*_{23}}]$ (see Figure 8.2 a), still this is a *positive* confidence estimate $(\mu^*_{23}-\overline{\mu^*_{23}}>0)$, in contrast with the "pure uncertainty" case, induced by dichotomously trivial procedures.

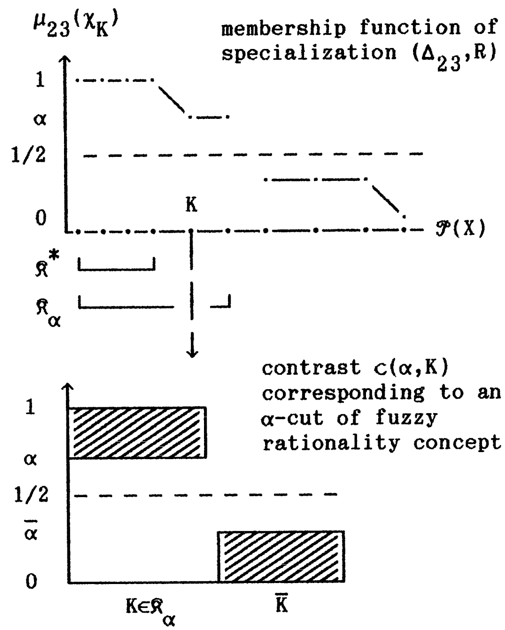


Fig.8.3. Ranking of crisp von Neumann - Morgenstern Solutions with respect to a *FDDP* $\Delta_{23}(\circ,I_5)$, and determining the corresponding contrasts

It should be noticed that for the $(\circ,I_5)$-based Fuzzy Stable Core the condition of dichotomous contensiveness coincides with he condition of ranking contensiveness, whereas in case of $(\circ,I_5)$-based fuzzy von Neumann - Morgenstern solution, a necessary and sufficient condition of ranking contensiveness is that $\mathfrak{R}^*$ contains single element (which must not necessarily be the Stable Core of either $R_{>1/2}$ or $R_{1/2}$).

Example 8.1. Let us suppose that R is a conventional "fuzzy preference relation", that is, a fuzzy [pre]ordering. With such *FR*, the subset of *crisply non-dominated alternatives* CND($R_{>0}$) is non-empty (see Note 8.3). At the face of it, this subset seems to be a good candidate for the induced crisp choice. However, this choice is due to *incontensive FDDP* Δ_{12} (see Theorem 8.2). In fact, the choice of CND($R_{>0}$) is "too resolute" (see Figure 8.4). Right choice is a (generally, greater) crisp subset CND($R_{>1/2}$)=CNMS($R_{>1/2}$)=SC($R_{>1/2}$), which is chosen with respect to each of the two *contensive FDDPs* Δ_{23} and Δ_{123}. It should be noticed that CND($R_{>0}$)

is the same as maximal decision with the above ranking FND, $CND(R_{>0})=FND_1$, whereas $CND(R_{>1/2})$ is a conventional median cut of FND, $CND(R_{>1/2})=FND_{1/2}$, as is demonstrated on Figure 8.4. ■

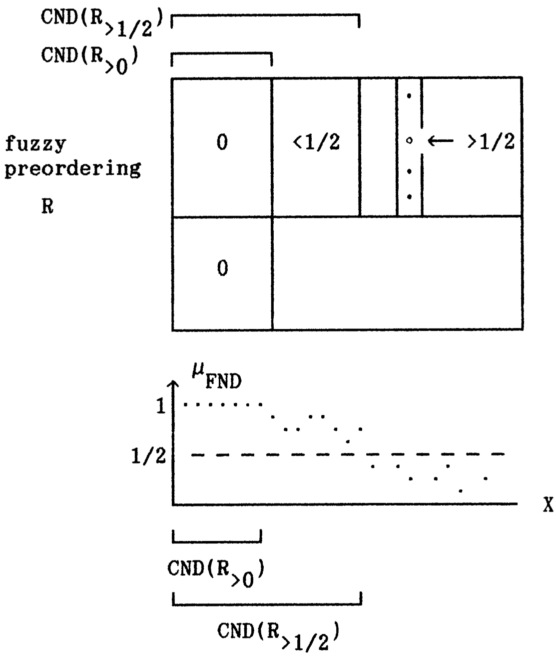


Fig. 8.4. Contensive, and incontensive choice with a fuzzy preordering

We complete this section with several applications of the above results to decision-making with concrete *FR*'s.

Example 8.2. Let $X=\{x_1,\ldots,x_5\}$, and let us consider an antireflexive fuzzy preference relation

$$R = \begin{vmatrix} 0.0 & 0.0 & 0.0 & 0.8 & 0.4 \\ 0.2 & 0.0 & 0.4 & 0.8 & 0.2 \\ 0.6 & 0.8 & 0.0 & 0.0 & 0.2 \\ 0.8 & 0.2 & 0.2 & 0.0 & 0.8 \\ 0.0 & 0.4 & 0.4 & 0.4 & 0.0 \end{vmatrix}$$

Strict median cut of R is the following crisp relation $R_{>1/2}$

$$R_{>1/2} = \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & \boxed{0} & \boxed{0} & 0 \\ 1 & 0 & \boxed{0} & \boxed{0} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Obviously, the only crisp von Neumann-Morgenstern Solution of $R_{>1/2}$ is a subset $K=\{x_3,x_4\}$; direct calculation yields $\mu^*_{23}=\mu_{23}(\chi_K)=0.80$; hence:

(Δ_{23},R) is *RC*; $\mathfrak{K}=\mathfrak{K}^*=\{K\}$; $\delta_{23}(\Delta_{23},R)=0.60$; $\mathfrak{D}(\Delta_{23},R)=c(0.80,K)$;

more explicitly, *MFC* includes all f.s.' of X in the interval

from (0.0 0.0 0.8 0.8 0.0)

to (0.2 0.2 1.0 1.0 0.2).

The induced crisp *ranking* ((Δ_{23},R) is *RC*!) can be represented as

$$[0.8,1.0]\big|\{x_3,x_4\}:[0.0,0.2]\big|\{x_1,x_2,x_5\}$$

Since the core $CND(R_{>1/2})=\{x_3\}$ is *not* the Stable Core, the specialization $(\Delta_{123}R)$ is *DT*. It should also be noticed that the ranking FND=(0.2 0.2 0.6 0.2 0.0) is *not* included in *MFC* ■

Example 8.3. Let $X=\{x_1,\dots,x_6\}$,

$$R = \begin{vmatrix} 0.00 & 0.16 & 1.00 & 1.00 & 0.84 & 0.84 \\ 0.16 & 0.00 & 1.00 & 1.00 & 0.84 & 0.84 \\ 0.60 & 0.60 & 0.00 & 0.40 & 0.60 & 0.76 \\ 0.60 & 0.60 & 0.12 & 0.00 & 0.28 & 0.76 \\ 0.76 & 0.60 & 0.40 & 0.40 & 0.00 & 0.60 \\ 0.76 & 0.60 & 0.84 & 0.72 & 1.00 & 0.00 \end{vmatrix}$$

We omit the construction of $\mathfrak{K}$, which turns to be consisting of two subsets,

$\mathfrak{K}=\{K_1,K_2\}$, $K_1=\{x_1,x_2\}$, $K_2=\{x_3,x_4\}$; $\mu_{23}(\chi_{K_1})=0.84$, $\mu^*_{23}=\mu_{23}(\chi_{K_2})=0.60$.

Hence, the induced crisp choice based on a specialization (Δ_{23},R) is sharpened, in comparison with the choice based on a crisp median cut of R (K_2 is rejected); here, *RC* case is in force, since $\mathfrak{K}^*$ contains the unique subset. The final results are as follows:

(Δ_{23},R) is *RC*; $\mathfrak{K}=\mathfrak{K}^*=\{K_1\}$; $\delta_{23}(\Delta_{23},R)=0.68$; $\mathfrak{D}(\Delta_{23},R)=c(0.84,K_1)$;

MFC includes all f.s.' of X in the interval

from (0.84 0.84 0.00 0.00 0.00)

to (1.00 1.00 0.16 0.16 0.16).

The induced crisp choice can be represented as

$$[0.84,1.00]\big|_{\{x_1,x_2\}}:[0.00,0.16]\big|_{\{x_3,x_4,x_5,x_6\}}.$$

Since $CND(R_{>1/2})$ is empty, the specialization $(\Delta_{123}R)$ is *DT*. With this *FR*, FND=(0.24 0.40 0.00 0.00 0.16) is again not included in *MFC*, though, in principle, it correctly reflects the rough structure of preferences ■

Example 8.4. $X=\{x_1,\dots,x_5\}$,

$$R=\begin{vmatrix} 0.0 & 0.7 & 0.9 & 0.4 & 0.1 \\ 0.6 & 0.0 & 0.8 & 0.3 & 0.5 \\ 0.9 & 0.8 & 0.0 & 0.2 & 0.6 \\ 0.2 & 0.3 & 0.1 & 0.0 & 0.9 \\ 0.4 & 0.1 & 0.0 & 1.0 & 0.0 \end{vmatrix}$$

Once more, we omit the construction of $\mathfrak{K}$: in fact, $R_{>1/2}$ possesses four crisp von Neumann - Morgenstern Solutions:

$\mathfrak{K}=\{K_1,K_2,K_3,K_4\}$; $K_1=\{x_1,x_4\}$, $K_2=\{x_1,x_5\}$, $K_1=\{x_2,x_5\}$, $K_2=\{x_3,x_4\}$;

in this case, $\mu^*_{23}=0.60$ is achieved for several subsets, and $\mathfrak{K}^*$ contains three of the above von Neumann - Morgenstern Solutions, $\mathfrak{K}^*=\{K_1,K_2,K_4\}$; hence,

(Δ_{23},R) is *DC* (but not *RC*!); $\delta_{23}(\Delta_{23},R)=0.20$;

$\mathfrak{D}(\Delta_{23},R)=c(0.60,K_1)\cup c(0.60,K_2)\cup c(0.60,K_4)$;

MFC includes all f.s.' of X belonging to one of the intervals:

from (0.6 0.0 0.0 0.6 0.0)

to (1.0 0.4 0.4 1.0 0.4).

from (0.6 0.0 0.0 0.0 0.6)

to (1.0 0.4 0.4 0.4 1.0).

from (0.0 0.0 0.6 0.6 0.0)

to (0.4 0.4 1.0 1.0 0.4).

The induced crisp choice is multifold: $[0.6,1.0]\big|_{K_i}:[0.0,0.4]\big|_{\overline{K}_i}$. As to the Fuzzy Stable Core, an observation that $\mathrm{CND}(R_{>1/2})=\emptyset$ implies dichotomous triviality of (Δ_{123},R).

In this example, FND=(0.1 0.2 0.1 0.0 0.1) is the worst possible ranking: it is not only very far from *MFC*, but offers the choice of the alternative x_2, which is not included even in the union of "partial" induced crisp choices! ■

Example 8.5. $X=\{x_1,\ldots,x_4\}$, R is a fuzzy ordering (antireflexive, perfectly antisymmetric, transitive, and weakly complete *FR*):

$$R = \left|\begin{array}{cccc} \boxed{0.0 \quad 0.4} & & 0.6 & 0.7 \\ \boxed{0.0 \quad 0.0} & & 0.3 & 0.8 \\ 0.0 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{array}\right|$$

Here, $\mathfrak{K}=\mathfrak{K}^*=\{x_1,x_2\}$; $\mu^*_{23}=\mu^*_{123}=0.6$; the induced crisp choice is

$$[0.6,1.0]\Big|\{x_1,x_2\}:[0.0,0.4]\Big|\{x_3,x_4\}.$$

It can be proved that, in this "classical" case, FND is always included in *MFC* (here, FND=(1.0 0.6 0.4 0.2)) ■

Example 8.6. $X=\{x_1,x_2,x_3\}$,

$$R = \left|\begin{array}{ccc} 0.0 & 0.6 & 0.5 \\ 0.1 & 0.0 & 0.7 \\ 0.8 & 0.3 & 0.0 \end{array}\right|$$

$$R_{>1/2} = \left|\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right|$$

In this case, strict median cut of an *FR* represents an "acknowledged" 3-cycle, possessing the worst possible properties; in particular, $\mathfrak{K}=\emptyset$, and (Δ_{23},R) is *DT* ■

Note 8.5. As a matter of fact, fuzzy preference relations in Examples 8.2 through 8.5 are "random *FR*'s", generated during a simulation tour with a Decision Support System DISPRIN (Kitainik [51]), developed by the author of the present volume for the purposes of verification of diverse theoretical concepts and estimating the efficiency of *FDDP*'s (more examples can be found in Chapter 12). One of the facilities of DISPRIN is the control of discreteness of the preference scale: thus, in Example 8.2 the step of the scale is 0.2, and the scale includes 6 "even" values $\{0.0, 0.2, \dots, 0.8, 1.0\}$; in Example 8.3, the step of the scale is 0.01, and the corresponding 101 values are $\{0.00, 0.01, \dots, 0.99, 1.00\}$; in the last two examples, the step is 0.1, with 11 preference values $\{0.0, 0.1, \dots, 0,9, 1.0\}$. It should be noticed that the "case studies" in the above four examples do not represent the whole scope of possible behavior of *FDDP*'s Δ_{23}, Δ_{123}; also, the probability of these cases is far from uniformity; asymptotically (when $n=|X|$ tends to infinity), most probable case is represented either by Example 8.3 (several crisp von Neumann - Morgenstern Solutions of the median cut, exactly one of them is included in the induced crisp choice) or by Example 8.4 (several crisp von Neumann - Morgenstern Solutions of the median cut, and the induced crisp choice is also multifold). The relative frequency of these items depends on the correlation between n, and the discreteness of preference scale; in case of purely continuous scale, the first result is predominant (see Chapter 12) ∎

8.5. PROCEDURES BASED ON THE DUAL COMPOSITION LAW

According to Theorem 7.1, all procedures in $\mathfrak{P}(\lceil\circ, I_5)$ are *DT*, and, with each specialization $(p,\tilde{R})$ of a procedure from this family, $p(R)$ is a normal f.s. of $\mathcal{P}(X)$. Hence, in virtue of Corollary 7.1, the structure of *MFC* associated with any *FDDP* of the dual family can easily be reconstructed from the *MFC* with basic dichotomies $\Delta_1(\lceil\circ, I_5)$, $\Delta_2(\lceil\circ, I_5)$, $\Delta_3(\lceil\circ, I_5)$. The study of basic dichotomies is, in its turn, considerably facilitated by Proposition 4.1 and Corollary 4.1, establishing cross-links between $\mathfrak{P}(\circ, I_5)$, and $\mathfrak{P}(\lceil\circ, I_5)$. The final form of *MFC* with basic dichotomies based on the dual composition law is as follows.

Theorem 8.5. (*MFC* with basic $(\lceil\circ, I_5)$-dichotomies).

(i) $\mathfrak{D}(\Delta_1, R) = \bigcup_{A \in \mathfrak{C}_*(R_{=0})} \ell(0, \bar{A})$.

(ii) $\mathfrak{D}(\Delta_2, R) = \bigcup_{A \in \mathfrak{Atnd}(R_{=0})} \ell(\eta_A)$,

where $\eta_A = \{A, R_{=0} \circ A \setminus A, \overline{R_{=0} \circ A}\}$.

(iii) $\mathfrak{D}(\Delta_3,R)=\bigcup\limits_{A\in\mathfrak{Inv}(R_{<1})} \mathcal{L}(\xi'_A)$,

where $\xi'_A=(\xi_A)'=\{R_{<1}\circ A, A\backslash R_{<1}\circ A,\bar{A}\}$ $(\xi_A=\{\bar{A},A\backslash R_{<1}\circ A,R_{<1}\circ A\}$ - see Lemma 8.2 (v) and Corollary 8.3) ■

Proof. (i) In virtue of Corollary 4.1,

$$\mathfrak{D}(\Delta_1(\lceil\circ,\mathrm{inc}),R,\mathfrak{E}) = \overline{\mathfrak{D}(\Delta_3(\circ,\mathrm{inc}),\bar{R},\mathfrak{E})}$$

By substituting explicit expression for the right-hand side *MFC* from Corollary 8.3, we arrive to

$$\mathfrak{D}(\Delta_1(\lceil\circ,I_5),R)=\bigcup_{A\in\mathfrak{C}_*((\bar{R})_1)} \overline{\mathcal{U}(1,A)}$$

Owing to Lemma 8.2 (v), the latter formula is equivalent to the following expression

$$\mathfrak{D}(\Delta_1(\lceil\circ,I_5),R)=\bigcup_{A\in\mathfrak{C}_*((\bar{R})_1)} \mathcal{L}(0,\bar{A})$$

Finally, Lemma 8.1 (ii) implies $(\bar{R})_1=(\overline{R_{>0}})=R_{=0}$, so that

$$\mathfrak{D}(\Delta_1(\lceil\circ,I_5),R)=\bigcup_{A\in\mathfrak{C}_*(R_{=0})} \mathcal{L}(0,\bar{A}).$$

(ii) Since $\Delta_2(\lceil\circ,I_5)(R)$ is a normal f.s. of $\tilde{\mathcal{P}}(X)$, *MFC* is defined as

$$\mathfrak{D}(\Delta_2(\lceil\circ,I_5),R)=\mu^{-1}_{\Delta_2(\lceil\circ,I_5)(R)}(1)$$

Writing out the explicit expression for $\Delta_2(\lceil\circ,I_5)(R)$, we come to

$$a\in\mathfrak{D}(\Delta_2(\lceil\circ,I_5),R) \Leftrightarrow \mathfrak{X}(\overline{R\lceil\circ a}\vee\bar{a})=1 \Leftrightarrow \mathfrak{X}(\bar{R}\circ a\vee\bar{a})=1.$$

In virtue of Lemma 8.1 (i), (iii), the latter equality implies the following chain of equivalencies:

$$(\bar{R})_1\circ a_1\cup(\bar{a})_1=X \Leftrightarrow R_{=0}\circ A_1\cup A_{=0}=X \Leftrightarrow R_{=0}\circ A_1\supseteq\overline{A_{=0}} \Leftrightarrow R_{=0}\circ A_1\supseteq A_{>0}.$$

Clearly, with any f.s. $a\in\tilde{\mathcal{P}}(X)$, $A_{>0}\supseteq A_1$, so that, in particular, the latter inclusion in the chain implies that $R_{=0}\circ A_1\supseteq A_1$, which is equivalent to $A_1\in\mathfrak{Ant}(R_{=0})$. The remaining of the proof is similar to that of Corollary 8.3 (i).

(iii) Using the same reasoning as in (i), we arrive to the following chain of equalities:

$$\mathscr{D}(\Delta_3(\circ,I_5),R,\mathscr{E}) = \lfloor \mathscr{D}(\Delta_1(\lceil\circ,I_5),\bar{R},\mathscr{E}) \rfloor = \bigcup_{A\in\mathfrak{Inv}((\bar{R})_{>0})} \lfloor \ell(\xi_A) \rfloor$$

$$= \bigcup_{A\in\mathfrak{Inv}((\bar{R})_{>0})} \ell((\xi_A)') = \bigcup_{A\in\mathfrak{Inv}(R_{<1})} \ell(\xi'_A) \blacksquare$$

Owing to the latter result, and to the general method of proofs, reconstructing *MFC* with other $(\lceil\circ,I_5)$-based *FDDP*'s turns to be a purely technical problem. Nevertheless, we present a curious result regarding "dual von Neumann - Morgenstern Solution" $(\Delta_2\wedge\Delta_3)(\lceil\circ,I_5)$.

Theorem 8.6. *MFC* with von Neumann - Morgenstern Solution based on a dual composition law $\lceil\circ$, and on Kleene - Dienes inclusion I_5 consists of *crisp* subsets and can be represented as

$$\mathscr{D}((\Delta_{23})(\lceil\circ,I_5),R)=\mathfrak{Eig}(\bar{R})\cap\mathscr{P}(X) \blacksquare$$

Proof. We propose a straightforward proof based on the explicit form of a membership function of rationality concept. In virtue of normality of $(\Delta_{23})(\lceil\circ,I_5)(R)$, $\mathscr{D}((\Delta_{23})(\lceil\circ,I_5),R)=\mu^{-1}_{(\Delta_2\wedge\Delta_3)(\lceil\circ,I_5)(R)}(1)$ or, equivalently,

$$a\in\mathscr{D}(\Delta_{23}(\lceil\circ,I_5),R) \Leftrightarrow (\mathring{\lambda}(\overline{R\lceil\circ a}\vee\bar{a}))\wedge(\mathring{\lambda}(a\vee R\lceil\circ a))=1 \Leftrightarrow (\mathring{\lambda}(\bar{R}\circ a\vee\bar{a}))\wedge(\mathring{\lambda}(a\vee\overline{\bar{R}\circ a}))=1$$

$$\Leftrightarrow (\bar{a}\vee\bar{R}\circ a=1)\&(a\vee\overline{\bar{R}\circ a}=1) \Leftrightarrow (\bar{a}\vee\bar{R}\circ a=1)\&(\bar{a}\wedge\bar{R}\circ a=0).$$

The latter assertion states that both the union, and the intersection of two fuzzy subsets $\bar{a}$, and $\bar{R}\circ a$ are crisp subsets; it is easy to prove that the only case when this assertion is satisfied is the case when the initial f.s.' are, in fact, crisp subsets. Taking into account that the union of the above subsets is the whole support, and the intersection is empty, we conclude that they are mutually complementary; finally, $a=A\in\mathscr{P}(X)$, and $\bar{R}\circ A=A$ which is the same with $a\in\mathfrak{Eig}(\bar{R})\cap\mathscr{P}(X)$ $\blacksquare$

Theorem 8.6 demonstrates a unique behavior of a *FDDP*: with any *fuzzy binary relation*, a multifold fuzzy choice with dual von Neumann - Morgenstern Solution includes only *crisp* subsets of a support! Similar properties of other *FDDP*'s with distinguished *FR*'s will be observed in Chapter 11.

Another consequence of Theorem 8.6 is that *eigen fuzzy subsets* (Sanchez [1]) can be used, in addition to well-known applications in fuzzy dynamic systems (Rovicki [1], Di Nola, Pedrycz, and Sessa [1]), and in fuzzy clustering (Jacas, Recasens [1]), also in decision-making; a more detailed discussion on the subject is presented in Chapter 10.

Chapter 9

Ranking and C-Spectral Properties of Fuzzy Relations (Fuzzy von Neumann – Morgenstern – Zadeh Solutions)

In the previous chapter, we studied in details the choice with *FR*'s associated with I_5-based *FDDP*'s. Generally speaking, a *DC* I_5-*FDDP* gives rise to a collection of contrasts, each possessing the same distinguishing power (dichotomousness) $\mu^* - \overline{\mu^*} > 0$, and based on some crisp von Neumann - Morgenstern Solution of the median cut of an original *FR*, "best fitting" the fuzzy version of the rationality concept $\Delta_{23}(\circ, I_5)(R)$. In case when the mentioned subset $\mathfrak{K}^*$ of the "best fitting" crisp von Neumann - Morgenstern Solutions of $R_{>1/2}$ includes the only element K (in particular, this case is in force when $R_{>1/2}$ has the Stable Core M, and $\mu_{123}(M) > 1/2$), the induced choice is also unique, thus providing not only dichotomous, but also a stronger ranking contensiveness. However, thus induced "choice-ranking" turns to be rather poor: it is nothing but a dichotomy $\{K, \overline{K}\}$. Therefore, a natural question can be set: can *FDDP*'s produce more detailed rankings? In other words, is ranking contensiveness a really working concept?

In this chapter, we to give positive answers to both questions. The idea of "switching" *FDDP*'s from choice to ranking is very simple: just changing a model of fuzzy inclusion, underlying the rationality concept, from Kleene - Dienes inclusion I_5 to conventional L.Zadeh' inclusion $\subseteq$. Since Theorem 7.1 still remains in force, only two $\subseteq$-based *FDDP*'s are of interest, namely, fuzzy version of von Neumann - Morgenstern Solution - $\Delta_{23}(\circ,\subseteq)$, and of the Stable Core - $\Delta_{123}(\circ,\subseteq)$. In fact, the results of the study of $\subseteq$-based *FDDP*'s remain valid for a wide class of reflexive fuzzy inclusions (see Theorem 9.3 and Corollary 9.2 at the end of this chapter). So, our main efforts are devoted to the research of the above two procedures; in the last section of this chapter, we also study incontensive *FDDP*'s based on L.Zadeh' inclusion.

It should be noticed that, unless the idea of the ranking method is easy, adequate technique for the study of $\subseteq$-*FDDP*'s is much more difficult than in the case of Kleene - Dienes inclusion; this technique is completely different from the apparatus of Chapter 8. We suggest two explanations of this effect. One reason is that α-cut method in case of L.Zadeh inclusion is more complicated than in case of I_5. Another reason is discontinuity of a *FI* $\subseteq$, causing additional problems.

9.1. BASIC CHARACTERISTICS. $\varkappa$-MAPPING

Let us write down a general formula of fuzzy von Neumann - Morgenstern solution (FNMS) based on arbitrary composition law $\odot$, and any *FI* inc ($P=\langle X,\mathcal{R},p\rangle$=FNMS [1])

$$\mu_{p(R)}(a)=\mu_{eq}(R\odot a,\bar{a}),$$

with eq being *fuzzy equivalence*,

$$\mu_{eq}(a,b)=\mu_{inc}(a,b)\wedge\mu_{inc}(b,a).$$

This formula represents, so to say, "anti-eigen" fuzzy subsets, satisfying the approximate equation $R\circ a\approx\bar{a}$. If we suppose that inc $\subseteq$, eq turns into exact crisp equality. So far, a *FDDP* Fuzzy von Neumann - Morgenstern - Zadeh Solution (FNMZS) can be defined as characteristic

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[1] In this chapter, only universal environment is considered; for this reason, the notation $\mathcal{E}$ is omitted in all formulas, and the term "R is *RC*" is understood as "a specialization $(p,R,\mathcal{E})$ is *RC*".

function of the corresponding *MFC*:

$$\mu_{p(R)}(a)=\begin{cases} 1, & R\odot a=\bar{a} \\ 0, & \text{otherwise} \end{cases}$$

In other words, FNMZS can be identified with its *MFC*. The technique of this chapter is based on a simple observation that FNMZS, when considered as crisp subset of $\tilde{\mathcal{P}}(X)$, forms the set of all stable points of a mapping

$$\varkappa_R:\tilde{\mathcal{P}}(X)\longrightarrow\tilde{\mathcal{P}}(X), \qquad \varkappa_R(a)=\overline{R\odot a}$$

Owing to Theorem 7.1, only $\vee$-$\wedge$-composition $\circ$ is of interest; as to the dual law $\ulcorner\circ$, the corresponding FNMZS can be written as

$$R\ulcorner\circ a=\bar{a} \Leftrightarrow \overline{\bar{R}\circ a=\bar{a}} \Leftrightarrow \bar{R}\circ a=a,$$

so that "$\ulcorner\circ$-FNMZS" coincides with the set of all *eigen fuzzy subsets* of the relation $\bar{R}$ [2] (see Theorem 8.6). In what follows, the term FNMZS is used in three meanings:

(1) to label *FDDP* $\Delta_{23}(\circ,\subseteq)$;

(2) to name the corresponding *MFC*;

(3) to call any f.s, included in *MFC*, that is, satisfying the equation $R\circ a=\bar{a}$.

The main problem in the study of FNMZS is that the exhaustive description of *MFC* is extremely complicated. An intrinsic reason for this complexity is the fact that FNMZS, when considered as a collection of f.s.' of X, has no semilattice properties (see Proposition 9.7 (i) below), in contrast with solutions of conventional relational equations - see, e.g. (Di Nola, Pedrycz, Sanchez, and Sessa [1]). Luckily, the approach under development does not require the search of *all* solutions; what we need is to verify ranking contensiveness, and to reconstruct the canonical ranking in *RC*-case (see Chapter 4 for definitions). To this end, we introduce two-sided estimates of the solution, and work out tools for indirect research of ranking properties, and of ranking contensiveness of FNMZS in terms of $\varkappa$-mapping.

First, let us investigate basic characteristics of $\varkappa_R$.

Lemma 9.1. With any $R\in\tilde{\mathcal{P}}_0(X^2)$,

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[2] Getting ahead of our story, we mark out that the corresponding *MFC* does *not* represent a well-defined ranking.

(i) $\varkappa_R$ is a $\subseteq$-antitone mapping.

(ii) $\varkappa_R^2=\varkappa_R\square\varkappa_R$ is a $\subseteq$-monotone mapping ■

Proof. (i) $a\subseteq b \Rightarrow R\circ a\subseteq R\circ b \Leftrightarrow \overline{R\circ b}\subseteq\overline{R\circ a} \Leftrightarrow \varkappa_R(a)\subseteq\varkappa_R(b)$.

(ii) Directly follows from (i) ■

Lemma 9.2 (α-cut equalities for $\varkappa_R$). With any $\alpha\in]0,1]$, $R\in\tilde{\mathscr{P}}_0(X^2)$, $a\in\tilde{\mathscr{P}}(X)$, the following equalities for the conventional, and the strict α-cuts of $\varkappa$-mapping are satisfied:

$$(\varkappa_R(a))_\alpha=\varkappa_{R_{>\bar{\alpha}}}(a_{>\bar{\alpha}})$$

$$(\varkappa_R(a))_{>\alpha}=\varkappa_{R_{\bar{\alpha}}}(a_{\bar{\alpha}}) \ ■$$

Proof. By definition of $\varkappa$-mapping, $(\varkappa_R(a))_\alpha=(\overline{R\circ a})_\alpha$. In virtue of Lemma 8.1 (ii), the right-hand side term is nothing but $\overline{(R\circ a)_{>\bar{\alpha}}}$; using Lemma 8.1 (i), we obtain $\overline{(R\circ a)_{>\bar{\alpha}}}=\overline{R_{>\bar{\alpha}}\circ a_{>\bar{\alpha}}}=\varkappa_{R_{>\bar{\alpha}}}(a_{>\bar{\alpha}})$. The second equality can be verified similarly ■

9.2. BOUNDS OF MULTIFOLD FUZZY CHOICE

Ranking properties of FNMZS can be discovered through the analysis of the behavior of "unity orbit" of $\varkappa$-mapping: $\Omega_R=\{\varkappa_R^j(1)\}$ ($\varkappa_R^0(1)=1$, $\varkappa_R^j(1)=\varkappa_R(\varkappa_R^{j-1}(1))$). More specifically, these properties prove to be determined by disposition of two attractors of Ω_R.

Let us denote $d_j=\varkappa_R^j(1)$ (from now on, for brevity, the symbol of an *FR* 'R' in subscripts is mostly omitted). Basic features of Ω are established in Propositions 9.1-9.3.

Proposition 9.1. (α-cut equalities in Ω). With any $\alpha\in]0,1]$, $j>0$, the following recurrent formulas for α-cuts of the members of unity orbit are satisfied:

$$(d_j)_\alpha=\varkappa_{R_{>\bar{\alpha}}}(d_{j-1})_{>\bar{\alpha}}$$

$$(d_j)_{>\alpha}=\varkappa_{R_{\bar{\alpha}}}(d_{j-1})_{\bar{\alpha}} \ ■$$

Proof. A straightforward consequence of Lemma 9.2 ■

Proposition 9.1 offers a convenient tool for investigating unity orbit Ω, and of its stable points through examining simultaneous "action" of two crisp relations R_α, and $R_{>\bar{\alpha}}$. Indeed, this proposition easily implies the following formulas:

$$(d_{2k+1})_\alpha=\varkappa_{R_{>\bar{\alpha}}}\square\varkappa_{R_\alpha}\square\ldots\square\varkappa_{R_\alpha}\square\varkappa_{R_{>\bar{\alpha}}}(X) \text{ (2k+1 members)}$$

$$(d_{2k})_\alpha=\varkappa_{R_{>\bar{\alpha}}}\square\varkappa_{R_\alpha}\square\ldots\square\varkappa_{R_\alpha}(X) \text{ (2k members).}$$

A substantial part of the subsequent results (Propositions 9.3, 9.6, 9.11, Theorem 9.2, etc.) is based on the properties of alternated composition chains of this type.

Proposition 9.2. (i) "Even members" of Ω form a non-increasing, "odd members" - a non-decreasing subsequence:

$$(\forall\ j>0)((d_{2j}\subseteq d_{2j-2})\&(d_{2j-1}\subseteq d_{2j+1}))$$

(ii) Any odd member does not exceed any even member,

$$(\forall\ j,l\geq 0)(d_{2j+1}\subseteq d_{2l})\ ■$$

Proof. First, let us write out two evident inequalities

$$1=d_0\supseteq d_1 \quad (9\text{-}1)$$

$$1=d_0\supseteq d_2. \quad (9\text{-}2)$$

Applying $\varkappa$ to (9-2), we derive, due to Lemma 9.1, (i), the inclusion

$$d_1\subseteq d_3. \quad (9\text{-}3)$$

Next, applying $\varkappa^2$ (j-1) times to (9-2), (9-3), (9-1), we obtain, with respect to Lemma 9.1 (ii),

$$d_{2j-2}\supseteq d_{2j} \quad (9\text{-}4)$$

$$d_{2j-1}\subseteq d_{2j+1} \quad (9\text{-}5)$$

$$d_{2j-2}\supseteq d_{2j-1}. \quad (9\text{-}6)$$

Hence, the statement (i) which is exactly (9-4)&(9-5), is true. Now, let us suppose that $j\geq l$; the inclusions (9-4), (9-6) show (with evident transposition of subscripts) that $d_{2l+1}\subseteq d_{2j+1}\subseteq d_{2l}$, thus proving also (ii) (verification in the case $j<l$ is similar) ■

Proposition 9.2 gives rise to significant definitions of a lower, and of an upper bound of *MFC*. To make these definitions more transparent, we resort to a well-known crisp analogy.

Example 9.1. Let R be a crisp binary relation on X, that is, a digraph. A collection of crisp subsets D_j of X, constituting a unity orbit Ω, can be described in the following way:

D_0 is the very support X;

D_1 is the Core of R (the set of all non-dominated alternatives);

D_2 is the set of all alternatives except for those which are worse than at least one alternative included in D_1;

D_3 is the Core of the induced relation $R|_{D_2}$;

. . . .

D_{2k+1} is the Core of the induced relation $R|_{D_{2k}}$;

D_{2k+2} is the set of all alternatives except for those which are worse than at least one alternative included in D_{2k+1};

. . . .

The corresponding decomposition of X is shown on Figure 9.1. Let us

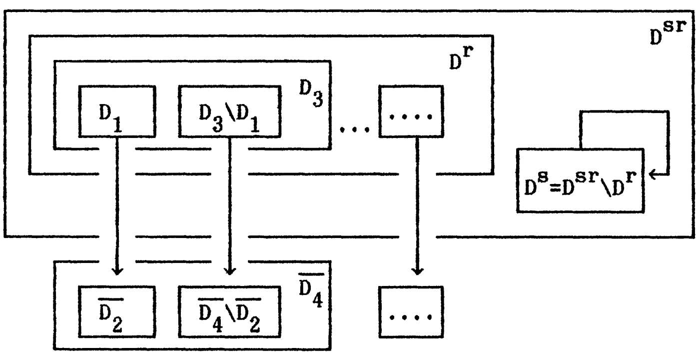


Fig. 9.1. Decomposition of a support on a regular and a singular-regular component of a binary relation

denote by D^r the greatest among the "odd" members D_{2k+1}, by D^{sr} - the smallest among the "even" members D_{2k}. Obviously, the convergence of the above sequence $\{D_j\}$ is equivalent to the condition $D^r=D^{sr}$, which, in its turn, is equivalent to the fact that $D^r=D^{sr}$ is the only crisp von Neumann - Morgenstern Solution of R. In that way, D^r represents, so to say, a *regular* component of crisp von Neumann - Morgenstern Solution of R. In case of non-convergence of $\{D_j\}$, the sequence is oscillating between D^r and D^{sr}. The difference $D^s=D^r\setminus D^{sr}$ can be considered as a kind of "singular body" of R. All alternatives of the induced relation $R|_{D^s}$ are dominated; nevertheless, none of the alternatives included in D^s is worse than at least one alternative included in D^r. This is the reason to call an "upper bound" D^{sr} a *singular-regular* component of von Neumann - Morgenstern Solution of a relation R.

Moreover, in crisp case, there exists a very simple decomposition of von Neumann - Morgenstern Solutions of R: with *each* von Neumann - Morgenstern Solution K of an induced relation $R|_{D^s}$, a subset $D^r\cup K$ is a von Neumann - Morgenstern Solution of R, and vice versa ∎

In the next statement, the length of unity orbit for fuzzy and crisp binary relations is estimated.

Proposition 9.3. (length of orbit).

(i) With arbitrary *FR*, $|\Omega|\leq 2|X|$. The estimate is exact.

(ii) With crisp *FR*, $|\Omega|\leq|X|$. The estimate is also exact ∎

Proof. (i) Let us denote $\tau_\alpha=\varkappa_{R_{>\bar{\alpha}}}\square\varkappa_{R_\alpha}$. With respect to Lemma 9.1 (i), both $\varkappa_{R_{>\bar{\alpha}}}$, and $\varkappa_{R_\alpha}$ are antitone; hence, τ_α is monotone, and $X\supseteq\tau_\alpha(X)\supseteq\tau^2_\alpha(X)\supseteq\dots$ is a decreasing sequence of subsets of a finite set X. It follows that either

$$\text{(a) } \tau^n_\alpha(X)=\emptyset \qquad \text{or} \qquad \text{(b) } \tau^n_\alpha(X)=\tau^{n-1}_\alpha(X)$$

holds. Next, Proposition 9.1 implies (by induction) the equality $(d_{2j})_\alpha=\tau^j_\alpha(X)$. Therefore, in the (a)-case, we come to

$$\emptyset=(d_{2n})_\alpha\supseteq(d_{2n+2})_\alpha\supseteq(d_{2n+1})_\alpha\supseteq(d_{2n-1})_\alpha$$

(all inclusions are implied by Proposition 9.2), so that both $(d_{2n+2})_\alpha=(d_{2n})_\alpha$, and $(d_{2n+1})_\alpha=(d_{2n-1})_\alpha$ are satisfied. In the (b)-case, $(d_{2n-2})_\alpha=(d_{2n})_\alpha$ holds. Hence, $(d_{2n-1})_\alpha=(d_{2n+1})_\alpha$. Finally,

$$(\forall\ \alpha\in]0,1])((d_{2n+2})_\alpha=(d_{2n})_\alpha\&(d_{2n+1})_\alpha=(d_{2n-1})_\alpha)$$

It follows that $(d_{2n+2}=d_{2n})\&(d_{2n+1}=d_{2n-1})$, so that $|\Omega|\leq 2n$.

The exactness of this estimate is demonstrated in the following example with $|X|=3$:

$$R = \begin{vmatrix} 0 & 0.2 & 0.8 \\ 0.4 & 0 & 0.3 \\ 0.3 & 0.7 & 0 \end{vmatrix}$$

Direct calculation of Ω yields

$$d_0=(1.0 \quad 1.0 \quad 1.0)$$

$$d_1=(0.6 \quad 0.3 \quad 0.2)$$

$$d_2=(0.7 \quad 0.8 \quad 0.4)$$

$$d_3=(0.6 \quad 0.6 \quad 0.3)$$

$$d_4=(0.6 \quad 0.7 \quad 0.4)$$

$$d_5=d_6=\ldots=d^r=d^{sr}=(0.6 \quad 0.6 \quad 0.4).$$

Hence, $|\Omega|=6=2|X|$ (analogous example can be constructed in any dimension).

(ii) With crisp R, all d_i's are crisp subsets of X. In such case, the length of the chain $d_1\subseteq d_3\subseteq\ldots\subseteq d_2\subseteq d_0=X$ cannot exceed $|X|$. With any 2x2 relation possessing empty Core, $\Omega=\{\emptyset,X\}$, so that $|\Omega|=|X|$, thus confirming the exactness of the estimate ∎

Following the idea of Example 9.1, let us denote by $d^r=\vee_j d_{2j+1}$, $d^{sr}=\wedge_j d_{2j}$ the lower, and the upper bounds of "odd", and of "even" subsequences of Ω. In Propositions 9.4, 9.5, we prove that d^r, and d^{sr} represent both the attractors of Ω, and the *bounds* of FNMZS (in the sequel, d^r is called a *regular bound*, d^{sr} - a *singular-regular bound* of FNMZS).

Proposition 9.4. (i) $d^r\subseteq d^{sr}$.

(ii) Let us suppose that $j\geq|\Omega|$; with odd j, $d_j=d^r$; with even j, $d_j=d^{sr}$.

(iii) $\varkappa(d^r)=d^{sr}$; $\varkappa(d^{sr})=d^r$ ∎

Proof. (i) Follows from Proposition 9.2 (ii).

(ii) Follows from Proposition 9.2 (i), Proposition 9.3 (i), and definitions of d^r, d^{sr}.

(iii) Let us select any odd $j\geq|\Omega|$; according to (ii), $d_j=d_{j+2}=d^r$, $d_{j+1}=d^{sr}$, so that $\varkappa(d^r)=\varkappa(d_j)=d_{j+1}=d^{sr}$, and $\varkappa(d^{sr})=\varkappa(d_{j+1})\stackrel{\Delta}{=}d_{j+2}=d^r$ ∎

Proposition 9.5. FNMZS (multifold fuzzy choice) is contained in the interval $[d^r, d^{sr}] \subseteq \tilde{\mathcal{F}}(X)$ ■

Proof. With any $a \in$FNMZS, $\varkappa(a)=a$. Hence,

$$1 \supseteq \varkappa(a)=a \Rightarrow \varkappa(1)=d_1 \subseteq \varkappa^2(a)=a \Rightarrow \varkappa(d_1)=d_2 \supseteq \varkappa^3(a)=a,$$

i.e. $d_1 \subseteq a \subseteq d_2$. Exactly as in the proof of Proposition 9.2, we derive that $d_{2j-1} \subseteq a \subseteq d_{2j}$, so that $d^r \subseteq a \subseteq d^{sr}$ ■

A natural question is whether the interval $[d^r, d^{sr}]$ contains anything valuable. In crisp case ($R \in \mathcal{F}(X^2)$, $\mathcal{E}=\mathcal{F}(X)$), CNMS may well be empty. A canonical example of a crisp relation without von Neumann - Morgenstern Solutions is an oriented 3-cycle

$$R = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

The corresponding digraph can be depicted as $R=\{x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_1\}$ (we recall that a sufficient condition of the existence of crisp von Neumann - Morgenstern Solution is the absence of oriented cycles of odd length, see Berge [1]). But again, the "completion effect" of fuzzy scale makes itself felt: in *universal* fuzzy environment the answer to the above question is positive, and FNMZS always yields a non-empty *MFC*!

Let us construct a concrete solution. We introduce a fuzzy subset k^f, $k^f = 1/2 |\neg| [d^r, d^{sr}]$ (see Chapter 2 for the definition of an operation $|\neg|$). α-cuts of this subset are studied in the next assertion.

Lemma 9.3. With $\alpha<1/2$, $(k^f)_\alpha=(d^{sr})_\alpha$, and $(k^f)_{>\alpha}=(d^{sr})_{>\alpha}$;
with $\alpha>1/2$, $(k^f)_\alpha=(d^r)_\alpha$, and $(k^f)_{>\alpha}=(d^r)_{>\alpha}$;
with $\alpha=1/2$, $(k^f)_\alpha=(d^{sr})_\alpha$, and $(k^f)_{>\alpha}=(d^r)_{>\alpha}$ ■

Proof. According to definitions of operation $|\neg|$, and of k^f,

$$\mu_{k^f}(x)=\begin{cases} \mu_{d^r}(x), & \mu_{d^r}(x)>1/2 \\ 1/2, & \mu_{d^r}(x)\le 1/2 \le \mu_{d^{sr}}(x), \\ \mu_{d^{sr}}(x), & \mu_{d^{sr}}(x)<1/2 \end{cases}$$

Hence, $(\forall\ \alpha<1/2)(\mu_{k^f}(x)\geq\alpha \Leftrightarrow \mu_{d^{sr}}(x)\geq\alpha)$, so that $(\forall\ \alpha<1/2)((k^f)_\alpha=(d^{sr})_\alpha)$. Next, $(\forall\ \alpha>1/2)(\mu_{k^f}(x)\geq\alpha \Leftrightarrow \mu_{k^f}(x)=\mu_{d^r}(x))$; hence, $(\forall\ \alpha>1/2)((k^f)_\alpha=(d^r)_\alpha)$; the remaining equalities are verified in the same way ■

These α-cut equalities result in the following

Proposition 9.6. k^f is FNMZS ■

Proof. The equality of f.s.' $\varkappa(k^f)=k^f$ is equivalent to a collection of crisp equalities $(\forall\ \alpha\in I)((\varkappa(k^f))_\alpha=(k^f)_\alpha)$. With respect to Lemma 9.2, $(\varkappa(k^f))_\alpha=\varkappa_{R_{>\bar\alpha}}((k^f)_{>\bar\alpha})$. First, let us set $\alpha<1/2$; Lemma 9.3 yields $(k^f)_\alpha=(d^{sr})_\alpha$, and $(k^f)_{>\bar\alpha}=(d^r)_{>\bar\alpha}$. Hence, $(\varkappa(k^f))_\alpha=\varkappa_{R_{>\bar\alpha}}((d^r)_{>\bar\alpha})$. Combining Lemma 9.2 with Proposition 9.4 (iii), we obtain $\varkappa_{R_{>\bar\alpha}}((d^r)_{>\bar\alpha})=(d^{sr})_\alpha$, so that $(\varkappa(k^f))_\alpha=(d^{sr})_\alpha=(k^f)_\alpha$. With $\alpha>1/2$, $\alpha=1/2$, the proof is similar ■

In particular, Proposition 9.6 implies the "existence theorem".

Theorem 9.1. For any *FR*, FNMZS is non-empty subset of $[d^r,d^{sr}]$ ■

In order to graphically illustrate the behavior of unity orbit, and of its attractors let us interpret $\tilde{\mathcal{P}}(X)$ as a n-dimensional cube. On the plain figures, we always assume n=2 and represent $\tilde{\mathcal{P}}(X)$ in the form of a square. Bottom left vertex of the square stands for 0, top right vertex for $\tilde{1}$, the center for 1/2; subrectangles of the main square depict intervals in $\tilde{\mathcal{P}}(X)$. The first illustration of this type is Figure 9.2.

In fact, k^f is not a "chance FNMZS", but the *fuzziest* one (the superscript 'f' in the notation indicates this property). We recall that a fuzzy subset a' is a *sharpened version* of f.s. a (Dubois, Prade [1]) iff

$$(\forall\ \alpha<1/2)(a_\alpha\subseteq a'_\alpha)\&(\forall\ \alpha>1/2)(a'_\alpha\subseteq a_\alpha).$$

In the axiomatic study of measures of fuzziness (see, e.g., De Luca, Termini [1]), any *index of fuzziness* decreases with respect to sharpening.

Proposition 9.7. (i) Any two FNMZS's are $\subseteq$-incomparable.

(ii) Any FNMZS is a sharpened version of k^f ■

Proof. (i) $a\subseteq b \Rightarrow b=\varkappa(b)\subseteq\varkappa(a)=a \Rightarrow a=b$.

(ii) It suffices to apply Proposition 9.5 to the definition of k^f ■

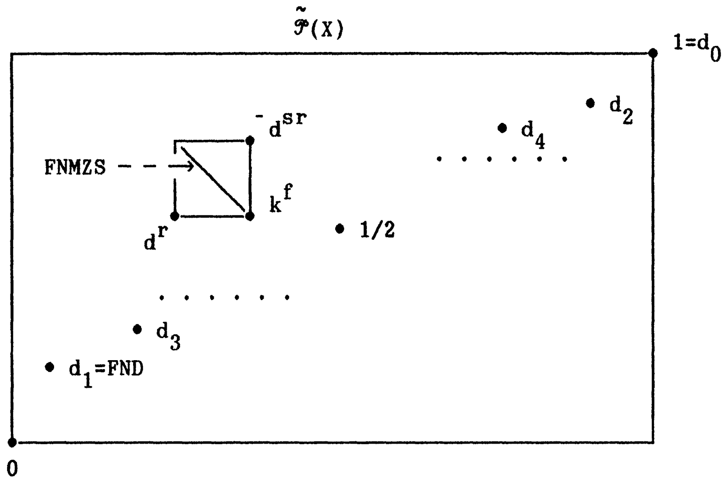


Fig. 9.2. Unity orbit, its attractors, and FNMZS

However, ranking of the support X according to membership values of a f.s. k^f is, generally speaking, incorrect, as is demonstrated in the following example.

Example 9.2. Let us consider an *FR* R_{E2} on a support $X=\{x_1,\dots,x_6\}$. Direct calculation results in the following evaluation of d^r, d^{sr}, and k^f:

$$R_{E2} = \begin{vmatrix} 0.0 & 0.7 & 0.6 & 0.0 & 0.7 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.7 & 0.3 & 0.6 \\ 0.3 & 0.8 & 0.0 & 0.1 & 0.7 & 0.0 \\ 0.0 & 0.9 & 0.1 & 0.0 & 0.0 & 0.4 \\ 0.3 & 0.1 & 0.5 & 0.1 & 0.0 & 1.0 \\ 0.1 & 1.0 & 0.8 & 0.9 & 0.9 & 0.0 \end{vmatrix}$$

$$d^r=(0.7\ \ 0.1\ \ 0.2\ \ 0.1\ \ 0.1\ \ 0.6)$$

$$d^{sr}=(0.8\ \ 0.3\ \ 0.4\ \ 0.4\ \ 0.3\ \ 0.9)$$

$$k^f=(0.7\ \ 0.3\ \ 0.4\ \ 0.4\ \ 0.3\ \ 0.6)$$

Hence, crisp ordering of X with respect to membership values of k^f is $x_1 > x_6 > x_3 = x_4 > x_2 = x_5$, and the corresponding partition can be written as

$$\xi_{\{k^f\}}=\{\{x_1\},\{x_6\},\{x_3,x_4\},\{x_2,x_5\}\}.$$

However, in case of R_{E2}, there exists not only the fuzziest FNMZS, but also the "sharpest", that is, the most determinant FNMZS

$$k^c=\chi_{X^*}|\neg|[d^r,d^{sr}]=(0.8\ \ 0.1\ \ 0.2\ \ 0.1\ \ 0.1\ \ 0.9),$$

where $X^*=\{x_1,x_6\}$ is the unique CNMS of $(R_{E1})_{>1/2}$ (see Lemma 9.6 below). The corresponding crisp order, and the induced partition are respectively $x_6>x_1>x_3>x_2=x_4=x_5$, and

$$\xi_{\{k^c\}}=\{\{x_6\},\{x_1\},\{x_3\},\{x_2,x_4,x_5\}\}.$$

It is of interest that the usage of conventional FND,

$$\text{FND}=\varkappa_{R_{E2}}(1)=(0.7\ \ 0.1\ \ 0.2\ \ 0.1\ \ 0.1\ \ 0.4)$$

results in the third ordering $x_1>x_6>x_3>x_2=x_4=x_5$, differing from both the k^f-, and the k^c-orderings. It follows that the induced crisp ranking ξ^o associated with the corresponding *MFC* (see Definition 5.2) can be neither $\xi_{\{k^f\}}$ nor $\xi_{\{k^c\}}$. Actually, C-ranking with respect to R_{E2} is

$$[0.6,0.9]/\{x_1,x_6\}+[0.1,0.4]/\{x_2,x_3,x_4,x_5\},$$

and the induced crisp partition ξ^o is the lattice supremum of $\xi_{\{k^f\}}$, and $\xi_{\{k^c\}}$ in the lattice Π_X of crisp partitions of X, $\xi^o=\xi_{\{k^f\}}\vee\xi_{\{k^c\}}$ ∎

Our next task is to investigate more closely the correspondence between the FNMS, and the FND $\varkappa_R(1)$. In case of antireflexive *FR*, FND is nothing but the first element in a unity orbit Ω; hence, Proposition 9.2 implies the inclusion FND$\subseteq d^r$, so that generally FND is a "lower estimate" of FNMZS, and it must not *necessarily belong to MFC.* This mutual "disposition" of FND and FNMZS is resembling the crisp case: CND is always included in any CNMS. The analogy is, however, even more complete: in crisp case, if CND is CNMS, then it is the *only* CNMS. The same result takes place in fuzzy environment.

Proposition 9.8. (i) If FNMZS contains FND, then FNMZS coincides with FND, FNMZS=$\{$FND$\}$.

(ii) FNMZS contains FND iff, for any given $\alpha\in]0,1]$, $R_\alpha\circ\text{CND}(R_{>\bar{\alpha}})=R_\alpha\circ X$ ∎

Proof. (i) FND=d_1 is FNMZS $\Leftrightarrow$ $\chi(d_1)=d_1$ $\Leftrightarrow$ $d_2=d_1$ $\Leftrightarrow$ $d_1=d_2=d_3=\ldots=d^r=d^{sr}$. In such case, Proposition 9.5 implies the equality FNMZS=$\{d^r\}=\{d^{sr}\}=\{$FND$\}$ ∎

(ii) From (i), FND∈FNMZS ⇒ FNMZS=$\{$FND$\}$ ⇒ $d_2=d_1$, that is,

$$(\forall\alpha\in]0,1])(((d_2)_\alpha=(d_1)_\alpha)\ \&\ ((d_2)_{>\alpha}=(d_1)_{>\alpha})).$$

Next, using Proposition 9.1, we obtain consequently

$$(d_1)_{>\bar{\alpha}}=\chi_{R_\alpha}((d_0)_\alpha)=\chi_{R_\alpha}(X);$$

$$(d_2)_{>\bar{\alpha}}=\chi_{R_\alpha}((d_1)_\alpha)=\chi_{R_\alpha}(\chi_{R_{>\bar{\alpha}}}(d_0)_{>\bar{\alpha}})=\chi_{R_\alpha}(\chi_{R_{>\bar{\alpha}}}(X))=\chi_{R_\alpha}(\mathrm{CND}_{R_{>\bar{\alpha}}}).$$

So, the already proved equality $(d_2)_{>\bar{\alpha}}=(d_1)_{>\bar{\alpha}}$ implies $\chi_{R_\alpha}(X)=\chi_{R_\alpha}(\mathrm{CND}_{R_{>\bar{\alpha}}})$; this is equivalent to $\overline{R_\alpha\circ\mathrm{CND}_{R_{>\bar{\alpha}}}}=\overline{R_\alpha\circ X}$, and $R_\alpha\circ\mathrm{CND}_{R_{>\bar{\alpha}}}=R_\alpha\circ X$ ∎

The following example shows that the latter "consistency condition" for FND is rather restrictive; even in case of *fuzzy* orderings, this condition fails very frequently (unlike the crisp case - see Chapter 13).

Example 9.3. $X=\{x_1,x_2,x_3,x_4\}$, a fuzzy ordering R_{E3} is defined as

$$R_{E3}=\begin{vmatrix} 0.0 & 0.3 & 0.7 & 0.8 \\ 0.0 & 0.0 & 0.8 & 0.9 \\ 0.0 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{vmatrix}$$

Here, FND=(1.0 0.7 0.2 0.1);

FNMZS=χ(FND)=$\{$(1.0 0.7 0.3 0.2)$\}$.

It is easy to calculate that with R_{E3} the condition of Proposition 9.8 (ii) fails for any α>0.7. The difference between the FND, and the FNMS demonstrates that preference values $x_1\xrightarrow{0.7}x_3$, $x_1\xrightarrow{0.8}x_4$ giving rise to $\mu_{FNMZS}(x_3)$, $\mu_{FNMZS}(x_4)$ are "more essential" than the values $x_2\xrightarrow{0.8}x_3$, $x_2\xrightarrow{0.9}x_4$, resulting in $\mu_{FND}(x_3)$, $\mu_{FND}(x_4)$. It should also be noticed that the formula FNMZS=χ(FND) is always in force for fuzzy preorderings (see Chapter 13) ∎

9.3. CONNECTED SPECTRUM, AND SPECTRAL PROPERTIES OF A FUZZY RELATION

Developing the ideas of Chapter 4, and of the above Example 9.1, we introduce new concepts of *singular body*, and of *connected spectrum* of an arbitrary *FR*.

Definition 9.1. (i) A *singular body* of an *FR* R is a f.s.

$$d^s=d^{sr}\setminus d^r=d^{sr}\wedge\overline{d^r}.$$

(ii) An *FR* R is called:

regular iff $d^s\subset\subset 1/2$ ($\mu_{d^s}(x)<1/2$ for all $x\in X$);

irregular iff $d^s\supseteq 1/2$;

otherwise, it is called *semiregular*;

(iii) d-*connected spectrum* (dC-*spectrum*) of an *FR*

$$\Lambda^d=\{J_d^i=[\alpha^i,\beta^i]\}=\Lambda_{[d^r,d^{sr}]}\ ;$$

d-*canonical partition* (dC-*partition*)

$$\xi^d=\{X_d^i\}=\xi_{[d^r,d^{sr}]}$$

(see Chapter 5); an interval fuzzy ranking (Λ^d,ξ^d) is called dC-*ranking* ∎

In the spirit of a general approach to ranking contensiveness stated in Chapter 5, our current problem is to characterize C-*ranking* (which is derived from an *unknown MFC* with a very complicated structure) in the terms of dC-*ranking* (which can be easily calculated). A one-sided estimate is easily implied by the above-stated results.

Proposition 9.9. $\xi^o \prec \xi^d$ (we recall that $\prec$ is a partial order in a lattice Π_X of crisp partitions of X- see Chapter 5) ∎

Proof. Follows from Proposition 9.5 (FNMZS$\subseteq[d^r,d^{sr}]$), and from Propositions 3.2, 3.3 ∎

So, dC-partition is generally less detailed than C-partition. To obtain more profound results, we examine very precisely *regularity properties* of an *FR*, more exactly, the "size" of its singular body. It turned out that, the "smaller" is d^s (the "more regular" is *FR*), the more efficient is FNMZS, and the closer it is connected with dC-ranking (we mark out the analogy with crisp considerations in Example 9.1).

First, let us investigate spectral properties of an *FR*.

Lemma 9.4. The interior $\overset{\circ}{\lambda}_{[d^r,d^{sr}]}$ of the union of the intervals included in a connected spectrum of an arbitrary *FR* is symmetric with respect to the "median" 1/2: $\overline{\overset{\circ}{\lambda}_{[d^r,d^{sr}]}}=\overset{\circ}{\lambda}_{[d^r,d^{sr}]}$ ■

Proof. $\alpha\in\overset{\circ}{\lambda}_{[d^r,d^{sr}]} \Leftrightarrow (\exists i)(\alpha\in\overset{\circ}{J}{}^i) \Leftrightarrow (d^r)_\alpha\neq(d^{sr})_\alpha \Leftrightarrow$

$\Leftrightarrow \chi_{R_\alpha}((d^{sr})_{>\bar{\alpha}})\neq\chi_{R_\alpha}((d^r)_{>\bar{\alpha}}) \Rightarrow (d^{sr})_{>\bar{\alpha}}\neq(d^r)_{>\bar{\alpha}} \Leftrightarrow$

$\Leftrightarrow (\exists m)(\bar{\alpha}\in\overset{\circ}{J}{}^m) \Leftrightarrow \bar{\alpha}\in\overset{\circ}{\lambda}_{[d^r,d^{sr}]}$ ■

Corollary 9.1. With any $J^i=[\alpha^i,\beta^i]\in\Lambda^d$, one of the following statements is fulfilled:

either $\overline{J^i}=[\overline{\beta^i},\overline{\alpha^i}]$ is also in Λ^d

or $\alpha^i=\beta^i$ (so that J^i is a pointwise component) ■

Proof. If $\overset{\circ}{J}{}^i=\emptyset$, then $\alpha^i=\beta^i$; otherwise, let us denote by J' the conjugation of J^i, $J'=\overline{J^i}$. According to Lemma 9.2, there exists a number m, satisfying the condition $\overset{\circ}{J'}\subseteq J^m$. If $J'\neq J^m$, then $\overset{\circ}{J}{}^m\setminus\overset{\circ}{J}\neq\emptyset$, so that $\overline{J^m}\supset J^i$. Applying Lemma 9.2 to J^m, we come to $J^i\subset\overline{\overset{\circ}{J}{}^m}\subseteq\overset{\circ}{\lambda}$, in contradiction with the condition $J^i\in\Lambda^d$. Hence, $\overline{J^i}=J'=J^m\in\Lambda^d$ ■

The following example demonstrates that pointwise components can actually represent the asymmetry of C-spectrum.

Example 9.4. $X=\{x_1,\dots,x_4\}$;

$$R_{E4}=\begin{vmatrix} 0.0 & 1.0 & 0.3 & 0.4 \\ 1.0 & 0.0 & 0.1 & 0.5 \\ 0.4 & 0.5 & 0.0 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.0 \end{vmatrix}$$

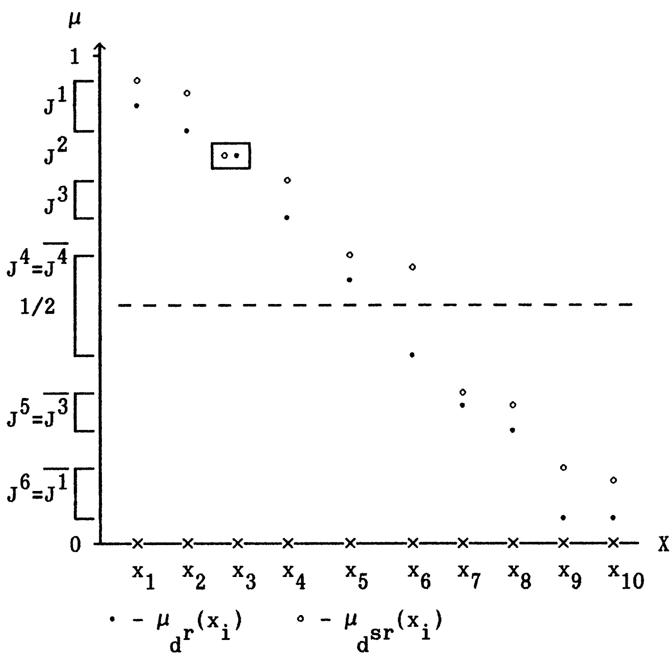
Here,
$$d^r=(0.5\quad 0.4\quad 0.7\quad 0.5)$$
$$d^{sr}=(0.6\quad 0.5\quad 0.7\quad 0.6)$$
$$\Lambda^d=\{\{0.7\},[0.4,0.6]\},$$

so that $J_d^1=\{0.7\}$ is "non-symmetric", whereas $J_d^2=[0.4,0.6]$ is "self-symmetric" ■

It should be noticed that symmetry property concerns only *connected components* of spectrum, not the initial intervals $[\mu_{d^r}(x),\mu_{d^{sr}}(x)]$; thus, in Example 9.2, there exists no conjugated interval for

$$[\mu_{d^r}(x_3),\mu_{d^{sr}}(x_3)]=[0.2,\ 0.4].$$

Spectral properties of FNMZS are illustrated on Figure 9.3.



Components: J^1-J^3 - dominating, J^5-J^6 - dominated, J^4 - neutral;

J^2 - 0-dimensional, all the rest - 1-dimensional

$\Lambda_d=\{J^1,\dots,J^6\}$

$\xi^d=\{\{x_1,x_2\},\{x_3\},\{x_4\},\{x_5,x_6\},\{x_7,x_8\},\{x_9,x_{10}\}$

Fig. 9.3. Connected spectrum, and dC-ranking of an FR

9.4. CLASSIFICATION OF MULTIFOLD FUZZY CHOICES

We underline the fact that the introduced spectral and regularity properties represent *invariant* characteristics of an *FR*; they must not *a priori* depend on the interpretation of an *FR* as of preference relation as well as to decision procedures. But, as a matter of fact, these concepts prove to be basic tools in the research of ranking facilities of a fuzzy preference relation.

In the next statement, we establish simple ties between spectral behavior and regularity.

Proposition 9.10. (i) An *FR* R is regular iff neutral component is absent; alternatively, regularity is equivalent to the following equalities:

$$(d^r)_{>1/2}=(d^r)_{1/2}=(d^{sr})_{>1/2}=(d^{sr})_{1/2}$$

(ii) In irregular case, there exists (clearly, unique) neutral component.

(iii) In semiregular case, either $|\xi^d|=1$ or there exist both neutral, and dominating (or dominated) components; moreover, if there exists a 1-dimensional dominating (or dominated) component, then all three types of components are present ∎

Proof. (i) R is regular $\Leftrightarrow d^s \subset\subset 1/2$

$$\Leftrightarrow (\forall x)(\mu_{d^{sr}}(x) \wedge \overline{\mu_{d^r}(x)} < 1/2)$$

$$\Leftrightarrow (\forall x)((\mu_{d^{sr}}(x) < 1/2) \vee (\mu_{d^r}(x) > 1/2))$$

$$\Leftrightarrow (\forall x)(1/2 \notin [\mu_{d^r}(x), \mu_{d^{sr}}(x)]) \Leftrightarrow 1/2 \notin \bigcup J_d^i \ .$$

The latter expression is equivalent to the equalities

$$(d^r)_{>1/2}=(d^r)_{1/2}=(d^{sr})_{>1/2}=(d^{sr})_{1/2}.$$

(ii) R is irregular $\Leftrightarrow d^s \supseteq 1/2 \Leftrightarrow (\forall x)(\mu_{d^{sr}}(x) \wedge \overline{\mu_{d^r}(x)} \geq 1/2)$

$$\Leftrightarrow (\forall x)((\mu_{d^{sr}}(x) \geq 1/2) \wedge (\mu_{d^r}(x) \leq 1/2))$$

$$\Leftrightarrow (\forall x)(1/2 \in [\mu_{d^r}(x), \mu_{d^{sr}}(x)]) \Rightarrow 1/2 \in \bigcap J_d^i \ .$$

(iii) Let us suppose that $|\xi^d|>1$; since R is neither regular nor irregular, there exists at least one neutral component, and at least one non-neutral component. Let J_d^1 be a one-dimensional non-neutral (say, dominating) component. In virtue of Corollary 9.1, $\overline{J_d^1}$, which is, clearly, a dominated component is also included in Λ^d, so that Λ^d contains all three types of components ■

Let us assume that R is a regular *FR*. Combining the results of Proposition 9.5, and of Proposition 9.10 (i), we see that a FNMZS with this *FR* is contained in an interval f.s. $[d^r,d^{sr}]$, and that the pointwise intervals $[\mu_{d^r}(x),\mu_{d^{sr}}(x)]$ are polarized with respect to "indifference line" 1/2. If we were in a choice situation, this disposition would have been sufficient for establishing dichotomous contensiveness. A preliminary estimate of dichotomousness of *MFC* could be determined as the difference between the lowest membership value of d^r in the interval $[1/2,1]$ and the highest membership value of d^{sr} in the interval $[0,1/2]$,

$$\delta(\Delta_{23}(\circ,\subseteq),R) \geq \lambda d^r\Big|_{(d^r)_{>1/2}} - \lambda d^{sr}\Big|_{(d^{sr})_{<1/2}}$$

But we are interested in a more detailed ranking. From a viewpoint of ranking contensiveness, we cannot maintain positively (at the moment) that dC-ranking is the one we need. According to Proposition 9.9, we can only assert that the induced partition ξ^o associated with FNMZS is at least as detailed as the partition ξ^d associated with $[d^r,d^{sr}]$. Can it be that the final ranking ξ^o is really more detailed? In *regular* case, the answer to this question is negative, and dC-ranking coincides with C-ranking. To establish this fact, we will construct a sufficiently large body of particular FNMZS' which are "covering", in certain meaning, the whole interval $[d^r,d^{sr}]$, thus making C-ranking equivalent to dC-ranking. It is of special interest that the announced collection of members of *MFC* is based, similarly to the case of $(\circ,I_5)$-procedures, on the CNMS of the nearest to R crisp binary relation, that is, on the strict median cut $R_{>1/2}$.

Lemma 9.5. In regular case, a subset of alternatives $X^*=(d^r)_{>1/2}$ is a crisp von Neumann - Morgenstern Solution of a binary relation $R_{>1/2}$ ■

Proof. With respect to Proposition 9.10,

$$X^*=(d^r)_{>1/2}=(d^r)_{1/2}=(d^{sr})_{>1/2}=(d^{sr})_{1/2}.$$

Next, Proposition 9.1 together with Proposition 9.4 (iii) implies the chain of equalities

$$\varkappa_{R_{>1/2}}(\chi_{X^*})=\varkappa_{R_{>1/2}}((d^r)_{>1/2})=(d^{sr})_{1/2}=\chi_{X^*};$$

hence, X^* is CNMS of $R_{>1/2}$ ∎

Now, everything is prepared for building the announced collection of FNMZS'. The only additional tool required for this construction is the Hamming projection $|\nearrow|$ [1].

Lemma 9.6. Given a $\beta\geq 1/2$, let us introduce two f.s.' of X:

$$c_0(\beta,X^*)=\beta/X^*+\overline{\beta}/\overline{X^*};\qquad {}^{\beta}k=c_0(\beta,X^*)|\nearrow|[d^r,d^{sr}].$$

Under these conditions, ${}^{\beta}k$ is FNMZS ∎

Proof. The method is similar to verification of Lemma 9.3, and of Proposition 9.6 in the case of k^f: α-cuts of ${}^{\beta}k$ coincide with alternated α-cuts of d^r, and of d^{sr} on the "conjugated" intervals J, $\overline{J}$ of the connected spectrum of R. More precisely,

$$({}^{\beta}k)_\alpha=\begin{cases}(d^{sr})_\alpha\ , & \alpha\in[0,\overline{\beta}]\cup]1/2,\beta[\\ X^*=(d^r)_{1/2}=(d^{sr})_{1/2}, & \alpha=1/2\\ (d^r)_\alpha\ , & \alpha\in]\overline{\beta},1/2[\cup[\beta,1]\end{cases}$$

The same is for strict α-cuts $({}^{\beta}k)_{>\alpha}$ (the proof repeats that of Lemma 9.3). With respect to these formulas, characteristic equations for FNMZS, $(\forall\ \alpha\in I)(\varkappa_{R_{>\overline{\alpha}}}(({}^{\beta}k)_{>\overline{\alpha}}))=({}^{\beta}k)_\alpha)$ are transformed, exactly as in the proof of Proposition 9.6, into a family of equalities $\varkappa_{R_{>\overline{\alpha}}}((d^r)_{>\overline{\alpha}}))=(d^{sr})_\alpha$, which are established in Lemma 9.2, and in Proposition 9.4 (iii) ∎

The latter construction can be interpreted as a one-dimensional version of choice analysis presented in Chapter 8. Instead of interval fuzzy

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[1] We accentuate the fact that the nearest crisp relation $R_{>1/2}$ itself is a Hamming projection of an *FR* on a "subspace" of crisp binary relations. It seems that the "Hamming projection method" is intrinsically related to our considerations.

subsets, namely, of fuzzy contrasts $c(\beta,K)$, which are responsible for contensive fuzzy choice with $(\circ,I_5)$-FNMS, we consider the "inner bounds" of all contrasts $c(\beta,X^*)$, that is, a parametric family $\{c_0(\beta,X^*)\}_{\beta\geq 1/2}$. Hamming projection of this family on the interval $[d^r,d^{sr}]$ forms the desired collection $\{^{\beta}k\}$, which is completely contained in *MFC* with a $(\circ,\subseteq)$-FNMS. In other words, we can consider this process as a subsequent "sharpening" of test rankings, starting with the *fuzziest* FNMZS k^f (which is nothing but $^{1/2}k$) and ending with the *most determinant* FNMZS $k^c={}^1k$ - see Example 9.2.

Theorem 9.2 (classification of FNMZS').

(i) A *regular FR* is *RC*, with the only exception $d^r=d^{sr}=\alpha\cdot 1$, $\alpha>1/2$); C-ranking coincides with dC-ranking.

(ii) In *semiregular case*, the condition $|\xi^d|>1$ is sufficient for *RC*-ness of an *FR*; however, C-ranking must not coincide with dC-ranking.

(iii) In *irregular case*, an *FR* is not only *RT*, but also *DT*; it induces no choice at all - $\xi^{\circ}=1_{\Pi}$ ■

Proof. (i) Let us consider separately two cases:

$$\text{(a) } d^r=d^{sr}, \qquad \text{and} \qquad \text{(b) } d^r\neq d^{sr}.$$

In the (a)-case, FNMZS$=\{d^r\}=\{d^{sr}\}$, and all components $J_d^i=\{\alpha^i\}$ of C-spectrum are pointwise - just the values of μ_{d^r}. Hence, $\xi^{\circ}=\xi^d$ is the very partition, generated by this unique f.s. $(X^i=\mu_{d^r}^{-1}(\alpha^i))$. A particular condition $|\xi^{\circ}|=|\xi^d|=1$ corresponds to the unique $\alpha^i=\alpha^*$, that is, to the statement $d^r=d^{sr}=\alpha^*\cdot 1$ (under this condition, it is easy matter to prove that $\alpha^*\geq 1/2$, and $R\subseteq 1/2$). Otherwise, $\xi^{\circ}=\xi^d$ is a non-trivial partition, so that R is *RC*.

In the (b)-case, there exists at least one 1-dimensional component $J_d^i=[\alpha^i,\beta^i]\in\Lambda^d$. With respect to Corollary 9.1, there exists a number m, such that $J_d^m=\overline{J_d^i}$ is also in Λ^d. Proposition 9.10 affirms that neither J_d^i

nor J_d^m is neutral, so that $J_d^i \cap J_d^m = \emptyset$, $X_d^i \cap X_d^m = \emptyset$, and ξ_d is a non-trivial partition, containing at least two different members, namely, the above-defined X_d^i, and X_d^m . According to Proposition 9.9, $\xi^o \prec \xi^d$, so that $\xi^o \neq 1_\Pi$, and R is *RC*.

Now, let us denote FNMZS by $\mathfrak{D}$, and let $\alpha = \mu_{d^r}(x)$, $\beta = \mu_{d^{sr}}(x)$; according to Lemma 9.6, ${}^{\alpha}k, {}^{\beta}k \in \mathfrak{D}$. Moreover, by definition of ${}^{\alpha}k$, ${}^{\beta}k$,

$$\mu_{({}^{\alpha}k)}(x) = \mu_{d^r}(x), \quad \mu_{({}^{\beta}k)}(x) = \mu_{d^{sr}}(x);$$

together with Proposition 9.5 ($\mathfrak{D} \subseteq [d^r, d^{sr}]$), this implies the equalities $1_{\mathfrak{D}} = d^r$; $t_{\mathfrak{D}} = d^{sr}$ (see Chapter 5). It follows that $\xi^o = \xi_{\mathfrak{D}} = \xi_{[d^r, d^{sr}]} = \xi^d$; therefore, C-ranking coincides with dC-ranking.

(ii) The proof of *RC*-ness in a particular case $|\xi^d| > 1$ is the same as in (i). The subsequent Example 9.5 demonstrates that, unlike the regular case, ξ^o must not coincide with ξ^d.

(iii) Following the proof of Proposition 9.10 (ii), we find that

$$1/2 \in \bigcap_{x \in X} [\mu_{d^r}(x), \mu_{d^{sr}}(x)];$$

this is equivalent to the statement $k^f = 1/2$, thus leading to dichotomous triviality (so much the more, to ranking triviality) ■

A geometric illustration of the above classification is presented on Figure 9.4.

Example 9.5 (non-identity of C-ranking, and of dC-ranking in semiregular case). Let $X = \{x_1, \ldots, x_5\}$, R_{E5} - a crisp relation

$$R_{E5} = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

Here,

$$d^r = (1 \quad 0 \quad 0 \quad 0 \quad 0);$$

$$d^{sr} = (1 \quad 0 \quad 1 \quad 1 \quad 1);$$

$$k^f = (1 \quad 0 \quad 1/2 \quad 1/2 \quad 1/2);$$

Hence, $\Lambda^d=\{[0,1]\}$, and the induced partition is trivial, $\xi^d=1_\Pi$. With respect to Proposition 9.5, any FNMZS can be written as

$$k=(1\quad 0\quad \alpha\quad \beta\quad \gamma).$$

Direct computation yields the explicit expression for $\varkappa(k)$:

$$\varkappa(k)=(1\quad 0\quad \bar{\gamma}\quad \bar{\alpha}\quad \bar{\beta}),$$

so that the condition "k is FNMZS", which is the same with the equality $\varkappa(k)=k$, is equivalent to the following: $(\alpha=\bar{\gamma})\&(\beta=\bar{\alpha})\&(\gamma=\bar{\beta})$; it is easy to derive that, under these conditions, $\alpha=\beta=\gamma=1/2$. It follows that FNMZS$=\{k^f\}$, so that $\Lambda^o=\{\{1\},\{1/2\},\{0\}\}$, and $\xi^o=\{\{x_1\},\{x_3,x_4,x_5\},\{x_2\}\}\neq\xi^d$ ■

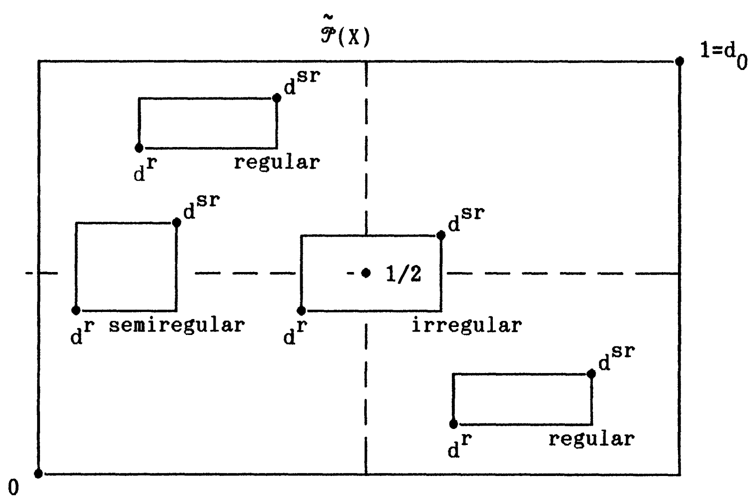

Fig. 9.4. Classification of FNMZS'

The first "practical" reason to use FNMZS simultaneously with a less rigid I_5-based FNMS is that the first one leads, when successfully applied (in a regular case, and often in a semiregular case), to a more definite structure of preferences. Of course, *ranking* of the support is generally more informative than a *dichotomy* of a support, always resulting from I_5-FNMS. Another reason is that the algorithm for the search of the bounds of FNMZS, and of dC-ranking is very simple, whereas the exhaustive description of *MFC* with I_5-FNMS is a difficult algorithmic problem (see Chapter 12).

A one more attractive feature of the above considerations regarding fuzzy version of von Neumann - Morgenstern Solution based on L.Zadeh' inclusion $\subseteq$ is that all results remain in force not only for this specific *FI*, but for a much wider class of reflexive fuzzy inclusions. We call a fuzzy inclusion $inc\in\tilde{\mathcal{P}}(\tilde{\mathcal{P}}^2(X))$ *strictly reflexive* iff $(\mu_{inc}(a,b)=1 \Leftrightarrow a\subseteq b)$. In particular, this property holds for all residuated fuzzy implications associated with a continuous t-norm (see Chapter 6), and for a number of fuzzy inclusions satisfying the axiomatics of Chapter 6.

Theorem 9.3. (i) For any reflexive fuzzy inclusion inc, and arbitrary *FR* R, *MFC* with ($\circ$,inc)-based FNMS contains FNMZS.

(ii) For any strictly reflexive fuzzy inclusion inc, and arbitrary *FR* R, *MFC* with ($\circ$,inc)-based FNMS coincides with FNMZS ■

Proof. (i) In virtue of Theorem 9.1, $FNMZS=\mathfrak{D}((\Delta_2\wedge\Delta_3)(\circ,\subseteq),R)$ is non-empty, in addition, for any $a\in\mathfrak{D}((\Delta_2\wedge\Delta_3)(\circ,\subseteq),R)$, $R\circ a=\bar{a}$; hence, for any fuzzy inclusion inc, and an arbitrary FNMZS a,

$$\mu_{(\Delta_2\wedge\Delta_3)(\circ,inc)(R)}(a)=\mu_{eq}(R\circ a,\bar{a})=\mu_{inc}(R\circ a,\bar{a})\wedge\mu_{inc}(\bar{a},R\circ a)=\mu_{inc}(\bar{a},\bar{a});$$

it follows that, for any reflexive inc, and arbitrary FNMZS a, $\mu_{(\Delta_2\wedge\Delta_3)(\circ,inc)(R)}(a)=1$, so that $a\in\mathfrak{D}((\Delta_2\wedge\Delta_3)(\circ,inc),R)$, which is equivalent to the set inclusion $FNMZS\subseteq\mathfrak{D}((\Delta_2\wedge\Delta_3)(\circ,inc),R)$.

(ii) Let inc be a strictly reflexive *FI*, and let us select $a\notin FNMZS$; under these assumptions, $R\circ a\neq\bar{a}$, so that either $R\circ a\subseteq\bar{a}$ or $\bar{a}\subseteq R\circ a$ is not satisfied; according to definition of strict reflexivity, this means that either $\mu_{inc}(R\circ a,\bar{a})$ or $\mu_{inc}(\bar{a},R\circ a)$ is smaller than 1, thus implying strict inequality $\mu_{(\Delta_2\wedge\Delta_3)(\circ,inc)(R)}(a)=\mu_{eq}(R\circ a,\bar{a})<1$; together with (i), this means that $a\notin\mathfrak{D}((\Delta_2\wedge\Delta_3)(\circ,inc),R)$; using (i) once more, we arrive to

$$FNMZS=\mathfrak{D}((\Delta_2\wedge\Delta_3)(\circ,inc),R) \blacksquare$$

Corollary 9.2. In universal environment, irregularity of an *FR* R is sufficient for both th dichotomous, and the ranking triviality of a specialization $(\Delta_{23}(\circ,inc),R)$ with an arbitrary reflexive fuzzy inclusion inc ■

Proof. Immediately follows from Theorem 9.3 (i).

9.5. FUZZY L.ZADEH' STABLE CORE

Similarly to the case of FNMZS, an *FDDP* "Fuzzy L.Zadeh Stable Core" (FZSC) $\Delta_{123}(\circ,\subseteq)$, that is, a FDDP based on the fuzzy rationality concept Stable Core with Boolean ($\vee$-$\wedge$) composition and L.Zadeh inclusion can be identified with the corresponding *MFC*:

$$\mu_{\Delta_{123}(\circ,\subseteq)(R)}(a)=\begin{cases} 1, & R\circ a=\bar{a} \ \& \ R\circ\bar{a}\subseteq\bar{a} \\ 0, & \text{otherwise} \end{cases}$$

A more invariant form of this expression is as follows:

$$\mathrm{FZSC}=\mathrm{FNMZS}\cap\overline{\text{Зnɒ}(R)}\ .$$

Therefore, FZSC is included in FNMZS. However, the author did not succeed in detecting a simple characterization of ranking contensiveness. Even more, we know that, in contrast with FNMZS, FZSC may well be empty; but we do not know a necessary and sufficient condition of non-emptiness of FZSC. A particular necessary condition is presented in the next statement.

Proposition 9.11. If FZSC is non-empty then $d^r\subseteq\varkappa(\overline{d^{sr}})$ ■

Proof. Let us suppose that FZSC$\neq\emptyset$, and let us select $a\in\mathrm{FZSC}\subseteq\mathrm{FNMZS}$. Then $\varkappa(a)=a\in[d^r,d^{sr}]$; therefore, $\bar{a}\in[\overline{d^{sr}},\overline{d^r}]$. Next, the additional condition $R\circ\bar{a}\subseteq\bar{a}$ is equivalent to the inclusion $a\subseteq\varkappa(\bar{a})$. Combining these conditions and using antimonotonicity of $\varkappa$ (see Lemma 9.1 (i)), we arrive to

$$d^r\subseteq a\subseteq k(a)\subseteq\varkappa(\overline{d^{sr}})\ ■$$

There also exists a tie between FZSC of an *FR* and FNMZS of its transitive closure $R_\tau=\bigvee_{m>0} R^m$ [2].

Theorem 9.4. (i) With any transitive *FR*, FZSC=FNMZS.

(ii) With an arbitrary *FR* R,

$$\mathfrak{D}(\Delta_{123}(\circ,\subseteq),R)\subseteq\mathfrak{D}(\Delta_{23}(\circ,\subseteq),R)\cap\mathfrak{D}(\Delta_{12}(\circ,\subseteq),R_\tau)\ ■$$

---

[2] We point out that the transitive closure of an antireflexive *FR* ***must not*** be a fuzzy preordering: it can well be neither antisymmetric nor antireflexive!

Proof. (i) If R is transitive, then, for each $a \in FNMZS$, combining two conditions: $R \circ a = \bar{a}$, and $R^2 \subseteq R$, we conclude that $R \circ \bar{a} = R^2 \circ a \subseteq R \circ a = \bar{a}$, so that $a \in FZSC$.

(ii) From $R \circ \bar{a} \subseteq \bar{a}$ we derive, for each integer $m>0$, $R^m \circ \bar{a} \subseteq \ldots \subseteq R^2 \circ \bar{a} \subseteq R \circ \bar{a} \subseteq \bar{a}$; it follows that $R_\tau \circ \bar{a} \subseteq \bar{a}$. On the other hand, $R \circ a = \bar{a}$ implies $R^2 \circ a = R \circ \bar{a} \subseteq \bar{a}$, so that $R^m \circ a \subseteq \bar{a}$, and hence, $R_\tau \circ a = \bar{a}$. In other words, each FZSC of R is also a FZSC of R_τ: $\mathcal{D}(\Delta_{123}(\circ,\subseteq),R) \subseteq \mathcal{D}(\Delta_{123}(\circ,\subseteq),R_\tau) = \mathcal{D}(\Delta_{23}(\circ,\subseteq),R_\tau) \cap \mathcal{D}(\Delta_{12}(\circ,\subseteq),R_\tau)$. Together with (i), this is exactly what is needed ■

It is edifying to compare the analyses of the collection of *FR*'s basing on contensive I_5-procedures (Examples 8.2 to 8.6) with the analyses of the same collection on the basis of FNMZS, and of FZSC (Examples 9.6 to 9.10).

Example 9.6. $X = \{x_1, \ldots, x_5\}$,

$$R = \begin{vmatrix} 0.0 & 0.0 & 0.0 & 0.8 & 0.4 \\ 0.2 & 0.0 & 0.4 & 0.8 & 0.2 \\ 0.6 & 0.8 & 0.0 & 0.0 & 0.2 \\ 0.8 & 0.2 & 0.2 & 0.0 & 0.8 \\ 0.0 & 0.4 & 0.4 & 0.4 & 0.0 \end{vmatrix}$$

Here, $d^r = (0.2 \quad 0.2 \quad 0.6 \quad 0.6 \quad 0.2)$

$$d^{sr} = (0.4 \quad 0.4 \quad 0.8 \quad 0.8 \quad 0.4)$$

$$d^s = 0.4 \cdot 1 \subset\subset 1/2$$

Hence, this *FR* is regular, and

$$\Lambda^d = \{[0.6,0.8],[0.2,0.4]\}; \quad X^* = \{x_2,x_4\} \quad \xi^d = \{\{x_2,x_4\},\{x_1,x_3,x_5\}\}.$$

Finally, a canonical ranking is

$$[0.6,0.8]\Big|\{x_3,x_4\} + [0.2,0.4]\Big|\{x_1,x_2,x_5\},$$

differing from the result with I_5-FNMS

$$[0.8,1.0]\Big|\{x_3,x_4\} : [0.0,0.2]\Big|\{x_1,x_2,x_5\}$$

(see Example 8.2) only in the more determinant interval estimates.

As to FZSC, a necessary condition $d^r \subseteq \chi(\overline{d^{sr}})$ in this example fails so that FZSC is empty. A FND is also inconsistent, since FNMZS is non-unique ■

Example 9.7. $X=\{x_1,\dots,x_6\}$,

$$R = \begin{vmatrix} 0.00 & 0.16 & 1.00 & 1.00 & 0.84 & 0.84 \\ 0.16 & 0.00 & 1.00 & 1.00 & 0.84 & 0.84 \\ 0.60 & 0.60 & 0.00 & 0.40 & 0.60 & 0.76 \\ 0.60 & 0.60 & 0.12 & 0.00 & 0.28 & 0.76 \\ 0.76 & 0.60 & 0.40 & 0.40 & 0.00 & 0.60 \\ 0.76 & 0.60 & 0.84 & 0.72 & 1.00 & 0.00 \end{vmatrix}$$

In this example, $d^r=(0.40\ \ 0.40\ \ 0.16\ \ 0.16\ \ 0.16\ \ 0.16)$

$$d^{sr}=(0.84\ \ 0.84\ \ 0.60\ \ 0.60\ \ 0.60\ \ 0.60)$$

$$d^s=0.6\cdot 1\supset 1/2$$

So, R is irregular, and a non-trivial ranking of X does not exist. Though FZSC is non-empty, it obviously contains 1/2, and FZSC is *DT*. Therefore, both $\subseteq$-procedures fail with this binary relation, in contrast with I_5-FNMS, resulting in the choice

$$[0.84,1.00]\big|_{\{x_1,x_2\}} : [0.00,0.16]\big|_{\{x_3,x_4,x_5,x_6\}}$$

(see Example 8.3) ■

Example 9.8. $X=\{x_1,\dots,x_5\}$,

$$R = \begin{vmatrix} 0.0 & 0.7 & 0.9 & 0.4 & 0.1 \\ 0.6 & 0.0 & 0.8 & 0.3 & 0.5 \\ 0.9 & 0.8 & 0.0 & 0.2 & 0.6 \\ 0.2 & 0.3 & 0.1 & 0.0 & 0.9 \\ 0.4 & 0.1 & 0.0 & 1.0 & 0.0 \end{vmatrix}$$

Here, $d^r=0.2\cdot 1$; $d^{sr}=d^s=0.8\cdot 1$, so that R is irregular, and both the FNMZS and the FZSC are *DT*, unlike the I_5-case, providing a multifold crisp choice:

$$[0.6,1.0]\big|_{K_i} : [0.0,0.4]\big|_{\overline{K}_i} \qquad (i=1,\dots 4)$$

(see Example 8.4) ■

Example 9.9. $X=\{x_1,\dots,x_4\}$,

$$R = \begin{vmatrix} 0.0 & 0.4 & 0.6 & 0.7 \\ 0.0 & 0.0 & 0.3 & 0.8 \\ 0.0 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{vmatrix}$$

In this case, alternatives are uniquely ranked:

$$d^r = d^{sr} = (1.0 \quad 0.6 \quad 0.4 \quad 0.3); \qquad FNMZS=FZSC=\{d^r\}$$

It should be noticed that the result of *choice* on the basis of FNMZS (according to dichotomousness of *MFC*) is $\{x_1\}$, thus differing from the choice with I_5-FNMS:

$$[0.6,1.0]\Big|\{x_1,x_2\}; [0.0,0.4]\Big|\{x_3,x_4\}$$

(see Example 8.5). FND=(1.0 0.6 0.4 0.2) does *not* belong to FNMZS (though in the above Example 8.5 it *was* included in *MFC* with I_5-FNMS) ■

Example 9.10. $X=\{x_1,x_2,x_3\}$,

$$R = \begin{vmatrix} 0.0 & 0.6 & 0.5 \\ 0.1 & 0.0 & 0.7 \\ 0.8 & 0.3 & 0.0 \end{vmatrix}$$

Here, $d^r=0.4\cdot 1$; $d^{sr}=d^s=0.6\cdot 1$ - irregular case, assuring the unfitness of this *FR* for any ranking (in Example 8.6, a negative result was obtained during the choice analysis) ■

9.6. INCONTENSIVE PROCEDURES BASED ON L.ZADEH' INCLUSION

In the spirit of the former study of I_5-based incontensive procedures (Sections 8.2, 8.3, 8.5), we would like to establish intrinsic reasons for incontensiveness of $\subseteq$-based FDDP's in universal environment. Since Corollary 7.1 still remains in force, providing simple reconstruction of *MFC* with arbitrary *DT* procedure by means of combinations of *MFC*'s with basic dichotomies, we start with the study of basic $(\odot,\subseteq)$-dichotomies.

Theorem 9.5 (*MFC* with basic $(\odot,\subseteq)$-dichotomies).

(i) $\mathfrak{D}(\Delta_1(\circ,\subseteq),R) = \lfloor \mathfrak{Inv}(R) \rfloor$.

(ii) $\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$ is an ideal (in a lattice $\tilde{\mathcal{F}}(X)$) containing a subset $\bigcup_{k\in FNMZS} [0,k]\cup[0,1/2]$.

(iii) $\mathfrak{D}(\Delta_3(\circ,\subseteq),R)$ is a dual ideal containing a subset $\bigcup_{k\in FNMZS} [k,1]$.

(iv) $\mathfrak{D}(\Delta_1(\lceil\circ,\subseteq),R)$ is an ideal in $\tilde{\mathcal{F}}(X)$ containing a subset $\bigcup_{k\in FNMZS(\bar{R})} [0,\bar{k}]$ (here, for brevity, $FNMZS(\bar{R})$ stands for $\mathfrak{D}(\Delta_{23}(\circ,\subseteq),\bar{R})$).

(v) $\mathfrak{D}(\Delta_2(\lceil\circ,\subseteq),R)=\mathfrak{Ainv}(\bar{R})$.

(vi) $\mathfrak{D}(\Delta_3(\lceil\circ,\subseteq),R)=\mathfrak{Inv}(\bar{R})$ ∎

Proof. (i) $a\in\mathfrak{D}(\Delta_1(\circ,\subseteq),R) \Leftrightarrow R\circ\bar{a}\subseteq\bar{a} \Leftrightarrow \bar{a}\in\mathfrak{Inv}(R) \Leftrightarrow a\in \overline{\mathfrak{Inv}(R)}$.

(ii) $a\in\mathfrak{D}(\Delta_2(\circ,\subseteq),R) \Leftrightarrow R\circ a\subseteq\bar{a}$. Let us prove that $\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$ is an ideal in $\tilde{\mathcal{P}}(X)$. Indeed, $(\forall\ b\subseteq a)(R\circ b\subseteq R\circ a\subseteq\bar{a}\subseteq\bar{b})$, so that, with each $a\in\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$, $[0,a]\subseteq\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$. Next, with any $a\in\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$, $b\in\tilde{\mathcal{P}}(X)$, $R\circ(a\wedge b)\subseteq R\circ a\wedge R\circ b\subseteq\bar{a}\wedge R\circ b\subseteq\bar{a}\subseteq\bar{a}\vee\bar{b}=\overline{(a\wedge b)}$; therefore, $a\wedge b\in\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$, and we conclude that $\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$ is an ideal in $\tilde{\mathcal{P}}(X)$. Clearly, any FNMZS k, as well as 1/2, belong to $\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$; hence, $\bigcup_{k\in \mathrm{FNMZS}}[0,k]\cup[0,1/2]$ is also included in $\mathfrak{D}(\Delta_2(\circ,\subseteq),R)$.

(iii) The proof is dual to (ii).

(iv) Directly follows from Corollary 4.1.

(v) $a\in\mathfrak{D}(\Delta_2(\lceil\circ,\subseteq),R) \Leftrightarrow R\lceil\circ a\subseteq\bar{a} \Leftrightarrow \overline{\bar{R}\circ a}\subseteq\bar{a} \Leftrightarrow \bar{R}\circ a\supseteq a \Leftrightarrow a\in\mathfrak{Ainv}(\bar{R})$.

(vi) $a\in\mathfrak{D}(\Delta_3(\lceil\circ,\subseteq),R) \Leftrightarrow \bar{a}\subseteq R\lceil\circ a \Leftrightarrow \bar{a}\subseteq\overline{\bar{R}\circ a} \Leftrightarrow \bar{R}\circ a\subseteq a \Leftrightarrow a\in\mathfrak{Inv}(\bar{R})$ ∎

So far, three of the basic dichotomies with L.Zadeh inclusion, namely, $\Delta_2(\circ,\subseteq)$, $\Delta_3(\circ,\subseteq)$, $\Delta_1(\lceil\circ,\subseteq)$ are "organically" incontensive: the corresponding *MFC*'s contain "massive" neighborhoods of constants 0/1, thus making confident choice impossible. As to the remaining three basic dichotomies, all of them refer to one of the families $\mathfrak{Inv}(\cdot)$, $\mathfrak{Ainv}(\cdot)$ of invariant/antiinvariant fuzzy subsets of an appropriate relation. Together with some results of Chapter 8, this attracts attention to the above families as well as to the family $\mathfrak{Eig}(\cdot)$ of eigen subsets of a binary relation (see Theorem 8.6). In the next chapter, the structure and the properties of these families will be studied in details.

Chapter 10

Invariant, Antiinvariant and Eigen Fuzzy Subsets. Mainsprings of Cut Technique in Fuzzy Relational Systems

In the previous study of fuzzy dichotomous decision procedures, multifold fuzzy choice was often determined by three families of *invariant*, *antiinvariant*, and *eigen* fuzzy subsets, associated with a crisp/fuzzy binary relation (see Theorems 8.1, 8.5, 8.6, 9.5, Corollary 8.3, etc.). We recall that eigen fuzzy subsets were originally introduced and studied by E.Sanchez [1]. Applications of eigen f.s.' can be found in the theory of fuzzy automata (Rovicki [1]), of fuzzy dynamic systems (Di Nola, Pedrycz, and Sessa [1]), in fuzzy clustering (Jacas [1], Jacas and Recasens [1]), etc. In this chapter, we propose an exhaustive algorithmic description of all three classes of f.s.', and present new results on topological properties, and on minimal/maximal elements of these classes.

The families $\mathfrak{Inv}(R)$, $\mathfrak{Ainv}(R)$, and $\mathfrak{Eig}(R)$ can be considered as solutions of relational inequalities $R\circ a\subseteq a$, $R\circ a\supseteq a$, and of a relational equation $R\circ a=a$. However, direct resolution of this equation, and inequalities using conventional method of "quasi-inverse relation" (see, e.g., Sanchez [2], Di Nola, Pedrycz, Sanchez, and Sessa [1]) is difficult. The reason is that the left-hand side of each inequality/equation is not independent with the right-hand side, and the latter contains the unknown variable.

In this chapter, we suggest an alternative technique for solving diverse problems in fuzzy set theory, including resolution of general systems of conventional and non-conventional fuzzy relational equations and inequalities. This straightforward technique is based on the consistent use of α-*cut defuzzification* (descending-to-crisp step), and of the reverse *canonical decomposition fuzzification* (*ascending-to-fuzzy* step). A noticeable advantage of "cut technique" is the possibility to involve highly developed crisp apparatus, in particular, that of graph theory. The technique under consideration combines algebraic properties of α-*cut mappings* (in the spirit of Lemma 8.1) with a generalized notion of *cut stability* (the original concept of *cutworthiness* was introduced by W.Bandler and L.Kohout [2]), and with common graphtheoretical methods. In order to avoid essential deviation from the main route of this book, we do not develop cut technique in complete generality (see Kitainik [13]), making remarks to those statements concerning invariant, antiinvariant, and eigen f.s.' which remain true in a more general situation.

Let us distinguish "crisp bodies" of the families under consideration by a subscript '0': $\mathfrak{Inv}_0(R)=\mathfrak{Inv}(R)\cap\mathcal{P}(X)$; $\mathfrak{Ainv}_0(R)=\mathfrak{Ainv}(R)\cap\mathcal{P}(X)$; $\mathfrak{Eig}_0(R)=\mathfrak{Eig}(R)\cap\mathcal{P}(X)$ (see Chapter 2).

Proposition 10.1 (Kitainik and Krystev [1]). Each of the families under consideration is *stable with respect to* α-*cuts* (*cut-stable*) in the sense that

$$(\forall\ a\in\tilde{\mathcal{P}}(X))(a\in\mathfrak{Inv}(R) \Leftrightarrow (\forall\alpha\in I)(a_\alpha\in\mathfrak{Inv}_0(R_\alpha)))$$

$$(\forall\ a\in\tilde{\mathcal{P}}(X))(a\in\mathfrak{Ainv}(R) \Leftrightarrow (\forall\alpha\in I)(a_\alpha\in\mathfrak{Ainv}_0(R_\alpha)))$$

$$(\forall\ a\in\tilde{\mathcal{P}}(X))(a\in\mathfrak{Eig}(R) \Leftrightarrow (\forall\alpha\in I)(a_\alpha\in\mathfrak{Eig}_0(R_\alpha)))$$

Proof. Direct implication $(a\in\mathfrak{Inv}(R) \Rightarrow (\forall\alpha\in I)(a_\alpha\in\mathfrak{Inv}(R_\alpha))$, and the corresponding statements for $\mathfrak{Ainv}$, and $\mathfrak{Eig}$) immediately follows from Lemma 8.1 (i).

We prove the reverse implication in the case of $\mathfrak{Inv}(R)$; the remaining proofs are analogous. Let $a\in\tilde{\mathcal{P}}(X)$, and let us suppose that, for all $\alpha\in I$, $a_\alpha\in\mathfrak{Inv}(R_\alpha)$. If we denote by a' the result of composition $R\circ a$, then, from Lemma 8.1 (i), we derive that, for each $\alpha\in I$, $(a')_\alpha=(R\circ a)_\alpha=R_\alpha\circ a_\alpha$. Since, by condition, $a_\alpha\in\mathfrak{Inv}(R_\alpha)$, it follows that $(a')_\alpha\subseteq R_\alpha\circ a_\alpha\subseteq a_\alpha$. In virtue of Lemma 8.1 (iv), this collection of inclusions implies $a'\subseteq a$, that is, $R\circ a\subseteq a$; hence, $a\in\mathfrak{Inv}(R)$ ■

The above result represents a "descending" fuzzy-to-crisp step (defuzzification): given a "fuzzy system", determine the collection of crisp "cut systems", and solve the corresponding crisp problem for all such partial systems. As to the "ascending" crisp-to-fuzzy step, the result of Proposition 10.1 is, for the moment, too implicit. It assures that, in principle, all fuzzy solutions can be reconstructed from α-cuts, but the method of synthesis is not clear.

In what follows, we show that the reverse ascending construction (fuzzification) is nothing but a generalized *canonical decomposition* $a=\bigvee_{\alpha}\alpha\cdot a_{\alpha}$, representing a f.s. a in terms of its α-cuts (Zadeh [1]), and propose general consideration of the "cut mapping", and of the canonical decomposition (similar ideas for valued algebraic systems were developed by U.Swami, and D.Raju [1,2]). For this purpose, let us consider the collection of α-cuts of a f.s. $a\in\tilde{\mathcal{F}}(X)$ as a mapping from I into $\mathcal{F}(X)$.

Definition 10.1. (i) A *structured cut mapping*

$$c:\tilde{\mathcal{F}}(X)\longrightarrow\tilde{\mathcal{F}}_{\mathcal{F}(X)}(I),\qquad c(a)(\alpha)=a_{\alpha},$$

assigns to each f.s. of X a set-valued f.s. of a unit interval I.

(ii) A *canonical decomposition mapping* $\mathfrak{d}:\tilde{\mathcal{F}}_{\mathcal{F}(X)}(I)\longrightarrow\tilde{\mathcal{F}}(X)$ is defined by the formula

$$\mathfrak{d}(f)(x)=\bigvee\{\alpha\,|\,x\in f(\alpha)\},$$

where $\bigvee$ stands for sup ∎

Of course, each $c(a)$ is an antitone mapping from I into $\mathcal{F}(X)$, $\beta>\alpha\Rightarrow a_{\beta}\subseteq a_{\alpha}$. In more abstract terms, each $c(a)$ is a homomorphism of a $\vee$-semilattice $I_{\vee}$ into a $\wedge$-semilattice of crisp subsets of X $(\mathcal{F}(X))_{\wedge}$, $c(a)(\alpha\vee\beta)=c(a)(\alpha)\wedge c(a)(\beta)$. Actually, $c(a)$ is an "infinite" homomorphism. From now on, the symbols $\bigvee$ (resp., $\bigwedge$) stand for the exact upper bound (resp., for the exact lower bound) in an arbitrary poset. We recall that an infinite homomorphism $f:L_{\vee}\longrightarrow M_{\vee}$ of a $\vee$-semilattice $L_{\vee}$ into a $\wedge$-semilattice $M_{\vee}$ is characterized by a condition:

for each subset $\mathcal{S}\subseteq L_{\vee}$, if $\bigvee\mathcal{S}$ exists, then there also exists $\bigwedge f(\mathcal{S})\in M_{\wedge}$, and $f(\bigvee\mathcal{S})=\bigwedge f(\mathcal{S})$.

We mark out that I is a complete lattice, so that $\mathfrak{d}(f)$ exists for all $f\in\mathcal{H}(I,X)$ (and even for all $f\in\tilde{\mathcal{F}}_{\mathcal{F}(X)}(I)$). Let us denote by $\mathcal{H}(I,X)$ the set of all *infinite homomorphisms* of a $\vee$-semilattice $I_{\vee}$ into a $\wedge$-semilattice

$(\mathscr{P}(X))_{\wedge}$ of crisp subsets of X. It should be noticed that, $\mathscr{H}(I,X)$ is nothing but the set of all lower semi-continuous (in order topology, see Birkhoff [1]) antitone functions from I to $\mathscr{P}(X)$. In what follows, we will prove that the image of $\mathfrak{c}$ is exactly $\mathscr{H}(I,X)$.

Lemma 10.1. (i) $\mathfrak{c}$ is injective homomorphism.

(ii) $\mathfrak{c}(\tilde{\mathscr{P}}(X))\subseteq\mathscr{H}(I,X)$ ∎

Proof (i) To prove injectivity, let us suppose that $a\neq b$ are two f.s.' of X. In such case, there exists $x\in X$ with $\alpha=a(x)\neq b(x)=\beta$. Next, $\alpha\neq\beta$ is equivalent to (either $\alpha>\beta$ or $\beta>\alpha$). In case when $\alpha>\beta$, we have $x\in a_{\alpha}\backslash b_{\alpha}$; in case when $\beta>\alpha$, we have $x\in b_{\beta}\backslash a_{\beta}$; in any case, either $a_{\alpha}\neq b_{\alpha}$ or $a_{\beta}\neq b_{\beta}$ or, equivalently, $\mathfrak{c}(a)\neq\mathfrak{c}(b)$.

Homomorphness was already proved in Lemma 8.1 (i).

(ii) Let $\mathscr{S}\subseteq I$, $\beta=\bigvee\mathscr{S}$, $f=\mathfrak{c}(a)$; we use $\wedge$ instead of $\bigcap$ as a semilattice operation in $\mathscr{P}(X)$. With these notations,

$$x\in f(\beta) \Leftrightarrow a(x)\geq\beta \Rightarrow (\forall\ \alpha\leq\beta)(a(x)\geq\alpha) \Rightarrow (\forall\ \alpha\leq\beta)(x\in f(\alpha)) \Rightarrow x\in \bigwedge_{\alpha\leq\beta} f(\alpha)\subseteq \bigwedge_{\alpha\in\mathscr{S}} f(\alpha).$$

Hence, $f(\beta)\subseteq \bigwedge_{\alpha\in\mathscr{S}} f(\alpha)$.

Conversely, $x\in \bigwedge_{\alpha\in\mathscr{S}} f(\alpha) \Leftrightarrow (\forall\ \alpha\in\mathscr{S})(x\in f(\alpha)) \Rightarrow (\forall\ \alpha\in\mathscr{S})(a(x)\geq\alpha)$

$\Rightarrow a(x)$ is upper bound of $\mathscr{S} \Rightarrow a(x)\geq\bigvee\mathscr{S}=\beta \Rightarrow x\in a_{\beta}$ ∎

Let ι be the identity mapping (for brevity, we will use ι without subscripts indicating the domain; this set can easily be recovered from the context of formulas).

Lemma 10.2 (primary ties between $\mathfrak{c}$ and $\mathfrak{b}$).

(i) $\mathfrak{b}\square\mathfrak{c}=\iota$.

(ii) $\mathfrak{c}\square\mathfrak{b}\geq\iota$ (that is, $(\forall\ f\in\mathscr{H}(I,X))(\forall\ \alpha\in I)(\mathfrak{c}(\mathfrak{b}(f))(\alpha)\supseteq f(\alpha))$ ∎

Proof. (i) $(\forall\ a\in\tilde{\mathscr{P}}(X),\ x\in X)((\mathfrak{b}\square\mathfrak{c})(a)(x)=\bigvee\{\alpha\,|\,a(x)\geq\alpha\}=a(x)$.

(ii) $x\in f(\alpha) \Rightarrow \bigvee\{\beta\,|\,x\in f(\beta)\}\geq\alpha \Leftrightarrow \mathfrak{b}(f)(x)\geq\alpha \Leftrightarrow x\in(\mathfrak{c}\square\mathfrak{b})(f)(\alpha)$, i.e. $f(\alpha)\subseteq(\mathfrak{c}\square\mathfrak{b})(f)(\alpha)$, so that $\mathfrak{c}\square\mathfrak{b}\geq\iota$ ∎

Basing on the above lemmas, one can precisely describe the image of $\mathfrak{c}$.

Theorem 10.1. Structured cut mapping $\mathfrak{c}$ is an isomorphism between $\tilde{\mathscr{P}}(X)$ and $\mathscr{H}(I,X)$, with $\mathfrak{b}$ being the reciprocal isomorphism ∎

Proof. Owing to Lemma 10.1 (ii), Lemma 10.2 (i), (ii), $c(\tilde{\mathcal{F}}(X))\subseteq\mathcal{H}(I,X)$, and, for each $f\in\mathcal{H}(I,X)$, $(c\Box\mathfrak{d})(f)\geq f$. It suffices to prove that, as a matter of fact, the latter inequality is an equality. Indeed, let $x\in(c\Box\mathfrak{d})(f)(\alpha)$, and let γ be $\mathfrak{d}(f)(x)$. By definition of c, and of $\mathfrak{d}$, $\gamma=\bigvee\{\beta\mid x\in f(\beta)\}\geq\alpha$. Since $f\in\mathcal{H}(I,X)$, we derive that $f(\gamma)=\bigwedge_{x\in f(\beta)} f(\beta)$, so that $x\in f(\gamma)$ also holds. Once more using homomorphness of f, we come to $x\in f(\gamma)=f(\alpha\vee\gamma)=f(\alpha)\wedge f(\gamma)$. It follows that $x\in f(\alpha)$, so that $(c\Box\mathfrak{d})(f)(\alpha)\subseteq f(\alpha)$, and finally, $(c\Box\mathfrak{d})(f)=f$ ■

Note 10.1. Lemmas 10.1, 10.2, and even a stronger version of Theorem 10.1 remain in force for arbitrary "product L-fuzzy spaces" $\mathfrak{F}=\prod_{j\in J}\tilde{\mathcal{F}}_L(X_j)$, and the corresponding "product crisp spaces" $\mathfrak{C}=\prod_{j\in J}\mathcal{F}(X_j)$, with L being an arbitrary lattice. Only the domain of $\mathfrak{d}$ should be slightly modified in case when L is incomplete (see Kitainik [13] for details) ■

In the light of cut technique, a general method of synthesis of the above three families (and of a wide class of cut-stable families) is very easy:

(1) (Analysis) Reformulate the problem for crisp α-cuts; solve the collection of crisp problems for all α-cuts;

(2) (Synthesis) Construct all functions from $\mathcal{H}(I,X)$, satisfying an additional condition: for all $\alpha\in I$, the value of a function belongs to the corresponding crisp 'α-solution'. Then use $\mathfrak{d}$ to convert all these functions to fuzzy subsets of X.

To clarify the structure of functions mentioned in the formulation of step (2), let us introduce the notions of *absorber*, and of *absorbed f.s.*

Definition 10.2. (i) Given a collection $\mathfrak{X}\subseteq\tilde{\mathcal{F}}(X)$ of f.s.', an *absorber* $a(\mathfrak{X})$ is defined as a mapping $a(\mathfrak{X}):\mathcal{F}(X)\longrightarrow\mathcal{F}(I)$, assigning to each *crisp* subset A of X a subset of all such α's that A is an α-cut A_α of an appropriate element $a\in\mathfrak{X}$:

$$a(\mathfrak{X})(A)=\{\alpha\in L\mid(\exists\ a\in\mathfrak{X})(A=A_\alpha)\}$$

(ii) Given an arbitrary mapping $a:\mathcal{F}(X)\longrightarrow\mathcal{F}(I)$, a class $\mathfrak{A}(a)\subseteq\mathcal{H}(I,X)$ of a-*absorbed mappings* is defined as

$$\mathfrak{A}(a)=\{f\in\mathcal{H}(I,X)\mid a\Box f\geq\iota\}=\{f\in\mathcal{H}(I,X)\mid(\forall\ \alpha\in I)(\alpha\in a(f(\alpha)))\}$$ ■

Theorem 10.2. Structured cut mapping $\mathfrak{c}$, and decomposition mapping $\mathfrak{d}$ represent a pair of reciprocal isomorphisms between the following sets:

with $\alpha=\alpha(\mathfrak{Inv}(R))$ - between $\mathfrak{Inv}(R)$ and $\mathfrak{A}(\alpha)$;

with $\alpha=\alpha(\mathfrak{Ainv}(R))$ - between $\mathfrak{Ainv}(R)$ and $\mathfrak{A}(\alpha)$;

with $\alpha=\alpha(\mathfrak{Eig}(R))$ - between $\mathfrak{Eig}(R)$ and $\mathfrak{A}(\alpha)$ ■

Proof. For brevity, let us use a symbol $\mathfrak{X}$ to denote each of the families. The inclusion $\mathfrak{c}(\mathfrak{X})\subseteq\mathfrak{A}(\alpha(\mathfrak{X}))$ obviously holds for each $\mathfrak{X}\subseteq\tilde{\mathcal{F}}(X)$. Owing to Theorem 10.1, $\mathfrak{c}$, and $\mathfrak{d}$ represent a reciprocal pair of isomorphisms between $\tilde{\mathcal{F}}(X)$, and $\mathcal{K}(I,X)$. So, we need to prove only the dual inclusion $\mathfrak{d}(\mathfrak{A}(\alpha(\mathfrak{X})))\subseteq\mathfrak{X}$. By definition of an absorber $\alpha(\mathfrak{X})$, for each $\alpha\in I$, and an arbitrary $f\in\mathfrak{A}(\alpha(\mathfrak{X}))$, $f(\alpha)=a_\alpha$ for some $a\in\mathfrak{X}$. From Proposition 10.1, we derive that $\{f(\alpha)\,|\,f\in\mathfrak{A}(\alpha(\mathfrak{X}))\}$ belongs to the corresponding crisp collection $(\mathfrak{Inv}_0(R_\alpha)$, $\mathfrak{Ainv}_0(R_\alpha)$ $\mathfrak{Eig}_0(R_\alpha))$. Let $f\in\mathfrak{A}(\alpha(\mathfrak{X}))$; then $a=\mathfrak{d}(f)$ satisfies the condition $\mathfrak{c}(a)=f$; then, for each α, a_α belongs to the above crisp collection of "cut solutions", and hence, in virtue of Proposition 10.1, $a\in\mathfrak{X}$ ■

Note 10.2. We underline that all the above results, and the general cut technique remain in force for a wide class of lattice-valued relational systems. In particular, with a linear, but not necessarily complete lattice L, we arrive to alternative technique for solving general systems of relational equations and inequalities of the type $\{\mathfrak{l}_i(R_q,R_v)\vartheta_i\mathfrak{r}_i(R_q,R_v))\}$, where $\mathfrak{l}_i,\mathfrak{r}_i$ are "cut commuting mappings" (in particular, abstract "compositional polynomials"), the "coefficients" R_q, and the "variables" R_v are collections of L-fuzzy n-ary relations, and $\vartheta\in\{=,\subseteq\}$. Thus, all types of conventional relational equations, studied in (Di Nola, Pedrycz, Sanchez and Sessa [1]), can be efficiently investigated using this technique ■

Before we go over to the solution of crisp problems for invariant, antiinvariant, and eigen f.s.', let us establish primary algebraic properties of fuzzy families.

Proposition 10.2. $\mathfrak{Inv}(R)$ is a complete sublattice of $\tilde{\mathcal{F}}(X)$; $\mathfrak{Ainv}(R)$ and $\mathfrak{Eig}(R)$ are complete $\vee$-subsemilattices of $\tilde{\mathcal{F}}(X)$ ■

Proof. The fact that all three families are $\vee$-subsemilattices of $\tilde{\mathcal{F}}(X)$ is directly implied by Lemma 8.1 (i). An obvious inequality $R\circ(a\wedge b)\subseteq R\circ a\wedge R\circ b$

shows that $\mathfrak{Inv}(R)$ is also a $\wedge$-subsemilattice of $\tilde{\mathcal{P}}(X)$. The completeness of the families is implied by continuity of a mapping $a \longrightarrow R \circ a$, so that each family proves to be a closed subset of n-dimensional cube $\tilde{\mathcal{P}}(X)$ ■

In the rest of this chapter, we denote by R^*_τ a "reflexive transitive closure" of a *FR* R, $R^*_\tau = E \vee R_\tau$ (R_τ is transitive closure of, E is identity relation). As in Chapter 3, given a crisp binary relation $R \in \mathcal{P}(X^2)$, we denote by $\hat{X}$ *condensation* of R, by $\hat{R}$ - the corresponding factor-relation, by $\pi: X \longrightarrow \hat{X}$ - the canonical projection; a subset $\pi^{-1}(\hat{x})$ ($\hat{x} \in \hat{X}$) is called a *bicomponent* (see, e.g., Theorem 3.2).

Proposition 10.3 (Kitainik and Krystev [1]). With crisp R,

$$\mathfrak{Inv}_0(R) = \pi_R^{-1}(\hat{R}^*_\tau \circ \mathcal{P}(\hat{X}_R)) \ ■$$

Proof. $R \circ A \subseteq A \Rightarrow (\forall\ m)(R^m \circ A \subseteq \ldots \subseteq R^2 \circ A \subseteq R \circ A \subseteq A \subseteq A) \Rightarrow R^*_\tau \circ A \subseteq A$. It follows that if a bicomponent contains any vertex from A, then it is included in A, so that $\pi_R^{-1}(\hat{R}^*_\tau \circ \pi_R(A)) \subseteq A$. A reverse inclusion is obvious, so that $\pi_R^{-1}(\hat{R}^*_\tau \circ \pi_R(A)) = A$. Hence, $A = \pi_R^{-1}(\hat{R}^*_\tau \circ Y)$, with Y being $\pi_R(A)$.

Conversely, let $Y \in \mathcal{P}(\hat{X}_R)$ be any subset of bicomponents; set $A = \pi_R^{-1}(\hat{R}^*_\tau \circ Y)$; then $R \circ A \subseteq \pi_R^{-1}(\hat{R} \circ \hat{R}^*_\tau \circ Y) \subseteq \pi_R^{-1}(\hat{R}^*_\tau \circ Y) = A$ ■

A bicomponent $Y = \pi_R^{-1}(\hat{y})$ is called *trivial* iff $|Y| = 1$ ($Y = \{y\}$), and $\mu_R(y,y) = 0$. Let us denote by $\hat{X}^t_R$ the set of all trivial bicomponents, by $\hat{X}^c_R = \hat{X}_R \setminus \hat{X}^t_R$ the set of all non-trivial bicomponents.

Proposition 10.4 (Kitainik and Krystev [1]). With crisp R,

$$\mathfrak{Eig}_0(R) = \pi_R^{-1}(\hat{R}^*_\tau \circ \mathcal{P}(\hat{X}^c_R)) \ ■$$

Proof. Unessential modification of the previous proof ■

Corollary 10.1 (Kitainik and Krystev [1]). The mappings $\mathfrak{I}(\hat{R}) \longrightarrow \mathfrak{Inv}_0(R)$, $Y \longrightarrow \pi_R^{-1}(\hat{R}^*_\tau \circ Y)$, and $\mathfrak{I}(R|_{\hat{X}^c_R}) \longrightarrow \mathfrak{Eig}_0(R)$, $Y \longrightarrow \pi_R^{-1}(\hat{R}^*_\tau \circ Y)$ are isomorphisms ■

Proof. Injectivity of these mappings is obvious. Surjectivity is implied by an equality $\hat{R}^*_\tau \circ Y = \hat{R}^*_\tau \circ (Y \setminus \hat{R}^*_\tau \circ Y)$ (which holds for any transitive binary relation), taking into consideration that $Y\backslash \hat{R}^*_\tau \circ Y \in \mathfrak{Z}(\hat{R}^*_\tau) \subseteq \mathfrak{Z}(\hat{R})$ ∎

These considerations lead to a two-step search algorithm for invariant and eigen crisp subsets of a crisp binary relation:

(1) find all bicomponents of R; build condensation $\hat{R}$ and its transitive closure $\hat{R}^*_\tau$; in case of eigen subsets, determine $\hat{X}^c_R$;

(2) construct all internally stable subsets of an acyclic digraph $\hat{R}^*_\tau$.

From the algorithmic viewpoint, the first step is polynomially complex (see, e.g., Swami, Thulasiraman [1]). Total number of subsets at the second step can be exponential, but can be constructed using a very simple (say, lexicographic) algorithm, owing to the fact that $\hat{R}^*_\tau$ is acyclic.

Corollary 10.2 (Kitainik and Krystev [1]). With crisp R, $\mathfrak{Znv}_0(R)=\mathfrak{Eig}_0(R)$ iff R has no trivial bicomponents ∎

Proof. Follows from Propositions 10.3, 10.4 ∎

Proposition 10.5 (Kitainik and Krystev [1]).For crisp R, $\mathfrak{Ainv}_0(R)$ contains all subsets $Y\in\mathcal{P}(X)$, satisfying the condition: the induced relation $R|_Y$ has no non-dominated vertices ∎

Proof. Obvious ∎

In order to obtain more information on the structure of absorbers, let us follow the behavior of the above families of f.s.' when varying the cut level α.

Proposition 10.6 (Kitainik and Krystev [1]). $\mathfrak{Znv}_0(R_\alpha)$ is α-monotone, that is, $(\forall\ \beta>\alpha)(\mathfrak{Znv}_0(R_\beta)\supseteq\mathfrak{Znv}_0(R_\alpha))$, whereas $\mathfrak{Ainv}_0(R_\alpha)$ is α-antitone: $(\forall\ \beta>\alpha)$ $(\mathfrak{Ainv}_0(R_\beta)\subseteq\mathfrak{Ainv}_0(R_\alpha))$ ∎

Proof. $\beta>\alpha$ implies $R_\beta\subseteq R_\alpha$; hence, $A\in\mathfrak{Znv}_0(R_\alpha) \Rightarrow R_\beta\circ A\subseteq R_\alpha\circ A\subseteq A \Rightarrow A\in\mathfrak{Znv}_0(R_\beta)$; it follows that $\mathfrak{Znv}_0(R_\beta)\supseteq\mathfrak{Znv}_0(R_\alpha)$. For $\mathfrak{Ainv}$, the proof is similar ∎

Corollary 10.3 (structure of absorbers, Kitainik and Krystev [1]).

(i) $\alpha(\mathfrak{Znv}(R))(\emptyset)=\alpha(\mathfrak{Ainv}(R))(\emptyset)=\alpha(\mathfrak{Eig}(R))(\emptyset)=\alpha(\mathfrak{Znv}(R))(X)=I$;

$\alpha(\mathfrak{Ainv}(R))(X)=\alpha(\mathfrak{Eig}(R))(X)$ is either I or a semi-open interval $[0,\alpha[$ for an appropriate α.

For each $R\in\tilde{\mathcal{P}}(X^2)$, $A\in\mathcal{P}(X)\backslash\{\emptyset,X\}$, the following assertions hold:

(ii) $\alpha(\mathfrak{Inv}(R))(A)$ is either empty set or a semi-open interval $]\alpha,1]$ for an appropriate α.

(iii) $\alpha(\mathfrak{Ainv}(R))(A)$ is either empty set or an interval $[0,\alpha]$ for an appropriate α.

(iv) $\alpha(\mathfrak{Eig}(R))(A)$ is either empty set or a semi-open interval $]\alpha,\beta]$ for appropriate α, and β ■

Proof. Directly follows from Proposition 10.6 ■

Corollary 10.4. (i) With each $A\in\mathfrak{Inv}_0(R_{>0})$, $\chi_A\in\mathfrak{Inv}(R)$.

(ii) With each $A\in\mathfrak{Ainv}_0(R_1)$, $\chi_A\in\mathfrak{Ainv}(R)$.

Proof. Directly follows from Propositions 10.1, 10.6 ■

So far, the families of fuzzy subsets $\mathfrak{Inv}(R)$, $\mathfrak{Ainv}(R)$ commonly contain a considerable number of *crisp* subsets.

In order to describe the process of synthesis of *fuzzy* solutions, we introduce an additional notion of *associated digraph*. From now on, the symbol $\mathfrak{X}$ stands for any of the families $\mathfrak{Inv}(R)$, $\mathfrak{Ainv}(R)$, $\mathfrak{Eig}(R)$. Let us denote by $c_0(\mathfrak{X})=\{c(a)(\alpha)\,|\,a\in\mathfrak{X},\alpha\in I\}\setminus\{\emptyset\}$ the set of all non-empty α-cuts of all f.s.' included in $\mathfrak{X}$; let also $N=|c_0(\mathfrak{X})|$.

Definition 10.3. (i) An *associated digraph* of a family $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$ is a digraph $\mathfrak{G}(\mathfrak{X})=(c_0(\mathfrak{X}),\mathcal{U}(\mathfrak{X}))$, where a dart leads from a vertex A into a vertex B, $(A,B)\in\mathcal{U}(\mathfrak{X})$ iff $B\subset A$ and $\vee\alpha(\mathfrak{X})(B)>\wedge\alpha(\mathfrak{X})(A)$.

(ii) A chain $(A_1,A_2,\dots,A_m)$, $(A_i,A_{i+1})\in\mathcal{U}(\mathfrak{X})$ in $\mathfrak{G}(\mathfrak{X})$ is called *complete* iff the union of the values of absorber $\bigcup_1^m\alpha(A_i)\subseteq I$ is a subinterval of I containing 0 ■

Let us denote by $\mathcal{T}(\mathfrak{X})$ the set of all *complete transitive chains* of $\mathfrak{G}(\mathfrak{X})$, by $\mathcal{S}_m$ $(m=1,\dots,N)$ - the interior of a unit simplex in I^m, $\mathcal{S}_m=\{(\alpha_1,\dots,\alpha_m\,|\,\alpha_1<\dots<\alpha_m\}$; given a collection $\tau=(A_1,\dots,A_m)$ of subsets of X, we set $\mathcal{S}(\tau)=(\prod_{i=0}^m\alpha(\mathfrak{X})(A_i))\cap\mathcal{S}_m$.

In the next assertion, we propose alternative characterization of the families under consideration in the terms of associated digraph.

Theorem 10.3. $\mathfrak{X}$ is isomorphic to the set $\mathfrak{B}(\mathfrak{X})=\bigcup_{\tau\in\mathcal{T}(\mathfrak{X})}\mathcal{S}(\tau)$ ■

Proof. Let $a\in\mathfrak{X}\setminus\{0\}$ be any non-zero f.s. in $\mathfrak{X}$, and let us denote by $\alpha_i=\mu_a(x_{k_i})$, $i=0,\dots,m$, the sequence of strictly increasing values of a membership function of a. Furthermore, let $A_i=A_{\alpha_i}$ be the set of the corresponding α-cuts of a f.s. a. Obviously, $A_0=X$, and $j<k$ implies $A_k\subset A_j$. In these notations, the function $c(a)$ can be expressed by the formula:

$$c(a)\big|_{]0,\alpha_0]}=A_0=X;\ c(a)\big|_{]\alpha_{i-1},\alpha_i]}=A_i;\ c(a)\big|_{]\alpha_m,1]}=\emptyset;$$

According to Theorem 10.1, $\alpha_i\in\alpha(\mathfrak{X})(A_i)$ for all $i=0,\dots,m$. It follows that, for each two integers $j<k$ in the range $0,\dots,m$,

$$\wedge\alpha(\mathfrak{X})(A_j)\le\alpha_j<\alpha_k\le\vee\alpha(\mathfrak{X})(A_i);$$

hence, $(A_j,A_k)\in\mathcal{U}(\mathfrak{X})$, and $(A_0,\dots,A_m)$ forms a transitive chain in $\mathfrak{G}(\mathfrak{X})$. Next, the above expression for $c(a)$ demonstrates that, for an arbitrary $\alpha\in I$, $0\le\alpha\le\alpha_m$, A_α is equal to A_i for an appropriate i. Hence, by definition of an absorber, $\alpha\in\alpha(\mathfrak{X})(A_i)$. Therefore, $\bigcup_{i=0}^{m}\alpha(\mathfrak{X})(A_i)$ contains the interval $[0,\alpha_m]$ which has a non-empty intersection with each $\alpha(\mathfrak{X})(A_i)$. Combining this result with Corollary 10.3, we conclude that $\bigcup_{i=0}^{m}\alpha(\mathfrak{X})(A_i)$ is a connected subset of I, containing 0; it follows that a transitive chain $(A_0,\dots,A_m)$ is complete, thus belonging to $\mathcal{T}(\mathfrak{X})$. So far, the image of the mapping $\mathfrak{X}\longrightarrow I^m$, $a\longrightarrow(\alpha_i)$, belongs to $\mathfrak{B}(\mathfrak{X})$.

Conversely, let $\hat{\alpha}=(\alpha_i)\in\mathfrak{B}(\mathfrak{X})$. In such case, $\hat{\alpha}$ belongs to some $\prod_{i=0}^{m}\alpha(\mathfrak{X})(A_i)\cap\mathcal{S}_m$, where $(A_0,\dots,A_m)\in\mathcal{T}(\mathfrak{X})$. Let us define a function $f_{\hat{\alpha}}:I\longrightarrow\mathcal{P}(X)$ exactly as in the above formula for $c(a)$:

$$f_{\hat{\alpha}}\big|_{]0,\alpha_0]}=A_0=X;\ f_{\hat{\alpha}}\big|_{]\alpha_{i-1},\alpha_i]}=A_i;\ f_{\hat{\alpha}}\big|_{]\alpha_m,1]}=\emptyset;$$

Obviously, $f_{\hat{\alpha}}$ is lower semi-continuous and hence, owing to linearity and completeness of a lattice I, $f_{\hat{\alpha}}\in\mathcal{H}(I,X)$. Owing to definition of $\mathfrak{G}(\mathfrak{X})$, we have, for all $0\le i\le m$, $\alpha_i\in\alpha(\mathfrak{X})(A_i)=\alpha(\mathfrak{X})(f_{\hat{\alpha}}(\alpha_i))$. Using Corollary 10.3 and completeness of the chain $(A_0,\dots,A_m)$, we conclude that the same statement $\alpha\in\alpha(\mathfrak{X})(f_{\hat{\alpha}}(\alpha))$ is satisfied for all $\alpha\in I$. Hence, $f_{\hat{\alpha}}$ is absorbed by $\alpha(\mathfrak{X})$, so that $\mathfrak{d}(f_{\hat{\alpha}})\in\mathfrak{X}$. It follows that the mapping $\mathfrak{B}(\mathfrak{X})\longrightarrow\mathfrak{X}$, $\hat{\alpha}\longrightarrow\mathfrak{d}(f_{\hat{\alpha}})$ is reverse to the above mapping $\mathfrak{X}\longrightarrow I^N$ ■

Gathering together all results, we arrive to the following algorithm for an exhaustive description of the family $\mathfrak{X}\in\{\mathfrak{Inv}(R),\mathfrak{Ainv}(R),\mathfrak{Eig}(R)\}$:

(1) Calculate all α-cuts of an original relation R;

(2) Using Propositions 10.3-10.5, build the corresponding families of crisp subsets $\mathfrak{X}_0(R_\alpha)$ for all α-cuts;

(3) Form the collection of all crisp subsets constructed at the previous step, $\bigcup_\alpha\mathfrak{X}_0(R_\alpha)$ (the support of an absorber α);

(4) According to Proposition 10.6, for each crisp subset A in the above support, build an interval $\alpha(\mathfrak{X}(R))(A)$;

(5) Construct the associated digraph $\mathfrak{G}(\mathfrak{X})$;

(6) Select a complete transitive chain $(A_1,A_2,\dots,A_m)$ in $\mathfrak{G}(\mathfrak{X})$; with an increasing sequence of numbers $\alpha_1<\alpha_2<\dots<\alpha_m$, $\alpha_i\in\alpha(\mathfrak{X})(A_i)$, a f.s. $\vee\alpha_i\cdot A_i$ represents a "general formula" of a member of $\mathfrak{X}$.

One of the consequences of Theorem 10.3 is an easy formula for a topological dimension of $\mathfrak{X}$ as of a subspace of a n-dimensional cube I^n.

Theorem 10.4. Topological dimension of $\mathfrak{X}$ is equal to the length of a maximal transitive complete chain in $\mathfrak{G}(\mathfrak{X})$ ■

Proof. Immediately follows from Corollary 10.3 and Theorem 10.3 (the only non-trivial observation is that, owing to the specific form of intervals $\alpha(\mathfrak{X})(A)$, for any pair of integers i,j within the range 0,...,m, we have $\alpha(\mathfrak{X})(A_i)\times\alpha(\mathfrak{X})(A_j)\cap\mathcal{F}_2\neq\emptyset$) ■

Another topological result is important for the subsequent study of contensiveness of FDDP's in restricted environment (see Chapter 11).

Theorem 10.5 (Kitainik and Krystev [1]). Each of the families $\mathfrak{Inv}(R)$, $\mathfrak{Ainv}(R)$, $\mathfrak{Eig}(R)$ is an arcwise connected subspace of $\tilde{\mathcal{P}}(X)$ ■

Proof. Given a number $\alpha\in I$, and a f.s. $a\in\mathfrak{Inv}(R)$, let us define an α-*truncate* of a as $tr(a,\alpha)=a\wedge\alpha\cdot 1$. Obviously,

$$tr(a,\alpha)_\beta=\begin{cases} a_\beta, & \beta\le\alpha \\ \emptyset, & \text{otherwise}\end{cases}$$

It follows that $(\forall\ \alpha\in I)(\forall\ \beta\in I)(tr(a,\alpha)_\beta\in\mathfrak{Inv}(R_\beta))$, and hence, $(\forall\ \alpha\in I)(tr(a,\alpha)\in\mathfrak{Inv}(R))$. Then the mapping $TR(a):I\longrightarrow\tilde{\mathcal{P}}(X)$, $\alpha\longrightarrow tr(a,\alpha)$, represents a continuous path from $0\in\mathfrak{Inv}(R)$ to $a\in\mathfrak{Inv}(R)$; hence, $\mathfrak{Inv}(R)$ is arcwise connected. For $\mathfrak{Ainv}(R)$, $\mathfrak{Eig}(R)$, the proof is similar ■

We go over to the study of extremal elements of the families of invariant, antiinvariant, and eigen fuzzy subsets. We recall that a family $\mathfrak{X}\subseteq\tilde{\mathcal{P}}(X)$ is called *closeable* (resp., *interiorable*) iff, for each $a\in\tilde{\mathcal{P}}(X)$, a subset $[a,1]\cap\mathfrak{X}$ (resp., $[0,a]\cap\mathfrak{X}$) has the smallest (resp., the greatest) element (Bandler and Kohout [2], Kitainik [12]). We cannot go far into details of closeability/interiorability; so we mention only the basic result (Kitainik [12]):

closeable (resp., interiorable) properties in $\tilde{\mathcal{P}}(X)$ with a *finite* X are complete $\wedge$-subsemilattices (resp., $\vee$-subsemilattices) of $\tilde{\mathcal{P}}(X)$.

In virtue of this result, and of Proposition 10.2, for each $a\in\tilde{\mathcal{P}}(X)$, there exists the greatest fuzzy subset not exceeding a in each all the three families; let us denote these extremal elements by $i^\vee(a,R)$ (invariant), $a^\vee(a,R)$ (antiinvariant), $e^\vee(a,R)$ (eigen). In addition, there exists the smallest invariant subset $i^\wedge(a,R)$ which is not smaller than a. These f.s.' can be viewed respectively as the best lower invariant/antiinvariant/eigen approximations, and the best upper invariant approximation of an arbitrary f.s. Assigning to the latter subset the values 0, and 1, we arrive to the conventional smallest, and greatest elements of the whole families.

An explicit formula for the greatest eigen f.s. was obtained in (Sanchez [1]): $e^\vee(a,R)=\bigwedge_{m=0}^{\infty} R^m\circ 1$ $(R^0\circ 1=1)$. In order to comprehend this construction, let us introduce a mapping $s_R:\tilde{\mathcal{P}}(X)\longrightarrow\tilde{\mathcal{P}}(X)$, $s_R(a)=\bigwedge_{m=0}^{\infty} R^m\circ a$ $(R^0\circ a=a)$. In the next proposition, basic properties of this mapping are listed.

Lemma 10.3. (i) s_R is idempotent mapping, $s_R^2=s_R$.

(ii) The image of s is the set of all antiinvariant f.s.', $s_R(\tilde{\mathcal{F}}(X))=\mathfrak{Ainv}(R)$; the latter family coincides with the set of all stable points of s_R.

(iii) $s(\mathfrak{Inv}(R))=\mathfrak{Eig}(R)$ ∎

Proof. (i) On the one hand, $s_R(a)\subseteq a$, so that $(\forall\ m\geq 0)(R^m\circ s_R(a)\subseteq R^m\circ a)$, and hence, $s_R^2(a)=\bigwedge_{m=0}^{\infty} R^m\circ s_R(a)\subseteq \bigwedge_{m=0}^{\infty} R^m\circ a=s_R(a)$. On the other hand, $R\circ s_R(a)=\bigwedge_{m=1}^{\infty} R^m\circ a\supseteq s_R(a)$, so that $(\forall\ m\geq 0)(R^m\circ s_R(a)\supseteq s_R(a))$; therefore, $s_R^2(a)\supseteq s_R(a)$. Finally, $s_R^2(a)=s_R(a)$.

(ii) Exactly as in (i), we arrive to $(\forall\ a\in\tilde{\mathcal{F}}(X))(R\circ s_R(a)=\bigwedge_{m=1}^{\infty} R^m\circ a\supseteq s_R(a))$, which is equivalent to the inclusion $s_R(\tilde{\mathcal{F}}(X))\subseteq\mathfrak{Ainv}(R)$. With $a\in\mathfrak{Ainv}(R)$, $R^m\circ a\supseteq a$ for all m; hence, $s_R(a)=a$; it follows that $s_R(\tilde{\mathcal{F}}(X))=\mathfrak{Ainv}(R)$, and that $\mathfrak{Ainv}(R)$ is the set of all stable points of s_R.

(iii) If $a\in\mathfrak{Inv}(R)$, then $\{R^m\circ a\}$ is a decreasing chain of f.s.'; hence, for any integer j, $\bigwedge_{m=0}^{\infty} R^m\circ a=\bigwedge_{m=j}^{\infty} R^m\circ a$. This implies the inclusion $s_R(a)\subseteq a$, so that $s_R(a)\in\mathfrak{Inv}(R)$; combining this inclusion with the statement (ii), we come to $s_R(a)\in\mathfrak{Inv}(R)\cap\mathfrak{Ainv}(R)=\mathfrak{Eig}(R)$. Since $\mathfrak{Eig}(R)\subseteq\mathfrak{Ainv}(R)$ consists of stable points of s_R, we conclude that $s(\mathfrak{Inv}(R))=\mathfrak{Eig}(R)$ ∎

In that way, the mapping s_R is nothing but the "antiinvariant projector", or the "interior operator" of the set of all *antiinvariant* subsets of a *FR* R. The fact that $s_R(1)$ is the greatest *eigen* f.s. is, so to say, occasional: the reason is that 1 itself is an *invariant* f.s. This observation gives rise to diverse generalizations of Sanchez' formula. In the next statement, we present "analytic" description of "interiors" and "closures" related to the three families $\mathfrak{Inv}(R)$, $\mathfrak{Ainv}(R)$, $\mathfrak{Eig}(R)$.

Theorem 10.6. For any *FR* $R\in\tilde{\mathcal{F}}(X^2)$, and a f.s. $a\in\tilde{\mathcal{F}}(X)$:

(i) The *greatest invariant* fuzzy subset not exceeding a

$$i^{\vee}(a,R)=\mathfrak{d}(f_i^{\vee}(a,R)),$$

where $f_i^{\vee}(a,R)\in\mathcal{X}(I,X)$ can be calculated by a recursive formula:

$$f_{i}^{\vee}(a,R)(0)=X; \quad f_{i}^{\vee}(a,R)(\alpha)=i^{\vee}(A_{\alpha}\cap(\bigcap_{\beta<\alpha} f_{i}^{\vee}(a,R)(\beta)),R_{\alpha})$$

(ii) The *greatest antiinvariant* fuzzy subset, not exceeding a

$$a^{\vee}(a,R)=s_{R}(a).$$

(iii) The *greatest eigen* fuzzy subset not exceeding a

$$e^{\vee}(a,R)=s_{R}(i^{\vee}(a,R)).$$

(iv) the *smallest invariant* fuzzy subset, which is not smaller than a

$$i^{\wedge}(a,R)=R_{\tau}^{*}\circ a \;\blacksquare$$

Proof. (i) Let $b\in[0,a]\cap\mathfrak{Inv}(R)$. Clearly, $B_0=A_0=X$. Next, it is easy to derive from Proposition 10.1, and from antimonotonicity of b_α that

$$(\forall\ \alpha>0)(b_{\alpha}\in[\emptyset,A_{\alpha}]\cap(\bigcap_{\beta<\alpha} b_{\beta})\cap\mathfrak{Inv}(R_{\alpha})).$$

Taking for b_α the greatest crisp subset in $\mathfrak{Inv}(R_\alpha)$ belonging to $[\emptyset,A_{\alpha}]\cap(\bigcap_{\beta<\alpha} b_{\beta})$, we arrive to the given formula for $f_{i}^{\vee}$ [1].

(ii) Directly follows from Lemma 10.3 (ii).

(iii) $b\in[0,a]\cap\mathfrak{Eig}(R)=[0,a]\cap\mathfrak{Inv}(R)\cap\mathfrak{Ainv}(R) \Rightarrow b\subseteq i^{\vee}(a,R)$

$$\Rightarrow b\in[0,i^{\vee}(a,R)]\cap\mathfrak{Ainv}(R) \Rightarrow b\subseteq s_{R}(i^{\vee}(a,R)).$$

Since, owing to Lemma 10.3 (iii), $s_{R}(i^{\vee}(a,R))\in\mathfrak{Eig}(R)$, the latter f.s. proves to be the greatest eigen f.s. in $[0,a]$ ■

(iv) Obviously, $R_{\tau}^{*}(a)\in\mathfrak{Inv}(R)$; next, if $a\subseteq b$, and $b\in\mathfrak{Inv}(R)$, then $R_{\tau}^{*}(a)\subseteq R_{\tau}^{*}(b)$; hence, $i^{\wedge}(a,R)=R_{\tau}^{*}(a)$ ■

Corollary 10.5 (Kitainik and Krystev [11]). $\mathfrak{Inv}(R)=\mathfrak{Inv}(R_{\tau})$; $\mathfrak{Eig}(R)=\mathfrak{Eig}(R_{\tau})$ ■

Proof. Owing to Theorem 10.1, and to the fact that R is transitive iff all α-cuts of R are transitive, it is sufficient to prove the statements only for crisp relations, in which case both equalities are easily implied by Propositions 10.3, 10.4. ■

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[1] Owing to finiteness of X, this formula can be written in a discrete form, since, for all $f\in\mathcal{H}(I,X)$, $\{f(\alpha)\}_{\alpha\in I}=\{f(\alpha)\}_{\alpha\in\{\mu_{R}(x,y)\}}$.

So far, not only the *greatest* element of the family of eigen fuzzy subsets is the same for the original relation, and for its transitive closure (see Sanchez [11]); actually, the entire families are also equal.

Example 10.1. Let us apply the above results to determining the families $\mathfrak{Inv}(R)$, $\mathfrak{Eig}(R)$ with the following *FR* R on a support $X=\{x_1,\dots,x_4\}$

$$R = \begin{vmatrix} 0.7 & 0.6 & 0.2 & 1.0 \\ 0.2 & 0.7 & 1.0 & 0.2 \\ 0.3 & 0.7 & 0.0 & 0.6 \\ 0.2 & 0.2 & 0.2 & 0.0 \end{vmatrix}$$

First, we write out bicomponents, and condensations of α-cuts of R:

α	Bicomponents, and darts of condensation of R_α
$[0.0,0.2]$	X
$]0.2,0.3]$	$\{x_1,x_2,x_3\}\longmapsto\{x_4\}$
$]0.3,0.6]$	$\{x_1\}\longmapsto\{x_2,x_3\}$, $\{x_1\}\longmapsto\{x_4\}$, $\{x_2,x_3\}\longmapsto\{x_4\}$
$]0.6,0.7]$	$\{x_1\}\longmapsto\{x_2,x_3\}$, $\{x_1\}\longmapsto\{x_4\}$
$]0.7,1.0]$	$\{x_1\}\longmapsto\{x_4\}$, $\{x_2\}\longmapsto\{x_3\}$

According to Propositions 10.3, 10.4, and Corollary 10.1, we reconstruct the families $\mathfrak{Inv}_0(R_\alpha)$, $\mathfrak{Eig}_0(R_\alpha)$:

α	$\mathfrak{Inv}_0(R_\alpha)$	$\mathfrak{Eig}_0(R_\alpha)$
$[0.0,0.2]$	$\{\emptyset,X\}$	$\{\emptyset,X\}$
$]0.2,0.3]$	$\{\emptyset,X,\{x_4\}\}$	$\{\emptyset,X\}$
$]0.3,0.6]$	$\{\emptyset,X,\{x_2,x_3,x_4\},\{x_4\}\}$	$\{\emptyset,X,\{x_2,x_3,x_4\}\}$
$]0.6,0.7]$	$\{\emptyset,X,\{x_2,x_3,x_4\},\{x_4\},\{x_1,x_4\},\{x_2,x_3\}\}$	$\{\emptyset,X,\{x_1,x_4\},\{x_2,x_3\}\}$
$]0.7,1.0]$	$\{\emptyset,X,\{x_2,x_3,x_4\},\{x_4\},\{x_1,x_4\},\{x_2,x_3\},\{x_3\},\{x_1,x_3,x_4\}\}$	$\{\emptyset\}$

Set $y_1=X$, $y_2=\{x_4\}$, $y_3=\{x_2,x_3,x_4\}$, $y_4=\{x_1,x_4\}$, $y_5=\{x_2,x_3\}$, $y_6=\{x_3\}$, $y_7=\{x_1,x_3,x_4\}$;

then absorbers of the families can be represented as

$$\alpha(\mathfrak{Ind}(R))=[0.0,1.0]/y_1+[0.2,1.0]/y_2+[0.3,1.0]/y_3+[0.6,1.0]/y_4+[0.6,1.0]/y_5$$
$$+[0.7,1.0]/y_6+[0.7,1.0]/y_7$$

$$\alpha(\mathfrak{Eig}(R))=[0.0,0.7]/y_1+[0.3,0.6]/y_3+[0.6,0.7]/y_4+[0.6,0.7]/y_5$$

Associated digraphs:

$\mathfrak{G}(\mathfrak{Ind}(R))$ is a transitive digraph on a support

$$c_0(\mathfrak{Ind}(R))=\{y_1,\dots,y_7\},$$

adjacency relation (in accordance with order function of $\mathfrak{G}$):

$$\begin{array}{c|ccccccc|}
 & y_1 & y_3 & y_4 & y_7 & y_5 & y_2 & y_6 \\
y_1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
y_3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
y_4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
y_7 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
y_5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

The set of all complete transitive chains

$$\mathfrak{T}(\mathfrak{Ind}(R))=\{\{y_1,y_i\}_{i=2,\dots,7},\{y_1,y_3,y_i\}_{i=2,5,6},\{y_1,y_4,y_2\},\{y_1,y_5,y_6\},$$
$$\{y_1,y_7,y_i\}_{i=2,6},\{y_1,y_3,y_5,y_6\}\}.$$

$\mathfrak{G}(\mathfrak{Eig}(R))$ is a transitive digraph on a support

$$c_0(\mathfrak{Eig}(R))=\{y_1,y_3,y_4,y_5\},$$

adjacency relation -

$$\begin{array}{c|cccc|}
 & y_1 & y_3 & y_4 & y_5 \\
y_1 & 0 & 1 & 1 & 1 \\
y_3 & 0 & 0 & 0 & 1 \\
y_4 & 0 & 0 & 0 & 0 \\
y_7 & 0 & 0 & 0 & 0
\end{array}$$

The set of all complete transitive chains

$$\mathfrak{T}(\mathfrak{X})=\{\{y_1,y_i\}_{i=3,4,5},\{y_3,y_5\},\{y_1,y_3,y_5\}\}.$$

In virtue of Theorems 10.3, 10.4, $\mathfrak{Inv}(R)$ is a 4-dimensional subset of $\tilde{\mathcal{F}}(X)$, containing unique 4-dimensional simplex associated with the chain $\{y_1,y_3,y_5,y_6\}$:

$$\{\alpha_1/y_1+\alpha_2/y_3+\alpha_3/y_5+\alpha_4/y_6\}=\{\alpha_1/x_1+\alpha_3/x_2+\alpha_4/x_3+\alpha_2/x_4\},$$

with α_i satisfying the conditions

$$\alpha_1<\alpha_2<\alpha_3<\alpha_4; \quad \alpha_2\in]0.3,1.0]; \quad \alpha_3\in]0.6,1.0]; \quad \alpha_4\in]0.7,1.0].$$

$\mathfrak{Eig}(R)$ is a 3-dimensional subset, with a single simplex of maximal dimension, associated with a chain $\{y_1,y_3,y_5\}$:

$$\{\alpha_1/y_1+\alpha_2/y_3+\alpha_3/y_5\}=\{\alpha_1/x_1+\alpha_3/x_2+\alpha_3/x_3+\alpha_2/x_4\},$$

where $\alpha_1<\alpha_2<\alpha_3$; $\alpha_2\in]0.3,0.6]$; $\alpha_3\in]0.6,0.7]$. We mark out that the latter simplex forms a part of the corresponding "maximal simplex" of $\mathfrak{Inv}(R)$.

The greatest eigen fuzzy subset of R is a constant f.s $e^{\vee}(1,R)=0.7\cdot 1$. Given a f.s. $a=0.4/x_1+0.8/x_2+0.6/x_3+0.9/x_4$, let us determine all the four lower, and upper approximations:

the greatest invariant f.s. not exceeding a,

$$i^{\vee}(a,R)=0.4/x_1+0.6/x_2+0.6/x_3+0.9/x_4;$$

the smallest invariant f.s. which is not smaller than a,

$$i^{\wedge}(a,R)=0.4/x_1+0.8/x_2+0.8/x_3+0.9/x_4;$$

the greatest eigen f.s. not exceeding a,

$$e^{\vee}(a,R)=0.4/x_1+0.6/x_2+0.6/x_3+0.6/x_4;$$

the greatest antiinvariant f.s. not exceeding a,

$$a^{\vee}(a,R)=0.4/x_1+0.7/x_2+0.6/x_3+0.6/x_4.$$

In that way, each of the four approximations differs from the remaining ones ■

Chapter 11

Contensiveness of Fuzzy Decision Procedures in Restricted Environment

In Chapters 7-9, we investigated contensiveness of *FDDP*'s and described the structure of *MFC* in universal environment, that is, under the assumption that trial rankings can be chosen freely among all f.s.' of the support X. A simple explanation why the prevalent majority of *FDDP*'s is identically incontensive (*DT*) in this environment was proposed in Chapter 7. The reason is that *MFC* with all these incontensive fuzzy choice rules includes "trivial constant choices" 0 or/and 1, independently of a specific *FR* (Theorem 7.1). Later on, we studied more precisely the shape of *MFC* with these *FDDP*'s and demonstrated that, for some fuzzy procedures, the above constant choices do not represent *isolated points* of *MFC*. Thus, *MFC* with all specializations based on rationality concepts $\Delta_2(\circ,I_5)$, $\Delta_3(\circ,I_5)$, $\Delta_1(\lceil\circ,I_5)$ is combined of lower/upper steps $\mathcal{L}(0,A)/\mathcal{U}(1,A)$ (Corollary 8.3, Theorem 8.5); each of these steps, being considered as an affine subspace of a $|X|$-dimensional cube $\tilde{\mathcal{F}}(X)$, contains a $|\bar{A}|$-dimensional neighborhood of 0 (lower step) or of 1 (upper step). For this reason, the three mentioned basic dichotomies, together with all *FDDP*'s in $\mathfrak{P}(\circ,I_5)$, $\mathfrak{P}(\lceil\circ,I_5)$, which contain each of these dichotomies as a separate mini-term of rationality concept, are "intrinsically incontensive": nothing can help to turn the corresponding *MFC* to a contensive mode.

With the remaining I_5-*FDDP*'s among those which are incontensive in universal environment, *MFC* is subdivided into a disjoint union of triangulations. One or two of these triangulations are always degenerate and coincide with constants 0, 1. But the answer to the question whether these constants are isolated or not in the whole *MFC* is more difficult.

In a common framework of fuzzy decision procedures, the above reasoning can be reformulated as follows. Let us suppose that one decided to prohibit trivial choices 0, and 1, that is, to erase these constants from a ranking domain (see Chapter 4 for other motivations). Can this action influence contensiveness of *FDDP*'s? We recall that, in conventional choice theory, the requirement of non-empty choice is rather common and can essentially effect the choice (see, e.g. Danilov, Sotskov [1]).

In this chapter, we discuss the behavior of *MFC* in restricted environment $\mathscr{E}_0=\tilde{\mathscr{F}}(X)\setminus\{0,1\}$, for all those *FDDP*'s in the families $\mathfrak{P}(\circ,I_5)$, $\mathfrak{P}(\lceil\circ,I_5)$, $\mathfrak{P}(\circ,\subseteq)$, $\mathfrak{P}(\lceil\circ,\subseteq)$, which were *DT* in universal environment. Obviously enough, contensive *FDDP*'s are insensitive to this transformation of ranking domain: indeed, *MFC* with any contensive specialization of these *FDDP*'s consists of contrasts $c(\alpha,A)$ and never contains 0 or 1.

First of all, we establish simple ties between *MFC* in universal environment $\tilde{\mathscr{F}}(X)$, and *MFC* in restricted environment $\mathscr{E}_0$.

Lemma 11.1. (i) Let inc be any continuous *FI*. In such case, with any *FDDP* $p\in\mathfrak{P}^{DTU}(\circ,\mathrm{inc})\cup\mathfrak{P}(\lceil\circ,\mathrm{inc})$, the following equality holds:

$$\mathscr{D}(p,R,\mathscr{E}_0)=\mathscr{D}(p,R)\cap\mathscr{E}_0$$

(ii) With inc=$\subseteq$, $p\in\mathfrak{P}^{DTU}(\circ,\mathrm{inc})\cup\mathfrak{P}(\lceil\circ,\mathrm{inc})$, $\mathscr{D}(p,R,\mathscr{E}_0)$ is either $\mathscr{E}_0$ or $\mathscr{D}(p,R)\cap\mathscr{E}_0$.

(iii) (Cf. Corollary 7.1). If inc is either a continuous *FI* or $\subseteq$, then, with any specialization $(p(\Delta_1,\Delta_2,\Delta_3),R,\mathscr{E}_0)$ of a *FDDP* $p\in\mathfrak{P}^{DTU}(\circ,\mathrm{inc})\cup\mathfrak{P}(\lceil\circ,\mathrm{inc})$, the following equality holds:

$$\mathscr{D}(p(\Delta_1,\Delta_2,\Delta_3),R,\mathscr{E}_0)=p(\mathscr{D}(\Delta_1,R),\mathscr{D}(\Delta_2,R),\mathscr{D}(\Delta_3,R),\mathscr{E}_0),$$

where the right-hand side polynomial is obtained from a rationality concept (that is, from a $\vee$-$\wedge$ polynomial) $p(\Delta_1,\Delta_2,\Delta_3)$ by changing operations $\vee$, $\wedge$ on fuzzy subsets of $\tilde{\mathcal{P}}(X)$ for operations $\bigcup$, $\bigcap$ on crisp subsets of $\tilde{\mathcal{P}}(X)$ ∎

Proof. (i) In virtue of Theorem 7.1 (ii), in universal environment, with any specialization (p,R) ($p\in\mathfrak{P}^{DTU}(\circ,inc)\bigcup\mathfrak{P}(\ulcorner\circ,inc)$), p(R) is a normal f.s. of $\tilde{\mathcal{P}}(X)$. It follows that $\mathcal{D}(p,R)\bigcap\mathcal{E}_0=\{a\in\mathcal{E}_0|\mu_{p(R)}(a)=1\}$. Next, continuity of inc implies that the mapping $p(R):\tilde{\mathcal{P}}(X)\longrightarrow I$ is also continuous. Owing to Theorem 7.1 (i), with each $p\in\mathfrak{P}^{DTU}(\circ,inc)\bigcup\mathfrak{P}(\ulcorner\circ,inc)$, either 0 or 1 belongs to $\mathcal{D}(p,R)$. It follows that $\lim_{a\longrightarrow 0}\mu_{p(R)}(a)=1$ (resp, $\lim_{a\longrightarrow 1}\mu_{p(R)}(a)=1$), so that $\mu^*(p,R,\mathcal{E})=1$. Hence, if $p(R)|_{\mathcal{E}_0}$ attains its maximum value (which is the same with non-emptiness of $\mathcal{D}(p,R,\mathcal{E}_0)$), then this maximum value is 1, so that $\mathcal{D}(p,R,\mathcal{E}_0)=\{a\in\mathcal{E}_0|\mu_{p(R)}(a)=1\}$, thus coinciding with the above expression for $\mathcal{D}(p,R)\bigcap\mathcal{E}_0$.

(ii) Owing to the fact that $\subseteq$ is crisp binary relation on $\tilde{\mathcal{P}}(X)$,

$$\mathcal{D}(p,R,\mathcal{E}_0)=\{a\in\mathcal{E}_0|\mu_{p(R)}(a)=\mu^*(p,R,\mathcal{E})\in\{0,1\}\}.$$

If $\mu^*(p,R,\mathcal{E})=0$, then $\mathcal{D}(p,R,\mathcal{E}_0)=\mathcal{E}_0$; otherwise,

$$\mathcal{D}(p,R,\mathcal{E}_0)=\{a\in\mathcal{E}_0|\mu_{p(R)}(a)=1\}=\mathcal{D}(p,R)\bigcap\mathcal{E}_0$$

(iii) Directly follows from (i), (ii), and Corollary 7.1 ∎

The above result causes a considerable reduction of "candidates for contensiveness" even in the restricted environment $\mathcal{E}_0$, as is demonstrated in the following theorem.

Theorem 11.1. *FDDP*'s based on the following fuzzy dichotomous rationality concepts are *dichotomously trivial* in environment $\mathcal{E}_0$.

(i) In the family $\mathfrak{P}(\circ,I_5)$:
Δ_{12}, Δ_2, Δ_3, $\Delta_{12}\vee\Delta_{23}$, $\Delta_{12}\vee\Delta_{13}$, $\Delta_{12}\vee\Delta_{13}\vee\Delta_{23}$, $\Delta_3\vee\Delta_{12}$, $\Delta_2\vee\Delta_{13}$, $\Delta_1\vee\Delta_2$, $\Delta_1\vee\Delta_3$, $\Delta_2\vee\Delta_3$, $\Delta_1\vee\Delta_2\vee\Delta_3$.

(ii) In the family $\mathfrak{P}(\ulcorner\circ,I_5)$:
Δ_1, $\Delta_1\vee\Delta_{23}$, $\Delta_1\vee\Delta_2$, $\Delta_1\vee\Delta_3$, $\Delta_1\vee\Delta_2\vee\Delta_3$.

(iii) In the families $\mathfrak{P}(\circ,\subseteq)$, $\mathfrak{P}(\ulcorner\circ,\subseteq)$: all *FDDP*'s, except for the $\Delta_{23}(\circ,\subseteq)$, $\Delta_{123}(\circ,\subseteq)$ ■

Proof. (i) With $(\circ,I_5)$-Core $\Delta_{12}(\circ,I_5)$, in virtue of Theorem 8.2, we have $\mathfrak{D}(\Delta_{12},R)=\mathcal{L}(0,\mathrm{CND}(R_{>0}))$. If $\mathrm{CND}(R_{>0})$ is empty, then $\mathfrak{D}(\Delta_{12},R)=\{0\}$; owing to Lemma 11.1 (i), $\mathfrak{D}(\Delta_{12},R,\mathcal{E}_0)=\emptyset$; hence, Δ_{12} is *DT* in environment $\mathcal{E}_0$. If $\mathrm{CND}(R_{>0})$ is non-empty, then $\mathfrak{D}(\Delta_{12},R,\mathcal{E}_0)=\mathcal{L}(0,\mathrm{CND}(R_{>0}))\setminus\{0\}$, thus containing $\varepsilon\cdot\chi_{\mathrm{CND}(R_{>0})}$ with an arbitrarily small ε (see also Section 8.3). Under these conditions, $\delta(\Delta_{12},R,\mathcal{E}_0)$ is obviously 0, and again, Δ_{12} is *DT* in environment $\mathcal{E}_0$.

With basic $(\circ,I_5)$-dichotomies $\Delta_2(\circ,I_5)$, and $\Delta_3(\circ,I_5)$, the proof repeats the case of Δ_{12} (here, we refer to Corollary 8.3 (ii), (iii) instead of Theorem 8.2). The only difference with the case of Δ_{12} is that $\mathfrak{D}(\Delta_2,R,\mathcal{E}_0)$, and $\mathfrak{D}(\Delta_3,R,\mathcal{E}_0)$ are always non-empty; also, in the case of Δ_3, dichotomous triviality is caused by the presence of "almost constant" f.s.' $\chi_A\vee(1-\varepsilon)\cdot\chi_{\bar{A}}$ (instead of $\varepsilon\cdot\chi_A$ in the case of Δ_2) in *MFC*.

With any other *FDDP* listed in the condition (i) of the present theorem, the $\vee$-$\wedge$ polynomial representing the corresponding rationality concept, contains one of the *DT FDDP*'s Δ_{12}, Δ_2, Δ_3 as a separate mini-term. We conclude from Lemma 11.1 (iii) that *MFC* with any of these remaining *FDDP*'s includes *MFC* with $\Delta_{12}/\Delta_2/\Delta_3$. Clearly, dichotomous triviality is "monotone" in the sense that, with a *DT* specialization of a FDP, any other specialization with a greater *MFC* is also *DT* (see Definition 5.1 (i)). This is sufficient to assure dichotomous triviality of any of the combined rationality concepts in the environment $\mathcal{E}_0$.

(ii) The proof repeats (i), with due respect to the facts that $\mathfrak{D}(\Delta_1(\ulcorner\circ,I_5),\bar{R},\mathcal{E})=\mathfrak{D}(\Delta_3(\circ,I_5),R,\mathcal{E})$ (Corollary 4.1), and that the supplement of a f.s. has the same dichotomousness as the original f.s.

(iii) In virtue of Theorem 9.5, *MFC*'s with specializations of *FDDP*'s base on rationality concepts $\Delta_2(\circ,\subseteq)$, $\Delta_3(\circ,\subseteq)$, $\Delta_1(\ulcorner\circ,\subseteq)$, are either ideals or dual ideals in $\tilde{\mathcal{P}}(X)$, thus containing $|X|$-dimensional neighborhoods of either 0 or 1. To prove dichotomous triviality of $\Delta_1(\circ,\subseteq)$, $\Delta_2(\ulcorner\circ,\subseteq)$, $\Delta_3(\ulcorner\circ,\subseteq)$, we observe that, in virtue of Theorem 9.5 (i), (v), (vi), *MFC* with $\Delta_1(\circ,\subseteq)$ is $\overline{\mathfrak{Inv}(R)}$, *MFC* with $\Delta_2(\lceil\circ,\subseteq)$ is $\mathfrak{Uinv}(\bar{R})$, and *MFC* with $\Delta_3(\lceil\circ,\subseteq)$ is $\mathfrak{Inv}(\bar{R})$. Theorem 10.5 states that each of the families $\mathfrak{Inv}(R)$, $\mathfrak{Uinv}(R)$, $\mathfrak{Eig}(R)$ is an arcwise connected subspace of $\tilde{\mathscr{P}}(X)$. Since 0 belongs to any of the above crisp subsets of $\tilde{\mathscr{P}}(X)$, we conclude that each of the above *MFC*'s contains a representative in any neighborhood of 0. Owing to Lemma 11.1 (ii), *MFC* in the restricted environment $\mathscr{E}_0$ is either $\mathscr{E}_0$ itself (which clearly implies dichotomous triviality of the corresponding specialization) or has 0 as a limit point; in the latter case,

$$\delta(\mathfrak{Inv}(R)\cap\mathscr{E}_0)=\delta(\mathfrak{Uinv}(R)\cap\mathscr{E}_0)=\delta(\mathfrak{Eig}(R)\cap\mathscr{E}_0)=0,$$

so that $\Delta_2(\lceil\circ,\subseteq)$, $\Delta_3(\lceil\circ,\subseteq)$ are *DT* in the environment $\mathscr{E}_0$. With $\Delta_1(\circ,\subseteq)$, argumentation is the same as in (ii). Dichotomous triviality of all other rationality concepts in the families $\mathfrak{P}(\circ,\subseteq)$, $\mathfrak{P}(\lceil\circ,\subseteq)$ is proved exactly as in (i).

Next, we propose the proof for the smallest rationality concept in the family $\mathfrak{P}(\lceil\circ,\subseteq)$, namely, $p=\Delta_{123}(\lceil\circ,\subseteq)$; the proof for all the remaining mini-terms Δ_{ij} can be achieved in a similar way. By definition, $\Delta_{123}(\lceil\circ,\subseteq)=\Delta_{23}(\lceil\circ,\subseteq)\wedge\Delta_1(\lceil\circ,\subseteq)$. As was already mentioned in Section 9.1, $\mathfrak{D}(\Delta_{23}(\lceil\circ,\subseteq),R)=\mathfrak{Eig}(\bar{R})$. As to the second term,

$$\mathfrak{D}(\Delta_1(\lceil\circ,\subseteq),R)=\{a\in\tilde{\mathscr{P}}(X)\,|\,R\ \lceil\circ\ \bar{a}\subseteq\bar{a}\}=\{a\in\tilde{\mathscr{P}}(X)\,|\,\overline{\bar{R}\circ\bar{a}}\subseteq\bar{a}\}=\{a\in\tilde{\mathscr{P}}(X)\,|\,a\subseteq\bar{R}\circ\bar{a}\}.$$

Finally, $\mathfrak{D}(p,R)=\mathfrak{Eig}(\bar{R})\cap\{a\in\tilde{\mathscr{P}}(X)\,|\,a\subseteq\bar{R}\circ\bar{a}\}$. Let us suppose that $\mathfrak{D}(p,R)\neq\emptyset$, and let $a\in\mathfrak{D}(p,R)$. On the one hand, according to Theorem 10.5, with each $\alpha\in I$, an α-*truncate* of a $tr(a,\alpha)=a\wedge\alpha\cdot 1$ belongs to $\mathfrak{Eig}(\bar{R})$. On the other hand, since $a\subseteq\bar{R}\circ\bar{a}$ also holds, we conclude from the monotonicity of $\vee$-$\wedge$ composition, and from the anti-monotonicity of negation, that $tr(a,\alpha)\subseteq a\subseteq\bar{R}\circ\bar{a}\subseteq\bar{R}\circ\overline{tr(a,\alpha)}$. Hence, $tr(a,\alpha)\in\mathfrak{D}(p,R)$, and the path $\{tr(a,\alpha)\,|\,\alpha>0\}$ converges to 0, thus implying dichotomous triviality of p ■

We see that a potentially non-trivial behavior in a restricted environment is possible only with *FDDP*'s including basic dichotomies $\Delta_1(\circ,I_5)$, $\Delta_2(\lceil\circ,I_5)$, $\Delta_3(\lceil\circ,I_5)$. Since $\Delta_1(\circ,I_5)$, and $\Delta_3(\lceil\circ,I_5)$ are mutually dual (see Corollary 4.1), a specialization of one of these procedures with an *FR* R is contensive/incontensive simultaneously with a specialization of a dual procedure based on a supplement *FR* $\overline{R}$. Hence, contensiveness conditions for a specialization of $\Delta_1(\circ,I_5)$ can be easily reformulated for the case of $\Delta_3(\ulcorner\circ,I_5)$, and *vise versa*. Therefore, we dwell on two basic dichotomies from different families, namely, on $\Delta_1(\circ,I_5)$, and on $\Delta_2(\ulcorner\circ,I_5)$. From Corollary 8.3 (i), and Theorem 8.5 (ii), (iii), we know that the corresponding *MFC*'s are the unions of disjoint triangulations. With $\Delta_1(\circ,I_5)$, *MFC* in universal environment contains both trivial choice constants 0, and 1; in case of $\Delta_2(\ulcorner\circ,I_5)$, 0 always belongs to *MFC*, whereas 1 generally does not. In fact, triangulations include a wide spectrum of ordinary and interval f.s.' (see Note 8.2). More precisely, a triangulation $t(\xi)$ with a base $\xi=\langle A^1,A^2,A^3\rangle\in\Pi_X$ can represent: a crisp subset χ_{A^1} $(A^2=\emptyset)$; a lower (0,A)-step $\ell(0,A^2)$ $(A^1=\emptyset)$; an upper (1,A)-step $u(1,A^1)$ $(A^3=\emptyset)$. It is important that, in the latter two cases, $\delta(t(\xi))=0$. Therefore, if a triangulation of this kind is included in *MFC*, then the corresponding specialization of a *FDDP* is incontensive. Our former analysis of dichotomousness of each of the elements of *MFC* with I_5-procedures (see Section 8.1) shows that the necessary and sufficient condition of positive dichotomousness of a triangulation is non-emptiness of both the left-hand side $(A^1\neq\emptyset)$, and of the right-hand side $(A^3\neq\emptyset)$ of a base ξ. Under this condition, dichotomousness can be expressed as (see Section 8.1)

$$\delta(t(\xi))=1/(|A^2|+1)$$

In particular, when $A^2=\emptyset$, $t(\xi)$ is reduced to a single crisp subset χ_{A^1}, so that $\delta(t(\xi))=1$.

It can be easily calculated that a non-degenerate triangulation is isolated not only from 0, and 1, but also from *any constant f.s.* $\alpha\cdot 1$. Indeed, Hamming distance $\rho_H(\alpha\cdot 1,t(\xi))$ can be expressed as

$$\rho_H(\alpha\cdot 1,t(\xi))=\bigwedge_{a\in t(\xi)}\rho_H(\alpha\cdot 1,a)=\rho_H(\alpha\cdot 1,\chi_{A^1}\vee\alpha\cdot\chi_{A^2})$$

$$=(\bar{\alpha}\cdot|A^1|+\alpha\cdot|A^3|)/n=(|A^1|-\alpha\cdot(|A^1|-|A^3|))/n\geq 1/n.$$

So, we need to determine the conditions under which specializations of $\Delta_1(\circ,I_5)$, $\Delta_2(\ulcorner\circ,I_5)$ result in non-degenerate or crisp triangulations, thus assuring contensiveness of these *FDDP*'s in restricted environment.

Theorem 11.2. (i) Basic dichotomy $\Delta_1(\circ,I_5)$ is *DC* in the environment $\mathscr{E}_0$ iff $n=|X|\geq 4$. More precisely, a specialization $(\Delta_1(\circ,I_5),R,\mathscr{E}_0)$ is *DC* iff the following three conditions are simultaneously fulfilled:

(1) the Core of strict zero-cut $R_{>0}$ of a *FR* R is empty, $\mathrm{CND}(R_{>0})=\emptyset$;

(2) strict zero-cut $R_{>0}$ of a *FR* R has proper bicomponents (different from X);

(3) each non-dominating bicomponent of $R_{>0}$ is non-trivial (that is, contains not less than two points).

(ii) Basic dichotomy $\Delta_3(\lceil\circ,I_5)$ is *DC* in the environment $\mathscr{E}_0$ iff $n\geq 4$. A specialization $(\Delta_1(\circ,I_5),R,\mathscr{E}_0)$ is *DC* iff the conditions (i) (1)-(3) hold for an inverse cut $R_{<1}$ of an *FR* R.

(iii) Basic dichotomy $\Delta_2(\lceil\circ,I_5)$ is *DC* in the environment $\mathscr{E}_0$. A sufficient condition of dichotomous contensiveness of a specialization $(\Delta_1(\circ,I_5),R,\mathscr{E}_0)$ can be formulated as follows:

$$\text{the crisp subset } e^{\vee}(1,R_{=0})=\bigwedge_{j=0}^{\infty} R_{=0}^{j}\circ X \text{ is a proper}$$

$$\text{(non-empty and not equal to X) subset of X}$$

(here, $R_{=0}$ is a zero level set of a *FR* R; for the definition of $e^{\vee}(a,R)$, see Chapter 10) ■

Proof. (i) First, let us derive the conditions (1)-(3) from the assumption of dichotomous contensiveness of a specialization $\mathscr{S}=(\Delta_1(\circ,I_5),R,\mathscr{E}_0)$. In virtue of Corollary 8.3 (i), and of Lemma 11.1 (i),

$$\mathfrak{D}(\Delta_1(\circ,I_5),R,\mathscr{E}_0)=\mathfrak{D}(\Delta_1(\circ,I_5),R)\cap\mathscr{E}_0=\bigcup_{A\in\mathfrak{Jnv}(R_{>0})}\ell(\xi_A)\backslash\{0,1\},$$

where $\xi_A=\{\bar{A},A\backslash R_{>0}\circ A,R_{>0}\circ A\}$. It follows that a specialization $\mathscr{S}$ is contensive iff there exists a non-trivial $\ell(\xi_A)$ with $A\in\mathfrak{Jnv}(R_{>0})$, and all such $\ell(\xi_A)$'s are non-degenerate, that is, both the $\bar{A}$, and the $R_{>0}\circ A$ are non-empty. However, the support X itself always belongs to $\mathfrak{Jnv}(R_{>0})$, thus violating the requirement $\bar{A}\neq\emptyset$. In fact, $\ell(\xi_X)$ is nothing but $\mathcal{L}(0,\mathrm{CND}(R_{>0}))$, an acknowledged "contensiveness breaker" (see Section 8.3).

Hence, the only possibility to preserve dichotomous contensiveness of a specialization is to assume $CND(R_{>0})=\emptyset$; in such case, $\iota(\xi_X)=\{0\}\notin\mathscr{E}_0$ is not included in $\mathfrak{D}(\Delta_1(\circ,I_5),R,\mathscr{E}_0)$. Hence, the necessity of (1) for contensiveness of a specialization is established. Next, since $X\in\mathfrak{Inv}_0(R_{>0})$ is unfit, non-emptiness of *MFC* in restricted environment presupposes the existence of at least one non-trivial invariant crisp subset of $R_{>0}$. Using Proposition 10.3, we arrive to $\mathfrak{Inv}_0(R_{>0})=\pi^{-1}_{R_{>0}}((\hat{R}_{>0})^*_\tau\circ\mathscr{P}(\hat{X}_{R_{>0}}))$, so that $\mathscr{P}(\hat{X}_{R_{>0}})$ must contain more than one point. This condition is nothing but the requirement of the existence of a proper bicomponent of $R_{>0}$. In that way, (2) is also obtained as a necessary condition of contensiveness of $\mathscr{S}$ in restricted environment. Finally, with each proper invariant crisp subset $A\in\mathfrak{Inv}(R_{>0})$, we need non-emptiness of $R_{>0}\circ A$. Clearly, it is necessary and sufficient to check the inequality $R_{>0}\circ A\neq\emptyset$ only for minimal invariant subsets. In virtue of Proposition 10.3, these subsets are nothing but "bottom level" nodes of condensation, that is, all *non-dominating* bicomponents of $R_{>0}$. If A is a non-dominating bicomponent of $R_{>0}$, then the requirement $R_{>0}\circ A\neq\emptyset$ is equivalent to the equality $R_{>0}\circ A=A$. Since $R_{>0}$ is antireflexive together with R, the equality $R_{>0}\circ A=A$ is equivalent, in its turn, to the inequality $|A|\geq 2$. So, the latter condition (3) is also implied by contensiveness of a specialization $(\Delta_1(\circ,I_5),R,\mathscr{E}_0)$. Sufficiency of (1)-(3) for contensiveness of a specialization can be proved by reversing of the above argumentation (the existence of a non-trivial invariant subset is guaranteed, and with each $A\in\mathfrak{Inv}(R_{>0})$, a triangulation $\iota(\xi_A)$ is non-degenerate).

Of course, there exist antireflexive *FR*'s satisfying (1)-(3); say, a crisp binary relation R on $X=\{x_1,x_2,x_3,x_4\}$:

$$R = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

has bicomponents $\hat{x}_1=\{x_1,x_2\}$, $\hat{x}_2=\{x_3,x_4\}$, $CND(R)=\emptyset$, and the only non-dominating bicomponent $\hat{x}_2$ contains two alternatives. Clearly, $R_{>0}=R$; using Proposition 10.3, it is easy to verify that $\mathfrak{Inv}(R_{>0})=\{X, \hat{x}_2\}$; hence,

$\mathfrak{D}(\Delta_1(\circ,I_5),R,\mathcal{E}_0)=\iota(\xi_{\hat{x}_2})$, so that $\Delta_1(\circ,I_5)$ is really *DC* in the environment $\mathcal{E}_0$. Similar examples can be easily built with any support *X*, containing more than four elements.

Now, let us prove that with $n\le3$, conditions (1)-(3) can not be simultaneously fulfilled. With n=2, the only possible structure of a crisp binary relation $(R_{>0})$ with an empty Core (which is required by Theorem 11.2 (i) (1)) is a cycle $x_1\longleftrightarrow x_2$; in such case, $R_{>0}$ has the only bicomponent $\{x_1,x_2\}$ which is in contradiction with (2). With n=3, let us consider the condensation $\hat{R}_{>0}$ of $R_{>0}$. Let us suppose that (1)-(3) are satisfied. In such case, (2) implies that at least one of the bicomponents is a singletone. If this bicomponent represents a non-dominated node of $\hat{R}_{>0}$, then it also represents a non-dominated alternative for $R_{>0}$ itself, which is in contradiction with (1). If the above one-point bicomponent is non-dominating, then we come into contradiction with (3). If a single-point bicomponent represents neither a non-dominated nor a non-dominating node of $\hat{R}_{>0}$, then there exist at least three bicomponents of $R_{>0}$, so that $\hat{R}_{>0}$ has three nodes, and hence, all of them are singletons, which is the contradiction with both (1), and (3).

(ii) Directly follows from the reciprocal duality of $\Delta_1(\circ,I_5)$, and $\Delta_3(\overline{\circ},I_5)$ (Corollary 4.1).

(iii) Using Theorem 8.5 and Lemma 11.1 (i), we can express *MFC* with $\Delta_2(\overline{\circ},I_5)$ in restricted environment $\mathcal{E}_0$ in the form

$$\mathfrak{D}(\Delta_2(\overline{\circ},I_5),R,\mathcal{E}_0)=\mathfrak{D}(\Delta_2(\overline{\circ},I_5),R)\cap\mathcal{E}_0=\bigcup_{A\in\mathfrak{A}\mathit{inv}(R_{=0})}\iota(\eta_A)\setminus\{0,1\},$$

where $\eta_A=\{A,R_{=0}\circ A\setminus A,\overline{R_{=0}\circ A}\}$ (it should be noticed that 1 must not necessarily belong to $\mathfrak{D}(\Delta_2(\overline{\circ},I_5),R)$). Similarly to (i), a necessary and sufficient condition of contensiveness of a specialization $(\Delta_2(\overline{\circ},I_5),R,\mathcal{E}_0)$ includes the requirement of non-emptiness of *MFC*, and the condition of non-degeneracy of each triangulation. The first condition is equivalent to the existence of a non-trivial antiinvariant subset of $R_{=0}$; the second can be reformulated as $(\forall\ A\in\mathfrak{A}\mathit{inv}(R_{=0}))(\overline{R_{=0}\circ A}\ne\emptyset)$, which is the same as $(\forall\ A\in\mathfrak{A}\mathit{inv}(R_{=0}))(R_{=0}\circ A\ne X)$.

Since the crisp subset $e^{\vee}(1,R_{=0})=\bigwedge_{j=0}^{\infty} R_{=0}^{j}\circ X$ in the formulation of (iii) is nothing but the greatest antiinvariant (and the greatest eigen - see Theorem 10.6) subset of a crisp binary relation $R_{=0}$, the condition $(e^{\vee}(1,R)\neq\emptyset)\&(e^{\vee}(1,R)\neq X)$ fulfills the first of the above requirements (existence of a non-trivial antiinvariant subset of $R_{=0}$). In addition, from the inequality $e^{\vee}(1,R)\neq X$ we derive that a subset $R_{=0}\circ e^{\vee}(1,R)=e^{\vee}(1,R)$ is strictly included in X; it follows that, with each $A\in\mathfrak{A}\mathrm{inv}(R_{=0})$, $R_{=0}\circ A\subseteq R_{=0}\circ e^{\vee}(1,R)=e^{\vee}(1,R)\neq X$, so that both the left-hand side, and the right-hand side terms of the base η_A are non-empty; hence, $\iota(\eta_A)$ is non-degenerate.

To complete the proof, let us analyze the example of a crisp reflexive relation R on a support $X=\{x_1,x_2,x_3\}$:

$$R=\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

Here, as with any crisp binary relation, $R_{=0}=\bar{R}$, so that

$$R_{=0}=\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

It is easy matter to find $e^{\vee}(1,R)=\{x_2,x_3\}$, which is a proper subset of X. Hence, a specialization $(\Delta_2(\ulcorner\circ,I_5),R,\mathfrak{E}_0)$ is *DC*, so that $\Delta_2(\ulcorner\circ,I_5)$ is dichotomously contensive in restricted environment ■

The latter theorem demonstrates rather exotic behavior of *GOCHA* and *GETCHA* rationality concepts. To be contensive in a restricted environment $\mathfrak{E}_0$, they require that both the top, and the bottom levels of a condensation of the corresponding extremal α-cuts $(R_{>0}/R_{<1})$ of a *FR* should contain only non-trivial bicomponents. We emphasize that *MFC* is completely determined by these cuts, so that the only "fuzzy information" that influences both the *MFC*, and the induced crisp choice, is the disposition of non-zero/non-unit elements in the matrix of R.

Let us derive more consequences of Theorem 11.2.

Corollary 11.1. (i) The following assertions (1), and (2) are equivalent:

(1) *MFC* with a specialization $(\Delta_1(\circ,I_5),R,\mathfrak{E}_0)$ is either empty or

contains only crisp subsets of X;

(2) each crisp invariant subset of a strict zero cut of R turns to be eigen subset: $\mathfrak{Inv}(R_{>0})=\mathfrak{Eig}(R_{>0})$.

(ii) The following assertions (1), and (2) are equivalent:

(1) *MFC* with a specialization $(\Delta_3(\lceil\circ,I_5),R,\mathcal{E}_0)$ is either empty or contains only crisp subsets of X;

(2) $\mathfrak{Inv}(R_{<1})=\mathfrak{Eig}(R_{<1})$ ■

Proof. (i) First, we note that, because $X\in\mathfrak{Inv}(R_{>0})$, the equality $\mathfrak{Inv}(R_{>0})=\mathfrak{Eig}(R_{>0})$ implies the equality $R_{>0}\circ X=X$, which is the same with $CND(R_{>0})=\emptyset$. Hence, the condition (1) of Theorem 11.2 (i) is satisfied. If X is the only crisp invariant (and eigen) subset of $R_{>0}$, then $\mathfrak{D}(\Delta_1(\circ,I_5),R)=\{0,1\}$, and $\mathfrak{D}(\Delta_1(\circ,I_5),R,\mathcal{E}_0)=\emptyset$; otherwise, with any $A\in\mathfrak{Inv}(R_{>0})$, we have $A\in\mathfrak{Eig}(R_{>0})$ so that $R_{>0}\circ A=A$, and the middle term of ξ_A, $A^2=A\backslash R_{>0}\circ A=\emptyset$; hence, ξ_A is reduced to a crisp subset. The converse statement is proved by reversing this argumentation.

(ii) The proof is similar ■

Corollary 11.2. If a crisp binary relation $R_{=0}$ possesses a non-empty Core then either *MFC* with a specialization $((\Delta_2(\lceil\circ,I_5),R,\mathcal{E}_0)$ is empty or the specialization is *DC* ■

Proof. $CND(R_{=0})\neq\emptyset$ is equivalent to $R_{=0}\circ X\neq X$. In such case, $R_{=0}\circ e^{\vee}(1,R)=e^{\vee}(1,R)\subseteq R_{=0}\circ X\neq X$, so that $e^{\vee}(1,R)\neq X$. If *MFC* with $((\Delta_2(\lceil\circ,I_5),R,\mathcal{E}_0)$ is non-empty, then the dichotomous contensiveness of this specialization follows from Theorem 11.2 (iii) ■

It is of interest that both the *GOCHA*, and the *GETCHA* rules "dislike" the Core, whereas dual internal stability is satisfied by the existence of the Core, though this Core $CND(R_{=0})$, representing the right-hand side term of a triangulation η_A, is never included in the induced crisp choice.

Based on the above results, let us write out the complete list of contensive *FDDP*'s in a restricted environment $\mathcal{E}_0$.

Theorem 11.3. *FDDP*'s based on the following fuzzy dichotomous rationality concepts are *DC* in the environment $\mathcal{E}_0$:

(i) In the family $\mathfrak{P}(\circ,I_5)$: Δ_{123}, $\Delta_1\vee\Delta_{23}$, Δ_{13}, Δ_{23}, Δ_1 $(n\geq 4)$, $\Delta_{13}\vee\Delta_{23}$.

(ii) In the family $\mathfrak{P}(\lceil\circ,I_5)$: Δ_{123}, Δ_{12}, Δ_{13}, Δ_{23}, $\Delta_{12}\vee\Delta_{13}$, $\Delta_{12}\vee\Delta_{23}$, $\Delta_{13}\vee\Delta_{23}$, $\Delta_{12}\vee\Delta_{13}\vee\Delta_{23}$, Δ_2, Δ_3 $(n\geq 4)$, $\Delta_2\vee\Delta_3$, $\Delta_2\vee\Delta_{13}$, $\Delta_3\vee\Delta_{12}$.

(iii) In the family $\mathfrak{P}(\circ,\subseteq)$: Δ_{123}, Δ_{23}.

(iv) In the family $\mathfrak{P}(\lceil\circ,\subseteq)$: none ■

Proof. (i) *FDDP*'s based on $\Delta_{123}(\circ,I_5)$, $\Delta_{23}(\circ,I_5)$ are contensive even in universal environment (see Theorem 8.4). With $\Delta_1\vee\Delta_{23}$, there exist two possibilities. If a specialization $\mathcal{S}=(\Delta_1\vee\Delta_{23},R,\mathcal{E}_0)$ provides the normality of Δ_{23}, $\mu^*(\Delta_{23},R,\mathcal{E}_0)=1$, then $\mathcal{D}(\Delta_1\vee\Delta_{23},R,\mathcal{E}_0)=\mathcal{D}(\Delta_1,R,\mathcal{E}_0)\cup\mathcal{D}(\Delta_{23},R)$; hence, $\mathcal{S}$ is *DC* iff the condition of Theorem 11.2 (i) is satisfied. If $\mu^*(\Delta_{23},R,\mathcal{E}_0)<1$ (in particular, if the "sub-specialization" (Δ_{23},R) is *DT*), then $\mathcal{D}(\Delta_1\vee\Delta_{23},R,\mathcal{E}_0)=\mathcal{D}(\Delta_1,R,\mathcal{E}_0)$, and again, contensiveness condition for $\mathcal{S}$ is due to Theorem 11.2 (i). Finally, a specialization based on $(\Delta_1\vee\Delta_{23})(\circ,I_5)$ is contensive iff a sub-specialization based on $\Delta_1(\circ,I_5)$ is contensive.

Using Lemma 11.1 (iii), and Theorem 11.2 (i), we arrive to a conclusion that $\mathcal{D}(\Delta_{13},R,\mathcal{E}_0)=\mathcal{D}(\Delta_1,R,\mathcal{E}_0)\cap\mathcal{D}(\Delta_3,R,\mathcal{E}_0)\subseteq\mathcal{D}(\Delta_1,R,\mathcal{E}_0)$.

Next, we mark out an obvious fact that the property "to possess dichotomous contensiveness" is antitone with respect to *MFC* in the meaning that, the smaller is the collection of fuzzy trial rankings, the greater is its dichotomousness, provided that the smaller collection is non-empty. Based on this observation, we conclude that, with a *DC* sub-specialization $\mathcal{S}'=(\Delta_1,R,\mathcal{E}_0)$, a specialization $\mathcal{S}=(\Delta_{13},R,\mathcal{E}_0)$ either results in empty *MFC* or is also *DC*. The following example of a crisp binary relation R on $X=\{x_1,x_2,x_3,x_4\}$ (a slightly modified binary relation from the proof of Theorem 11.2 (i)) shows that non-emptiness of *MFC* is possible.

$$R = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

The graph of R is depicted on Figure 11.1.

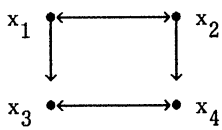


Fig. 11.1. Graph of the relation R

In fact, R falls under the condition of Corollary 11.1, and $\mathfrak{D}(\Delta_1,R,\mathcal{E}_0)=\{\chi_{\{x_1,x_2\}}\}$. At the same time, $\{x_1,x_2\}$ is a minimal externally stable subset of R_1, $\{x_1,x_2\}\in\mathfrak{C}_*(R_1)$; hence, according to Corollary 8.3, $\chi_{\{x_1,x_2\}}\in\mathfrak{D}(\Delta_3,R,\mathcal{E}_0)$. Finally, $\mathfrak{D}(\Delta_{13},R,\mathcal{E}_0)=\chi_{\{x_1,x_2\}}$, so that $\Delta_{13}(\circ,I_5)$ is *DC* together with $\mathcal{S}$.

With $\Delta_{13}\vee\Delta_{23}$, argumentation is the same as in the case of $\Delta_1\vee\Delta_{23}$, with due respect to the previous result.

(ii) Since all *FDDP*'s in the family $\mathfrak{P}(\lceil\circ,I_5)$ are normal in universal environment, using the same argumentation as in (i), we arrive to a conclusion that, in order to prove dichotomous contensiveness of all listed *FDDP*'s, it suffices to give examples of contensive specializations in restricted environment for all mini-terms included in the corresponding polynomials, namely, for mini-terms Δ_{123}, Δ_{12}, Δ_{13}, Δ_{23}, Δ_2, Δ_3. Contensiveness of Δ_2, Δ_3 was proved in Theorem 11.2 (ii), (iii); contensiveness of Δ_{23} follows from Theorem 8.6. An example of a crisp relation

$$R = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

($X=\{x_1,x_2,x_3\}$) demonstrates dichotomous contensiveness of the remaining $(\lceil\circ,I_5)$-*FDDP*'s Δ_{123}, Δ_{12}, Δ_{13}, Δ_{23} in a restricted environment. Indeed, using Lemma 11.1 and Theorem 8.5, it is easy matter to calculate that

$$\mathfrak{D}(\Delta_{123},R,\mathcal{E}_0)=\mathfrak{D}(\Delta_{12},R,\mathcal{E}_0)=\mathfrak{D}(\Delta_{13},R,\mathcal{E}_0)=\mathfrak{D}(\Delta_{23},R,\mathcal{E}_0)=\{\chi_{\{x_2,x_3\}}\}$$

(iii), (iv) Reformulation of Theorem 11.1 (iii) ∎

In that way, the study of contensiveness of all *FDDP*'s in two different environments is completed. On Figures 11.2, 11.3, the "maps" of the two "dichotomous countries" are depicted.

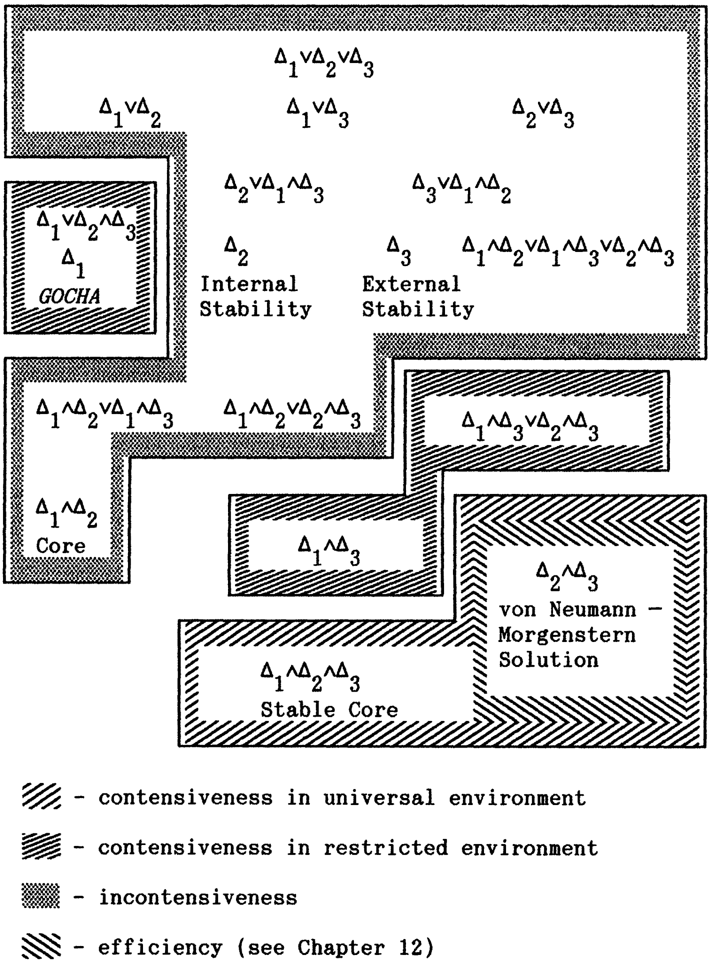
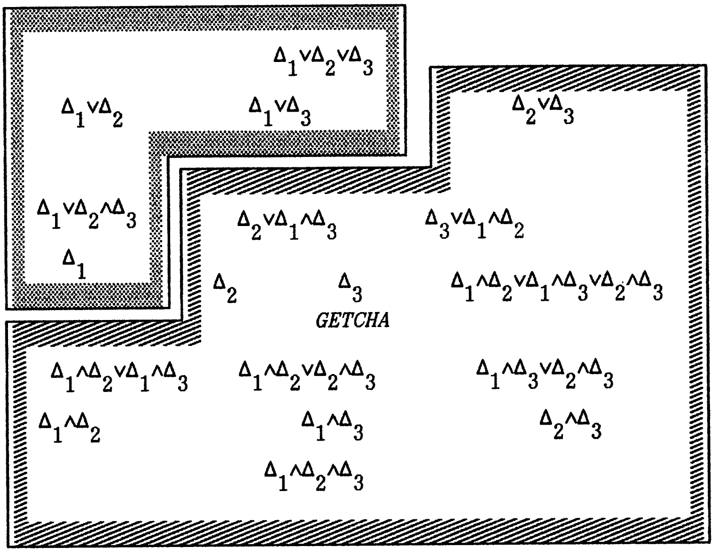


Fig. 11.2. Map of the "dichotomous country" $\mathfrak{P}(\circ, I_5)$



- contensiveness in restricted environment

- incontensiveness

Fig. 11.3. Map of the "dichotomous country" $\mathfrak{P}(\ulcorner\circ, I_5)$

Chapter 12

Efficiency of Fuzzy Decision Procedures

The results of the Chapters 7-11 predetermine considerable reduction of the number of well-defined choice rules with binary relations. Only the limited number of procedures turn to be contensive in one of the two environments under consideration. However, even if a *FDDP* is contensive, we can never guarantee that it is "identically contensive" with any *FR*. Let us assume, for example, that the "best" procedures, namely, the $(\circ, I_5)$-, and the $(\circ, \subseteq)$-based fuzzy von Neumann - Morgenstern solution (FNMS), and the Fuzzy Stable Core (FSC) are implemented in an applied *DSS*. Why not prefer FSC, resulting in a much more determinant crisp choice/ranking? Actually, the answer to this question essentially depends on a concrete *preference domain*. More specifically, returning to the above consideration of a decision procedure as of "decision-making machine" (see Preface, Chapter 1), one can associate its *efficiency* with the *probability of success* in "real-life situations". The concept of "success" can be analyzed in multiple aspects. In this book, we deal with two significant aspects of efficiency. First, we consider the confident result of the analysis with a given preference relation, namely, the *contensiveness*, as invariable condition of its successful application. Next, given a contensive specialization, one can be interested whether the resulting fuzzy and/or crisp choice is definite enough. This definiteness, in its turn, can be associated with the "sharpness" of either the *MFC* or the induced crisp choice.

As was already mentioned in Chapter 1, modern approaches to aggregation of preferences are so various and sophisticated, that the assumption of randomness of the input *FR* seems to be rather sensitive. Under this assumption, it is natural to involve conventional probabilistic considerations in the research of efficiency of *FDDP*'s. It should be noticed that ideas of this type were successfully used in cluster-analysis with weighted graphs (Matula [21]). Motivated by this reasoning, we consider three aspects of efficiency: with respect to preference domain, with respect to ranking domain, and with respect to induced crisp choice.

Let $P=(X,\mathfrak{R},\mathfrak{E},p)$ be a *FDDP*, and let us suppose that a preference domain $\mathfrak{R}$ is supplied with a probability measure ν, $\nu(\mathfrak{R})=1$. Under these assumptions, *efficiency* of a *FDDP* P with respect to preference domain is defined as

$$\mathrm{Eff}_{\mathrm{Pref}}(P,\nu,\mathfrak{E})=\mathrm{Prob}((p,R,\mathfrak{E}) \text{ is } DC\ (RC))=\nu(\{R\,|\,(p,R,\mathfrak{E}) \text{ is } DC\ (RC)\}) \qquad (12\text{-}1)$$

(we will omit the symbol $\mathfrak{E}$ in universal environment - see Chapters 4, 9).

Given a *contensive specialization* $(p,R,\mathfrak{E})$ of a *FDDP*, one can also measure its efficiency with respect to *ranking domain* (*environment*) $\mathfrak{E}$. In contrast with the case of $\mathrm{Eff}_{\mathrm{Pref}}$, the less is the range of *MFC*, the more efficient is the choice. Informally, one can identify relative measure of *MFC* with respect to the whole environment $\mathfrak{E}$ with a probability of "guessing" most rational ranking in a single random tossing in $\mathfrak{E}$; the less is this probability, the less trivial is the result (for incontensive *MFC*, this reasoning is meaningless). So, we define efficiency measure of a specialization with respect to *MFC* as

$$\mathrm{Eff}_{\mathrm{MFC}}(p,R,\mathfrak{E},\nu)=1-\nu(\mathfrak{D}(p,R,\mathfrak{E})), \qquad (12\text{-}2)$$

with ν being a probability measure on $\mathfrak{E}$, $\nu(\mathfrak{E})=1$.

Similarly, one can define efficiency measure of a contensive specialization with respect to induced crisp choice as a mean portion of rejected alternatives:

$$\mathrm{Eff}_{\mathrm{Ch}}(p,R,\mathfrak{E},\nu)=1-\frac{1}{|X|\,|\mathcal{K}^*|}\cdot\sum_{K\in\mathcal{K}^*}|K|, \qquad (12\text{-}3)$$

with $\mathcal{K}^*$ being the induced (generally, multifold) crisp choice, that is, the collection of all alternative crisp choices resulting from *MFC* (see Chapter 5 for definitions, Chapters 8, 11 for description of induced crisp choices with diverse *FDDP*'s).

Of course, if a *FDDP*/specialization is contensive, then any of its efficiency estimates, calculated in accordance with (12-1)-(12-3), yields some positive value; the higher is this value, the more favorable is our judgment. But what is the final conclusion? In other words, how can we answer the overall question:

"Is a *FDDP* (resp., a *MFC*, an induced crisp choice) efficient or not?"

In our opinion, the answer should be based on *asymptotic* characteristics of efficiency. In fact, it concerns not a single procedure, but the variety of procedures with the same rationality concept applied to *FR*'s of growing dimension. Thus, a decision procedure P deserves the name of an efficient procedure with respect to preference domain if the corresponding sequence of estimates $\mathrm{Eff}_{\mathrm{Pref}}(P,\nu,\mathcal{E})$ guarantees positive estimate independent of the number of alternatives.

Definition 12.1 (Kitainik [41]). A rationality concept p is called *efficient with respect to preference domain* in a class $\mathcal{Q}=\{P=(X,\mathcal{R},\mathcal{E},p),\nu\}$ of *FDDP*'s iff $\bigwedge_{P\in\mathcal{Q}} \mathrm{Eff}_{\mathrm{Pref}}(P,\nu,\mathcal{E})>0$ ∎

Following the main route of our research, we concentrate on general properties of decision procedures. For this reason, let us estimate the efficiency of *FDDP*'s in the greatest preference domain suitable for all *FDDP*'s, that is, in the class $\tilde{\mathcal{F}}_0(X^2)$ of all antireflexive *FR*'s for $\circ$-based *FDDP*'s, and in the class $\tilde{\mathcal{F}}_1(X^2)$ of all reflexive *FR*'s for $\ulcorner\!\circ$-based *FDDP*'s. We assume that each of these classes is supplied with a uniform distribution υ, and that the dimension of *FR*'s $n=|X|$ infinitely increases. So far, we identify an overall efficiency of a rationality concept p with its efficiency in the corresponding "universal class" $\mathcal{Q}_0^u$ or $\mathcal{Q}_1^u$:

$$\text{with } p\in\mathfrak{P}(\circ,\mathrm{inc}),\ \text{ in the class } \mathcal{Q}_0^u=\{(X=\{1,\dots,n\},\tilde{\mathcal{F}}_0(X^2),p),\upsilon\};$$
$$\text{with } p\in\mathfrak{P}(\ulcorner\!\circ,\mathrm{inc}),\ \text{ in the class } \mathcal{Q}_1^u=\{(X=\{1,\dots,n\},\tilde{\mathcal{F}}_1(X^2),p),\upsilon\},$$

where n is an arbitrary positive integer.

First, let us study four representatives of *FDDP*'s, which are contensive in universal environment: $\Delta_{123}(\circ,I_5)$, $\Delta_{123}(\circ,\subseteq)$, $\Delta_{23}(\circ,I_5)$, and $\Delta_{23}(\circ,\subseteq)$.

We begin calculations with two simple results concerning random graphs.

In the following lemma, a straightforward proof is proposed, though the subsequent formulas can be easily derived using common technique of random graphs (see, e.g., Erdos, Spencer [1]). From now on, $\vec{G}_{n,q}=(X,U_{n,q})$ stands for a crisp random digraph with n vertices and with random independent darts; the probability of the existence of a dart between any two different vertices is q.

Lemma 12.1. (i) $\text{Prob}(\vec{G}^{\,q}_{n,q}$ has non-empty Core$)\xrightarrow[n\to\infty]{}0$.

(ii) $\text{Prob}(\vec{G}_{n,q}$ has a proper bicomponent$)\xrightarrow[n\to\infty]{}0$.

Proof. (i) $\text{Prob}(\vec{G}_{n,q}$ has non-empty Core)

$$=1-\text{Prob}((\forall\ y\in X)(\exists\ x\in X)((x,y)\in U_{n,q}))\le 1-(1-q^n)^n=O(nq^n)\xrightarrow[n\to\infty]{}0.$$

(ii) Since condensation of a digraph is an acyclic digraph, the existence of a proper bicomponent is equivalent to the existence of a proper non-dominated bicomponent, which is, in its turn, equivalent to the existence of a subset $Y\subset X$ of vertices (say, with k elements), satisfying the condition: no dart of $\vec{G}_{n,q}$ has its tail in $\bar{Y}$ and its head in Y. Hence,

$$\text{Prob}(\vec{G}_{n,q} \text{ has a proper bicomponent})$$
$$=\text{Prob}(\vec{G}_{n,q} \text{ has a proper non-dominated bicomponent})$$
$$=\text{Prob}((\exists\ Y\subset X)(\forall\ y\in Y,\ x\in\bar{Y})((x,y)\notin U_{n,q}))$$
$$<2\sum_{k=1}^{n'}\binom{n}{k}q^{k(n-k)}=2\sum_{k=1}^{n'}\prod_{j=0}^{k-1}\frac{n-j}{k-j}\cdot q^{k(n-k)}<2\sum_{k=1}^{n'}((n-k+1)q^{n-k})^k$$

(in the above expressions, n' is the greatest integer which is less or equal to n/2). With a sufficiently large n, the latter expression does not exceed $4(n/2+1)q^{n/2}$ thus converging to 0 when n infinitely increases ∎

This elementary result suffices to prove inefficiency of three in the four "universally contensive" *FDDP*'s.

Theorem 12.1. (i) $(\circ,I_5)$-based Stable Core $\Delta_{123}(\circ,I_5)$ is inefficient.

(ii) Both the FZSC $\Delta_{123}(\circ,\subseteq)$, and the FNMZS $\Delta_{23}(\circ,\subseteq)$ are inefficient ∎

Proof. (i) Owing to Theorem 8.4 (ii), contensiveness of a specialization $(\Delta_{123}(\circ,I_5),R)$ requires the existence of the non-trivial Stable Core of a crisp relation $R_{>1/2}$. With uniform distribution on $\tilde{\mathcal{P}}_0(X^2)$, $R_{>1/2}$ is nothing but $\overrightarrow{G}_{n,1/2}$; hence, in virtue of Lemma 12.1 (i), $Eff_{Pref}(\Delta_{123}(\circ,I_5),\upsilon)=o(2^{-n})\xrightarrow[n\to\infty]{}0$.

(ii) It suffices to prove inefficiency of FNMZS, because $\{R|(FZSC,R)$ is $DC\}\subset\{R|(FNMZS,R)$ is $DC\}$. First, let us prove that the emptiness of the crisp Core of strict 1/2-cut of R implies dichotomous triviality of a specialization of FNMZS. Indeed, let us suppose that $CND(R_{>1/2})=\emptyset$. Using Proposition 9.1, we consequently derive the following equalities:

$$(d_1)_{1/2}=\varkappa_{R_{>1/2}}((d_0)_{>1/2})=\varkappa_{R_{>1/2}}(X)=CND(R_{>1/2})=\emptyset$$

$$(d_2)_{>1/2}=\varkappa_{R_{1/2}}((d_1)_{1/2})=\varkappa_{R_{1/2}}(\emptyset)=X$$

$$(d_3)_{1/2}=\varkappa_{R_{>1/2}}((d_2)_{>1/2})=\varkappa_{R_{>1/2}}(X)=CND(R_{>1/2})=\emptyset$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(d^r)_{1/2}=\ldots=\emptyset$$

$$(d^{sr})_{>1/2}=\ldots=X$$

Owing to Lemma 8.1 (iii), the latter two equalities are equivalent to a pair of inequalities $\mathcal{G}d^r\leq 1/2\leq\mathcal{R}d^{sr}$ or, alternatively, to a pair of inclusions $d^r\subseteq 1/2\subseteq d^{sr}$, so that $d^s=d^{sr}\wedge\overline{d^r}\supseteq 1/2$; hence, in virtue of Theorem 9.2, a specialization (FNMZS,R) is *DT*.

The remaining of the proof repeats (i) ■

We emphasize that, even with small dimensions, say, with $n\geq 10$, each of the listed *FDDP*'s has a vanishing probability of dichotomous contensiveness.

The only remaining thing is to verify the efficiency of $(\circ,I_5)$-FNMS with respect to preference domain. In virtue of Theorem 8.4 (i) (3),

$$Eff_{Pref}((\Delta_{23})(\circ,I_5),\mathcal{R},\upsilon)=Prob(R_{>1/2}\text{ has a crisp von Neumann - Morgenstern Solution }K\in\mathfrak{Z}(R_{1/2}))$$

We refer to the following unpublished result communicated to the author by A.Thomason: "with a random digraph $\overrightarrow{G}_{n,q}$, there almost always exist von Neumann - Morgenstern Solutions of order m=(log n-loglog n)/L+d for a non-zero bounded number of values of d (here, L=log(1/(1-p)))". Since $R_{>1/2}$ is nothing but $\overrightarrow{G}_{n,1/2}$, the existence of von Neumann - Morgenstern Solutions is almost surely guaranteed. Next, with continuous scale of preferences

$I=[0,1]$, each von Neumann - Morgenstern Solution K of $R_{>1/2}$ almost surely belongs to $\mathfrak{Z}(R_{1/2})$. So, the overall result on efficiency of *FDDP*'s with respect to preference domain is as follows.

Theorem 12.2. The only efficient *FDDP* in both families $\mathfrak{P}(\lceil\circ,\mathrm{inc})$, $\mathfrak{P}(\circ,\mathrm{inc})$ in universal classes $\mathcal{Q}_0^u$ or $\mathcal{Q}_1^u$ is the $(\circ,I_5)$-FNMS $\Delta_{23}(\circ,I_5)$ ■

Efficiency estimates in low dimensions n=10-30 were calculated in a simulation tour with DISPRIN (see Note 8.5 and Kitainik [5] for details). The resultant empiric estimate $\widehat{\mathrm{Eff}}_{\mathrm{Pref}}((\Delta_{23})(\circ,I_5),\mathfrak{R},\upsilon)=0.90\text{-}0.96$.

It should also be noticed that non-uniqueness of the induced crisp choice is to a considerable extent seeming effect: again, with continuous scale [0,1], the probability of unique crisp choice is 1. In real-life situations, this probability depends on the degree of discreteness of a membership scale, that is, on the number k of its gradations. Thus, for n=10-30 the empiric estimate of the mathematical expectation of the probability of unique crisp choice increases from 0.42-0.60 (k=11 - fuzzy scale 0.0, 0.1,...,1.0) to 0.79-0.85 (k=101 - fuzzy scale 0.00, 0.01,...,1.00).

As to the efficiency in ranking domain, the results for both $(\Delta_{23})(\circ,I_5)$, and $(\Delta_{123})(\circ,I_5)$ are favorable. Let us denote by p any of these *FDDP*s, and let $\mu^*>1/2$ be the corresponding maximum value $\mu^*(p,R,\mathfrak{F})$ (see Theorem 8.3). Geometrically, each (μ^*,K)-contrast $C(\mu^*,K)$ is a n-dimensional cube with side length $\overline{\mu^*}<1/2$; its volume is $\overline{\mu^*}^n<2^{-n}$. Furthermore, *MFC* is the union of $|\mathcal{K}^*|$ disjoint contrasts (in case of Δ_{123}, the contrast is unique - see Theorem 8.4 (ii)). Maximal number of crisp von Neumann - Morgenstern solutions is attained with Moon - Moser graphs (Moon, Moser [1]) and makes $\approx 3^{n/3}$. So, even in the worst case when $R_{>1/2}$ is a Moon - Moser graph, and $\mathcal{K}^*$ includes all its crisp von Neumann - Morgenstern solutions, the efficiency estimate is

$$\mathrm{Eff}_{\mathrm{MFC}}(p,R,\mathfrak{F},\nu) > 1-3^{n/3}\cdot 2^{-n} = 1-(3^{1/3}/2)^n \xrightarrow[n\to\infty]{} 1$$

(even with n=10, the result is sufficiently high).

Efficiency estimate of the induced crisp choice $\mathrm{Eff}_{\mathrm{Ch}}$ with $(\Delta_{23})(\circ,I_5)$ can be derived from well-known results concerning cliques in random graphs. Indeed, each CNMS of a digraph $\overrightarrow{G}_{n,1/2}$ is obviously a maximal clique in the underlying graph G (see Tutte [1] for definitions). According to (Matula [1]), number of elements in "almost all" maximal cliques $K \in \mathcal{K}$ grows much slower than n; hence, Eff_{Ch} rapidly increases to 1 when n increases. Thus, in a simulation tour with DISPRIN, Eff_{Ch} increased from 0.67-0.70 (n=10) to 0.83-0.84 (n=30) - see Table 12.1. In this computer experiment, the size of the sample was 1000; the step 1.00 of discrete preference scale corresponds to random *crisp* relations.

So, the $(\circ, I_5)$-based version of fuzzy von Neumann - Morgenstern Solution is the "absolutely best" *FDDP*: it is the only procedure which is efficient in all three meanings. This result seems to be of special interest for applied *DSS*' (see Chapter 15 for more details).

Table 12.1

Efficiency estimates of the induced crisp choice

(procedure $p=\Delta_{23}(\circ, I_5)$; computer simulation)

Number of alternatives n	Step of discrete scale $\frac{1}{k-1}$	Percentage of rejected alternatives: distribution within intervals]0;50[	[50;75[	[75;100[	mean	standard deviation	Resolution $\delta(p,R)$: mean	standard deviation
10	0.10	0.10	0.41	0.49	67.6	22.8	0.16	0.13
10	0.01	0.07	0.36	0.57	70.6	20.3	0.15	0.10
10	1.00	0.07	0.29	0.64	70.5	19.9	1.00	0.00
20	0.10	0.10	0.00	0.90	75.8	25.0	0.10	0.11
20	0.01	0.08	0.00	0.92	77.6	23.8	0.10	0.07
20	1.00	0.06	0.00	0.94	79.2	21.1	1.00	0.03
30	0.10	0.05	0.00	0.95	83.2	19.7	0.10	0.10
30	0.01	0.05	0.00	0.95	84.3	19.4	0.09	0.05
30	1.00	0.05	0.00	0.95	83.8	20.2	1.00	0.03

We point out that an applied *DSS ELECTRE* based on the use of von Neumann - Morgenstern Solutions with weighted preference relations was proposed by B.Roy [1] 20 years ago. On the one hand, the results of the present research can be considered as an extended theoretical motivation of advantages of this method. On the other hand, these results enable one to propose a modification of decision rules implemented in *ELECTRE*. In this system, an overall choice is based on scanning *diverse* level sets of a preference relation. According to considerations of Chapter 8, it is sufficient to confine oneself to a *single* strict median cut.

We cannot miss saying a few words about a number of procedures in the family $\mathfrak{P}(\circ, I_5)$ which are contensive only in the restricted environment $\mathscr{E}_0$. With evaluation of *FR*'s in the continuous scale I, all these *FDDP*'s are inefficient with respect to preference domain (see Kitainik [41]). Indeed, these procedures require certain "favorable" properties of extremal α-cuts of an original *FR* (see Chapters 8, 11). However, with a uniformly distributed (on the cube $\tilde{\mathscr{F}}_0(X^2)$ or $\tilde{\mathscr{F}}_1(X^2)$) continuous *FR*, an extremal cut $R_?$ is almost always either a strongly complete relation $(\forall\ x \neq y)(xR_?y)$ or a zero relation $(\forall\ x,y)(\neg(xR_?y))$: thus, $R_{>0}$, and $\bar{R}_{>0}=R_{<1}$ are strongly complete, whereas R_1, and $\bar{R}_1=R_{=0}$ are zero relations. Moreover, it turns out that just the required properties are not satisfied with the corresponding cuts. Thus, a necessary condition of contensiveness of $(\circ, I_5)$-*GOCHA* rule $\Delta_1(\circ, I_5)$ is the existence of a non-trivial bicomponent of a strict zero cut $R_{>0}$ (see Theorem 11.2. (i) (2)). Clearly, with a strongly complete digraph, this condition is almost never fulfilled.

The result remains unchanged with a discrete scale of preference values (which is often the case in software implementations): none of the mentioned procedures gains efficiency. Say, with the above family of scales $\{i/k \mid i=0,\dots,k\}$, and with preference values having equal probabilities 1/k, asymptotic inefficiency of $\Delta_1(\circ, I_5)$ follows from Lemma 12.1 (ii), because $R_{>0}$ is a random graph $\vec{G}_{n,q}$ with q=1-1/k (see also Kitainik [41]).

We conclude this chapter with a note on computational aspects of *FDDP*'s. Condensation of a graph, underlying the search of *MFC* for a prevailing number of *FDDP*'s (*GOCHA*, *GETCHA*, etc.), as well as the Stable Core, can be constructed in polynomial time (see, e.g., Swami, Thulasiraman [1] for quick algorithms). The same is for unity orbit, and its attractors in the case of FNMZS, and FZSC. In contrast with these simple algorithms, the problem of constructing of von Neumann - Morgenstern Solutions is NP-complete (Garey, Johnson [1], Problem GT57). Is it an occasional effect that well-defined procedures have difficult algorithmic implementations? Or, maybe, it is manifestation of a general principle? For the present, the answer to this question remains open.

Chapter 13

Decision-Making with Special Classes of Fuzzy Binary Relations

In this chapter, we consider "universally contensive" fuzzy dichotomous decision procedures, namely, the von Neumann - Morgenstern Solution Δ_{23}, and the Stable Core Δ_{123}, based on conventional composition law $\circ$, and on fuzzy inclusions I_5, $\subseteq$, with two special preference domains. These domains include two conventional classes of binary preference relations widely used in decision-making:

- the class of *fuzzy preorderings* (Zadeh [2]) (antireflexive, perfectly antisymmetric and transitive *FR*'s), and
- the class of "*fuzzy tournaments*", that is, *reciprocal FR*'s (Bezdek, B.Spillman, and R.Spillman [1,2]), defined, for non-diagonal pairs of alternatives, by an equation $R^{-1}=\bar{R}$ (reciprocal *FR*'s are supposed to be antireflexive, $\mu_R(x,x)\equiv0$).

13.1. FUZZY PREORDERINGS

Basic results on decision-making with fuzzy preorderings were obtained by L.Zadeh [2], by S.Orlovski [1], and by S.Ovchinnikov [5]. A well-known result of S.Orlovski [1] says that a fuzzy preordering always possesses *crisply non-dominated alternatives* (clearly, these alternatives form the

Core of a crisp preordering $R_{>0}$ - CND($R_{>0}$)). As was already mentioned in Example 8.1, this crisp subset, as it often happens with diverse "Core-like" collections of fuzzy subsets, cannot be considered as a well-defined crisp choice (see Section 8.3, Chapter 11). So, the first question is to discover adequate induced crisp choice with fuzzy preordering, together with its interval preference estimates. In the terms of our common approach, this is the same as to describe *MFC* with $\Delta_{23}(\circ,I_5)$, $\Delta_{123}(\circ,I_5)$ (see Section 8.4).

Another question concerns the possibility of ranking of alternatives according to *MFC*. On the one hand, Proposition 9.8 (ii), and Example 9.3 demonstrate that FND is unfit as a ranking concept with fuzzy preordering. On the other hand, Proposition 9.8 (i) shows that some of the "classical" interrelations between the Core, and the von Neumann - Morgenstern Solution remain valid in fuzzy environment. Therefore, we are going to recover the complete picture of mutual disposition of the Core, the von Neumann - Morgenstern Solution, and the Stable Core with a special case of fuzzy preorderings.

In the first place, we study choice properties of fuzzy preorderings based on $(\circ,I_5)$-procedures.

Theorem 13.1. Let R be a fuzzy preordering.

(i) The following four statements are equivalent:

(1) a specialization (Δ_{23},R) is *DC*;

(2) a specialization (Δ_{23},R) is *RC*;

(3) a specialization (Δ_{123},R) is *DC*;

(4) a specialization (Δ_{123},R) is *RC*.

(ii) In case of contensiveness, the following assertions are fulfilled:

(1) $\mu^*(\Delta_{23},R)=\mu^*(\Delta_{123},R)$;

(2) $\mathfrak{D}(\Delta_{23},R)=\mathfrak{D}(\Delta_{123},R)$ (*MFC* with fuzzy von Neumann - Morgenstern Solution is the same as with the Fuzzy Stable Core) ■

Proof. (i) All equivalencies follow from Theorem 8.4, with due respect to the fact that $R_{>1/2}$ is a crisp preordering thus possessing the Stable Core which is, in addition, the only von Neumann - Morgenstern Solution.

(ii) (1) Owing to (i) and to Theorems 8.3, 8.4, $\mu^*(\Delta_{23},R)=\mu_{\Delta_{23}}(\chi_M)$, $\mu^*(\Delta_{123},R)=\mu_{\Delta_{123}}(\chi_M)$; therefore, the equality $\mu^*(\Delta_{23},R)=\mu^*(\Delta_{123},R)$ is equivalent to the inequality $\mu_{\Delta_1}(\chi_M)\geq\mu_{\Delta_{23}}(\chi_M)$. We will prove that, in case of dichotomous contensiveness of specializations $(\Delta_{23}(\circ,I_5),R)$, $(\Delta_{123}(\circ,I_5),R)$, the inequality $\mu_{\Delta_1}(\chi_M)\geq\mu_{\Delta_2}(\chi_M)$ holds. Let us write out explicit formulas for $\mu_{\Delta_1}(\chi_M)$, $\mu_{\Delta_2}(\chi_M)$, $\mu_{\Delta_3}(\chi_M)$. Direct calculation yields

$$\mu_{\Delta_1}(\chi_M)=\overline{\overset{\circ}{\vee}R\big|_{\bar{M}\times M}>1/2};\quad \mu_{\Delta_2}(\chi_M)=\overline{\overset{\circ}{\vee}R\big|_{M\times M}>1/2};\quad \mu_{\Delta_3}(\chi_M)=\bigwedge_{y\in\bar{M}}\ \bigvee_{x\in M}\mu_R(x,y)>1/2.$$

So far, the inequality $\mu_{\Delta_1}(\chi_M)\geq\mu_{\Delta_2}(\chi_M)$ is, in its turn, equivalent to the inequality $\overset{\circ}{\vee}R\big|_{\bar{M}\times M}\leq\overset{\circ}{\vee}R\big|_{M\times M}$. Let us suppose that $1/2>\overset{\circ}{\vee}R\big|_{\bar{M}\times M}=\mu_R(y_0,x_0)$ with some $y_0\in\bar{M}$, $x_0\in M$ Using the formula for $\mu_{\Delta_3}(\chi_M)$, we derive that, owing to contensiveness of $(\Delta_{23}(\circ,I_5),R)$, there exists an $x\in M$ with $\mu_R(x,y_0)>1/2$. Next, transitivity of R implies the inequality

$$\mu_R(x,x_0)\geq\mu_R(x,y_0)\wedge\mu_R(y_0,x_0)=\mu_R(y_0,x_0);$$

hence, $\overset{\circ}{\vee}R\big|_{M\times M}\geq\mu_R(x,x_0)\geq\overset{\circ}{\vee}R\big|_{\bar{M}\times M}$.

(2) Immediately follows from (i) (1), and Theorem 8.3:

$$\mathfrak{D}(\Delta_{23},R)=\mathfrak{D}(\Delta_{123},R)=c(\mu^*,M),\ \text{with}\ \mu^*=\mu^*(\Delta_{23},R)=\mu^*(\Delta_{123},R)\ \blacksquare$$

This theorem can be considered as a fuzzy generalization of the above classical result: in case of crisp preordering, the Stable Core coincides with the only von Neumann - Morgenstern Solution. We discover that, even using a *fuzzy* preordering and admitting *multifold* fuzzy choice, we arrive to the same collections of fuzzy trial rankings constituting the corresponding *MFC*'s, be that *MFC*'s obtained on the basis of fuzzy version of von Neumann - Morgenstern Solution, or of a fuzzy Stable Core. However, the significant difference of fuzzy versions with the crisp prototypes is that the above fuzzy solutions have nothing to do with fuzzy version of the Core (fuzzy Core is always incontensive), whereas the "ordinary" Core of a crisp preordering coincides with the above two choices.

The same behavior can be observed with contensive FDDP's based on L.Zadeh's inclusion $\subseteq$. We precede our study of $(\circ,\subseteq)$-based procedures FNMZS, and FZSC by a very simple nonfuzzy statement.

Lemma 13.1. Let R be a crisp preordering on X, and let Y be a subset of X, satisfying the condition $CND(R)\subseteq Y\subseteq X$. Then $\varkappa(Y)=\varkappa(CND(R))=CND(R)$ $\blacksquare$

Proof. It is common knowledge that, with a crisp preordering R, CND(R) is simultaneously the unique CNMS, $\varkappa(\mathrm{CND}(R))=\mathrm{CND}(R)$. Next, an inclusion $\mathrm{CND}(R)\subseteq Y\subseteq X$ implies $R\circ\mathrm{CND}(R)\subseteq R\circ Y\subseteq R\circ X$ or, equivalently,

$$\varkappa(X)=\mathrm{CND}(R)\supseteq\varkappa(Y)\supseteq\varkappa(\mathrm{CND}(R))=\mathrm{CND}(R) \blacksquare$$

Theorem 13.2. With a fuzzy preordering R, the following statements hold:

(i) Specializations $(\Delta_{12}(\circ,\subseteq),R)$, $(\Delta_{123}(\circ,\subseteq),R)$ are ranking contensive, except for the case $R=0$ ($\mu_R(x,y)\equiv 0$); moreover, $d^r=d^{sr}$, so that *MFC* with both the FNMZS, and the FZSC is reduced to a single f.s.

(ii) Unity orbit Ω includes at most 3 elements; more precisely, $d_2=d_3$ ■

Proof. In virtue of Propositions 9.2, 9.4, the condition presented in (ii) implies the equality $d^r=d^{sr}$. Therefore, let us begin with verifying the statement (ii). The equality $d_2=d_3$ is equivalent to a family of α-cut equalities $(\forall\,\alpha)((d_2)_\alpha=(d_3)_\alpha)$. With respect to Proposition 9.1,

$$(d_2)_\alpha=\varkappa_{R_{>\bar\alpha}}\square\varkappa_{R_\alpha}(X), \text{ and } (d_3)_\alpha=\varkappa_{R_{>\bar\alpha}}\square\varkappa_{R_\alpha}\square\varkappa_{R_{>\bar\alpha}}(X)$$

(in the above expressions, some of the formally necessary brackets are omitted). Obviously, both the R_α, and the $R_{>\bar\alpha}$ are crisp preorderings.

First, let us suppose that $\alpha>1/2$; it follows that $R_\alpha\subseteq R_{>\bar\alpha}$; hence, $(\forall\, Z\subseteq X)(\varkappa_{R_{>\bar\alpha}}(Z)\subseteq\varkappa_{R_\alpha}(Z))$. In particular, $\varkappa_{R_{>\bar\alpha}}(X)\subseteq\varkappa_{R_\alpha}(X)$. It follows from the anti-monotonicity of the mapping $\varkappa_{R_\alpha}$ (see Lemma 9.1 (i)), that $\varkappa_{R_\alpha}\square\varkappa_{R_\alpha}(X)\subseteq\varkappa_{R_\alpha}\square\varkappa_{R_{>\bar\alpha}}(X)$. With respect to Lemma 13.1, a left-hand side term of the latter expression is nothing but $\varkappa_{R_\alpha}(X)$; combining two inclusions, we derive $\varkappa_{R_{>\bar\alpha}}(X)\subseteq\varkappa_{R_\alpha}(X)\subseteq\varkappa_{R_\alpha}\varkappa_{R_{>\bar\alpha}}(X)$. Applying Lemma 13.1 to a crisp preordering $R_{>\bar\alpha}$, we arrive to an equality

$$\varkappa_{R_{>\bar\alpha}}\square\varkappa_{R_\alpha}(X)=\varkappa_{R_{>\bar\alpha}}\square\varkappa_{R_\alpha}\square\varkappa_{R_{>\bar\alpha}}(X)=\varkappa_{R_{>\bar\alpha}}(X);$$

it follows that both α-cuts $(d_2)_\alpha$ and $(d_3)_\alpha$ coincide with a crisp subset $\varkappa_{R_{>\bar\alpha}}(X)=\mathrm{CND}(R_{>\bar\alpha})$.

In case when $\alpha\le 1/2$, the proof is slightly different:

$$(\alpha\le 1/2) \Leftrightarrow (R_\alpha\supseteq R_{>\bar\alpha}) \Leftrightarrow (\varkappa_{R_\alpha}(X)\subseteq\varkappa_{R_{>\bar\alpha}}(X)) \Rightarrow (\varkappa_{R_\alpha}(X)=\varkappa_{R_\alpha}\square\varkappa_{R_\alpha}(X)\supseteq\varkappa_{R_\alpha}\square\varkappa_{R_{>\bar\alpha}}(X))$$

$$\Rightarrow (\varkappa_{R_{>\bar{\alpha}}} \square \varkappa_{R_\alpha}(X) \subseteq \varkappa_{R_{>\bar{\alpha}}} \square \varkappa_{R_\alpha} \square \varkappa_{R_{>\bar{\alpha}}}(X)) \Rightarrow ((d_2)_\alpha \subseteq (d_3)_\alpha).$$

Since d_3 is always included in d_2 (Proposition 9.2 (ii)), the opposite inclusion $(d_3)_\alpha \subseteq (d_2)_\alpha$ also holds; finally, $(d_2)_\alpha = (d_3)_\alpha$.

In that way, FNMZS contains the only element $d^r = d^{sr}$ (see Proposition 9.5), and all components of the connected spectrum of R are pointwise (0-dimensional - see Section 9.3). Since R possesses crisply non-dominated alternatives, d^r is normal; hence, the only possible case of incontensiveness of a specialization $(\Delta_{12}(\circ,\subseteq),R)$ is $d^r=1$, which is obviously equivalent to R=0. Since R is transitive, application of Theorem 9.4 shows that the specialization $(\Delta_{123}(\circ,\subseteq),R)$ is also contensive, thus completing the proof ■

So, a well-defined choice (resp., ranking) of alternatives with a fuzzy preordering is provided by fuzzy versions of the Stable Core $\Delta_{123}(\circ,I_5)$ (resp., $\varkappa$(FND)). The result of the choice *always* differs from fuzzy Core $\Delta_{12}(\circ,I_5)$; the rule for ranking can sometimes be the same with an alternative fuzzy Core, FND=$\varkappa(1)$ (see Proposition 9.8 for a necessary and sufficient condition of coincidence of these rules).

13.2. RECIPROCAL RELATIONS

Extensive study of reciprocal *FR*'s in fuzzy decision-making was undertaken by J.Bezdek, B.Spillman, and R.Spillman [1,2]. It should be mentioned that reciprocal relation represent a valued version of *tournament* relation, widely used in voting theory (see a survey by V.Volskiy [1] for recent results), in multipurpose decision-making (Kacprzyk [1], Fodor and Roubens [1,2], Podinovski [1], etc). The subsequent results also lie in the framework of classical approach.

In case of a reciprocal *FR* R, *MFC*, as well as the fact of contensiveness, can be recovered using a single number - the height of FND(R) $\alpha^*=\overset{\circ}{\vee}$FND(R). First, let us prove a simple auxiliary "singletone lemma" (see also Bezdek, B.Spillman, and R.Spillman [1]).

Lemma 13.2. With reciprocal *FR*, either $\alpha^* \le 1/2$ (FND$\subseteq$1/2) or there exists unique $x^* \in X$, satisfying the condition $\mu_{\overline{FND}}(x^*) > 1/2$; in this case, $\mu_{FND}(x^*) = \alpha^*$ and $\vee_{y \ne x^*} \mu_{FND}(y) \le \alpha^*$ ■

Proof. (i) Clearly, the condition $\alpha^{*}\leq 1/2$ is equivalent to the inclusion $FND\subseteq 1/2$. Let us suppose that $\alpha^{*}>1/2$, and that the maximum value of FND is achieved with an alternative x^{*}, $\mu_{FND}(x^{*})=\alpha^{*}$. From definition of FND we derive that $\overline{\bigvee_{y\neq x^{*}} \mu_R(y,x^{*})}=\alpha^{*}$, so that $\bigvee_{y\neq x^{*}} \mu_R(y,x^{*})=\overline{\alpha^{*}}$. Owing to reciprocality of R, with $y\neq x^{*}$, $\mu_R(y,x^{*})\equiv\overline{\mu_R(x^{*},y)}$, thus implying the equality

$$\overline{\bigvee_{y\neq x^{*}} \mu_R(x^{*},y)}=\overline{\bigwedge_{y\neq x^{*}} \mu_R(x^{*},y)}=\overline{\alpha^{*}},$$

which is equivalent to the equality $\bigwedge_{y\neq x^{*}} \mu_R(x^{*},y)=\alpha^{*}$. The latter equality implies, in its turn, that $(\forall y\neq x^{*})(\mu_R(x^{*},y)\geq\alpha^{*})$, and $(\forall y\neq x^{*})(\bigvee_x \mu_R(x,y)\geq\alpha^{*})$. Hence, $(\forall y\neq x^{*})(\mu_{FND}(y)\leq\overline{\alpha^{*}}<1/2<\alpha^{*})$; in particular, x^{*} is unique ■

In fact, x^{*} is nothing but a fuzzy version of classical "winner in the sense of Condorcet" (see, e.g., Aizerman [1]), supplemented with a prevalence estimate. We will show that fuzzy versions of von Neumann - Morgenstern Solution, and of the Stable Core, (be that I_5- or $\subseteq$-FDDP's) with a reciprocal *FR* are contensive only in the case of existence of such winner ("winning cycles" are ignored by these decision procedures). Let us begin with $(\circ,I_5)$-FDDP's.

Theorem 13.3. With a reciprocal *FR* R, specializations based on fuzzy von Neumann - Morgenstern Solution, and the Fuzzy Stable Core are *DC* iff $\alpha^{*}>1/2$. Under this condition,

$$\mu^{*}(\Delta_{23},R)=\mu^{*}(\Delta_{123},R)=\alpha^{*}, \text{ and } \mathcal{D}(\Delta_{23},R)=\mathcal{D}(\Delta_{123},R)=c(\alpha^{*},\{x^{*}\}) \blacksquare$$

Proof. Directly follows from Theorems 8.3, 8.4 and Lemma 13.2, with due respect to the facts that:

crisp binary relation $R_{1/2}$ is weakly complete, and hence, any internally stable crisp subset of this relation is a singletone;

exactly one of these singletons, namely, the $\{x^{*}\}$, represents an externally stable subset of crisp relation $R_{>1/2}$; in addition, $\{x^{*}\}$ is the Stable Core of $R_{>1/2}$ ■

We precede the study of $\subseteq$-based procedures with the following result.

Lemma 13.3. With reciprocal *FR*, the following statements hold:

(i) If $\alpha^*>1/2$, then both the conventional and the strict α^*-, and $\overline{\alpha^*}$-cuts of R satisfy the properties (a)-(c):

(a) $R_{>\alpha^*}$ possesses non-dominated alternatives including x^* and has no dominating alternatives;

(b) R_{α^*} and $R_{>\overline{\alpha^*}}$, possess a single-point Stable Core x^*;

(c) $R_{\overline{\alpha^*}}$ possesses unique dominating alternative x^* and has no non-dominated alternatives.

(ii) If $\alpha^*<1/2$, then the following properties of α^*- and of $\overline{\alpha^*}$-cuts are satisfied:

(a) none of crisp relations $R_{>\alpha^*}$, $R_{\overline{\alpha^*}}$ possesses non-dominated or dominating alternatives;

(b) R_{α^*} has no non-dominated alternatives; the set of all dominating alternatives coincides with $FND^{-1}(\alpha^*)$;

(c) $R_{>\overline{\alpha^*}}$ has no dominating alternatives; the set of all non-dominated alternatives coincides with $FND^{-1}(\alpha^*)$.

(iii) For $\alpha^*=1/2$:

(a) $R_{>\alpha^*}=R_{\overline{\alpha^*}}$

(b) $R_{\alpha^*}=R_{>\overline{\alpha^*}}$ ■

Proof. We omit routine details of verification of these propositions. The common idea of all proofs is a well-known duality between the concepts of a non-dominated, and of a dominating alternative (see Fodor and Roubens [2], Ovchinnikov and Roubens [1,2] Roubens [1,2] for the extended study of this duality). Let us define $R^*=\overline{R^{-1}}$ (except for the diagonal elements - $\mu_{R^*}(x,x)\equiv 0$; in the above-mentioned papers, this *FR* is denoted by R^d). With any crisp relation, $CND(R)=CD(R^*)$, with CD being crisp domination - $CD(R)=\{x\in X \mid (\forall y\in X\setminus\{x\})(xRy)\}$ (the proof is elementary). Clearly, reciprocal *FR*'s are characterized by the equality $R^*=R$. This characteristic equality is equivalent to a family of α-cut equalities $(\forall\alpha)(R^*_\alpha=R_\alpha)$. It easily follows from Lemma 8.1 (ii) that $R^*_\alpha=(R_{>\bar{\alpha}})^*$. Hence, the characteristic equality for reciprocal *FR*'s implies $(\forall\alpha)(\mathrm{CND}(R_\alpha)=\mathrm{CD}(R_{>\bar{\alpha}}))$. The latter family of equalities underlies the proofs of all results stated in items (i)-(iii) ■

Let α^* be greater than 1/2; under this assumption, we denote by k^{rec} a f.s. $\alpha^*/x^*+\overline{\alpha^*}/\overline{\{x^*\}}$, and formulate the following result completely describing FNMZS, and ranking contensiveness of reciprocal *FR*'s.

Theorem 13.2. (i) Specializations $(\Delta_{23}(\circ,\subseteq),R)$ and $(\Delta_{123}(\circ,\subseteq),R)$ are *RC* in universal environment iff $\alpha^*>1/2$; in such case, *MFC* with FNMZS, and with FZSC is reduced to a single f.s. k^{rec}. In addition, unity orbit Ω includes at most 5 elements.

(ii) R is *RT* iff $\alpha^*\le 1/2$; under this condition, both d^r and d^{sr} are constants: $d^r=\alpha^*\circ 1$, $d^{sr}=\overline{\alpha^*}\circ 1$; unity orbit Ω includes at most 6 elements ■

Proof. (i) In Table 13.1, strict and non-strict $\bar{\alpha}^*$-cuts, and α^*-cuts of the members d_0-d_5 of the unity orbit Ω of an *FR* R are presented (see Chapter 9 for definitions). All cells in this table are filled according to "$\varkappa$-mapping technique" (Lemma 9.2, Proposition 9.1), with due respect to Lemma 13.3 (i). We comment upon three less obvious values $(d_2)_{>\alpha^*}$, $(d_3)_{\overline{\alpha^*}}$, and $(d_4)_{>\alpha^*}$. According to Lemma 13.3 (i) (a), $x^*\in(d_1)_{\overline{\alpha^*}}=\mathrm{CND}(R_{>\alpha^*})$. Owing to anti-monotonicity of $\varkappa$-mapping and taking into account Lemma 13.3 (i) (c), we arrive to the inclusion

$$(d_2)_{>\alpha^*}=\varkappa_{R_{\overline{\alpha^*}}}(\mathrm{CND}(R_{>\alpha^*}))\subseteq\varkappa_{R_{\overline{\alpha^*}}}(\{x^*\})=\{x^*\}.$$

Therefore, $(d_2)_{>\alpha^*}$ is either $\emptyset$ or $\{x^*\}$. In the first case, $(d_3)_{\overline{\alpha^*}}=X$; in the second case,

$$(d_3)_{\overline{\alpha^*}}=\varkappa_{R_{>\alpha^*}}(\{x^*\})=\overline{R_{>\alpha^*}\circ\{x^*\}}=\{y\in X\,|\,\mu_R(x^*,y)\le\alpha^*\}$$
$$=\{x^*\}\cup\{y\in X\,|\,\mu_R(y,x^*)\ge\overline{\alpha^*}\}=\{x^*\}\cup R^{-1}_{\overline{\alpha^*}}\circ\{x^*\}.$$

Table 13.1

Critical cuts of members d_0-d_5 of unity orbit Ω

i	$(d_i)_{\overline{\alpha^*}}$	$(d_i)_{>\overline{\alpha^*}}$	$(d_i)_{\alpha^*}$	$(d_i)_{>\alpha^*}$
0	X	X	X	X
1	$\mathrm{CND}(R_{>\alpha^*})$	$\{x^*\}$	$\{x^*\}$	$\emptyset$
2	X	$\{x^*\}$	$\{x^*\}$	$\emptyset$ or $\{x^*\}$
3	X or $\{x^*\} \cup R^{-1}_{\overline{\alpha^*}} \circ \{x^*\}$	$\{x^*\}$	$\{x^*\}$	$\emptyset$
4	X	$\{x^*\}$	$\{x^*\}$	$\emptyset$
5	X	$\{x^*\}$	$\{x^*\}$	$\emptyset$

It follows that

$$(d_4)_{>\alpha^*} = \chi_{R_{\overline{\alpha^*}}}((d_3)_{\overline{\alpha^*}}) = \overline{R_{\overline{\alpha^*}} \circ \{x^*\} \cup R_{\overline{\alpha^*}} \circ R^{-1}_{\overline{\alpha^*}} \circ \{x^*\}}. \tag{13-1}$$

Owing to Lemma 13.3 (i) (c), $R_{\overline{\alpha^*}} \circ \{x^*\} = \overline{\{x^*\}}$; next, $R_{\overline{\alpha^*}} \circ R^{-1}_{\overline{\alpha^*}} \circ \{x^*\}$ obviously includes x^*; hence, the union of "compositional" members under the sign of supplement in (13-1) is the whole support X, so that $(d_4)_{>\alpha^*} = \emptyset$.

It easily follows from the analysis of the above α-cuts (see Lemma 8.1 (iii)) that $d_4 = d_5 = k^{rec}$. In virtue of Propositions 9.3-9.5, $d^r = d^{sr} = k^{rec}$ also holds. Hence, unity orbit contains at most 5 elements $d_0, \ldots, d_4$, and FNMZS contains single f.s. k^{rec}. Direct calculation shows that $\overline{R \circ k^{rec}} = \overline{R} \circ (\overline{\alpha^*}/x^* + \alpha^*/\overline{\{x^*\}}) \subseteq \overline{k^{rec}}$; hence, k^{rec} is also the only FZSC of R ∎

So far, in contrast with the above considerations regarding fuzzy preorderings, decision-making with fuzzy tournaments yields a one-to-one generalization of classical results.

Chapter 14

Applications to Crisp Choice Rules

The results of comparative study of fuzzy dichotomous decision procedures obtained in Chapters 7-12, (Theorems 7.1, 8.1-8.4, etc.) can be applied to conventional decision-making based on crisp preference relations. In the framework of the above approach, crisp relations represent nothing but the specific *preference domain* $\mathcal{R}=\mathcal{P}(X^2)$. The only difference with the traditional consideration is the extension of "choice area" (*ranking domain*) from crisp subsets to fuzzy subsets of the support. This extension involves multiple new effects that were "invisible" in crisp case. Two of them are of special interest. One is reduction of the number of well-defined crisp choice rules due to sifting the original list of the rules through contensiveness criteria. Another is the appearance of uncertainty domains, "absorbing" conventional crisp choice. The only two procedures in both families $\mathfrak{P}_\Delta(\circ)$, $\mathfrak{P}_\Delta(\lceil\circ)$ that avoid such absorption are the von Neumann - Morgenstern solution, and the Stable Core: it easily follows from Theorems 8.3, 8.4 that, given a crisp relation, *MFC* with any of these procedures contains only crisp subsets. In that way, we arrive to a solid motivation enabling one to reduce the conventional list of choice rules with binary relations.

A more detailed study of the above-mentioned uncertainty domains for some procedures brings other interesting results.

14.1. ADJUSTING CRISP CHOICE

We recall that, owing to the results of Chapter 11, contensiveness of a considerable number of choice rules in the families $\mathfrak{P}(\circ, I_5)$, $\mathfrak{P}(\lceil\circ, I_5)$ can be achieved only in the restricted environment $\mathcal{E}_0=\tilde{\mathcal{P}}(X)\backslash\{0,1\}$ (prohibited trivial choice). Let us consider a model example of *GOCHA* procedure $\Delta_1(\circ, I_5)$ in the crisp preference domain $\mathcal{P}(X^2)$, and in the environment $\mathcal{E}_0$, in order to discover necessary changes in the crisp prototype $\Delta_1(\circ)$. In virtue of Corollary 8.3 (i), *MFC* with this procedure is represented as a collection of "interval triangulations" $1/A_1+[0,1]/A_2+0/A_3$, with $A_1=\bar{A}$, $A_2=A\backslash R\circ A$, $A_3=R\circ A$, where $A\in\mathfrak{Inv}(R)$, $R\circ A\subseteq A$, is an arbitrary invariant subset of R (see Figure 14.1; actually, in Corollary 8.3 (i), there appeared $R_{>0}\circ A$, but with crisp relation, all non-trivial α-cuts coincide with the relation itself, so that $R_{>0}=R$). According to Theorem 11.2, a necessary condition of dichotomous contensiveness of a specialization is the non-existence of a crisp Core CND(R). (once more, we changed $CND(R_{>0})$ for CND(R)). In comparison with Theorem 3.2, this requirement seems a somewhat paradoxical: on the one hand, conventional choice with *GOCHA* rule proves to be very close to graphodominant choice (in fact, the chosen subset is nothing but the Core of condensation of R); on the other hand, fuzzy version of *GOCHA* procedure does not "admit" the proper Core of the original relation. The explanation of this seeming paradox lies in the uncertainty domain, "swallowing" crisp cores. It looks as if *GOCHA* rule has been developed for the analysis of a complicated situation when the acknowledged Core rule fails (in our opinion, this explanation is acceptable from the "historical" viewpoint).

Now, let us suppose that contensiveness conditions listed in Theorem 11.2 are satisfied. Then, what is the "right" induced crisp choice with a crisp relation? It is an easy matter to derive from Corollary 8.3 (i), that *MFC* with a crisp *FR* is represented as a union $\bigcup_{A\in\mathfrak{Inv}^*(R)} \iota(\xi_A)$, with $\mathfrak{Inv}^*(R)$ being the set of maximal with respect to inclusion crisp invariant subsets of R. Using Proposition 10.3, we conclude that each subset in $\mathfrak{Inv}^*(R)$ is a supplement to exactly *one* among the non-dominated bicomponents of R; in addition, the latter is required to be a proper subset of X containing at least two alternatives. So, the union of "incontestably chosen" crisp subsets $\{\bar{A}|A\in\mathfrak{Inv}^*(R)\}$ represents the result of the conventional *GOCHA* choice (see Schwartz [1], and Theorem 3.2); the only difference with the original definition is that each particular member $\bar{A}$ of this collection should be considered as an *alternative* choice. However, this collection is far from covering the overall induced crisp choice. Indeed, with each $A\in\mathfrak{Inv}^*(R)$, any part of the subset $A\backslash R\circ A=CND(R|_A)$, that is, of the Core of the induced relation $R|_A$, can arbitrarily be either included in or excluded from the resulting crisp choice. In other words, $CND(R|_A)$ represents the *uncertainty domain.*

The nature of the above uncertainty can be understood not only in the terms of "extended scale of preferences", "fuzzy decision procedures", etc. Actually, it has clear interpretation in the bounds of classical Choice Theory, representing the "domain of violation" of one or another axiom of rational choice. In the case of *GOCHA* choice rule, it is the Independence axiom that fails in the neighborhood of uncertainty domain. We recall that the Independence axiom (Chernoff [1], postulate 5*; see also Aizerman [1], Plott [1]) for a choice function $C:\mathcal{P}(X)\longrightarrow\mathcal{P}(X)$ can be formulated as

$$(\forall\ Y\subseteq X)(\forall\ Y'\subseteq X)(C(Y)\subseteq Y'\subseteq Y\Rightarrow C(Y')=C(Y))$$

Let us consider the choice function C_G associated with *GOCHA* rule:

with $R\in\mathcal{P}_0(X^2)$, $Y\in\mathcal{P}(X)$,

$C_G(R)(Y)$={the union of minimal *GOCHA*-rational subsets of the induced relation $R|_Y$}.

We assume, for simplicity, that $Y=X$, $C_G(R)(Y)=\bar{A}$, and that the triangulation $\mathfrak{t}(\xi_A)$ with $A=\overline{C_G(R)(X)}$ is non-degenerate, that is, $CND(R|_A)\neq\emptyset$. If we set $Y'=C_G(R)(Y)\cup CND(C_G(R)(Y))$, then $C_G(R)(Y)\subset Y'\subset Y$. Next, it can be easily derived from the definition of *GOCHA* choice rule, and from Proposition 10.3 that $C_G(R)(Y')=Y'\neq C_G(R)(Y)$, and hence, *GOCHA* choice $C_G(R)$ does not satisfy Independence axiom (see Figure 14.1).

To get free from uncertainty domains, crisp preference relation must possess more special structure. A necessary and sufficient condition of emptiness of all uncertainty domains can be represented as

$$(\forall\ A\in\mathfrak{Inv}(R))(A\backslash R\circ A=\emptyset)\Leftrightarrow(\forall\ A\in\mathfrak{Inv}(R))(R\circ A=A),$$

which is equivalent to the equality $\mathfrak{Inv}(R)=\mathfrak{Eig}(R)$ (see Chapter 10).

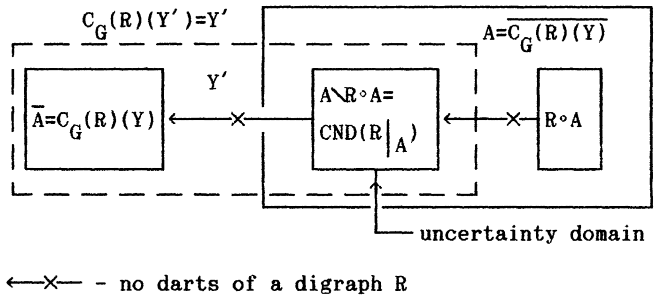


Fig. 14.1. Uncertainty domain, and violation of Independence axiom

In Corollary 10.2, we established a necessary and sufficient condition of faithfulness of this equality: to guarantee a definite choice, preference relation should be free of trivial bicomponents.

Example 14.1. Let R be the following crisp relation on $X=\{x_1,\dots,x_8\}$:

$$R = \begin{vmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

Direct calculation of bicomponents of R yields: $\hat{X}_R=\{\hat{x}_1,\dots,\hat{x}_5\}$; $\hat{x}_1=\{x_1,x_3\}$; $\hat{x}_2=\{x_2,x_5\}$; $\hat{x}_3=\{x_4\}$; $\hat{x}_4=\{x_6\}$; $\hat{x}_5=\{x_7,x_8\}$; condensation of R is the relation

$$\hat{R} = \begin{vmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} 1 & 0 \\ 1 & 1 \\ \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}} \end{matrix} & \begin{matrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{matrix} \end{vmatrix}$$

GOCHA choice   uncertainty domain

Hence, non-dominated bicomponents of R are $\hat{x}_1$, and $\hat{x}_2$, so that *GOCHA* choice is their union $\bar{A}=C_G(X)=\{x_1,x_2,x_3,x_5\}$, with $A=\{x_4,x_6,x_7,x_8\}\in\Im n\mathfrak{D}(R)$. A relation $R|_A$ has the Core $CND(R|_A)=\{x_4,x_6\}$, representing the uncertainty domain ■

Similar results can be obtained for a considerable number of choice rules in both $\circ$- and $\lceil\circ$-based families. Thus, with *GETCHA* rationality concept $\Delta_3(\lceil\circ,I_5)$, triangulation has the form

$$\tau_A=\{A_1,A_2,A_3\}=\{A\backslash R\lceil\circ A,A\cap R\lceil\circ A,\bar{A}\}$$

with A being arbitrary conventional "R-dominating" subset of X ($\bar{A}\subseteq R\lceil\circ A$); with this rule, the existence of uncertainty domain is also related to the failure of Independence axiom.

So far, formal application of FDDP's to well-known rationality concepts in a conventional crisp preference domain leads to a more profound understanding of the nature of crisp choice rules with binary relations, and motivates considerable changes in some rules.

14.2. PRODUCING NEW CHOICE RULES (FNMZS AND DIPOLE DECOMPOSITION)

In this section, we demonstrate another example of instrumentality of *fuzzy* decision procedures in *crisp* environment by discovering "ranking properties" of the conventional crisp von Neumann - Morgenstern Solution.

Let us suppose that the Core of a crisp binary relation R is empty (which is almost always true for common relations, see Chapter 12). We know that crisp von Neumann - Morgenstern Solutions of R are nothing but the solutions of a relational equation $R\circ A=\bar{A}$ (see Chapter 9). Following the same idea as in the previous section, let us solve this equation with a *crisp* relation R, but in the *fuzzy environment* . (X). In other words, let us consider *MFC* associated with a specialization (FNMZS,R). The solution can be easily reconstructed using the technique of Chapter 9 (we omit routine calculations) and gives rise to a new ranking concept - we call it *dipole decomposition.* A dipole decomposition $\mathfrak{D}(R)\in\Pi$ is a partition of the support X

$$\mathfrak{D}(R)=\{Z_1^+,\ldots,Z_m^+,Z_0,Z_m^-,\ldots,Z_1^-\},$$

with its members satisfying the system of equations:

$$R\circ Z_j^+=Z_j^-, \qquad R^{-1}\circ Z_j^+\subseteq \bigcup_{i=1}^{j} Z_i^-.$$

The structure of darts of a digraph R associated with this system of equations and inclusions is shown on Figure 14.2. Furthermore, *MFC* with a specialization (FNMZS,R) can be represented as

$$\mathfrak{D}(p,R)=\{\sum\alpha_j/Z_j^+ + \tfrac{1}{2}/Z_0+\sum\bar{\alpha}_j/Z_j^- \mid \mathfrak{D}(R)=\{Z_j^+,Z_0,Z_{m-j+1}^-\}_1^m;\ \alpha_m\geq\ldots\geq\alpha_1\geq 1/2\}.$$

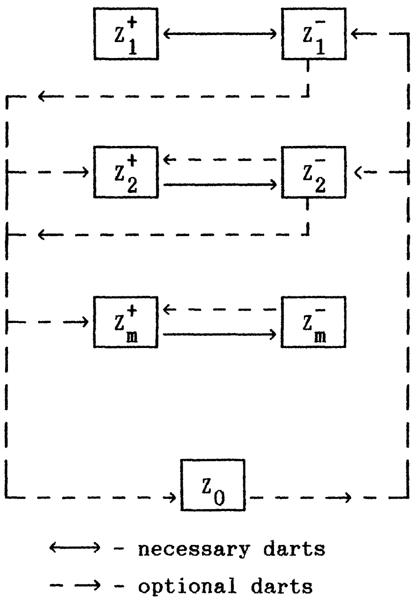


Fig. 14.2. Dipole decomposition

According to the latter expression, the members of the above partition $\mathfrak{D}(R)$ can be ordered with respect to their enumeration Z_j^+,Z_0,Z_{m-j+1}^-.

So, the fuzzy FNMZS procedure aimed at producing rankings, discovers a specific crisp "block structure" in a digraph R, which is responsible for ranking of crisp alternatives, though the original notion of von Neumann - Morgenstern Solution has nothing to do with rankings. In fact, dipole decomposition is both the generalization and the refinement of ordinary von Neumann - Morgenstern solution. Indeed, if $Z_0=\emptyset$, then the union of "positive components" $\bigcup_{i=1}^{m} Z_i^+$ forms an ordinary von Neumann - Morgenstern solution of R; conversely, any crisp von Neumann - Morgenstern solution N of a binary relation (with empty Core) determines a dipole decomposition $(N,\bar{N})$. One can also observe that dipole decomposition is a generalization of the above crisp $\times$-construction (see Example 9.1). The difference with the original sequence $\{D_i\}$ in the above example is that dipole decomposition must not be unique. Nevertheless, dipole decompositions can be used for at least two purposes:

- splitting of a singular body D^s of a relation aimed at building von Neumann - Morgenstern Solutions;
- direct ranking of crisp alternatives in case when preference scale can be considered as *order* scale (see Pfanzagl [1]); we recall that, under the condition of emptiness of the Core of $R=R_{>0}$, FNMZS itself is *DT* (see Theorem 12.1).

We cannot go far in this problem; the only thing that deserves mentioning is that, combining the search of dipole decompositions with a special technique, close to that of quasi-inverse relations (Sanchez [2], Di Nola, Pedrycz, Sanchez, and Sessa [1]), one can obtain a complete (though very tedious) description of FNMZS, that is, of all solutions of a relational equation $R \circ a=\bar{a}$ with an arbitrary *FR* R.

Chapter 15

Applications to Decision Support Systems and to Multipurpose Decision-Making

In this concluding chapter, we present an outline of applications of the above results to *DSS*' of general type, and to selected problems of multipurpose decision-making.

15.1. GENERAL APPLICATIONS TO DECISION SUPPORT SYSTEMS

The total of our study of fuzzy and crisp decision procedures undertaken in Chapters 7-14 can be expressed in the form of a "list of advise and cautions", addressed to a designer of an applied *DSS*, provided that the *DSS* is intended to support decision-making with binary preference relations, be that crisp or valued models of individual or/and collective preferences.

1 (Preference Domain should be examined in the first place).
Irrespective of whether the general concept of choice and/or ranking in a *DSS* is based on *prior* or on *posterior* decisions (see Fodor and Roubens [1]), the design of a Decision Rules module of the System essentially depends on the presupposed breadth of Preference Domain. The broader is the class of preference relations admitted in the System (i.e. the more flexible are the tools of modeling and aggregating of individual/group

preferences), the smaller is the collection of adequate decision rules. Therefore, the Preference Modeling subsystem, and the Decision Rules subsystem should not be implemented separately.

2 (The case of narrow Preference Domain).
In case when Preference Modeling subsystem is "monitoring" the decision process in such a way as to guarantee that the final preference relation should be a preordering, ranking module can be included in the Decision Rules subsystem. The only additional requirement concerns the case of valued relations: in general, not the Core $FND(R)=\varkappa_R(1)$, but the Stable Core $\varkappa(FND(R))=\varkappa_R^2(1)$ represents the correct ranking (see Chapters 9, 13).

3 (The case of broad Preference Domain).
In case when Preference Modeling subsystem includes non-restricted aggregating tools, so that the resultant Preference Domain can be considered as a "universal" one, two advices can be of value for a designer of a Decision Rules module:

- ranking procedures can be omitted without prejudice to the quality of decision-making; all the same, they would almost always be inconsistent (see Chapter 12);
- in a rather extended list of conventional, and modern choice rules, von Neumann - Morgenstern Solution (in case of valued relations, $\Delta_{23}(\circ,I_5)$) is the only efficient decision procedure; all the remaining choice rules in the families $\mathfrak{P}(\cdot.\cdot)$ would also fail too frequently (see Chapters 8, 12).

4 (Requirements to choice and ranking rules).
In principle, the collection of four ranking and choice rules $\Delta_{123}(\circ,\subseteq)$, $\Delta_{23}(\circ,\subseteq)$, $\Delta_{123}(\circ,I_5)$, $\Delta_{23}(\circ,I_5)$ can well exhaust the Decision Rules module of a *DSS* with any non-specific Preference Domain. Nevertheless, if, for some reasons, other choice rules from the families $\mathfrak{P}(\cdot.\cdot)$ are also implemented in a *DSS*, special efforts should be made in order to recognize, display, and interpret *uncertainty domains*, and *triangulations* resulting from the majority of additional rules (Chapters 8, 11).

5 (The Core should be avoided).
One should systematically discern between the Stable Core, and the Core. The latter should be invariably avoided for both the choice and the ranking purpose (Sections 8.3, Chapters 9, 11, 12).

6 (Miscellaneous)
The following results of the above study can be useful in algorithmic implementation of a *DSS*:

- explicit formulas for contensiveness conditions of specializations (Chapters 8, 9, 11, 13, 14);
- description of *MFC*, and of induced crisp choice and/or ranking (Chapters 8-11, 13);
- duality formulas for "cross-linked" *FDDP*'s (Chapter 4).

15.2. APPLICATIONS TO MULTIPURPOSE DECISION-MAKING

In this section, we consider a special model of aggregation of preferences, and applications of the developed approach to the problem of co-ordination of preferences in multipurpose decision-making.

We are mainly interested in *individual* decision-making. Let us suppose that X is, as above, the original set of crisp alternatives; let $\mathbb{C}=\{C\}$ be a collection of crisp or fuzzy preference relations on X, considered as partial criteria. "Subjective" information can include decision-maker's evaluations of alternatives for some of the criteria, and the notion of *significant criterion*; we suppose that it can be represented as a f.s. $\widetilde{sgc}\in\mathcal{P}(\mathbb{C})$. In addition, let $\widetilde{sfn}\in\mathcal{P}(\overline{1,|\mathbb{C}|})$ be the fuzzy notion of *sufficient number* of criteria (from now on, $\overline{k,l}$ denotes the interval of integers from k to l, with bounds included). Also, let us assume that a *DSS* is designed in such a manner that a decision-maker is allowed to give explicit *a priori* estimates of the original alternatives; we denote by $\widetilde{apr}\in\mathcal{P}(X)$ the corresponding f.s. "good alternative". It is known in descriptive decision theory (see, e.g., Larichev [11]), that *a priori* estimates of alternatives often vary from decision-maker's choice in the "criteria space" as well as from the results of "objective" decision procedures. The study of such deviations is very important and forms the subject of "concordance analysis".

In that way, we consider a special type of Multicriteria Decision Problem (*MCDP*, see Chapter 2) as an aggregate $\mathbb{D}=\{X,\mathbb{C},sgc,sfn,apr,\dots\}$ [1]

---

[1] For brevity, we write sgc(x), sfn(x), apr(x) instead of $\mu_{sgc(x)}$, ...

(other notions will be introduced later on). The subsequent steps of decision analysis are the Aggregation of Preferences, the application of Decision Rules, and the Concordance Analysis.

The problem of aggregation of criteria in multicriteria decision-making was studied very extensively (Dubois and Prade [2,3], Fodor [2], Fodor and Roubens [1], Kacprzyk [1], Kacprzyk and Yager [1], Roubens [2], Roy [1], Zimmerman [1], and many others). We confine ourselves to a particular case when the notion of significance of criteria is explicated in the above form of f.s. Under this assumption, we propose both the list of empiric aggregation concepts, and a more speculative "fuzzy majority approach".

15.2.1. Empiric Aggregation Rules

A family of empiric aggregation rules representing five natural "aggregating predicates" is given in Table 15.1. It is of interest that that only the first of the presented rules is "transitivity preserving". All the remaining rules commonly lead to intransitive preference relations even in the case when all local criteria are crisp/fuzzy orderings (see Chapter 1).

15.2.2. Fuzzy Majority Approach

Majority relation is an acknowledged object in Decision Theory (see, e.g., Berezovski, Borzenko, and Kempner [1], Fishburn [1,2], Miller [1], Richelson [1], Podinovski [1]; modern problems can be found in a survey Lezina [2]). In fuzzy decision-making, we mark out the papers by J.Kacprzyk [1], and by H.Nurmi [1].

In the present study, we do not consider majority approach with voting problems, collective decision-making, etc. We dwell on an alternative axiomatic basis of majority relations, attempting to get over the excessive "locality" of conventional majority method. So far, we do not distinguish between "voters" and "criteria", and interpret majority in the framework of multipurpose decision problems.

We recall that, at the preliminary step of crisp majority approach the following sets $I^{+}(x,y)$ are determined (see, e.g., Roy [1])

$$I^{+}(x,y)=\{C\in\mathfrak{C} \mid x \text{ is better than } y \text{ with respect to } C\}$$

With the "majority tradition", significances of criteria are accumulated

Table 15.1

Empiric aggregation rules for preference relations

("x is preferred to y iff <proposition>..." $\Leftrightarrow$ $\mu_R(x,y)$=<formula>)

proposition	1. it is preferred for at least one criterion: $(\exists\ C\in\mathfrak{C})(x \geq_C y)$
formula	$\bigvee_{C\in\mathfrak{C}} \mu_C(x,y)$
proposition	2. it is preferred for all criteria: $(\forall\ C\in\mathfrak{C})(x \geq_C y)$
formula	$\bigwedge_{C\in\mathfrak{C}} \mu_C(x,y)$
proposition	3. it is preferred for at least one significant criterion: $(\exists\ C\in\mathfrak{C})((C \text{ is sgc})\&(x \geq_C y))$
formula	$\bigvee_{C\in\mathfrak{C}} \mu_C(x,y) \wedge \mu_{sgc}(C)$
proposition	4. it is preferred for all significant criteria: $(\forall\ C\in\mathfrak{C})((C \text{ is sgc}) \Rightarrow (x \geq_C y))$
formula	$\bigwedge_{C\in\mathfrak{C}} (\mu_{sgc}(C) \Longrightarrow \mu_C(x,y))$
proposition	5. it is preferred for a sufficient number of significant criteria: $(\exists\ Z\subseteq\mathfrak{C})((\lvert Z\rvert \text{ is sfn})\&((\forall\ C\in Z)((C \text{ is sgc})\&(x \geq_C y))))$
formula	$\bigvee_{Z\in\mathcal{P}(\mathfrak{C})} \mu_{sfn}(\lvert Z\rvert) \wedge \bigwedge_{C\in Z} (\mu_C(x,y) \wedge \mu_{sgc}(C))$

The properties of the Model 4 "for all significant criteria" depend on the choice of fuzzy implication.

as additive "weights", so that the aggregated crisp preference relation M can be defined as (see, e.g., Podinovski [1])

$$xMy \Leftrightarrow \sum_{C\in I^+(x,y)} sgc(C) > \sum_{C\in I^+(y,x)} sgc(C) \qquad (15\text{-}1)$$

Let us denote $\sum_{C\in I^+(x,y)} sgc(C)$ by $m^+(x,y)$, $\bigvee_{C\in I^+(x,y)} sgc(C)$ by $m^\vee(x,y)$. Several fuzzy refinements of the formula (15-1) are immediate:

$$\mu_M(x,y)=m^+(x,y)/(m^+(x,y)+m^+(y,x)) \qquad (15\text{-}2)$$

$$\mu_M(x,y)=m^\vee(x,y)/(m^\vee(x,y)\vee m^\vee(y,x)) \qquad (15\text{-}3)$$

$$\mu_M(x,y)=m^+(x,y)/\bigvee_{x\neq y} m^+(x,y) \qquad (15\text{-}4)$$

$$\mu_M(x,y)=m^\vee(x,y)/\bigvee_{x\neq y} m^\vee(x,y) \qquad (15\text{-}5)$$

The formula (15-2) represents the most straightforward generalization of a crisp majority relation, resulting in a reciprocal *FR* M. Formally, all the above expressions differ only in specific "rating coefficients". There exists, however, a more essential distinction between (15-2), (15-3), on the one hand, and (15-4), (15-5), on the other hand. In the first two formulas, only the comparison of the two given alternatives influences the aggregated measure of preference. Thus, with equally significant 110 criteria, the score 100:10 for x as against y, and the score 10:1 for u as against v, yields the same result $\mu_M(x,y)=\mu_M(u,v)=10/11$. Is this approach adequate in the framework of a single decision problem, taking into account that u, and v are indistinguishable for a great majority of criteria? We refer to Fishburn [2] for extended discussion on the subject; we adhere to an opinion that these ratios are far from being equivalent. An alternative approach is provided by formulas (15-4), (15-5), where rating coefficients absorb the overall "scoring" results.

15.2.2.1. Pre-majority relation

Motivated by the above discussion, let us use a more general majority method. We observe that a union of k-dimensional standard cubes $\mathscr{D}^n = \bigcup_{k=1}^{n} I^k$ can be supplied with a natural structure of a *symmetric Pareto ordering* $\overset{S}{\geq}$:

for $\xi\in I^k$, $\eta\in I^l$,

$$\xi\overset{S}{\geq}\eta \Leftrightarrow \text{(there exists an injective mapping } \psi:\overline{1,l}\longrightarrow\overline{1,k}\text{,}$$
$$\text{satisfying the condition } \xi_{\psi(i)}\geq\eta_i \text{ for all } i\in\overline{1,l}\text{)}$$

Thus, with $\xi=(0.6\ 0.1\ 0.3\ 0.7)\in I^4$, $\eta=(0.1\ 0.5\ 0.6)\in I^3$, $\xi\overset{S}{\geq}\eta$ is satisfied: the desired injective mapping ψ is, for example, $\psi(1)=2$, $\psi(2)=1$, $\psi(3)=4$.

Next, instead of subsets $I^+(x,y)$, let us consider a more general $\mathscr{D}^n$*-fuzzy pre-majority relation* $\mathcal{M}_0$:

$$\mu_{\mathcal{M}_0}(x,y)=\{sgc(C)\,|\,C\in I^+(x,y)\}$$

In case when $\mathfrak{C}$ contains fuzzy local preference relations, $I^+(x,y)$ itself should also be transformed: $I^+(x,y)=\{C\in\mathfrak{C}\,|\,\mu_C(x,y)>\mu_C(y,x)\}$.

It is of interest that in particular case when all local criteria turn to be the crisp ones (and hence, $\mathcal{M}_0$ is a $\mathscr{P}(X)$-fuzzy relation, $\mu_{\mathcal{M}_0}(x,y)=I^+(x,y)$), pre-majority relation preserves transitivity.

Proposition 15.1. If all local criteria are crisp and transitive, then $\mathcal{M}_0$ is a transitive $\mathscr{P}(X)$-fuzzy relation ■

Proof. Let x, y, z be three alternatives. According to definition of $\mathcal{M}_0$, $C\in\mu_{\mathcal{M}_0}(x,y)\cap\mu_{\mathcal{M}_0}(y,z) \Leftrightarrow (x\underset{C}{\geq}y)\&(y\underset{C}{\geq}z)$; transitivity of C implies $x\underset{C}{\geq}z$, so that $C\in\mu_{\mathcal{M}_0}(x,z)$, and finally, $\mu_{\mathcal{M}_0}(x,y)\cap\mu_{\mathcal{M}_0}(y,z)\subseteq\mu_{\mathcal{M}_0}(x,z)$ ■

So far, violation of transitivity, which is a generally known property of majority relations, occurs on the way from a pre-majority to the "proper" majority relation, and is caused by the fact that a "majority order" $Y\succ Z \Leftrightarrow |Y|>|Z|$ does not induce a homomorphism from $\mathscr{P}(X)$ into $\{0,1\}$.

15.2.2.2. Axioms for aggregation

We consider the problem of "majority aggregation" as a problem of transformation of a *pre-majority relation* $\mathcal{M}_0$ into a [crisp or fuzzy] *majority relation* $\mathcal{M}$. What could be the adequate axioms for such transformation? The first axiom seems to be out of suspicion: membership values of $\mathcal{M}$ should be monotone with respect to symmetric Pareto ordering on the elements of $\mathcal{M}_0$. Let us denote by X_0^2 the subset $X^2\setminus\{(x,x)\}$; we formulate the first axiom of fuzzy majority:

FM$_1$. For any two pairs of alternatives (x,y), $(z,t)\in X_0^2$,

$$\mu_{\mathcal{M}_0}(x,y)\overset{S}{\geq}\mu_{\mathcal{M}_0}(z,t) \Rightarrow \mu_{\mathcal{M}}(x,y)\geq\mu_{\mathcal{M}}(z,t)$$

This axiom determines fuzzy majority relation up to an arbitrary monotone transformation. We emphasize that the axiom is much weaker than the "additive assumption" of ordinary majority; of course, all formulas (15-1)-(15-5) satisfy this axiom. But again, a "rating axiom" is needed. We did not succeed too much in formulating as natural rating conditions as is the monotonicity axiom *FM*$_1$. Yet in particular case when sgc is sufficiently "polarized", the following rating axiom *FM*$_2$ can be useful:

FM$_2$. For any pair of alternatives $(x,y)\in X_0^2$, membership value $\mu_{\mathcal{M}}(x,y)$ is included in the corresponding limits of values of $\mu_{\mathcal{M}_0}(x,y)$:

$$\mu_{\mathcal{M}}(x,y)\in[\overset{S}{\wedge}\mu_{\mathcal{M}_0}(x,y),\overset{S}{\vee}\mu_{\mathcal{M}_0}(x,y)]$$

15.2.2.3. Fuzzy majority relations

Clearly, there can exist more that one aggregated fuzzy majority relation satisfying *FM*$_1$, *FM*$_2$. Given a *MCDP* $\mathbb{D}=\langle X,\mathfrak{C},\mathrm{sgc},\mathrm{sfn},\mathrm{apr},\ldots\rangle$, let us denote by $\mathfrak{Maj}=\mathfrak{Maj}(\mathbb{D})$ the set of all fuzzy majority relations satisfying these axioms. Primary properties of the family $\mathfrak{Maj}$ are established in Theorem 15.1. Before formulating the results, we notice that a symmetric Pareto ordering $\overset{S}{\geq}$, when applied to $\mathcal{M}_0$, induces crisp transitive relation G on X_0^2, $(x,y)G(z,t) \Leftrightarrow \mu_{\mathcal{M}_0}(x,y)\overset{S}{\geq}\mu_{\mathcal{M}_0}(z,t)$. Let us denote by $X_0,\ldots X_k$ "order levels" of G: X_k contains all non-dominating vertices; with $i<k$, X_i is defined recurrently as $G^{-1}(X_{i+1})\setminus(G^{-1})^2(X_{i+1})$.

Theorem 15.1. (i) With any *MCDP* $\mathbb{D}$, $\mathfrak{Maj}(\mathbb{D})$ is non-empty; moreover, it has both the smallest ($\mathcal{M}_\vee$), and the greatest ($\mathcal{M}_\wedge$) element.

(ii) $\mu_{\mathcal{M}_\vee}(x,y)= \bigvee\limits_{C\in I^+(x,y)} \mathrm{sgc}(C)$

(iii) $\mathcal{M}_\wedge$ is defined by the following recurrent formulas:

$$(\forall\ (x,y)\in\mathfrak{X}_k)(\mu_{\mathcal{M}_\wedge}(x,y)=\overset{S}{\wedge}\ \mu_{\mathcal{M}_0}(x,y))$$

$$(\forall\ i<k,\ (x,y)\in\mathfrak{X}_i)(\mu_{\mathcal{M}_\wedge}(x,y)=\overset{S}{\wedge}\mu_{\mathcal{M}_0}(x,y)\vee(\bigvee_{(z,t)\in G(\{(x,y)\})}\mu_{\mathcal{M}_\wedge}(z,t)))$$

(iv) $\mathfrak{Maj}$ is a convex subset of $[\mathcal{M}_\wedge, \mathcal{M}_\vee]$ (with the common understanding of convexity, $(\mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{Maj},\ \lambda \in I) \Rightarrow \lambda\mathcal{M}_1 + \bar{\lambda}\mathcal{M}_2 \in \mathfrak{Maj})$ ■

Proof. $FM_2 \Rightarrow (\forall\ R \in \mathfrak{Maj})(R \subseteq \mathcal{M}_\vee)$; since $\mathcal{M}_\vee$ clearly satisfies both the FM_1, and the FM_2, it follows that $\mathcal{M}_\vee$ is the greatest element in $\mathfrak{Maj}$. The proof for $\mathcal{M}_\wedge$ is similar; hence (i)-(iii) are true. (iv) is fulfilled because of "convexity" of both axioms ■

Example 15.1. In Table 15.2, primary data for a *MCDP* of selecting machines and technology for wood-road engineering are presented (see Kitainik [2]). $\mathcal{M}_\wedge$, and $\mathcal{M}_\vee$ are given in Table 15.3. Owing to Lemma 8.4, it is easy to verify that $\bigcap_{\mathcal{M} \in \mathfrak{Maj}} \mathfrak{K}(\mathcal{M}) = \{\{x_1, x_2\}, \{x_6\}\}$. Hence, there are no transitive relations in $\mathfrak{Maj}$ (for such a relation $\mathcal{M}$, $\mathfrak{K}(\mathcal{M})$ should contain unique subset). $\mathfrak{K}^*(\mathcal{M}_\vee) = \{\{x_6\}\}$ determines unique choice, whereas for $\mathcal{M}_\wedge$ the picture is most ambiguous: $\mathfrak{K}^*(\mathcal{M}_\wedge) = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}\}$ forms the partition of X, so that any alternative can be chosen on the basis of $\mathcal{M}_\wedge$.

15.2.3. Concordance Analysis on the Basis of a Priori Preferences

15.2.3.1. A priori, and a posteriori preferences

It should be noticed that the situation under discussion is potentially contradictory. Indeed, if *a priori* preferences, represented by a f.s. apr, considerably differ from the resultant *MFC*, then we come to a conflict between apr, on the one hand, and sgc, sfn, on the other hand. This conflict leads to a natural conclusion that either a decision-maker is not enough competent in the problem or the data contained in $\mathbb{D}$ are incomplete, so that apr was actually based on some implicit reasoning. In what follows, we propose a model of Concordance Analysis based on the successive study of this controversy.

Given a *MCDP* $\mathbb{D}$, let us suppose that, on the basis of aggregation rules, we determine a preference relation R, which is studied by means of a *FDDP* P. We define the *measure of concordance* between *a priori* and *a posteriori* preferences as a fuzzy notion con($\mathbb{D}$), $\mathrm{con}(\mathbb{D}) = \mathrm{con}(\mathrm{apr}, R, P) = \mu_{p(R)}(\mathrm{apr})$ (in

Table 15.2

Primary Data for a Multicriteria Decision Problem

Attributes (C_i)		Alternatives (x_j)					
denomination	significance (*sgc*)	x_1: 2B+T+M+R	x_2: 3B+T+R	x_3: 2B+E+T+M	x_4: 2B+E+T+R	x_5: 3B+2R	x_6: 3B+R
I. Quantitative - calculated *relative* values							
C_1: specific layout	0.3	6	0	64	46	100	50
C_2: cost price	0.8	0	7	89	94	100	62
C_3: labour productivity	0.8	0	0	100	96	98	75
C_4: labour mechanization level	0.7	0	0	100	97	99	71
C_5: specific metal content	0.4	0	15	33	48	100	65
C_6: cpecific energy capacity	0.6	1	0	95	100	57	26
II. Qualitative - decision-maker's estimate							
C_7: equipment unification	0.4	sf	gd	isf	isf	gd	exc
A priori preferences of decision-maker (*apr*)		0.6	0.5	0.7	0.8	0.3	0.5

Abbreviations:

B - bulldozer, T - tip-lorry, M - motor-grader, R - roller,

E - excavating machine;

isf - insufficient, sf - sufficient, gd - good, exc - excellent;

Calculation accuracy: C_1-C_3 - to 10%, C_4-C_6 - to 4%
(thus, x_1 and x_2 are indistinguishable due to C_1, C_2, and C_6)

Table 15.3

Membership function of $\mathcal{M}_{\wedge}$: $\mathcal{M}_{\vee}$

Alter-natives	Alternatives					
	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	0.4:0.4	0.6:0.8	0.6:0.8	0.6:0.8	0.6:0.8
x_2	0.4:0.4	0	0.6:0.8	0.6:0.8	0.6:0.8	0.6:0.8
x_3	0.7:0.8	0.7:0.8	0	0.4:0.6	0.4:0.8	0.7:0.7
x_4	0.7:0.8	0.7:0.8	0.3:0.3	0	0.4:0.4	0.7:0.8
x_5	0.7:0.8	0.7:0.8	0.4:0.6	0.4:0.6	0	0.7:0.8
x_6	0.7:0.8	0.7:0.8	0.6:0.8	0.6:0.8	0.6:0.8	0

Kitainik [2], another measure was proposed). This notion of concordance can be interpreted as a degree of "anticipation" by the decision-maker of the result of "objective analysis". In this sense, a f.s. con is considered as an empiric evaluation of decision-maker's competence in a specific problem $\mathbb{D}$.

One can also discover analogy between dichotomousness and "resoluteness": the more polarized are decision-maker's *a priori* preferences, the more resolute is his position. So, we can define fuzzy notion res (resoluteness) on a class of *MCDP*'s: $\text{res}(\mathbb{D})=\delta(\text{apr}(\mathbb{D}))$. Let us consider the case when P is $(\circ, I_5)$-von Neumann - Morgenstern Solution. An "ideal" decision-maker should select χ_K with $K\in\mathfrak{K}^*$, thus achieving $\text{con}(\mathbb{D})=\mu^*(p,R,\mathfrak{S})$, $\text{res}(\mathbb{D})=1$. However, a softer estimate $\text{con}(\mathbb{D})=\mu^*(p,R,\mathfrak{S})$, $\text{res}(\mathbb{D})\in[\delta(p,R,\mathfrak{S}),1]$ is also completely compatible with the data contained in $\mathbb{D}$.

In case of indeterminate aggregated relation, $R\in\mathcal{R}\subseteq\tilde{\mathcal{P}}(X^2)$ (say, with fuzzy majority relations), we define $\text{con}_* = \bigwedge_{R\in\mathcal{R}} \text{con}(\text{apr},R,P)$, $\text{con}^* = \bigvee_{R\in\mathcal{R}} \text{con}(\text{apr},R,P)$, and, given a concordance level γ, we consider a γ-cut of con with respect to Preference Domain, that is, a subdomain of preference relations $\mathcal{R}^*_{\gamma}=\{R\in\mathcal{R}\,|\,\text{con}(\text{apr},R,P)\geq\gamma\}$. The corresponding interval

$$\text{res}_{\gamma}=[\ \bigwedge_{R\in\mathcal{R}^*_{\gamma}} \delta(P,R,\mathfrak{S}),\ \bigvee_{R\in\mathcal{R}^*_{\gamma}} \delta(P,R,\mathfrak{S})\ ]$$

indicates the range of well founded resoluteness, which is compatible with decision-maker's competence.

Example 15.2. For a *MCDP* described in the previous example, $con^*=0.4$ corresponds to $K=\{x_1,x_4\}$, yielding a pointwise estimate res=0.2, and a dichotomous choice $0.6|_K:0.4|_{\bar{K}}$. However, both the resoluteness and the competence of decision-maker are under question. Indeed, an "ideal decision-maker" can determine apr as $\chi_{\{x_6\}}$, thus achieving $con^*=1.0$, res=0.6, and a dichotomy $0.8|_{\{x_6\}}:0.2|_{\overline{\{x_6\}}}$. Therefore, one can hardly regard the original estimates (sgc,apr) as a well-defined couple ∎

15.2.3.2. The "Retrospective picture" of decision-maker

In our opinion, joint analysis of a pair of estimates $(con(\mathbf{D}),res(\mathbf{D}))$ in comparison with the same pair of estimates achieved by an *ideal decision-maker* opens interesting perspectives in the study of psychology of decision-making. Thus, studying mutual disposition of empiric, and of ideal estimates, we discovered two polar patterns of decision-maker's behavior (see Alyabjev, Kitainik, and Perelmuter [1]):

- a "self-assured" decision-maker ($con(\mathbf{D})$ is considerably smaller than $\mu^*(p,R,\mathcal{E})$, whereas $res(\mathbf{D})$ is much greater than $\delta(p,R,\mathcal{E})$);
- a "hesitating" decision-maker (the smaller is $res(\mathbf{D})$, remaining in the admissible bounds $[\delta(p,R,\mathcal{E}),1]$, the greater is $con(\mathbf{D})$).

Using the above technique of estimating competence and resoluteness of decision-maker based on the study of his interaction with an applied *DSS*, one can systematically observe the quality of decisions achieved by decision-maker in diverse problems, thus moving to a "retrospective picture" of decision-maker.

This approach can be used for decision analysis as well as for training, and also for specifying decision problems that can be successfully solved by a decision-maker.

15.3. EXPERT ASSISTANT FICCKAS †

In this section, we briefly discuss a computer system FICCKAS - Fuzzy Information Cluster, Choice, and Knowledge Acquisition System designed for involved structural analyses of information of a mixed quantitative/ qualitative nature and of a numeric/linguistic form in many applied fields. Such information can be interpreted in terms of "measurements" of two types: empiric observations and expert judgments. This interpretation in turn naturally implies the use of the unifying concept of the measurement theory - that of scale (Pfanzagl [1]). The application of this concept to a real-world problem logically leads to the simultaneous consideration of a variety of interrelated scales.

The principal paradigm underlying the concept of the system considered is based on the understanding that any applied comprehensive analysis of the above wide classes of data should not *a priori* be formulated as a problem of a specific class, for example, as a problem of choice, clustering, pattern recognition, knowledge acquisition, etc. These classes are, in fact, not classes of problems but rather of methods that help to view informational situations analyzed from different alternative perspectives. Therefore, in real-life problems it appears natural to dynamically combine these methods in a common informational framework thus forming higher level data processing technologies. In our opinion, an advanced Expert Assistant System should comply with this concept. Certainly, developing such combined technologies is itself a research problem, and the system outlined here can also be used for such type of research.

Analytic procedures implemented in this system involve a scope of methods of clustering, choice, and knowledge acquisition with fuzzy data (see Kitainik, Orlovski, Roubens [1] for details).

15.3.1. Data Representation

Data analyzed have the form of OBJECT/ATTRIBUTE relations. "Measurements" of objects with respect to each attribute can be expressed in a number of scales of different types: continuous/discrete, numeric/linguistic, nominal/ordered, crisp/fuzzy.

Correspondences between scales are described as fuzzy relations resulting from representing linguistic gradations as trapezoidal fuzzy

† This section was written in collaboration with Sergei Orlovski.

numbers. This type of correspondence is used for converting values from numeric to linguistic scales. It is also applied to "translate" similarity/preference relations from one scale to another.

15.3.2. Clustering

Clustering involves the following basic steps:

(1) Building similarity relations for attributes based on pairwise comparisons of fuzzy subsets. The following similarity measures are implemented: Hamming, Baldwin-Pilsworth (see Norris, Pilsworth, Baldwin [1]), Max-Min. In case of objects, local similarities (with respect to separate attributes) are first constructed taking into account types of respective scales with their subsequent aggregation. The aggregation can be based on the use of atomic and non-atomic significance measures on a specified set of attributes (scales) which allow for a variety of types of resultant similarity relations, close to that presented in Table 15.1.

(2) Two search algorithms are included in the system: single connection method, aimed at building classical dendrogram related to level sets of the similarity relation; complete connection method, applied to level sets of the similarity relation. While the component method produces a unique partition of the support, the complete connection (maximal clique) method requires additional analysis: (a) selecting clusters of a desired size; (b) eliminating dominated clusters; (c) determining cluster priorities for constructing coverings or partitions.

(3) Constructing coverings/partitions by clusters using dissimilarity relation between primary clusters.

15.3.3. Choice

The following basic steps are included here: (1) constructing partial preferences, (2) constructing aggregated preference, (3) constructing resulting choices and/or rankings. Preferences can be treated in two different ways: using concepts of optimality, and using binary preference relations.

(1) Fuzzy optimality with respect to a scale of an attribute is associated with "fuzzy optimal value" constructed using max-min or implication-based extensions. Preference relation with respect to a scale semantically reflects the formula: "optimal value is maximum/minimum".

(2) The rules for aggregating local optimalities and preference

relations follow the ideas of subsection 15.2.1.

(3) In case of an aggregated optimality, fuzzy ranking (choice) is defined by the membership function of the aggregated fuzzy goal set. In case of an aggregated preference relation, *FDDP*'s are used.

15.3.4. Knowledge acquisition

This type of analysis is based on the use of combinatorial algorithms and designed for the generation of logical rules explaining differences between classes of objects (considered as alternative diagnoses or more general outcomes). Resultant rules can be incorporated in knowledge bases for expert systems, or used for the classification of new objects. Three basic steps are included here:

- generation of collections of primary rules with their validity estimates;
- selection of collections of "efficient" primary rules;
- verification of resultant rules with test objects.

15.3.5. Design, Software, and Applications

FICCKAS includes four basic units: EDITOR, CLUSTER, CHOICE, and KNOWLEDGE. The interface between these units is based on a two-level file system including Data Files, and Protocols. Data Files contain OBJECT/ATTRIBUTE relations, and interdependencies between scales. Protocols represent a wide scope of "viewpoints" of the data corresponding to intermediate or final results of analyses in each functional unit. A multi-screen User Interface provides numerous flexible facilities for loading, processing and saving all types of data implemented in the system.

The current version FICCKAS 2.1 is implemented for IBM AT and compatibles. Source code includes about 100000 instructions.

Among most interesting applications of FICCKAS, we mention medical diagnostics, and the study of inheriting excellent properties of racing horses. The authors were pleasantly surprised by an enthusiasm of experts in the latter field: after a short training period, they used FICCKAS independently and succeeded in solving diverse practical and scientific problems in horse breeding.

Of course, this broad outline cannot exhaust the features of this extended software system. We plan to devote a special monograph to FICCKAS.

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