Fuzzy Relations

MSc IS/ISR

Overview

- Relations can be used to model dependencies, correlations or connections between variables, quantities or attributes.
- We generalize the concept of relations to fuzzy relations. The extension principle is key in this process.
- Fuzzy relations are useful for representing and understanding fuzzy controllers that describe a vague connection between input and output values.
- Furthermore, we can establish an interpretation of fuzzy sets and membership degrees on the basis of special fuzzy relations called similarity relations.
- This interpretation plays a crucial role in the context of fuzzy controllers.

Objectives

- To know the concept of a fuzzy relation
- To learn how to compose fuzzy relations

Crisp Relations I

Before we introduce the definition of a fuzzy relation, we briefly review the fundamental concepts and mechanisms of crisp relations that are needed for understanding fuzzy relations.

Example 1: A house has six doors and each of them has a lock which can be unlocked by certain keys.

- Let the set of doors be $T = \{t_1, t_2, t_3, t_4, t_5, t_6\}$, and the set of keys S$= \{s_1, s_2, s_3, s_4, s_5\}$.
- The relation between doors and keys is:
  - key s_5 is the main key and fits to all doors; key s_1 fits only to door t_1; key $s_2$fits to t_1 and t_2; s_3 fits to t_3 and t_4, s_4 to t_5.

This situation can be formally described by the relation $R \subseteq S \times T$ ("fits to"). The pair $(s, t) \in S \times T$ is an element of R if and only if key s fits to door t

- This means that

$$R = \{(s_1, t_1), (s_2, t_1), (s_2, t_2), (s_3, t_3), (s_3, t_4), (s_4, t_5), (s_5, t_1), (s_5, t_2), (s_5, t_3), (s_5, t_4), (s_5, t_5), (s_5, t_6)\}$$

Crisp Relations II

Another way of describing the relation R is shown in the Table below.

The relation R: "key fits to door"

R	t_1	t_2	t_3	t_4	t_5	t_6
s_1	1	0	0	0	0	0
s_2	1	1	0	0	0	0
s_3	0	0	1	1	0	0
s_4	0	0	0	0	1	0
s_5	1	1	1	1	1	1

The entry 1 at position (s_i, t_j) indicates that $(s_i, t_j) \in R$ holds, i.e. key s_i opens door t_j.

0 stands for $(s_i, t_j) \notin R$, i.e key s_i does not open door t_j.

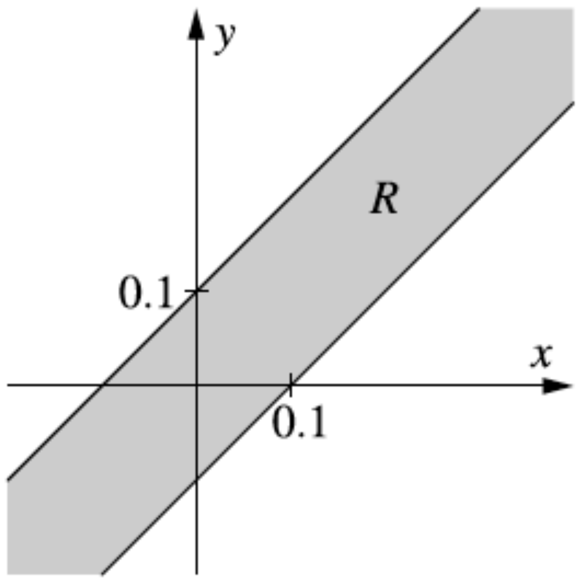
Crisp Relations III

Example 2: Let us consider a measuring instrument which can measure a quantity $y \in \mathbb{R}$ with a precision of ± 0.1.

- If x_0 is the measured value, we know the true value y_0 lies within the interval $[x_0 - 0.1, x_0 + 0.1]$.
- This can be described by the relation

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \leq 0.1\}$$

A graphical representation of this relation is given



The relation $y \hat{=} x \pm 0.1$

Crisp Relations IV

Mappings or their graphs can be considered as special cases of relations.

- If the function $f : X \rightarrow Y$ is a mapping of X to Y, the graph of f is the relation

$$graph(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y .$$

To be able to interpret a relation $R \subseteq X \times Y$ as a graph of a function we need that for each $x \in X$ there exists exactly one $y \in Y$ such that the pair (x, y) is contained in R.

Application of Relations and Deduction I

- So far we have used relations in a merely descriptive way.
- But similar to functions, relations can also be applied to elements or sets.
  - If $R \subseteq X \times Y$ is a relation between the sets X and Y and $M \subseteq X$ is a subset of X, the image of M under R is the set

$$R[M] = \{y \in Y \mid \exists x \in X : (x, y) \in R \text{ and } x \in M\} \quad (1)$$

  - $R[M]$ contains the elements from Y which are related to at least one element of the set M.

Application of Relations and Deduction II

If $f : X \rightarrow Y$ is a mapping, then applying the relation $graph(f)$ to a one-element set $\{x\} \subseteq X$ we obtain the one-element set which contains the image of x under the function f:

$$graph(f)\{x\}=f(x).$$

More generally, for arbitrary subsets $M \subseteq X$ we have

$$graph(f)[M] = f[M] = \{y \in Y \mid \exists x \in X : x \in M \wedge f(x) = y\}$$

Application of Relations and Deduction III

Example 3: Now we use the relation R from Example 1 in order to determine which doors can be unlocked if we have keys $\{s_1, s_2, s_3, s_4\}$.

All we have to do is to calculate all elements (doors) which are related (relation “fits to”) to at least one of the keys in $\{s_1, s_2, s_3, s_4\}$.

The set of doors we want to know is

$$R[\{s_1, s_2, s_3, s_4\}] = \{t_1, t_2, t_3, t_4, t_5\}$$

Application of Relations and Deduction IV

Example 4: We follow up Example 2 and assume that we have the information that the measuring instrument indicated a value between *0.2* and *0.4*.

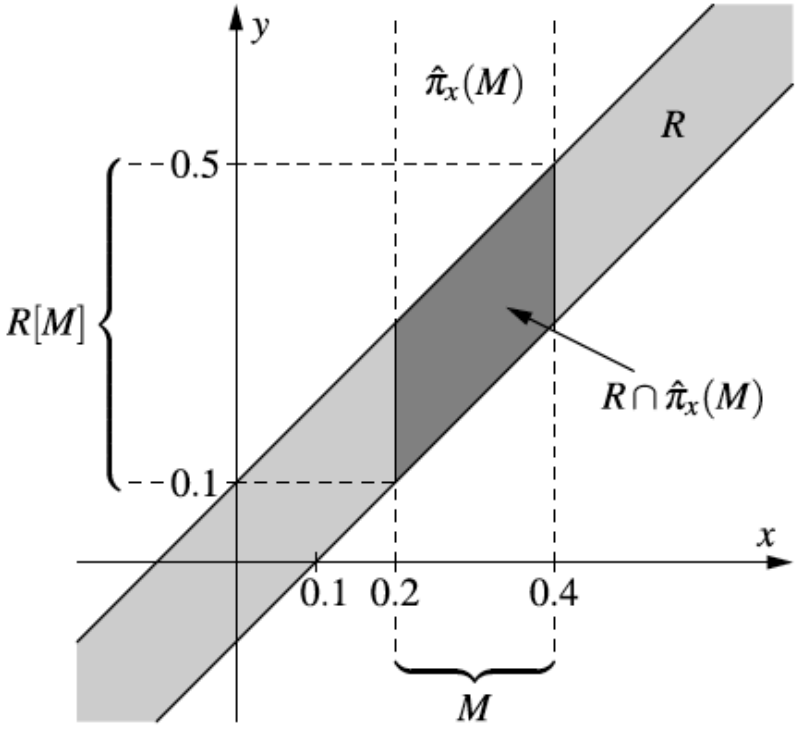
From this we can conclude that the true value is contained in the set

$$R[[0.2, 0.4]] = [0.1, 0.5]$$

The figure shows how to obtain the set *R[M]:* is the projection of the intersection of the relation, R, with the cylindrical extension of the set *M*, $\hat{\pi}_x(M)$:

$$R[M] = \pi_y(R \cap \hat{\pi}_x(M))$$

Note that $\hat{\pi}_x(M) = \{(x, y) \in X \times Y | x \in M\}$



How to determine the set $R[M]$ graphically

Application of Relations and Deduction V

Example 5: Logical deduction based on an **implication** of the form

$$x \in A \ \rightarrow y \in B$$

can be modelled and computed by relations, too.

All we have to do is to encode the rule $x \in A \ \rightarrow y \in B$ by the relation

$$R = \{(x, y) \in X \times Y | x \in A \ \rightarrow y \in B\} = (A \times B) \cup (\bar{A} \times Y) \quad (3)$$

where *X* and *Y* are the sets of possible values that *x* and *y* can attain.

Application of Relations and Deduction V

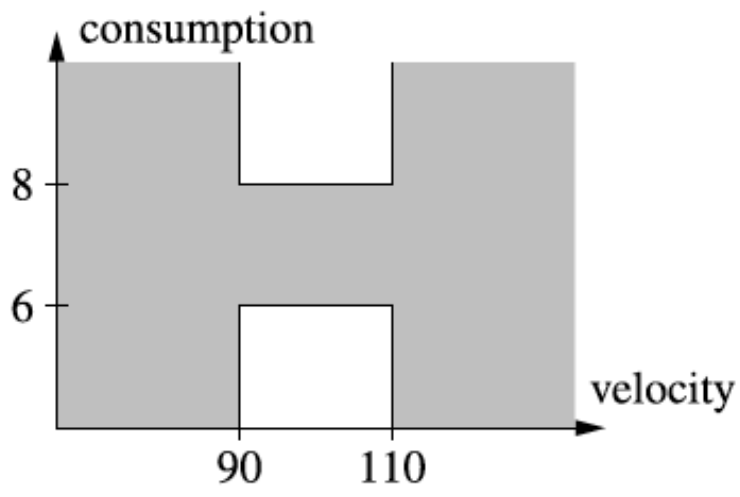
The rule

"If the velocity is between 90 km/h and 100 km/h, then the fuel consumption is between 6 and 8 litres"

as a logical formula is

$$v \in [\,90,\; 110] \rightarrow b \in [\,6,\; 8]$$

and it has the relation shown in the right figure.



Relation for the rule $v \in [90, 110] \rightarrow b \in [6, 8]$

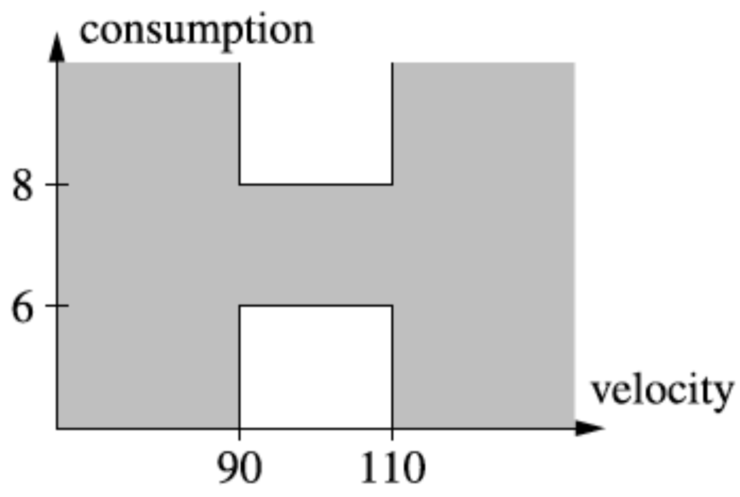
Application of Relations and Deduction V

If we know that the velocity has the value v, in the case *$90 \leq v \leq 110$* we can conclude that for the consumption b we must have *$6 \leq b \leq 8$*.

Otherwise, and without further knowledge and any pieces of information than just the rule and the value for *v*, we cannot say anything about the value for the consumption which means that we obtain $b \in [0, \infty)$.

This relation applied to the set M becomes:

$$R[M] = \begin{cases} [6,8] & if\ M = [90,110], \\ \emptyset & if\ M = \emptyset, \\ [0,\infty) & otherwise. \end{cases}$$



Relation for the rule $v \in [90, 110] \rightarrow b \in [6, 8]$

Chains of Deductions – Composition of Relations I

- The previous example shows how logical deduction can be represented in terms of a relation.
- Inferring new facts from rules and known facts usually means that we deal with chained deduction steps in the form of:

From

$\phi_1 \rightarrow \phi_2$ and $\phi_2 \rightarrow \phi_3$

we can derive

$\phi 1 \rightarrow \phi 3$.

- A similar principle can be formulated in the context of relations using the concept of composition of relations

Chains of Deductions – Composition of Relations II

- Consider the relations $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$.
  - An element $x \in X$ is indirectly related to an element $z \in Z$ if there exists an element $y \in Y$ such that x and y are in the relation R_1 and y and z are in the relation R_2 .
  - We can say that we go from x to z via y.
- This composition of the relations R_1 and R_2 define a relation between X and Z

$$R_2 \circ R_1 = \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in R_1 \land (y, z) \in R_2\} \quad (4)$$

- Then, we have for all $M \subseteq X$

$$(R_2 \circ R_1)[M] = R_2\big[R_1[M]\big]$$

Chains of Deductions – Composition of Relations III

Example 6: We extend Example 1 of the keys and doors by considering a set $P = \{p_1, p_2, p_3\}$ of three people owning various keys.

This is expressed by the relation

$$R' = \{(p_1, s_1), (p_1, s_2), (p_2, s_3), (p_2, s_4), (p_3, s_5)\} \subseteq P \times T.$$

$(p_i, s_j) \in R'$ means that person p_i owns the key s_j.

Chains of Deductions – Composition of Relations IV

Example 6: The composition of the relations *R* and *R'* contains the pair *(p, t)* $\in$ *P x T* if and only if person *p* can unlock door *t*.

$$R{\circ}R' = \{(p_1, t_1), (p_1, t_2), (p_2, t_3), (p_2, t_4), (p_2, t_5),$$

$$(p_3, t_1), (p_3, t_2), (p_3, t_3), (p_3 t_4), (p_3, t_5), (p_3, t_6)\}$$

Using the relation $R{\circ}R'$ we can determine which doors can be unlocked if the people p_1 and p_2 are present.

$$(R{\circ}R')[\{p_1, p_2\}] = R[R'[\{p_1, p_2\}]]=\{t_1, t_2, t_3, t_4, t_5\}$$

Chains of Deductions – Composition of Relations V

Example 7: In Example 2, we used the relation

$$R = \{(x, y) \in \mathbb{R}\times\mathbb{R} \,;\, |x - y| \leq 0.1\}$$

to model the fact that the measured value x represents the true value y with a precision of 0.1.

When we can determine the quantity z from the quantity y with a precision of 0.2, we obtain the relation

$$R' = \{(y, z) \in \mathbb{R}\times\mathbb{R} \,;\, |y - z| \leq 0.2\}.$$

Chains of Deductions – Composition of Relations VI

Example 7: The composition of R and R' results in the relation

$$R' \circ R = \{(x, z) \in \mathbb{R}\times\mathbb{R}\ ;\ |x - z| \leq 0.3\}.$$

If the measuring instrument indicates the value x_0, we can conclude that the value of the quantity z is in the set

$$(R' \circ R)\{x_0\} = [x_0 - 0.3, x_0 + 0.3].$$

Chains of Deductions – Composition of Relations VII

Example 8: Previous Example 5 demonstrated how an implication of the form

$$x \in A \rightarrow y \in B$$

can be represented by a relation.

When another rule

$$y \in C \rightarrow z \in D$$

is known, in the case of $B \subseteq C$ we can derive the rule $x \in A \rightarrow z \in D$.

Otherwise, knowing x does not provide any information about z in the context of these two rules.

Chains of Deductions – Composition of Relations VII

Example 8: Composition of the relation *R'* representing the implication

$$x \in A \rightarrow y \in B$$

and the relation *R* representing the implication

$$y \in C \rightarrow z \in D$$

becomes the relation $R' \circ R$ for the implication

$$x \in X \rightarrow z \in Z$$

$$R' \circ R = \begin{cases} (A \times D) \cup (\bar{A} \times Z) & if\ B \subseteq C, \\ (A \times Z) \cup (\bar{A} \times Z) = X \times Z & otherwise. \end{cases}$$

Fuzzy Relations I

A fuzzy relation is a generalized crisp relation where two elements can be gradually related to each other.

Definition 1: A fuzzy set $\rho \in \mathcal{F}(X \times Y)$ is called a fuzzy relation between the reference sets X and Y.

The greater the membership degree $\rho(x, y)$ the stronger is the relation between x and y.

Fuzzy Relations II

Example 9: Set *X = {s, f, e}* denote a set of financial funds, devoted to *shares (s)*, *fixed-interest stocks (f)* and *real estates (e)*.

Set *Y = {l, m, h}* contains the elements *low (l), medium (m)* and *high (h)* risk.

The fuzzy relation $\rho \in \mathcal{F}(X \times Y)$ in the right-hand side table shows for every pair

$$(x, y) \in X \times Y$$

how much the fund x is considered having the risk factor y.

$$\rho(e, m) = 0.5$$

means the fund dedicated to real estates is considered to have a medium risk with a degree of 0.5

ρ	l	m	h
s	0.0	0.3	1.0
f	0.6	0.9	0.1
e	0.8	0.5	0.2

Fuzzy Relations III

Example 10: The measuring instrument from Example 2 had a precision of ± 0.1.

It is not very realistic to assume that, given that the instrument shows the value x_0, all values from the interval $[x_0 - 0.1, x_0 + 0.1]$ are equally likely to represent the true value of the measured quantity, y.

Instead of the crisp relation R from Example 2 for representing this fact, we could use a fuzzy relation to model the above.

For instance

$$\rho(x, y) = 1 - \min\{10|x - y|, 1\}$$

yields the membership degree of 1 for when the reading a and the true value y are equal $x = y$.

For other measurement reading of the instrument the membership degree to the relation decreases linearly with increasing distance $|x - y|$ until the difference between x and y exceeds the value 0.1.

Image of Fuzzy Set under a Fuzzy Relation I

We extend Eq. (1) to the framework of fuzzy sets and fuzzy relations by applying the Extension Principle.

Definition 2: For a fuzzy relation $\rho \in \mathcal{F}(X \times Y)$ and a fuzzy set $\mu \in \mathcal{F}(X)$, the image of μ under ρ is the fuzzy set $\rho[\mu] \in \mathcal{F}(Y)$

$$\rho[\mu](y) = \sup_{x \in X} \min\{\rho(x, y), \mu(x)\} \qquad (5)$$

Image of Fuzzy Set under a Fuzzy Relation II

Example 11: On the basis of the fuzzy relations from Example 9, we want to estimate the risk of a mixed fund which concentrates on shares but also invests a smaller part of its money into real estates.

We can represent this mixed fund over $\{s, r, f\}$ as the following fuzzy set μ

$$\mu(s) = 0.8,\ \mu(f) = 0,\ \mu(r) = 0.2.$$

Image of Fuzzy Set under a Fuzzy Relation III

Example 11: To determine the risk of this mixed fund, we compute the image of the fuzzy set μ under the fuzzy relation ρ.

In other words, we apply definition 2

$$\rho[\mu](l) = 0.2,\ \rho[\mu](m) = 0.3,\ \rho[\mu](h) = 0.8.$$

- Notice: The calculation of the image of a fuzzy set μ under a fuzzy relation ρ is similar to multiplication of a matrix with a vector where the multiplication of the components is replaced by the minimum and the addition by the maximum.

Image of Fuzzy Set under a Fuzzy Relation IV

Example 12: We have the information that the measuring instrument from Example 10 indicated a value of "*about 0.3*".

This is assumed to be represented by the triangular fuzzy set

$$\mu = \Lambda_{0.2,0.3,0.4}$$

The image of the fuzzy set $\mu = \Lambda_{0.2,0.3,0.4}$ under the relation $\rho(x, y) = 1 - \min\{10|x - y|, 1\}$ from Example 10 gives the fuzzy set of the true value y

$$\rho[\mu](y) = 1 - \min\{10|y - 0.3|, 1\}$$

Image of Fuzzy Set under a Fuzzy Relation V

Example 13: Example 5 illustrated how logic deduction on the basis of an implication of the form $x \in A \rightarrow y \in B$ can be represented using a relation.

$$R = \{(x, y) \in X \times Y | x \in A \rightarrow y \in B\} = (A \times B) \cup (\bar{A} \times Y)$$

We generalize this method for the case that the sets *A* and *B* are replaced by the fuzzy sets μ or *v*.

- Using the equivalence $\left[[(x, y) \in \rho]\right] \equiv [[x \in \mu \rightarrow y \in v]]$
- If we choose the Gödel implication as truth function for the implication

$$w_{\rightarrow}(\alpha, \beta) = \begin{cases} 1 & if\ \alpha \leq \beta \\ \beta & otherwise \end{cases},$$

then we define the fuzzy relation

$$\rho(x, y) = \begin{cases} 1 & if\ \mu(x) \leq v(y), \\ v(y) & otherwise \end{cases}$$

Image of Fuzzy Set under a Fuzzy Relation IV

If "*about 2*" is modelled with the triangular fuzzy set $\mu = \Lambda_{1,2,3}$ and "*about 3*" by the triangular fuzzy set $v = \Lambda_{2,3,4}$, then the rule

"If *x* is *about 2*, then *y* is *about 3*"

is modelled with the fuzzy relation

$$\rho(x,y) = \begin{cases} 1 & if\ 1 - \min\{|3-y|,1\} \leq |2-x|, \\ 1 - \min\{|3-y|,1\} & otherwise \end{cases}$$

Knowing "*x is about 2.5*" represented by the fuzzy set $\mu' = \Lambda_{1.5,2.5,3.5}$ we obtain for y the fuzzy set

$$\rho(x,y) = \begin{cases} y - 1.5 & 2 \leq y \leq 2.5 \\ 1 & 2.5 \leq y \leq 3.5 \\ 4.5 - y. & 3.5 \leq y \leq 4 \\ 0.5. & otherwise \end{cases}$$

Composition of Fuzzy Relations I

We define the composition of fuzzy relations based on Eq. (4)

$$R_2 \circ R_1 = \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in R_1 \land (y, z) \in R_2\}$$

that describes composition in the case of crisp relations.

This is again done by applying the Extension principle.

Definition 3: Let $\rho_1 \in \mathcal{F}(X \times Y)$ and $\rho_2 \in \mathcal{F}(Y \times Z)$ be fuzzy relations. The composition of the two fuzzy relations is the fuzzy relation $\rho_2 \circ \rho_1 \in \mathcal{F}(Y \times Z)$

$$(\rho_2 \circ \rho_1)(x, z) = \sup_{y \in Y} \min\{\rho_1(x, y), \rho_2(y, z)\} \quad (7)$$

Composition of Fuzzy Relations II

Example 14: Let us come back to Example on the risk of financial funds.

We extend the risk estimation of funds with set

Z = {hl, ll, lp, hp}

with elements standing for

"high loss", "low loss", "low profit", "high profit"

and right-hand side fuzzy relation

$\rho' \in \mathcal{F}(Y \times Z)$

determines for each tuple *(y, z)* the possibility to have a profit or loss of *z* under the risk *y*.

ρ'	hl	ll	lp	hp
l	0.0	0.4	1.0	0.0
m	0.3	1.0	1.0	0.4
h	1.0	1.0	1.0	1.0

Composition of Fuzzy Relations III

The fuzzy relation resulting from the composition of the fuzzy relations ρ and ρ' is shown on the right hand side.

For the mixed fond from Example 11 which was represented by the fuzzy set μ:

$$\mu(s) = 0.8, \mu(f) = 0, \mu(r) = 0.2$$

we obtain

$(\rho' \circ \rho)[\mu](hl) = 0.8;$

$(\rho' \circ \rho)[\mu](ll) = 0.8;$

$(\rho' \circ \rho)[\mu](lp) = 0.8;$

$(\rho' \circ \rho)[\mu](hp) = 0.8$

as fuzzy set describing the possible profit or loss

ρ'	hl	ll	lp	hp
s	1.0	1.0	1.0	1.0
f	0.3	0.9	0.9	0.4
r	0.3	0.5	0.8	0.4

Summary

- In this session we have extended the concept of crisp relations and their composition to the case of fuzzy sets.

- In doing so, the Extension Principle was applied.

- We have seen how logical deduction based on an implication can be modelled with a relation.

- This plays a crucial role the context of fuzzy controllers.