FUZZY LOGIC MATHEMATICS FOR FUZZY LOGIC

Professor Francisco Chiclana¹

¹Institute of Artificial Intelligence School of Computer Science and Informatics Faculty of Computing, Engineering and Media De Montfort University – UK



- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

DEFINITION

A set is a collection of things that can be distinguished from one another as individuals and that share some property.

Each individual in this collection is called a member, or element, of the set.

NOTATION

a is an element of (or belongs to) the set A: $a \in A$

a does not belong to A: $a \notin A$

5/44

A set can be described in different ways:

LIST METHOD: Enumerate the elements that belong to the set (finite sets)

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

RULE METHOD Describe the set analytically by stating conditions for membership

$$A = \{x | P(x)\}$$

"the set C is composed of elements x, such that (every) x has the property P "

$$A = \{x | 0 \le x \le 1\}$$

Universal and Empty Sets

UNIVERSAL SET Set that consists of all the individuals that are of interest in the context of a particular application

 Empty Set Which contains no elements and is denoted by the symbol \emptyset

SET INCLUSION

 $A\subseteq B\Leftrightarrow {\sf every}\ {\sf element}\ {\sf of}\ A$ is also an element of B

$$A\subseteq B\Leftrightarrow [\forall x\in A\Rightarrow x\in B]$$

$$A=B\Leftrightarrow [A\subseteq B\subseteq A]$$

$$A \subset B \Leftrightarrow [A \subseteq B \land A \neq B]$$

POWER SET

The set which consists of all possible subsets of a given set \boldsymbol{X}

$$\mathcal{P}(X) = \{A | A \subseteq X\}$$

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

9/44

The complement of a set A, denoted by \overline{A} is the set of all the elements in the universal set X which are not in A

$$\overline{A} = \{x \in X | x \notin A\}$$

$$\overline{\overline{A}} = A$$

$$\overline{X} = \emptyset$$

$$\overline{\emptyset} = X$$

The union of set A and set B is the set containing all the elements belonging to A or B, or to both

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

$$A \cup X = X$$
If $A \subseteq B \Longrightarrow A \cup B = B$

$$A \cup A = A$$

$$A \cup \overline{A} = X$$

The intersection of set A and set B is the set containing all the elements belonging to both sets A and B simultaneously

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

Two sets are A and B are disjoints if they have no element in common, i.e. $A \cap B = \emptyset$

$$A\cap X=A$$
 If $A\subseteq B\Longrightarrow A\cap B=A$
$$A\cap A=A$$

$$A\cap \overline{A}=\emptyset$$

The difference of sets A and B is a set that consists of all the elements which belong to A, but which do not belong to B

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

$$A - B \neq B - A$$

$$X - A = \overline{A}$$

$$A \subseteq B \Longrightarrow A - B = \emptyset$$

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

Commutativity

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

DISTRIBUTIVITY

$$A\cap (B\cup C)=(A\cap B)\cup (A\cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DE MORGAN'S LAWS

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

1 Classical Set Theory

- Concept of a Set
- Set Operations
- Fundamental Properties
- Characteristic Functions
- Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

Another Set Representation Method

CHARACTERISTIC FUNCTION

A set $A = \{x | P(x)\}$ can be represented using the following characteristic function:

$$\delta_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

1 indicates membership (property P(x) is true) and 0 non-membership (property P(x) is false).

This method is particularly important in FL since, contrary to the other two methods, it can be generalised via the concept of membership function to fuzzy sets.

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions
- 2 Some Additional Mathematical Concepts
 - Real Numbers Intervals
 - Cartesian Product
 - Straight line: Equation
 - Interval Arithmetic

SET INCLUSION

Let X be the universal set:

$$A \subseteq B \Leftrightarrow \delta_A(x) \le \delta_B(x) \ \forall x \in X$$

$$A \cap B = \{x | \delta_A(x) = 1 \land \delta_B(x) = 1\}$$

In terms of the characteristic functions

$$\delta_{A \cap B}(x) = \min\{\delta_A(x), \delta_B(x)\}\$$

$$A \cup B = \{x | \delta_A(x) = 1 \lor \delta_B(x) = 1\}$$

In terms of the characteristic functions

$$\delta_{A \cup B}(x) = \max\{\delta_A(x), \delta_B(x)\}$$

Complement

$$\delta_{\overline{A}}(x) = 1 - \delta_A(x) \ \forall x \in X$$

Intersection – Union: Other options

Intersection

$$\delta_{A\cap B}(x) = \delta_A(x) \cdot \delta_B(x)$$
 product

$$\delta_{A \cap B}(x) = \max\{0, \delta_A(x) + \delta_B(x) - 1\}$$

Union

$$\delta_{A \cup B}(x) = \delta_A(x) + \delta_B(x) - \delta_A(x) \cdot \delta_B(x)$$

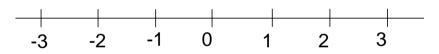
$$\delta_{A \cup B}(x) = \min\{1, \delta_A(x) + \delta_B(x)\}\$$

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

Real number set \mathbb{R}

Often represented by an x-axis with each point on the axis corresponding to a real number in $\mathbb R$



The set of all points between given points a and b on the real line ($a \le b$) is called an interval

- Closed $[a,b] = \{x \in \mathbb{R} | a \le x \le b\}$
- Open $(a, b) = \{x \in \mathbb{R} | a < x < b\}$ (or [a, b])
- Half-open

$$(a,b] = \{x \in \mathbb{R} | a < x \le b\} (or[a,b])$$

 $[a,b) = \{x \in \mathbb{R} | a < x < b\} (or[a,b])$

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic



The cartesian product of two real numbers is the two-dimensional Euclidean space, usually called a plane

Can be represented by the so-called Cartesian Coordinates or $\boldsymbol{x}-\boldsymbol{y}$ axes

Given A and B two sets, the cartesian product $A\times B$ is the set of *ordered* pairs composed from elements $a\in A$ and $b\in B$

In general $(a,b) \neq (b,a)$

When A=B notation used is: $A\times A=A^2$

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

33/44

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$
$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0} \cdot (x - x_0)$$

IN PARTICULAR

$$(x_0, 0), (x_1, 1) \Rightarrow y = \frac{x - x_0}{x_1 - x_0}$$

 $(x_0, 1), (x_1, 0) \Rightarrow y = \frac{x - x_1}{x_0 - x_1}$

Represent

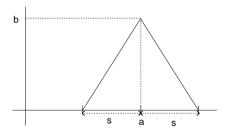
$$A(x) = \begin{cases} x - 5, & 5 \le x \le 6; \\ 7 - x, & 6 < x \le 7; \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate

- *A*(3)
- $\blacksquare A(5)$
- A(5.5)
- $\blacksquare A(6)$
- A(6.5)
- *A*(7)
- $\blacksquare A(9)$

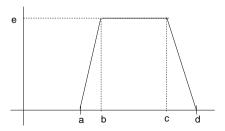
SYMMETRIC TRIANGULAR-SHAPED FUNCTION

$$A(x) = \begin{cases} b\left(1 - \frac{|x - a|}{s}\right), & a - s \le x \le a + s; \\ 0, & \text{otherwise.} \end{cases}$$



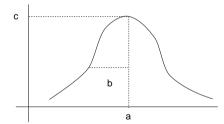
TRAPEZODIAL SHAPED FUNCTION

$$A(x) = \begin{cases} \frac{(a-x) \cdot e}{a-b}, & a \leq x \leq b; \\ e, & b \leq x \leq c; \\ \frac{(d-x) \cdot e}{d-c}, & c \leq x \leq d; \\ 0, & \text{otherwise.} \end{cases}$$



Bell Shaped function - Gaussian

$$A(x) = c \cdot e^{-\frac{(x-a)^2}{2b^2}}$$



Plot the following two functions

$$A(x) = \begin{cases} 1 - \frac{|x-3|}{2}, & 1 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

$$B(x) = \begin{cases} 1 - \frac{|x-4|}{2}, & 2 \le x \le 6; \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the analytic expression of

- $\overline{A}(x) = 1 A(x)$
- $\overline{B}(x) = 1 B(x)$
- \blacksquare min{A(x), B(x)}

- 1 Classical Set Theory
 - Concept of a Set
 - Set Operations
 - Fundamental Properties
 - Characteristic Functions
 - Set Operations and Characteristic Functions

- Real Numbers Intervals
- Cartesian Product
- Straight line: Equation
- Interval Arithmetic

CLASSICAL INTERVAL ARITHMETIC

42/44

To extend the elementary operations of addition, subtraction, multiplication, and division for interval-valued operands

$$[a_1, b_1] \star [a_2, b_2] = \{x_1 \star x_2 | a_1 \le x_1 \le b_1 \land a_2 \le x_2 \le b_2\};$$
$$\star \in \{+, -, \cdot, /\};$$

and the expression $[a_1,b_1]/[a_2,b_2]$ is not defined if $0\in [a_2,b_2]$.

CLASSICAL INTERVAL ARITHMETIC

To extend the elementary operations of addition, subtraction, multiplication, and division for interval-valued operands

Since

$$\star : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$$

is a continuous function defined in a compact set, then it takes its minimum and maximum values as well as all the values in between.

So, it is

$$[a_1, b_1] \star [a_2, b_2] = [\min B, \max B]$$

$$B = \{a_1 \star a_2, a_1 \star b_2, b_1 \star a_2, b_1 \star b_2\}.$$

If $a_1 = b_1$ and $a_2 = b_2$, then $\min B = \max B$ and interval arithmetic becomes number arithmetic.

CLASSICAL INTERVAL ARITHMETIC

To extend the elementary operations of addition, subtraction, multiplication, and division for interval-valued operands

$$[2,4] + [-3,-1] = [-1,3]$$

$$[2,4] - [-3,-1] = [3,7]$$

$$[2,4] \cdot [-3,-1] = [-12,-2]$$

$$[2,4]/[-3,-1] = [-4,-2/3]$$