FUZZY DECISION MAKING – OWA OPERATOR – SOFT MAJORITY

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- First Approach to Fuzzy Decision Making: Bellman and Zadeh's 1970 proposal
- More general approach based on Linguistic Quantifiers and OWA operator
- Soft majority concept
- Classical resolution process of a group decision making problem
- Concept of non-dominance degree

- Decision-making is a part of almost every conceivable human task
- It can be described in simple terms as a 'cognitive process consisting in selecting an alternative amid multiple options.'
- The aim is to choose *the best* alternative taking into account information available
- In many cases, we face 'simple' decision making problems
  - No need of any logical process to support us
  - We make 'good' decision
- However, many situations are not so simple
  - Helpful and important to have a decision support system
- What do we mean by a complex decision-making situation?
  - One in which we have a set of multiple criteria or objectives to meet

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IN MOST CASES

Comparison of different actions according to their desirability cannot be done using a single criterion or a unique expert

First proposal of Fuzzy Sets in decision making I

R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment," in Management Science vol. 17, no. 4, 1970, pp. 141–164

Three decision-making environments:

- *certainty* where information on each alternative is clearly known;
- *risk* where knowledge about alternatives is probabilistic;
- *uncertainty* where there exists non-probabilistic uncertainty about alternatives and they need to be evaluated in an approximate fashion.

Within uncertainty scenarios, *fuzzy decision-making* problems, where the information about the decision-making problem is vague and imprecise, has become a prolific research field in the last 50 years, primarily due to Zadeh's fuzzy set theory (1965) but mainly since the pioneering study on fuzzy decision-making by Bellman and Zadeh (1970).

First proposal of Fuzzy Sets in decision making II

R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment," in Management Science vol. 17, no. 4, 1970, pp. 141–164

- Bellman and Zadeh proposed Fuzzy Sets as a tool to develop and model multicriteria decision problem
- Relevant goals and constraints are expressed in terms of fuzzy sets over the set of decision alternatives
  - a set X of possible actions;
  - a set of goals $G_i (i = 1, \ldots, n)$, each of which is expressed in terms of a fuzzy set defined on X;
  - a set of constraints $C_i (i = 1, \ldots, m)$, each of which is also expressed by a fuzzy set defined on X.
- A decision is determined by an appropriate aggregation of these fuzzy sets

- Bellman and Zadeh suggested the use of an intersection operator
  - The the objective is to find the alternative that satisfies best (maximise) both goals and constraints
  - This means that goals and constraints are treated equally: 'symmetry between goals and constraints' assumption
  - Goals and constraints can be considered all at same level of criteria to use to find the best alternative
  - Implicitly implies all criteria be satisfied by a solution to a problem
- So, how do we find the best alternative in Bellman and Zadeh's fuzzy decision making?

- We have a set $X = \{x_1, \ldots, x_l\}$ of possible actions;
- We have a set $G = \{G_1, , \ldots, G_n\}$ of goals
  - $\mu_{G_i}(x_k)$ measures how well goal G_i is verified by alternative $x_k \in X$;
- We have a set $C = \{C_1, \ldots, C_m\}$ of constraints,
  - $\mu_{C_j}(x_k)$ measures how well C_j is verified by alternative $x_k \in X$.

1. $G(x_k) = \min\{\mu_{G_1}(x_k), \ldots, \mu_{G_n}(x_k)\}$ measures how well alternative $x_k \in X$ verifies all the goals;
2. $C(x_k) = \min\{\mu_{C_1}(x_k), \ldots, \mu_{C_m}(x_k)\}$ measures how well alternative $x_k \in X$ verifies all the constraints;
3. $D(x_k) = \min\{G(x_k), C(x_k)\}$ measures how well $x_k \in X$ verifies all the goals and all the criteria
4. The best alternative $x \in X$ is defined as the alternative with

$$D(x) = \max\{D(x_1), \ldots, D(x_l)\}.$$

- Bellman and Zadeh's fuzzy decision making model implicitly implies all criteria be satisfied by a solution to a problem
- This condition may not always be the appropriate relationship between the criteria
  - Decision maker may be satisfied if most of the criteria are satisfied
- Classical Logic only uses all and there exist
- Natural language is richer: almost all, most, many ...
- these are known as linguistic quantifiers
- There is a need to define Quantifier Guided Aggregation decision functions
- These aggregation operators allow to implement soft majority concept

ABSOLUTE such as 'about 10'
Fuzzy subsets of the non-negative reals

$$x \in \mathbb{R}^+ : Q(x)$$

degree to which x satisfies the concept conveyed by the linguistic quantifier Q

RELATIVE OR PROPORTIONAL such as 'most', 'few', 'about α'
Fuzzy subsets of the unit interval

$$y \in \mathbb{R}^+ : Q(y)$$

same meaning as above

RIM Increasing such as 'all', 'most', 'many', 'at least α'

RDM Decreasing such as 'at most one', 'few', 'none', 'at most α'

RUM Unimodal such as 'about α'

RIM QUANTIFIERS

The more criteria satisfied the better the solution

- Decision problem with n criteria $A_1, \ldots, A_n$
- X set of possible alternative solutions
- $x \in X \Longrightarrow A_i(x) \in [0,1]$ degree to which x satisfies criteria A_i
- Overall Evaluation or Decision Function

$$D(x) = Agg(A_1(x), ..., A_n(x))$$

- Best solution is that with highest overall evaluation

DECISION FUNCTION

$$Agg(A_1(x), \ldots, A_n(x)) = \min_i(A_1(x), \ldots, A_n(x))$$

DECISION FUNCTION

$$Agg(A_1(x),\ldots,A_n(x)) = \phi(A_1(x),\ldots,A_n(x))$$

ϕ Ordered Weighted Averaging (OWA) Operator

$$\phi \colon \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i \cdot b_i$$

1. $W = (w_1, \ldots, w_n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$
2. b_i is the i-highest value in the set $\{a_1, \ldots, a_n\}$

- Let $S_k = \sum_{j=0}^{k} w_j$

  Note that: $S_n = 1, S_0 = 0$
- Assume an input B such that $b_j = 1$ if $j \leq k$ and $b_j = 0$ if $j > k$
- This indicates that k criteria are completely satisfied and the rest are completely unsatisfied
- In this situation

$$\phi(B) = S_k$$

- $S_k = \sum_{j=0}^{k} w_j$ degree of satisfaction decision maker has if k (k/n) criteria are satisfied
- Because $S_k = S_{k-1} + w_k$

INTERPRETATION OF w_k

Increase in satisfaction decision maker has when move from the satisfaction of $k-1$ criteria to the satisfaction of k criteria

IF FUNCTION DEGREE OF SATISFACTION S_k IS KNOWN, THEN WEIGHTS ARE

$$w_k = S_k - S_{k-1} \ \ with \ S_0 = 0 \ \ and \ \ S_n = 1$$

- RIM quantifiers have the same properties than function degree of satisfaction S_k
- If decision maker states that he/she desires Q of the criteria to be satisfied
  - $Q(0) = Q(0/n) = 0$ and $Q(1) = Q(n/n) = 1$
  - The more criteria satisfied the greater the value
- We can use the previous result using Q as the satisfaction function

IF RIM QUANTIFIER Q IS KNOWN

In this case $Q(\frac{k}{n})$ means degree of satisfaction attained when satisfying any k of the n criteria with that quantifier Q

$$w_k = Q\left(\frac{k}{n}\right) - Q\left(\frac{k-1}{n}\right)$$

Find k criteria satisfied by alternative x

$$G_k(x) = f_k(A_1(x), \ldots, A_n(x))$$

Combine Steps 1 & 2

$$D(x) = \sum_{i=1}^{n} w_k \cdot G_k(x)$$

- Recall that $G_k(x)$ stipulates that we need to find k criteria that are satisfied
  - Only need to consider the k most satisfied

$$G_k(x) = f(b_1, ..., b_k)$$

with b_j the j-th largest of the criteria scores

- $G_k(x)$ requires that all the k criteria are satisfied: f is t-norm
  - Using *min* t-norm we have: $G_k(x) = b_k$

NOTE

This is a quantifier guided OWA operator

$$D(x) = \sum_{i=1}^{n} w_k \cdot G_k(x) = \sum_{i=1}^{n} w_k \cdot b_k$$

where b_k is the k-highest value to aggregate

1. Use Q to generate set of OWA weights $(w_1, w_2, \ldots, w_n)$

$$w_i = Q(i/n) - Q((i-1)/n)$$

2. For each alternative $x \in X$ we calculate the overall evaluation

$$D(x) = \phi_Q(A_1(x), A_2(x), \ldots, A_n(x))$$

ϕ_Q is OWA operator guided by linguistic quantifier Q

$A_i(x) \in [0,1]$ degree to which the alternative x satisfied the criteria A_i

NOTE

The linguistic quantifier represents the soft majority concept to implement in the decision making problem, which is done via its use in the weighting vector computation described in step 1 above.

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p_{ij} preference degree of x_i over x_j $\quad p_{ji}$ preference degree of x_j over x_i

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$$d(x_i, x_j) = \max\{p_{ji} - p_{ij}, 0\}$$

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The degree up to which x_i is not dominated by any of the ('all') elements of X is

$$\mu_{ND}(x_i)=\min_{x_j\in X}(1-d(x_i,x_j))=1-\max_{x_j\in X}d(x_i,x_j)$$

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The maximum non-dominated elements in X

$$X^{ND}=\{x_i\in X|\mu_{ND}(x_i)=\max_{x_j\in X}\mu_{ND}(x_j)\}$$

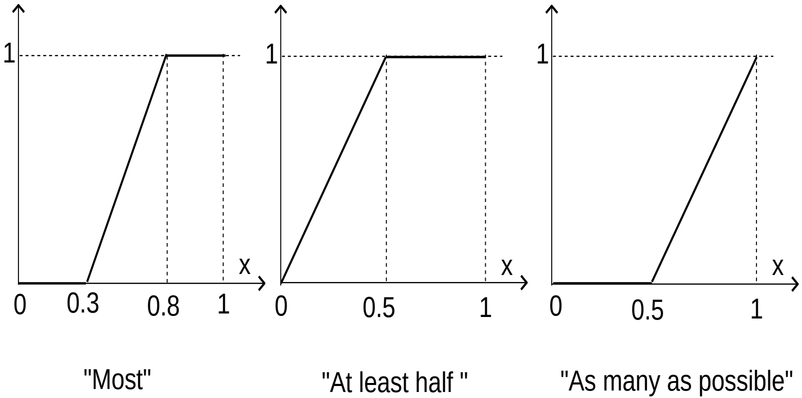
$$\{P^1, P^2, \ldots, P^m\} \longrightarrow P^c = \left(p^c_{ij}\right)$$

$$p^c_{ij} = \phi_Q\left(p^1_{ij}, \ldots, p^m_{ij}\right) = \sum_{k=1}^{n} w_k \cdot p^k_{ij}$$

ϕ_Q OWA Operator guided by fuzzy linguistic quantifier Q representing a 'soft majority' concept: 'all', 'most', 'many', 'at least α', etc.

$$Q(r) = \begin{cases} 0 & \text{if } 0 \le r < a \\ \frac{r-a}{b-a} & \text{if } a \le r \le b \\ 1 & \text{if } b < r \le 1 \end{cases}$$

- P^c – preference degree of one alternative over another for the "majority" of the experts. For example: "majority" here could be "as many as possible" of the experts/criteria
- From P^c apply non-dominance degree to obtain the alternative not dominated by a "majority" of alternatives. For example: "majority" here could be "most" alternatives.
- Together, we would aim to obtain the "best" alternative, where best means being *not dominated* by "most" alternatives according to "as many experts as possible."


1
X
0
0.3
0.8
1
"Most"
1
X
0
0.5
1
"At least half "
1
X
0
0.5
1
"As many as possible"

- Assume we have a set of 3 alternatives:

$$X = \{x_1, x_2, x_3\}$$

- Assume we have a set of 4 experts:

$$E = \{e_1, e_2, e_3, e_4\}$$

- Each expert compares alternatives in pairs and preference degrees of one alternative over another are provided:

$$\{P^1, P^2, P^3, P^4\}$$

- We want to find the "best" alternative, where best means being *not dominated* by "most" alternatives according to "as many experts as possible."

Assuming experts provide the following individual fuzzy preference relations

$$P^1 = \begin{pmatrix} 0.5 & 0.3 & 0.1 \\ 0.7 & 0.5 & 0.2 \\ 0.9 & 0.8 & 0.5 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0.5 & 0.3 & 0.8 \\ 0.7 & 0.5 & 0.4 \\ 0.2 & 0.6 & 0.5 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.5 & 0.6 & 0.6 \\ 0.4 & 0.5 & 0.6 \\ 0.4 & 0.4 & 0.5 \end{pmatrix} \quad P^4 = \begin{pmatrix} 0.5 & 0.8 & 0.6 \\ 0.2 & 0.5 & 0.7 \\ 0.4 & 0.3 & 0.5 \end{pmatrix}$$

First, we compute the collective preference relation using an OWA operator guided by the linguistic quantifier "as many as possible"

1. The fuzzy linguistic quantifier for "as many as possible" is the following

$$Q(r) = \begin{cases} 0 & \text{if } 0 \leq r < 0.5 \\ 2r - 1 & \text{if } 0.5 \leq r \leq 1 \end{cases}$$

2. Use Q to generate the OWA operator weighting vector (w_1, w_2, w_3, w_4).

$$w_i = Q(i/n) - Q((i-1)/n)$$

We need to find i/n with $n = 4$ and $i = 0, 1, 2, 3, 4$. Then we need find the value $Q(i/n)$.

i/n	0	0.25	0.5	0.75	1
$Q(i/n)$	0	0	0	0.5	1

Then we have:

$$(w_1, w_2, w_3, w_4) = (0, 0, 0.5, 0.5).$$

3. The collective fuzzy preference relations the set of alternatives for "as many as possible" experts is:

$$P^c = \begin{pmatrix} 0.5 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.35 & 0.5 \end{pmatrix}$$

We show how p^c_{12} is computed from $\{p^1_{12}, p^2_{12}, p^3_{12}, p^4_{12}\} = \{0.3, 0.3, 0.6, 0.8\}$

- Ordered values are: $(0.8, 0.6, 0.3, 0.3)$.
- Apply weighting vector: $(w_1, w_2, w_3, w_4) = (0, 0, 0.5, 0.5)$

$$p^c_{12} = 0 * 0.8 + 0 * 0.6 + 0.5 * 0.3 + 0.5 * 0.3 = 0.3$$

1. The fuzzy linguistic quantifier for "most" is

$$Q(r) = \begin{cases} 0 & \text{if } 0 \le r < 0.3 \\ 2r - 0.6 & \text{if } 0.3 \le r \le 0.8 \\ 1 & \text{if } 0.8 < r \le 1 \end{cases}$$

2. Use Q to generate the OWA operator weighting vector (w_1, w_2, w_3).

$$w_i = Q(i/n) - Q((i-1)/n)$$

We need to find i/n with $n = 3$ and $i = 0,1,2,3$. Then we need find the value $Q(i/n)$.

i/n	0	1/3	2/3	1
$Q(i/n)$	0	1/15	11/15	1

Then we have:

$$(w_1, w_2, w_3) = (1/15, 10/15, 4/15).$$

3. Computation of matrix with elements $nd_{ij} = 1 - d_{ij} = 1 - d(x_i, x_j) = 1 - \max\{p_{ji} - p_{ij}, 0\}$.

$$P^c = (p_{ij}) \qquad D^c = (d^c_{ij}) \qquad ND^c = 1 - D^c$$

$$P^c = \begin{pmatrix} 0.5 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.35 & 0.5 \end{pmatrix} \quad D^c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.05 \\ 0.2 & 0 & 0 \end{pmatrix} \quad ND^c = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0.95 \\ 0.8 & 1 & 1 \end{pmatrix}$$

4. Reordering of row elements matrix ND^c: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0.95 \\ 1 & 1 & 0.8 \end{pmatrix}$

5. Application of weighting vector $(w_1, w_2, w_3) = (1/15, 10/15, 4/15)$:

$$\begin{aligned} 1/15 * 1 + 10/15 * 1 + 4/15 * 1 &= 1 \\ 1/15 * 1 + 10/15 * 1 + 4/15 * 0.95 &= 0.987 \\ 1/15 * 1 + 10/15 * 1 + 4/15 * 0.8 &= 0.947 \end{aligned}$$

6. The 'best' alternative is : x_1. So, x_1 is the best alternative as it has the maximum *non dominance* degree by "most" alternatives according to "as many experts as possible."

- Comparison of different actions cannot be done using a single criterion or a unique expert
- Decision support systems important for complex situations
- The best alternative is obtained in two phases or steps: aggregation (fusing individual information into collective information) + exploitation (applying a choice function – like non-dominance degree)
- We have cover the first approach to fuzzy decision making by Bellman and Zadeh and also an extension of this based on the use of linguistic quantifiers.
- Solution should be the best for the majority of experts/criteria
- Majority is a fuzzy concept
- Implemented by OWA guided by linguistic quantifier

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