# Outline

# Introduction

- Particle swarm optimization (PSO)
  - It is motivated from foraging and social behavior of swarm.
- PSO is proposed by Eberhart and Kennedy (1995), "A new optimizer using particle swarm theory". In: Proceedings of the 6th international symposium on micromachine and human science, pp 3943, Nagoya, Japan, Mar 1316, 1995.
- PSO was first proposed for continuous nonlinear functions.
- PSO is developed using two methodologies
  - Artificial life: mimicking bird flocking, fish schooling, and swarming theory
  - Evolutionary computation

# Introduction

- PSO is developed using two methodologies
  - Artificial life: mimicking bird flocking, fish schooling, and swarming theory
  - Evolutionary computation
- The swarm searches for the food in a cooperative way
- Each member in the swarm learns from its experience and also from other members for changing the search patten to locate the food.
- PSO is developed using the simple concepts and primitive operators.
- PSO is computationally inexpensive both in memory and speed, and also can be easily implemented using computer programming.

# Particle swarm optimization (PSO)

- PSO starts with initializing population randomly similar to GA.
- Unlike GA operators, solutions are assigned with randomized velocity to explore the search space.
- Each solution in PSO is referred to as particle.

**Three distinct features of PSO**

| Best fitness of each particle | Best fitness of swarm | Velocity and position update of each particle |
|---|---|---|

- pbest$_i$: the best solution (fitness) achieved so far by particle $i$
- gbest: the best solution (fitness) achieved so far by any particle in the swarm
- Velocity and position update: for exploring and exploiting the search space to locate the optimal solution.

# Position and Velocity

- Position of particle $(i)$ is adjusted as

$$x_i^{(t+1)} = x_i^{(t)} + v_i^{(t+1)} \tag{1}$$

- Velocity of particle $(i)$ is updated as follows:

$$v_i^{(t+1)} = \underbrace{wv_i^{(t)}}_{1} + \underbrace{c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)})}_{2} + \underbrace{c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})}_{3} \tag{2}$$

- $i$ is the $i-th$ particle.
- $t$ is the generation counter.
- $v_i^{(0)}$ set randomly.
- $w$ adds to the inertia of the particle.
- $c_1$ and $c_2$ are the acceleration coefficients.
- $r_1$ and $r_2$ are random numbers $\in [0, 1]$.
- $p_{(i,lb)}^{(t)}$ is the local best of $i - th$ particle.
- $p_{gb}^{(t)}$ is the global best.

# Velocity Components

$$v_i^{(t+1)} = wv_i^{(t)} + c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)}) + c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})$$

- **Momentum part**, $wv_i^{(t)}$
  - inertia component
  - memory of previous flight direction
  - prevents particle from drastically changing direction

$V_i$

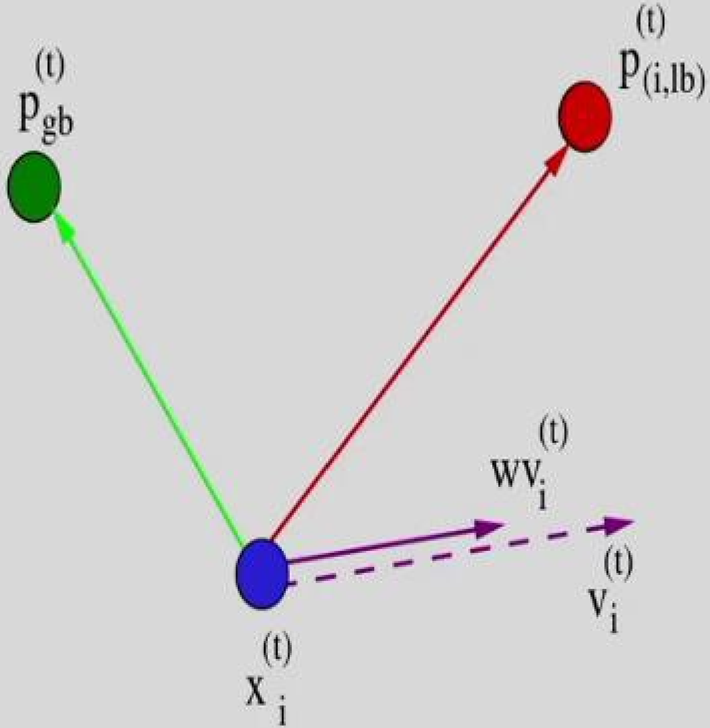
$wv_i$

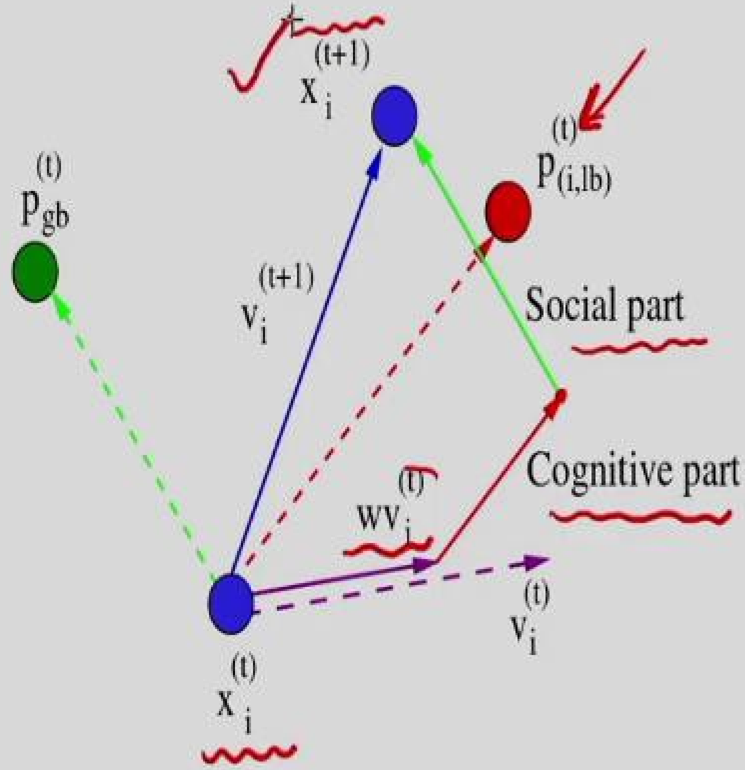
# Velocity Components

$$v_i^{(t+1)} = wv_i^{(t)} + c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)}) + c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})$$

- **Momentum part**, $wv_i^{(t)}$
  - inertia component
  - memory of previous flight direction
  - prevents particle from drastically changing direction
- **Cognitive part**, $c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)})$
  - quantifies performance relative to past performances
  - memory of previous best position
  - nostalgia
- **Social part**, $c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})$
  - quantifies performance relative to neighbors
  - envy

# Geometrical Illustration of Velocity Components




- **Momentum part**, $wv_i^{(t)}$
- **Cognitive part**, $c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)})$
- **Social part**, $c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})$

# Local and Global Best Positions

- $p_{(i,lb)}^{(t)}$ is the personal best position of $i-th$ particle in $t$ generation. Assume minimization problem.

$$p_{(i,lb)}^{(t+1)} = \begin{cases} x_i^{(t+1)} & \text{if } f(x_i^{(t+1)}) < f(p_{(i,lb)}^{(t)}) \\ \\ p_{(i,lb)}^{(t)} & \text{Otherwise.} \end{cases} \tag{3}$$

- $p_{gb}^{(t)}$ is the global best position in $t$ generation which is calculated as

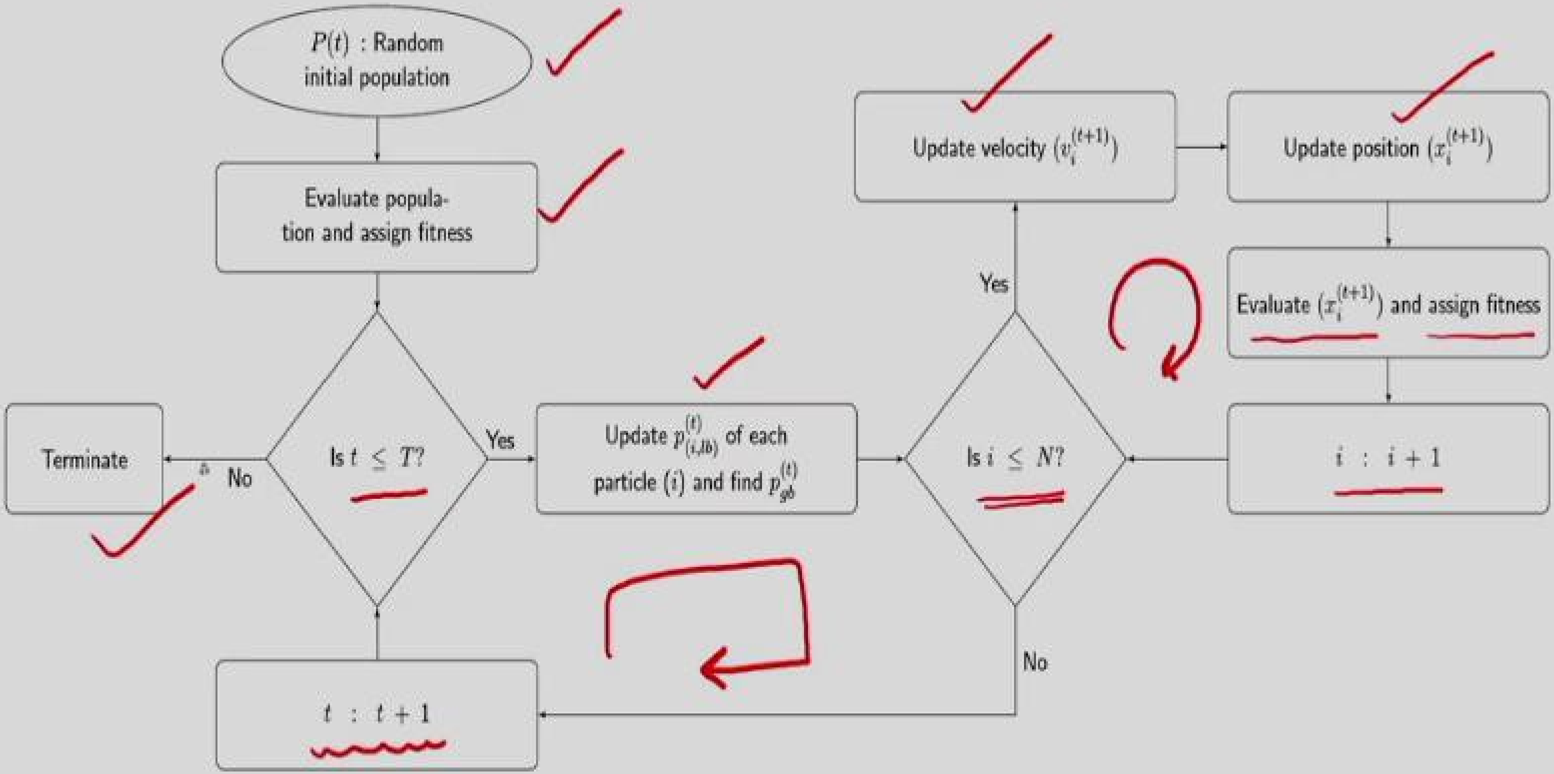
$$p_{gb}^{(t)} \in \{p_{(1,best)}^{(t)}, \ldots, p_{(N,best)}^{(t)}\} | f(p_{gb}^{(t)}) = \min\{f(p_{(1,best)}^{(t)}), \ldots, f(p_{(N,best)}^{(t)})\}$$

or,

$$p_{gb}^{(t)} \in \{x_1^{(t)}, \ldots, x_N^{(t)}\} | f(p_{gb}^{(t)}) = \min\{f(x_1^{(t)}), \ldots, f(x_N^{(t)})\}, \tag{4}$$

where $N$ is the number of particles in the swarm.

# Flowchart of ~~DE~~ PSO



# Generalized Framework of EC Techniques

**Algorithm 1** Generalized Framework

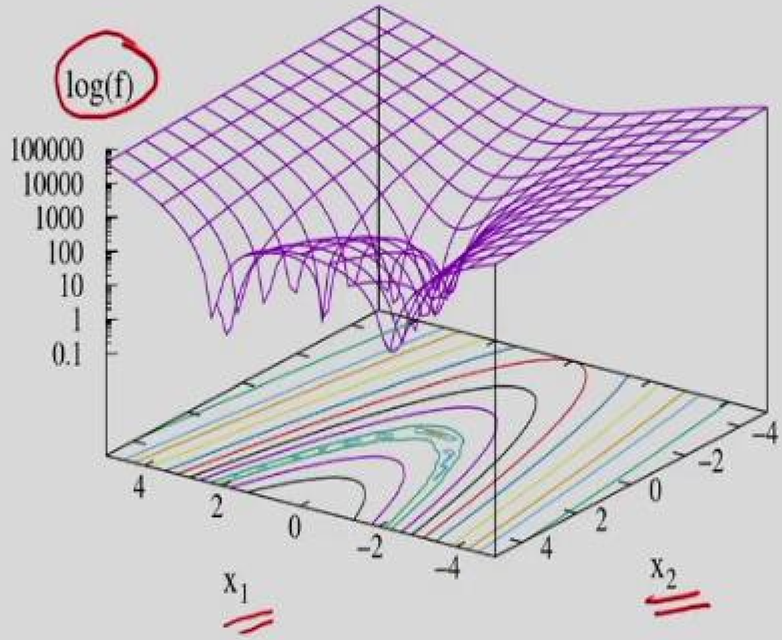
Solution representation — % Genetics
**Input**: $t := 1$ (Generation counter), Maximum allowed generation $= T$
Initialize random swarm $(P(t))$; — % Swarm
Evaluate $(P(t))$; — % Evaluate objective, constraints and assign fitness
**while** $t \leq T$ **do**
    Update $p^{(t)}_{(i,lb)}$ of each particle $(i)$ and find $p^{(t)}_{gb}$; — % New step
    **for** $(i = 1; i \leq N, i++)$ **do** — % For each particle $i$
        ~~$M(t) :=$ Selection$(P(t))$;~~ Update velocity $(v_i^{(t+1)})$;
        ~~$Q(t) :=$ Variation$(M(t))$;~~ Update position $(x_i^{(t+1)})$; — % Variation
        Evaluate $(x_i^{(t+1)})$ and include it in $P(t+1)$;
    **end for**
    ~~$P(t+1) :=$ Survivor$(P(t), Q(t))$;~~
    $t := t + 1$;
**end while**

# Working Principles Through An Example

## Rosenbrock Function

Minimize $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$,

bounds $-5 \le x_1 \le 5$ and $-5 \le x_2 \le 5$.




- Optimum solution is $x^* = (1,1)^T$ and $f(x) = 0$

# Initial Swarm

- Let the population size is $N = 8$ and $t = 1$.

Initial swarm

| Index($i$) | $x_i^{(1)}$ | $v_i^{(1)}$ |
|---|---|---|
| 1 | $(2.212, 3.009)^T$ | $(0.0, 0.0)^T$ |
| 2 | $(-2.289, -2.396)^T$ | $(0.0, 0.0)^T$ |
| 3 | $(-2.393, -4.790)^T$ | $(0.0, 0.0)^T$ |
| 4 | $(-0.639, 1.692)^T$ | $(0.0, 0.0)^T$ |
| 5 | $(-3.168, 0.706)^T$ | $(0.0, 0.0)^T$ |
| 6 | $(0.215, -2.350)^T$ | $(0.0, 0.0)^T$ |
| 7 | $(-0.742, 1.934)^T$ | $(0.0, 0.0)^T$ |
| 8 | $(-4.563, 4.791)^T$ | $(0.0, 0.0)^T$ |

# Evaluate Population

- We calculate objective function $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ for each solution.
- For Solution 1: $x^{(1)} = (2.212, 3.009)^T$ and $f(x^{(1)}) = 357.154$.

Initial swarm

| Index($i$) | $x_i^{(1)}$ | $f(x_i^{(1)})$ |
|---|---|---|
| 1 | $(2.212, 3.009)^T$ | 357.154 |
| 2 | $(-2.289, -2.396)^T$ | 5843.569 |
| 3 | $(-2.393, -4.790)^T$ | 11066.800 |
| 4 | $(-0.639, 1.692)^T$ | 167.414 |
| 5 | $(-3.168, 0.706)^T$ | 8718.166 |
| 6 | $(0.215, -2.350)^T$ | 574.796 |
| 7 | $(-0.742, 1.934)^T$ | 194.618 |
| 8 | $(-4.563, 4.791)^T$ | 25731.235 |

- Let us consider the fitness value same as the function value.

# Local best of each particle

- This is the first generation so the local best of each particle is itself.

| Index($i$) | $x_i^{(1)}$ | $f(x_i^{(1)})$ | $p_{(i,lb)}^{(1)}$ |
|---|---|---|---|
| 1 | $(2.212, 3.009)^T$ | 357.154 | $(2.212, 3.009)^T$ |
| 2 | $(-2.289, -2.396)^T$ | 5843.569 | $(-2.289, -2.396)^T$ |
| 3 | $(-2.393, -4.790)^T$ | 11066.800 | $(-2.393, -4.790)^T$ |
| 4 | $(-0.639, 1.692)^T$ | 167.414 | $(-0.639, 1.692)^T$ |
| 5 | $(-3.168, 0.706)^T$ | 8718.166 | $(-3.168, 0.706)^T$ |
| 6 | $(0.215, -2.350)^T$ | 574.796 | $(0.215, -2.350)^T$ |
| 7 | $(-0.742, 1.934)^T$ | 194.618 | $(-0.742, 1.934)^T$ |
| 8 | $(-4.563, 4.791)^T$ | 25731.235 | $(-4.563, 4.791)^T$ |

## Global Best of Swarm

- The global best of the swarm is

$$p_{gb}^{(t)} \in \{x_1^{(t)}, \ldots, x_N^{(t)}\} | f(p_{gb}^{(t)}) = \min\{f(x_1^{(t)}), \ldots, f(x_N^{(t)})\},$$

| Index($i$) | $x_i^{(1)}$ | $f(x_i^{(1)})$ | $p_{(i,lb)}^{(1)}$ | $p_{gb}^{(1)}$ |
|---|---|---|---|---|
| 1 | $(2.212, 3.009)^T$ | 357.154 | $(2.212, 3.009)^T$ | $(-0.639, 1.692)^T$ |
| 2 | $(-2.289, -2.396)^T$ | 5843.569 | $(-2.289, -2.396)^T$ | $(-0.639, 1.692)^T$ |
| 3 | $(-2.393, -4.790)^T$ | 11066.800 | $(-2.393, -4.790)^T$ | $(-0.639, 1.692)^T$ |
| 4 | $(-0.639, 1.692)^T$ | 167.414 | $(-0.639, 1.692)^T$ | $(-0.639, 1.692)^T$ |
| 5 | $(-3.168, 0.706)^T$ | 8718.166 | $(-3.168, 0.706)^T$ | $(-0.639, 1.692)^T$ |
| 6 | $(0.215, -2.350)^T$ | 574.796 | $(0.215, -2.350)^T$ | $(-0.639, 1.692)^T$ |
| 7 | $(-0.742, 1.934)^T$ | 194.618 | $(-0.742, 1.934)^T$ | $(-0.639, 1.692)^T$ |
| 8 | $(-4.563, 4.791)^T$ | 25731.235 | $(-4.563, 4.791)^T$ | $(-0.639, 1.692)^T$ |

# Velocity Update

- The velocity of each particle is updated using

$$v_i^{(t+1)} = wv_i^{(t)} + c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)}) + c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})$$

- Assume, $w = 0.75$, $c_1 = 1.5$ and $c_2 = 2.0$.
- Let the random numbers for each particle are

| Particle | $r_1$ | $r_2$ |
|---|---|---|
| 1 | 0.661 | 0.312 |
| 2 | 0.919 | 0.271 |
| 3 | 0.782 | 0.824 |
| 4 | 0.299 | 0.055 |
| 5 | 0.874 | 0.595 |
| 6 | 0.133 | 0.582 |
| 7 | 0.031 | 0.736 |
| 8 | 0.366 | 0.954 |

- For particle 1, $x_1^{(1)} = (2.212, 3.009)^T$ $v_1^{(1)} = (0.0, 0.0)^T$, $p_{(1,lb)}^{(1)} = (2.212, 3.009)^T$, $p_{gb}^{(1)} = (-0.639, 1.692)^T$, $r_1 = 0.661$ and $r_2 = 0.312$.
- $v_{1,1}^{(2)} =$ $0.75 \times 0 + 1.5 \times 0.661(2.212 - 2.212) + 2.0 \times 0.312(-0.639 - 2.212) = -1.779$.
- $v_{1,2}^{(2)} =$ $0.75 \times 0 + 1.5 \times 0.661(3.009 - 3.009) + 2.0 \times 0.312(1.692 - 3.009) = -0.822$.

# Velocity Update

- For particle 2, $x_2^{(1)} = (-2.289, -2.396)^T$

$v_2^{(1)} = (0.0, 0.0)^T$,

$p_{(2,lb)}^{(1)} = (-2.289, -2.396)^T$,

$p_{gb}^{(1)} = (-0.639, 1.692)^T$, $r_1 = 0.919$ and

$r_2 = 0.271$.

# Velocity Update

- For particle 2, $x_2^{(1)} = (-2.289, -2.396)^T$
  $v_1^{(1)} = (0.0, 0.0)^T$,
  $p_{(1,lb)}^{(1)} = (-2.289, -2.396)^T$,
  $p_{gb}^{(1)} = (-0.639, 1.692)^T$, $r_1 = 0.919$ and
  $r_2 = 0.271$.
- $v_{2,1}^{(2)} =$
  $0.75\times 0 + 1.5\times 0.919(-2.289-(-2.289))+$
  $2.0 \times 0.212(-0.639 - (-2.289)) = 0.893.$
- $v_{2,2}^{(2)} =$
  $0.75\times 0 + 1.5\times 0.271(-2.396-(-2.396))+$
  $2.0 \times 0.212(1.692 - (-2.396)) = 2.212.$

Updated velocity of each particle

| Particle | $v_{i,1}^{(2)}$ | $v_{i,2}^{(2)}$ |
|---|---|---|
| 1 | $-1.779$ | $-0.882$ |
| 2 | 0.893 | 2.212 |
| 3 | 2.890 | 10.683 |
| 4 | 0.000 | 0.000 |
| 5 | 3.010 | 1.174 |
| 6 | $-0.994$ | 4.704 |
| 7 | 0.151 | $-0.357$ |
| 8 | 7.491 | $-5.916$ |

# Position Update

- The position of each particle is updated as

$$x_i^{(t+1)} = x_i^{(t)} + v_i^{(t+1)}$$

| Particle | $x_i^{(1)}$ | $v_i^{(2)}$ |
|---|---|---|
| 1 | $(2.212, 3.009)^T$ | $(-1.779, -0.822)^T$ |
| 2 | $(-2.289, -2.396)^T$ | $(0.893, 2.212)^T$ |
| 3 | $(-2.393, -4.790)^T$ | $(2.890, 10.683)^T$ |
| 4 | $(-0.639, 1.692)^T$ | $(0.000, 0.000)^T$ |
| 5 | $(-3.168, 0.706)^T$ | $(3.010, 1.174)^T$ |
| 6 | $(0.215, -2.350)^T$ | $(-0.994, 4.704)^T$ |
| 7 | $(-0.742, 1.934)^T$ | $(0.151, -0.357)^T$ |
| 8 | $(-4.563, 4.791)^T$ | $(7.491, -5.916)^T$ |

| Particle | $x_i^{(2)}$ |
|---|---|
| 1 | $(0.433, 2.187)^T$ |
| 2 | $(-1.396, -0.184)^T$ |
| 3 | $(0.498, 5.893)^T$ |
| 4 | $(-0.639, 1.692)^T$ |
| 5 | $(-0.157, 1.879)^T$ |
| 6 | $(-0.779, 2.354)^T$ |
| 7 | $(-0.590, 1.577)^T$ |
| 8 | $(2.928, -1.125)^T$ |

- The limit on $x_2$ is $[-5, 5]$. Therefore, we keep $x_2$ of solution 3 on the bound, that is, 5.

# Evaluate Swarm

Fitness of particles after position update

| Index($i$) | $x_i^{(2)}$ | $f(x_i^{(2)})$ |
|---|---|---|
| 1 | $(0.433, 2.187)^T$ | 399.984 |
| 2 | $(-1.396, -0.184)^T$ | 460.648 |
| 3 | $(0.498, 5.000)^T$ | 2258.514 |
| 4 | $(-0.639, 1.692)^T$ | 167.414 |
| 5 | $(-0.157, 1.879)^T$ | 345.375 |
| 6 | $(-0.779, 2.354)^T$ | 308.580 |
| 7 | $(-0.590, 1.577)^T$ | 153.484 |
| 8 | $(2.928, -1.125)^T$ | 9406.994 |

Initial swarm

| Index($i$) | $x_i^{(1)}$ | $f(x_i^{(1)})$ |
|---|---|---|
| 1 | $(2.212, 3.009)^T$ | 357.154 |
| 2 | $(-2.289, -2.396)^T$ | 5843.569 |
| 3 | $(-2.393, -4.790)^T$ | 11066.800 |
| 4 | $(-0.639, 1.692)^T$ | 167.414 |
| 5 | $(-3.168, 0.706)^T$ | 8718.166 |
| 6 | $(0.215, -2.350)^T$ | 574.796 |
| 7 | $(-0.742, 1.934)^T$ | 194.618 |
| 8 | $(-4.563, 4.791)^T$ | 25731.235 |

- Increase the generation counter by 1, meaning, $t = t + 1 = 2$.

# Velocity Update: $2^{nd}$ generation

- The velocity of each particle is updated using

$$v_i^{(t+1)} = wv_i^{(t)} + c_1 r_1 (p_{(i,lb)}^{(t)} - x_i^{(t)}) + c_2 r_2 (p_{gb}^{(t)} - x_i^{(t)})$$

- Assume $w = 0.75$, $c_1 = 1.5$ and $c_2 = 2.0$.
- Let the random numbers for each particle are

| Particle | $r_1$ | $r_2$ |
|---|---|---|
| 1 | 0.127 | 0.531 |
| 2 | 0.653 | 0.225 |
| 3 | 0.533 | 0.472 |
| 4 | 0.739 | 0.048 |
| 5 | 0.309 | 0.837 |
| 6 | 0.148 | 0.057 |
| 7 | 0.110 | 0.308 |
| 8 | 0.343 | 0.320 |

- For particle 1, $x_1^{(2)} = (2.212, 3.009)^T$ $v_1^{(1)} = (-1.779, -0.822)^T$, $r_1 = 0.127$ and $r_2 = 0.531$.
- Find the local best of particle 1: Since $f(x_1^{(1)}) = 357.153 < f(x_1^{(2)}) = 399.984$, the local best of the particle is its previous position, that is, $p_{(1,lb)}^{(2)} = (2.212, 3.009)^T$
- For finding the global best of the swarm we need to find the local best of each particle.

# Velocity Update: $2^{nd}$ generation

Find the local best of each particle

| Particle | $f(x_i^{(1)})$ | $f(x_i^{(2)})$ | $p_{(i,lb)}^{(2)}$ |
|---|---|---|---|
| 1 | 357.154 | 399.984 | $(2.212, 3.009)^T$ |
| 2 | 5843.569 | 460.648 | $(-1.396, -0.184)^T$ |
| 3 | 11066.800 | 2258.514 | $(0.498, 5.000)^T$ |
| 4 | 167.414 | 167.414 | $(-0.639, 1.692)^T$ |
| 5 | 8718.166 | 345.375 | $(-0.157, 1.879)^T$ |
| 6 | 574.796 | 308.580 | $(-0.779, 2.354)^T$ |
| 7 | 194.618 | 153.484 | $(-0.590, 1.577)^T$ |
| 8 | 25731.235 | 9406.994 | $(2.928, -1.125)^T$ |

- The global best particle of the swarm is particle '7', that is, $p_{gb}^{(2)} = (-0.590, 1.577)^T$.

Updated velocity of each particle

| Particle | $v_{i,1}^{(3)}$ | $v_{i,2}^{(3)}$ |
|---|---|---|
| 1 | $-2.083$ | $-1.107$ |
| 2 | 1.033 | 2.452 |
| 3 | 1.140 | 3.934 |
| 4 | 0.005 | $-0.011$ |
| 5 | 1.533 | 0.374 |
| 6 | $-0.724$ | 3.439 |
| 7 | 0.114 | $-0.268$ |
| 8 | 3.364 | $-2.705$ |

# Position Update: $2^{nd}$ generation

- The position of each is updated as

$$x_i^{(t+1)} = x_i^{(t)} + v_i^{(t+1)}$$

| Particle | $x_i^{(2)}$ | $v_i^{(3)}$ |
|---|---|---|
| 1 | $(0.433, 2.187)^T$ | $(-2.083, -1.107)^T$ |
| 2 | $(-1.396, -0.184)^T$ | $(1.033, 2.452)^T$ |
| 3 | $(0.498, 5.000)^T$ | $(1.140, 3.934)^T$ |
| 4 | $(-0.639, 1.692)^T$ | $(0.005, -0.011)^T$ |
| 5 | $(-0.157, 1.879)^T$ | $(1.533, 0.374)^T$ |
| 6 | $(-0.779, 2.354)^T$ | $(-0.724, 3.439)^T$ |
| 7 | $(-0.590, 1.577)^T$ | $(0.114, -0.268)^T$ |
| 8 | $(2.928, -1.125)^T$ | $(3.364, -2.705)^T$ |

| Particle | $x_i^{(3)}$ |
|---|---|
| 1 | $(-1.649, 1.079)^T$ |
| 2 | $(-0.363, 2.268)^T$ |
| 3 | $(1.638, 9.827)^T$ |
| 4 | $(-0.634, 1.681)^T$ |
| 5 | $(1.376, 2.254)^T$ |
| 6 | $(-1.503, 5.793)^T$ |
| 7 | $(-0.477, 1.309)^T$ |
| 8 | $(6.291, -3.830)^T$ |

- The variable $x_2$ of particles '3' and '6', and also variable $x_1$ of particle '8' are out of the bound.
- Put them on the bound, that is, 5.

# Evaluate Swarm: $2^{nd}$ generation

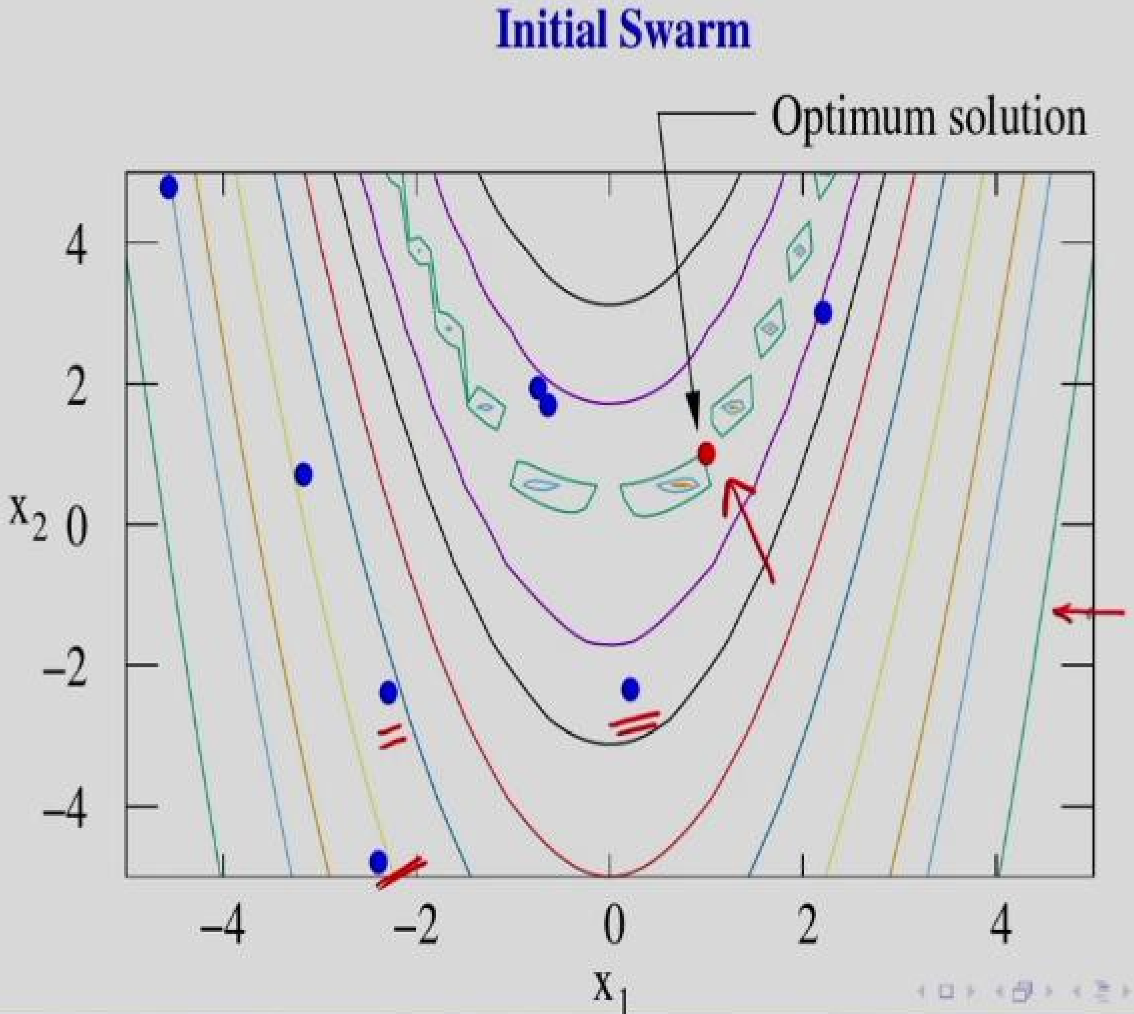
Fitness of particles after second generation

| Index($i$) | $x_i^{(3)}$ | $f(x_i^{(3)})$ |
|---|---|---|
| 1 | $(-1.649, 1.079)^T$ | 276.367 |
| 2 | $(-0.363, 2.268)^T$ | 458.327 |
| 3 | $(1.638, 5.000)^T$ | 537.876 |
| 4 | $(-0.634, 1.681)^T$ | 166.098 |
| 5 | $(1.376, 2.254)^T$ | 13.222 |
| 6 | $(-1.503, 5.000)^T$ | 575.777 |
| 7 | $(-0.477, 1.309)^T$ | 119.231 |
| 8 | $(5.000, -3.830)^T$ | 83134.582 |

Fitness of particles after first generation

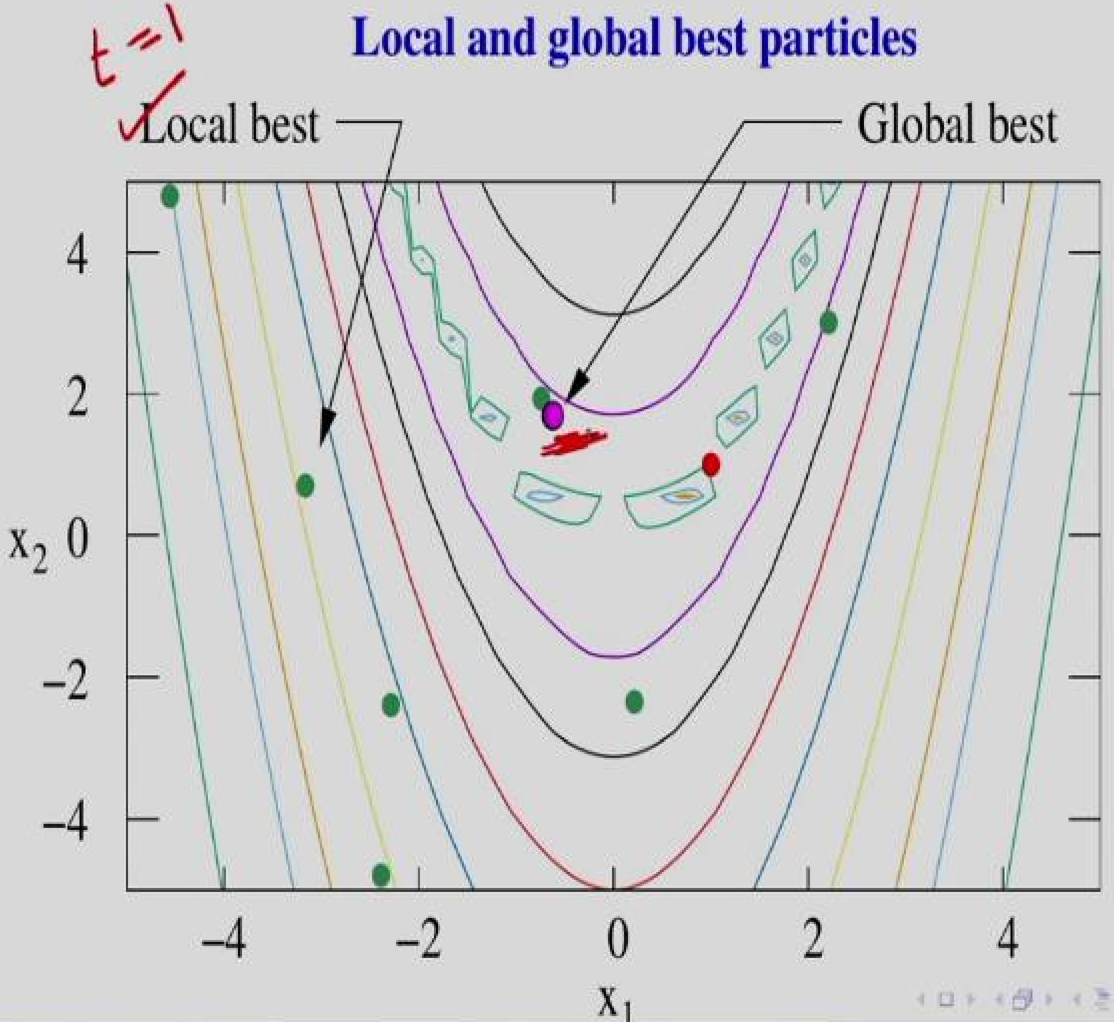
| Index($i$) | $x_i^{(2)}$ | $f(x_i^{(2)})$ |
|---|---|---|
| 1 | $(0.433, 2.187)^T$ | 399.984 |
| 2 | $(-1.396, -0.184)^T$ | 460.648 |
| 3 | $(0.498, 5.000)^T$ | 2258.514 |
| 4 | $(-0.639, 1.692)^T$ | 167.414 |
| 5 | $(-0.157, 1.879)^T$ | 345.375 |
| 6 | $(-0.779, 2.354)^T$ | 308.580 |
| 7 | $(-0.590, 1.577)^T$ | 153.484 |
| 8 | $(2.928, -1.125)^T$ | 9406.994 |

# Graphical Example

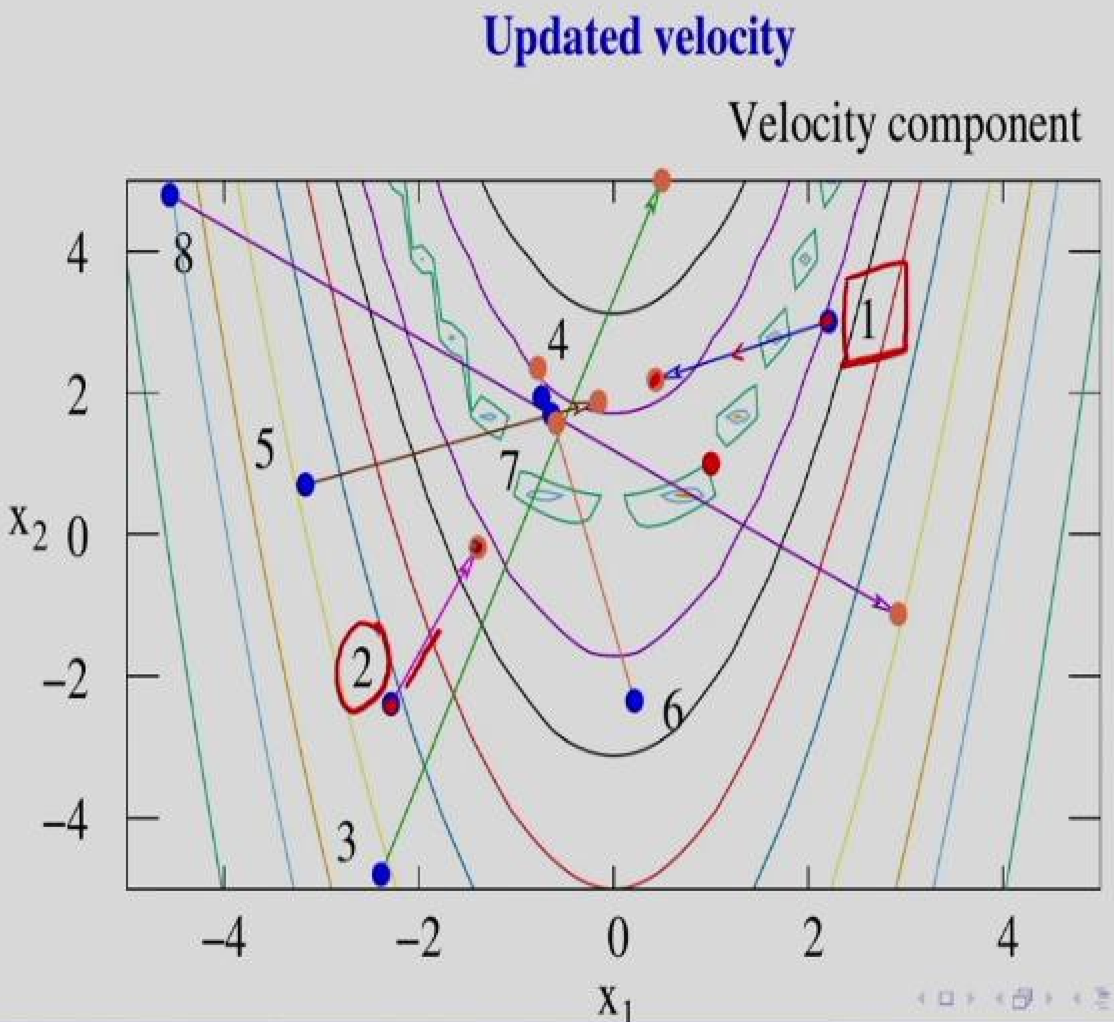


# Graphical Example

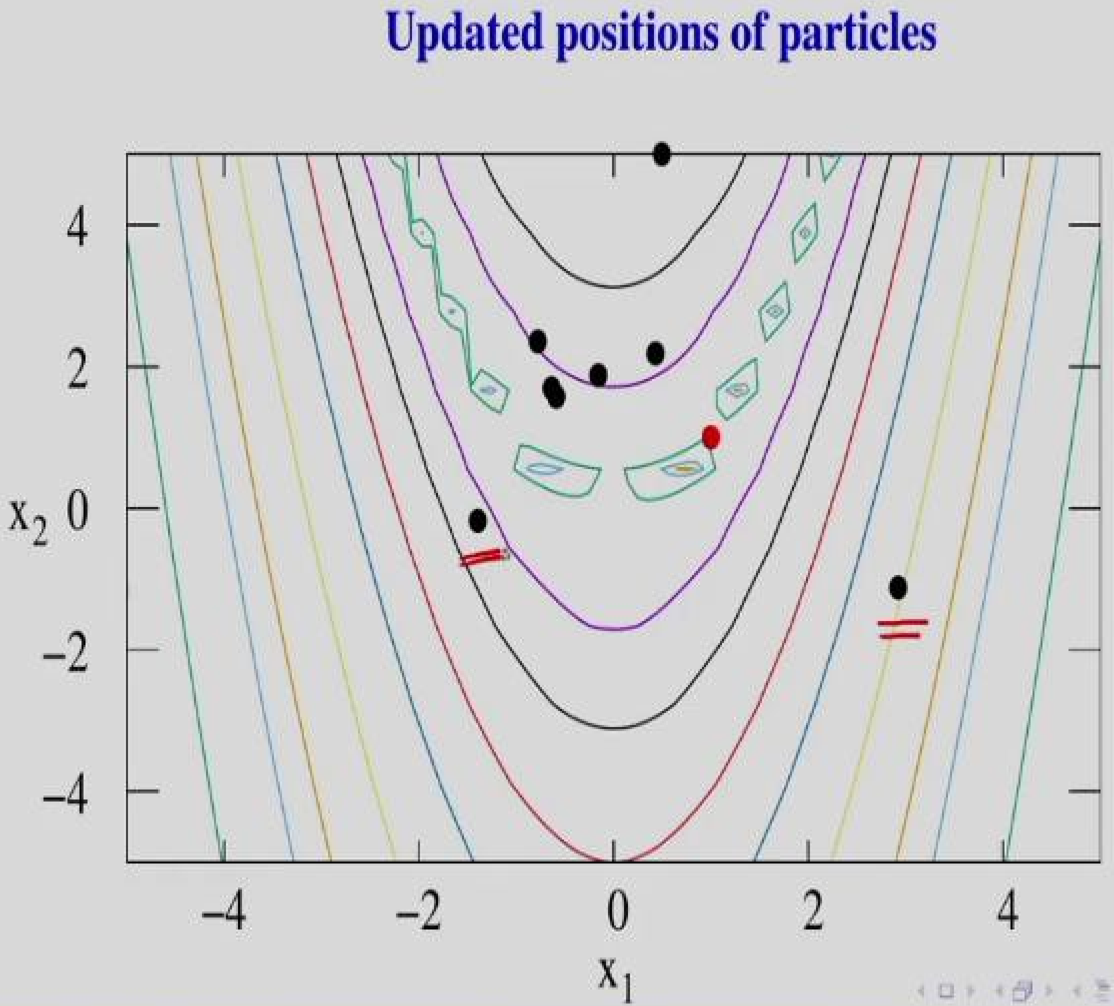
$t = 1$



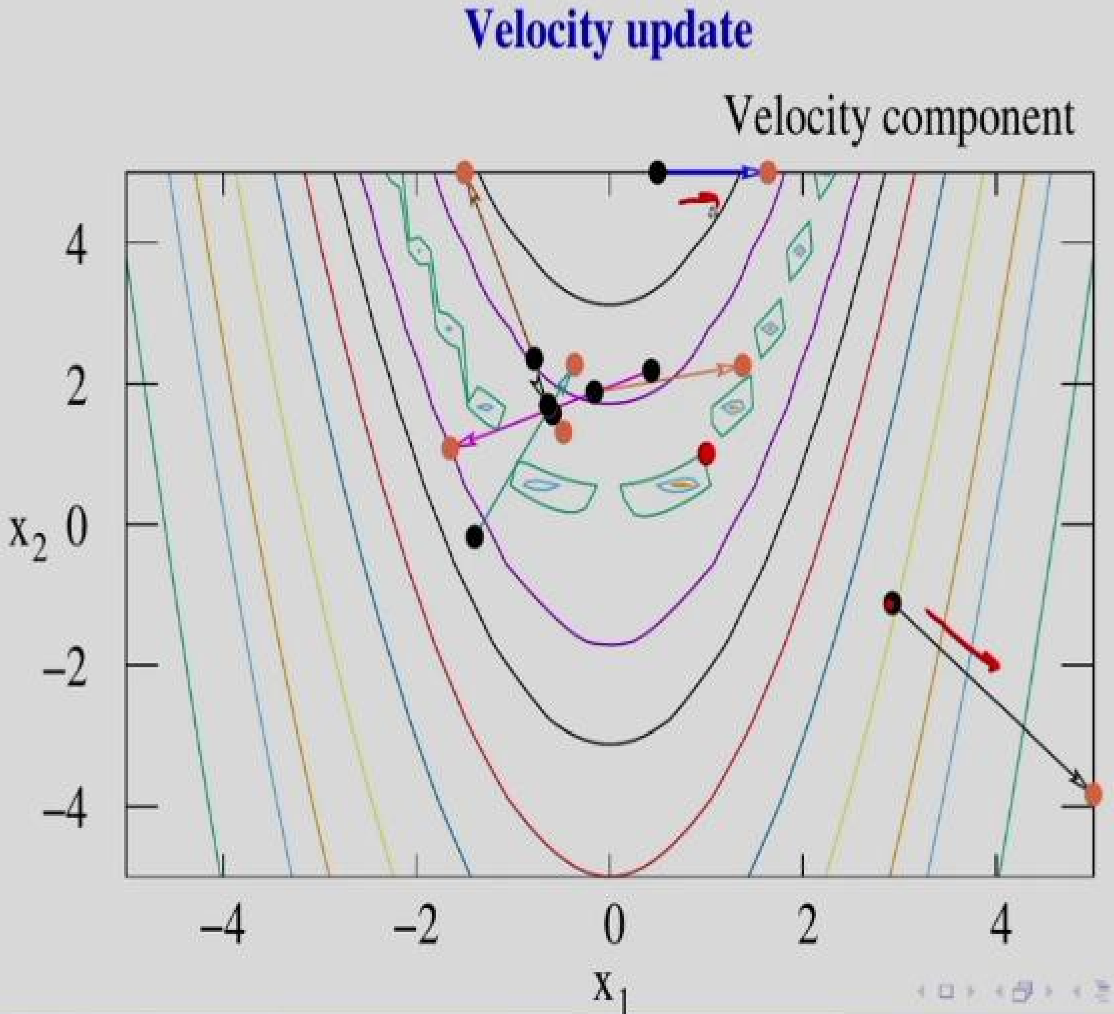
# Graphical Example



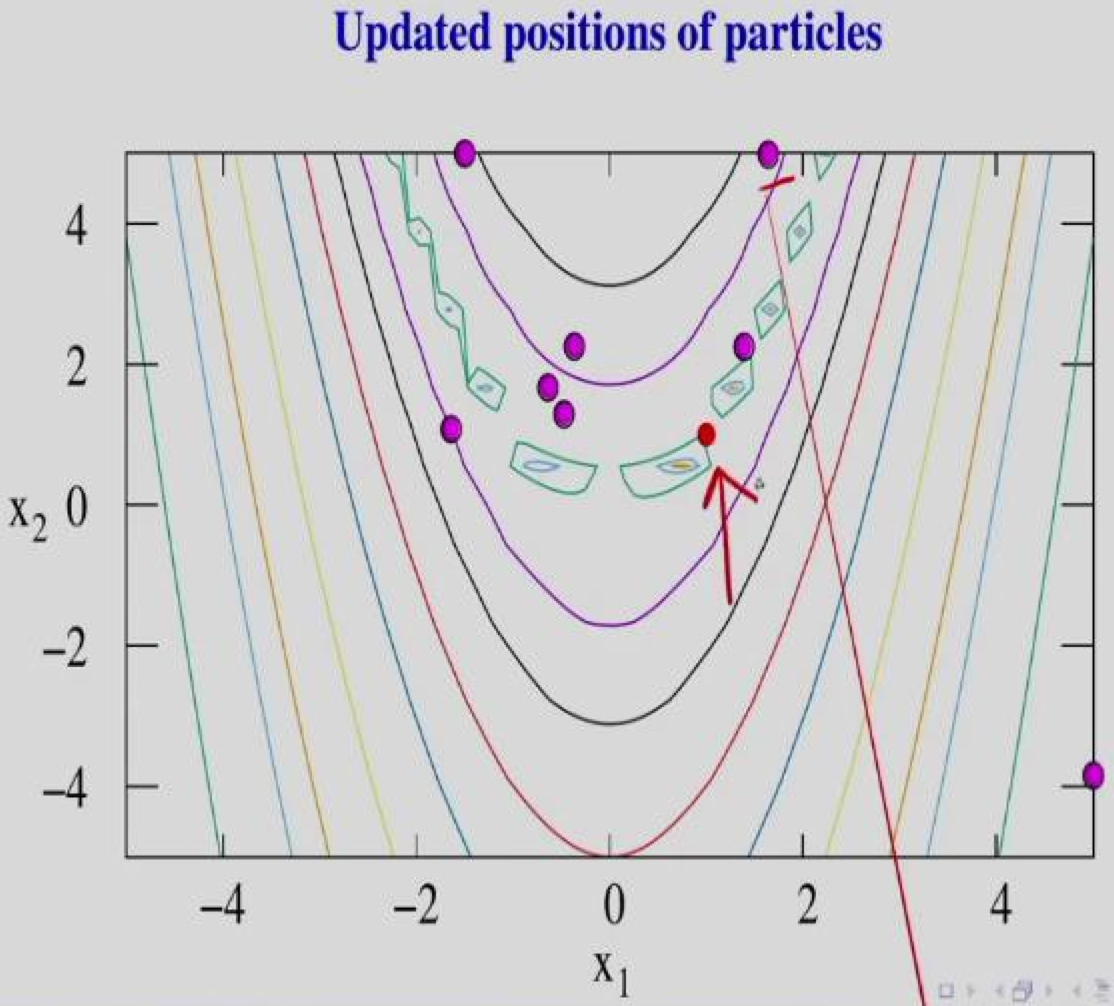
# Graphical Example



# Graphical Example

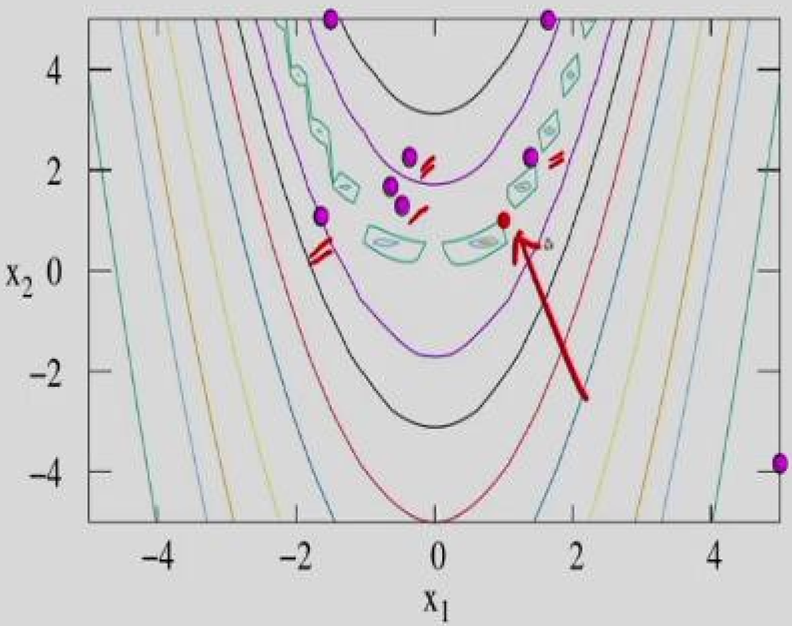


# Graphical Example

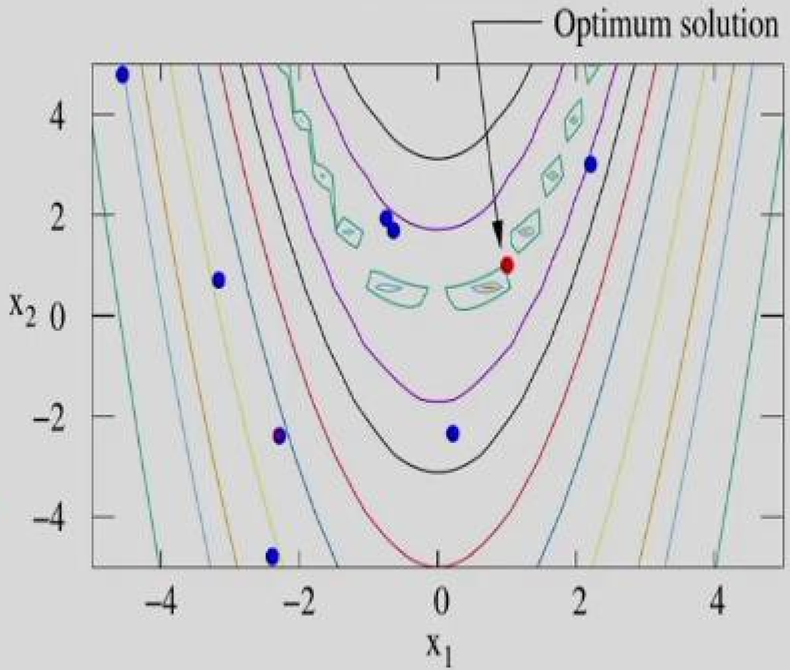


# Graphical Example

**Updated positions of particles**




**Initial Swarm**



# Closure

- Introduction of PSO
- Distinct feature of PSO
  - global best, local best, velocity and position updates
- Velocity update
  - Velocity components: Momentum, cognitive and social parts
  - Graphical illustration
- Position update
- Flowchart of PSO
- PSO on the generalized framework
- Working principles of PSO through Rosenbrock function
- Graphical example