Introduction

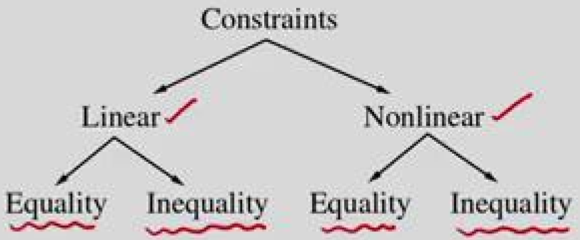
Constrained Optimization

- A constrained optimization problem can be written as

$$\begin{aligned} \text{Minimize} \quad & f(x), & \\ \text{subject to,} \quad & g_j(x) \geq 0, & j = 1, 2, \ldots, J, \\ & h_k(x) = 0, & k = 1, 2, \ldots, K, \\ & x_i^{(L)} \leq x_i \leq x_i^{(U)}, & i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

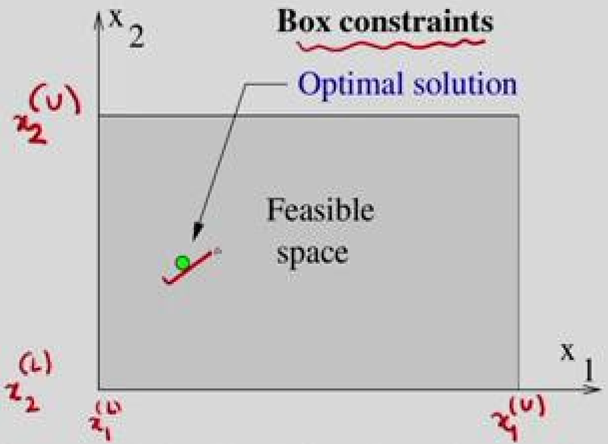
- $x = (x_1, x_2, \ldots, x_N)^T$ is the vector of decision variables.
- $f(x)$ is the objective function.
- $g_j(x)$ is the inequality constraint.
- $h_k(x)$ is the equality constraint.

Types of Constraints



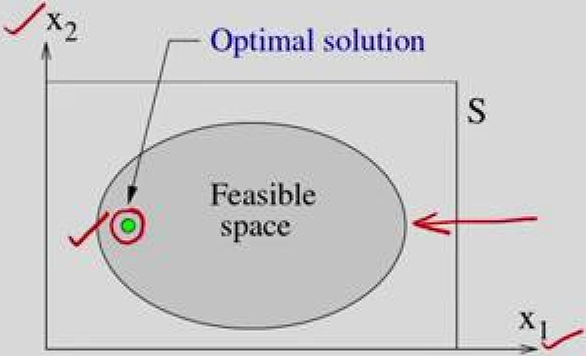
- Linear constraints are relatively easy to deal with.
- Nonlinear equality constraints can be hard to handle.

Graphical Example



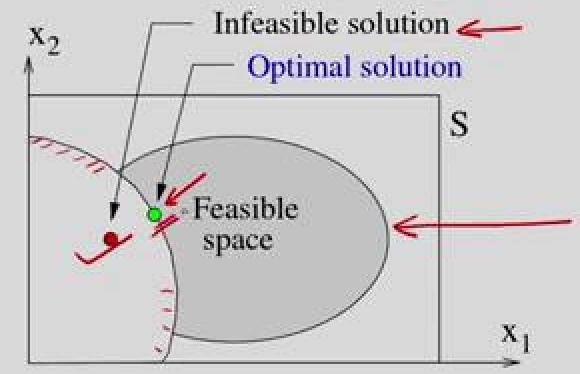
- Feasible search space is defined by the bounds on the variables.

Graphical Example



- Feasible search space is defined by the common space among constraints and the variables bounds.
- The constraints do not change the previous optimal solution.

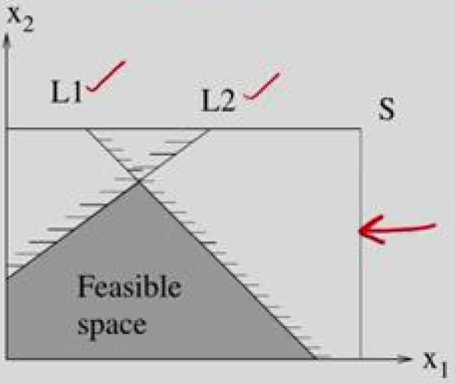
Graphical Example



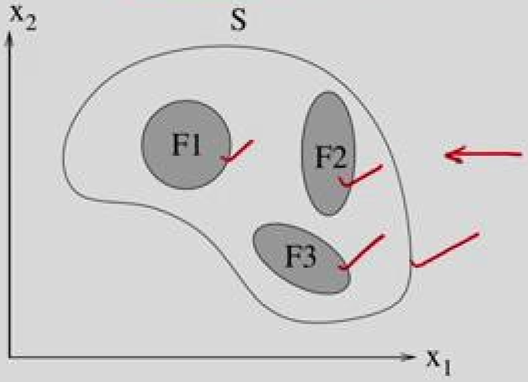
- Some constraints can make the previous optimal solution infeasible.
- Mostly, the optimum solution is found on the boundary of feasible space.

Visualization

Continuous search space

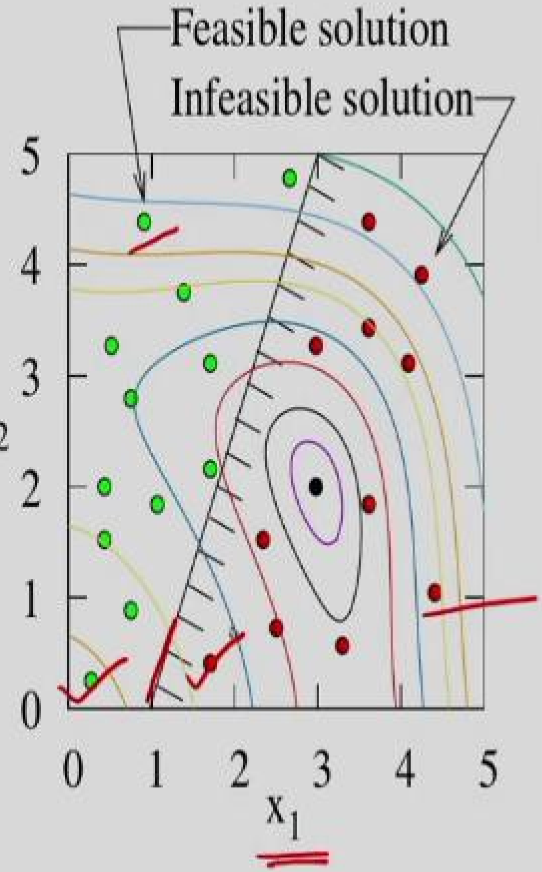


Disconnected search space



Why Constraint Handling is needed?

- EC techniques are population-based techniques.
- When EC techniques are used for constrained optimization problem, some solutions can be feasible and infeasible.
- In order to identify them and assign fitness, we need constraint handling techniques.
- Moreover, almost every practical optimization problem involves constraint(s).



Types of Constraint Handling Techniques

- Carlos A Coello Coello, "Theoretical and numerical constraint-handling techniques used with evolutionary algorithms: a survey of the state of the art," Computer Methods in Applied Mechanics and Engineering, Volume 191, Issues 1112, 2002, Pages 1245-1287.

Types

- Penalty Function Methods
- Special representations and operators
- Separation of constraints and objectives
- Hybrid Methods

Penalty Function Methods

- These methods are found be simple and most popular methods for handling constraints.
- If a solution violates any constraint, the solution is penalized by adding penalty with the objective function.
- The constraint problem is transformed to an unconstrained problems by adding penalty term of each constraint violation to the objective function value.

Penalty Function Methods

- Interior penalty methods: These methods work for feasible points and penalize points that are close to the constraint boundary.
- Exterior penalty methods: These methods penalize infeasible points but not the feasible solutions.

Penalty Function Methods

- A constrained optimization problem can be written as

$$
\begin{aligned}
\text{Minimize} \quad & f(x), & & \\
\text{subject to,} \quad & g_j(x) \geq 0, & & j = 1, 2, \ldots, J, \\
& h_k(x) = 0, & & k = 1, 2, \ldots, K, \\
& x_i^{(L)} \leq x_i \leq x_i^{(U)}, & & i = 1, 2, \ldots, N.
\end{aligned} \tag{1}
$$

- EC techniques normally adopt exterior penalty methods.
- The penalty function method can be written as

$$
P(x, R) = f(x) + \Omega(R, g(x), h(x)), \tag{2}
$$

where R is a set of penalty parameters, Ω is the penalty term chosen to favor the selection of feasible point over infeasible point.

Penalty Function Methods

Parabolic Penalty

$$\Omega = R\{h(x)\}^2 \tag{3}$$

- It is used for handling equality constraints only.
- Since all infeasible points are penalized, it is an exterior penalty term.
- It starts with small value of R which increases gradually.

$g(x) \geq 0$

Bracket Operator Penalty

$$\Omega = R\,\langle g(x)\rangle^2 \tag{4}$$

where $\langle \alpha \rangle = \alpha$, when α is negative; zero otherwise.

$\alpha = -1.23$

$\langle \alpha \rangle = -1.23$

$\alpha = 0.27$

$\langle \alpha \rangle = 0$

- Since it assigns positive value to infeasible point, it is an exterior penalty term.
- It starts with small value of R which increases gradually.
- It is one of the commonly used methods.

Penalty Function Methods

- The penalty function method can be written as

$$P(x,R) = f(x) + \sum_{k=1}^{K} R_k \{h_k(x)\}^{\gamma} + \sum_{j=1}^{J} R_j \ \langle g_j(x) \rangle^{\beta}, \tag{5}$$

where R_k and R_j are the penalty parameters. Normally, β and γ are kept 1 and 2, respectively.

Handling Equality Constraints

- The penalty function methods can handle both equality and inequality constraints.
- One of the suggested ways to handle the equality constraints is

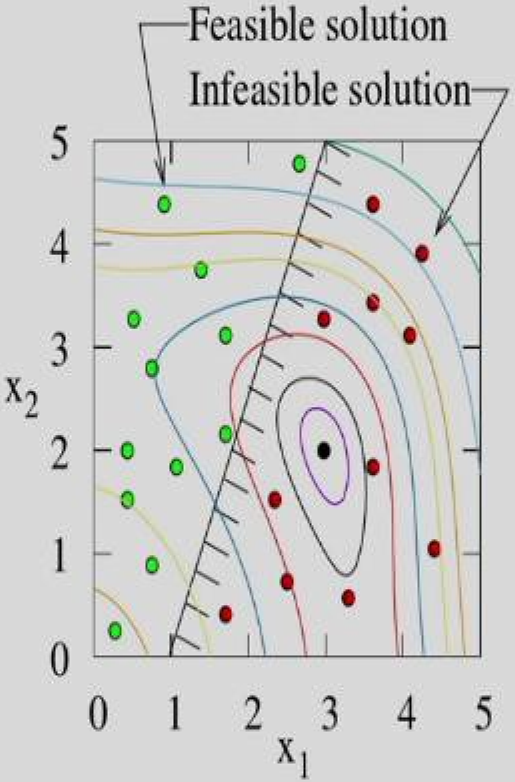
  $|h(x)| - \epsilon \leq 0$, where ϵ is the tolerance allowed (a small value).

Types of Penalty Functions

Types of Penalty Functions

- Death Penalty
- Static Penalty
- Dynamic Penalty
- Adaptive Penalty
- Other Approaches
  - Self-Adaptive Fitness Formulation
  - Stochastic Ranking, etc

Death Penalty

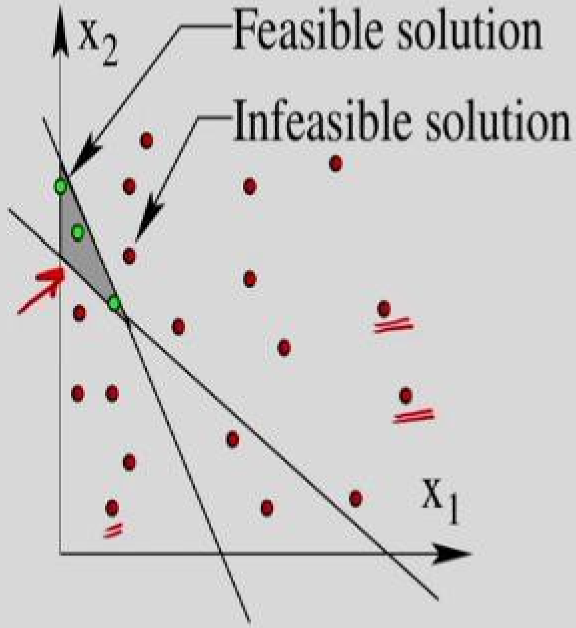


- Death Penalty: Infeasible solution is rejected and re-generated again.
- This is the easiest way to handle the constraints.
- It is considered as computationally efficient.

- It does not require to estimate the degree of violation for assigning the fitness to a solution.

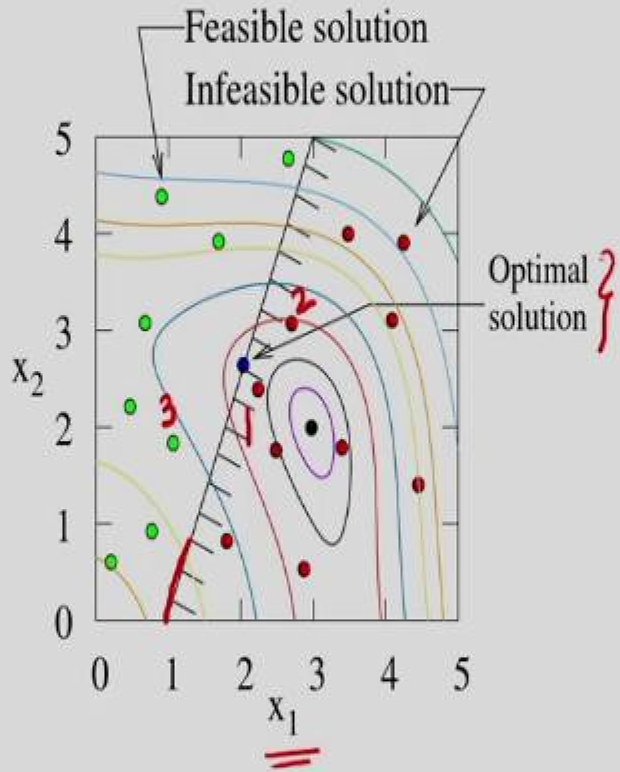
Limitations of Death Penalty

- It is not advisable for those problems in which the feasible space is limited.
- Thereby, it can stagnate search of EC techniques for small feasible spaced problems.



Limitations of Death Penalty

- Since an infeasible solution can be closer to the optimal solution in the variable-space than a feasible solution, the method cannot take advantage of such information.



Static Penalty

- The penalty function method can be written as

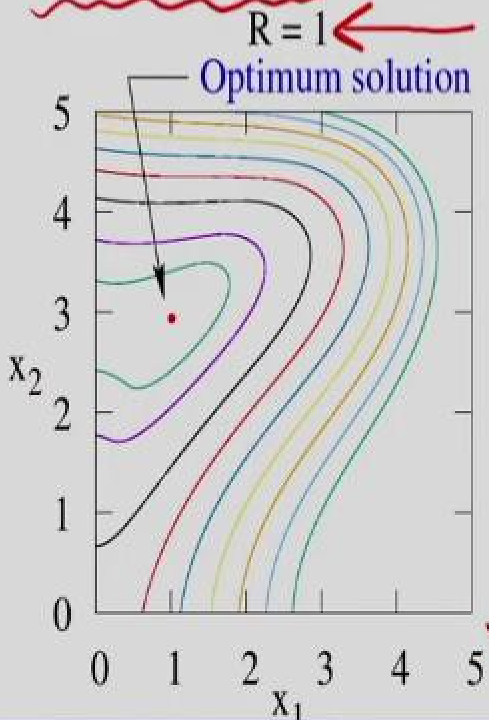
$$P(x, R) = f(x) + \sum_{k=1}^{K} R_k \{h(x)\}^{\gamma} + \sum_{j=1}^{J} R_j \ \langle g_j(x) \rangle^{\beta}, \tag{6}$$

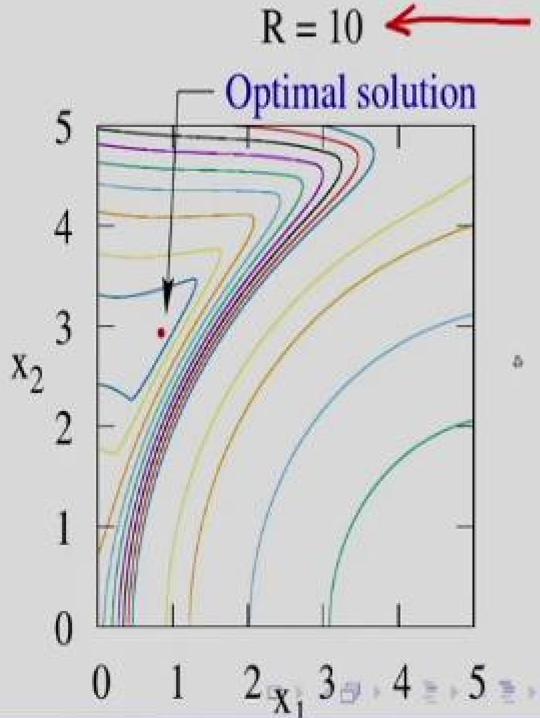
where R_k and R_j are the penalty parameters. Normally, β and γ are kept 1 and 2, respectively.

- The penalty parameters remain constant throughout evolutionary process.

Limitations of Static Penalty

- Penalty factors are problem dependent.
  - Small values of penalty factors may not be able to differentiate feasible and infeasible solutions in the populations.
  - Large values of penalty factors can distort the penalty function that can result in artificial local optimal solutions.





Dynamic Penalty

- The penalty factors include the current generation counter (t) in its computation.

$$P(x, R) = f(x) + (C \times t)^{\alpha} \left[\sum_{k=1}^{K} \{h(x)\}^{\gamma} + \sum_{j=1}^{J} \langle g_j(x) \rangle^{\beta} \right], \tag{7}$$

  where C, α, β and γ are the user defined constants.
- It suggests that the penalty term $((C \times t)^{\alpha})$ is increasing with the generation counter (t).
- Joines and Houck [1994] used $C = 0.5$, $\alpha = 1$ or 2, β and γ are kept 1 and 2, respectively.
- Here, $\langle g_j(x) \rangle = g_j(x)$, when $g_j(x) < 0$, otherwise, it is zero.
- We convert an equality constraint into two inequality constraints as $|h(x)| - \epsilon \leq 0$.

Limitation

- Although it is considered good for many EC techniques, it is difficult to produce good dynamic penalty factors for static functions.

Adaptive Penalty

- Similar to dynamic penalty, adaptive penalty also changes with generation counter (t)
- Bean and Hadj-Alouane [1992,1997] developed such an adaptive penalty

$$P(x,R) = f(x) + \lambda(t)\left[\sum_{k=1}^{K}|h(x)| + \sum_{j=1}^{J}\langle g_j(x)\rangle^2\right], \tag{8}$$

where $\lambda(t)$ is getting updated in every generation.

- It is calculated as

$$\lambda(t+1) = \begin{cases} (1/\beta_1)\cdot\lambda(t), & \text{if case \# 1} \\ \beta_2\cdot\lambda(t), & \text{if case \# 2} \\ \lambda(t), & \text{otherwise,} \end{cases}$$

where case # 1 denotes situation where the best solution in the last k generations was always feasible, case # 2 denotes situation where the best individual in the last k generations was never feasible, $\beta_1, \beta_2 > 1$, $\beta_1 > \beta_2$.

Adaptive Penalty

$$\lambda(t+1) = \begin{cases} (1/\beta_1) \cdot \lambda(t), & \text{if case \# 1} \\ \beta_2 \cdot \lambda(t), & \text{if case \# 2} \\ \lambda(t), & \text{otherwise,} \end{cases}$$

- The penalty term $\lambda(t+1)$ decreases if all the best solutions in the last k generations were feasible
- It increases if they were all infeasible.
- If there are some feasible and infeasible solutions tied as best in the population, then the penalty does not change.

Limitation

- The major issue with this penalty is the setting of the parameters that can be difficult sometimes.

Major Issues with Penalty Function Methods

- Choosing the best penalty term is not known a priori for any arbitrary problem.
- Large value of penalty terms can distort the function with artificial optimal solutions.
  - If the optimal solution is on the constraint boundary, EC techniques will push all solutions inside the feasible search space and may find difficulty to move toward the boundary.
- Small penalty terms will not penalize the infeasible solution as compared to the objective function value.
  - EC techniques will waste their effort in searching the solution in the infeasible space.

Hand Calculations

Himmelblau Function

$$\begin{aligned} \text{Minimize} \quad & f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2, \\ \text{subject to} \quad & (x_1 - 5)^2 + x_2^2 \leq 26, \\ & 4x_1 + x_2 \leq 20, \\ & x_1, x_2 \geq 0. \end{aligned}$$

- Let us convert the inequality constraints in the form of $g_j(x) \geq 0$.

$$\begin{aligned} g_1(x) = & \; 26 - (x_1 - 5)^2 - x_2^2 \geq 0, \\ g_2(x) = & \; 20 - 4x_1 - x_2 \geq 0. \end{aligned}$$

Hand Calculations

- Let us generate random solutions in the range of $0 \leq x_1, x_2 \leq 6$.

Index(i)	$(x_1, x_2)^T$
1	$(3.660, 4.595)^T$
2	$(2.380, 5.561)^T$
3	$(4.698, 3.219)^T$
4	$(3.755, 5.151)^T$
5	$(1.976, 1.754)^T$
6	$(3.654, 5.160)^T$
7	$(0.100, 3.858)^T$
8	$(2.446, 0.880)^T$

- Consider solution 1 $x^{(1)} = (3.660, 4.595)^T$.
- Calculate $f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2 = 364.823$
- Calculate $g_1(x_1, x_2) = 26 - (x_1 - 5)^2 - x_2^2 = 3.089$
- Calculate $g_2(x_1, x_2) = 20 - 4x_1 - x_2 = 0.765$
- Since both the constraints are satisfied, solution 1 is a feasible solution.
- Death Penalty: it will keep this solution because it is feasible.

Hand Calculations

- For solution 1, $f(x^{(1)}) = 364.823$, $g_1(x^{(1)}) = 3.089$ and $g_2(x^{(1)}) = 0.765$.

- Consider Static Penalty: $P(x, R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x) \rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
$P(x^{(1)}, R) = 364.823 + R_1 \ \langle 3.089 \rangle^2 + R_2 \ \langle 0.765 \rangle^2 = 364.823 + 0 + 0 = 364.823$

$\alpha \geq 0 \quad \langle \alpha \rangle = 0$

Hand Calculations

- For solution 1, $f(x^{(1)}) = 364.823$, $g_1(x^{(1)}) = 3.089$ and $g_2(x^{(1)}) = 0.765$.
- Consider Static Penalty: $P(x, R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x) \rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
  $P(x^{(1)}, R) = 364.823 + R_1 \ \langle 3.089 \rangle^2 + R_2 \ \langle 0.765 \rangle^2 = 364.823 + 0 + 0 = 364.823$
- Consider Dynamic Penalty: $P(x) = f(x) + (C \times t)^{\alpha} \left[\sum_{j=1}^{2} \langle g_j(x) \rangle^2 \right]$
- Considering $C = 0.5$, and $\alpha = 1$. Assuming it is the first generation, that is, $t = 1$.

Hand Calculations

- For solution 1, $f(x^{(1)}) = \boxed{364.823}$ $g_1(x^{(1)}) = 3.089$ and $g_2(x^{(1)}) = 0.765$.
- Consider Static Penalty: $P(x, R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x) \rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
  $P(x^{(1)}, R) = 364.823 + R_1 \ \langle 3.089 \rangle^2 + R_2 \ \langle 0.765 \rangle^2 = 364.823 + 0 + 0 = \boxed{364.823}$
- Consider Dynamic Penalty: $P(x) = f(x) + (C \times t)^\alpha \left[\sum_{j=1}^{J} \langle g_j(x) \rangle^2 \right]$
- Considering $C = 0.5$, and $\alpha = 1$. Assuming it is the first generation, that is, $t = 1$.
- The penalty function value is
  $P(x^{(1)}) = f(x) + (0.5 \times 1) \left[\langle 3.089 \rangle^2 + \langle 0.765 \rangle^2 \right] = 364.823 + 0 + 0 = \boxed{364.823}$
- Since it is a feasible solution, the penalty function value remains the same for static and dynamic penalty functions.

Hand Calculations

- For solution 2, we calculate $f(x^{(2)} = (2.380, 5.561)^T) = 692.216$, $g_1(x^{(2)}) = -11.791$ and $g_2(x^{(2)}) = 4.917$.
- The constraint $g_1(x^{(2)})$ is not satisfied, hence the solution is infeasible.

- Consider Static Penalty: $P(x,R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x)\rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
  $P(x^{(2)}, R) = 692.216 + \langle -11.791\rangle^2 + 5.0 \ \langle 4.917\rangle^2 = 692.216 + 139.021 + 0 = \boxed{831.236}$

- Consider Dynamic Penalty: $P(x) = f(x) + (C \times t)^\alpha \left[\sum_{j=1}^{J} \langle g_j(x)\rangle^2\right]$
- Considering $C = 0.5$, and $\alpha = 1$. Assuming it is the first generation, that is, $\boxed{t = 1.}$
- The penalty function value is $P(x^{(2)}) = 692.216 + (0.5 \times 1)\left[\langle -11.791\rangle^2 + \langle 4.917\rangle^2\right] =$
  $692.216 + 0.5[139.021 + 0] = \boxed{761.726}$
- If we consider the tenth generation, that is, $\boxed{t = 10}$, the penalty function value becomes
  $P(x^{(2)}) = 692.216 + \boxed{(0.5 \times 10)}[139.021 + 0] = \boxed{1387.319}$

Hand Calculations

- For solution 3, we calculate $f(x^{(3)} = (4.698, 3.219)^T) = 269.112$, $g_1(x^{(3)}) = 15.548$ and $g_2(x^{(3)}) = -2.010$.
- The constraint $g_2(x^{(3)})$ is not satisfied, hence the solution is infeasible.

- Consider Static Penalty: $P(x, R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x) \rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
  0
  $P(x^{(3)}, R) = 269.112 + \langle 15.548 \rangle^2 + 5.0 \ \langle -2.010 \rangle^2 = 269.112 + 0 + 20.208 = \boxed{289.320}$

Hand Calculations

- For solution 3, we calculate $f(x^{(3)} = (4.698, 3.219)^T) = 269.112$, $g_1(x^{(3)}) = 15.548$ and $g_2(x^{(3)}) = -2.010$.
- The constraint $g_2(x^{(3)})$ is not satisfied, hence the solution is infeasible.

- Consider Static Penalty: $P(x, R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x) \rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
$P(x^{(3)}, R) = 269.112 + \langle 15.548 \rangle^2 + 5.0 \ \langle -2.010 \rangle^2 = 269.112 + 0 + 20.208 = 289.320$

- Consider Dynamic Penalty: $P(x) = f(x) + (C \times t)^{\alpha} \left[\sum_{j=1}^{J} \langle g_j(x) \rangle^2 \right]$
- Considering $C = 0.5$, and $\alpha = 1$. Assuming it is the first generation, that is, $t = 1$.
- The penalty function value is $P(x^{(3)}) = 269.112 + (0.5 \times 1) \left[\langle 15.548 \rangle^2 + \langle -2.010 \rangle^2 \right] =$
$269.112 + 0.5[0 + 4.042] = \boxed{271.133}$

0

Hand Calculations

- For solution 4, we calculate $f(x^{(4)} = (3.755, 5.151)^T) = 610.196$, $g_1(x^{(4)}) = -2.081$ and $g_2(x^{(4)}) = -0.169$.
- Both constraints are not satisfied, hence the solution is infeasible.

- Consider Static Penalty: $P(x,R) = f(x) + \sum_{j=1}^{2} R_j \ \langle g_j(x) \rangle^2$
- Let us take $R_1 = 1$ and $R_2 = 5$.
- The penalty function value is
  $P(x^{(4)}, R) = 610.196 + \langle -2.081 \rangle^2 + 5.0 \ \langle -0.169 \rangle^2 = 614.671$

- Consider Dynamic Penalty: $P(x) = f(x) + (C \times t)^{\alpha} \left[\sum_{j=1}^{J} \langle g_j(x) \rangle^2 \right]$
- Considering $C = 0.5$, and $\alpha = 1$. Assuming it is the first generation, that is, $t = 1$.
- The penalty function value is
  $P(x^{(4)}) = 610.196 + (0.5 \times 1) \left[\langle -2.081 \rangle^2 + \langle -0.169 \rangle^2 \right] = 612.376$

Hand Calculations

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$	Static Penalty	Dynamic Penalty
1	$(3.660, 4.595)^T$	364.823	3.089	0.765	364.823	364.823
2	$(2.380, 5.561)^T$	692.216	−11.791	4.917	831.236	761.726
3	$(4.698, 3.219)^T$	269.112	15.548	−2.010	289.320	271.133
4	$(3.755, 5.151)^T$	610.196	−2.081	−0.169	614.671	612.376
5	$(1.976, 1.754)^T$	32.329	13.780	10.342	32.329	32.329
6	$(3.654, 5.160)^T$	598.194	−2.434	0.225	604.120	601.157
7	$(0.100, 3.858)^T$	114.638	−12.900	15.743	281.046	197.842
8	$(2.446, 0.880)^T$	31.385	18.704	9.335	31.385	31.385

Separation of Objective Function and Constraints

- These approaches handle objective function and constraints separately.

Superiority of Feasible Solutions over Infeasible

Powell and Skolnick Approach

- Considering minimization problem, fitness $F(x)$ of a solution is calculated as

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is feasible} \\ f(x) + R\left(\displaystyle\sum_{j=1}^{J} |\langle g_j(x)\rangle| + \sum_{k=1}^{K} |h_k(x)|\right) + \lambda(t, x), & \text{otherwise,} \end{cases} \tag{3}$$

Powell and Skolnick Approach

- The fitness is given as

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is feasible} \\ f(x) + R\left(\sum_{j=1}^{J} |\langle g_j(x)\rangle| + \sum_{k=1}^{K} |h_k(x)|\right) + \lambda(t,x), & \text{otherwise,} \end{cases}$$

- Here, R is the penalty factor, and $\lambda(t,x)$ is the difference between the worst feasible solution and the best static penalized function value among all infeasible solutions.
- The significance is that the best infeasible solution in the population will have the same fitness value as that of the worst feasible solution in the population.

Himmelblau Function

$$\begin{aligned} \text{Minimize} \quad & f(x_1,x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2, \\ \text{subject to} \quad & (x_1-5)^2 + x_2^2 \le 26, \\ & 4x_1 + x_2 \le 20, \\ & 0 \le x_1, x_2 \le 6. \end{aligned}$$

Hand Calculations

- Let is consider the following solutions.

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765
2	$(2.380, 5.561)^T$	692.216	-11.791	4.917
3	$(4.698, 3.219)^T$	269.112	15.548	-2.010
4	$(3.755, 5.151)^T$	610.196	-2.081	-0.169
5	$(1.976, 1.754)^T$	32.329	13.780	10.342
6	$(3.654, 5.160)^T$	598.194	-2.434	0.225
7	$(0.100, 3.858)^T$	114.638	-12.900	15.743
8	$(2.446, 0.880)^T$	31.385	18.704	9.335

- Let us consider $R = 2$.
- For calculating the fitness, the static penalty function is
  $F_s(x) = f(x) + R\left(\sum_{j=1}^{J} |\langle g_j(x)\rangle| + \sum_{k=1}^{K} |h_k(x)|\right)$

Hand Calculations

- Let us consider solution 1, $x^{(1)} = (3.660, 4.595)^T$, $f(x^{(1)}) = 364.823$, $g_1(x^{(1)}) = 3.089$ and $g_2(x^{(1)}) = 0.765$.
- Since it is a feasible solution, $F_s(x^{(1)}) = f(x^{(1)}) = 364.823$.
- Let us consider solution 2, $x^{(2)} = (2.380, 5.561)^T$, $f(x^{(2)}) = 692.216$, $g_1(x^{(2)}) = -11.791$ and $g_2(x^{(2)}) = 4.917$.
- It is an infeasible solution.
- The penalty function value is $F_s(x^{(2)}) = f(x^{(2)}) + R(|\langle g_1(x^{(2)})\rangle| + |\langle g_2(x^{(2)})\rangle|)$ $= 692.216 + 2(11.791) + 0 = 715.797$.
- Let us consider solution 3, $x^{(3)} = (4.698, 3.219)^T$, $f(x^{(3)}) = 269.112$, $g_1(x^{(3)}) = 15.548$ and $g_2(x^{(3)}) = -2.010$.
- It is an infeasible solution.
- The penalty function value is $F_s(x^{(3)}) = f(x^{(3)}) + R(|\langle g_1(x^{(3)})\rangle| + |\langle g_2(x^{(3)})\rangle|)$ $= 269.112 + 0 + 2(2.010) = 273.133$.

Hand Calculations

- Let us consider $R = 2$.

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$	Static Penalty, $F_s(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765	364.823
2	$(2.380, 5.561)^T$	692.216	−11.791	4.917	715.797
3	$(4.698, 3.219)^T$	269.112	15.548	−2.010	273.133
4	$(3.755, 5.151)^T$	610.196	−2.081	−0.169	614.697
5	$(1.976, 1.754)^T$	32.329	13.780	10.342	32.329
6	$(3.654, 5.160)^T$	598.194	−2.434	0.225	603.063
7	$(0.100, 3.858)^T$	114.638	−12.900	15.743	140.438
8	$(2.446, 0.880)^T$	31.385	18.704	9.335	31.385

- The infeasible solutions are '2', '3', '4', '6' and '7'.
- The best fitness among the infeasible solutions in the population is 140.438 corresponding to solution '7'.
- The worst fitness of feasible solution in the population is 364.823 corresponding to solution '1'.

Hand Calculations

- Therefore, $\lambda(t,x) = 364.823 - 140.438 = 224.385$.
- Since solution 1 is feasible, the fitness will remain same as objective function value.
- Let us consider solution 2, which has static penalty function value 715.797.
- The fitness of solution 2 is $F(x^{(2)}) = 715.797 + \lambda(t,x) = 940.182$
- The fitness assigned to each solution by Powell and Skolnick approach is

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$	Penalty function	$F(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765	364.823	364.823
2	$(2.380, 5.561)^T$	692.216	−11.791	4.917	715.797	940.182
3	$(4.698, 3.219)^T$	269.112	15.548	−2.010	273.133 $+\lambda$	497.518
4	$(3.755, 5.151)^T$	610.196	−2.081	−0.169	614.697 $+\lambda$	839.082
5	$(1.976, 1.754)^T$	32.329	13.780	10.342	32.329	32.329
6	$(3.654, 5.160)^T$	598.194	−2.434	0.225	603.063 $+\lambda$	827.448
7	$(0.100, 3.858)^T$	114.638	−12.900	15.743	140.438 $+\lambda$	364.823
8	$(2.446, 0.880)^T$	31.385	18.704	9.335	31.385	31.385

- The fitness of the best infeasible solution '7' and the fitness of the worst feasible solution '1' are the same.

Hand Calculations

- Let us consider large value of $R = 100$.

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$	Static Penalty, $F(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765	364.823
2	$(2.380, 5.561)^T$	692.216	−11.791	4.917	1871.286
3	$(4.698, 3.219)^T$	269.112	15.548	−2.010	470.149
4	$(3.755, 5.151)^T$	610.196	−2.081	−0.169	835.214
5	$(1.976, 1.754)^T$	32.329	13.780	10.342	32.329
6	$(3.654, 5.160)^T$	598.194	−2.434	0.225	841.635
7	$(0.100, 3.858)^T$	114.638	−12.900	15.743	1404.632
8	$(2.446, 0.880)^T$	31.385	18.704	9.335	31.385

- The best fitness among the infeasible solutions in the population is 470.149 corresponding to solution '3'.
- The worst fitness of feasible solution in the population is 364.823 corresponding to solution '1'.
- Therefore, $\lambda(t, x) = 364.823 - 470.149 = -105.326$.

Hand Calculations

- The fitness assigned to each solution by Powell and Skolnick approach for $R = 100$ is

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$	$F(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765	364.823
2	$(2.380, 5.561)^T$	692.216	−11.791	4.917	1765.959
3	$(4.698, 3.219)^T$	269.112	15.548	−2.010	364.823
4	$(3.755, 5.151)^T$	610.196	−2.081	−0.169	729.888
5	$(1.976, 1.754)^T$	32.329	13.780	10.342	32.329
6	$(3.654, 5.160)^T$	598.194	−2.434	0.225	736.309
7	$(0.100, 3.858)^T$	114.638	−12.900	15.743	1299.305
8	$(2.446, 0.880)^T$	31.385	18.704	9.335	31.385

- The fitness of the best infeasible solution '3' and the fitness of the worst feasible solution '1' are the same.

Deb's Approach

- Deb's approach is similar to Powell and Skolnick approach. However, it does not require any penalty parameter, R and $\lambda(t, x)$.

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is feasible;} \\ f_{max} + \sum_{j=1}^{J} |\langle g_j(x) \rangle| + \sum_{k=1}^{K} |h_k(x)|, & \text{Otherwise.} \end{cases} \quad (4)$$

- Here, f_{max} is the objective function value of the worst feasible solution in the population.
- Therefore, this approach is considered as penalty parameter-less approach.

Hand Calculations

- Let us consider the following solutions

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765
2	$(2.380, 5.561)^T$	692.216	-11.791	4.917
3	$(4.698, 3.219)^T$	269.112	15.548	-2.010
4	$(3.755, 5.151)^T$	610.196	-2.081	-0.169
5	$(1.976, 1.754)^T$	32.329	13.780	10.342
6	$(3.654, 5.160)^T$	598.194	-2.434	0.225
7	$(0.100, 3.858)^T$	114.638	-12.900	15.743
8	$(2.446, 0.880)^T$	31.385	18.704	9.335

- The feasible solutions in the population are '1', '5' and '8'.
- Among them, the worst feasible solution is '1' with the objective function value 364.823.
- It means that $f_{max} = 364.823$.

Hand Calculations

- Let us consider solution 1, $x^{(1)} = (3.660, 4.595)^T$, $f(x^{(1)}) = 364.823$, $g_1(x^{(1)}) = 3.089$ and $g_2(x^{(1)}) = 0.765$.
- Since it is a feasible solution, $F(x^{(1)}) = f(x^{(1)}) = 364.823$.
- Let us consider solution 2, $x^{(2)} = (2.380, 5.561)^T$, $f(x^{(2)}) = 692.216$, $g_1(x^{(2)}) = -11.791$ and $g_2(x^{(2)}) = 4.917$.
- It is an infeasible solution.
- Fitness of solution 2 is $F(x^{(2)}) = |\langle g_1(x^{(2)})\rangle| + |\langle g_2(x^{(2)})\rangle| + f_{max} = 11.791 + 364.823 = 376.614$.
- Let us consider solution 3, $x^{(3)} = (4.698, 3.219)^T$, $f(x^{(3)}) = 269.112$, $g_1(x^{(3)}) = 15.548$ and $g_2(x^{(3)}) = -2.010$.
- It is an infeasible solution.
- The fitness solution 3 is $F(x^{(3)}) = \underbrace{|\langle \overset{0}{g_1(x^{(3)})}\rangle| + |\langle g_2(x^{(3)})\rangle|} + \underbrace{f_{max}}$ $= 2.010 + 364.823 = \underline{366.833}$.

Hand Calculations

- The fitness of each solution using Deb's approach is

Index(i)	$(x^{(i)})^T$	$f(x^{(i)})$	$g_1(x^{(i)})$	$g_2(x^{(i)})$	$F(x^{(i)})$
1	$(3.660, 4.595)^T$	364.823	3.089	0.765	364.823
2	$(2.380, 5.561)^T$	692.216	−11.791	4.917	376.614
3	$(4.698, 3.219)^T$	269.112	15.548	−2.010	366.833
4	$(3.755, 5.151)^T$	610.196	−2.081	−0.169	367.073
5	$(1.976, 1.754)^T$	32.329	13.780	10.342	32.329
6	$(3.654, 5.160)^T$	598.194	−2.434	0.225	367.257
7	$(0.100, 3.858)^T$	114.638	−12.900	15.743	377.723
8	$(2.446, 0.880)^T$	31.385	18.704	9.335	31.385

- The best infeasible solution '3' has more fitness value than the worst feasible solution '1'.
- A feasible solution is always better than any infeasible solution in the population.

Closure

- Constraint handling via separation of objective function and constraints
  - Powell and Skolnick's approach
  - Deb's approach
- Hand calculations for both the approaches