# Introduction to Toplogy I: Homework 5

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**Exercise 4.4**. Let $(X, \mathrm{d})$ be a metric space, let $f : X \to X$, suppose there is a "Lipschitz constant" $r \in [0, 1)$ such that for all $p, q \in X$ we have

$$\mathrm{d}(f(p), f(q)) \leq r\mathrm{d}(p, q).$$

Prove that $f$ is continuous. Hint: Take $\delta = \varepsilon$.

*Solution.* Let $\varepsilon > 0$. Take $\delta = \varepsilon$. If $d(p, q) < \delta$ then

$$\mathrm{d}(f(p), f(q)) \leq r\mathrm{d}(p, q) < r\delta = r\varepsilon \leq \varepsilon,$$

since $0 \leq r < 1$. Thus for every $p \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ (namely $\delta = \varepsilon$) with $\mathrm{d}(p, q) < \delta$ implying $\mathrm{d}(f(p), f(q)) < \varepsilon$. Hence $f$ is continuous. □

**Exercise 4.5**. Here are two examples of how Theorem 4.13 can fail if the hypotheses are weakened.

(i) Instead of asking for a uniform constant $r < 1$ such that $\mathrm{d}(f(p), f(q)) \leq r \cdot \mathrm{d}(p, q)$ for all $p, q \in X$, we might just have asked that $\mathrm{d}(f(p), f(q)) < \mathrm{d}(p, q)$ whenever $p \neq q$. But let $X = [1, \infty)$ with the usual metric, and let $f : X \to X$ be defined by $f(x) = x + 1/x$; prove that $f$ satisfies this weaker condition, but does not have a fixed point.

(ii) The theorem can also fail if the space is not complete: let $X = [1, \infty) \cap \mathbb{Q}$ with the usual metric, and let $f : X \to X$ be defined by $f(x) = x/2 + 1/x$; prove that $f$ satisfies the hypothesis of the theorem with of $r = 1/2$, but does not have a fixed point

*Solution (i).* Let $X = [1, \infty)$ and $f(x) = x + 1/x$. For $x > y \geq 1$,

$$f(x) - f(y) = (x - y) + \left(\frac{1}{x} - \frac{1}{y}\right) = (x - y)\left(1 - \frac{1}{xy}\right).$$

Since $xy \geq 1$ we have $0 \leq 1 - 1/xy < 1$, hence

$$0 < f(x) - f(y) < x - y,$$

so $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$. Thus the weaker condition holds.

If $x$ were a fixed point then $x + 1/x = x$, so $1/x = 0$, which is impossible. Therefore $f$ has no fixed point. □

*Solution (ii).* Let $X = [1, \infty) \cap \mathbb{Q}$ and $f(x) = \dfrac{x}{2} + \dfrac{1}{x}$. For $x, y \geq 1$,

$$f(x) - f(y) = \frac{x - y}{2} + \left(\frac{1}{x} - \frac{1}{y}\right) = (x - y)\left(\frac{1}{2} - \frac{1}{xy}\right),$$

so

$$|f(x) - f(y)| = |x - y|\left|\frac{1}{2} - \frac{1}{xy}\right|.$$

Because $1/(xy) \in (0, 1]$ for $x, y \geq 1$, we have $|1/2 - 1/xy| \leq 1/2$. Thus

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|,$$

for all $x, y \in X$. Hence $f$ satisfies the hypothesis with $r = 1/2$. Note $f(X) \subset \mathbb{Q}$ since $x \mapsto x/2$ and $x \mapsto 1/x$ preserve rationals, so $f : X \to X$ is well defined.

A fixed point would satisfy $x/2 + 1/x = x$, i.e. $x^2 = 2$. This solution $x = \sqrt{2}$ is not rational, so there is no fixed point in $X$. Thus the theorem can fail when the space is not complete. □

**Exercise 5.3**. Let $(X, \mathrm{d})$ be any metric space (possibly incomplete), and let $U, V \subset X$ be two open, dense subsets. Prove that $U \cap V$ is again dense.

*Solution.* Let $W \subset X$ be any nonempty open set. Since $U$ is dense, $W \cap U \neq \emptyset$. As $U$ is open, $W \cap U$ is a nonempty open subset of $X$. Because $V$ is dense, $(W \cap U) \cap V \neq \emptyset$. Hence

$$W \cap (U \cap V) = (W \cap U) \cap V \neq \emptyset.$$

Thus every nonempty open $W$ meets $U \cap V$, so $\overline{U \cap V} = X$. Therefore $U \cap V$ is dense. □

**Exercise 5.4**.

(i) A point $p$ in a metric space $(X, \mathrm{d})$ is called *isolated* if there is some $r > 0$ such that $B_r(p) = \{p\}$. Use the Baire category theorem to prove that a complete metric space with no isolated points is uncountable.

(ii) Give an example of a countable, complete metric space.

*Solution to (i).* Suppose $X$ is complete, has no isolated points, and is countable. Write

$$X = \bigcup_{n=1}^{\infty} \{x_n\}.$$

Each singleton $\{x_n\}$ is closed. Because $x_n$ is not isolated, $\{x_n\}$ has empty interior, hence it is nowhere dense. Thus $X$ is a countable union of nowhere dense sets, so $X$ is meager. This contradicts the Baire Category Theorem (complete metric spaces are Baire). Therefore $X$ cannot be countable. □

*Solution to (ii).* Take $X = \mathbb{Z}$ with the usual metric $\mathrm{d}(m, n) = |m - n|$. Any Cauchy sequence in $\mathbb{Z}$ is eventually constant, so it converges in $\mathbb{Z}$. Hence $\mathbb{Z}$ is complete and countable. □

**Exercise 5.7**. Give examples to show that Proposition 5.6 can fail

(i) if the sets $F_n$ are not closed.

(ii) if the metric space $X$ is not complete.

(iii) if the diameters all finite, but do not go to zero.

(This is really tricky, because it can't happen in $\mathbb{R}^n$ with the Euclidean metric. But let $X = C([0,1])$ with the sup metric, and let $F_n$ be the set of continuous functions $f : [0,1] \to [0,1]$ with $f(0) = 1$ and $f(x) = 0$ for $x \geq 1/n$. We can see that $F_1 \supset F_2 \supset \cdots$ and the proof that each $F_n$ is closed is similar to Exercise 3.4. Prove that $\operatorname{diam}(F_n) = 1$, but that $F_1 \cap F_2 \cap \cdots$ is empty.)

*Solution to (i).* Take $X = \mathbb{R}$ with the usual metric and

$$F_n = \left(0, \frac{1}{n}\right).$$

Then $F_1 \supset F_2 \supset \cdots$, each $F_n$ is nonempty (but not closed), $\operatorname{diam}(F_n) = 1/n \to 0$, yet $\bigcap_{n=1}^{\infty} F_n = \emptyset$. This shows failure when the sets are not closed. □

*Solution to (ii).* Let $X = \mathbb{Q}$ (with the usual metric). For each $n$ set

$$F_n = \left[\sqrt{2} - \frac{1}{n} \sqrt{2} + \frac{1}{n}\right] \cap \mathbb{Q}.$$

Each $F_n$ is closed in $X = \mathbb{Q}$, nonempty, nested, and $\operatorname{diam}(F_n) \leq 2/n \to 0$. But $\bigcap_{n=1}^{\infty} F_n = \{\sqrt{2}\} \cap \mathbb{Q} = \emptyset$ since $\sqrt{2} \notin \mathbb{Q}$. This shows failure when $X$ is not complete. □

*Solution to (iii).* Let $X = C([0,1])$ with the sup metric and for each $n$ define $F_n$ to be the set of continuous $f : [0,1] \to [0,1]$ with $f(0) = 1$ and $f(x) = 0$ for all $x \geq 1/n$. Then $F_1 \supset F_2 \supset \cdots$ and each $F_n$ is closed.

If the diameter of $F_n$ is 1, then pick $0 < x_1 < x_2 < 1/n$. Define $f \in F_n$ with $f(0) = 1$, linear down to 0 at $x_1$, and $g \in F_n$ with $g(0) = 1$, constant 1 on $[0, x_2]$ then linear to 0 at $1/n$. At any $x \in (x_1, x_2)$ we have $f(x) = 0$ and $g(x) = 1$, so $\mathrm{d}_\infty(f, g) = 1$. Thus $\operatorname{diam}(F_n) \geq 1$, and since all functions take values in $[0,1]$ we have $\operatorname{diam}(F_n) \leq 1$, hence $\operatorname{diam}(F_n) = 1$.

Otherwise, if $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then $f$ belongs to every $F_n$. This means that for each $n$, $f(x) = 0$ for all $x \geq 1/n$. Hence $f(x) = 0$ for every $x > 0$, while $f(0) = 1$, contradicting continuity. So the intersection is empty. $\square$