# Introduction to Proof: Homework 2

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*Victor Ostrik 12:00*

**Hashem A. Damrah**
UO ID: 952102243

**Problem 1** In each of the following give a disjunction that is equivalent to the given proposition:

(i) $P \Rightarrow Q$.

(ii) $\neg P \Rightarrow Q$.

(iii) $P \Rightarrow \neg Q$.

*Solution to (i).* $\neg P \vee Q$. □

*Solution to (ii).* $P \vee Q$. □

*Solution to (iii).* $\neg P \vee \neg Q$. □

**Problem 2** Translate the following into a symbolic logic problem, then provide a proof:

Given: If Smith wins the nomination, he will be happy, and if he is happy, he is not a good campaigner. But if he loses the nomination, he will lose the confidence of the party. He is not a good campaigner if he loses the confidence of the party. If he is not a good campaigner, then he should resign from the party. Either Smith wins the nomination or he loses it.

Prove: Smith should resign from the party.

*Solution.* Let $W$ = "Smith wins the nomination", $H$ = "Smith is happy", $G$ = "Smith is a good campaigner", $C$ = "Smith has the confidence of the party", and $R$ = "Smith should resign from the party".

Translating the given information into symbolic logic

| | | | |
|---|---|---|---|
| 1. | $W \Rightarrow H$ | Hypothesis | □ |
| 2. | $H \Rightarrow \neg G$ | Hypothesis | |
| 3. | $\neg W \Rightarrow \neg C$ | Hypothesis | |
| 4. | $\neg G \Rightarrow R$ | Hypothesis | |
| 5. | $W \vee \neg W$ | Hypothesis | |
| 6. | $W$ | Dischargeable Hypothesis | |
| 7. | $H$ | MP, for 6, for 1 | |
| 8. | $\neg G$ | MP, for 7, for 2 | |
| 9. | $R$ | MP, for 8, for 4 | |
| 10 | $W \Rightarrow R$ | DT, discharge for 6 [(6) - (9) unusable] | |

**Problem 3** Fill in the blanks to give a proof of $R \vee [P \wedge Q], \neg Q \vdash R$.

1. $R \vee [P \wedge Q]$ ??
2. $\neg R$ ??
3. ?? Tautology
4. $\neg R \Rightarrow [P \wedge Q]$ ??
5. $P \wedge Q$ ??
6. ?? RCS, ??
7. ?? Hypothesis
8. $Q \wedge \neg Q$ ??
9. $R$ ??

*Solution.*

1. $R \vee [P \wedge Q]$ Hypothesis
2. $\neg R$ Dischargeable Hypothesis
3. $\neg R \vee R$ Tautology
4. $\neg R \Rightarrow [P \wedge Q]$ DI, for 2, for 1
5. $P \wedge Q$ MPD, for 2, for 4
6. $Q$ RCS, for 5
7. $\neg Q$ Hypothesis
8. $Q \wedge \neg Q$ CI, for 6, for 7
9. $R$ II, discharge for 2 [(2) - (8) unusable]

**Problem 4** Fill in the blanks to give a proof of $(P \wedge \neg Q) \Rightarrow (R \Rightarrow Q) \vdash P \Rightarrow [Q \vee \neg R]$. [Note: This proof uses both DT and Indirect Inference].

| | | |
|---|---|---|
| 1. | $(P \wedge \neg Q) \Rightarrow (R \Rightarrow Q)$ | ?? |
| 2. | $P$ | Dischargeable Hypothesis |
| 3. | $\neg[Q \vee \neg R]$ | Dischargeable Hypothesis |
| 4. | ?? | Tautology |
| 5. | $\neg Q \wedge R$ | ?? |
| 6. | $\neg Q$ | ?? |
| 7. | $P \wedge \neg Q$ | ?? |
| 8. | $R \Rightarrow Q$ | ?? |
| 9. | $\neg R$ | ?? |
| 10. | $R$ | ?? |
| 11. | $R \wedge \neg R$ | ?? |
| 12. | $Q \vee \neg R$ | ?? |
| 13. | $P \Rightarrow [Q \vee \neg R$ | ?? |

*Solution.*

| | | |
|---|---|---|
| 1. | $(P \wedge \neg Q) \Rightarrow (R \Rightarrow Q)$ | Hypothesis |
| 2. | $P$ | Dischargeable Hypothesis |
| 3. | $\neg[Q \vee \neg R]$ | Dischargeable Hypothesis |
| 4. | $\neg[Q \vee \neg R] \Leftrightarrow \neg Q \wedge R$ | Tautology |
| 5. | $\neg Q \wedge R$ | MPB, for 3, for 4 |
| 6. | $\neg Q$ | LCS, for 5 |
| 7. | $P \wedge \neg Q$ | CI, for 2, for 6 |
| 8. | $R \Rightarrow Q$ | MP, for 7, for 1 |
| 9. | $\neg R$ | MT, for 6, for 8 |
| 10. | $R$ | RCS, for 5 |
| 11. | $R \wedge \neg R$ | CI, for 10, for 9 |
| 12. | $Q \vee \neg R$ | II, discharge for 3 [(3) - (11) unusable] |
| 13. | $P \Rightarrow [Q \vee \neg R]$ | DT, discharge for 2 [(2) - (12) unusable] |

**Problem 5** Show that $[P \wedge Q] \Rightarrow R, \neg R, P \vdash \neg Q$.

*Solution.*

| | | |
|---|---|---|
| 1. | $[P \wedge Q] \Rightarrow R$ | Hypothesis |
| 2. | $\neg R$ | Hypothesis |
| 3. | $P$ | Hypothesis |
| 4. | $\neg[P \wedge Q]$ | MT, for 2, for 1 |
| 5. | $\neg[P \wedge Q] \Leftrightarrow [\neg P \vee \neg Q]$ | Tautology |
| 6. | $\neg P \vee \neg Q$ | MPB, for 4, for 5 |
| 7. | $\neg Q$ | DI, for 3, for 6 |

**Problem 6** Show that $P \Rightarrow Q$, $R$, $R \Rightarrow [Q \Rightarrow P] \vdash P \Leftrightarrow Q$.

*Solution.*

| | | |
|---|---|---|
| 1. | $P \Rightarrow Q$ | Hypothesis |
| 2. | $R$ | Hypothesis |
| 3. | $R \Rightarrow [Q \Rightarrow P]$ | Hypothesis |
| 4. | $Q \Rightarrow P$ | MP, for 2, for 3 |
| 5. | $P \Leftrightarrow Q$ | For 1, for 4 |

**Problem 7** Show that $P \Rightarrow \neg Q$, $\neg R \Rightarrow Q \vdash P \Rightarrow R$.

*Solution.*

| | | |
|---|---|---|
| 1. | $P \Rightarrow \neg Q$ | Hypothesis |
| 2. | $\neg R \Rightarrow Q$ | Hypothesis |
| 3. | $\neg Q \Rightarrow R$ | Contrapositive, for 2 |
| 4. | $P \Rightarrow R$ | SI, for 1, for 3 |

**Problem 8** Show that $\neg P \Rightarrow Q$, $T \Rightarrow \neg P$, $\neg[Q \vee R] \vdash \neg T$

*Solution.*

| | | |
|---|---|---|
| 1. | $\neg P \Rightarrow Q$ | Hypothesis |
| 2. | $T \Rightarrow \neg P$ | Hypothesis |
| 3. | $\neg[Q \vee R]$ | Hypothesis |
| 4. | $\neg[Q \vee R] \Leftrightarrow [\neg Q \wedge \neg R]$ | Tautology |
| 5. | $\neg Q \wedge \neg R$ | MPB, for 3, for 4 |
| 6. | $\neg Q$ | RCS, for 5 |
| 7. | $P$ | MT, for 6, for 1 |
| 8. | $\neg T$ | MT, for 7, for 2 |

**Problem 9** Show that $\neg P \Rightarrow Q$, $Q \Rightarrow [R \Rightarrow S]$, $\neg S \vdash R \Rightarrow P$.

*Solution.*

| | | |
|---|---|---|
| 1. | $\neg P \Rightarrow Q$ | Hypothesis |
| 2. | $Q \Rightarrow [R \Rightarrow S]$ | Hypothesis |
| 3. | $\neg S$ | Hypothesis |
| 4. | $\neg P$ | Dischargeable Hypothesis |
| 5. | $Q$ | MP, for 4, for 1 |
| 6. | $R \Rightarrow S$ | MP, for 5, for 2 |
| 7. | $\neg R$ | MT, for 3, for 6 |
| 8. | $\neg P \Rightarrow \neg R$ | DT, discharge for 4 [(4) - (7) unusable] |
| 9. | $R \Rightarrow P$ | Contrapositive |

**Problem 10** Show that $P \Rightarrow T$, $Q \Rightarrow T$, $R \Leftrightarrow [P \vee Q]$, $R \vdash T$.

*Solution.*

| | | |
|---|---|---|
| 1. | $P \Rightarrow T$ | Hypothesis |
| 2. | $Q \Rightarrow T$ | Hypothesis |
| 3. | $R \Leftrightarrow [P \vee Q]$ | Hypothesis |
| 4. | $R$ | Hypothesis |
| 5. | $P \vee Q$ | MPB, for 4, for 3 |
| 6. | $[P \vee Q] \Rightarrow T$ | IC, for 1, for 2 |
| 7. | $T$ | MP, for 5, for 6 |

**Problem 11** Show that $S \Rightarrow P$, $Q \Rightarrow R$, $S \vdash [P \Rightarrow Q] \Rightarrow R$.

*Solution.*

| | | |
|---|---|---|
| 1. | $S \Rightarrow P$ | Hypothesis |
| 2. | $Q \Rightarrow R$ | Hypothesis |
| 3. | $S$ | Hypothesis |
| 4. | $P$ | MP, for 3, for 1 |
| 5. | $P \Rightarrow Q$ | Dischargeable Hypothesis |
| 6. | $Q$ | MP, for 4, for 5 |
| 7. | $R$ | MP, for 6, for 2 |
| 8. | $[P \Rightarrow Q] \Rightarrow R$ | DT, discharge for 5 [(5) - (7) unusable] |

**Problem 12** Show that $R \Rightarrow T$, $\neg T \Leftrightarrow S$, $[R \wedge \neg S] \Rightarrow \neg Q \vdash R \Rightarrow \neg Q$.

*Solution.*

1. $R \Rightarrow T$ Hypothesis
2. $\neg T \Leftrightarrow S$ Hypothesis
3. $[R \wedge \neg S] \Rightarrow \neg Q$ Hypothesis
4. $R$ Dischargeable Hypothesis
5. $T$ MP, for 4, for 1
6. $\neg S$ MT, for 5, for 2
7. $R \wedge \neg S$ CI, for 4, for 6
8. $\neg Q$ MP, for 7, for 3
9. $R \Rightarrow \neg Q$ DT, discharge for 4 [(4) - (8) unusable]

**Problem 13** Show that $\neg P \Rightarrow Q$, $[R \Rightarrow Q] \Rightarrow S$, $\neg S \vee T$, $R \Rightarrow \neg P \vdash T \vee V$.

*Solution.*

1. $\neg P \Rightarrow Q$ Hypothesis
2. $[R \Rightarrow Q] \Rightarrow S$ Hypothesis
3. $\neg S \vee T$ Hypothesis
4. $R \Rightarrow \neg P$ Hypothesis
5. $R$ Dischargeable Hypothesis
6. $\neg P$ MP, for 5, for 4
7. $Q$ MP, for 6, for 1
8. $R \Rightarrow Q$ DT, discharge for 5 [(5) - (7) unusable]
9. $S$ MP, for 8, for 2
10. $T$ DI, for 9, for 3
11. $T \vee V$ CI, for 10

**Problem 14** Show that $[R \wedge \neg Q] \Rightarrow P$, $[T \Rightarrow S] \Leftrightarrow [R \Rightarrow Q]$, $R \vdash [\neg P \vee [T \Rightarrow S]] \Rightarrow Q$.

| | | |
|---|---|---|
| *Solution.* 1. | $[R \wedge \neg Q] \Rightarrow P$ | Hypothesis |
| 2. | $[T \Rightarrow S] \Leftrightarrow [R \Rightarrow Q]$ | Hypothesis |
| 3. | $R$ | Hypothesis |
| 4. | $\neg P \vee [T \Rightarrow S]$ | Dischargeable Hypothesis |
| 5. | $\neg P$ | Dischargeable Hypothesis |
| 6. | $R \wedge \neg Q$ | Dischargeable Hypothesis |
| 7. | $P$ | MP, for 1, for 6 |
| 8. | $P \wedge \neg P$ | CI, for 7, for 5 |
| 10. | $\neg[R \wedge \neg Q]$ | II, discharge for 6 [(6) - (9) unusable] |
| 11. | $\neg[R \wedge \neg Q] \Leftrightarrow [\neg R \vee Q]$ | Tautology |
| 12. | $\neg R \vee Q$ | MP, for 10, for 11 |
| 13. | $Q$ | DI, for 3, for 12 |
| 14. | $[\neg P \vee [T \Rightarrow S]] \Rightarrow Q$ | DT, discharge for 4 [(4) - (13) unusable] |

**Problem 15** Use the Euclidean Algorithm to find integers $a$ and $b$ such that $37a + 100b = 1$. Use this information to solve $37x + 42 = 15$ in $\mathbb{Z}_{100}$.

*Solution.* (i) $100 = 2 \cdot 37 + 26 \Rightarrow 100 - 2 \cdot 37 = 26$,

(ii) $37 = 1 \cdot 26 + 11 \Rightarrow 37 - 1 \cdot 26 = 11$,

(iii) $26 = 2 \cdot 11 + 4 \Rightarrow 26 - 2 \cdot 11 = 4$,

(iv) $11 = 2 \cdot 4 + 3 \Rightarrow 11 - 2 \cdot 4 = 3$,

(v) $4 = 1 \cdot 3 + 1 \Rightarrow 4 - 1 \cdot 3 = 1$.

Finally, write the equation for the greatest common divisor $1 = 4 - 1 \cdot 3$. Now, we back-substitute to express 1 as a linear combination of 37 and 100.

(i) Substitute $3 = 11 - 2 \cdot 4$: $1 = 4 - 1 \cdot (11 - 2 \cdot 4) = 3 \cdot 4 - 1 \cdot 11$.

(ii) Substitute $4 = 26 - 2 \cdot 11$: $1 = 3 \cdot (26 - 2 \cdot 11) - 1 \cdot 11 = 3 \cdot 26 - 7 \cdot 11$.

(iii) Substitute $11 = 37 - 1 \cdot 26$: $1 = 3 \cdot 26 - 7 \cdot (37 - 1 \cdot 26) = 10 \cdot 26 - 7 \cdot 37$.

(iv) Substitute $26 = 100 - 2 \cdot 37$: $1 = 10 \cdot (100 - 2 \cdot 37) - 7 \cdot 37 = 10 \cdot 100 - 27 \cdot 37$.

So, we find $a = -27, \quad b = 10$. Now we can solve the congruence.

First, subtract 42 from both sides: $37x \equiv -27 \pmod{100}$. Since $-27 \equiv 73 \pmod{100}$, we can rewrite this as $37x \equiv 73 \pmod{100}$. From Step 1, we found that $37 \cdot (-27) \equiv 1 \pmod{100}$, so the inverse of 37 modulo 100 is $-27 \equiv 73 \pmod{100}$. Now multiply both sides of the congruence by 73: $x \equiv 73 \cdot 73 \pmod{100}$ Calculate $73 \cdot 73 \pmod{100}$: $73 \cdot 73 = 5329$. Find $5329 \pmod{100}$: $5329 \pmod{100} = 29$

Thus, $x = 29$ is the solution to $37x + 42 \equiv 15 \pmod{100}$. □

**Problem 16** For what primes $p$ is the element $p - 1$ a perfect square in $\mathbb{Z}_p$? Investigate this question by working out the cases $p = 2$, $p = 3$, $p = 5$, $p = 7$, $p = 11$, $p = 13$, $p = 17$, and $p = 19$. See if you notice any patterns an try to make a conjecture.

*Solution.* To see, we'll use the equation $x^2 = p - 1 \pmod{p}$. I'll use the Legendre symbol $\left(\frac{a}{p}\right)$ to determine if $a$ is a quadratic residue module $p$. For $a = p - 1$, we have $\left(\frac{p-1}{p}\right)$ will determine if $p - 1$ is a quadratic residue of module $p$.

(i) For $p = 2$: $p - 1 = 2 - 1 = 1$, which is a perfect square since $1^2 \pmod{2} = 1 \pmod{2}$.

(ii) For $p = 3$: $p - 1 = 3 - 1 = 2$. To see if 2 is a perfect square in $\mathbb{Z}_2$, we need to check all perfect squares

$$0^2 = 0, \quad 1^2 = 1, \quad 2^2 = 0.$$

Therefore, 2 isn't a perfect square.

(iii) For $p = 5$: $p - 1 = 5 - 1 = 4$, which is clearly a perfect square since $2^2 \pmod{4} = 4 \pmod{2}$.

(iv) For $p = 7$: $p - 1 = 7 - 1 = 6$

$$0^2 = 0, \quad 1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 2, \quad 4^2 = 2, \quad 5^2 = 4, \quad 6^2 = 1.$$

Therefore, 6 isn't a perfect square.

(v) For $p = 11$: $p - 1 = 11 - 1 = 10$

$$\begin{aligned}&0^1 = 0, \quad 1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 5, \quad 5^2 = 3, \quad 6^2 = 3, \quad 7^2 = 5, \quad 8^2 = 9\\&9^2 = 4, \quad 10^2 = 1.\end{aligned}$$

Therefore, 10 isn't a perfect square.

(vi) For $p = 13$: $p - 1 = 13 - 1 = 12$

$$0^1 = 0, \quad 1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 1, \quad 5^2 = 12.$$

Therefore, 12 is a perfect square in $\mathbb{Z}_{12}$.

(vii) For $p = 17$: $p - 1 = 17 - 1 = 16$

$$0^2 = 0, \quad 1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 16.$$

Therefore, 16 is a perfect square in $\mathbb{Z}_{17}$.

(viii) For $p = 19$: $p - 1 = 19 - 1 = 18$. I don't want to typeset the entire list, but 18 is not a perfect square in $\mathbb{Z}_{19}$.

Here's a summary table of everything

| $p$ | $p-1$ | Quadratic Residue | $p \pmod 4$ |
|---|---|---|---|
| 2 | 1 | Yes | 2 |
| 3 | 2 | No | 3 |
| 5 | 4 | Yes | 1 |
| 7 | 6 | No | 3 |
| 11 | 10 | No | 3 |
| 13 | 12 | Yes | 1 |
| 17 | 16 | Yes | 1 |
| 19 | 18 | No | 3 |

From this, I've come to the following conjecture:

**Conjecture:** *Given a prime number $p$, $p-1$ is a perfect square in $\mathbb{Z}_p$ if and only if $p \equiv 1 \pmod 4$ (excluding the special case where $p = 2$). This is consistent with the property of the Legendre symbol*

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 1 \pmod 3). \end{cases}$$

□

**Problem 17** Find $2^{1000}$ in $\mathbb{Z}_7$. Then find $3^{1000}$ in $\mathbb{Z}_7$. Explain how you got your answers.

*Solution.* I'll write out all the squares of 2 in $\mathbb{Z}_7$ until we start seeing a pattern or we get $2^n = 1$.

$$2^0 = 0, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 1$$

Breaking $2^{1000} = 2^{3 \cdot 333} \cdot 2^1 = 2^{999} \cdot 2 = 2$. I got my answer because I saw that $2^3 = 1$. I used this to my advantage by taking $\left\lfloor \frac{1000}{3} \right\rfloor = 999$, where $\lfloor x \rfloor$ is the biggest whole integer, $z$ such that $z \leq x$. So, $2^{999} = 1$ and using rules of exponentiation, I got $2^{999+1} = 2^{999} \cdot 2^1 = 2$.

Now, the same process pretty much repeats for $3^{1000}$

$$3^0 = 0, \quad 3^1 = 3, \quad 3^2 = 2, \quad 3^3 = 6, \quad 3^4 = 4, \quad 3^5 = 5, \quad 3^6 = 1.$$

Breaking $3^{1000} = 3^{996} \cdot 3^4 = 3^4 = 4$. Again, just like with the previous answer, I found the biggest multiple of 6, $x$, such that $6x \leq 1000$. That number was 166, giving us $166 \cdot 6 = 996$. And we know that any number that's divisible by 6, then $3^{6x} = 1$. Therefore, $3^{1000} = 3^{996} \cdot 3^4 = 3^4 = 4$. □

**Problem 18** Consider a sum of three consecutive squares (like $7^2 + 8^2 + 9^2$). What do you get when you reduce this mod 3 (that is, when you compute the remainder when you divided by 3)? Pick another sum of three consecutive squares and try it again. Try it one more time. State a conjecture and see if you can prove it.

*Solution.* The sum of $7^2 + 8^2 + 9^2 = 194$. Finding $194 \pmod 3$, we get $194 - 3 \cdot 64 = 2$. The sum of another three consecutive squares $10^2 + 11^2 + 12^2 = 365$. Computing $365 \pmod 3$, we get $365 - 3 \cdot 121 = 2$. Another three consecutive squares $4^2 + 5^2 + 6^2 = 77$, and $77 \pmod 3 \equiv 77 - 3 \cdot 25 = 2$.

**Conjecture:** *The sum of the squares of three consecutive integers is congruent to* $2 \pmod 3$.

Let the three consecutive integers be $n-1$, $n$, and $n+1$. Then their squares are $(n-1)^2$, $n^2$, and $(n+1)^2$. We want to evaluate

$$(n-1)^2 + n^2 + (n+1)^2 \pmod 3.$$

Expanding each term, we get

$$(n-1)^2 = n^2 - 2n + 1, \quad n^2 = n^2, \quad (n+1)^2 = n^2 + 2n + 1.$$

Adding them together we get

$$(n^2 - 2n + 1) + (n^2) + (n^2 + 2n + 1) = 3n^2 + 2.$$

Since we are multiplying $n^2$ by 3, then $3n^2 \pmod 3 = 0$, leaving a remainder of 2. $\square$

**Problem 19** The following proof has a mistake. Find what is wrong, and explain. $(R \vee \neg S) \Rightarrow \neg P$, $Q \Rightarrow R$, $S \Rightarrow T \vdash (P \Rightarrow \neg R) \wedge (Q \Rightarrow T)$.

| | | |
|---|---|---|
| 1. | $(R \vee \neg S) \Rightarrow \neg P$ | Hypothesis |
| 2. | $Q \Rightarrow R$ | Hypothesis |
| 3. | $P$ | Dischargeable Hypothesis |
| 4. | $\neg(R \vee \neg S)$ | MT, for 1, for 3 |
| 5. | $\neg(R \vee \neg S) \Leftrightarrow (\neg R \wedge S)$ | Tautology |
| 6. | $\neg R \wedge S$ | MPB, for 5, for 4 |
| 7. | $S$ | RCS, for 6 |
| 8. | $\neg R$ | LCS for 6 |
| 9. | $P \Rightarrow \neg R$ | DT, discharge for 3 [(3) - (8) unusable] |
| 10. | $Q$ | Dischargeable Hypothesis |
| 11. | $S \Rightarrow T$ | Hypothesis |
| 12. | $T$ | MP, for 11, for 7 |
| 13. | $Q \Rightarrow T$ | DT, discharge for 10 [(10) - (12) unusable] |
| 14. | $(P \Rightarrow \neg R) \wedge (Q \Rightarrow T)$ | CI, for 9, for 13 |

*Solution.* The mistake is on line 12. The proof incorrectly uses the fact that $S$ is true from line 7. This is not valid though, because the deduction ending on line 8 was based on assuming that $P$ is true, which is not a given and was only a temporary assumption for a direct proof (DT). According to the rules of DT, once the assumption is discharged, all intermediate steps derived from that assumption (lines 3 to 8) become invalid outside the scope of the assumption. $\square$

**Problem 20** Show that $(R \vee \neg S) \Rightarrow \neg P$, $Q \Rightarrow R$, $S \Rightarrow T \vdash (P \Rightarrow S) \wedge (Q \Rightarrow (\neg P \wedge R))$.

*Solution.*

| | | |
|---|---|---|
| 1. | $(R \vee \neg S) \Rightarrow \neg P$ | Hypothesis |
| 2. | $Q \Rightarrow R$ | Hypothesis |
| 3. | $S \Rightarrow T$ | Hypothesis |
| 4. | $P$ | Dischargeable Hypothesis |
| 5. | $\neg(R \vee \neg S)$ | MT, for 4, for 1 |
| 6. | $\neg(R \vee \neg S) \Leftrightarrow (\neg R \wedge S)$ | Tautology |
| 7. | $\neg R \wedge S$ | MPB, for 5, for 6 |
| 8. | $S$ | RCS, for 7 |
| 9. | $P \Rightarrow S$ | DT, discharge for 4 [(4) - (8) unusable] |
| 10. | $Q$ | Dischargeable Hypothesis |
| 11. | $R$ | MP, for 10, for 2 |
| 12. | $\neg P$ | MP, for 11, for 1 |
| 13. | $\neg P \wedge R$ | CI, for 12, for 11 |
| 14. | $Q \Rightarrow (R \wedge \neg P)$ | DT, discharge for 10 [(10) - (13) unusable] |
| 15. | $(P = S) \wedge (Q \Rightarrow (\neg P \wedge R))$ | CI, for 9, for 14 |