$\Rightarrow$ $S \subseteq V$ is a subspace

4) $V = \mathbb{C}$ over $\mathbb{C}$, $S = \mathbb{R}$

$\Rightarrow$ S is not a subspace, because S is not closed under scalar multiplication. end of Jan 8

5). $V = \mathbb{R}^{n \times n}$, S = the set of all symmetric matrices

i.e.: $S = \{ A \in V : A^T = A \}$

(Homework: show that S is a subspace of V.

6). Definition: Let $S = \{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq V$ be a subset. The span of S is

$$\operatorname{Span} S = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k : c_1, c_2, \cdots, c_k \in \mathbb{R} \text{ (or } \mathbb{C}) \}.$$

Then $\operatorname{Span} S$ is a subspace of V.

6'). Definition: Let $S \subseteq V$ be an infinite set. The span of S is

$$\operatorname{Span} S = \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k . \text{ for } \{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq S \text{ any finite subset, and } \forall c_1, \cdots, c_k \in F \}$$

Linear independence / dependent relations

Definition: A finite set of vectors $\{\vec{x}_1, \cdots, \vec{x}_k\} \subseteq V$ is called linearly dependent if there exists $c_1, c_2, \cdots, c_n$, which are not all zeros, such that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}.$$

Definition: A finite set of vectors $\{\vec{x}_1, \cdots, \vec{x}_k\} \subseteq V$ is called linearly independent if $c_1\vec{x}_1 + \cdots + c_k\vec{x}_k = \vec{0}$ implies $c_1 = c_2 = \cdots = c_k = 0$.

Examples: 1). $\{\vec{x}\} \subseteq V$ is linearly independent if and only if $\vec{x} \neq \vec{0}$.

2) $\{\vec{x}_1, \vec{x}_2\} \subseteq V$ is linearly dependent if and only if $\vec{x}_1$ and $\vec{x}_2$ is scalar multiples of each other

3). Let $V = \mathbb{R}^n$ (or $\mathbb{C}^n$). Let $S = \{\vec{x}_1, \cdots, \vec{x}_m\}$ where $m > n$, then S is linearly dependent.

<u>Proof</u>: Denote $A = (\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_m) \in \mathbb{R}^{n\times m}$ (or $\mathbb{C}^{n\times m}$)

$\{\vec{x}_1, \cdots, \vec{x}_m\}$ linearly dep. $\Leftrightarrow \exists c_1, \cdots, c_m$ not all zero such that $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_m\vec{x}_m = \vec{0}$

$\Leftrightarrow A\vec{x} = \vec{0}$ has non-trivial solutions $\Leftrightarrow Null(A) \neq \{\vec{0}\} \Leftrightarrow \dim(Null(A)) > 0 \Leftrightarrow \underline{nullity(A) > 0}$.

proved below!

By rank-nullity thm: $rank(A) + nullity(A) = m$. as $rank(A) \le n$, $nullity(A) \ge m - n > 0$.

4). A set containing the zero vector is automatically linearly dependent.

Jan 10.

Definition: A set $S \subseteq V$ is linearly dependent if S contains a linearly dependent finite set.

Definition: A set $S \subseteq V$ is linearly independent if every finite subset of S is linearly independent.

<u>Basis, Dimension</u>

Definition: A subset $S \subseteq V$ is a basis of V if it satisfies the following two conditions:

1) S is linearly independent, and

2) $V = \operatorname{Span} S$, i.e. $\forall \vec{v} \in V$: $\exists \vec{x}_1, \cdots, \vec{x}_n \in S$ and scalars $a_1, \cdots, a_n$ such that $\vec{v} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n$.

Corollary. Let $S = \{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\} \subseteq V$ be a basis of V. Then $\forall \vec{v} \in V$, there exists a unique n-tuples $a_1, \cdots, a_n$ such that $\vec{v} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n$.

<u>Proof</u>. Suppose there exists scalars $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ such that

$$\vec{x} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n = b_1\vec{x}_1 + b_2\vec{x}_2 + \cdots + b_n\vec{x}_n$$

Then $a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n = b_1\vec{x}_1 + b_2\vec{x}_2 + b_n\vec{x}_n$