Midterm 1, Math 433. April 25 2025. One page of notes, front and back is allowed. Must be handwritten.

**Problem 1** *Suppose that*
$$f : \mathbb{R}^3 \to \mathbb{R}$$
*is a function with nowhere vanishing differential. Suppose $p \in \mathbb{R}^3$ and that $f(p) = a$. Let $S = \{f^{-1}(a)\}$. Suppose that $\vec{V}$ in $T_pS$. Argue that*
$$df(V) = 0.$$

**Solution 2** *By definition, $\vec{V} = \frac{d}{dt}(\vec{x} \circ \alpha)(t)|_{t=0}$ for some $\alpha(t)$ a path in a chart $u$ with $\vec{x} \circ \alpha(0) = p$. By definition of the differential of df*
$$df(V) = \frac{d}{dt} f(\vec{x} \circ \alpha)(t)|_{t=0} = 0$$
*because $f$ is constant on the image of $\vec{x}$*

**Problem 3** *Fix a value a and consider the curve*
$$\alpha(t) = (a\cos(t)), a\sin(t), t)$$

*Compute the curvature in terms of a.*

**Solution 4**
$$\frac{d}{dt}\alpha(t) = (-a\sin(t)), a\cos(t), 1)$$
*and we see that*
$$\left|\frac{d}{dt}\alpha(t)\right|^2 = a^2 + 1$$
*so this is a constant speed curve. Dividing by the speed gives a unit tangent vector*
$$\vec{T} = \frac{1}{\sqrt{a^2+1}} \frac{d}{dt}\alpha(t)$$
*So to compute the derivative of $\vec{T}$ with respect to arclength, simply compute w.r.t $t$ and then divide by speed, that is*
$$\vec{T}' = \frac{1}{\sqrt{a^2+1}} \frac{d}{dt}\left[\frac{1}{\sqrt{a^2+1}} \frac{d}{dt}\alpha(t)\right]$$
$$= \frac{1}{a^2+1}(-a\cos(t)), -a\sin(t), 0)$$
*Since this is just $k\vec{N}$ we pull out the unit vector*
$$\vec{N} = (-\cos(t)), -\sin(t), 0)$$
*and conclude*
$$\kappa = \frac{a}{a^2+1}$$

**Problem 5** *Consider the shape defined by*

$$\vec{x}(u,v) = (v\sin(u), v\cos(u), v)$$

$$U \subset \mathbb{R}^2 = \left\{ (u,v) : \begin{array}{c} 0 < u < \pi, \\ 0 < v < 1 \end{array} \right\}$$

1. *Argue this is a smooth parametrization of a regular surface*

2. *Compute the metric components $E, F,$ and $G$ in terms of the coordinate vectors $\vec{x}_u, \vec{x}_v$.*

3. *Write down a double integral that gives the area of the surface and compute the area.*

**Solution 6** *All components are smooth in terms of $u, v$ so function is smooth. The differential is given by*

$$d\vec{x} = \begin{pmatrix} v\cos u & \sin u \\ -v\sin u & \cos u \\ 0 & 1 \end{pmatrix}$$

*the top 2 by 2 has determinant $v > 0$ so this is rank 2. Finally, it's 1-1: $(\sin u, \cos u)$ is 1-1 on $(0, \pi)$. These together with Prop 4 section 2.2 give the result.*

*We then compute*

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} v\cos u & \sin u \\ -v\sin u & \cos u \\ 0 & 1 \end{pmatrix}^T \cdot \begin{pmatrix} v\cos u & \sin u \\ -v\sin u & \cos u \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} v^2 & 0 \\ 0 & 1 \end{pmatrix}$$

*Thus the area is*

$$\int_0^1 \int_0^\pi \sqrt{v^2}\, dudv = \int_0^1 \int_0^\pi v\, dudv = \frac{\pi}{2}$$

**Problem 7** *For which values $a \in \mathbb{R}$ is the surface defined by*

$$x^4 + y^4 = \frac{e^z + e^{-z}}{2} + a$$

*a regular surface?*

**Solution 8** *Let $f = x^4 + y^4 - \frac{e^z + e^{-z}}{2}$ then*

$$df = \left( 4x^3, 4y^3, \frac{e^z - e^{-z}}{2} \right)$$

The only place this vanishes is if $x = 0, y = 0$ and $z = 0$. So we just need the level set $a$ to avoid the origin. The level set containing the origin is $f(0,0,0) = -1$. So the answer is all $a$ except $a = -1$; provided it's non-empty. But for any $a < -1$ simply solve $\frac{e^z + e^{-z}}{2} = -a$ and then let $x, y = 0$. For $a > -1$ we can let $z = 0, y = 0$ and solve $x^4 - 1 = a$ to get this level set is non-empty.

**Problem 9** *Suppose that $C$ is a curve where both curvature and torsion are constants $\kappa_0$ and $\tau_0$ along the curve. Show that the normal vector $\vec{N}$ is periodic, and find it's period (as function of arclength.) (Hint: Find some $\lambda$ such that*

$$\vec{N}'' = -\lambda^2 \vec{N}$$

*and then you may use free of charge the fact that if $f''(t) = -\lambda^2 f(t)$ then $f$ is a period function of $t$ with period $\frac{2\pi}{\lambda}$.*

**Solution 10** *Recall that*

$$\vec{T}' = \kappa \vec{N}$$
$$\vec{N}' = -\kappa \vec{T} - \tau \vec{B}$$
$$\vec{B}' = \tau \vec{N}$$

*So take two derivatives of $\vec{N}$*

$$[N']' = \left[ -\kappa \vec{T} - \tau \vec{B} \right]'$$
$$= -\kappa_0 \vec{T}' - \tau_0 \vec{B}'$$

*having used the fact that $\kappa_0, \tau_0$ are constant .. but then plug in Frenet equations again*

$$[N']' = -\kappa_0 \vec{T}' - \tau_0 \vec{B}$$
$$\quad - \kappa_0 \kappa_0 \vec{N} - \tau_0 \tau_0 \vec{N}$$
$$= -\left( \kappa_0^2 + \tau_0^2 \right) \vec{N}$$

*that is, each of the components of $N$ must satisfy*

$$f'' = -\left( \kappa_0^2 + \tau_0^2 \right) f$$

*so must be periodic with period*

$$\frac{2\pi}{\sqrt{\kappa_0^2 + \tau_0^2}}$$