Abstract Linear Algebra: Homework 10

Due on March 16, 2025 at 23:59

Xiaojing Chen–Murphy

Hashem A. Damrah
UO ID: 952102243

Problem 1. Classify by Jordan Canonical form (i.e. up to similarity and up to the order of the Jordan blocks) for all 6×6 matrices which have characteristic polynomial $(x-2)^2(x+3)^4$.

Solution. We first need to identify the eigenvalues and their algebraic multiplicities

(i) $\lambda = 2$ with algebraic multiplicity 2.

(ii) $\lambda = -3$ with algebraic multiplicity 4.

Now, we determine the possible Jordan block structures. Since the algebraic multiplicity of $\lambda = 2$ is 2, the only possible Jordan forms are

(i) $J_1(2) \oplus J_1(2)$, both blocks are size 1,

(ii) $J_2(2)$, a single block of size 2.

Since the algebraic multiplicity of $\lambda = -3$ is 4, the possible Jordan block structures are

(i) $J_1(-3) \oplus J_1(-3) \oplus J_1(-3) \oplus J_1(-3)$, all blocks are size 1.

(ii) $J_2(-3) \oplus J_1(-3) \oplus J_1(-3)$, one block of size 2, two blocks of size 1.

(iii) $J_3(-3) \oplus J_1(-3)$, one block of size 3, one block of size 1.

(iv) $J_2(-3) \oplus J_2(-3)$, two blocks of size 2.

(v) $J_4(-3)$, a single block of size 4.

Each Jordan form consists of a combination of the Jordan blocks for $\lambda = 2$ and $\lambda = -3$:

(i) $J_1(2) \oplus J_1(2) \oplus J_1(-3) \oplus J_1(-3) \oplus J_1(-3) \oplus J_1(-3)$

(ii) $J_1(2) \oplus J_1(2) \oplus J_2(-3) \oplus J_1(-3) \oplus J_1(-3)$

(iii) $J_1(2) \oplus J_1(2) \oplus J_3(-3) \oplus J_1(-3)$

(iv) $J_1(2) \oplus J_1(2) \oplus J_2(-3) \oplus J_2(-3)$

(v) $J_1(2) \oplus J_1(2) \oplus J_4(-3)$

(vi) $J_2(2) \oplus J_1(-3) \oplus J_1(-3) \oplus J_1(-3) \oplus J_1(-3)$

(vii) $J_2(2) \oplus J_2(-3) \oplus J_1(-3) \oplus J_1(-3)$

(viii) $J_2(2) \oplus J_3(-3) \oplus J_1(-3)$

(ix) $J_2(2) \oplus J_2(-3) \oplus J_2(-3)$

(x) $J_2(2) \oplus J_4(-3)$

Therefore, there are 10 distinct Jordan forms for matrices with the given characteristic polynomial. $\square$

Problem 2. Find the Jordan canonical form of the matrix $\begin{pmatrix} a & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{pmatrix}$.

Solution. We first need to compute the characteristic polynomial. The matrix A has a along the main diagonal and ones along the second superdiagonal. This structure suggests that A is a companion-like matrix shifted by aI.

To determine the characteristic polynomial, consider the determinant of $A - xI$

$$A - xI = \begin{pmatrix} a-x & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a-x & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a-x & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a-x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a-x & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a-x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a-x \end{pmatrix}.$$

By expanding the determinant recursively, we see that the characteristic polynomial satisfies $\chi_A(x) = (a-x)^n$ This means the only eigenvalue of A is a with algebraic multiplicity n.

To determine the Jordan form, we examine the generalized eigenspaces. Since there is a single eigenvalue a and the superdiagonal structure extends throughout the matrix, we check the size of the largest Jordan block by computing the rank of successive powers of $A - aI$.

The matrix $A - aI$ is nilpotent. The key observation is that $(A - aI)^{n-1}$ has only a single nonzero entry, meaning the generalized eigenspace has a single block of size n.

Thus, the Jordan form of A is

$$J(A) = J_n(a) = \begin{pmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}.$$

□

Problem 3. Let $A = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}$.

(i) Find a basis for each generalized eigenspace of A consisting of a union of disjoint cycles of generalized eigenvectors.

(ii) Find a Jordan canonical form J of A using your basis in Part (i).

(iii) Find the minimal polynomial of A.

Solution to (i). Using Wolfram Alpha, we get the characteristic polynomial

$$f_A(\lambda) = -(\lambda - 2)^2(\lambda + 1),$$

giving us eigenvalues of $\lambda_1 = -1$ with algebraic multiplicity 1 and $\lambda_2 = 2$ with algebraic multiplicity 2.

An eigenvalue will have a generalized eigenvector if its algebraic multiplicity is greater than the geometric multiplicity. If so, then the eigenvalue will have n generalized eigenvectors, where n is the difference between the algebraic multiplicity and the geometric multiplicity.

First, we work with $\lambda_1 = -1$. We need to first compute the eigenvectors, giving us

$$\begin{aligned} \underline{\lambda = -1}: \quad A - \lambda_1 I &= \begin{pmatrix} 12 & -4 & -5 \\ 21 & -9 & -11 \\ 3 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \Rightarrow \ker(A - \lambda_2 I) &= \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}. \end{aligned}$$

Since the algebraic multiplicity of λ_1 is 1, there are no generalized eigenvectors.

Now, we find the generalized eigenvectors for $\lambda_2 = 2$ with multiplicity of 2.

$$\begin{aligned} \underline{\lambda = -2}: \quad A - \lambda_2 I &= \begin{pmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ \Rightarrow \ker(A - \lambda I) &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Since the algebraic multiplicity of λ_2 is 2, there exists a single generalized eigenvector. To find it, we solve $(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{v}_1$. Therefore, we have

$$(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{v}_1 \Rightarrow \begin{pmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Therefore, the generalized eigenspaces are given by

$$G_{\lambda_1}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad G_{\lambda_2}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \qquad \square$$

Solution to (ii). The generalized eigenspace for $\lambda_1 = -1$ has dimension 1, so there will be a 1×1 Jordan block for -1.

The generalized eigenspace for $\lambda_2 = 2$ has dimension 2. Since the geometric multiplicity is 1 (only one eigenvector for 2), there will be a single 2×2 Jordan block for 2.

The matrix P is constructed from the combinations of the basis for the generalized eigenspaces for λ_1 and λ_2. Therefore, A can be written as

$$A = PJP^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 0 & 0 & 1 \\ 3 & -1 & -2 \end{pmatrix}. \qquad \square$$

Solution to (iii). The eigenvalue $\lambda_1 = -1$ has a 1×1 Jordan block, so it appears in the minimal polynomial as $(x + 1)$. The eigenvalue $\lambda_2 = 2$ has a 2×2 Jordan block, so it appears in the minimal polynomial as $(x - 2)^2$. Thus, the minimal polynomial of A is

$$m_A(x) = (x + 1)(x - 2)^2. \qquad \square$$

Problem 4. Let D be the differential linear operator on the vector space $V = \text{Span}\{1, t, t^2, e^t, te^t\}$, i.e.,

$$D(f(x)) = \frac{\mathrm{d}}{\mathrm{d}x} f(x), \quad \text{for any} \quad f(x) \in V.$$

(i) Let $\mathcal{B} = \{1, t, t^2, e^t, te^t\}$. Find the matrix representation $A = [D]_{\mathcal{B}}$.

(ii) Find a basis for each generalized eigenspace of D consisting of a union of disjoint cycles of generalized eigenvectors.

(iii) Find a Jordan canonical form J of D using your basis in Part (ii).

(iv) Find the minimal polynomial of D.

Solution to (i). To find the matrix representation of D with respect to the basis $\mathcal{B}$, we need to compute D applied to each basis element and express the result as a linear combination of the basis elements.

$$D(1) = 0, \quad D(t) = 1, \quad D(t^2) = 2t, \quad D(e^t) = e^t, \quad \text{and} \quad D(te^t) = e^t + te^t.$$

Now, we can express these results in terms of the basis $\mathcal{B}$:

$$\begin{aligned} D(1) &= 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot e^t + 0 \cdot te^t \\ D(t) &= 1 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot e^t + 0 \cdot te^t \\ D(t^2) &= 0 \cdot 1 + 2 \cdot t + 0 \cdot t^2 + 0 \cdot e^t + 0 \cdot te^t \\ D(e^t) &= 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 1 \cdot e^t + 0 \cdot te^t \\ D(te^t) &= 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 1 \cdot e^t + 1 \cdot te^t. \end{aligned}$$

Therefore, the matrix representation $A = [D]_{\mathcal{B}}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \square$$

Solution to (ii). First, we need to find the characteristic polynomial of A, giving us

$$\det(A-\lambda I)=\det\begin{pmatrix}-\lambda & 1 & 0 & 0 & 0\\ 0 & -\lambda & 2 & 0 & 0\\ 0 & 0 & -\lambda & 0 & 0\\ 0 & 0 & 0 & 1-\lambda & 1\\ 0 & 0 & 0 & 0 & 1-\lambda\end{pmatrix}=-\lambda^3(\lambda-1)^2.$$

The eigenvalues are $\lambda_1=0$ with algebraic multiplicity 3 and $\lambda_2=1$ with algebraic multiplicity 2. Next, we find the eigenvectors and generalized eigenvectors for each eigenvalue.

For $\lambda_1=0$, we have

$$A\mathbf{v}_1=0\Rightarrow\begin{pmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\Rightarrow\mathbf{v}_1=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$$

Since the algebraic multiplicity of λ_1 is 3 and the geometric multiplicity is 1, there are 2 generalized eigenvectors. To find them like we did in problem 3, we solve $(A-\lambda_1 I)\mathbf{v}_2=\mathbf{v}_1$ and $(A-\lambda_1 I)\mathbf{v}_3=\mathbf{v}_2$.

$$(A-\lambda_1 I)\mathbf{v}_2=\mathbf{v}_1\Rightarrow\begin{pmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\Rightarrow\mathbf{v}_2=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$$

$$(A-\lambda_1 I)\mathbf{v}_3=\mathbf{v}_2\Rightarrow\begin{pmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\Rightarrow\mathbf{v}_3=\begin{pmatrix}0\\0\\1/2\\0\\0\end{pmatrix}.$$

For $\lambda_2 = 1$, we have

$$(A - \lambda_2 I) = \mathbf{v}_1 \Rightarrow \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since the algebraic multiplicity of λ_2 is 2 and the geometric multiplicity is 1, there is 1 generalized eigenvector. To find it, we solve $(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{v}_1$.

$$(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{v}_1 \Rightarrow \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the generalized eigenspaces are given by

$$G_{\lambda_1}(D) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad G_{\lambda_2}(D) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$ $\square$

Solution to (iii). The generalized eigenspace for $\lambda_1 = 0$ has dimension 3, so there will be a single 3×3 Jordan block for 0.

The generalized eigenspace for $\lambda_2 = 1$ has dimension 2. Since the geometric multiplicity is 1 (only one eigenvector for 1), there will be a single 2×2 Jordan block for 1.

The matrix P is constructed from the combinations of the basis for the generalized eigenspaces for λ_1

and λ_2. Therefore, A can be written as

$$A = PJP^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \square$$

Solution to (iv). The eigenvalue $\lambda_1 = 0$ has a 3×3 Jordan block, so it appears in the minimal polynomial as x^3. The eigenvalue $\lambda_2 = 1$ has a 2×2 Jordan block, so it appears in the minimal polynomial as $(x-1)^2$. Thus, the minimal polynomial of D is

$$m_D(x) = x^3(x-1)^2. \qquad \square$$

Problem 5. Let $A \in \mathbb{C}^{n\times n}$. Assume $(A + I_n)^m = 0$. Prove that A is invertible and find $\det(A)$

Solution. To show that A is invertible, we must prove that $\det(A) \neq 0$, or equivalently, that A is non-singular. The given condition $(A + I_n)^m = 0$ means that $A + I_n$ is nilpotent, i.e., all its eigenvalues are zero. That is, for every eigenvalue λ of $A + I_n$, we have $\lambda^m = 0$, implying $\lambda = 0$. This means that the only eigenvalue of $A + I_n$ is 0, so the eigenvalues of A satisfy

$$\mu + 1 = 0 \Rightarrow \mu = -1.$$

Thus, all eigenvalues of A are -1, which means the matrix A is diagonalizable (if it is already diagonalizable) or similar to a Jordan block form with eigenvalues -1. Since -1 is never zero, all eigenvalues of A are nonzero, which implies that A is invertible.

Since the determinant of a matrix is the product of its eigenvalues and all eigenvalues of A are -1, we obtain $\det(A) = (-1)^n$. $\square$

Problem 6. Let V be the vector space of all polynomials over $\mathbb{R}$. Let D be the differentiation operator on the vector space V. Find the minimal polynomial of D on V or prove that D has no minimal polynomial on V.

Solution. We consider the minimal polynomial $m_D(x)$, which is the monic polynomial of least degree such that $m_D(D) = 0$. This means there must exist a nonzero polynomial $m_D(x) = a_0 + a_1x + \cdots + a_kx^k$ such that

$$m_D(D) = a_0I + a_1D + \cdots + a_kD^k = 0.$$

In other words, applying this polynomial expression of D to any polynomial $p(x)$ must yield 0.

Now, we check if D Has a Minimal Polynomial. If D had a minimal polynomial of degree k, then for every polynomial $p(x)$, applying D repeatedly up to the k-th derivative should result in zero. However, differentiation does not satisfy such a condition for all polynomials. Specifically, the space of polynomials is infinite-dimensional. We also know that there exist polynomials of arbitrarily high degree, and differentiation reduces the degree by one at each step. Therefore, no finite power D^k annihilates all polynomials; for any k, there exist polynomials of degree higher than k, whose k-th derivative is nonzero.

Thus, there is no nonzero polynomial $m_D(x)$ such that $m_D(D)$ annihilates all polynomials.

Since D does not satisfy a minimal polynomial equation for all polynomials in V, we conclude that D has no minimal polynomial on V. $\square$

Problem 7. Suppose $A \in \mathbb{C}^{n\times n}$ satisfies $A^2 = A$. Prove that A is diagonalizable.

Solution. Since A satisfies $A^2 - A = 0$, the minimal polynomial of A, which is the monic polynomial of least degree that annihilates A, must divide $x^2 - x$. The polynomial $x^2 - x = x(x-1)$. Thus, the minimal polynomial of A must be one of

$$x, \quad x-1, \quad \text{or} \quad x(x-1).$$

This means that the only possible eigenvalues of A are 0 and 1.

A matrix is diagonalizable if and only if its minimal polynomial splits completely into distinct linear factors. We have already determined that the minimal polynomial of A is one of

$$x, \quad x-1, \quad \text{or} \quad x(x-1),$$

all of which are products of distinct linear factors. This guarantees that A is diagonalizable.

Since the only possible eigenvalues are 0 and 1, we decompose the space $\mathbb{C}^n$ as a direct sum of the eigenspaces corresponding to 0 and 1

$$\mathbb{C}^n = \ker(A) \oplus \operatorname{Im}(A).$$

From $A^2 = A$, it follows that if $\mathbf{v}$ is in the image of A, then $A\mathbf{v} = \mathbf{v}$, meaning all vectors in $\operatorname{Im}(A)$ are eigenvectors corresponding to eigenvalue 1. Similarly, if $\mathbf{v} \in \ker(A)$, then $A\mathbf{v} = \mathbf{0}$, meaning all vectors in $\ker(A)$ are eigenvectors corresponding to eigenvalue 0.

Since the generalized eigenspaces span $\mathbb{C}^n$ and contain only true eigenvectors (i.e., there are no nontrivial Jordan blocks), A is diagonalizable. □

Problem 8. Give an example of two matrices who have the same characteristic polynomial but distinct minimal polynomials.

Solution. Define A and B as the following matrices

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Their characteristic polynomials are

$$\det(A) = -(\lambda - 3)(\lambda - 2)^2 = \det(B),$$

but their characteristic polynomials are

$$m_A(x) = (x-2)(x-3) \quad \text{and} \quad m_B(x) = (x-3)(x-2)^2.$$

□