Math 307, Homework #7
Due Wednesday, November 20
SOLUTIONS TO SELECTED PROBLEMS

1. Given $Q \Rightarrow R$, prove $[P \Rightarrow T] \Rightarrow [(Q \vee \sim T) \Rightarrow (\sim P \vee R)]$.

   Proof:

1.	$Q \Rightarrow R$	hyp.
2.	$P \Rightarrow T$	dis. hyp.
3.	$Q \vee \sim T$	dis. hyp.
4.	$\sim(\sim P \vee R)$	dis. hyp.
5.	$P \wedge \sim R$	GSP, For 4 (de Morgan taut.)
6.	$\sim R$	RCS, For 5
7.	$\sim Q$	MT For 1, For 6
8.	$\sim T$	DI, For 3, For 7
9.	$\sim P$	MT, For 2, For 8
10.	P	LCS, For 5
11.	$P \wedge \sim P$	CI, For 10, For 9
12.	$\sim P \vee R$	II, discharge For 4
13.	$(Q \vee \sim T) \Rightarrow (\sim P \vee R)$	DT, discharge For 3
14.	$(P \Rightarrow T) \Rightarrow [(Q \vee \sim T) \Rightarrow (\sim P \vee R)]$	DT, discharge For 2

2. (a) If $C \subseteq A$ and $D \subseteq B$ then prove $D - A \subseteq B - C$.

   Proof: Let $x \in D - A$. Then $x \in D$ and $x \notin A$. Since $x \in D$ and $D \subseteq B$, $x \in B$. Since $C \subseteq A$ and $x \notin A$, this implies $x \notin C$. So $x \in B - C$, hence $D - A \subseteq B - C$.

   (b) Prove $A = X \cap A$ if and only if $A \subseteq X$.

   Proof:

   Assume $A = X \cap A$, and let $a \in A$. Then $a \in X \cap A$, so $a \in X$. Thus, $A \subseteq X$.

   Now assume $A \subseteq X$. Obviously $X \cap A \subseteq A$, so let $a \in A$. Since $A \subseteq X$, $a \in X$. So a is in both X and A, hence $a \in X \cap A$. This shows $A \subseteq X \cap A$, therefore the two sets are equal.

   (c) Prove $A = X \cup A$ if and only if $X \subseteq A$.

   Proof:

   Assume $A = X \cup A$, and let $x \in X$. Then $x \in X \cup A$, so $x \in A$. Hence $X \subseteq A$.

   Now assume $X \subseteq A$. Surely $A \subseteq X \cup A$, so we only need to prove the subset in the opposite direction. Let $x \in X \cup A$. Then $x \in X$ or $x \in A$. But if $x \in X$ then $x \in A$ since $X \subseteq A$, so both cases lead to $x \in A$. Hence, we have shown $X \cup A \subseteq A$; so the two sets are equal.

3. Let $f \colon S \to T$ be a function. Prove that if $X \subseteq T$ and $Y \subseteq T$ then $f^{-1}(X) - f^{-1}(Y) = f^{-1}(X - Y)$.

   Proof:

   Let $z \in f^{-1}(X) - f^{-1}(Y)$. Then $f(z) \in X$ and $f(z) \notin Y$. So $f(z) \in X - Y$, which means $z \in f^{-1}(X - Y)$. Thus, $f^{-1}(X) - f^{-1}(Y) \subseteq f^{-1}(X - Y)$.

   Now let $z \in f^{-1}(X - Y)$. Then $f(z) \in X - Y$. So $f(z) \in X$ and $f(z) \notin Y$. The former says that $z \in f^{-1}(X)$, and the latter says that $z \notin f^{-1}(Y)$. So $z \in f^{-1}(X) - f^{-1}(Y)$. Therefore $f^{-1}(X - Y) \subseteq f^{-1}(X) - f^{-1}(Y)$, and the two sets are equal.

4. Let $f \colon S \to T$ be a function, let $A \subseteq S$ and $B \subseteq S$.

(a) Prove $f(A) - f(B) \subseteq f(A - B)$.

Let $z \in f(A) - f(B)$. Then $z \in f(A)$, so $z = f(u)$ for some $u \in A$. If $u \in B$ then $f(u) \in f(B)$, but this would contradict $z \notin f(B)$. So $u \notin B$, which means $u \in A - B$, which in turn gives $z \in f(A - B)$. Hence, $f(A) - f(B) \subseteq f(A - B)$.

(b) If f is one-to-one, prove $f(A - B) \subseteq f(A) - f(B)$.

Assume f is one-to-one, and let $x \in f(A - B)$. Then $x = f(u)$ for some $u \in A - B$. Then $u \in A$, so $x \in f(A)$. Suppose $x \in f(B)$. Then $x = f(b)$ for some $b \in B$. But then $f(u) = x = f(b)$, so because f is one-to-one this gives $u = b$. But $u \notin B$ and $b \in B$, so this is a contradiction. We have therefore shown $x \notin f(B)$, so $x \in f(A) - f(B)$. Hence, $f(A - B) \subseteq f(A) - f(B)$.

(c) Create an example of an S, T, f, A, and B such that $f(A) - f(B) \neq f(A - B)$.

Let $S = \{0, 1\}$ and $T = \{5\}$, and $f : S \to T$ be the function given by $f(0) = f(1) = 5$. Let $A = \{0\}$ and $B = \{1\}$. Then $A - B = \{0\}$, so $f(A - B) = \{5\}$. But $f(A) = \{5\} = f(B)$, so $f(A) - f(B) = \emptyset$.

5. Suppose $f : A \to B$, $X \subseteq A$, $W \subseteq B$, $f(X) \cap W = \emptyset$, and $f(X) \cup W = B$.

(a) Prove that $X \cap f^{-1}(W) = \emptyset$.

Suppose $x \in X \cap f^{-1}(W)$. Then $x \in X$ and $f(x) \in W$. But since $x \in X$, $f(x) \in f(X)$. So $f(x) \in f(X) \cap W$, which is a contradiction since $f(X) \cap W = \emptyset$. We conclude that $X \cap f^{-1}(W)$ does not contain any elements, therefore $X \cap f^{-1}(W) = \emptyset$.

(b) If f is one-to-one, prove that $A = X \cup f^{-1}(W)$.

Proof:

Since $X \subseteq A$ and $f^{-1}(W) \subseteq A$ (by definition), we clearly have $X \cup f^{-1}(W) \subseteq A$. So let $u \in A$. Then $f(u) \in B$, so $f(u) \in f(X) \cup W$.

<u>Case 1</u>: $f(u) \in f(X)$.

Then we have $f(u) = f(v)$ for some $v \in X$. Since f is one-to-one, $u = v$. Hence $u \in X$, and so $u \in X \cup f^{-1}(W)$.

<u>Case 2</u>: $f(u) \in W$.

In this case we have $u \in f^{-1}(W)$, and so $u \in X \cup f^{-1}(W)$ again.

Both cases lead to $u \in X \cup f^{-1}(W)$, so we have proven $A \subseteq X \cup f^{-1}(W)$. Hence, the two sets are equal.

(c) If $f(A - X) = W$ prove that f is onto.

Proof:

Let $b \in B$. Since $B = f(X) \cup W$, either $b \in f(X)$ or $b \in W$. If $b \in f(X)$, then $b = f(u)$ for some $u \in X$; in particular, notice that $u \in A$. If $b \in W$ then $b \in f(A - X)$, so again $b = f(v)$ for some $v \in A - X$; again, note in particular that $v \in A$. So in both cases we have that $b = f(a)$ for some $a \in A$. Therefore f is onto.

In questions 6–13 below, prove the indicated statement by induction.

6. $1 + 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2$, for all $n \geq 0$.

Proof: For an integer $n \geq 0$ consider the following statement:

$$P(n): \ 1 + 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2.$$

I. When $n = 0$ the statement $P(0)$ says that $1 = (0 + 1)^2$, and this is true.

II. Assume that $P(n)$ is true for some integer $n \geq 0$, that is $1+3+5+\cdots+(2n+1)=(n+1)^2$. Add $2n+3$ to both sides to get

$$\begin{aligned} 1+3+5+\cdots+(2n+1)+(2(n+1)+3) &= 1+3+5+\cdots+(2n+1)+(2n+3) \\ &= (n+1)^2+(2n+3) \\ &= n^2+2n+1+2n+3 \\ &= n^2+4n+4 \\ &= (n+2)^2 \\ &= ((n+1)+1)^2. \end{aligned}$$

Thus we proved conditional statement $P(n) \Rightarrow P(n+1)$.

III. By PMI, we have shown that $1+3+\cdots+(2n+1)=(n+1)^2$ for all $n \geq 0$.

7. $1^3+2^3+\cdots+n^3=[\frac{n(n+1)}{2}]^2$ for all $n \geq 1$.

Proof: For $n \in \mathbb{N}$ consider the following statement: $P(n):\ 1^3+2^3+\cdots+n^3=[\frac{n(n+1)}{2}]^2$.

I. When $n=1$ the statement $P(1)$ is $1^3=\left(\frac{1\cdot 2}{2}\right)^2$, which is true.

II. Assume $P(n)$ is true for some $n \in \mathbb{N}$, that is $1^3+2^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2$. Add $(n+1)^3$ to both sides to get

$$\begin{aligned} 1^3+2^3+\cdots+(n+1)^3=\left[\frac{n(n+1)}{2}\right]^2+(n+1)^3 &= (n+1)^2\cdot\left[\frac{n^2}{4}+n+1\right] \\ &= \frac{(n+1)^2}{4}\cdot\left[n^2+4n+4\right] \\ &= \frac{(n+1)^2}{2^2}\cdot(n+2)^2 \\ &= \left[\frac{(n+1)(n+2)}{2}\right]^2. \end{aligned}$$

Thus $P(n) \Rightarrow P(n+1)$.

III. By PMI, we conclude that $1^3+2^3+\cdots+n^3=[\frac{n(n+1)}{2}]^2$ for all $n \geq 1$.

8. For all $n \geq 1$, $\sum\limits_{k=1}^{n}(-1)^k k^2=(-1)^n\frac{n(n+1)}{2}$.

Proof: For $n \in \mathbb{N}$ consider the following statement: $P(n):\ \sum\limits_{k=1}^{n}(-1)^k k^2=(-1)^n\frac{n(n+1)}{2}$.

I. When $n=1$ the statement $P(1)$ is that $(-1)1^2=(-1)^1\cdot\frac{1\cdot 2}{2}$, which is clearly true.

II. Assume $n \in \mathbb{N}$ and $P(n)$ is true, i.e. $(-1)1^2+(-1)^2 2^2+\cdots+(-1)^n n^2=(-1)^n\frac{n(n+1)}{2}$. Add $(-1)^{n+1}(n+1)^2$ to both sides of the induction hypothesis to get

$$\begin{aligned} (-1)1^2+(-1)^2 2^2+\cdots+(-1)^n n^2+(-1)^{n+1}(n+1)^2 &= (-1)^n\frac{n(n+1)}{2}+(-1)^{n+1}(n+1)^2 \\ &= (-1)^n(n+1)\left[\frac{n}{2}-(n+1)\right] \\ &= (-1)^n\frac{(n+1)}{2}\left[n-2(n+1)\right] \\ &= (-1)^n\frac{(n+1)}{2}\left[n-2n-2\right] \\ &= (-1)^n\frac{(n+1)}{2}\left[-n-2\right] \\ &= (-1)^{n+1}\frac{(n+1)}{2}(n+2). \end{aligned}$$

Thus $P(n) \Rightarrow P(n+1)$.

III. By PMI, we can conclude that $\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$ for all natural numbers $n \geq 1$.

9. $\frac{(2n)!}{n! \cdot 2^n}$ is an odd number, for every $n \in \mathbb{N}$.

   Proof: For $n \in \mathbb{N}$ consider the following statement: $P(n): \frac{(2n)!}{n! \cdot 2^n}$ is an odd integer.

   I. $\frac{(2 \cdot 1)!}{1! \cdot 2^1} = \frac{2 \cdot 1}{1 \cdot 2} = 1$, which is odd integer, so $P(1)$ is true.

   II. Assume $n \in \mathbb{N}$ and $\frac{(2n)!}{n! \cdot 2^n}$ is odd integer, i.e. $P(n)$ is true.

   Then
$$\frac{(2(n+1))!}{(n+1)! \cdot 2^{n+1}} = \frac{(2n+2)!}{(n+1) \cdot n! \cdot 2 \cdot 2^n} = \frac{(2n+2)(2n+1)}{(n+1) \cdot 2} \cdot \frac{(2n)!}{n! \cdot 2^n} = (2n+1) \cdot \frac{(2n)!}{n! \cdot 2^n}.$$

   By the induction hypothesis, $\frac{(2n)!}{n! \cdot 2^n}$ is odd integer. Surely $2n+1$ is odd integer, so $\frac{(2(n+1))!}{(n+1)! \cdot 2^{n+1}}$ is the product of two odd numbers—hence, it is odd integer. Hence $P(n) \Rightarrow P(n+1)$.

   III. By PMI, we conclude that $\frac{(2n)!}{n! \cdot 2^n}$ is an odd for all $n \in \mathbb{N}$.

10. For all natural numbers $n > 4$, $2^n > n^2$.

    Proof: For $n \in \mathbb{N}$ consider the following statement: $P(n): 2^n > n^2$.

    I. $2^5 = 32$ and $5^2 = 25$, and $32 > 25$. So $P(5)$ is true.

    II. Assume $n \in \mathbb{N}$, $n \geq 5$, and $2^n > n^2$. Multiply by 2 to get $2^{n+1} > 2n^2$. Since $n \geq 5$, $n-2 \geq 3$. So $n(n-2) \geq 15 > 1$. Then $n^2 - 2n > 1$, so $n^2 > 2n+1$. Add n^2 to both sides to get $2n^2 > n^2 + 2n + 1 = (n+1)^2$.

    We now have $2^{n+1} > 2n^2 > (n+1)^2$, so we proved $P(n) \Rightarrow P(n+1)$.

    III. By PMI, we conclude that $2^n > n^2$ for all natural numbers $n \geq 5$.

11. Consider the sequence given recursively by $a_0 = 0$ and $a_n = \sqrt{2 + a_{n-1}}$ for all $n \geq 1$. So $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$, and so forth. Then $a_n \leq 2$ for all $n \geq 0$.

    Proof: For $n \in \mathbb{Z}_{\geq 0}$ consider the following statement: $P(n): a_n \leq 2$.

    I. We have $a_0 = 0 \leq 2$, so $P(0)$ holds true.

    II. Assume $n \in \mathbb{N}$ and $a_n \leq 2$. Then $2 + a_n \leq 4$, and so $\sqrt{2+a_n} \leq \sqrt{4} = 2$. But $a_{n+1} = \sqrt{2+a_n}$, so $a_{n+1} \leq 2$. Thus $P(n) \Rightarrow P(n+1)$.

    III. By PMI, $a_n \leq 2$ for all $n \in \mathbb{Z}_{\geq 0}$.

12. $(1 + \frac{1}{2})^n > 1 + \frac{n}{2}$ for all $n \geq 2$.

    Proof: For $n \in \mathbb{N}$ consider the following statement: $P(n): (1+\frac{1}{2})^n > 1 + \frac{n}{2}$.

    I. $(1+\frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4}$ and $1 + \frac{2}{2} = 2$. Clearly $\frac{9}{4} > 2$, so $P(2)$ holds true.

    II. Assume $n \in \mathbb{N}$, $n \geq 2$, and $(1+\frac{1}{2})^n > 1 + \frac{n}{2}$.

    Multiply both sides of the induction hypothesis by $1 + \frac{1}{2}$: this gives
$$(1+\tfrac{1}{2})^{n+1} > (1+\tfrac{n}{2}) \cdot (1+\tfrac{1}{2}) = 1 + \tfrac{n}{2} + \tfrac{1}{2} + \tfrac{n}{4} = 1 + \tfrac{n+1}{2} + \tfrac{n}{4} > 1 + \tfrac{n+1}{2}.$$

    This completes the induction step, i.e. we proved $P(n) \Rightarrow P(n+1)$.

    III. By PMI we conclude that $(1+\frac{1}{2})^n > 1 + \frac{n}{2}$ for all $n \in \mathbb{Z}_{\geq 2}$.

13. For all $n \in \mathbb{N}$, if $n \geq 1$ then $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Proof: For $n \in \mathbb{N}$ consider the following statement: $P(n): \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

I. When $n = 1$ the statement $P(1)$ is that $1 \leq 2 - \frac{1}{2}$, which is true.

II. Assume $n \in \mathbb{N}$ and $P(n)$ holds, i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Add $\frac{1}{(n+1)^2}$ to both sides of the induction hypothesis to get

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$

Surely $n^2 + 2n + 1 > n^2 + 2n$, so $(n+1)^2 > n(n+2)$. Therefore

$$\tfrac{1}{n} > \tfrac{n+2}{(n+1)^2} = \tfrac{n+1}{(n+1)^2} + \tfrac{1}{(n+1)^2} = \tfrac{1}{n+1} + \tfrac{1}{(n+1)^2}.$$

Then $-\frac{1}{n+1} > \frac{1}{(n+1)^2} - \frac{1}{n}$, so $2 - \frac{1}{n+1} > 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$. At this point we have a chain

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}$$

which proves $P(n) \Rightarrow P(n+1)$.

III. By PMI, we conclude that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all natural numbers n.

14. Fill in each box below with a mathematical proposition that makes the biconditional true, and is not a tautology (for example, I don't want you to write "$A \subseteq B$" in the first box, even though this makes the biconditional true). Copy the complete biconditional statements into your homework; do not actually write in the boxes on this worksheet.

$$\begin{aligned}
A \subseteq B &\Leftrightarrow (\forall x)[x \in A \Rightarrow x \in B] \\
A = B &\Leftrightarrow [A \subseteq B \wedge B \subseteq A] \\
x \in f(A) &\Leftrightarrow (\exists a \in A)[x = f(a)] \\
y \in f^{-1}(B) &\Leftrightarrow f(y) \in B \\
x \in A \cup B &\Leftrightarrow (x \in A \vee x \in B) \\
x \in A \cap B &\Leftrightarrow (x \in A \wedge x \in B) \\
x \in A - B &\Leftrightarrow (x \in A \wedge x \notin B) \\
f\colon S \to T \text{ is onto} &\Leftrightarrow (\forall t \in T)(\exists s \in S)[t = f(s)] \\
f\colon S \to T \text{ is one-to-one} &\Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b] \\
x \in A \cap (B - C) &\Leftrightarrow (x \in A \wedge x \in B \wedge x \notin C) \\
X = \emptyset &\Leftrightarrow (\forall x)[x \notin X]
\end{aligned}$$

15. Write definitions for the following sets, using set-builder notation. The first one is done for you. For the last two, $f\colon S \to T$.

$$\begin{aligned}
A \cap B &= \{x \mid x \in A \wedge x \in B\} & A \cup B &= \{x \mid x \in A \vee x \in B\} \\
X - A &= \{x \mid x \in X \wedge x \notin B\} & f(A) &= \{x \mid (\exists a \in A)[x = f(a)]\} \\
f^{-1}(B) &= \{x \in S \mid f(x) \in B\} & C \cap f^{-1}(B) &= \{x \in S \mid x \in C \wedge f(x) \in B\}
\end{aligned}$$