Funds of Anal I: Homework 4

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Exercise 2.3.5

Let (x_n) and (y_n) be given, and define (z_n) to be "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Solution 2.3.5

Proof.

(i) **Forward Direction:** Suppose (z_n) is convergent. Let $\lim_{n\to\infty} z_n = L$ for some real number L.

The sequence (z_n) is constructed by alternating elements from (x_n) and (y_n), so

$$(\forall n \in \mathbb{N})[z_{2n-1} = x_n \wedge z_{2n} = y_n].$$

Since (z_n) converges to L, both subsequences (z_{2n-1}) and (z_{2n}) must also converge to L (by the property that every subsequence of a convergent sequence converges to the same limit).

(a) *Convergence of* (x_n)*:* Since $z_{2n-1} = x_n$ for each n, the sequence (x_n) is the subsequence (z_{2n-1}). Thus, (x_n) converges to L.

(b) *Convergence of* (y_n)*:* Similarly, since $z_{2n} = y_n$ for each n, the sequence (y_n) is the subsequence (z_{2n}). Thus, (y_n) also converges to L.

Therefore, both (x_n) and (y_n) are convergent, and we have $\lim_{n\to\infty} x_n = L = \lim_{n\to\infty} y_n$.

(ii) **Reverse Direction:** Suppose (x_n) and (y_n) are both convergent with

$$\lim_{n\to\infty} x_n = L \quad \text{and} \quad \lim_{n\to\infty} y_n = L.$$

We need to show that (z_n) converges to L.

For any $\varepsilon > 0$, since $x_n \to L$ and $y_n \to L$, there exists an integer N such that for all $n \geq N$,

$$|x_n - L| < \varepsilon \quad \text{and} \quad |y_n - L| < \varepsilon.$$

In the sequence (z_n), every x_n and y_n appears as an element, specifically

$$z_{2n-1} = x_n \quad \text{and} \quad z_{2n} = y_n.$$

Thus, for all $m \geq 2N$, each term z_m is either x_n or y_n for some $n \geq N$. Therefore, for $m \geq 2N$,

$$|z_m - L| < \varepsilon.$$

This shows that $z_n \to L$ as $n \to \infty$, so (z_n) converges to L.

Therefore, we conclude that

$$\lim_{n\to\infty} z_n = L \Leftrightarrow \lim_{n\to\infty} x_n = L = \lim_{n\to\infty} y_n.$$ □

Exercise 2.3.9

(i) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim_{n\to\infty} b_n = 0$. Show that $\lim_{n\to\infty} (a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(ii) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(iii) Use (i) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution 2.3.9

(i) *Proof.* Let $\varepsilon > 0$ be arbitrary. Let $|b_n| < \varepsilon/M$. We can't use the ALT since a_n might not converge. However, since a_n is bounded, we have

$$(\exists M \in \mathbb{N})(\forall n \in \mathbb{N})[|a_n b_n| \leq M|b_n| < \varepsilon]. \qquad \square$$

(ii) No, since all we know about a_n is that it's bounded, not necessarily convergent.

(iii) *Proof.* In part (i), we showed that if (a_n) is bounded and $(b_n) \to 0$, then $(a_n b_n) \to 0$. If $a = 0$, then $(a_n) \to 0$, so we can apply the Algebraic Limit Theorem to conclude that $(a_n b_n) \to 0$. $\square$

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(i) If $\lim_{n\to\infty}(a_n - b_n) = 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(ii) If $(b_n) \to b$, then $|b_n| \to |b|$.

(iii) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(iv) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

Solution 2.3.10

(i) False. Counterexample, let $a_n = n$ and $b_n = -n$.

(ii) True, since if $|b_n - b| < \varepsilon$, then $||b_n| - |b|| \leq |b_n - b| < \varepsilon$.

(iii) True by ALT since $\lim_{n\to\infty}(b_n - a_n) + \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$.

(iv) True, since $0 \leq |b_n - b| \leq a_n$, we have $a_n \geq 0$. Let $\varepsilon > 0$. Choose N such that $(\forall n \geq N)[a_n < \varepsilon]$. Therefore, $|b_n - b| \leq a_n < \varepsilon$.

Exercise 2.4.1

(i) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(ii) Now that we know $\lim_{n\to\infty} x_n$ exists, explain why $\lim_{n\to\infty} x_{n+1}$ must also exist and equal the same value.

(iii) Take the limit of each side of the recursive equation in part (i) to explicitly compute $\lim_{n\to\infty} x_n$.

Solution 2.4.1

(i) *Proof.* Let L be the limit of the sequence, assuming it exists. If $x_n \to L$ as $n \to \infty$, then $x_{n+1} \to L$ as well, and we can take limits on both sides of the recurrence relation

$$L = \frac{1}{4 - L} \Leftrightarrow L(4 - L) = 1 \Leftrightarrow L^2 - 4L + 1 = 0 \Rightarrow x = 2 \pm \sqrt{3}.$$

Base Case: For $n = 1$, we have $x_1 = 3$, and indeed $2 - \sqrt{3} \approx 0.27 < 3 < 2 + \sqrt{3} \approx 4.73$.

Inductive Step: Suppose $2 - \sqrt{3} < x_n < 2 + \sqrt{3}$ for some $n \geq 1$. Then

$$4 - (2 + \sqrt{3}) < 4 - x_n < 4 - (2 - \sqrt{3}),$$

which simplifies to

$$2 - \sqrt{3} < 4 - x_n < 2 + \sqrt{3}.$$

Taking reciprocals, we get

$$\frac{1}{2 + \sqrt{3}} < \frac{1}{4 - x_n} < \frac{1}{2 - \sqrt{3}}.$$

Since $x_{n+1} = \frac{1}{4 - x_n}$, it follows that $2 - \sqrt{3} < x_{n+1} < 2 + \sqrt{3}$.

By induction, $2 - \sqrt{3} < x_n < 2 + \sqrt{3}$ for all $n \geq 1$, so the sequence is bounded.

Next, we analyze whether the sequence (x_n) is increasing or decreasing. Consider the difference $x_{n+1} - x_n$

$$x_{n+1} - x_n = \frac{1}{4 - x_n} - x_n.$$

Since the sequence is bounded and has only one possible limit within the bounds (namely $2 - \sqrt{3}$), any accumulation point must be $2 - \sqrt{3}$. $\square$

(ii) Skipping a single term does not change the limit of the sequence.

(iii) Since $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$, we get

$$x = \frac{1}{4 - x} \Leftrightarrow x^2 - 4x + 1 = 0 \Leftrightarrow (x - 2)^2 = 3 \Leftrightarrow x = 2 \pm \sqrt{3}.$$

Since $2 + \sqrt{3} > 3$ is impossible, we conclude $\lim_{n\to\infty} x_n = 2 - \sqrt{3}$.

Exercise 2.4.2

(i) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim_{n \to \infty} y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim_{n \to \infty} y_n = 3/2$. What is wrong with this argument?

(ii) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (i) be applied to compute the limit of this sequence.

Solution 2.4.2

(i) The sequence $y_n = (1, 2, 1, 2, 1, \dots)$ is not convergent, so the argument is invalid.

(ii) Yes, y_n converges by the monotone convergence theorem, since (y_n) is bounded between $0 < y_n < 3$ and y_n is increasing.

Exercise 2.4.6

(i) Explain why $\sqrt{xy} \leq \frac{x+y}{2}$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(ii) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim\limits_{n\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$ both exist and are equal.

Solution 2.4.6

(i) *Proof.* Since $x, y > 0$, we have

$$\sqrt{xy} \leq \frac{x+y}{2} \Leftrightarrow 2\sqrt{xy} \leq x + y \Leftrightarrow 4xy \leq x^2 + 2xy + y^2 \Leftrightarrow 0 \leq (x-y)^2.$$

The last inequality is always true, so $\sqrt{xy} \leq {}^{x+y}/_{2}$. □

(ii) *Proof.* The inequality $x_1 \leq y_1$ is always true, since

$$\sqrt{x_n y_n} \leq \frac{x_n + y_n}{2} \Rightarrow x_{n+1} \leq y_{n+1}.$$

Also $x_n \leq y_n$ implies $\frac{x_n+y_n}{2} = y_{n+1} \leq y_n$. Similarly, $\sqrt{x_n y_n} = x_{n+1} \geq x_n$. By MCT, both converge since they are bounded and monotone. □