Fundamentals of Analysis II: Homework 5

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Exercise 7.2.3.

(i) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying
$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0.,$$
and in this case $\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$

(ii) For each n, let P_n be the partition $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = n(n+1)/2$ will be useful.

(iii) Use the sequential criterion for integrability from (i) to show directly that $f(x) = x$ is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution to (i). Assume f is integrable on $[a, b]$. Let $(P_n)_{n=1}^{\infty}$ be a sequence of partitions such that $0 \leq U(f, P_n) - L(f, P_n) < \varepsilon_n = 1/n$, as this is possible since f is integrable on $[a, b]$. By Squeeze Theorem, we have
$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0.$$

Assume that there exists a partition P_n such that
$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0.$$
Let $\varepsilon > 0$. Then there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, $0 \leq U(f, P_n) - L(f, P_n) < \varepsilon$. Let $P_\varepsilon = P_n$ such that f is integrable on $[a, b]$. Hence,
$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

Therefore, f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying
$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0. \qquad \square$$

Solution to (ii). Break $[0, 1]$ into n equal subintervals. Then $P_n = \{0, 1/n, 2/n, \cdots, n-1/n, 1\}$. Let $f(x) = x$. For each subinterval $[i/n, i+1/n]$ of P_n with $0 \leq i \leq n - 1$, let
$$\begin{aligned} m_i &= \inf\{f(x) \mid x \in [i/n, i+1/n]\} = i/n \\ M_i &= \sup\{f(x) \mid x \in [i/n, i+1/n]\} = i+1/n. \end{aligned}$$

Hence, the upper sum is
$$\begin{aligned} U(f, P_n) &= \sum_{i=0}^{n-1} M_i \Delta x_i = \sum_{i=0}^{n-1} \frac{i+1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=0}^{n-1}(i+1) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n} \\ L(f, P_n) &= \sum_{i=0}^{n-1} m_i \Delta x_i = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=0}^{n-1} i \\ &= \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n}. \end{aligned} \qquad \square$$

Solution to (iii). As $n \to \infty$, $U(f, P_n)$ and $L(f, P_n)$ both approach $1/2$. Therefore,
$$\int_0^1 f = \lim_{n\to\infty} U(f, P_n) = \frac{1}{2} = \lim_{n\to\infty} L(f, P_n). \qquad \square$$

Exercise 7.2.4. Let g be bounded on $[a, b]$ and assume that there exists a partition P with $L(g, P) = U(g, P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

Solution. Suppose g is a bounded function on $[a, b]$, and there exists a partition $P = \{x_0, x_1, \cdots, x_n\}$ such that the lower and upper sums satisfy

$$L(g, P) = U(g, P).$$

By definition, the lower and upper sums are given by

$$L(g, P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad U(g, P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where

$$\begin{aligned} m_i &= \inf\{g(x) \mid x \in [x_{i-1}, x_i]\} \\ M_i &= \sup\{g(x) \mid x \in [x_{i-1}, x_i]\}. \end{aligned}$$

Since $L(g, P) = U(g, P)$, it follows that $m_i = M_i$ for each subinterval $[x_{i-1}, x_i]$. This implies that $g(x)$ is constant on each subinterval, meaning that g is a piecewise constant function with respect to P.

Since g is piecewise constant on a finite partition, it follows that g is Riemann integrable. The Riemann integral of g over $[a, b]$ is given by

$$\int_a^b g(x)\, dx = L(g, P) = U(g, P),$$

which simplifies to

$$\int_a^b g(x)\, dx = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i = c(b - a),$$

where c is the constant value of g on each subinterval $[x_{i-1}, x_i]$. □

Exercise 7.2.7. Let $f : [a, b] \to \mathbb{R}$ be increasing on the set $[a, b]$ (i.e., $f(x) \leq f(y)$ whenever $x < y$). Show that f is integrable on $[a, b]$.

Solution. Let P_n be a partition of $[a, b]$ into n equal subintervals. Then

$$P_n = \left\{a, a + \frac{b-a}{n}, \cdots, a + \frac{k(b-a)}{n}, \cdots, a + \frac{(n-1)(b-a)}{n}, b\right\}.$$

For each subinterval $\left[a + \frac{k(b-a)}{n}, a + \frac{(k+1)(b-a)}{n}\right]$, define

$$\begin{aligned} m_k &= \inf\left\{f(x) \;\middle|\; x \in \left[a + \frac{k(b-a)}{n}, a + \frac{(k+1)(b-a)}{n}\right]\right\} = f\left(a + \frac{k(b-a)}{n}\right) \\ M_k &= \sup\left\{f(x) \;\middle|\; x \in \left[a + \frac{k(b-a)}{n}, a + \frac{(k+1)(b-a)}{n}\right]\right\} = f\left(a + \frac{(k+1)(b-a)}{n}\right). \end{aligned}$$

Since f is increasing, we have $m_k \leq M_k$ for all k, ensuring that

$$\begin{aligned} U(f, P_n) &= \sum_{k=0}^{n-1} M_k \Delta x_k = \sum_{k=0}^{n-1} f\left(a + \frac{(k+1)(b-a)}{n}\right) \cdot \frac{b-a}{n} \\ L(f, P_n) &= \sum_{k=0}^{n-1} m_k \Delta x_k = \sum_{k=0}^{n-1} f\left(a + \frac{k(b-a)}{n}\right) \cdot \frac{b-a}{n}. \end{aligned}$$

The difference between the upper and lower sums is

$$U(f, P_n) - L(f, P_n) = \sum_{k=0}^{n-1} \left(f\left(a + \frac{(k+1)(b-a)}{n} \right) - f\left(a + \frac{k(b-a)}{n} \right) \right) \cdot \frac{b-a}{n}.$$

Since f is increasing, the terms inside the summation are nonnegative, and summing over all intervals gives a telescoping sum

$$U(f, P_n) - L(f, P_n) = (f(b) - f(a)) \cdot \frac{b-a}{n}.$$

Taking the limit as $n \to \infty$, we observe that

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = \lim_{n\to\infty} (f(b) - f(a)) \cdot \frac{b-a}{n} = 0.$$

Since the difference between the upper and lower sums can be made arbitrarily small, it follows that f is Riemann integrable on $[a, b]$. □

Exercise 7.3.1. Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } x = 1 \end{cases},$$

over the interval $[0, 1]$.

(i) Show that $L(f, P) = 1$ for every partition P of $[0, 1]$.

(ii) Construct a partition P for which $U(f, P) < 1 + 1/10$.

(iii) Given $\varepsilon > 0$, construct a partition P_ε for which $U(f, P_\varepsilon) < 1 + \varepsilon$.

Solution to (i). Let $P = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[0, 1]$ with $x_0 = 0$ and $x_n = 1$. The lower sum is given by

$$L(h, P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad m_i = \inf\{h(x) \mid x \in [x_{i-1}, x_i]\}.$$

Since $h(x) = 1$ for all $x < 1$, and the infimum over any subinterval $[x_{i-1}, x_i]$ with $x_i \leq 1$ is simply 1, we conclude that $m_i = 1$ for all i. Therefore,

$$L(h, P) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1.$$

Thus, $L(h, P) = 1$ for every partition P. □

Solution to (ii). The upper sum is given by

$$U(h, P) = \sum_{i=1}^{n} M_i \Delta x_i, \quad M_i = \sup\{h(x) \mid x \in [x_{i-1}, x_i]\}.$$

Since $h(x) = 1$ everywhere except at $x = 1$, the supremum M_i is equal to 1 for all subintervals except the one containing $x = 1$. If we choose a partition where the last subinterval is small, we can make $U(h, P)$ arbitrarily close to 1.

Let P be a partition such that $x_{n-1} = 0.99$ and $x_n = 1$. Then,

$$U(h, P) = \sum_{i=1}^{n-1} 1 \cdot (x_i - x_{i-1}) + 2 \cdot (1 - 0.99).$$

Since the sum of all subintervals must be 1, we compute:

$$U(h, P) = (1 - 0.01) + 2(0.01) = 1.01 < 1 + \frac{1}{10}.$$

Thus, we have constructed a partition satisfying the given condition. □

Solution to (iii). Given $\varepsilon > 0$, we wish to construct a partition P_ε such that

$$U(h, P_\varepsilon) < 1 + \varepsilon.$$

From part (ii), we see that making the last subinterval $[x_{n-1}, 1]$ sufficiently small will reduce the contribution of the term $2 \cdot (1 - x_{n-1})$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and define a partition P_ε where $x_{n-1} = 1 - \frac{1}{N}$. Then,

$$U(h, P_\varepsilon) = \sum_{i=1}^{n-1} 1 \cdot (x_i - x_{i-1}) + 2 \cdot (1 - x_{n-1}).$$

Since $\sum_{i=1}^{n-1}(x_i - x_{i-1}) = 1 - (1 - \frac{1}{N}) = \frac{1}{N}$, we get

$$U(h, P_\varepsilon) = (1 - \frac{1}{N}) + 2 \cdot \frac{1}{N} = 1 + \frac{1}{N}.$$

By construction, $\frac{1}{N} < \varepsilon$, so

$$U(h, P_\varepsilon) < 1 + \varepsilon.$$

Hence, we have successfully constructed the desired partition. □

Exercise 7.3.7. Assume $f : [a, b] \to \mathbb{R}$ is integrable.

(i) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a, b]$, then g in integrable as well.

(ii) Find an example to show that g may fail to be integrable if it differers from f at a countable number of points.

Solution to (i). Define $h(x) = g(x) - f(x)$. By assumption, there exist finitely many points $x_1, x_2, \cdots, x_n \in [a, b]$ such that $g(x) \neq f(x)$ at these points, and $g(x) = f(x)$ elsewhere. Thus, we can write $h(x)$ as

$$h(x) = \begin{cases} 0, & x \in [a, b] \setminus \{x_1, x_2, \cdots, x_n\} \\ g(x) - f(x), & x \in \{x_1, x_2, \cdots, x_n\}. \end{cases}.$$

Since $h(x)$ is nonzero at only finitely many points, its set of discontinuities is finite.

Now, recall that the sum of two integrable functions is integrable. Since f is integrable by assumption and $g = f + h$, it suffices to show that h is integrable.

Consider the upper and lower sums of h with respect to any partition P

$$L(h, P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad U(h, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Since $h(x) = 0$ everywhere except at finitely many points, we can make the contribution of these points arbitrarily small by refining the partition. Specifically, for any $\varepsilon > 0$, we can choose a partition where the subintervals containing the exceptional points are small enough such that

$$U(h, P) - L(h, P) < \varepsilon.$$

This implies that the difference between the upper and lower sums of h can be made arbitrarily small, which shows that h is integrable.

Finally, since $g = f + h$ is the sum of two integrable functions, it follows that g is also integrable. □

Solution to (ii). Define f and g as

$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b] \\ 0 & \text{otherwise} \end{cases}.$$

Since the rationals are dense in the reals, f and g differ at a countable number of points. However, g is not integrable on $[a, b]$ since it is discontinuous at every irrational point in $[a, b]$. Hence, f is integrable but g is not. □

Exercise 7.4.1. Let f be a bounded function on a set A, and set

$$M = \sup\{f(x) \mid x \in A\}, \quad m = \inf\{f(x) \mid x \in A\}$$
$$M' = \sup\{|f(x)| \mid x \in A\}, \quad \text{and} \quad m' = \inf\{|f(x)| \mid x \in A\}.$$

(i) Show that $M - m \geq M' - m'$.

(ii) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

(iii) Provide the details for the argument that in this case we have $|\int_a^b f| \leq \int_a^b |f|$.

Solution to (i). Since M and m are the largest and smallest values that f attains on A, we immediately have

$$-M \leq f(x) \leq M \quad \text{for all} \quad x \in A.$$

Similarly,

$$m \leq f(x) \leq M \quad \text{for all} \quad x \in A.$$

Taking absolute values, we obtain

$$\begin{aligned} m' &= \inf\{|f(x)| \mid x \in A\} \geq \inf\{m, -m\} = |m| \\ \text{and} \quad M' &= \sup\{|f(x)| \mid x \in A\} \leq \sup\{M, -m\} = \max\{M, -m\}. \end{aligned}$$

Comparing $M - m$ and $M' - m'$ gives us

$$M - m = \sup f(x) - \inf f(x).$$

Meanwhile, using the inequalities for M' and m' above,

$$M' - m' \leq \max\{M, -m\} - |m|.$$

Since $\max\{M, -m\} \leq M$ and $m \leq |m|$, it follows that

$$M' - m' \leq M - m.$$ □

Solution to (ii). We need to show that if f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$.

A function is Riemann integrable if and only if it is bounded and the set of its discontinuities has measure zero. Since f is integrable, it is bounded, so there exists some constant $K > 0$ such that $|f(x)| \leq K$ for all $x \in [a, b]$.

Now, consider the discontinuities of $|f|$. A discontinuity of $|f|$ occurs only if f is discontinuous at some point x or if $f(x) = 0$ and f changes sign at x. The former case contributes at most a measure zero set (since f is integrable). The latter case also forms a measure zero set because it consists of isolated points where sign changes occur.

Since the set of discontinuities of $|f|$ is contained within the measure zero set of discontinuities of f, it follows that $|f|$ has measure zero discontinuities. Thus, $|f|$ is Riemann integrable. □

Solution to (iii). The integral is a limit of Riemann sums, so for any partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with sample points $c_i \in [x_{i-1}, x_i]$, we approximate the integral as

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^n f(c_i)\Delta x_i.$$

Taking absolute values and applying the triangle inequality

$$\left|\sum_{i=1}^n f(c_i)\Delta x_i\right| \le \sum_{i=1}^n |f(c_i)|\Delta x_i.$$

Passing to the limit as the partition is refined, we obtain

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

□