Math 307, Homework #6
Due Wednesday, November 13
SOLUTIONS TO SELECTED PROBLEMS

1. In each case, use mathematical notation to write the negation of the given statement, in such a way that no quantifier is immediately preceded by a negation sign. For parts (a)–(d) decide which is true: the given statement or its negation.
   (a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y = 0]$ TRUE
   $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y \neq 0]$
   (b) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = 0]$
   $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y \neq 0]$ TRUE
   (c) $(\exists x, y \in \mathbb{R})[x^2 + y^2 = -1]$
   $(\forall x, y \in \mathbb{R})[x^2 + y^2 \neq -1]$ TRUE
   (d) $(\forall x \in \mathbb{R})[x > 0 \Rightarrow (\forall y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \Rightarrow y = z]]$ TRUE
   $(\exists x \in \mathbb{R})[x > 0 \wedge (\exists y, z \in \mathbb{R})[y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x \wedge y \neq z]]$
   (e) $(\forall \epsilon \in \mathbb{R})[\epsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})[0 < \ \delta \wedge (\forall x \in \mathbb{R})[1 - \delta < x < 1 + \delta \Rightarrow |f(x) - 5| < \epsilon]]]$
   $(\exists \epsilon \in \mathbb{R})[\epsilon > 0 \wedge (\forall \delta \in \mathbb{R})[0 < \delta \Rightarrow (\exists x \in \mathbb{R})[1 - \delta < x < 1 + \delta \wedge |f(x) - 5| \geq \epsilon]]]$
   (f) $(\forall a, b \in \mathbb{R})(a < b) \Rightarrow (\exists c \in \mathbb{R})[a < c < b \wedge f'(c) = \frac{f(b)-f(a)}{b-a}]$.
   $(\exists a, b \in \mathbb{R})(a < b) \wedge (\forall c \in \mathbb{R})[a < c < b \Rightarrow f'(c) \neq \frac{f(b)-f(a)}{b-a}]$

   Part (f) is a statement which is true for differentiable functions $f \colon \mathbb{R} \to \mathbb{R}$, and it is a well-known theorem taught in every calculus class. What is the common name of this theorem?

   Part (f) is the Mean Value Theorem.

2. In each part below I give the definition for a mathematical concept we have encountered, but using the shorthand notation in quantifiers. Fill in each box with the appropriate mathematical term or phrase that best completes the definition. In parts (b)–(d), $f \colon S \to T$ and $A \subseteq S$.
   (a) $A \subseteq B \Leftrightarrow (\forall x \in A)[x \in B]$
   (b) f is one-to-one $\Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$
   (c) f is onto $\Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$.
   (d) $f(A) = \{z | (\exists v \in A)[z = f(v)]\}$

3. Suppose $f \colon S \to T$ is one-to-one, $A \subseteq S$, and $B \subseteq S$. Give a line proof showing that $f(A) \cap f(B) \subseteq f(A \cap B)$.

   Proof:
   1. Let $x \in f(A) \cap f(B)$.
   2. Then $x = f(a)$ for some $a \in A$.
   3. And $x = f(b)$ for some $b \in B$.
   4. Since $f(a) = f(b)$ and f is one-to-one, $a = b$.
   5. Therefore $a \in A$ and $a \in B$, so $a \in A \cap B$.
   6. So $x = f(a)$ and $a \in A \cap B$, hence $x \in f(A \cap B)$.
   7. Thus, $f(A) \cap f(B) \subseteq f(A \cap B)$.

4. Give a line proof showing that if $A \cap B = \emptyset$ and $B \cup C = A \cup D$ then $B \subseteq D$.

   Proof:

   1. Let $x \in B$.
   2. Then $x \in B \cup C$, so $x \in A \cup D$ as well.
   3. That is, $x \in A$ or $x \in D$.
   4. But if $x \in A$ then $x \in A \cap B$, whereas $A \cap B = \emptyset$. So $x \notin A$.
   5. Therefore $x \in D$.
   6. Hence, $B \subseteq D$.

5. Let $f \colon S \to T$, and suppose that f is onto. Let $A \subseteq S$. Give a line proof that $T - f(A) \subseteq f(S - A)$.

   Proof:

   1. Let $x \in T - f(A)$.
   2. Then $x \in T$ and $x \notin f(A)$.
   3. Since f is onto, $x = f(s)$ for some $s \in S$.
   4. If $s \in A$ then $x \in f(A)$, but we know the latter is false. So $s \notin A$.
   5. Hence $s \in S - A$, so $x \in f(S - A)$.
   6. Thus, $T - f(A) \subseteq f(S - A)$.

6. Give a line proof showing $A \cap (X - B) = (A \cap X) - (A \cap B)$.

   Proof:

   1. Let $x \in A \cap (X - B)$.
   2. Then $x \in A$ and $x \in X - B$.
   3. So $x \in X$ and $x \notin B$.
   4. Since x is in both A and X, $x \in A \cap X$.
   5. And since x is not in B, x cannot be in $A \cap B$.
   6. Thus, $x \in (A \cap X) - (A \cap B)$.
   7. We have therefore shown $A \cap (X - B) \subseteq (A \cap X) - (A \cap B)$.
   8. Now assume $y \in (A \cap X) - (A \cap B)$.
   9. Then $y \in A \cap X$, and $y \notin A \cap B$.
   10. So $y \in A$ and $y \in X$.
   11. Since $y \in A$ and $y \notin A \cap B$, we must have $y \notin B$.
   12. So $y \in X - B$.
   13. Since we also had $y \in A$, we get $y \in A \cap (X - B)$.
   14. Therefore $(A \cap X) - (A \cap B) \subseteq A \cap (X - B)$.
   15. Since we have now shown the subsets in both directions, the two sets are equal.

7. Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2e^{-x} + 5$.

   (a) Prove that f is one-to-one.

Proof:
1. Assume $x, y \in \mathbb{R}$ and $f(x) = f(y)$.
2. Then $2e^{-x} + 5 = 2e^{-y} + 5$.
3. Subtract 5 from both sides to get $2e^{-x} = 2e^{-y}$.
4. Divide by 2 to get $e^{-x} = e^{-y}$.
5. Take natural log of both sides: $-x = -y$.
6. So $x = y$.
7. Hence, we have proven that f is injective.

(b) Is f onto? Justify your answer.

The values e^z are always positive, so $2e^{-5} + 5$ is always larger than 5. In particular, there is no $x \in \mathbb{R}$ such that $f(x) = -1$. So f is not onto.

(c) Let $g \colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 5x^3 + 41$. Prove that g is onto.

Proof:
1. Assume $t \in \mathbb{R}$.
2. Let $s = \sqrt[3]{\frac{t-41}{5}}$ (this exists because every real number has a cube root).
3. Then $5s^3 = t - 41$, so $5s^3 + 41 = t$. That is, $f(s) = t$.
4. Hence, f is onto.

8. Suppose $f \colon A \to B$ and $g \colon B \to C$.

(a) If f and g are one-to-one, prove that $g \circ f$ is one-to-one.

Proof:
1. Assume $x, y \in A$ and $(g \circ f)(x) = (g \circ f)(y)$.
2. Then $g(f(x)) = g(f(y))$.
3. Because g is one-to-one, $f(x) = f(y)$.
4. Because f is one-to-one, $x = y$.
5. Hence, $g \circ f$ is one-to-one.

(b) If f and g are both onto, prove that $g \circ f$ is onto.

Proof:
1. Let $x \in C$.
2. Since g is onto, $x = g(u)$ for some $u \in B$.
3. Since f is onto, $u = f(r)$ for some $r \in A$.
4. Then $x = g(u) = g(f(r)) = (g \circ f)(r)$.
5. Hence, $g \circ f$ is onto.

9. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ has the property that $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) < f(y)]$.

(a) Prove that f is one-to-one.

Proof:
1. Let $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$.
2. Assume $x \neq y$.
3. Then $x < y$ or $y < x$.
4. If $x < y$ then $f(x) < f(y)$, which contradicts $f(x) = f(y)$.
5. If $y < x$ then $f(y) < f(x)$, which again contradicts $f(x) = f(y)$.
6. Since both cases lead to contradictions, we conclude that $x = y$.
7. Henace, f is one-to-one.

(b) Give an example of a function f satisfying the given property but which is not onto.

The function $f(x) = e^x$ satisfies the property, but it only takes positive values. So for example, there is no x such that $f(x) = 0$. Hence, f is not onto.

10. (a) If $f \colon \mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = x^2 - 1$ and $g \colon \mathbb{Z} \to \mathbb{Z}$ is given by $g(x) = 3x + 2$, determine $(g \circ f)(0)$ and $(g \circ f)(2)$. Determine an algebraic formula for $(g \circ f)(x)$ for any integer x.

$(g \circ f)(0) = g(f(0)) = g(-1) = -3 + 2 = -1$. Likewise, $(g \circ f)(2) = g(f(2)) = g(3) = 3(3) + 2 = 11$.
$(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 3(x^2 - 1) + 2 = 3x^2 - 1$.

(b) Suppose that $f\colon S \to T$ and $g\colon T \to U$. If $A \subseteq S$, give a line proof that $(g \circ f)(A) = g(f(A))$.

Proof:

1. Assume $x \in (g \circ f)(A)$. .
2. Then $x = (g \circ f)(a)$ for some $a \in A$.
3. Then $x = g(f(a))$.
4. Since $f(a) \in f(A)$, we have $g(f(a)) \in g(f(A))$.
5. That is, $x \in g(f(A))$.
6. So $(g \circ f)(A) \subseteq g(f(A))$.
7. Now assume $y \in g(f(A))$.
8. Then $y = g(u)$ for some $u \in f(A)$.
9. And we have $u = f(v)$ for some $v \in A$.
10. So $y = g(u) = g(f(v)) = (g \circ f)(v)$.
11. Since $v \in A$, we have $y \in (g \circ f)(A)$.
12. We have now shown that $g(f(A)) \subseteq (g \circ f)(A)$, so the two sets are equal.

11. Consider the function $f\colon \mathbb{Z}_7 \to \mathbb{Z}_7$ given by $f(x) = x^3 + 1$. Answer the following questions:

(a) Is f one-to-one? Explain why or why not.

f is not one-to-one: $f(1) = 2 = f(2)$, but $1 \neq 2$.

(b) Is f onto? Explain why or why not.

Here we need to examine all the values of f:

$$f(0) = 1,\ f(1) = 2,\ f(2) = 2,\ f(3) = 0,\ f(4) = 2,\ f(5) = 0,\ f(6) = 0.$$

So f is not onto: for example, there is no $x \in \mathbb{Z}_7$ such that $f(x) = 3$.

(c) Determine $f(S)$ where $S = \{0, 2, 4, 6\}$.

$f(S) = \{f(0), f(2), f(4), f(6)\} = \{0, 1, 2\}$.

(d) What is $f^{-1}(\{0\})$?

$f^{-1}(\{0\}) = \{3, 5, 6\}$.

(e) If $A = \{1, 2, 3, 4\}$ and $B = \{0, 4, 5, 6\}$, determine $f^{-1}(A)$ and $f^{-1}(B)$. Also determine $f^{-1}(A \cap B)$.

$f^{-1}(A) = \{1, 2, 4\}$ and $f^{-1}(B) = \{3, 5, 6\}$.

$A \cap B = \{4\}$, so $f^{-1}(A \cap B) = \emptyset$.

12. Suppose $f\colon S \to T$, $A \subseteq S$, and $B \subseteq T$. Give line proofs for each of the following:

(a) $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

Proof:

1. Assume $f(A) \subseteq B$.
2. Let $x \in A$.
3. Then $f(x) \in f(A)$, so $f(x) \in B$ as well.
4. This means $x \in f^{-1}(B)$.
5. Hence, $A \subseteq f^{-1}(B)$.

(b) $f(A) \cap B = \emptyset \Rightarrow A \subseteq S - f^{-1}(B)$.

Proof:

1. Assume $f(A) \cap B = \emptyset$.
2. Let $x \in A$.
3. Then $f(x) \in f(A)$.
4. Since $f(A) \cap B = \emptyset$, we must have $f(x) \notin B$.
5. So $x \notin f^{-1}(B)$.
6. Since $A \subseteq S$ and $x \in A$, we also have $x \in S$.
7. So $x \in S - f^{-1}(B)$, hence $A \subseteq S - f^{-1}(B)$.

13. Suppose $f\colon S \to T$, $A \subseteq T$, $B \subseteq T$, and $C \subseteq S$. Give a line proof of each of the following:

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Proof:

1. Assume $x \in f^{-1}(A \cap B)$.
2. Then $f(x) \in A \cap B$.
3. So $f(x) \in A$, hence $x \in f^{-1}(A)$.
4. Likewise $f(x) \in B$, hence $x \in f^{-1}(B)$.
5. So $x \in f^{-1}(A) \cap f^{-1}(B)$.
6. We have therefore shown $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.
7. Now let $y \in f^{-1}(A) \cap f^{-1}(B)$.
8. Then $y \in f^{-1}(A)$, so $f(y) \in A$.

9 Likewise $y \in f^{-1}(B)$, so $f(y) \in B$.

10. Then $f(y) \in A \cap B$.
11. So $y \in f^{-1}(A \cap B)$.
12. We have now shown $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$, so the two sets are equal.

(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(c) $f(f^{-1}(A)) \subseteq A$

Proof:

1. Let $x \in f(f^{-1}(A))$.
2. Then $x = f(u)$ for some $u \in f^{-1}(A)$.
3. This means $f(u) \in A$, so $x \in A$.
4. Hence $f(f^{-1}(A)) \subseteq A$.

(d) $C \subseteq f^{-1}(f(C))$.

Proof:

1. Let $x \in C$.
2. Then $f(x) \in f(C)$.
3. Since for all u we know $u \in f^{-1}(f(C)) \Leftrightarrow f(u) \in f(C)$, we can deduce $x \in f^{-1}(f(C))$.
4. Hence $C \subseteq f^{-1}(f(C))$.

(e) If f is onto then $f(f^{-1}(A)) = A$.

Proof:

1. We have already shown $f(f^{-1}(A)) \subseteq A$ in (c).
2. So assume $x \in A$.
3. Since f is onto, $x = f(u)$ for some $u \in S$.
4. Then $f(u) \in A$, so $u \in f^{-1}(A)$.
5. Hence $f(u) \in f(f^{-1}(A))$.
6. That is, $x \in f(f^{-1}(A))$.
7. So $A \subseteq f(f^{-1}(A))$, hence the two sets are equal.

(f) If f is one-to-one then $C = f^{-1}(f(C))$.

Proof:

1. We have already shown $C \subseteq f^{-1}(f(C))$ in (d).
2. So assume $x \in f^{-1}(f(C))$.
3. Then $f(x) \in f(C)$.
4. So $f(x) = f(c)$ for some $c \in C$.
5. Since f is one-to-one, $x = c$.
6. So $x \in C$.
7. Hence $f^{-1}(f(C)) \subseteq C$, therefore the two sets are equal.

14. Construct an example of a function $f\colon \{0,1,2\} \to \{0,1\}$ and a subset $A \subseteq \{0,1\}$ where $f(f^{-1}(A)) \neq A$. Also, construct an example of a function $g\colon \{0,1,2\} \to \{0,1\}$ and a subset $C \subseteq \{0,1,2\}$ where $C \neq g^{-1}(g(C))$.

Let $f\colon \{0,1,2\} \to \{0,1\}$ be given by $f(0) = f(1) = f(2) = 0$. Let $A = \{1\}$. Then $f^{-1}(A) = \emptyset$ and so $f(f^{-1}(A)) = f(\emptyset) = \emptyset$. Note that $f(f^{-1}(A)) \neq A$.

For the second part, let $g\colon \{0,1,2\} \to \{0,1\}$ be given by $g(0) = g(1) = 0$ and $g(2) = 1$. Let $C = \{0\}$. Then $g(C) = \{0\}$, and so $g^{-1}(g(C)) = g^{-1}(\{0\}) = \{0,1\}$. Note that $g^{-1}(g(C)) \neq C$.

15. Suppose $f, g\colon \mathbb{Z} \to \mathbb{Z}$ are two functions such that $f(M_3) \subseteq M_6$, $g(M_2) \subseteq M_7$, and $g^{-1}(M_5) = M_3$. Prove that for all $x \in \mathbb{Z}$, if $3|x$ then $35|g(f(x))$.

    Proof:

    1. Assume $x \in \mathbb{Z}$ and $3|x$.
    2. Then $x \in M_3$, so $f(x) \in f(M_3)$.
    3. By assumption $f(M_3) \subseteq M_6$, so $f(x) \in M_6$.
    4. But $M_6 \subseteq M_2$, so $f(x) \in M_2$. Then $g(f(x)) \in g(M_2)$.
    5. Since we have assumed $g(M_2) \subseteq M_7$, we have $g(f(x)) \in M_7$.
    6. So $7|g(f(x))$.
    7. But we also have $M_6 \subseteq M_3$, so $f(x) \in M_3$.
    8. By assumption $M_3 = g^{-1}(M_5)$, so $f(x) \in g^{-1}(M_5)$.
    9. That is, $g(f(x)) \in M_5$.
    10. So $5|g(f(x))$.
    11. Since $7|g(f(x))$ we have $g(f(x)) = 7y$ for some $y \in \mathbb{Z}$.
    12. Since $5|g(f(x)) = 7y$ we have $5|7$ or $5|y$ by property (P).
    13. Since $5 \nmid 7$ we have $5|y$, so $y = 5z$ for some $z \in \mathbb{Z}$.
    14. Thus $g(f(x)) = 7y = 7 \cdot (5z) = 35z$, hence $35|g(f(x))$.

16. Given $[Q \wedge S] \Rightarrow R$ and $\sim S \Rightarrow T$, prove $[P \Rightarrow Q] \Rightarrow [\sim T \Rightarrow [\sim P \vee R]]$.

    Proof:

1.	$[Q \wedge S] \Rightarrow R$	hyp.
2.	$\sim S \Rightarrow T$	hyp.
3.	$P \Rightarrow Q$	dis. hyp.
4.	$\sim T$	dis. hyp.
5.	$\sim(\sim P \vee R)$	dis. hyp.
6.	$P \wedge \sim R$	GSP, For 5, de Morgan's law
7.	S	MT, For 2, For 4
8.	P	LCS, For 6
9.	Q	MP, For 3, For 8
10.	$Q \wedge S$	CI, For 9, For 7
11.	R	MP For 1, For 10
12.	$\sim R$	RCS, For 6
13.	$R \wedge \sim R$	CI, For 11, For 12
14.	$\sim P \vee R$	II, discharge For 5
15.	$\sim T \Rightarrow [\sim P \vee R]$	DT, discharge For 4
16.	$[P \Rightarrow Q] \Rightarrow [\sim T \Rightarrow [\sim P \vee R]]$	DT, discharge For 3