Introduction to Toplogy I: Homework 7

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Nicolas Addington

Hashem A. Damrah
UO ID: 952102243

Exercise 7.3. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called *lower semi-continuous* at a point $p \in X$ if $\liminf_{q \to p} f(q) \geq f(p)$, or equivalently, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\mathrm{d}(p, q) < \delta$ implies $f(q) > f(p) - \varepsilon$. The idea is that in the limit, f can only jump down. You can guess what *upper semi-continuous means.*

(i) Let $X = \mathbb{R}$ with the usual metric, and consider the floor function $\lfloor x \rfloor$, which returns the greatest integer $\leq x$, and the ceiling function $\lceil x \rceil$, which returns the least integer $\geq x$. Which one is lower semi-continuous, and which one is upper semi-continuous?

(You don't have to prove it.)

(ii) Prove that $f : X \to \mathbb{R}$ is lower semi-continuous (at every point) if and only if it is continuous as a map of topological spaces when the codomain $\mathbb{R}$ is given the lower semi-continuous topology from Example 7.3(c).

Solution to (i). The ceiling function is lower semi-continuous, and the floor function is upper semi-continuous. $\square$

Solution to (ii). Assume $f : X \to \mathbb{R}$ is lower semi-continuous. Let $U \subset \mathbb{R}$ is open in the lower semi-continuous topology. Then for every $y \in U$, there exists an $\varepsilon > 0$ such that $(y - \varepsilon, \infty) \subset U$. Now, for any $p \in f^{-1}(U)$, we have $f(p) \in U$, so there exists an $\varepsilon > 0$ such that $(f(p) - \varepsilon, \infty) \subset U$. By the lower semi-continuity of f at p, there exists a $\delta > 0$ such that for all $q \in X$ with $\mathrm{d}(p, q) < \delta$, we have $f(q) > f(p) - \varepsilon$. This implies that $f(q) \in (f(p) - \varepsilon, \infty) \subset U$. Therefore, $f^{-1}(U)$ contains the open ball $B_\delta(p)$ showing that $f^{-1}(U)$ is open in X. Hence, f is continuous as a map of topological spaces.

Conversely, assume $f : X \to \mathbb{R}$ is continuous as a map of topological spaces when $\mathbb{R}$ is given the lower semi-continuous topology. Let $p \in X$ and $\varepsilon > 0$. Consider the open set $U = (f(p) - \varepsilon, \infty)$ in the lower semi-continuous topology. By continuity, $f^{-1}(U)$ is open in X, and since $p \in f^{-1}(U)$, there exists a $\delta > 0$ such that the open ball $B_\delta(p)$ is contained in $f^{-1}(U)$. This means that for all $q \in X$ with $\mathrm{d}(p, q) < \delta$, we have $f(q) \in U$, which implies $f(q) > f(p) - \varepsilon$. Therefore, f is lower semi-continuous at p. Since p was arbitrary, f is lower semi-continuous at every point in X.

Thus, $f : X \to \mathbb{R}$ is lower semi-continuous if and only if it is continuous as a map of topological spaces when the codomain $\mathbb{R}$ is given the lower semi-continuous topology. $\square$

Exercise 7.5. The countable complement topology on $\mathbb{R}$ is like the finite complement topology, but we say that a subset $U \subset \mathbb{R}$ is open if either the complement $\mathbb{R} \setminus U$ is countable, or $U = \emptyset$.

(i) Prove that in this topology, a sequence $p_1, p_2, p_3, \ldots \in \mathbb{R}$ converges to a limit $\ell \in \mathbb{R}$ if and only if it is eventually constant, meaning that there is an N such that for all $n \geq N$ we have $p_n = \ell$.

(ii) Give an example of a subset $A \subset \mathbb{R}$ that is not closed in this topology, but nonetheless for every sequence $p_1, p_2, \ldots \in A$ converging to a limit $\ell \in \mathbb{R}$, we have $\ell \in A$. (Thus Proposition 2.10 does not hold in a general topological space.)

(iii) Give an example of a metric space (Y, d) and a map $f : \mathbb{R} \to Y$ that is not continuous with respect to the co-countable topology on $\mathbb{R}$ and the metric topology on Y, but nonetheless for every convergent $p_n \to \ell$ in $\mathbb{R}$, we have $f(p_n) \to f(\ell)$ in Y . (Thus Proposition 1.10 does not hold in a general topological space.)

Solution to (i). Assume $p_n \to \ell$. Suppose, towards a contradiction, that (p_n) is not eventually constant equal to ℓ. Then for every $N \in \mathbb{N}$ there exists $n \geq N$ with $p_n \neq \ell$. In particular there are infinitely many indices n for which $p_n \neq \ell$. Let

$$S := \{p_n \mid n \in \mathbb{N} \text{ and } p_n \neq \ell\},$$

be the set of distinct terms of the sequence different from ℓ. Since S is a subset of the countable set $\{p_1, p_2, \dots\}$, the set S is countable.

Consider the set $U = \mathbb{R} \setminus S$. Because S is countable, U is open in the countable-complement topology, and $\ell \in U$ (by construction $\ell \notin S$). By the definition of convergence, there exists N_0 such that for all $n \geq N_0$ we have $p_n \in U$. But $U = \mathbb{R} \setminus S$ contains no point of S, so this implies there are no indices $n \geq N_0$ with $p_n \neq \ell$. This contradicts the assumption that there are infinitely many indices with $p_n \neq \ell$. Therefore our assumption was false, and the sequence must be eventually constant equal to ℓ.

Conversely, assume that there exists N such that $p_n = \ell$ for all $n \geq N$. Let U be any open neighborhood of ℓ in the countable-complement topology. By definition of convergence in a topological space, we must find N' so that $p_n \in U$ for all $n \geq N'$. But since $p_n = \ell \in U$ for every $n \geq N$, taking $N' = N$ works. Thus $p_n \to \ell$.

Thus, a sequence converges to a limit ℓ in the countable complement topology if and only if it is eventually constant at ℓ. □

Solution to (ii). Pick any point $x_0 \in \mathbb{R}$ such that $p_n \neq x_0$ for all n. Let $A = \mathbb{R} \setminus \{x_0\}$. Then, A is not closed in the countable complement topology since its complement $\{x_0\}$ is finite (and hence countable). However, for any sequence $p_1, p_2, \dots \in A$ that converges to a limit $\ell \in \mathbb{R}$, the sequence must be eventually constant at ℓ. Since all terms of the sequence are in A, it follows that ℓ must also be in A. Thus, A satisfies the required condition. □

Solution to (iii). Let $Y = \mathbb{R}$ with the usual metric, and define the function $f : \mathbb{R} \to Y$ by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

The function f is not continuous at $x = 0$ with respect to the co-countable topology on $\mathbb{R}$ and the metric topology on Y. To see this, consider the open set $U = (0.5, 1.5)$ in Y. The preimage $f^{-1}(U) = \{0\}$, which is not open in the co-countable topology since its complement $\mathbb{R} \setminus \{0\}$ is uncountable. However, for any convergent sequence $p_n \to \ell$ in $\mathbb{R}$, the sequence must be eventually constant at ℓ. If $\ell \neq 0$, then $f(p_n) = 0$ for all sufficiently large n, so $f(p_n) \to f(\ell) = 0$. If $\ell = 0$, then $f(p_n) = 1$ for all sufficiently large n, so $f(p_n) \to f(\ell) = 1$. Thus, for every convergent sequence $p_n \to \ell$ in $\mathbb{R}$, we have $f(p_n) \to f(\ell)$ in Y. □

Exercise 8.1. Let X be a topological space. Let $Y \subset X$, and give Y the subspace topology. Let $A \subset Y$.

(i) Prove that if A is closed in Y and Y is closed in X, then A is closed in X.

(ii) Give two examples to show that if A is closed in Y and Y is not closed in X, then A may or may not be closed in X.

(iii) Prove that if A is open in Y and Y is open in X, then A is open in X.

(iv) Give two examples to show that if A is open in Y and Y is not open in X, then A may or may not be open in X.

(v) Let $A \subset Y$, let $\mathrm{cl}_X(A)$ denote the closure of A in X, and let $\mathrm{cl}_Y(A)$ denote the closure of A in Y. Prove that
$$\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y.$$

(vi) Let $A \subset Y$, let $\mathrm{int}_X(A)$ denote the interior of A as a subset of X, and let $\mathrm{int}_Y(A)$ denote the interior of A as a subset of Y. Prove that
$$\mathrm{int}_X(A) \subset \mathrm{int}_Y(A).$$

(vii) Give an example where the inclusion in part (vi) is strict.

Solution to (i). Assume A is closed in Y and Y is closed in X. Then, there exists a closed set C in X such that $A = C \cap Y$. Since Y is closed in X, the intersection of two closed sets, C and Y, is also closed in X. Therefore, A is closed in X. □

Solution to (ii). Let $X = \mathbb{R}$ with the usual topology and let $Y = (0, 1) \subset X$. Define $A = (0, 1) \subset Y$. Then A is closed in Y because its complement in Y is empty, which is open in Y. However, A is not closed in X, since its closure in X is $[0, 1]$, and thus $A \neq \mathrm{cl}_X(A)$.

For another example, let $X = \mathbb{R}$ and let $Y = (0, 1) \subset X$ as before, and define $A = \{1/2\} \subset Y$. Then A is closed in Y because singletons are closed in any subspace. In this case, A is also closed in X, since $\{1/2\}$ is closed in the real topology. This shows that when Y is not closed in X, a set may be closed in Y while either failing to be closed in X or still being closed in X. □

Solution to (iii). Assume A is open in Y and Y is open in X. Then, by definition, we have that $A = V \cap Y$ for some open set V in X. Since Y is open in X, the intersection of two open sets, V and Y, is also open in X. Therefore, A is open in X. □

Solution to (iv). Suppose we have $x \in \mathrm{cl}_Y(A)$. By definition of closure in Y, every open subset $U \subset Y$ containing x intersects A. Then, by Proposition 8.1, there is an open subset $V \subset X$ such that $U = V \cap Y$. Since U intersects A, we have that $V \cap Y \cap A \neq \emptyset$, and thus V intersects A as well. Therefore, every open subset $V \subset X$ containing x intersects A, and so $x \in \mathrm{cl}_X(A)$. Thus, $\mathrm{cl}_Y(A) \subseteq \mathrm{cl}_X(A) \cap Y$.

Conversely, suppose we have $x \in \mathrm{cl}_X(A) \cap Y$. Using the same logic as above, we can show that every open subset $U \subset Y$ containing x intersects A. Therefore, $x \in \mathrm{cl}_Y(A)$. Thus, $\mathrm{cl}_X(A) \cap Y \subseteq \mathrm{cl}_Y(A)$.

Therefore, we have $\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y$. □

Solution to (v). Suppose we have $x \in \mathrm{int}_X(A)$. By definition of interior of X, there exists an open subset $U \subset X$ such that $x \in U$ and $U \subset A$. Construct the open subset $V = U \cap Y$. By Proposition 8.1, V is open in Y, and since $x \in U$ and $U \subset A$, we have $x \in V$ and $V \subset A$, since $A \subset Y$. Hence, $x \in \mathrm{int}_Y(A)$. Therefore, we have $\mathrm{int}_X(A) \subset \mathrm{int}_Y(A)$. □

Solution to (vi). Let $X = \mathbb{R}$ with the usual topology and let $Y = [0, 2) \subset X$. Define $A = [0, 1] \subset Y$. Then the interior of A in X is $(0, 1)$, since the endpoint 0 does not admit an open neighborhood in X contained in A. However, the interior of A in Y is $[0, 1)$, because sets of the form $U \cap Y$ with U open in X give a neighborhood basis in Y, and any such set containing 0 is contained in A. Thus $\mathrm{int}_X(A) = (0, 1)$ is strictly contained in $\mathrm{int}_Y(A) = [0, 1)$. □

Solution to (vii). Consider again the sets $X = \mathbb{R}$, $Y = [0, 2)$, and $A = [0, 1]$. As shown in part (vi), we have $\mathrm{int}_X(A) = (0, 1)$ while $\mathrm{int}_Y(A) = [0, 1)$. Since $(0, 1)$ is a proper subset of $[0, 1)$, this provides an example where the inclusion $\mathrm{int}_X(A) \subset \mathrm{int}_Y(A)$ is strict. □

Exercise 8.2.

(i) Let X be a topological space, and suppose that we can write $X = F_1 \cup \cdots \cup F_n$, where each F_i is closed. Let Y be another topological space, let $f : X \to Y$, and let $f_i : F_i \to Y$ be the restriction of f to F_i: that is, for $x \in F_i$ we set $f_i(x) = f(x)$. Prove that f is continuous if and only if f_i is continuous for all i.

Hint: Use the fact that a map is continuous if and only if the preimage of every closed set is closed.

(ii) This is usually applied to show that a piecewise function is continuous. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1/3, \\ 3x - 1 & \text{if } 1/3 \leq x \leq 2/3, \\ 1 & \text{if } x \geq 2/3. \end{cases}$$

If we wanted to apply part (i) to show that f is continuous, which should sets should we take for the F_i?

(iii) Give an example of how part (i) can fail if we allow countably many closed sets F_i.

Solution to (i). Assume first that $f : X \to Y$ is continuous. Fix an index i and let $U \subset Y$ be open. Then

$$f_i^{-1}(U) = \{x \in F_i \mid f_i(x) \in U\} = \{x \in F_i \mid f(x) \in U\} = f^{-1}(U) \cap F_i.$$

Since $f^{-1}(U)$ is open in X, its intersection with F_i is open in the subspace topology on F_i. Hence f_i is continuous.

Conversely, assume that each restriction $f_i : F_i \to Y$ is continuous. Let $C \subset Y$ be closed. For each i, the set $f_i^{-1}(C)$ is closed in the subspace F_i, and because F_i is closed in X, it follows that $f_i^{-1}(C)$ is closed in X. Since

$$f^{-1}(C) = \bigcup_{i=1}^{n} f_i^{-1}(C),$$

we see that $f^{-1}(C)$ is a finite union of closed sets in X, and is therefore closed in X. Thus the preimage of every closed set in Y is closed in X, which shows that f is continuous. This completes the proof. □

Solution to (ii). To apply part (i), we decompose the domain $[0, 1]$ into closed sets on which f is defined by a single expression. The appropriate choice is

$$F_1 = [0, 1/3], \qquad F_2 = [1/3, 2/3], \qquad F_3 = [2/3, 1].$$

Each of these sets is closed in the subspace topology on $[0, 1]$, and the restriction of f to each F_i is continuous. By part (i), it follows that f is continuous on $[0, 1]$. □

Solution to (iii). To demonstrate the failure of part (i) when countably many closed sets are used, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

This function is not continuous at 0, since the left-hand limit equals 0 while the right-hand limit equals 1. However, we may write $\mathbb{R}$ as a countable union of closed sets on which the restriction of f is continuous. Define $F_1 = (-\infty, 0]$ and, for each integer $n \geq 2$, define $F_n = [1/n, \infty)$. Each F_n is closed in $\mathbb{R}$, and $\bigcup F_n = \mathbb{R}$. On F_1, the function f is constantly equal to 0 and is therefore continuous, while on every F_n with $n \geq 2$, the function f is constantly equal to 1 and is again continuous. Thus each restriction $f|_{F_n}$ is continuous, but the function f itself is discontinuous. This example shows that the conclusion of part (i) does not extend to countable unions of closed sets. □

Exercise 9.3.

(i) Let X and Y be topological spaces, and let $p : X \times Y \to X$ be the map given by $p(x, y) = x$. Prove that p is continuous.

(ii) Write "Similarly, the map $q : X \times Y \to Y$ be the map given by $q(x, y) = y$ is continuous."

Solution to (i). Let $U \subset X$ be open. Then

$$p^{-1}(U) = \{(x, y) \in X \times Y \mid p(x, y) \in U\} = \{(x, y) \in X \times Y \mid x \in U\} = U \times Y.$$

By definition of the product topology, the set $U \times Y$ is open in $X \times Y$ whenever U is open in X and Y is open in itself. Hence $p^{-1}(U)$ is open in $X \times Y$. Since the preimage of every open set is open, the map p is continuous. □

Solution to (ii). Similarly, the map $q : X \times Y \to Y$ given by $q(x, y) = y$ is continuous. □