In 3D, two lines are parallel, intersect at a point, or are skew.

Ex: Determine if the lines are parallel, skew, or intersect

$\vec{r}_1(t) = \langle 1, 2, 0 \rangle + t\langle -2, 7, 1 \rangle$

$\vec{r}_2(s) = \langle 2, 3, 1 \rangle + s\langle -2, 1, -1 \rangle$

Direction Vectors: $\vec{v}_1 = \langle -2, 7, 1 \rangle$

$\vec{v}_2 = \langle -2, 1, -1 \rangle$

Since the direction vectors are not parallel, the lines are not parallel.

Find t and s such that $\vec{r}_1(t) = \vec{r}_2(s)$.

If there is a solution, the lines intersect.

If there is not a solution, the lines do not intersect and are not parallel $\Rightarrow$ skew lines.

$$\left\{\begin{array}{l} 1-2t = 2-2s \\ 2+7t = 3+s \\ t = 1-s \end{array}\right.$$

$$\begin{array}{rl} 1-2t &= 2-2s \\ 4+14t &= 6+2s \\ \hline 5+12t &= 8 \Rightarrow t = \frac{1}{4} \end{array}$$

$s = 7t - 1 = \frac{7}{4} - 1 = \frac{3}{4}$

Check 3rd equation: $1-s = 1-\frac{3}{4} = \frac{1}{4} = t$

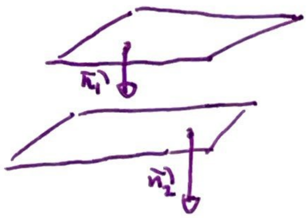
Lines intersect when $t=\frac{1}{4}$, $s=\frac{3}{4}$

$\vec{r}_1\left(\frac{1}{4}\right) = \left\langle \frac{1}{2}, \frac{15}{4}, \frac{1}{4} \right\rangle$

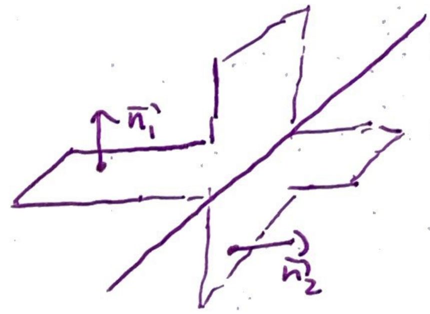
Point of Intersection: $\left(\frac{1}{2}, \frac{15}{4}, \frac{1}{4}\right)$

Two planes are either parallel or intersect in a line.

Parallel:



Intersecting



Parallel normal vectors determine parallel planes.

$$\begin{cases} x + 2y - 3y = 1 \\ 2x + 4y - 6y = 5 \end{cases} \qquad \begin{aligned} \vec{n}_1 &= \langle 1, 2, -3 \rangle \\ \vec{n}_2 &= \langle 2, 4, -6 \rangle \\ &= 2\vec{n}_1 \end{aligned}$$

Planes intersect in a line.
Since line lies in both planes, the direction vector is
$\vec{v} = \vec{n}_1 \times \vec{n}_2$

Ex: Find line of ~~intersection~~ intersection of planes

$$x - 2y + 3z = 0 \qquad \vec{n}_1 = \langle 1, -2, 3 \rangle$$
$$x + y - 2z = 6 \qquad \vec{n}_2 = \langle 1, 1, -2 \rangle$$

Line lies in both planes. Therefore direction vector is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$.

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 1 & -2 \end{vmatrix} = \langle 1, 5, 3 \rangle$$

Point of intersection: Find a point on both planes

Suppose $z = 0$, then planes reduce to system

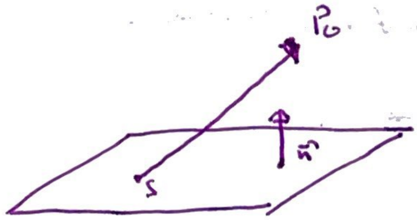
$$\left.\begin{aligned} x - 2y &= 0 \\ x + y &= 6 \end{aligned}\right\} \quad 3y = 6 \Rightarrow y = 2 \Rightarrow x = 4$$

A point on line is $P(4, 2, 0)$

Line of intersection is $\vec{r}(t) = \langle 4, 2, 0 \rangle + t\langle 1, 5, 3 \rangle$

Distance

Find distance from the point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$.



The distance from P_0 to plane is distance from P_0 to nearest point that lies in the plane.

Let S be any point in plane.
By orthogonality, distance $= \left|\text{comp}_{\vec{n}} \overrightarrow{P_0S}\right| = \left| \dfrac{\overrightarrow{P_0S} \cdot \vec{n}}{|\vec{n}|} \right|$

Ex: Find distance from $P_0(1, 4, -2)$ to

$2x - 3y + z = 5$

$S(1, -1, 0)$ is a point on the plane.

$\vec{n} = \langle 2, -3, 1 \rangle$ is the normal.

$\overrightarrow{P_0S} = \langle 0, -5, 2 \rangle$

$$\text{Distance} = \left|\text{comp}_{\vec{n}} \overrightarrow{P_0S}\right|$$

$$= \left| \frac{\overrightarrow{P_0S} \cdot \vec{n}}{|\vec{n}|} \right|$$

$$= \left| \frac{15 + 2}{\sqrt{14}} \right| = \qquad D = \frac{17}{\sqrt{14}}$$