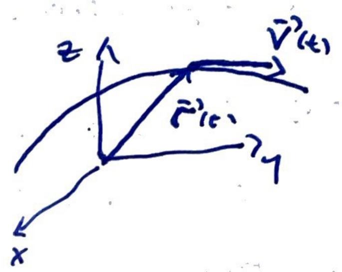
§13.4 : Motion in Space

Suppose an object moves along a space curve defined by $\vec{r}(t)$.

Its position at time t is $\vec{r}(t)$.



The velocity is $\vec{v}(t) = \vec{r}\,'(t)$

The speed is $|\vec{v}(t)|$

The acceleration is $\vec{a}(t) = \vec{v}\,'(t)$

Ex: Find the velocity, speed, and acceleration given the position $\vec{r}(t) = \langle t^2 - t, t^2 + t, t^3 \rangle$

Velocity : $\vec{v}(t) = \vec{r}\,'(t) = \langle 2t-1, 2t+1, 3t^2 \rangle$

Speed : $|\vec{v}(t)| = \sqrt{(2t-1)^2 + (2t+1)^2 + 9t^4}$

$= \sqrt{2 + 8t^2 + 9t^4}$

Acceleration: $\vec{a}(t) = \vec{v}\,'(t)$

$= \langle 2, 2, 6t \rangle$

Projectile Motion

Suppose an object is fired with an initial speed $|\vec{v}_0|$ and angle of elevation α from a height h.

Find the position of the object, neglecting air resistence

Newton's Second Law of Motion: The net force, $\vec{F}(t)$, exerted on an object with mass m is the product of its mass and acceleration, $\vec{a}(t)$.

$\vec{F}(t) = m\vec{a}(t)$.

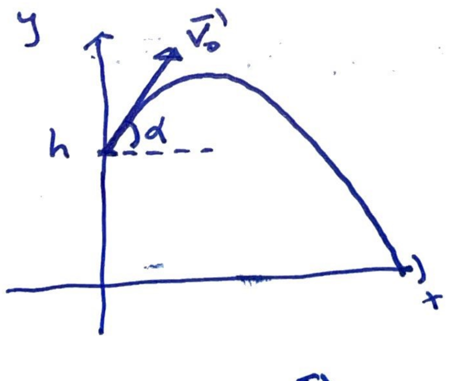
Only considering force due to gravity,

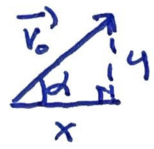
$\vec{F}(t) = -mg\hat{j}$ where $g = 9.8\ m/s^2$

Therefore $\vec{a}(t) = -g\hat{j}$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = -gt\hat{j} + \vec{c}$$

$$\vec{v}(0) = \vec{c} = \vec{v}_0 \quad : \text{Initial Velocity}$$





$\cos\alpha = \dfrac{x}{|\vec{v}_0|}$

$\sin\alpha = \dfrac{y}{|\vec{v}_0|}$

$$\vec{v}(0) = (|\vec{v}_0|\cos\alpha)\hat{i} + (|\vec{v}_0|\sin\alpha)\hat{j}$$

$$\vec{v}(t) = (|\vec{v}_0|\cos\alpha)\hat{i} + (|\vec{v}_0|\sin\alpha - gt)\hat{j}$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= (|\vec{v}_0|\cos\alpha)t\,\hat{i} + \left((|\vec{v}_0|\sin\alpha)t - \frac{g}{2}t^2\right)\hat{j} + \vec{C}$$

$\vec{r}(0) = \vec{C} = \langle 0, h\rangle$ Initial position

$$\vec{r}(t) = (|\vec{v}_0|\cos\alpha)t\,\hat{i} + \left((|\vec{v}_0|\sin\alpha)t - \frac{g}{2}t^2 + h\right)\hat{j}$$

Note $x = |\vec{v}_0|\cos\alpha\, t \Rightarrow t = \dfrac{x}{|\vec{v}_0|\cos\alpha}$

$$y = |\vec{v}_0|\sin\alpha\, t - \frac{g}{2}t^2 + h$$

$$y(x) = (\tan\alpha)x - \frac{g}{2|\vec{v}_0|^2\cos^2\alpha}x^2 + h$$

The motion of the projectile follows a downward parabola in the xy-plane.

Ex: A projectile is fired with initial speed 200 m/s with an angle of elevation of 60° from a height of 100 m.

a) Find the position

$$\vec{r}(t) = 100t\,\hat{i} + (100\sqrt{3}\,t - 4.9t^2 + 100)\,\hat{j}$$

b) What is the distance traveled along the ground?

Find time of impact.

$$y(t) = 100\sqrt{3}\,t - 4.9t^2 + 100 = 0$$

$$t = \frac{-100\sqrt{3} \pm \sqrt{30000 + 4(100)(4.9)}}{-9.8}$$

Time of impact

$$t^* = \frac{-100\sqrt{3} - \sqrt{30000 + 4(100)(4.9)}}{-9.8} \approx 35.92 \text{ seconds}$$

$$X(t^*) = 3592 \text{ meters}$$

c) Find maximum height of projectile.

$$y'(t) = 100\sqrt{3} - 9.8t = 0$$

$$t^{**} = \frac{100\sqrt{3}}{9.8}$$

max height is $y(t^{**}) = 1630.61$ meters

d) Find speed at impact.

$$\vec{v}(t) = 100\hat{i} + (100\sqrt{3} - 9.8t)\hat{j}$$

$$\text{Speed} = \sqrt{100^2 + (100\sqrt{3} - 9.8t)^2}$$

At impact, speed is $|\vec{v}(t^*)| \approx 204.84$ m/s