Functional Complex Variables I: Homework 3

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Exercise 2.20.9. Let f denote the function whose values are

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}.$$

Show that if $z = 0$, then $\Delta w / \Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x \Delta y$, plane. Then, show that $\Delta w / \Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Example 2, Sec 19.)

Solution. If this limit depends on the path, then the derivative does not exist. Now, we examine

$$f(z) = \frac{\overline{z}^2}{z}.$$

We want to compute

$$\frac{f(z)}{z} = \frac{\overline{z}^2}{z^2}.$$

Now, we evaluate along different paths to see if the limit exists. Let $z = x + iy$, so $\overline{z} = x - iy$. Then

$$\frac{f(z)}{z} = \frac{(x - iy)^2}{(x + iy)^2} = \left(\frac{\overline{z}}{z}\right)^2.$$

So we can simplify our analysis to evaluating $\left(\frac{\overline{z}}{z}\right)^2$ as $z \to 0$ along various paths.

The first path will be along the real axis $z = x$ where $y = 0$. This gives us $\overline{z} = x$, $z = x$, so

$$\left(\frac{\overline{z}}{z}\right)^2 = \left(\frac{x}{x}\right)^2 = 1.$$

So along the real axis, $\frac{f(z)}{z} \to 1$.

The second path will be along the imaginary axis $z = iy$ where $x = 0$. This gives us $\overline{z} = -iy$, $z = iy$, so

$$\left(\frac{\overline{z}}{z}\right)^2 = \left(\frac{-iy}{iy}\right)^2 = (-1)^2 = 1.$$

So along the imaginary axis, $\frac{f(z)}{z} \to 1$.

The final path will be along the line $y = x$. This gives us $z = x + ix = x(1 + i)$, so as $x \to 0$, $z \to 0$. Then $\overline{z} = x(1 - i)$, and

$$\frac{\overline{z}}{z} = \frac{1 - i}{1 + i} = \frac{(1 - i)^2}{(1 + i)(1 - i)} = \frac{1 - 2i + i^2}{1 - i^2} = \frac{1 - 2i - 1}{1 + 1} = \frac{-2i}{2} = -i.$$

Then

$$\left(\frac{\overline{z}}{z}\right)^2 = (-i)^2 = -1.$$

So along the line $y = x$, $\frac{f(z)}{z} \to -1$.

Since we have different values for $\frac{f(z)}{z}$ along different paths to the origin, we conclude that the limit

$$\lim_{z \to 0} \frac{f(z)}{z}.$$

does not exist. Therefore, the derivative $f'(0)$ does not exist. ☐

Exercise 2.23.4. Use the theorem in Sec. 23 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(i) $f(z) = 1/z^4$.

(ii) $f(z) = \sqrt{r}e^{i\theta/2}$.

(iii) $f(z) = e^{-\theta}\cos(\ln(r)) + ie^{-\theta}\sin(\ln(r))$.

Solution to (i). We write $f(z) = \frac{1}{z^4}$. Since $z \neq 0$, we can use the power rule for complex functions, which gives

$$f'(z) = \frac{d}{dz}(z^{-4}) = -4z^{-5} = -\frac{4}{z^5}.$$

This function is differentiable everywhere except at $z = 0$, and so it is differentiable in any domain that excludes 0. □

Solution to (ii). We are given $f(z) = \sqrt{r}e^{i\theta/2}$, where $z = re^{i\theta}$, and the domain is $r > 0$, $\alpha < \theta < \alpha + 2\pi$ (so the function is single-valued and continuous).

Let

$$u(r,\theta) = \sqrt{r}\cos\left(\frac{\theta}{2}\right) \quad \text{and} \quad v(r,\theta) = \sqrt{r}\sin\left(\frac{\theta}{2}\right).$$

Computing the partial derivatives, we have

$$\begin{aligned} u_r &= \frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right), \quad u_\theta = -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) \\ v_r &= \frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right), \quad v_\theta = \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right). \end{aligned}$$

Since $ru_r = v_\theta$ and $u_\theta = -rv_r$, the derivative exists. Then

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + i\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cdot\frac{e^{i\theta/2}}{\sqrt{r}} = \frac{1}{2}f(z).$$ □

Solution to (iii). Let $f(z) = e^{-\theta}\cos(\ln(r)) + ie^{-\theta}\sin(\ln(r))$. Define

$$u(r,\theta) = e^{-\theta}\cos(\ln(r)) \quad \text{and} \quad v(r,\theta) = e^{-\theta}\sin(\ln(r)).$$

Computing the partial derivatives, we have

$$\begin{aligned} u_r &= e^{-\theta}\cdot\frac{-\sin(\ln(r))}{r}, \quad u_\theta = -e^{-\theta}\cos(\ln(r)) \\ v_r &= e^{-\theta}\cdot\frac{\cos(\ln(r))}{r}, \quad v_\theta = -e^{-\theta}\sin(\ln(r)). \end{aligned}$$

Since $ru_r = v_\theta$ and $u_\theta = -rv_r$, the derivative exists. Then

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + i\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cdot\frac{e^{i\theta/2}}{\sqrt{r}} = \frac{1}{2}f(z).$$ □

Exercise 2.26.1. Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$ when

(i) $u(x,y) = 2x(1-y)$.

(ii) $u(x,y) = 2x - x^3 + 3xy^2$.

(iii) $u(x,y) = \sinh(x)\sin(y)$.

(iv) $u(x,y) = y/x^2 + y^2$.

Solution to (i). Computing the second partial derivatives, we have

$$u_{xx} = 0 \quad \text{and} \quad u_{yy} = 0.$$

Thus, $\Delta u = u_{xx} + u_{yy} = 0$, so u is harmonic.

To find a harmonic conjugate v, we use the Cauchy–Riemann equations

$$u_x = 2(1-y), \quad u_y = -2x \Rightarrow v_y = u_x = 2(1-y) \quad \text{and} \quad v_x = -u_y = 2x.$$

Integrate v_y with respect to y to get

$$v(x,y) = \int 2(1-y)\,\mathrm{d}y = 2y - y^2 + h(x).$$

Differentiate this with respect to x to get $v_x = h'(x)$. But from earlier, $v_x = 2x$, so $h'(x) = 2x$, which implies that $h(x) = x^2$. Therefore, we have

$$v(x,y) = 2y - y^2 + x^2.$$

□

Solution to (ii). Computing the second partial derivatives, we have

$$u_{xx} = -6x \quad \text{and} \quad u_{yy} = 6x.$$

So $\Delta u = u_{xx} + u_{yy} = -6x + 6x = 0$. Hence u is harmonic.

Again, using the Cauchy-Riemann equations, we have

$$u_x = 2 - 3x^2 + 3y^2, \quad u_y = 6xy \Rightarrow v_y = u_x \quad \text{and} \quad v_x = -u_y = -6xy.$$

Integrate v_y with respect to y to get

$$v(x,y) = \int (2 - 3x^2 + 3y^2)\,dy = 2y - 3x^2y + y^3 + h(x).$$

Then differentiate with respect to x to get $v_x = -6xy + h'(x)$, and compare to $v_x = -6xy$ to get $h'(x) = 0 \Rightarrow h(x) = C$. Therefore, we have

$$v(x,y) = 2y - 3x^2y + y^3.$$

□

Solution to (iii). Computing the second partial derivatives, we have

$$u_{xx} = \sinh(x)\sin(y) \quad \text{and} \quad u_{yy} = -\sinh(x)\sin(y).$$

So, $\Delta u = u_{xx} + u_{yy} = \sinh(x)\sinh(y) - \sinh(x)\sinh(y) = 0$. Hence, u is harmonic.

Using the Cauchy-Riemann equations, we have

$$\begin{aligned} u_x &= \cosh(x)\sin(y), & u_y &= \sinh(x)\cos(y) \\ v_y = u_x &= \cosh(x)\sin(y), & v_x &= -u_y = -\sinh(x)\cos(y). \end{aligned}$$

Integrate v_y with respect to y to get

$$v(x,y) = -\cosh(x)\cos(y) + h(x).$$

Then $v_x = -\sinh(x)\cos(y) + h'(x)$, and since $v_x = -\sinh(x)\cos(y)$, we get $h'(x) = 0 \Rightarrow h(x) = C$. Therefore, we have

$$v(x,y) = -\cosh(x)\cos(y).$$

□

Solution to (iv). Simplifying u, we have $u = y/r^2$. Computing the second partial derivatives, we have

$$\begin{aligned} u_x &= \frac{-2xy}{(x^2+y^2)^2} \\ u_y &= \frac{x^2-y^2}{(x^2+y^2)^2}, \\ u_{xx} &= \frac{-2y(x^2+y^2)^2 + 8x^2y(x^2+y^2)}{(x^2+y^2)^4} = \frac{2y(-(x^2+y^2)^2 + 4x^2(x^2+y^2))}{(x^2+y^2)^4}, \\ u_{yy} &= \frac{2y(x^2+y^2)^2 - 8y^2(x^2+y^2)}{(x^2+y^2)^4} \qquad = \frac{2y((x^2+y^2)^2 - 4y^2(x^2+y^2))}{(x^2+y^2)^4} \end{aligned}$$

We could simplify $u_{xx} + u_{yy}$, but instead note this is the imaginary part of $f(z) = 1/z$. Then

$$f(z) = \frac{x-iy}{x^2+y^2} \Rightarrow \operatorname{Im}(f(z)) = -\frac{y}{x^2+y^2} = -u(x,y),$$

so u is harmonic on $\mathbb{R}^2 \setminus \{(0,0)\}$, and a harmonic conjugate is

$$v(x,y) = \frac{x}{x^2+y^2}.$$

□

Exercise 3.29.10.

(i) Show that if e^z is real, then $\operatorname{Im}(z) = n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$.

(ii) If e^z is pure imaginary, what restriction is placed on z?

Solution to (i). Let $z = x + iy$ where $x, y \in \mathbb{R}$. Then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

So $\operatorname{Im}(e^z) = e^x \sin(y)$. If e^z is real, then $\operatorname{Im}(e^z) = 0$, so we must have

$$\sin(y) = 0 \Rightarrow y = n\pi,$$

where $n \in \mathbb{Z}$. Since $y = \operatorname{Im}(z)$, we conclude that $\operatorname{Im}(z) = n\pi$, for some $n \in \mathbb{Z}$. □

Solution to (ii). Again, let $z = x + iy$. Then as above, $e^z = e^x(\cos(y) + i\sin(y))$, If e^z is pure imaginary, then its real part must vanish $\operatorname{Re}(e^z) = e^x\cos(y) = 0$. Since $e^x \neq 0$ for all $x \in \mathbb{R}$, it must be that

$$\cos(y) = 0 \Rightarrow y = \frac{\pi}{2} + n\pi \quad (n \in \mathbb{Z}).$$

So $\operatorname{Im}(z) = \frac{\pi}{2} + n\pi$ for some integer n. In other words,

$$\operatorname{Im}(z) = \left(n + \frac{1}{2}\right)\pi,$$

for $n \in \mathbb{Z}$. □

Exercise 3.33.1. Show that

(i) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln(2)}{2}\right)$ $(n = 0, \pm 1, \pm 2, \cdots)$.

(ii) $(-1)^{1/\pi} = e^{(2n+1)i}$ $(n = 0, \pm 1, \pm 2, \cdots)$.

Solution to (i). Let us compute $(1+i)^i$ using the identity $z^w = \exp[w\log(z)]$, where $\log(z)$ is the complex logarithm

$$(1+i)^i = \exp\left[i\log(1+i)\right].$$

To compute $\log(1+i)$, we write $1+i$ in polar form as

$$1+i = \sqrt{2}\cdot\exp\left[i\frac{\pi}{4} + 2n\pi i\right],$$

where $n \in \mathbb{Z}$. Hence,

$$\log(1+i) = \ln(|1+i|) + i\arg(1+i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2n\pi\right).$$

Therefore,

$$(1+i)^i = \exp\left(i\left[\ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2n\pi\right)\right]\right) = \exp\left(i\ln(\sqrt{2}) - \left(\frac{\pi}{4} + 2n\pi\right)\right).$$

Note that $\ln(\sqrt{2}) = \ln(2)/2$, so we obtain

$$(1+i)^i = \exp\left(-\frac{\pi}{4} - 2n\pi\right)\exp\left(i\frac{\ln(2)}{2}\right).$$

The expression $\exp(-\pi/4 - 2n\pi)$ can be written equivalently as $\exp\left(-\pi/4 + 2n\pi\right)$ by letting $n \mapsto -n$, we get

$$(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln(2)}{2}\right),$$

for $n \in \mathbb{Z}$ □

Solution to (ii). We compute $(-1)^{1/\pi}$ using the identity $z^w = \exp[w\log(z)]$ to get

$$(-1)^{1/\pi} = \exp\left[\frac{1}{\pi}\log(-1)\right].$$

Since $\log(-1) = i\pi + 2n\pi i = (2n+1)\pi i$ for $n \in \mathbb{Z}$ (principal value $i\pi$), we have

$$(-1)^{1/\pi} = \exp\left[\frac{1}{\pi}\cdot(2n+1)\pi i\right] = \exp\left[(2n+1)i\right].$$

Thus, $(-1)^{1/\pi} = e^{(2n+1)i}$, for $n \in \mathbb{Z}$. □

Exercise 3.33.3. Use definition (1), Sec. 33, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

Solution. We use the principal branch definition of exponentiation for complex numbers

$$z^c = \exp\left[c\log(z)\right] \quad \text{where} \quad \log(z) = \ln(|z|) + i\,\mathrm{Arg}(z).$$

Let $z = -1+\sqrt{3}i$. First, compute its modulus,

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$$

Next, we compute its argument. Note that z lies in the second quadrant, since $\mathrm{Re}(z) = -1$ and $\mathrm{Im}(z) = \sqrt{3} > 0$. Hence,

$$\mathrm{Arg}(z) = \pi - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Now, we compute the logarithm, to get

$$\log(z) = \ln(2) + i\frac{2\pi}{3}.$$

So,

$$z^{3/2} = \exp\left(\frac{3}{2}\log(z)\right) = \exp\left(\frac{3}{2}\ln(2) + i\cdot\frac{3}{2}\cdot\frac{2\pi}{3}\right) = \exp\left(\ln(2^{3/2}) + i\pi\right).$$

Then,

$$z^{3/2} = 2^{3/2}\cdot e^{i\pi} = 2\sqrt{2}\cdot(-1) = -2\sqrt{2}.$$

This corresponds to the principal value. But since the logarithm is multivalued, the full set of values is given by:

$$z^{3/2} = \exp\left(\frac{3}{2}(\ln 2 + i\,\mathrm{Arg}(z) + 2\pi i n)\right) = 2\sqrt{2}\cdot e^{i(\pi+3n\pi)} = \pm 2\sqrt{2}$$

depending on whether n is even or odd. Therefore,

$$(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}. \qquad \square$$

Exercise (Extra). Derive the Cauchy–Riemann equations in polar coordinates.

Solution. Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined in a region where $z = x + iy$ is represented in polar form as

$$z = re^{i\theta} \quad \text{where } x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

Define $u(r,\theta) = u(x(r,\theta), y(r,\theta))$ and similarly for $v(r,\theta)$. The goal is to express the Cauchy–Riemann equations in terms of r and θ.

Recall the standard Cauchy–Riemann equations in Cartesian coordinates:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

By the chain rule, we compute u_x and u_y in terms of u_r and u_θ, we have

$$\begin{aligned} u_x &= \partial_u r\,\partial_r x + \partial_u\theta\,\partial_\theta x \\ u_y &= \partial_u r\,\partial_r y + \partial_u\theta\,\partial_\theta y. \end{aligned}$$

Since $r = \sqrt{x^2+y^2}$, and $\theta = \tan^{-1}(y/x)$, we compute

$$\begin{aligned} r_x &= \frac{x}{\sqrt{x^2+y^2}} = \cos(\theta), \quad & \partial_r y &= \frac{y}{\sqrt{x^2+y^2}} = \sin(\theta) \\ \theta_x &= \frac{-y}{x^2+y^2} = -\frac{\sin(\theta)}{r}, \quad & \partial_\theta y &= \frac{x}{x^2+y^2} = \frac{\cos(\theta)}{r}. \end{aligned}$$

Substituting into the chain rule expressions, we obtain

$$\begin{aligned} u_x &= u_r\cos(\theta) - \frac{1}{r}u_\theta\sin(\theta) \\ u_y &= u_r\sin(\theta) + \frac{1}{r}u_\theta\cos(\theta). \end{aligned}$$

Similarly,

$$\begin{aligned} v_x &= v_r\cos(\theta) - \frac{1}{r}v_\theta\sin(\theta) \\ v_y &= v_r\sin(\theta) + \frac{1}{r}v_\theta\cos(\theta). \end{aligned}$$

Now substitute into the Cauchy–Riemann equations

$$u_x = v_y \Rightarrow u_r\cos(\theta) - \frac{1}{r}u_\theta\sin(\theta) = v_r\sin(\theta) + \frac{1}{r}v_\theta\cos(\theta)$$

$$u_y = -v_x \Rightarrow u_r \sin(\theta) + \frac{1}{r} u_\theta \cos(\theta) = -\left(v_r \cos(\theta) - \frac{1}{r} v_\theta \sin(\theta) \right).$$

Now multiply both equations by r and reorganize terms

$$\begin{aligned} r u_r \cos(\theta) - u_\theta \sin(\theta) &= r v_r \sin(\theta) + v_\theta \cos(\theta), \\ r u_r \sin(\theta) + u_\theta \cos(\theta) &= -r v_r \cos(\theta) + v_\theta \sin(\theta). \end{aligned}$$

Now isolate terms involving $r u_r$ and $r v_r$. Multiply the first equation by $\cos(\theta)$ and the second by $\sin(\theta)$, then add

$$\begin{aligned} & r u_r (\cos^2 \theta + \sin^2 \theta) + u_\theta(-\sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)) \\ & = r v_r (\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta)) + v_\theta(\cos^2 \theta + \sin^2 \theta) \\ \Rightarrow \quad & r u_r = v_\theta. \end{aligned}$$

Similarly, multiply the first equation by $\sin(\theta)$ and the second by $\cos(\theta)$, then subtract

$$\begin{aligned} & r u_r (\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta)) + u_\theta(-\sin^2 \theta - \cos^2 \theta) \\ & = r v_r (\sin^2 \theta + \cos^2 \theta) + v_\theta(\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta)) \\ \Rightarrow \quad & -u_\theta = r v_r. \end{aligned}$$

Thus, the Cauchy–Riemann equations in polar coordinates are

$$r u_r = v_\theta \quad \text{and} \quad r v_r = -u_\theta. \qquad \square$$