Problem 1 *Suppose that S is a surface such that on a chart*

$$U = \left\{ (u,v) \mid \begin{array}{c} -1 < u < 1 \\ -1 < v < 1 \end{array} \right\}$$

the metric components are given by

$$\begin{aligned} E &= e^{2u} \\ F &= 0 \\ G &= 1. \end{aligned}$$

Compute the Gauss curvature as a function of u and v. (Hint: You don't know the surface S, but can you construct a surface with the same Gauss curvature?)

We know that if we have a chart with the same metric component E, F, G, this must describe an isometric surface, and so has the same Gauss curvature. So construct by hand

$$(u,v) \mapsto (e^u, v, 0)$$

which gives

$$\begin{aligned} E &= e^{2u} \\ F &= 0 \\ G &= 1 \end{aligned}$$

but this map is just a diffeomorphism of the plane, where $K = 0$. So $K = 0$.

Problem 2 *For both of the following two parameterizations: determine if the maps given are* ***conformal*** *maps from the flat plane to the surface S*

1. $$(u,v) \mapsto (e^u v, e^v u, 2uv)$$

2. $$(u,v) \mapsto \left(\sqrt{2} e^u \sin(v), e^u \cos(v), e^u \cos(v)\right)$$

For 1) we get

$$\begin{aligned} x_u &= (e^u v, e^v, 2v) \\ x_v &= (e^u, e^v u, 2u) \end{aligned}$$

so

$$F = (e^u v, e^v, 2v) \cdot (e^u, e^v u, 2u) \neq 0 \text{ for any where with } u, v > 0$$

For 2)

$$\begin{aligned}
x_u &= \left(\sqrt{2}e^u \sin(v), e^u \cos(v), e^u \cos(v)\right) \\
x_v &= \left(\sqrt{2}e^u \cos(v), -e^u \sin(v), e^u \sin(v)\right) \\
x_u \cdot x_v &= \left(\sqrt{2}\right)^2 \sin(v)\cos(v) - 2\sin(v)\cos(v) = 0 \\
x_u \cdot x_u &= 2e^{2u} = x_v \cdot x_v \text{ so conformal}
\end{aligned}$$

Problem 3 *Suppose that on a chart*

$$\begin{aligned}
E &= e^{2u} \\
F &= 0 \\
G &= (1+v^2)e^{2u}
\end{aligned}$$

Find

$$\Gamma^1_{11} \text{ and } \Gamma^2_{21}$$

We have

$$x_{u,u} = \Gamma^1_{11}x_u + \Gamma^2_{11}x_v + L_1 N$$

so

$$\begin{aligned}
x_{u,u} \cdot x_u = \frac{1}{2}E_u = \frac{1}{2}2e^{2u} &= x_u \cdot \left(\Gamma^1_{11}x_u + \Gamma^2_{11}x_v + L_1 N\right) = x_u \cdot x_u \Gamma^1_{11} + 0 \text{ (as } F = 0) \\
\implies \Gamma^1_{11} &= \frac{1}{2}2e^{2u}/E = 1
\end{aligned}$$

similarly

$$x_{uv} = \Gamma^1_{12}x_u + \Gamma^2_{21}x_v + L_2 N$$

and

$$\begin{aligned}
x_{uv} \cdot x_v &= \frac{1}{2}G_u \\
&= \Gamma^2_{21}G
\end{aligned}$$

so

$$\Gamma^2_{21} = \frac{1}{2}\frac{G_u}{G} = 1$$

Problem 4 *For two real constants $A > B > 0$ consider the (not closed) curve given by*

$$\alpha(x) = \left(x, 0, A^2 - \frac{x^2}{2}\right)$$
$$x \in (-B, B)$$

Rotate this around the y axis to get a surface S (which is topologically an annulus).

1. *If we fix A does the total image of the Gauss map of S depend on B? Explain.*
2. *If we fix B does the total image of the Gauss map of S depend on A? Explain*

Rotate around the x-axis (as I intended) we can a parameterization. For

$$-b \leq u \leq b$$
$$-\pi \leq v < \pi$$
$$(u, v) \mapsto (u, 0, 0) + \left(A^2 - \frac{u^2}{2}\right)(0, \sin v, \cos v)$$

which gives tangent vectors

$$x_u = (1, 0, 0) - u(0, \sin v, \cos v)$$
$$x_v = \left(A^2 - \frac{u^2}{2}\right)(0, \cos v, -\sin v)$$

Notice that the normal, which is going to be determined by the direction (ignoring length)

$$(1, -u \sin v, -u \cos v) \wedge (0, \cos v, -\sin v)$$

has no dependence whatsoever on the parameter A. On the other hand, this definitely does depend on u, so allowing B to change is going to change the range of N.

Problem 5 *Consider the ruled surface given by*

$$(t, v) \mapsto \alpha(t) + v\omega(t)$$

for

$$t \in (-\pi, \pi)$$
$$v \in (-1, 1)$$

with

$$\alpha(t) = (\cos(t), \sin(t), 0)$$
$$\omega(t) = \left(\frac{3}{5}\cos(14t), \frac{3}{5}\sin(14t), \frac{4}{5}\right)$$

1. *Identify a point on the ruled surface (OK to use coordinates (t, v) of the point) where $K = 0$, or argue no such points exists.*
2. *Identify a point on the ruled surface (OK to use coordinates (t, v) of the point) where $K < 0$, or argue no such points exists .*
3. *Identify a point on the ruled surface (OK to use coordinates (t, v) of the point) where $K > 0$, or argue no such points exists .*

   (Hint; for the positive results, you can stick to $v = 0$)

For this problem, the equation

$$K = \frac{eg - f^2}{EG - F^2}$$

is your friend. We are interested in knowing the sign of K and we know that $EG - F^2 > 0$, so we can focus on the sign of

$$eg - f^2$$

To compute this, recall that

$$\begin{aligned} e &= \langle N, x_{tt} \rangle \\ g &= \langle N, x_{vv} \rangle \\ f &= \langle N, x_{tv} \rangle \end{aligned}$$

These are not horrible to compute

$$\begin{aligned} x_t &= \alpha'(t) + v\omega'(t) \\ x_{tt} &= \alpha''(t) + v\omega''(t) \\ x_{tv} &= \omega'(t) \\ x_v &= \omega(t) \\ x_{vv} &= 0 \end{aligned}$$

Immediately we see that g is always zero. Thus

$$eg - f^2 = -f^2$$

is either 0 or strictly negative depending on whether f is not zero. This answers negatively the existence of a point with $K > 0$.

Up to a scale

$$\begin{aligned} f &= \frac{x_t \wedge x_v}{|x_t \wedge x_v|} \cdot x_{tv} \\ &= \frac{1}{|x_t \wedge x_v|} \det(x_t, x_v, x_{tv}) \end{aligned}$$

restrict this to $v = 0$ for easy computations,

$$f = \frac{1}{|x_t \wedge x_v|} \det(\alpha'(t), \omega(t), \omega'(t))$$

so f is 0 iff the three vector are linearly independent. To find where $f = 0$, find a point where

$$\alpha' \parallel \omega'(t)$$

which happens at $t = 0, v = 0$, so here $K = 0$

To find where $K \neq 0$, pick any t such that α' and $\omega'(t)$ are independent; this will be most t.