Decomposition of Acceleration:

Consider a moving object with position $\vec{r}(t)$ at time $t$.

Its velocity is $\vec{v}(t) = \vec{r}\,'(t)$

Its speed is $\alpha(t) = |\vec{v}(t)|$

The unit tangent vector to the curve is

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}(t)}{\alpha(t)} \qquad \frac{\text{Velocity}}{\text{Speed}}$$

Therefore $\vec{V}(t) = \alpha(t)\,\hat{T}(t)$

The acceleration is

$$\begin{aligned}
\vec{a}(t) &= \alpha'(t)\,\hat{T}(t) + \alpha(t)\,\hat{T}'(t) \\
&= \alpha'(t)\,\hat{T}(t) + \alpha(t)\,|\hat{T}'(t)|\,\frac{\hat{T}'(t)}{|\hat{T}'(t)|} \\
&= \alpha'(t)\,\hat{T}(t) + \alpha(t)\,|\hat{T}'(t)|\,\hat{N}(t) \\
&= \alpha'(t)\,\hat{T}(t) + (\alpha(t))^2 K(t)\,\hat{N}(t)
\end{aligned}$$

since curvature $K(t) = \dfrac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|}$

$$\Rightarrow |\hat{T}'(t)| = |\vec{r}\,'(t)|\,K(t)$$

$$= \alpha(t)\,K(t)$$

magnitude of $\hat{T}'(t)$ is product of speed and curvature.

Scalar Projection:

$$\operatorname{comp}_{\hat{T}} \vec{a}(t) = \frac{\vec{a}(t) \cdot \hat{T}(t)}{|\hat{T}(t)|} = \vec{a}(t) \cdot \hat{T}(t)$$

$$= \left(\alpha'(t)\,\hat{T}(t) + (\alpha(t))^2 K(t)\,\hat{N}(t)\right) \cdot \hat{T}(t)$$

$$= \alpha'(t)\,\hat{T}(t) \cdot \hat{T}(t) + (\alpha(t))^2 K(t)\,\hat{N}(t) \cdot \hat{T}(t)$$

$= \alpha'(t)$ since $\hat{T} \cdot \hat{T} = 1$ and $\hat{N} \cdot \hat{T} = 0$.

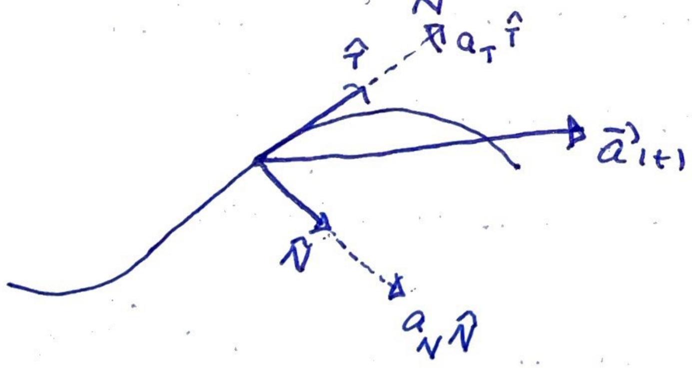
$= a_T(t)$ is the tangential component of acceleration.

Similarly,

$$\operatorname{comp}_{\hat{N}} \vec{a}(t) = \frac{\vec{a}(t) \cdot \hat{N}(t)}{|\hat{N}(t)|} = \vec{a}(t) \cdot \hat{N}(t)$$

$$= (\alpha(t))^2 K(t)$$

$= a_N(t)$ normal component of acceleration




The acceleration vector lies in the osculating plane containing $\hat{T}$ and $\hat{N}$.
It is decomposed in component acting in direction of curve and component acting in direction of curvature.

Note: $\vec{v}(t)\cdot\vec{a}(t) = \alpha(t)\,\hat{T}(t)\cdot\left(a_T\,\hat{T}(t) + a_N\,\hat{N}(t)\right)$

$$= \alpha(t)a_T\,\hat{T}\cdot\hat{T} + \alpha(t)a_N\,\hat{T}\cdot\hat{N}$$

$$= \alpha(t)a_T$$

$$a_T = \frac{\vec{v}(t)\cdot\vec{a}(t)}{\alpha(t)} = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{|\vec{r}\,'(t)|}$$

$$a_N = (\alpha(t))^2\kappa(t) = |\vec{r}\,'(t)|^2\,\frac{|\vec{r}\,'(t)\times r''(t)|}{|\vec{r}\,'(t)|^3}$$

$$= \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$$

Ex: Find tangential and normal components of acceleration given $\vec{r} = \langle \ln t, t^2+3t, 4\sqrt{t}\rangle$

$\vec{r}\,' = \langle \frac{1}{t}, 2t+3, \frac{2}{\sqrt{t}}\rangle$

$\vec{r}\,'' = \langle \frac{-1}{t^2}, 2, -\frac{1}{t^{3/2}}\rangle$

$\vec{r}\,'\times\vec{r}\,'' = \langle -2t^{-1/2} - 3t^{-3/2} - 4t^{-1/2}, t^{-5/2} - 2t^{-5/2}, 2t^{-1}+2t^{-1}+3t^{-2}\rangle$

$= \langle -6t^{-1/2} - 3t^{-3/2}, -t^{-5/2}, 4t^{-1}+3t^{-2}\rangle$

$= t^{-5/2}\langle -6t^2-3t, -1, 4t^{-3/2}+3t^{-1/2}\rangle$

$$a_T = \frac{\vec{r}\,'\cdot\vec{r}\,''}{|\vec{r}\,'|} = \frac{-t^{-3}+4t+6-2t^{-2}}{(t^{-2}+4t^2+12t+9+4t^{-1})^{1/2}}$$

$$a_N = \frac{|\vec{r}\,'\times r''|}{|r'|} = \frac{t^{-5/2}\left(9t^2(2t+1)^2+1+t^{-1}(4t^{-1}+3)^2\right)^{1/2}}{(t^{-2}+4t^2+12t+9+4t^{-1})^{1/2}}$$