Math 307, Homework #7
Due Wednesday, November 20

1. Given $Q \Rightarrow R$, prove $[P \Rightarrow T] \Rightarrow [(Q \vee \sim T) \Rightarrow (\sim P \vee R)]$.
2. (a) If $C \subseteq A$ and $D \subseteq B$ then prove $D - A \subseteq B - C$.
   (b) Prove $A = X \cap A$ if and only if $A \subseteq X$.
   (c) Prove $A = X \cup A$ if and only if $X \subseteq A$.
3. Let $f \colon S \to T$ be a function. Prove that if $X \subseteq T$ and $Y \subseteq T$ then $f^{-1}(X) - f^{-1}(Y) = f^{-1}(X - Y)$.
4. Let $f \colon S \to T$ be a function, let $A \subseteq S$ and $B \subseteq S$.
   (a) Prove $f(A) - f(B) \subseteq f(A - B)$.
   (b) If f is one-to-one, prove $f(A - B) \subseteq f(A) - f(B)$.
   (c) Create an example of an S, T, f, A, and B such that $f(A) - f(B) \neq f(A - B)$.
5. Suppose $f \colon A \to B$, $X \subseteq A$, $W \subseteq B$, $f(X) \cap W = \emptyset$, and $f(X) \cup W = B$.
   (a) Prove that $X \cap f^{-1}(W) = \emptyset$.
   (b) If f is one-to-one, prove that $A = X \cup f^{-1}(W)$.
   (c) If $f(A - X) = W$ prove that f is onto.

In questions 6–13 below, prove the indicated statement by induction.

6. $1 + 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2$, for all $n \geq 0$.
7. $1^3 + 2^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2$ for all $n \geq 1$.
8. For all $n \geq 1$, $\sum\limits_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$.
9. $\frac{(2n)!}{n! \cdot 2^n}$ is an odd number, for every $n \in \mathbb{N}$.
10. For all $n > 4$, $2^n > n^2$.
11. Consider the sequence given recursively by $a_0 = 0$ and $a_n = \sqrt{2 + a_{n-1}}$ for all $n \geq 1$. So $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, $a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, and so forth. Then $a_n \leq 2$ for all $n \geq 0$.
12. $(1 + \frac{1}{2})^n > 1 + \frac{n}{2}$ for all $n \geq 2$.
13. For all $n \in \mathbb{N}$, if $n \geq 1$ then $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

14. Fill in each box below with a mathematical proposition that makes the biconditional true, and is not a tautology (for example, I don't want you to write "$A \subseteq B$" in the first box, even though this makes the biconditional true). Copy the complete biconditional statements into your homework; do not actually write in the boxes on this worksheet.

$A \subseteq B \Leftrightarrow$ $\boxed{}$

$A = B \Leftrightarrow$ $\boxed{}$

$x \in f(A) \Leftrightarrow$ $\boxed{}$

$y \in f^{-1}(B) \Leftrightarrow$ $\boxed{}$

$x \in A \cup B \Leftrightarrow$ $\boxed{}$

$x \in A \cap B \Leftrightarrow$ $\boxed{}$

$x \in A - B \Leftrightarrow$ $\boxed{}$

$f\colon S \to T$ is onto $\Leftrightarrow$ $\boxed{}$

$f\colon S \to T$ is one-to-one $\Leftrightarrow$ $\boxed{}$

$x \in A \cap (B - C) \Leftrightarrow$ $\boxed{}$

$X = \emptyset \Leftrightarrow$ $\boxed{}$

15. Write definitions for the following sets, using set-builder notation. The first one is done for you.

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

$A \cup B =$ $\boxed{}$

$X - A =$ $\boxed{}$

$f(A) =$ $\boxed{}$

$f^{-1}(B) =$ $\boxed{}$

$C \cap f^{-1}(B) =$ $\boxed{}$