

Clairaut's Theorem: Suppose f is defined on a disk D in \mathbb{R}^2 that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then $f_{xy}(a, b) = f_{yx}(a, b)$

Ex: Consider $f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

For $(x, y) \neq (0, 0)$, then

$$\begin{aligned} f_x &= \frac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \end{aligned}$$

At $(0, 0)$, use difference quotient

$$\begin{aligned} f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{0}{h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0 \end{aligned}$$

$$f_x = \begin{cases} \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Using similar steps,

$$f_y = \begin{cases} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ by using the difference quotient.

$$\begin{aligned} f_{xy}(0, 0) &= \lim_{h \rightarrow 0} \frac{f_x(0, h) - f_x(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h^5}{h^4} - 0}{h} = \lim_{h \rightarrow 0} -1 = -1 \end{aligned}$$

$$\begin{aligned} f_{yx}(0, 0) &= \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{h^5}{h^4} - 0}{h} = \lim_{h \rightarrow 0} 1 = 1 \end{aligned}$$

Therefore f is an example of a function where $f_{xy} \neq f_{yx}$

At origin $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

f_{xy} and f_{yx} are discontinuous at $(0, 0)$.

$$\text{Ex: Consider } yz^2 + x \ln y = z^3$$

For all (x, y, z) that satisfy the equation, the point lies on a surface and the equation implicitly defines a surface. If x and y are treated independent, then as either x or y changes, z must change to remain on the surface.

Find rate of change of z with respect to x treating $z = f(x, y)$, a function of (x, y) .

$$\text{Find } \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x} (yz^2 + x \ln y) = \frac{\partial}{\partial x} (z^3)$$

$$2yz \frac{\partial z}{\partial x} + \ln y = 3z^2 \frac{\partial z}{\partial x}$$

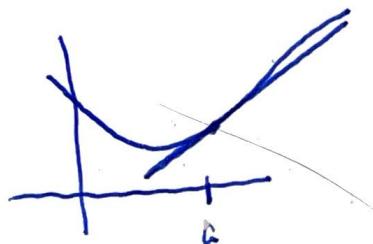
$$\ln y = 3z^2 \frac{\partial z}{\partial x} - 2yz \frac{\partial z}{\partial x}$$

$$= (3z^2 - 2yz) \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\ln y}{3z^2 - 2yz}$$

§14.4 : Tangent Plane

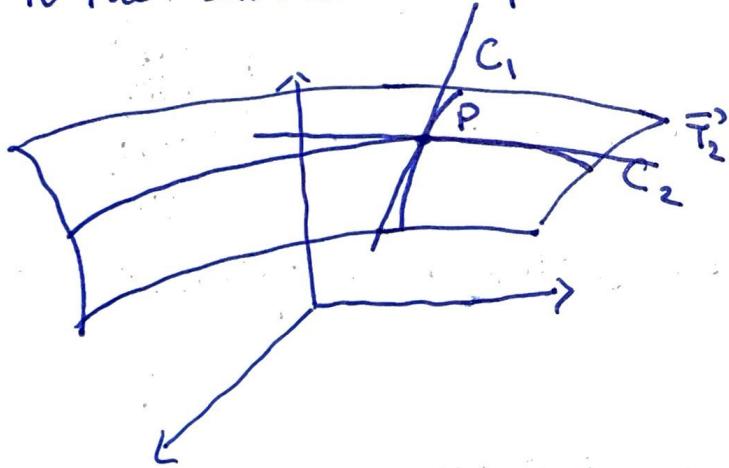
Recall, if $y = f(x)$ is differentiable at $x=a$, then the tangent line to $y = f(x)$ at $(a, f(a))$ is $y = f(a) + f'(a)(x-a)$. It is the best fit line to the curve.



$$f(x) \approx f(a) + f'(a)(x-a) \quad \begin{matrix} \text{Linear} \\ \text{approximation} \end{matrix}$$

for x near a .

Now consider $z = f(x, y)$. Find tangent plane to the surface at $\vec{P}(a, b, f(a, b))$.



The tangent plane to $z = f(x, y)$ at P contains all tangent vectors to curves that pass through P at the point P .

Let C_1 be the trace in $y=b$.

Curves on surface through P .

Let C_2 be the trace in $x=a$.

$$\vec{T}_1 = \langle a, b, f(a, b) \rangle + t \langle 1, 0, f_x(a, b) \rangle$$

Tangent lines to C_1 and C_2 at P .

$$\vec{T}_2 = \langle a, b, f(a, b) \rangle + t \langle 0, 1, f_y(a, b) \rangle$$

The direction vectors of the tangent lines lie in the tangent plane.

$$\vec{v}_1 = \langle 1, 0, f_x(a, b) \rangle$$

$$\vec{v}_2 = \langle 0, 1, f_y(a, b) \rangle$$

Normal vector of tangent plane is

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -f_x(a, b), -f_y(a, b), 1 \rangle$$

$P(a, b, f(a, b))$ point in plane

$Q(x, y, z)$ arbitrary point in plane

Then $\vec{n} \cdot \vec{PQ} = 0$ by orthogonality

$$-f_x(a, b)(x-a) - f_y(a, b)(y-b) + z - f(a, b) = 0$$

$$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

is the tangent plane to $z = f(x, y)$ at P

The tangent plane is the best fit plane to the surface at the point. It is also called the linearization of $f(x, y)$ at (a, b) .

$$L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

$L(x, y) \approx f(x, y)$ for (x, y) near (a, b) .

Ex: Find the tangent plane to $z = \frac{x-y}{3y+2x}$ at $(2, -1)$

$$z_x = \frac{(3y+2x)(1) - (x-y)(2)}{(3y+2x)^2} = \frac{5y}{(3y+2x)^2}$$

$$z_y = \frac{(3y+2x)(-1) - (x-y)(3)}{(3y+2x)^2} = \frac{-5x}{(3y+2x)^2}$$

$$z_x(2, -1) = -5 \quad z(2, -1) = 3$$

$$z_y(2, -1) = -10$$

$$\text{Tangent Plane is } z = 3 - 5(x-2) - 10(y+1)$$