Functional Complex Variables I: Homework 6

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Exercise 4.49.1. Apply the Cauchy-Goursat theorem to show that

$$\int_c f(z)\, \mathrm{d}z = 0,$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(i) $f(z) = \dfrac{z^2}{z-3}$; (ii) $f(z) = ze^{-z}$; (iii) $f(z) = \dfrac{1}{z^2+2z+2}$;

(iv) $f(z) = \operatorname{sech}(z)$; (v) $f(z) = \tan(z)$; (vi) $f(z) = \operatorname{Log}(z+2)$.

Solution to (i). The function is analytic everywhere except at $z = 3$, which lies *outside* the unit circle $|z| = 1$. Therefore, $f(z)$ is analytic on and inside the contour C. Thus, by Cauchy-Goursat,

$$\int_C f(z)\, \mathrm{d}z = 0. \qquad \square$$

Solution to (ii). Both the exponential function and the identity function are entire (analytic on all of $\mathbb{C}$), so $f(z)$ is entire as well. Since f is analytic everywhere, it is in particular analytic on and inside C. Thus, by Cauchy-Goursat,

$$\int_C f(z)\, \mathrm{d}z = 0. \qquad \square$$

Solution to (iii). Factor the denominator

$$z^2 + 2z + 2 = (z + 1 + i)(z + 1 - i).$$

The singularities are at $z = -1 \pm i$, both of which satisfy $|z| = \sqrt{1^2 + 1^2} = \sqrt{2} > 1$, so they lie outside the unit circle. Hence, $f(z)$ is analytic on and inside C. Thus, by Cauchy-Goursat,

$$\int_C f(z)\, \mathrm{d}z = 0. \qquad \square$$

Solution to (iv). Take $f(z) = \operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}$. This is the reciprocal of an entire function $\cosh(z)$, whose zeros occur at $z = (2n+1)\pi i/2$. The closest singularities of $\operatorname{sech}(z)$ are at $z = \pm\pi i/2$, and since

$$\left|\frac{\pi i}{2}\right| = \frac{\pi}{2} > 1,$$

these lie outside the unit circle. Hence, $f(z)$ is analytic on and inside C. Thus, by Cauchy-Goursat,

$$\int_C f(z)\, \mathrm{d}z = 0. \qquad \square$$

Solution to (v). Let $f(z) = \tan(z) = \sin(z)/\cos(z)$. The function $\tan(z)$ has singularities where $\cos(z) = 0$, i.e., at $z = \pi/2 + n\pi$, $n \in \mathbb{Z}$. The smallest modulus of such a point is $\pi/2 > 1$, so all singularities are outside the unit circle. Hence, $f(z)$ is analytic on and inside C. Thus, by Cauchy-Goursat,

$$\int_C f(z)\, \mathrm{d}z = 0. \qquad \square$$

Solution to (vi). Let $f(z) = \operatorname{Log}(z+2)$, where Log denotes the principal branch of the complex logarithm, which is analytic on $\mathbb{C} \setminus (-\infty, 0]$. The branch point of $\operatorname{Log}(z+2)$ is at $z = -2$, and the branch cut lies along $(-\infty, -2]$. The unit circle $|z| = 1$ lies entirely to the right of -2, so the function is analytic on and inside C. Therefore:

$$\int_C f(z)\, \mathrm{d}z = 0. \qquad \square$$

Exercise 4.49.2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 1). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} fz\,\mathrm{d}z,$$

when

(i) $f(z) = \dfrac{1}{3z^2 + 1}$; (ii) $f(z) = \dfrac{z+2}{\sin(z/2)}$; (iii) $f(z) = \dfrac{z}{1 - e^z}$.

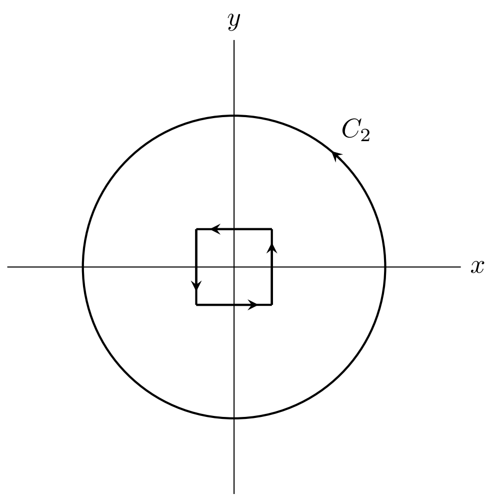

Figure 1

Solution to (i). Factor the denominator to get

$$3z^2 + 1 = 3\left(z - \frac{i}{\sqrt{3}}\right)\left(z + \frac{i}{\sqrt{3}}\right),$$

so the function has singularities at $z = \pm i/\sqrt{3} \approx \pm 0.577i$, both of which lie *inside* the square C_1 and the circle C_2. The function is analytic *everywhere* in the region between C_1 and C_2, since the singularities are enclosed by both contours.

Therefore, by the corollary to the Cauchy-Goursat Theorem (which states that if f is analytic in the region between two positively oriented simple closed curves, then the integrals over both curves are equal),

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z. \qquad \square$$

Solution to (ii). The singularities of this function occur where $\sin(z/2) = 0$, i.e., at

$$\frac{z}{2} = n\pi \Rightarrow z = 2n\pi, \qquad n \in \mathbb{Z}.$$

The singularities are therefore located at $z = 0, \pm 2\pi, \pm 4\pi, \ldots$. Since $2\pi \approx 6.28$, the only singularity inside the circle $|z| = 4$ is at $z = 0$. This singularity also lies within the square C_1.

Since $f(z)$ is analytic everywhere in the annular region between C_1 and C_2, the conditions of the corollary are satisfied. Therefore,

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z. \qquad \square$$

Solution to (iii). This function has singularities where $e^z = 1$, i.e., $z = 2\pi i n$, $n \in \mathbb{Z}$. These are isolated singularities along the imaginary axis at $z = 0, \pm 2\pi i, \pm 4\pi i, \dots$. Since $2\pi \approx 6.28$, the singularities at $z = \pm 2\pi i$ lie outside the circle $|z| = 4$, and the only singularity inside C_2 is at $z = 0$, which also lies within C_1.

The function is analytic in the entire region between the two contours C_1 and C_2, so by the corollary to the Cauchy-Goursat Theorem

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z. \qquad \square$$

Exercise 4.49.3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z - z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \dots \\ 2\pi i & \text{when } n = 0 \end{cases},$$

according to Exercise 10(b), Sec. 42. Use that result and the corollary in Sec. 49 to show that if C is the boundary of the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$, described in the positive sense, then

$$\int_{C} (z - 2 - i)^{n-1}\,\mathrm{d}z = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \cdots \\ 2\pi i & \text{when } n = 0 \end{cases}.$$

Solution. This is a function of the form $f(z) = (z - z_0)^{n-1}$ where $z_0 = 2 + i$. The rectangle defined by $0 \leq x \leq 3$, $0 \leq y \leq 2$, and oriented positively (counterclockwise), forms a simple closed contour C that contains the point $z_0 = 2 + i$ in its interior.

Since the function $f(z)$ is analytic everywhere inside and on both C and any such circle C_0, except possibly at z_0, and since both contours positively enclose z_0, the corollary to the Cauchy-Goursat Theorem guarantees that

$$\int_{C} f(z)\,\mathrm{d}z = \int_{C_0} f(z)\,\mathrm{d}z.$$

Therefore, applying the result from Exercise 10(b), Sec. 42, we obtain

$$\int_{C} (z - 2 - i)^{n-1}\,\mathrm{d}z = \begin{cases} 0 & \text{if } n = \pm 1, \pm 2, \dots \\ 2\pi i & \text{if } n = 0 \end{cases}. \qquad \square$$

Exercise 4.49.4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos(2bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

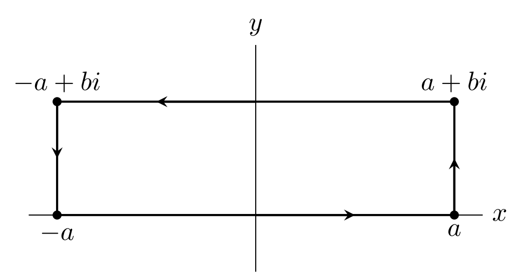

Figure 2

(i) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 2 can be written

$$2\int_0^a e^{-x^2}\,\mathrm{d}x - 2e^{b^2} \int_0^a e^{-x^2} \cos(2bx)\,\mathrm{d}x,$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}\,\mathrm{d}y - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}\,\mathrm{d}y.$$

Thus, with the aid of the Cauchy-Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos(2bx)\,\mathrm{d}x = e^{-b^2}\int_0^a e^{-x^2}\,\mathrm{d}x + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)\,\mathrm{d}y.$$

(ii) By accepting the fact that

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2},$$

and observing that

$$\left|\int_0^b e^{y^2}\sin(2ay)\,\mathrm{d}y\right| \leq \int_0^b e^{y^2}\,\mathrm{d}y,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (i).

Solution to (i). Let $f(z) = e^{-z^2}$ and consider the rectangular contour shown in Fig. 2 with vertices at $-a$, a, $a+bi$, and $-a+bi$, traversed counterclockwise.

By the Cauchy-Goursat Theorem, since $f(z) = e^{-z^2}$ is entire (analytic everywhere), the integral around the closed contour is zero, we have

$$\oint_R e^{-z^2}\,\mathrm{d}z = 0,$$

where R is the rectangular contour.

We now compute the integral along each leg of the rectangle. For the, lower horizontal leg (from $-a$ to a along the real axis), we have

$$\int_{-a}^a e^{-x^2}\,\mathrm{d}x = 2\int_0^a e^{-x^2}\,\mathrm{d}x \quad \text{(by symmetry)}.$$

For the upper horizontal leg, (from $a+bi$ to $-a+bi$), let $z = x+bi$, $\mathrm{d}z = \mathrm{d}x$, and note that x goes from a to $-a$. Therefore, we have

$$\begin{aligned}\int_{a+bi}^{-a+bi} e^{-z^2}\,\mathrm{d}z &= -\int_{-a}^a e^{-(x+bi)^2}\,\mathrm{d}x = -\int_{-a}^a e^{-x^2-2bix-b^2}\,\mathrm{d}x \\ &= -e^{-b^2}\int_{-a}^a e^{-x^2}e^{-2ibx}\,\mathrm{d}x.\end{aligned}$$

Then, using Euler's formula $e^{2ibx} = \cos(2bx) - i\sin(2bx)$ and noting that the integrand is even in the real part and odd in the imaginary part. So the imaginary part integrates to zero, the leaving

$$\int_{a+bi}^{-a+bi} e^{-z^2}\,\mathrm{d}z = -2e^{-b^2}\int_0^a e^{-x^2}\cos(2bx)\,\mathrm{d}x.$$

For the right vertical leg (from a to $a+bi$), let $z = a+iy$, $dz = i\,\mathrm{d}y$, $y\in[0,b]$, giving us

$$\int_a^{a+bi} e^{-z^2}\,\mathrm{d}z = \int_0^b e^{-(a+iy)^2}i\,\mathrm{d}y = i\int_0^b e^{-a^2-2aiy-y^2}\,\mathrm{d}y = ie^{-a^2}\int_0^b e^{-y^2}e^{-i2ay}\,\mathrm{d}y.$$

For the left vertical leg (from $-a+bi$ to $-a$), let $z = -a+iy$, $dz = -i\,\mathrm{d}y$, $y\in[0,b]$, giving us

$$\int_{-a+bi}^{-a} e^{-z^2}\,\mathrm{d}z = -i\int_0^b e^{-(-a+iy)^2}\,\mathrm{d}y = -i\int_0^b e^{-a^2-2aiy-y^2}\,\mathrm{d}y = -ie^{-a^2}\int_0^b e^{-y^2}e^{i2ay}\,\mathrm{d}y.$$

Now summing all legs, and using the fact that the total integral is zero

$$0 = 2\int_0^a e^{-x^2}\,\mathrm{d}x - 2e^{-b^2}\int_0^a e^{-x^2}\cos(2bx)\,\mathrm{d}x + ie^{-a^2}\int_0^b e^{-y^2}e^{-i2ay}\,\mathrm{d}y - ie^{-a^2}\int_0^b e^{-y^2}e^{i2ay}\,\mathrm{d}y.$$

Group the last two terms, we have

$$ie^{-a^2}\left(\int_0^b e^{-y^2}e^{-i2ay}\,\mathrm{d}y - \int_0^b e^{-y^2}e^{i2ay}\,\mathrm{d}y\right) = -2e^{-a^2}\int_0^b e^{-y^2}\sin(2ay)\,\mathrm{d}y.$$

So we can rewrite the equation as

$$0 = 2\int_0^a e^{-x^2}\,\mathrm{d}x - 2e^{-b^2}\int_0^a e^{-x^2}\cos(2bx)\,\mathrm{d}x - 2e^{-a^2}\int_0^b e^{-y^2}\sin(2ay)\,\mathrm{d}y.$$

Divide through by 2 and rearrange, we obtain

$$\int_0^a e^{-x^2}\cos(2bx)\,\mathrm{d}x = e^{-b^2}\int_0^a e^{-x^2}\,\mathrm{d}x + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)\,\mathrm{d}y. \qquad \square$$

Solution to (ii). I'm going to evaluate the Gaussian integral, as it's finally being covered and I've been waiting for this moment. Consider the full Gaussian integral over $(-\infty, \infty)$,

$$I := \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

This integral cannot be evaluated by elementary antiderivatives, so instead we compute I^2 by considering a double integral

$$\begin{aligned} I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \cdot \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \cdot \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Changing to polar coordinates, we have $x = r\cos(\theta), y = r\sin(\theta)$, so that $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ and $x^2 + y^2 = r^2$. Then

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Evaluating the polar integral, we have

$$\begin{aligned} \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}\mathrm{d}\theta \\ &= \int_0^{2\pi}\frac{1}{2}\,\mathrm{d}\theta \\ &= \pi. \end{aligned}$$

Thus, we have

$$I = \sqrt{\pi}.$$

Now, we can deduce the half-line integral. Since e^{-x^2} is an even function, we have

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = 2\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Use this in the expression from part (i), we have

$$\int_0^a e^{-x^2}\cos(2bx)\,\mathrm{d}x = e^{-b^2}\int_0^a e^{-x^2}\,\mathrm{d}x + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)\,\mathrm{d}y.$$

Let $a \to \infty$, we have

$$\int_0^a e^{-x^2}\,\mathrm{d}x \to \frac{\sqrt{\pi}}{2} \quad \text{and} \quad e^{-(a^2+b^2)} \to 0 \quad \text{exponentially fast.}$$

Notice that

$$\int_0^b e^{y^2}\sin(2ay)\,\mathrm{d}y,$$

is bounded, so its contribution vanishes in the limit. Hence

$$\int_0^\infty e^{-x^2}\cos(2bx)\,\mathrm{d}x = e^{-b^2}\cdot\frac{\sqrt{\pi}}{2}.$$

Finally, we have

$$\int_0^\infty e^{-x^2}\cos(2bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad (b > 0).$$

□

Exercise 4.52.2. Find the value of the integral of $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

(i) $g(z) = \dfrac{1}{z^2+4}$; (ii) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Solution to (i). Factoring the denominator, we have $z^2+4 = (z+2i)(z-2i)$. The singularities are at $z = 2i$ and $z = -2i$. The contour $|z-i| = 2$ is centered at $z = i$ and has radius 2. Therefore, $|2i - i| = 1 < 2$ and $|-2i - i| = 3 > 2$, so the singularity at $z = 2i$ lies *inside* the contour, while the singularity at $z = -2i$ lies *outside*.

Since only $z = 2i$ is inside, we write

$$g(z) = \frac{1}{(z-2i)(z+2i)} = \frac{1}{z+2i}\cdot\frac{1}{z-2i}.$$

The function $f(z) = 1/(z+2i)$ is analytic on and inside the contour (since $z = -2i$ is outside). So we can use Cauchy's Integral Formula for the simple pole at $z = 2i$

$$\int_{|z-i|=2}\frac{f(z)}{z-2i}\,\mathrm{d}z = 2\pi i\cdot f(2i) = 2\pi i\cdot\frac{1}{2i+2i} = 2\pi i\cdot\frac{1}{4i} = \frac{\pi}{2}.$$

□

Solution to (ii). Again, factoring the denominator, we have $(z^2+4)^2 = [(z-2i)(z+2i)]^2$. So the integrand has a pole of order 2 at $z = 2i$, which is *inside* the circle, and another at $z = -2i$, which is *outside* the circle. We can rewrite the integrand as

$$g(z) = \frac{1}{[(z-2i)^2(z+2i)^2]} = \frac{1}{(z+2i)^2}\cdot\frac{1}{(z-2i)^2}.$$

Let

$$f(z) = \frac{1}{(z+2i)^2},$$

which is analytic inside and on the circle (since $z = -2i$ is outside). We apply the Cauchy Integral Formula for derivatives, to get

$$\int_{|z-i|=2}\frac{f(z)}{(z-2i)^2}\,\mathrm{d}z = 2\pi i\cdot f'(2i).$$

Computing the derivative of f, we have $f'(z) = -2(z+2i)^{-3}$. So,

$$f'(2i) = -2(4i)^{-3} = -2 \cdot \frac{1}{64i^3} = -2 \cdot \frac{1}{64(-i)} = \frac{2}{64i} = \frac{1}{32i}.$$

Therefore, we have

$$\int_{|z-i|=2} \frac{1}{(z^2+4)^2}\,\mathrm{d}z = 2\pi i \cdot \frac{1}{32i} = \frac{\pi}{16}. \qquad \square$$

Exercise 4.52.3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,\mathrm{d}s \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

Solution. Assume that $|2| < 3$. Since $z = 2$ is strictly *inside* the contour C (because $|2| < 3$), we apply Cauchy's Integral Formula, to get

$$\int_C \frac{f(s)}{s-z}\,ds = 2\pi i \cdot f(z).$$

Therefore, we have

$$g(2) = 2\pi i \cdot f(2) = 2\pi i \cdot (2(2)^2 - 2 - 2) = 2\pi i \cdot (8 - 2 - 2) = 2\pi i \cdot 4 = 8\pi i.$$

Assume that $|z| > 3$, i.e., z is *outside* the contour C. In this case, the function $f(s)/(s-z)$ is analytic in s on and inside the contour C, because z is outside the region enclosed by C and f is entire.

Since the integrand is analytic inside and on C, and C is a closed curve, the Cauchy-Goursat Theorem implies that

$$g(z) = \int_C \frac{f(s)}{s-z}\,\mathrm{d}s = 0. \qquad \square$$