Differential Geometry I: Homework 3

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Exercise 1. Let M be $\mathbb{R}^2$, and let $x_0 \in \mathbb{R}^2$ and $f \in C^\infty(M)$. Given a metric g we can define a gradient flow for f, call this flow $\gamma_{g,x_0}(t)$, which satisfies

$$\begin{aligned}\gamma_{g,x_0}(0) &= x_0\\ \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{g,x_0}(t) &= -\nabla_g f.\end{aligned}$$

Suppose we have two different metrics g_1 and g_2. Let

$$\begin{aligned}a_1 &= \lim_{t\to\infty}\gamma_{g_1,x_0}(t)\\ a_2 &= \lim_{t\to\infty}\gamma_{g_2,x_0}(t).\end{aligned}$$

Give an example of a function f and two metrics such that a_1 and a_2 exist, but $a_1 \neq a_2$.

Solution. We can consider the function

$$f(x,y) = (x^2+y^2-1)^2 + (x-1)^2.$$

This function has two local minima: one at $(1,0)$ and another at approximately $(-0.5,0)$. Now, we can define two different metrics on $\mathbb{R}^2$:

$$g_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad g_2 = \begin{pmatrix}10 & 0\\ 0 & 1\end{pmatrix}.$$

The first metric g_1 is the standard Euclidean metric, while the second metric g_2 stretches the x-direction by a factor of 10. Now, let's consider the gradient flows starting from the point $x_0 = (0,0)$. Under the metric g_1, the gradient flow will move towards the local minimum at $(1,0)$, so we have

$$a_1 = (1,0).$$

On the other hand, under the metric g_2, the gradient flow will be influenced more heavily in the x-direction, causing it to move towards the local minimum at approximately $(-0.5,0)$. Thus, we have

$$a_2 \approx (-0.5,0).$$

Therefore, we have constructed a function f and two metrics g_1 and g_2 such that the limits of the gradient flows starting from the same point x_0 are different $a_1 \neq a_2$. □

Exercise 2. Show that every smooth manifold admits a Riemmanian metric.

Solution. Assume that M is a smooth manifold. By definition, M is locally diffeomorphic to $\mathbb{R}^n$. Therefore, for each point $p \in M$, there exists an open neighborhood U_p of p and a diffeomorphism $\varphi_p : U_p \to V_p \subset \mathbb{R}^n$. Since $\mathbb{R}^n$ has a standard Euclidean metric g_{std}, we can pull back this metric to U_p via the diffeomorphism φ_p. Specifically, we define a metric g_p on U_p by

$$g_p(X,Y) = g_{\text{std}}(D\varphi_p(X), D\varphi_p(Y)),$$

for any tangent vectors $X, Y \in T_qM$ with $q \in U_p$. Now, the collection $\{U_p\}_{p\in M}$ forms an open cover of M. Since M is a smooth manifold, it is paracompact, which means that there exists a partition of unity subordinate to this open cover. Let $\{\psi_p\}_{p\in M}$ be such a partition of unity, where each $\psi_p : M \to [0,1]$ is a smooth function with support contained in U_p and $\sum_{p\in M}\psi_p(q) = 1$ for all $q \in M$. We can now define a global Riemannian metric

$$g(X,Y) = \sum_{p\in M}\psi_p(q)g_p(X,Y),$$

for any tangent vectors $X, Y \in T_qM$. This metric g is smooth because it is a finite sum of smooth functions (due to the local finiteness of the partition of unity) and is positive-definite since each g_p is positive-definite and the weights $\psi_p(q)$ are non-negative and sum to 1. Therefore, we have constructed a Riemannian metric on the smooth manifold M. Thus, every smooth manifold admits a Riemannian metric. ☐

Exercise 3. Consider the function on $L^2(\mathbb{R})$

$$F(u) = \int_{\mathbb{R}} u^2 \, \mathrm{d}x.$$

Given a function $u_0 \in L^2$, explicitly write down the gradient flow for F starting at u_0.

Solution. Let $H = L^2(\mathbb{R})$ and $F(u) = \int_{\mathbb{R}} u^2 \, \mathrm{d}x$. Fix $u, v \in L^2(\mathbb{R})$ and compute

$$DF(u)[v] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F(u + \varepsilon v) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\mathbb{R}} (u + \varepsilon v)^2 \, \mathrm{d}x = \int_{\mathbb{R}} 2uv \, \mathrm{d}x = 2 \langle u, v \rangle_{L^2}.$$

Therefore, $DF(u)$ is a linear functional that sends $v \mapsto 2 \langle u, v \rangle_{L^2}$.

By the Riesz representation theorem, there exists a unique element $\nabla_{L^2} F(u) \in L^2(\mathbb{R})$ such that

$$DF(u)[v] = \langle \nabla_{L^2} F(u), v \rangle_{L^2} \text{ for all } v \in L^2(\mathbb{R}),$$

which is just simply $\nabla_{L^2} F(u) = 2u$.

Therefore, the gradient flow starting at u_0 is given by the ODE

$$\gamma_{u_0}(0) = u_0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{u_0}(t) = -\nabla_{L^2} F(\gamma_{u_0}(t)) = -2\gamma_{u_0}(t).$$

This is a simple ODE with solution

$$\gamma_{u_0}(t) = u_0 e^{-2t}.$$

Now, we can verify that this is indeed the gradient flow:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = \langle \nabla F(u(t)), \dot{u}(t) \rangle_{L^2} = -\|\nabla F(u(t))\|_{L^2}^2 = -4\|u(t)\|_{L^2}^2 = -4F(u(t)) \leq 0,$$

which shows that F is decreasing along the flow. ☐

Exercise 4. Let

$$\mathcal{M} = \{F : \mathbb{S}^1 \to \mathbb{R}^2, F \text{ is a smooth immersion}\}.$$

(i) Given F in $\mathcal{M}$, show that there is a metric on the tangent space defined as follows

$$g_F(V, W) = \int \langle V(\theta), W(\theta) \rangle \left| \frac{\mathrm{d}F}{\mathrm{d}\theta} \right| \mathrm{d}\theta,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$ (In this case, V, W are smooth functions $\mathbb{S}^1 : \mathbb{R}^2$ so that there is a path defined by

$$\gamma(t) = F(\theta) + tV(\theta),$$

which gives a velocity vector

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(0) = V(\theta).$$

(ii) Show that given any path $\gamma(t)$ in $\mathcal{M}$, that is, a path of maps

$$F(\theta, t) \to \mathbb{R}^2,$$

there is (at least for small t) a family of diffeomorphisms $\Phi_t : \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \to \mathbb{S}^1$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}F(\Phi(\theta, t), t) \perp \frac{\mathrm{d}F}{\mathrm{d}\theta} \text{ for all } \theta \text{ when } t = 0,$$

and $\Phi(\theta, 0) = \theta$.

(iii) Consider the arclength function on $\mathcal{M}$ given by

$$L(F) = \int_{\mathbb{S}^1} \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta.$$

Assume that F is a constant speed map,

$$\left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| = C \text{ for some constant } C,$$

and V is a normal tangent vector,

$$V \perp \frac{\mathrm{d}F}{\mathrm{d}\theta} \text{ for all } \theta.$$

Show that

$$VL(F) = -\int \left\langle \frac{\mathrm{d}^2F}{\mathrm{d}s^2}, V \right\rangle \mathrm{d}\theta,$$

where s is the arclength parameter: Hint: you may want to use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta}, \frac{\mathrm{d}F}{\mathrm{d}t} \right\rangle = 0 = \left\langle \frac{\mathrm{d}^2F}{\mathrm{d}\theta^2}, \frac{\mathrm{d}F}{\mathrm{d}t} \right\rangle + \left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta}, \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\mathrm{d}F}{\mathrm{d}t} \right\rangle.$$

Solution to (i). We need to show that g_F is a metric on the tangent space $T_F\mathcal{M}$. First, we show that g_F is bilinear. Let $V_1, V_2, W \in T_F\mathcal{M}$ and $a, b \in \mathbb{R}$. Then,

$$\begin{aligned} g_F(aV_1 + bV_2, W) &= \int \langle aV_1(\theta) + bV_2(\theta), W(\theta)\rangle \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta \\ &= a\int \langle V_1(\theta), W(\theta)\rangle \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta + b\int \langle V_2(\theta), W(\theta)\rangle \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta \\ &= ag_F(V_1, W) + bg_F(V_2, W). \end{aligned}$$

Similarly, we can show linearity in the second argument.

Next, we show symmetry. Let $V, W \in T_F\mathcal{M}$. Then,

$$\begin{aligned} g_F(V, W) &= \int \langle V(\theta), W(\theta)\rangle \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta \\ &= \int \langle W(\theta), V(\theta)\rangle \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta = g_F(W, V). \end{aligned}$$

Finally, we show positive-definiteness. Let $V \in T_F\mathcal{M}$. Then,

$$g_F(V, V) = \int \langle V(\theta), V(\theta)\rangle \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta = \int |V(\theta)|^2 \left|\frac{\mathrm{d}F}{\mathrm{d}\theta}\right| \mathrm{d}\theta \geq 0.$$

Moreover, if $g_F(V, V) = 0$, then $|V(\theta)|^2 = 0$ for all θ, which implies that $V(\theta) = 0$ for all θ. Thus, V is the zero vector in $T_F\mathcal{M}$.

Therefore, g_F is a metric on the tangent space $T_F\mathcal{M}$. □

Solution to (ii). Suppose we have a path $\gamma(t)$ in $\mathcal{M}$ given by $F(\theta, t)$. We want to find a family of diffeomorphisms $\Phi_t : \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \to \mathbb{S}^1$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}F(\Phi(\theta, t), t) \perp \frac{\mathrm{d}F}{\mathrm{d}\theta} \text{ for all } \theta \text{ when } t = 0.$$

To achieve this, we can define $\Phi(\theta, t)$ as the solution to the following ODE:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(\theta, t) = -\frac{\left\langle \frac{\mathrm{d}F}{\mathrm{d}t}(\Phi(\theta, t), t), \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right\rangle}{\left| \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right|^2} \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t),$$

with the initial condition $\Phi(\theta, 0) = \theta$. This ODE ensures that the derivative of F with respect to t at $\Phi(\theta, t)$ is orthogonal to the derivative of F with respect to θ at the same point when $t = 0$. To see this, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} F(\Phi(\theta, t), t) &= \frac{\mathrm{d}F}{\mathrm{d}t}(\Phi(\theta, t), t) + \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \frac{\mathrm{d}\Phi}{\mathrm{d}t}(\theta, t) \\
&= \frac{\mathrm{d}F}{\mathrm{d}t}(\Phi(\theta, t), t) - \frac{\left\langle \frac{\mathrm{d}F}{\mathrm{d}t}(\Phi(\theta, t), t), \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right\rangle}{\left| \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right|^2} \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t).
\end{aligned}$$

Taking the inner product with $\frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t)$, we get

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}t} F(\Phi(\theta, t), t), \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right\rangle &= \left\langle \frac{\mathrm{d}F}{\mathrm{d}t}(\Phi(\theta, t), t), \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right\rangle \\
&\quad - \frac{\left\langle \frac{\mathrm{d}F}{\mathrm{d}t}(\Phi(\theta, t), t), \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right\rangle}{\left| \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right|^2} \left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t), \frac{\mathrm{d}F}{\mathrm{d}\theta}(\Phi(\theta, t), t) \right\rangle \\
&= 0.
\end{aligned}$$

Thus, we have constructed the desired family of diffeomorphisms Φ_t. $\square$

Solution to (iii). We have

$$L(F) = \int_{\mathbb{S}^1} \left| \frac{\mathrm{d}F}{\mathrm{d}\theta} \right| \mathrm{d}\theta.$$

Therefore,

$$VL(F) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} L(F + \varepsilon V) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\mathbb{S}^1} \left| \frac{\mathrm{d}}{\mathrm{d}\theta}(F + \varepsilon V) \right| \mathrm{d}\theta.$$

Computing the derivative inside the integral, we get

$$\begin{aligned}
VL(F) &= \int_{\mathbb{S}^1} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \left(\left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta} + \varepsilon \frac{\mathrm{d}V}{\mathrm{d}\theta}, \frac{\mathrm{d}F}{\mathrm{d}\theta} + \varepsilon \frac{\mathrm{d}V}{\mathrm{d}\theta} \right\rangle^{1/2} \right) \mathrm{d}\theta \\
&= \int_{\mathbb{S}^1} \frac{\left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta}, \frac{\mathrm{d}V}{\mathrm{d}\theta} \right\rangle}{\left| \frac{\mathrm{d}F}{\mathrm{d}\theta} \right|} \mathrm{d}\theta.
\end{aligned}$$

Since F is a constant speed map, we have $\left| \frac{\mathrm{d}F}{\mathrm{d}\theta} \right| = C$. Thus,

$$VL(F) = \frac{1}{C} \int_{\mathbb{S}^1} \left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta}, \frac{\mathrm{d}V}{\mathrm{d}\theta} \right\rangle \mathrm{d}\theta.$$

Integrating by parts and using periodicity on $\mathbb{S}^1$ (so the boundary term vanishes) gives

$$\begin{aligned}
VL(F) &= \frac{1}{C} \left[\left\langle \frac{\mathrm{d}F}{\mathrm{d}\theta}, V \right\rangle \bigg|_{\mathbb{S}^1} - \int_{\mathbb{S}^1} \left\langle \frac{\mathrm{d}^2 F}{\mathrm{d}\theta^2}, V \right\rangle \mathrm{d}\theta \right] \\
&= -\frac{1}{C} \int \left\langle \frac{\mathrm{d}^2 F}{\mathrm{d}\theta^2}, V \right\rangle \mathrm{d}\theta.
\end{aligned}$$

The boundary term vanishes since $\mathbb{S}^1$ is a closed curve, and since $\mathrm{d}s = C\,\mathrm{d}\theta$, those cancel out. Therefore, we have the desired result

$$VL(F) = -\int \left\langle \frac{\mathrm{d}^2 F}{\mathrm{d}s^2}, V \right\rangle \mathrm{d}s. \qquad \square$$