Math 307, Homework #6
Due Wednesday, November 13

1. In each case, use mathematical notation to write the negation of the given statement, in such a way that no quantifier is immediately preceded by a negation sign. For parts (a)–(d) decide which is true: the given statement or its negation.
   (a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y = 0]$
   (b) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = 0]$
   (c) $(\exists x, y \in \mathbb{R})[x^2 + y^2 = -1]$
   (d) $(\forall x \in \mathbb{R})[x > 0 \Rightarrow (\forall y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \Rightarrow y = z]]$
   (e) $(\forall \epsilon \in \mathbb{R})[\epsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})[0 < \delta \wedge (\forall x \in \mathbb{R})[1 - \delta < x < 1 + \delta \Rightarrow |f(x) - 5| < \epsilon]]]$
   (f) $(\forall a, b \in \mathbb{R})(a < b) \Rightarrow (\exists c \in \mathbb{R})[a < c < b \wedge f'(c) = \frac{f(b)-f(a)}{b-a}]$.

   Part (f) is a statement which is true for differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$, and it is a well-known theorem taught in every calculus class. What is the common name of this theorem?

2. In each part below I give the definition for a mathematical concept we have encountered, but using the shorthand notation in quantifiers. Fill in each box with the appropriate mathematical term or phrase that best completes the definition. In parts (b)–(d), $f\colon S \to T$ and $A \subseteq S$.
   (a) $\boxed{} \Leftrightarrow (\forall x \in A)[x \in B]$
   (b) $\boxed{} \Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$
   (c) $\boxed{} \Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$.
   (d) $\boxed{} = \{z \mid (\exists v \in A)[z = f(v)]\}$

3. Suppose $f\colon S \to T$ is one-to-one, $A \subseteq S$, and $B \subseteq S$. Give a line proof showing that $f(A) \cap f(B) \subseteq f(A \cap B)$.

4. Give a line proof showing that if $A \cap B = \emptyset$ and $B \cup C = A \cup D$ then $B \subseteq D$.

5. Let $f\colon S \to T$, and suppose that f is onto. Let $A \subseteq S$. Give a line proof that $T - f(A) \subseteq f(S - A)$.

6. Give a line proof showing $A \cap (X - B) = (A \cap X) - (A \cap B)$.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2e^{-x} + 5$.
   (a) Prove that f is one-to-one.
   (b) Is f onto? Justify your answer.
   (c) Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 5x^3 + 41$. Prove that g is onto.

8. Suppose $f\colon A \to B$ and $g\colon B \to C$.
   (a) If f and g are one-to-one, prove that $g \circ f$ is one-to-one.
   (b) If f and g are both onto, prove that $g \circ f$ is onto.

9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has the property that $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) < f(y)]$.
   (a) Prove that f is one-to-one.

(b) Give an example of a function f satisfying the given property but which is not onto.

10. (a) If $f\colon \mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = x^2 - 1$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ is given by $g(x) = 3x + 2$, determine $(g \circ f)(0)$ and $(g \circ f)(2)$. Determine an algebraic formula for $(g \circ f)(x)$ for any integer x.

    (b) Suppose that $f\colon S \to T$ and $g\colon T \to U$. If $A \subseteq S$, give a line proof that $(g \circ f)(A) = g(f(A))$.

If $f\colon S \to T$ and $A \subseteq T$, define $f^{-1}(A) = \{x \in S \,|\, f(x) \in A\}$. This is called the **inverse image of A under f**. WARNING: Do not think that f^{-1} is a function here. The symbols "f^{-1}" have no meaning by themselves in this context.

Example: If $f\colon \mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = x^2$, then $f^{-1}(\{0,1,2,3,4,5,6\}) = \{0,1,2,-1,-2\}$. Also $f^{-1}(\{8,9,10\}) = \{3,-3\}$ and $f^{-1}(\{5,6,7\}) = \emptyset$. Note that $f^{-1}(16)$ doesn't make any sense (but $f^{-1}(\{16\})$ does and $f^{-1}(\{16\}) = \{-4,4\}$). We can only talk about $f^{-1}(S)$ where S is a set.

In proofs, the "move" you will always use is that the statement "$x \in f^{-1}(A)$" is equivalent to "$f(x) \in A$".

11. Consider the function $f\colon \mathbb{Z}_7 \to \mathbb{Z}_7$ given by $f(x) = x^3 + 1$. Answer the following questions:

    (a) Is f one-to-one? Explain why or why not.

    (b) Is f onto? Explain why or why not.

    (c) Determine $f(S)$ where $S = \{0,2,4,6\}$.

    (d) What is $f^{-1}(\{0\})$?

    (e) If $A = \{1,2,3,4\}$ and $B = \{0,4,5,6\}$, determine $f^{-1}(A)$ and $f^{-1}(B)$. Also determine $f^{-1}(A \cap B)$.

12. Suppose $f\colon S \to T$, $A \subseteq S$, and $B \subseteq T$. Give line proofs for each of the following:

    (a) $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

    (b) $f(A) \cap B = \emptyset \Rightarrow A \subseteq S - f^{-1}(B)$.

13. Suppose $f\colon S \to T$, $A \subseteq T$, $B \subseteq T$, and $C \subseteq S$. Give a line proof of each of the following:

    (a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

    (b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

    (c) $f(f^{-1}(A)) \subseteq A$

    (d) $C \subseteq f^{-1}(f(C))$.

    (e) If f is onto then $f(f^{-1}(A)) = A$.

    (f) If f is one-to-one then $C = f^{-1}(f(C))$.

14. Construct an example of a function $f\colon \{0,1,2\} \to \{0,1\}$ and a subset $A \subseteq \{0,1\}$ where $f(f^{-1}(A)) \neq A$. Also, construct an example of a function $g\colon \{0,1,2\} \to \{0,1\}$ and a subset $C \subseteq \{0,1,2\}$ where $C \neq g^{-1}(g(C))$.

15. Suppose $f, g\colon \mathbb{Z} \to \mathbb{Z}$ are two functions such that $f(M_3) \subseteq M_6$, $g(M_2) \subseteq M_7$, and $g^{-1}(M_5) = M_3$. Prove that for all $x \in \mathbb{Z}$, if $3|x$ then $35|g(f(x))$.

16. Given $[Q \wedge S] \Rightarrow R$ and $\sim S \Rightarrow T$, prove $[P \Rightarrow Q] \Rightarrow [\sim T \Rightarrow [\sim P \vee R]]$.