Differential Geometry: Homework 1

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Exercise 1.2.2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Solution. Let $\alpha : I \to \mathbb{R}$ be a parametrized curve, for some interval I. Let $f(t) = \|\alpha(t)\| = \alpha(t) \cdot \alpha(t)$. The derivative is given by

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\alpha(t) \cdot \alpha(t)] = 2\alpha(t) \cdot \alpha'(t).$$

Since $t_0 \in I$ is a global minimum, we have

$$f'(t_0) = 0 \Rightarrow \alpha(t_0) \cdot \alpha'(t_0) = 0.$$

Since $\alpha(t_0) \neq 0 \neq \alpha'(t_0)$, we have that $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. □

Exercise 1.2.4. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $\mathbf{v} \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$ and that $\alpha(0)$ is also orthogonal to $\mathbf{v}$. Prove that $\alpha(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$.

Solution. Let $f(t) = \alpha(t) \cdot \mathbf{v}$. Then

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\alpha(t) \cdot \mathbf{v}] = \alpha'(t) \cdot \mathbf{v}.$$

Since $\alpha'(t)$ is orthogonal to $\mathbf{v}$, we have that $f'(t) = 0$ for all $t \in I$. Thus, $f(t)$ is constant. Since $f(0) = \alpha(0) \cdot \mathbf{v} = 0$, we have that $f(t) = 0$ for all $t \in I$. Thus, $\alpha(t) \cdot \mathbf{v} = 0$ for all $t \in I$. □

Exercise 1.3.1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = \left\langle 3t, 3t^2, 2t^3 \right\rangle$ make a constant angle with the line $y = 0$, $z = x$.

Solution. The tangent line to the curve $\alpha(t)$ is given by

$$\alpha'(t) = \left\langle 3, 6t, 6t^2 \right\rangle.$$

The line $y = 0$, $z = x$ is given by $\beta(s) = \langle s, 0, s \rangle$. The direction vector of $\beta(s)$ is given by $\langle 1, 0, 1 \rangle$. The angle between the two lines is given by

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\alpha'(t) \cdot \langle 1, 0, 1 \rangle}{\|\alpha'(t)\|\|\langle 1, 0, 1 \rangle\|} = \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \cdot \sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Thus, the angle between the two lines is constant. □

Exercise 1.3.4. Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left\langle \sin(t), \cos(t) + \log\left(\tan\left(\frac{t}{2}\right)\right) \right\rangle,$$

where t is the angle that the y-axis makes with the vector $\alpha'(t)$. The trace of α is called the tractrix. Show that

(i) α is a differentiable parametrized curve, regular except at $t = \pi/2$.

(ii) The length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1.

Solution to (i). At $t = \pi/2$, we get
$$\alpha(\pi/2) = \langle 1, 1 \rangle.$$
Differentiating $\alpha(t)$, we obtain
$$\alpha'(t) = \left\langle \cos(t), -\sin(t) + \frac{1}{2}\csc\left(\frac{t}{2}\right)\sec^2\left(\frac{t}{2}\right) \right\rangle.$$
Evaluating at $t = \pi/2$, we find
$$\alpha'(\pi/2) = \langle 0, 0 \rangle.$$
Thus, α is not regular at $t = \pi/2$.

Let $I = (0, \pi) \setminus \{\pi/2\}$. For $t \in I$, we note that
$$|\cos(t)| > 0,$$
since $t \neq \pi/2$. Therefore, $\alpha'(t) \neq 0$ for all $t \in I$, meaning that α is regular on I. □

Solution to (ii). Let $P = \alpha(t)$ be a point on the tractrix, and let $\mathbf{v} = \alpha'(t)$ denote the tangent vector at P. Since t is the angle that the y-axis makes with $\mathbf{v}$, we have
$$\mathbf{v} = \|\mathbf{v}\| \left\langle \sin(t), \cos(t) \right\rangle.$$
This means the direction of the tangent vector is $\langle \sin(t), \cos(t) \rangle$. Let ℓ denote the tangent line at P. The line ℓ passes through $P = \alpha(t)$ and is in the direction of $\mathbf{v}$, so its parametric form is
$$\ell(s) = \alpha(t) + s\left\langle \sin(t), \cos(t) \right\rangle.$$
To find where ℓ intersects the y-axis, we set the x-component of $\ell(s)$ to 0
$$\sin(t) + s\sin(t) = 0 \Rightarrow s = -1.$$
Plugging back into $\ell(s)$, we find the y-coordinate of the point of intersection
$$\begin{aligned} y &= \cos(t) + \log\left(\tan\left(\frac{t}{2}\right)\right) - \cos(t) \\ &= \log\left(\tan\left(\frac{t}{2}\right)\right). \end{aligned}$$
So the point of intersection is
$$Q = \left\langle 0, \log\left(\tan\left(\frac{t}{2}\right)\right) \right\rangle.$$
Therefore, the segment of the tangent line between $P = \alpha(t)$ and the y-axis has length
$$\begin{aligned} \|\alpha(t) - Q\| &= \left\| \left\langle \sin(t), \cos(t) + \log\left(\tan\left(\frac{t}{2}\right)\right) - \log\left(\tan\left(\frac{t}{2}\right)\right) \right\rangle \right\| \\ &= \|\langle \sin(t), \cos(t) \rangle\| = \sqrt{\sin^2(t) + \cos^2(t)} = 1. \end{aligned}$$
Hence, the length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1. □

Exercise 1.3.6. Let $\alpha(t) = \left\langle ae^{bt}\cos(t), ae^{bt}\sin(t) \right\rangle$, $t \in \mathbb{R}$, where $a > 0$ and $b < 0$, be a parametrized curve.

(i) Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*).

(ii) Show that $\alpha'(t) \to (0,0)$ as $t \to +\infty$ and that

$$\lim_{t \to +\infty} \int_{t_0}^{t} \|\alpha'(x)\| \, \mathrm{d}x.$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

Solution to (i). Since $b < 0$, we have $e^{bt} \to 0$ as $t \to +\infty$. Therefore,

$$\alpha(t) = \left\langle ae^{bt}\cos(t), ae^{bt}\sin(t)\right\rangle \to (0,0) \quad \text{as} \quad t \to +\infty.$$

The presence of the $\cos(t)$ and $\sin(t)$ terms implies that $\alpha(t)$ winds around the origin as it decays in magnitude. Thus, α spirals into the origin as $t \to +\infty$. □

Solution to (ii). We compute

$$\begin{aligned}\alpha'(t) &= \left\langle \frac{d}{dt}(ae^{bt}\cos t), \frac{d}{dt}(ae^{bt}\sin t)\right\rangle \\ &= \left\langle a(be^{bt}\cos t - e^{bt}\sin t), a(be^{bt}\sin t + e^{bt}\cos t)\right\rangle.\end{aligned}$$

Since $e^{bt} \to 0$ as $t \to +\infty$, it follows that

$$\alpha'(t) \to (0,0) \quad \text{as} \quad t \to +\infty.$$

Now we compute the arc length

$$\begin{aligned}\int_{t_0}^{t} \|\alpha'(x)\| \, \mathrm{d}x &= \int_{t_0}^{t} \sqrt{(abe^{bx}\cos x - ae^{bx}\sin x)^2 + (abe^{bx}\sin x + ae^{bx}\cos x)^2} \, \mathrm{d}x \\ &= \int_{t_0}^{t} ae^{bx}\sqrt{(b\cos x - \sin x)^2 + (b\sin x + \cos x)^2} \, \mathrm{d}x.\end{aligned}$$

Now simplify the expression under the square root

$$\begin{aligned}(b\cos x - \sin x)^2 + (b\sin x + \cos x)^2 &= b^2\cos^2 x - 2b\cos x\sin x + \sin^2 x + b^2\sin^2 x + 2b\sin x\cos x + \cos^2 x \\ &= b^2(\cos^2 x + \sin^2 x) + (\sin^2 x + \cos^2 x) \\ &= b^2 + 1.\end{aligned}$$

So we have

$$\int_{t_0}^{t} \|\alpha'(x)\| \, \mathrm{d}x = \int_{t_0}^{t} ae^{bx}\sqrt{b^2+1} \, \mathrm{d}x = a\sqrt{b^2+1}\int_{t_0}^{t} e^{bx} \, \mathrm{d}x.$$

Evaluate the integral

$$a\sqrt{b^2+1}\int_{t_0}^{t} e^{bx} \, \mathrm{d}x = a\sqrt{b^2+1} \cdot \frac{1}{b}\left(e^{bt} - e^{bt_0}\right).$$

Taking the limit as $t \to +\infty$

$$\lim_{t \to +\infty} \int_{t_0}^{t} \|\alpha'(x)\| \, \mathrm{d}x = a\sqrt{b^2+1} \cdot \frac{1}{b}\left(0 - e^{bt_0}\right) = -\frac{a\sqrt{b^2+1}}{b}e^{bt_0} < \infty.$$

Therefore, the arc length of α over $[t_0, \infty)$ is finite. □

Exercise 1.3.10. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a,b] \subset I$ and set $\alpha(a) = \mathbf{p}$, $\alpha(b) = \mathbf{q}$.

(i) Show that, for any constant vector $\mathbf{v}$, $\|\mathbf{v}\| = 1$,

$$(\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} = \int_a^b \alpha'(t) \cdot \mathbf{v} \, \mathrm{d}t \leq \int_a^b \|\alpha'(t)\| \, \mathrm{d}t.$$

(ii) Set

$$\mathbf{v} = \frac{\mathbf{q} - \mathbf{p}}{\|\mathbf{q} - \mathbf{p}\|},$$

and show that

$$\|\alpha(b) - \alpha(a)\| \leq \int_a^b \|\alpha'(t)\| \, \mathrm{d}t;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Solution to (i). We first show the equality on the left-hand side. Since $\mathbf{v}$ is a constant vector, we can factor it out of the integral. Thus, we have

$$\int_a^b \alpha'(t) \cdot \mathbf{v} \, \mathrm{d}t = \int_a^b \alpha'(t) \, \mathrm{d}t \cdot \mathbf{v} = (\alpha(b) - \alpha(a)) \cdot \mathbf{v} = (\mathbf{q} - \mathbf{p}) \cdot \mathbf{v}.$$

Thus, we have shown the equality.

Now, we can show the inequality. Using the Cauchy-Schwartz inequality, we know that for any vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$. Applying this to $\alpha'(t)$ and $\mathbf{v}$, we have $\alpha'(t) \cdot \mathbf{v} \leq \|\alpha'(t)\|$. Therefore, we have

$$\int_a^b \alpha'(t) \cdot \mathbf{v} \, \mathrm{d}t \leq \int_a^b \|\alpha'(t)\| \, \mathrm{d}t.$$

Thus, we have shown the inequality. □

Solution to (ii). Computing the original integral with the new value of $\mathbf{v}$, we have

$$\int_a^b \alpha'(t) \cdot \mathbf{v} \, \mathrm{d}t = \int_a^b \alpha'(t) \, \mathrm{d}t \cdot \left(\frac{\mathbf{q} - \mathbf{p}}{\|\mathbf{q} - \mathbf{p}\|} \right) = (\mathbf{q} - \mathbf{p}) \cdot \frac{\mathbf{q} - \mathbf{p}}{\|\mathbf{q} - p\|}.$$

Since $(\mathbf{q} - \mathbf{p})(\mathbf{q} - \mathbf{p}) = \|\mathbf{q} - \mathbf{p}\|^2$, we get

$$\int_a^b \alpha'(t) \, \mathrm{d}t \cdot \mathbf{v} = \|\mathbf{q} - \mathbf{p}\|.$$

Since we've already established the inequality in part (i), we have

$$\int_a^b \alpha'(t) \cdot \frac{\mathbf{q} - \mathbf{p}}{\|\mathbf{q} - \mathbf{p}\|} \, \mathrm{d}t = \|\mathbf{q} - \mathbf{p}\| = \|\alpha(b) - \alpha(a)\| \leq \int_a^b \|\alpha'(t)\| \, \mathrm{d}t.$$

Thus, we have shown the inequality. □

Exercise 1.4.2. A plane P contained in $\mathbb{R}^3$ is given by the equation $ax + by + cz + d = 0$. Show that the vector $\mathbf{v} = \langle a, b, c \rangle$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

Solution. Essentially, this question is asking to show that the normal vector to the plane is the vector $\mathbf{v} = \langle a, b, c \rangle$.

Let P_1 and P_2 be points on the plane P. Then, we have $\mathbf{p} = \overline{P_1 P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. Since they are both on the plane, we have

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0. \end{aligned}$$

Subtracting the two equations, we have

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0.$$

Thus, we have

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \Rightarrow \mathbf{v} \cdot \mathbf{p} = 0.$$

Thus, we have shown that $\mathbf{v}$ is perpendicular to the plane.

We now compute the distance from the origin $(0, 0, 0)$ to the plane. Let $\mathbf{v} = \langle a, b, c \rangle$ be the normal vector to the plane and let $\mathbf{p}$ be any point on the plane P, so that $ax + by + cz + d = 0$. Then the vector $\mathbf{p}$ points from the origin to a point on the plane, and the distance from the origin to the plane is given by projecting $\mathbf{p}$ onto the normal vector

$$D = |\mathrm{proj}_{\mathbf{v}}(\mathbf{p})| = \frac{|\mathbf{p} \cdot \mathbf{v}|}{\|\mathbf{v}\|}.$$

Since $\mathbf{p} \cdot \mathbf{v} = -d$, we have

$$D = \frac{|-d|}{\|\mathbf{v}\|} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}},$$

as desired. This completes the proof. □

Exercise 1.4.10. The natural orientation of $\mathbb{R}^2$ makes it possible to associate a sign to the area A of a parallelogram generated by two linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. To do this, let $\{\mathbf{e}_i\}$, $i = 1, 2$, be the natural ordered basis of $\mathbb{R}^2$, and write $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$. Observe the matrix relation

$$\begin{pmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis $\{\mathbf{u}, \mathbf{v}\}$, we can say that A is positive or negative according to whether the orientation of $\{\mathbf{u}, \mathbf{v}\}$ is positive or negative. This is called the *orientated area* in $\mathbb{R}^2$.

Solution. We are given two linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ written in terms of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ as

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \quad \text{and} \quad \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2.$$

Then the matrix with u and v as rows is

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix},$$

and the matrix with u and v as columns is

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

Now consider the product of these two matrices

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{pmatrix}.$$

This matrix appears on the left-hand side of the problem statement and encodes the inner products of $\mathbf{u}$ and $\mathbf{v}$.

The area A of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ satisfies

$$A^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2,$$

which is the determinant of the Gram matrix above

$$A^2 = \begin{vmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{vmatrix}.$$

Substituting the matrix identity from earlier, we obtain

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the square of the determinant is always non-negative, we can use the sign of the determinant itself to determine the orientation of the basis $\{\mathbf{u}, \mathbf{v}\}$. Therefore, the signed or *oriented* area is given by

$$A = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix},$$

where $A > 0$ if $\{\mathbf{u}, \mathbf{v}\}$ is positively oriented, and $A < 0$ if it is negatively oriented. This completes the proof. $\square$

Exercise 1.4.11.

(i) Show that the volume V of a parallelepiped generated by three linearly independent vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is given by $V = |(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}|$, and introduce an *orientated volume* in $\mathbb{R}^3$.

(ii) Prove that

$$V^2 = \begin{vmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} & \mathbf{u} \cdot \mathbf{w} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{w} \cdot \mathbf{u} & \mathbf{w} \cdot \mathbf{v} & \mathbf{w} \cdot \mathbf{w} \end{vmatrix}.$$

Solution to (i). The volume of a parallelepiped is given by $V = \text{Base} \times \text{Height}$. The base is given by the area of the parallelogram formed by the vectors $\mathbf{u}$ and $\mathbf{v}$, which is given by $|\mathbf{u} \wedge \mathbf{v}|$. The height is given by the component of $\mathbf{w}$ in the direction of the normal vector $\mathbf{u} \wedge \mathbf{v}$, which is given by

$$h = \frac{|(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \wedge \mathbf{v}\|}.$$

Therefore, we have

$$V = \text{Base} \times \text{Height} = \|\mathbf{u} \wedge \mathbf{v}\| \cdot \frac{|(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \wedge \mathbf{v}\|} = |(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}|.$$

The oriented volume V_{oriented} carries a sign that depends on whether $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ form a right-handed or left-handed basis. If $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ follows the right-hand rule, then V_{oriented} is positive. Otherwise, it is negative. $\square$

Solution to (ii). Let $A = (\mathbf{u}\ \mathbf{v}\ \mathbf{w})$. Then, we have

$$\det(A)\det(A) = \det(A^2) = \det(A^T A) = \det(A^T) \cdot \det(A) = \det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \cdot \det(\mathbf{u}\ \mathbf{v}\ \mathbf{w}).$$

This gives us

$$\det(A^T A) = \begin{vmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} & \mathbf{u}\cdot\mathbf{w} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{w} \\ \mathbf{w}\cdot\mathbf{u} & \mathbf{w}\cdot\mathbf{v} & \mathbf{w}\cdot\mathbf{w} \end{vmatrix}.$$

Notice that $\det(A^T A) = |(\mathbf{u}\wedge\mathbf{v})\cdot\mathbf{w}|^2 = V^2$. Thus, we've shown the desired result. $\square$

Exercise 1.5.1. Given the parametrized curve (helix)

$$\alpha(s) = \left\langle a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right\rangle, \quad s\in\mathbb{R},$$

where $c^2 = a^2 + b^2$.

(i) Show that the parameter s is the arc length.

(ii) Determine the curvature and the torsion of α.

(iii) Determine the osculating plane of α.

(iv) Show that the lines containing $\mathbf{N}(s)$ and passing through $\alpha(s)$ meet the z-axis under a constant angle equal to $\pi/2$.

(v) Show that the tangent lines to α make a constant angle with the z-axis.

Solution to (i). If $\alpha(s)$ is parametrized by arc length, then the magnitude of the derivative of $\alpha(s)$ must be equal to 1. We compute

$$\alpha'(s) = \left\langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right\rangle.$$

The magnitude of the derivative is given by

$$\begin{aligned}\|\alpha'(s)\| &= \sqrt{\frac{a^2}{c^2}\sin^2\left(\frac{s}{c}\right) + \frac{a^2}{c^2}\cos^2\left(\frac{s}{c}\right) + \frac{b^2}{c^2}} \\ &= \sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}} = \sqrt{1} = 1.\end{aligned}$$

Therefore, we have shown that s is the arc length. $\square$

Solution to (ii). The curvature is given by $\kappa(s) = \|\alpha''(s)\|$. Computing the second derivative from part (i), we have

$$\alpha''(s) = \left\langle -\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right\rangle.$$

The magnitude of the second derivative is given by

$$\kappa(s) = \|\mathbf{T}'(s)\| = \sqrt{\frac{a^2}{c^4}\cos^2\left(\frac{s}{c}\right) + \frac{a^2}{c^4}\sin^2\left(\frac{s}{c}\right)} = \frac{a}{c^2}.$$

Thus, the curvature is given by $\kappa(s) = a/c^2$.

The torsion, $\tau(s)$, is given by $\mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$. Using the unit normal and the binormal vector from part (iii). Now, we compute the derivative of $\mathbf{B}(s)$,

$$\mathbf{B}'(s) = \left\langle \frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right\rangle.$$

Therefore, we get

$$\left\langle \frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right\rangle = \tau(s)\left\langle -\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right\rangle.$$

Thus, the torsion is given by $\tau(s) = -b/c^2$. $\square$

Solution to (iii). Since $\alpha(s)$ is parametrized by arc length, then $\mathbf{T}(s) = \alpha'(s)$. Now, we need to find the unit normal vector, $\mathbf{N}(s)$, which is given by

$$\mathbf{N}(s) = \frac{\mathbf{T}(s)}{\|\mathbf{T}(s)\|} = \left\langle -\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right\rangle.$$

The osculating plane at s is the plane through $\alpha(s)$ orthogonal to $\mathbf{B}(s)$, i.e., $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$. Now, we need to find the binormal vector, $\mathbf{B}(s)$,

$$\begin{aligned}
\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s) &= \left\langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right\rangle \wedge \left\langle -\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right\rangle \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right) & -\sin\left(\frac{s}{c}\right) & 0 \end{vmatrix} \\
&= \left\langle \frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\sin^2\left(\frac{s}{c}\right) + \frac{a}{c}\cos^2\left(\frac{s}{c}\right)\right\rangle \\
&= \left\langle \frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right\rangle.
\end{aligned}$$

Now, we can find the osculating plane. The osculating plane is given by the equation

$$0 = \mathbf{B}(s) \cdot (x - x_0, y - y_0, z - z_0),$$

where $\alpha(s) = \langle x_0, y_0, z_0\rangle$. Thus, the osculating plane is given by

$$\left\langle x - a\cos\left(\frac{s}{c}\right), y - a\sin\left(\frac{s}{c}\right), z - b\frac{s}{c}\right\rangle \cdot \left\langle \frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right\rangle = 0.$$

□

Solution to (iv). A line through $\alpha(s)$ in the direction of $\mathbf{N}(s)$ is given by

$$\ell_s(t) = \alpha(s) + t\mathbf{N}(s),$$

and the z-axis consists of all points of the form $(0, 0, z)$. To find where the line intersects the z-axis, set the x and y components of $\ell_s(t)$ to zero

$$\begin{aligned}
x(t) &= a\cos\left(\frac{s}{c}\right) - t\cos\left(\frac{s}{c}\right) = 0 \\
y(t) &= a\sin\left(\frac{s}{c}\right) - t\sin\left(\frac{s}{c}\right) = 0.
\end{aligned}$$

Solving either equation (assuming $\cos(s/c) \neq 0$ or $\sin(s/c) \neq 0$), we get $t = a$. Plugging into the z-component

$$z = b\frac{s}{c} + 0 = b\frac{s}{c}.$$

Thus, the intersection point with the z-axis is

$$\left(0, 0, b\frac{s}{c}\right).$$

Now, consider the vector from $\alpha(s)$ to this point

$$\mathbf{v} = \left(0, 0, b\frac{s}{c}\right) - \alpha(s) = \left\langle -a\cos\left(\frac{s}{c}\right), -a\sin\left(\frac{s}{c}\right), 0\right\rangle.$$

Since this vector is proportional to $\mathbf{N}(s)$, and lies entirely in the xy-plane, it is orthogonal to the z-axis. Therefore, the angle between this line and the z-axis is $\pi/2$.

Hence, we have shown that these lines intersect the z-axis at a constant angle of $\pi/2$. □

Solution to (v). Since $\alpha(s)$ is parametrized by arc length, the unit tangent vector is

$$\mathbf{T}(s) = \alpha'(s) = \left\langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c} \right\rangle.$$

The direction vector of the z-axis is $\hat{\mathbf{k}} = \langle 0, 0, 1\rangle$. Then the angle θ between $\mathbf{T}(s)$ and the z-axis satisfies

$$\cos(\theta) = \mathbf{T}(s)\cdot\hat{\mathbf{k}} = \frac{b}{c}.$$

Since this value is constant (independent of s), the angle between the tangent vector and the z-axis is constant, and is given by

$$\theta = \cos^{-1}\left(\frac{b}{c}\right).$$

□

Exercise 1.5.11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \le \theta \le b$.

(i) Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\,\mathrm{d}\theta,$$

where the prime denotes the derivative relative to θ.

(ii) Show that the curvature is

$$\kappa(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[(\rho')^2 + \rho^2]^{3/2}}.$$

Solution to (i). The curve in polar coordinates is given by $\alpha(\theta) = \langle \rho\cos(\theta), \rho\sin(\theta)\rangle$. The general formula for the arc length is given by

$$s = \int_a^b \|\alpha'(\theta)\|\,\mathrm{d}\theta = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

We compute the derivatives

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \rho'\cos(\theta) - \rho\sin(\theta) \quad\text{and}\quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \rho'\sin(\theta) + \rho\cos(\theta).$$

Thus, we have

$$\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= (\rho'\cos(\theta) - \rho\sin(\theta))^2 + (\rho'\sin(\theta) + \rho\cos(\theta))^2 \\
&= \cos^2(\theta)(\rho')^2 - 2\sin(\theta)\cos(\theta)\rho\rho' + \sin^2(\theta)\rho^2 \\
&\qquad + \sin^2(\theta)(\rho')^2 + 2\sin(\theta)\cos(\theta)\rho\rho' + \cos^2(\theta)\rho^2 \\
&= (\rho')^2 + \rho^2(\theta).
\end{aligned}$$

Therefore, we have

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \int_a^b \sqrt{(\rho')^2 + \rho^2}\,\mathrm{d}\theta.$$

Thus, we have shown it's the arc length. □

Solution to (ii). The curvature is given by

$$\kappa(\theta) = \frac{\|\alpha'(\theta) \wedge \alpha''(\theta)\|}{\|\alpha'(\theta)\|^3}.$$

We compute the first and second derivatives

$$\begin{aligned}\alpha'(\theta) &= \rho'\langle\cos(\theta), \sin(\theta)\rangle + \rho\langle-\sin(\theta), \cos(\theta)\rangle \\ &= \langle\rho'\cos(\theta) - \rho\sin(\theta), \rho'\sin(\theta) + \rho\cos(\theta)\rangle \\ \alpha''(\theta) &= \rho''\langle\cos(\theta), \sin(\theta)\rangle + 2\rho'\langle-\sin(\theta), \cos(\theta)\rangle + \rho\langle-\cos(\theta), -\sin(\theta)\rangle \\ &= \langle\rho''\cos(\theta) - 2\rho'\sin(\theta) - \rho\cos(\theta), \rho''\sin(\theta) + 2\rho'\cos(\theta) - \rho\sin(\theta)\rangle.\end{aligned}$$

Now, we compute the cross product

$$\begin{aligned}\alpha'(\theta) \wedge \alpha''(\theta) &= (\rho'\cos\theta - \rho\sin\theta)(\rho''\sin\theta + 2\rho'\cos\theta - \rho\sin\theta) \\ &\quad - (\rho'\sin\theta + \rho\cos\theta)(\rho''\cos\theta - 2\rho'\sin\theta - \rho\cos\theta) \\ &= \rho^2 + 2(\rho')^2 - \rho\rho''.\end{aligned}$$

Next, compute the norm of the first derivative

$$\|\alpha'(\theta)\|^2 = (\rho'\cos\theta - \rho\sin\theta)^2 + (\rho'\sin\theta + \rho\cos\theta)^2 = (\rho')^2 + \rho^2.$$

So the curvature is

$$\kappa(\theta) = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[(\rho')^2 + \rho^2]^{3/2}} = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[(\rho')^2 + \rho^2]^{3/2}}.$$

Thus, we have shown the curvature. □

Exercise 1.5.12. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve (not necessarily by arc length), and let $\beta : J \to \mathbb{R}^3$ be a reparameterization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$ (see Remark 2). Let $t = t(s)$ be the inverse function of s and set $\mathrm{d}\alpha/\mathrm{d}t = \alpha'$, $\mathrm{d}^2\alpha/\mathrm{d}t^2 = \alpha''$, etc. Prove that

(i) $\mathrm{d}t/\mathrm{d}s = 1/\|\alpha'\|$, $\mathrm{d}^2 t/\mathrm{d}s^2 = -(\alpha' \cdot \alpha''/\|\alpha'\|^4)$.

(ii) The curvature of α at $t \in I$ is

$$\kappa(t) = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}.$$

(iii) The torsion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{\|\alpha' \wedge \alpha''\|^2}.$$

(iv) If $\alpha : I \to \mathbb{R}^2$ is a plane curve $\alpha(t) = \langle x(t), y(t)\rangle$, the signed curvature (see Remark 1) of α at t is

$$\kappa(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{3/2}}.$$

Solution to (i). Since $\beta(s) = \alpha(t(s))$, that means that β is a reparameterization of α by arc length. Thus, we have $\|\beta'(s)\| = 1$. Since $s = s(t)$ is arc length from $t_0 \in I$ to t, we have

$$s = \int_{t_0}^{t} \|\alpha'(x)\|\,\mathrm{d}x.$$

Differentiating both sides with respect to t, we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\alpha'(t)\|.$$

Since $s = s(t)$ and $t = t(s)$ is its inverse, using the Inverse Function Theorem, we have

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\mathrm{d}s/\mathrm{d}t} = \frac{1}{\|\alpha'(t)\|}.$$

Computing the second derivative, we have

$$\begin{aligned}\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\|\alpha'(t(s))\|} \\ &= -\frac{1}{2\|\alpha'(t(s))\|^2}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\|\alpha'(t(s))\| \\ &= -\frac{1}{2\|\alpha'(t(s))\|^2}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left[\alpha'(t(s))\cdot\alpha'(t(s))\right] \\ &= -\frac{1}{2\|\alpha'(t(s))\|^2}\cdot\frac{2\alpha'(t(s))\cdot\alpha''(t(s))}{\|\alpha'(t(s))\|}\cdot\frac{\mathrm{d}t}{\mathrm{d}s} \\ &= -\frac{\alpha'(t(s))\cdot\alpha''(t(s))}{\|\alpha'(t(s))\|^4}.\end{aligned}$$

This completes the proof. □

Solution to (ii). Recall that $\beta(s) = \alpha(t(s))$ is the arc length reparameterization of α. Then

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{1}{\|\alpha'(t)\|}\cdot\alpha'(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

Taking the derivative again with respect to s, we apply the chain rule

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\alpha'(t)}{\|\alpha'(t)\|}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha'(t)}{\|\alpha'(t)\|}\right)\cdot\frac{1}{\|\alpha'(t)\|}.$$

Let us now compute the derivative inside

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha'(t)}{\|\alpha'(t)\|}\right) = \frac{\|\alpha'(t)\|\frac{\mathrm{d}}{\mathrm{d}t}\alpha'(t) - \alpha'(t)\frac{\mathrm{d}}{\mathrm{d}t}\|\alpha'(t)\|}{\|\alpha'(t)\|^2}.$$

Computing each term, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha'(t) = \alpha''(t) \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}t}\|\alpha'(t)\| = \frac{\alpha'(t)\cdot\alpha''(t)}{\|\alpha'(t)\|}.$$

Therefore, we have

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} = \frac{\|\alpha'(t)\|\alpha''(t) - \alpha'(t)\frac{\alpha'(t)\cdot\alpha''(t)}{\|\alpha'(t)\|}}{\|\alpha'(t)\|^3}.$$

We decompose $\alpha''(t)$ into two components: one parallel and one perpendicular to $\alpha'(t)$. The parallel component is given by the projection

$$\alpha_{\|}(t) = \frac{\alpha'(t)\cdot\alpha''(t)}{\|\alpha'(t)\|^2}\alpha'(t),$$

and the perpendicular component is

$$\alpha_{\perp}(t) = \alpha''(t) - \alpha_{\|}(t).$$

Since $\alpha_{\perp}(t)$ is perpendicular to $\alpha'(t)$, we have $\alpha'(t)\cdot\alpha_{\perp}(t) = 0$.

Substituting the decomposition of $\alpha''(t)$ into the second derivative, we get

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} = \frac{\|\alpha'(t)\|(\alpha_{\|}(t) + \alpha_{\perp}(t)) - \alpha'(t)\frac{\alpha'(t)\cdot\alpha''(t)}{\|\alpha'(t)\|}}{\|\alpha'(t)\|^3}.$$

The parallel components cancel out

$$\|\alpha'(t)\|\alpha_{\|}(t) = \alpha'(t)\frac{\alpha'(t)\cdot\alpha''(t)}{\|\alpha'(t)\|},$$

so we are left with
$$\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} = \frac{\|\alpha'(t)\|\alpha_\perp(t)}{\|\alpha'(t)\|^3} = \frac{\alpha_\perp(t)}{\|\alpha'(t)\|^2}.$$

Since $\alpha_\perp(t)$ is perpendicular to $\alpha'(t)$, we can express the magnitude of $\alpha_\perp(t)$ in terms of the cross product. Specifically,
$$\|\alpha_\perp(t)\| = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|}.$$

Thus, the second derivative simplifies to
$$\kappa(t) = \frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|^3}.$$

This completes the proof. □

Solution to (iii). We know that $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$ and that $\mathrm{d}\mathbf{B}/\mathrm{d}s = -\tau(s)\mathbf{N}(s)$, from the Frenet–Serret formulas. Our goal is to express the torsion $\tau(t)$ of the original parametrization $\alpha(t)$ in terms of its derivatives with respect to t.

Recall that $\beta(s) = \alpha(t(s))$, and so
$$\mathbf{T}(s) = \frac{\mathrm{d}\beta}{\mathrm{d}s} = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa(s)\mathbf{N}(s), \quad \text{and} \quad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau(s)\mathbf{N}(s).$$

We now compute $\mathrm{d}\mathbf{B}/\mathrm{d}s$ directly in terms of α. First, recall that $\mathbf{B} = \mathbf{T} \wedge \mathbf{N}$, and use the product rule for derivatives
$$\begin{aligned}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{T} \wedge \mathbf{N}) \\ &= \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \wedge \mathbf{N} + \mathbf{T} \wedge \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s}.\end{aligned}$$

Using the Frenet–Serret formulas again,
$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N} \quad \text{and} \quad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} + \tau\mathbf{B},$$

so
$$\begin{aligned}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} &= \kappa\mathbf{N} \wedge \mathbf{N} + \mathbf{T} \wedge (-\kappa\mathbf{T} + \tau\mathbf{B}) \\ &= 0 + \tau\mathbf{T} \wedge \mathbf{B}.\end{aligned}$$

Since $\mathbf{B} = \mathbf{T} \wedge \mathbf{N}$, and the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis, we know that $\mathbf{T} \wedge \mathbf{B} = -\mathbf{N}$. Hence,
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = \tau(-\mathbf{N}) = -\tau\mathbf{N}.$$

Now, to write τ in terms of α, recall that
$$\mathbf{B} = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|},$$

and
$$\begin{aligned}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}\right) \\ &= \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}\right).\end{aligned}$$

Differentiating the numerator

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha' \wedge \alpha'') = \alpha' \wedge \alpha''' + \alpha'' \wedge \alpha'' = \alpha' \wedge \alpha''',$$

since $\alpha'' \wedge \alpha'' = 0$. Then

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} &= \frac{1}{\|\alpha'\|} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}\right) \\
&= \frac{1}{\|\alpha'\|} \cdot \left(\frac{\alpha' \wedge \alpha'''}{\|\alpha' \wedge \alpha''\|} - \frac{(\alpha' \wedge \alpha'') \cdot (\alpha' \wedge \alpha''')}{\|\alpha' \wedge \alpha''\|^3}(\alpha' \wedge \alpha'')\right).
\end{aligned}$$

Taking the dot product with $\mathbf{N} = \mathbf{B} \wedge \mathbf{T}$, we isolate the component in the $\mathbf{N}$ direction. Since

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau \mathbf{N},$$

and $\mathbf{N}$ is perpendicular to $\mathbf{B}$ and $\mathbf{T}$, we can dot both sides with $\mathbf{N}$ to get

$$\begin{aligned}
\tau &= -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{N} \\
&= -\frac{1}{\|\alpha'\|} \cdot \left(\frac{\alpha' \wedge \alpha'''}{\|\alpha' \wedge \alpha''\|} \cdot \mathbf{N}\right).
\end{aligned}$$

Since $\mathbf{B} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}$ and $\mathbf{N} = \mathbf{B} \wedge \mathbf{T}$, this leads us to

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{\|\alpha' \wedge \alpha''\|^2}.$$

This completes the proof. □

Solution to (iv). Computing $\alpha' \wedge \alpha''$, we have

$$\alpha' \wedge \alpha'' = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = \langle 0, 0, x'y'' - x''y' \rangle.$$

The magnitude of the cross product is given by

$$\|\alpha' \wedge \alpha''\| = \sqrt{(x'y'' - x''y')^2} = |x'y'' - x''y'|.$$

The magnitude of the first derivative cubed is given by

$$\|\alpha'\|^2 = \left((x')^2 + (y')^2\right)^{3/2}.$$

Therefore, the curvature is given by

$$\kappa(t) = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3} = \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{3/2}}.$$

This completes the proof. □

Exercise 1.5.14. Let $\alpha : (a, b) \to \mathbb{R}^2$ be a regular parametrized plane curve. Assume that there exists t_0, $a < t_0 < b$, such that the distance $\|\alpha(t)\|$ from the origin to the trace of α will be a maximum at t_0. Prove that the curvature κ of α at t_0 satisfies $|\kappa(t_0)| \geq 1/\|\alpha(t_0)\|$.

Solution. Let $f(t) = \|\alpha(t)\|^2$, the square of the distance from the origin to the curve. Since f is maximized at t_0, we have

$$f'(t_0) = 0 \quad \text{and} \quad f''(t_0) \leq 0.$$

Compute the first derivative

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

so at t_0,

$$\alpha(t_0) \cdot \alpha'(t_0) = 0.$$

Thus, the position vector $\alpha(t_0)$ is perpendicular to the velocity vector $\alpha'(t_0)$, and lies in the direction of the unit normal vector $\mathbf{N}(t_0)$. Hence we can write

$$\alpha(t_0) = \|\alpha(t_0)\|\mathbf{N}(t_0).$$

Now compute the second derivative

$$f''(t) = 2\left(\|\alpha'(t)\|^2 + \alpha(t) \cdot \alpha''(t)\right).$$

At t_0, the condition $f''(t_0) \leq 0$ implies

$$\|\alpha'(t_0)\|^2 + \alpha(t_0) \cdot \alpha''(t_0) \leq 0.$$

Using $\alpha(t_0) = \|\alpha(t_0)\|\mathbf{N}(t_0)$, we compute

$$\begin{aligned} \alpha(t_0) \cdot \alpha''(t_0) &= \|\alpha(t_0)\|\mathbf{N}(t_0) \cdot \alpha''(t_0) \\ &= \|\alpha(t_0)\| \cdot \kappa(t_0)\|\alpha'(t_0)\|^2, \end{aligned}$$

by the Frenet-Serret formula in the plane.

Substituting into the inequality, we obtain

$$\|\alpha'(t_0)\|^2 + \|\alpha(t_0)\| \cdot \kappa(t_0)\|\alpha'(t_0)\|^2 \leq 0.$$

Factoring out $\|\alpha'(t_0)\|^2 > 0$, we get

$$1 + \|\alpha(t_0)\| \cdot \kappa(t_0) \leq 0 \Rightarrow \kappa(t_0) \leq -\frac{1}{\|\alpha(t_0)\|}.$$

Alternatively, if $\kappa(t_0) \geq 0$, then this same argument applies with the curve reflected through the origin (i.e., apply the same proof to $-\alpha(t)$). In either case, we conclude

$$|\kappa(t_0)| \geq \frac{1}{\|\alpha(t_0)\|}.$$

This completes the proof. □