Introduction to Proof: Homework 3

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Problem 1

Show that $[U \wedge P] \Rightarrow [Q \wedge R]$, $P \Leftrightarrow [S \vee T]$, $R \wedge T \vdash U \Rightarrow Q$.

Solution 1

	Statement	Explanation
1.	$[U \wedge P] \Rightarrow [Q \wedge R]$	Hypothesis
2.	$P \Leftrightarrow [S \vee T]$	Hypothesis
3.	$R \wedge T$	Hypothesis
4.	R	LCS, for 3
5.	T	RCS, for 3
6.	P	MPB, for 5, for 2
7.	Assume U	Dischargeable Hypothesis
8.	$U \wedge P$	CI, for 6, for 7
9.	$Q \wedge R$	MP, for 8, for 1
10.	Q	LCS, for 9
11.	$U \Rightarrow Q$	DT, discharge for 7 [(7) – (10) unusable]

Problem 2

Show that $\neg P \Rightarrow Q$, $P \Rightarrow \neg Q$, $P \Leftrightarrow R \vdash \neg Q \Leftrightarrow R$. (Hint: Recall that $X \Leftrightarrow Y$ is an abbreviation for something.)

Solution 2

	Statement	Explanation
1.	$\neg P \Rightarrow Q$	Hypothesis
2.	$P \Rightarrow \neg Q$	Hypothesis
3.	$P \Leftrightarrow R$	Hypothesis
4.	Assume $\neg Q$	Dischargeable Hypothesis
5.	P	MT, for 5, for 1
6.	R	MPB, for 5, for 3
7.	$\neg Q \Rightarrow R$	DT, discharge for 5 [(5) – (7) unusable]
8.	Assume R	Dischargeable Hypothesis
9.	P	MP, for 9, for 3
10.	$\neg Q$	MT, for 10, for 2
11.	$R \Rightarrow \neg Q$	DT, discharge for 9 [(9) – (11) unusable]
12.	$\neg Q \Leftrightarrow R$	For 8, for 12

Problem 3

Show that $R \vee S$, $\neg P$, $Q \vee \neg R$, $P \Leftrightarrow Q \vdash S$.

Solution 3

	Statement	Explanation
1.	$R \vee S$	Hypothesis
2.	$\neg P$	Hypothesis
3.	$Q \vee \neg R$	Hypothesis
4.	$P \Leftrightarrow Q$	Hypothesis
5.	$\neg Q$	MTB, for 2, for 4
6.	$\neg R$	DI, for 5, for 3
7.	Assume $\neg S$	Dischargeable Hypothesis
8.	R	DI, for 7, for 1
9.	$\neg R \wedge R$	CI, for 6, for 8
10.	S	II, for 7 [(7) – (9) unusable]

Problem 4

Translate each of the following into a normal English sentence. Also, identify the statement as True or False. For example, the proposition

$$(\forall x)[(x \in \mathbb{Z} \wedge x \mid 24) \Rightarrow x \mid 48].$$

says that "Every divisor of 24 is a divisor of 48". It is True.

(i) $(\forall x)[(x \in \mathbb{N} \wedge x \mid 13) \Rightarrow x \in \{1, 13\}]$.

(ii) $(\exists a)[a \in \mathbb{N} \wedge (a > 51 \wedge a < 52)]$.

(iii) $(\forall x)[(x \in \mathbb{N} \wedge x^2 = 2) \Rightarrow x = 5]$.

(iv) $(\forall x)[x \in \mathbb{N} \Rightarrow (\exists y)[y \in \mathbb{N} \wedge y \mid x]]$.

(v) $(\forall n)[(n \in \mathbb{N} \wedge (4 \mid n \wedge 6 \mid n)) \Rightarrow 24 \mid n]$.

Solution 4

(i) Every divisor of 13 is either 13 or 1, meaning 13 is prime, which is true.

(ii) There exists a natural number between 51 and 52, which is false.

(iii) Every natural number whose square is 2 is 5, which is false.

(iv) For every natural number, there exists a natural number which divides it, which is true.

(v) For every natural number that's a divisor of both 4 and 6, it is also a divisor of 24, which is true.

Problem 5

The phrase "the integer x is a perfect square" means $x \in \mathbb{Z} \wedge (\exists y \in \mathbb{Z})[x = y^2]$. Keeping this in mind, write out mathematical statements—using only quantifiers and other math symbols, no English words—which say the same thing as the following sentences. Do not worry about whether the statements are true or false!

(i) For every integer a, if a is even then a^2 is a multiple of 5.

(ii) 5 is smaller than every integer.

(iii) Every integer which is a perfect square is also a perfect cube.

(iv) Every nonzero integer is either a multiple of six or a multiple of seven.

(v) There exists an integer which is larger than every other integer.

(vi) There is an element of the set A having the property that every element of the set B divides it.

(vii) Every element of the set A is either even or a multiple of 13.

(viii) There exists an integer which is not divisible by any divisor of 240.

(ix) Every nonzero element of $\mathbb{Z}_5$ has a multiplicative inverse.

(x) For all integers x, if x is congruent to three mod eight then there exists an integer y such that $x \cdot y$ is congruent to one mod eight.

(xi) There exists an integer p with the following properties: p is even and for every pair of integers a and b, if p divides ab then it must divide either a or b.

Solution 5

(i) $(\forall a)([a \in \mathbb{Z} \wedge a \equiv_2 0] \Rightarrow 5 \mid a)$.

(ii) $(\forall x)(x \in \mathbb{Z} \Rightarrow 5 < x)$.

(iii) $(\forall x)([x \in Z \wedge \sqrt{x} \in \mathbb{Z}] \Rightarrow \sqrt[3]{x} \in \mathbb{Z})$.

(iv) $(\forall x)([x \in \mathbb{Z} \wedge x \neq 0] \Rightarrow [6 \mid x \vee 7 \mid x])$.

(v) $(\exists x)(\forall y)([x \in \mathbb{Z} \wedge y \in \mathbb{Z}] \Rightarrow x > y)$.

(vi) $(\exists a)(\forall b)([a \in A \wedge b \in A] \Rightarrow a \mid b)$.

(vii) $(\forall a)(a \in A \Rightarrow [a \equiv_2 0 \vee 13 \mid a])$.

(viii) $(\exists x)(x \in \mathbb{Z} \Rightarrow (\forall y)[y \in \mathbb{Z} \wedge y \mid 240 \Rightarrow x \nmid y])$.

(ix) $(\forall x)(x \in \mathbb{Z}_5 \wedge x \neq 0 \Rightarrow (\exists x^{-1})[x^{-1} \in \mathbb{Z}_5 \wedge xy = 1])$.

(x) $(\forall x)([x \in \mathbb{Z} \wedge x \equiv_8 3] \Rightarrow (\exists y)[y \in \mathbb{Z} \wedge xy \equiv_8 1])$.

(xi) $(\exists x)(x \in \mathbb{Z} \Rightarrow [(x \equiv_2 0 \wedge (\forall a)(\forall b)([a \in \mathbb{Z} \wedge b \in \mathbb{Z} \wedge x \mid ab] \Rightarrow [x \mid a \vee x \mid b]))])$.

Problem 6

In each part, identify the given set by listing all of its elements. For example:

Given the set $\{x \in \mathbb{Z} \mid x \equiv_4 3 \wedge 5 \leq x \wedge x \leq 20\}$ you would answer $\{7, 11, 15, 19\}$, as these two sets are equal.

(i) $\{x \in \mathbb{Z}_4 \mid (\exists y)[y \in \mathbb{Z}_4 \wedge y \neq 0 \wedge xy = 0]\}$.

(ii) $\{x^3 + 1 \mid x \in \mathbb{N}\} \cap \{y \mid y \in \mathbb{N} \wedge 1 \leq y \leq 30\}$.

(iii) $\{x \in \mathbb{N} \mid (\exists a)(\exists b)[a \in \mathbb{N} \wedge b \in \mathbb{N} \wedge x = a^2 + b^2]\} \cap \{x \mid x \in \mathbb{N} \wedge x \leq 20\}$.

(iv) $\{z \in \mathbb{Z}_{30} \mid (\exists u)[u \in \mathbb{Z}_{30} \wedge zu = 1]\}$.

(v) $\{a \mid a \in \mathbb{Z} \wedge a \equiv_4 1\} \cap \{b \mid b \in \mathbb{Z} \wedge b \equiv_2 0\}$.

(vi) $\{(a, b) \mid a \in \mathbb{Z}_2 \wedge b \in \mathbb{Z}_2 \wedge a + b = 1\}$.

(vii) $\{x \in \mathbb{N} \mid x \equiv_5 1 \wedge x \leq 40\} \cap \{x \in \mathbb{N} \mid x \equiv_6 4\}$.

Solution 6

(i) $\{0, 2\}$.

(ii) $\{2, 9, 28\}$.

(iii) $\{2, 5, 8, 10, 13, 17, 18, 20\}$. If we're including $0 \in \mathbb{N}$, then we get the set $\{0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20\}$.

(iv) $\{1, 2, 3, 4, 5, 6, 9, 10, 15\}$.

(v) $\emptyset$.

(vi) $\{(0, 1), (1, 0)\}$.

(vii) $\{16\}$.

Problem 7

In each part below, use mathematical notation to write the negation of the given statement in such a way that no quantifier is immediately preceded by a negation sign, every universal quantifier is applied to a conditional, and every existential quantifier is applied to a conjunction. In each part, decide which statement is true: the given statement or its negation.

(i) $(\forall y)[y \in \mathbb{Z} \Rightarrow y > 0]$.

(ii) $(\forall x)[(x \in \mathbb{Z} \wedge 2 \mid x) \Rightarrow 4 \mid x]$.

(iii) $(\forall x)[x \in \mathbb{Z}_6 \Rightarrow (\exists y)[y \in \mathbb{Z}_6 \wedge x +_6 y = 0]]$.

(iv) $(\exists x)[x \in \mathbb{Z}_6 \wedge (\forall y)[y \in \mathbb{Z}_6 \Rightarrow x +_6 y = 0]]$.

(v) $(\forall x)[x \in \mathbb{Z} \Rightarrow (\exists y)[y \in \mathbb{Z} \wedge x = y^2]]$.

(vi) $(\exists a)[a \in \mathbb{N} \wedge (\forall b)[[b \in \mathbb{N} \wedge a \neq b] \Rightarrow b > a]]$.

(vii) $(\exists m)(\exists n)[m, n \in \mathbb{N} \wedge m > n]$.

(viii) $(\forall m)(\forall n)[m, n \in \mathbb{N} \Rightarrow m > n]$.

(ix) $(\exists m)[m \in \mathbb{N} \wedge (\forall n)[n \in \mathbb{N} \Rightarrow m > n]$.

(x) $(\forall m)[m \in \mathbb{N} \Rightarrow (\exists n)[n \in \mathbb{N} \wedge m > n]$.

(xi) $(\forall a)(\forall b)[(a, b \in \mathbb{R} \wedge a < b) \Rightarrow (\exists c)[c \in \mathbb{R} \wedge [a < c \wedge c < b]]]$.

Solution 7

(i) $(\exists y)[y \in \mathbb{Z} \wedge y \leq 0]$. The negation is *true.*

(ii) $(\exists x)[(x \in \mathbb{Z} \wedge 2 \mid x) \wedge 4 \nmid x]$. The negation is *true.*

(iii) $(\exists x)[x \in \mathbb{Z}_6 \wedge (\forall y)[y \in \mathbb{Z}_6 \Rightarrow x +_6 \neq 0]]$. The negation is *false.*

(iv) $(\forall x)[x \in \mathbb{Z}_6 \Rightarrow (\exists y)[y \in \mathbb{Z}_6 \wedge x +_6 y \neq 0]]$. The negation is *true.*

(v) $(\exists x)[x \in \mathbb{Z} \wedge (\forall y)[y \in \mathbb{Z} \Rightarrow x \neq y^2]]$. The negation is *true.*

(vi) $(\forall a)[a \in \mathbb{N} \Rightarrow (\exists b)[(b \in \mathbb{N} \wedge a \neq b) \wedge b \leq a]]$. The negation is *true.*

(vii) $(\forall m)(\forall n)[m \in \mathbb{Z} \wedge n \in \mathbb{N} \Rightarrow m \leq n]$. The negation is *false.*

(viii) $(\exists m)(\exists n)[m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge m \leq n]$. The negation is *true.*

(ix) $(\forall m)[m \in \mathbb{N} \Rightarrow (\exists n)[n \in \mathbb{N} \wedge m \leq n]]$. The negation is *true.*

(x) $(\exists m)[m \in \mathbb{N} \wedge (\forall n)[n \in \mathbb{N} \Rightarrow m \leq n]]$. The negation is *true.*

(xi) $(\exists a)(\exists b)[(a \in \mathbb{R} \wedge b \in \mathbb{R} \wedge a < b) \wedge (\forall c)[c \in \mathbb{R} \Rightarrow [a \geq c \vee c \geq b]]]$. The negation is *false.*

Problem 8

Fill in the blanks in the outline below to prove that $(\forall a, b, k)[[a, b, k \in \mathbb{Z} \wedge (k \geq 1 \wedge a \mid b)] \Rightarrow a^k \mid b^k]$.
Proof:

1. Assume $a, b, k \in \mathbb{Z}$ and $k \geq 1$ and $a \mid b$.
2. $\boxed{}(y \in \mathbb{Z} \wedge b = y \cdot a)$.
3. $b = y \cdot a$ for some $\boxed{}$.
4. $b^k = \boxed{}$.
5. $(\exists u)(u \in \mathbb{Z} \wedge \boxed{})$.
6. $a^k \mid b^k$.
7. $\boxed{}$ DT, discharge for 1, [(1) − (6) unusable].
8. $(\forall a, b, k)([a, b, k \in \mathbb{Z} \wedge (k \geq 1 \wedge a \mid b)] \Rightarrow a^k \mid b^k)$.

Answer the following questions:

(i) The above proof used one IE step, one EI step, and one IU step. Label them in column 2 of your proof (the same column with DT in it).

(ii) The definition of $a \mid b$ used k for the variable in the existential statement. In step 2, why did we change and use y instead?

(iii) In step 5 we introduced the variable u in the existential statement. Could we have used k here instead? What about y?

Solution 8

Proof:

1.	Assume $a, b, k \in \mathbb{Z}$ and $k \geq 1$ and $a \mid b$.	Dischargeable Hypothesis
2.	$(\exists y)(y \in \mathbb{Z} \wedge b = y \cdot a)$.	IE
3.	$b = y \cdot a$ for some $y \in \mathbb{Z}$.	EI
4.	$b^k = (y \cdot a)^k$.	
5.	$(\exists u)(u \in \mathbb{Z} \wedge b^k = u \cdot a^k)$.	IE
6.	$a^k \mid b^k$.	
7.	$(a, b, k \in \mathbb{Z} \wedge k \geq 1 \wedge a \mid b) \Rightarrow a^k \mid b^k$	DT, discharge for 1, [(1) − (6) unusable].
8.	$(\forall a, b, k)([a, b, k \in \mathbb{Z} \wedge (k \geq 1 \wedge a \mid b)] \Rightarrow a^k \mid b^k)$.	IU

(i) They are lines number 2, 3, 5, and 7.

(ii) We changed to y because we needed to use a in the existential statement.

(iii) No, we could not have used k or y in step 5. We needed a new variable to represent the integer that divides b^k.

Problem 9

Fill in the blanks in the outline below to prove that $(\forall a, b, c)[[a, b, c \in \mathbb{Z} \wedge (a \mid b \wedge b \mid c)] \Rightarrow a \mid c]$. [Hint: Notice that p and q, from step 6, will have to appear in steps 1–5 somewhere.]
Proof:

1. Assume $\boxed{}$.
2. $(\exists k)[k \in \mathbb{Z} \wedge b = k \cdot a]$.
3. $\boxed{}$ for some $\boxed{}$.
4. $(\exists k)[k \in \mathbb{Z} \wedge c = k \cdot b]$.
5. $\boxed{}$ for some $\boxed{}$.
6. $c = q \cdot b = p \cdot (qa) = pq \cdot a$.
7. $(\exists r)[r \in \mathbb{Z} \wedge \boxed{}]$.
8. $a \mid c$.
9. $\boxed{}$. DT, discharge for 1
10. $\boxed{}$.

Answer the following questions:

(i) The above proof used two IE steps, one EI step, and one IU step. Label them in column 2 of your proof (the same column with DT in it). You do not have to give reasons for any of the other steps.

(ii) It is okay that we used k in both step 2 and step 4. Why? (This might be hard to explain in words, but at least try to come to some kind of understanding for yourself).

Solution 9

Proof:

1.	Assume $a, b, c \in \mathbb{Z}$ and $a \mid b$ and $b \mid c$.	Dischargeable Hypothesis
2.	$(\exists k)[k \in \mathbb{Z} \wedge b = k \cdot a]$.	IE
3.	$b = y \cdot a$ for some $y \in \mathbb{Z}$.	EI
4.	$(\exists k)[k \in \mathbb{Z} \wedge c = k \cdot b]$.	IE
5.	$c = q \cdot b$ for some $q \in \mathbb{Z}$.	EI
6.	$c = q \cdot b = p \cdot (qa) = pq \cdot a$.	
7.	$(\exists r)[r \in \mathbb{Z} \wedge c = r \cdot a]$.	IE
8.	$a \mid c$.	
9.	$(a, b, c \in \mathbb{Z} \wedge a \mid b \wedge b \mid c) \Rightarrow a \mid c$	DT, discharge for 1.
10.	$(\forall a, b, c)[[a, b, c \in \mathbb{Z} \wedge a \mid b \wedge b \mid c] \Rightarrow a \mid c]$	IU

Problem 9 continued on next page . . .

(i) They are lines number 2, 3, 4, 5, 7, and 9.

(ii) It is okay to use k in both steps 2 and 4. Because we wrote out the definition of $a \mid b$ and $b \mid c$ and k was just a placeholder, which we then changed to y and q respectively in the line right after each one.

Problem 10

Fill in the blanks in the outline below to prove that $(\forall n)[(n \in \mathbb{N} \wedge 3 \nmid n) \Rightarrow n^2 \equiv_3 1]$. Also:

(i) There are three uses of the Deduction Theorem in this proof. Label the appropriate DT steps.

(ii) There are two steps at the end of the proof with the phrase "Logical rule?" in bold. For these, label the appropriate rule from symbolic logic that is being used.

1. Assume $\boxed{}$.
2. $n \not\equiv_3 0$.
3. So either $n \equiv_3 1$ or $n \equiv_3 2$.
4. Assume $n \equiv_3 1$.
5. Then $3 \mid \boxed{}$.
6. So $n - 1 = 3 \cdot P$ for some $\boxed{}$.
7. $n = \boxed{}$.
8. $n^2 = \boxed{} = 3 \cdot \boxed{} + 1$.
9. $n^2 - 1 = \boxed{}$.
10. $(\exists y)[y \in \mathbb{Z} \wedge \boxed{}]$.
11. $3 \mid \boxed{}$.
12. $n^2 \equiv_3 1$.
13. So $n \equiv_3 1 \Rightarrow n^2 \equiv_3 1$.
14. Now assume $n \equiv_3 2$.
15. $\boxed{}$.
16. $\boxed{}$.
17. $\boxed{}$.
18. $\boxed{}$.
19. $\boxed{}$.
20. $\boxed{}$.
21. $\boxed{}$.
22. $\boxed{}$.
23. $\boxed{}$.
24. So $n \equiv_3 2 \Rightarrow n^2 \equiv_3 1$.
25. We therefore have $(n \equiv_3 1 \vee n \equiv_3 2) \Rightarrow n^2 \equiv_3 1$. **Logical rule?**
26. So $n^2 \equiv_3 1$. **Logical rule?**
27. $(n \in \mathbb{N} \wedge 3 \nmid n) \Rightarrow n^2 \equiv_3 1$.
28. $\boxed{}$.

Solution 10

1.	Assume $n \in N$ and $3 \nmid n$.	Dischargeable Hypothesis
2.	$n \not\equiv_3 0$.	
3.	So either $n \equiv_3 1$ or $n \equiv_3 2$.	
4.	Assume $n \equiv_3 1$.	Dischargeable Hypothesis
5.	Then $3 \mid (n-1)$.	
6.	So $n - 1 = 3 \cdot P$ for some $P \in \mathbb{N}$.	
7.	$n = 3 \cdot P + 1$.	
8.	$n^2 = (3 \cdot P + 1)^2 = 3(3P^2 + 2P) + 1$.	
9.	$n^2 - 1 = 3(3P^2 + 2P)$.	
10.	$(\exists y)[y \in \mathbb{Z} \wedge n^2 - 1 = 3 \cdot y]$.	
11.	$3 \mid (n^2 - 1)$.	
12.	$n^2 \equiv_3 1$.	
13.	So $n \equiv_3 1 \Rightarrow n^2 \equiv_3 1$.	DT, discharge for 4, [(4) – (12) unusable].
14.	Now assume $n \equiv_3 2$.	Dischargeable Hypothesis
15.	Then $3 \mid (n-2)$.	
16.	So $n - 2 = 3 \cdot Q$ for some $Q \in \mathbb{N}$.	
17.	$n = 3 \cdot Q + 2$.	
18.	$n^2 = (3 \cdot Q + 2)^2 = 3(3Q^2 + 4Q) + 4$	
19.	$n^2 - 2 = 3(3Q^2 + 4Q) + 2$.	
20.	$3 \mid (n^2 - 2)$.	
21.	$(\exists y)[y \in \mathbb{Z} \wedge n^2 - 2 = 3 \cdot y]$.	
22.	$3 \mid (n^2 - 2)$.	
23.	$n^2 \equiv_3 2$.	
24.	So $n \equiv_3 2 \Rightarrow n^2 \equiv_3 1$.	DT, discharge for 14, [(15) – (23) unusable].
25.	We therefore have $(n \equiv_3 1 \vee n \equiv_3 2) \Rightarrow n^2 \equiv_3 1$.	IC, for 13, for 24
26.	So $n^2 \equiv_3 1$.	MP, for 25, for 3
27.	$(n \in \mathbb{N} \wedge 3 \nmid n) \Rightarrow n^2 \equiv_3 1$.	DT, discharge for 1, [(1) – (26) unusable].
28.	$(\forall n)[(n \in \mathbb{N} \wedge 3 \nmid n) \Rightarrow n^2 \equiv_3 1]$.	IU

(i) The Deduction Theorem steps are lines 13, 24, and 27.

(ii) The logic rules used in lines 25 and 26 are Inference by Cases (IC) and Modus Ponens (MP), respectively.

Problem 11

Prove $(\forall n)[(n \in \mathbb{N} \wedge n \equiv_6 3) \Rightarrow n^2 + 2n + 10 \equiv_{12} 1]$.

Solution 11

1.	Assume $n \in \mathbb{N}$ and $n \equiv_6 3$.	Dischargeable Hypothesis
2.	Then $6 \mid (n-3)$.	
3.	So $n - 3 = 6 \cdot P$ for some $P \in \mathbb{N}$.	
4.	$n = 6 \cdot P + 3$.	
5.	$n^2 + 2n + 10 = (6 \cdot P + 3)^2 + 2(6 \cdot P + 3) + 10$	
6.	$n^2 + 2n + 10 = 36P^2 + 36P + 9 + 12P + 6 + 10$	
7.	$n^2 + 2n + 10 = 36P^2 + 48P + 25$	
8.	$n^2 + 2n + 10 = 12(3P^2 + 4P + 2) + 1$	
9.	$n^2 + 2n + 10 - 1 = 12(3P^2 + 4P + 2)$	
10.	$(\exists y)(y \in \mathbb{Z} \wedge n^2 + 2n + 10 - 1 = 12 \cdot y$.	IE
11.	$12 \mid (n^2 + 2n + 10 - 1)$.	
12.	$n^2 + 2n + 10 \equiv_{12} 1$.	
13.	So $(n \in \mathbb{N} \wedge n \equiv_6 3) \Rightarrow n^2 + 2n + 10 \equiv_{12} 1$.	DT, discharge for 1 [(1) – (12) unusable].
14.	$(\forall n)[(n \in \mathbb{N} \wedge n \equiv_6 3) \Rightarrow n^2 + 2n + 10 \equiv_{12} 1]$.	IU

Problem 12

Prove $(\forall n)[(n \in \mathbb{N} \land n \equiv_5 2) \Rightarrow (2 \nmid n \lor n^2 \equiv_{20} 4)]$.

[Hint: Remember that $(X \lor Y) \Leftrightarrow (\neg X \Rightarrow Y)$. Use DT twice in this proof.]

Solution 12

1.	Assume $n \in \mathbb{N}$ and $n \equiv_5 2$.	Dischargeable Hypothesis
2.	Then $5 \mid (n-2)$.	
3.	So $n - 2 = 5 \cdot P$ for some $P \in \mathbb{N}$.	
4.	$n = 5 \cdot P + 2$.	
5.	$n^2 = (5 \cdot P + 2)^2 = 25P^2 + 20P + 4$.	
6.	$n^2 = 5(5P^2 + 4P) + 4$.	
7.	$n^2 \equiv_{20} 4$.	
8.	Let $X = 2 \nmid n$ and $Y = n^2 \equiv_{20} 4$.	
9.	$(X \lor Y) \Leftrightarrow (\neg X \Rightarrow Y)$.	Tautology
10.	Assume $2 \mid n$.	Dischargeable Hypothesis
11.	Then $n = 2k$ for some $k \in \mathbb{N}$.	
12.	But $n \equiv_5 2$ by assumption, so n must take the form $n = 5P + 2$.	
13.	Using step 7, $n^2 \equiv_{20} 4$.	
14.	Thus, $2 \mid n \Rightarrow n^2 \equiv_{20} 4$.	
15.	$2 \nmid n \lor n^2 \equiv_{20} 4$.	Tautology from 9
16.	$(n \in \mathbb{N} \land n \equiv_5 2) \Rightarrow (2 \nmid n \lor n^2 \equiv_{20} 4)$.	DT, discharge for 1, [(1) – (15) unusable].
17.	$(\forall n)[(n \in \mathbb{N} \land n \equiv_5 2) \Rightarrow (2 \nmid n \lor n^2 \equiv_{20} 4)]$.	UI