Abstract Linear Algebra: Homework 3

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Problem 1. Let V be a n-dimensional vector space. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subset V$ is a spanning set of V, i.e. $\mathrm{Span}(S) = V$. Prove that S is a basis of V.

Solution. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and suppose that the vectors in S are not linearly independent. Then, there exists a nontrivial linear combination of the vectors in S that equals the zero vector. That is, there exist scalars $c_1, c_2, \cdots, c_n \in \mathbb{F}$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$

where not all c_i are zero.

If S is not linearly independent, then at least one vector in S can be written as a linear combination of the other vectors in S. This would imply that the number of linearly independent vectors in S is strictly less than n.

However, $\dim(V) = n$, and the number of linearly independent vectors in a spanning set cannot be less than n (because a spanning set must contain at least n linearly independent vectors to span an n-dimensional vector space).

This contradiction shows that our assumption that S is not linearly independent is false. Hence S must be linearly independent. □

Problem 2. Consider $V = \mathbb{R}^{n\times n}$ and let $S = \{A \in V \mid \mathrm{Tr}(A) = 0\}$.

(i) Prove that S is a subspace of V.

(ii) Find a basis for S. Make sure to justify that the set you give is a basis.

Solution to (i). The zero matrix $0 \in \mathbb{R}^{n\times n}$ has all entries equal to zero. Its trace is $\mathrm{Tr}(0) = 0$. Thus, $0 \in S$.

Let $A, B \in S$ and $c \in \mathbb{F}$. Then, by definition of S, $\mathrm{Tr}(A) = 0$ and $\mathrm{Tr}(B) = 0$. Consider the matrix $cA + B$. The trace of $cA + B$ is

$$\mathrm{Tr}(cA + B) = \mathrm{Tr}(cA) + \mathrm{Tr}(B) = c\mathrm{Tr}(A) + \mathrm{Tr}(B) = c(0) + 0 = 0.$$

Thus, $cA + B \in S$. Therefore, S is closed under scalar multiplication. Therefore, S is a subspace of V. □

Solution to (ii). The vector space $V = \mathbb{R}^{n\times n}$ has dimension n^2, since it consists of $n \times n$ matrices with n^2 independent entries.

The subspace $S \subset V$ imposes one linear condition on the matrices in V, the trace must be zero. The trace of a matrix $A = (a_{ij})$ is

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

This condition restricts the diagonal entries $a_{11}, a_{22}, a_{33}, \cdots, a_{nn}$ such that their sum is zero, reducing the degrees of freedom by 1. Thus $\dim(S) = n^2 - 1$.

To construct a basis for S, we need to find $n^2 - 1$ linearly independent matrices in S. We can construct these matrices by considering the following structure

(i) For each pair (i, j) with $i \neq j$, define E_{ij} to be the matrix with a 1 in the (i, j)-th entry and 0 elsewhere. These matrices are clearly linearly independent and satisfy $\mathrm{Tr}(E_{ij}) = 0$ since all diagonal entries are 0.

(ii) For the diagonal entries, we require matrices such that their trace is 0. We can construct $n - 1$ linearly independent matrices of this type by defining

$$D_k = E_{kk} - E_{nn}, \quad \text{for } k = 1, 2, \cdots, n - 1,$$

where E_{kk} is the matrix with a 1 in the (k, k)-entry and 0 elsewhere. The trace of D_k is

$$\mathrm{Tr}(D_k) = \mathrm{Tr}(E_{kk}) - \mathrm{Tr}(E_{nn}) = 1 - 1 = 0.$$

The total number of matrices in this set is $(n^2 - n) + (n - 1) = n^2 - 1$, which matches $\dim(S)$. The matrices E_{ij} and D_k are constructed to be linearly independent, since each matrix has a unique pattern of nonzero entries. Any $A \in S$ can be written as a linear combination of E_{ij} and D_k. For the diagonal entries of Aj, we use D_k to ensure the trace is 0. For the off-diagonal entries, we use E_{ij}. Therefore, the set of matrices $B = \{E_{ij} \mid i \neq j\} \cup \{D_k \mid k = 1, 2, \cdots, n-1\}$ is a basis for S and $\text{Span}(B) = S$. $\square$

Problem 3. Let W_1 and W_2 be subspaces of a vector space V. Let $\dim(W_1) = m$ and $\dim(W_2) = p$. Define $W_1 + W_2 = \{x_1 + x_2 \mid x_1 \in W_1, x_2 \in W_2\}$. Prove that

(i) $W_1 + W_2$ is a subspace of V.

(ii) $\dim(W_1 + W_2) = m + p - \dim(W_1 \cap W_2)$.

Solution to (i). The zero vector $\mathbf{0} \in V$ are in both $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$, as they are both subspaces. Thus, $\mathbf{0} \in W_1 + W_2$.

Let $\mathbf{u}, \mathbf{v} \in W_1 + W_2$ and $c \in \mathbb{F}$. Then, by definition of $W_1 + W_2$, there exists $\mathbf{x}_1, \mathbf{x}_2 \in W_1$ and $\mathbf{y}_1, \mathbf{y}_2 \in W_2$ such that $\mathbf{u} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{v} = \mathbf{x}_2 + \mathbf{y}_2$. Now, consider $c\mathbf{u} + \mathbf{v} = c(\mathbf{x}_1 + \mathbf{y}_1) + (\mathbf{x}_2 + \mathbf{y}_2) = (c\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + c\mathbf{y}_2)$. Since W_1 and W_2 are subspaces, $c\mathbf{x}_1 + \mathbf{x}_2 \in W_1$ and $\mathbf{y}_1 + c\mathbf{y}_2 \in W_2$. Thus, $\mathbf{u} + \mathbf{v} \in W_1 + W_2$. Therefore, $W_1 + W_2$ is a subspace of V. $\square$

Solution to (ii). The intersection $W_1 \cap W_2$ is also a subspace of V, and by definition, any element in $W_1 \cap W_2$ belongs to both W_1 and W_2. Let $\dim(W_1 \cap W_2) = k$, and let $\{z_1, z_2, \cdots, z_k\}$ be a basis for $W_1 \cap W_2$. Extend $\{z_1, z_2, \cdots, z_k\}$ to a basis of W_1. Let $\{z_1, z_2, \cdots, z_k, u_1, u_2, \cdots, u_{m-k}\}$ be a basis for W_1, where $m = \dim(W_1)$. Similarly, extend $\{z_1, z_2, \cdots, z_k\}$ to a basis of W_2. Let $\{z_1, z_2, \cdots, z_k, v_1, v_2, \cdots, v_{p-k}\}$ be a basis for W_2, where $p = \dim(W_2)$.

To construct a basis for $W_1 + W_2$, consider the union of the basis elements of W_1 and W_2.

(i) Start with the $m - k$ additional basis vectors from W_1, $\{u_1, u_2, \cdots, u_{m-k}\}$, which are linearly independent and not in W_2.

(ii) Add the $p - k$ additional basis vectors from W_2, $\{v_1, v_2, \cdots, v_{p-k}\}$, which are linearly independent and not in W_1.

(iii) Include the k basis vectors from $W_1 \cap W_2$, $\{z_1, z_2, \cdots, z_k\}$.

Thus, a basis for $W_1 + W_2$ is given by $\{z_1, z_2, \cdots, z_k, u_1, u_2, \cdots, u_{m-k}, v_1, v_2, \cdots, v_{p-k}\}$. The total number of basis vectors in $W_1 + W_2$ is $k + (m - k) + (p - k) = m + p - k$. Since $k = \dim(W_1 \cap W_2)$, we have $\dim(W_1 + W_2) = m + p - \dim(W_1 \cap W_2)$. Therefore, $W_1 + W_2$ is a subspace of V with dimension $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. $\square$

Problem 4. Consider the following subspaces of $\mathbb{R}^{2\times 2}$,

$$W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in \mathbb{R}^{2\times 2}, a, b, c \in \mathbb{R} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \in \mathbb{R}^{2\times 2}, a, b \in \mathbb{R} \right\}.$$

Compute the dimension of the subspace $W_1 + W_2$. Explain your answer. (Note: the definition of $W_1 + W_2$ is given in Problem 3).

Solution. Every matrix in W_1 can be written as

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $\dim(W_1) = 3$.

Every matrix in W_2 can be written as

$$a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $\dim(W_2) = 2$.

For a matrix $A \in W_1 + W_2$, it must satisfy both forms of W_1 and W_2, meaning

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix}.$$

Therefore, we have $a = b$, $b = a$, $c = -a$, and $a = b$. From these equations, we have that $a = b$ and $c = -a$. Thus, any matrix in $W_1 \cap W_2$ is of the form

$$\begin{pmatrix} a & a \\ -a & a \end{pmatrix}.$$

This means that $W_1 \cap W_2$ is spanned by

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Therefore, $\dim(W_1 \cap W_2) = 1$. Then, by Problem 3, we have

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 3 + 2 - 1 = 4. \qquad \square$$

Problem 5. Show that the polynomials 2, $1 + t$, $t + t^2$ form a basis for $\mathbb{P}^2(\mathbb{R})$. Then find the coordinate of $3 + t + 2t^2$ in this basis.

Solution. A set of vectors (or functions) is linearly independent if the only solution to a linear combination equaling zero is the trivial solution. Suppose $c_1(2) + c_2(1 + t) + c_3(t + t^2) = 0$. Expanding and grouping, we have $(2c_1 + c_2) + (c_2 + c_3)t + c_3t^2 = 0$. For this polynomial to be zero, the coefficients of each power of t must be zero. This gives us the system of equations

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0. \end{aligned}$$

From $c_3 = 0$, substitute into $c_2 + c_3 = 0$ to get $c_2 = 0$. Substitute $c_2 = 0$ into $2c_1 + c_2 = 0$ to get $c_1 = 0$. Thus, the only solution is the trivial solution, $c_1 = c_2 = c_3 = 0$, giving us $\{2, 1 + t, t + t^2\}$. So the set is linearly independent.

The space $\mathbb{P}(\mathbb{R}^2)$ consists of all polynomials of degree at most 2. Since $\{2, 1+t, t+t^2\}$ contains 3 linearly independent polynomials and $\dim(\mathbb{P}(\mathbb{R}^2)) = 3$, the set $\{2, 1+t, t+t^2\}$ spans $\mathbb{P}(\mathbb{R}^2))$. Therefore, $\{2, 1+t, t+t^2\}$ is a basis for $\mathbb{P}(\mathbb{R}^2)$.

The coordinate vector of $3 + t + 2t^2$ relative to the basis $\{2, 1 + t, t + t^2\}$ is the unique $(a, b, c) \in \mathbb{R}^3$ such that

$$a(2) + b(1 + t) + c(t + t^2) = 3 + t + 2t^2.$$

Solving the system of equations gives us $(a, b, c) = (2, -1, 2)$. $\square$

Problem 6. Let $V = \mathbb{R}^\infty = \{(a_1, a_2, \cdots) \mid a_1, a_2, \cdots \mathbb{R}\}$. Define $T : V \to V$ by

$$T((a_1, a_2, a_3, \cdots)) = (a_2, a_3, \cdots).$$

(i) Prove that T is a linear transformation on V.

(ii) Prove that T is onto, but not one-to-one.

Solution to (i). Let $\mathbf{a} = (a_1, a_2, \cdots) \in V$ and $\mathbf{b} = (b_1, b_2, \cdots) \in V$. Then, $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \cdots)$. Then,

$$T(\mathbf{a} + \mathbf{b}) = (a_2 + b_2, a_3 + b_3, \cdots) = (a_2, a_3, \cdots) + (b_2, b_3, \cdots) = T(\mathbf{a}) + T(\mathbf{b}).$$

Let $c \in \mathbb{F}$. Then,

$$T(c\mathbf{a}) = (ca_2, ca_3, \cdots) = c(a_2, a_3, \cdots) = cT(\mathbf{a}).$$

Therefore, T is a linear transformation on V. □

Solution to (ii). Let $\mathbf{w} = (w_1, w_2, w_3, \cdots)$ be any vector in V. Let $v_2 = w_1$, $v_3 = w_2$, $\cdots$. Thus, we can choose any v_1 to be any real number. Let $v_1 = 0$. Then, the vector $\mathbf{v} = (v_1, v_2, v_3, \cdots) = (0, w_2, w_3, \cdots)$ that satisfies $T(\mathbf{v}) = \mathbf{w}$. Since we can construct a pre-image for any $\mathbf{w} \in V$, it follows that T is onto.

Consider the two vectors $\mathbf{v} = (1, 0, 0, \cdots)$ and $\mathbf{w} = (0, 0, 0, \cdots)$. Then,

$$T(\mathbf{v}) = T((1, 0, 0, \cdots)) = (0, 0, 0, \cdots) \quad \text{and} \quad T(\mathbf{w}) = T((0, 0, 0, \cdots)) = (0, 0, 0, \cdots).$$

Clearly, $T(\mathbf{v}) = T(\mathbf{w})$, but $\mathbf{v} \neq \mathbf{w}$. Therefore, T is not one-to-one. □

Problem 7. Let $V = \mathbb{R}^\infty = \{(a_1, a_2, \cdots) \mid a_1, a_2, \cdots \mathbb{R}\}$. Define $T : V \to V$ by

$$T((a_1, a_2, a_3, \cdots)) = (0, a_1, a_2, \cdots).$$

(i) Prove that T is a linear transformation on V.

(ii) Prove that T is one-to-one, but not onto.

Solution to (i). Let $\mathbf{a} = (a_1, a_2, \cdots) \in V$ and $\mathbf{b} = (b_1, b_2, \cdots) \in V$. Then, $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \cdots)$. Then,

$$T(\mathbf{a} + \mathbf{b}) = (0, a_1 + b_1, a_2 + b_2, \cdots) = (0, a_1, a_2, \cdots) + (0, b_1, b_2, \cdots) = T(\mathbf{a}) + T(\mathbf{b}).$$

Let $c \in \mathbb{F}$. Then,

$$T(c\mathbf{a}) = (0, ca_1, ca_2, \cdots) = c(0, a_1, a_2, \cdots) = cT(\mathbf{a}).$$

Therefore, T is a linear transformation on V. □

Solution to (ii). Let $\mathbf{a} = (a_1, a_2, a_3, \cdots)$ and $\mathbf{b} = (b_1, b_2, b_3, \cdots)$ and assume $T(\mathbf{a}) = T(\mathbf{b})$. Then,

$$T(\mathbf{a}) = T(\mathbf{b}) \Rightarrow (0, a_1, a_2, \cdots) = (0, b_1, b_2, \cdots).$$

From this, we have $a_1 = b_1$, $a_2 = b_2$, and so on. Thus, $\mathbf{a} = \mathbf{b}$. Therefore, T is one-to-one.

Consider the sequence $\mathbf{w} = (1, 0, 0, 0, \cdots) \in V$. We need to check if there exists $\mathbf{v} = (a_1, a_2, a_3, \cdots) \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Using the definition of T, we have $T(\mathbf{v}) = (0, a_1, a_2, \cdots)$. For this to equal $\mathbf{w}$, we must have $0 = 1$, which is a contradiction. Therefore, T is not onto. □

Problem 8. Let V and W be vector spaces over $\mathbb{F}$. Let $\mathcal{L}(V, W)$ be the set of all linear transformations from V to W. For any $T, U \in \mathcal{L}(V, W)$, define $T + U$ by

$$(\forall \mathbf{x} \in V)[(T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x})].$$

For any $T \in \mathcal{L}(V, W)$ and $c \in \mathbb{F}$, define cT by

$$(\forall \mathbf{x} \in V)[(cT)(\mathbf{x}) = cT(\mathbf{x})].$$

Prove that $\mathcal{L}(V, W)$ with the above addition and scalar multiplication is a vector space over $\mathbb{F}$.

Solution. Let $T, U \in \mathcal{L}(V, W)$. For each $\mathbf{x} \in V$, we have

$$(T+U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x}).$$

Since $T(\mathbf{x})$ and $U(\mathbf{x})$ are in W, their sum is also in W. Therefore, $T + U \in \mathcal{L}(V, W)$. Therefore, $\mathcal{L}(V, W)$ is closed under addition.

Let $T \in \mathcal{L}(V, W)$ and $c \in \mathbb{F}$. For each $\mathbf{x} \in V$, we have

$$(cT)(\mathbf{x}) = cT(\mathbf{x}).$$

Since $T(\mathbf{x})$ is in W and W is a vector space, $cT(\mathbf{x})$ is also in W. Hence, $cT \in \mathcal{L}(V, W)$. Therefore, $\mathcal{L}(V, W)$ is closed under scalar multiplication.

Define $0 \in \mathcal{L}(V, W)$ by $0(\mathbf{x}) = 0_W$ for all $\mathbf{x} \in V$, where 0_W is the zero vector in W. For all $\mathbf{x} \in V$, we have

$$(T+0)(\mathbf{x}) = T(\mathbf{x}) + 0(\mathbf{x}) = T(\mathbf{x}) + 0_W = T(\mathbf{x}).$$

Therefore, $T + 0 = T$. Similarly, we have $0 + T = T$. Thus, 0 is the additive identity in $\mathcal{L}(V, W)$.

For any $T \in \mathcal{L}(V, W)$, define $-T \in \mathcal{L}(V, W)$ by $(-T)(\mathbf{x}) = -T(\mathbf{x})$ for all $\mathbf{x} \in V$. For all $\mathbf{x} \in V$, we have

$$(T + (-T))(\mathbf{x}) = T(\mathbf{x}) + (-T)(\mathbf{x}) = T(\mathbf{x}) + (-T(\mathbf{x})) = 0_W.$$

Therefore, $T + (-T) = 0$. Similarly, we have $(-T) + T = 0$. Thus, $-T$ is the additive inverse of T.

Let $T, U, V \in \mathcal{L}(V, W)$. For all $\mathbf{x} \in V$, we have

$$((T+U)+V)(\mathbf{x}) = (T+U)(\mathbf{x}) + V(\mathbf{x}) = (T(\mathbf{x}) + U(\mathbf{x})) + V(\mathbf{x}) = T(\mathbf{x}) + (U(\mathbf{x}) + V(\mathbf{x})) = T(\mathbf{x}) + (U+V)(\mathbf{x}).$$

Therefore, $(T + U) + V = T + (U + V)$. Thus, $\mathcal{L}(V, W)$ is associative with respect to addition.

Let $T, U \in \mathcal{L}(V, W)$. For all $\mathbf{x} \in V$, we have

$$(T+U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x}) = U(\mathbf{x}) + T(\mathbf{x}) = (U+T)(\mathbf{x}).$$

Therefore, $T + U = U + T$. Thus, $\mathcal{L}(V, W)$ is commutative with respect to addition.

Let $T, U \in \mathcal{L}(V, W)$ and $c, d \in \mathbb{F}$. For all $\mathbf{x} \in V$, we have

$$(c(dT))(\mathbf{x}) = c(dT(\mathbf{x})) = cdT(\mathbf{x}) = ((cd)T)(\mathbf{x}).$$

Therefore, $c(dT) = (cd)T$. Thus, $\mathcal{L}(V, W)$ is closed under scalar multiplication.

Let $T, U \in \mathcal{L}(V, W)$ and $c \in \mathbb{F}$. For all $\mathbf{x} \in V$, we have

$$(c(T+U))(\mathbf{x}) = c(T(\mathbf{x}) + U(\mathbf{x})) = cT(\mathbf{x}) + cU(\mathbf{x}) = (cT + cU)(\mathbf{x}).$$

Therefore, $c(T + U) = cT + cU$. Thus, $\mathcal{L}(V, W)$ is distributive with respect to scalar multiplication.

Let $T \in \mathcal{L}(V, W)$ and $c, d \in \mathbb{F}$. For all $\mathbf{x} \in V$, we have

$$((c+d)T)(\mathbf{x}) = (c+d)T(\mathbf{x}) = cT(\mathbf{x}) + dT(\mathbf{x}) = (cT + dT)(\mathbf{x}).$$

Therefore, $(c + d)T = cT + dT$. Thus, $\mathcal{L}(V, W)$ is distributive with respect to scalar multiplication.

Let $T \in \mathcal{L}(V, W)$. For all $\mathbf{x} \in V$, we have

$$(1T)(\mathbf{x}) = 1T(\mathbf{x}) = T(\mathbf{x}).$$

Therefore, $1T = T$. Thus, $\mathcal{L}(V, W)$ contains a multiplicative identity.

Therefore, $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$. $\square$

Problem 9. True or False. (No explanation needed)

(i) If S is a linear dependent set, then each vector in S is a linear combination of other vectors in S.

(ii) Any set containing the zero vector is a linearly dependent.

(iii) Subset of linearly independent set is linearly independent.

(iv) Let V be a vector space. Let $W \subseteq V$ be a subspace with $\dim(W) = \dim(V)$. Then $W = V$.

Solution to (i). False, a set being linearly dependent means that there exists at least one non-trivial linear combination of the vectors in S that equals the zero vector. However, this does not necessarily mean that every vector in the set is a linear combination of the others. $\square$

Solution to (ii). True, if a set contains the zero vector, then you can form a non-trivial linear combination where the zero vector is involved, and the result is the zero vector, which makes the set linearly dependent. $\square$

Solution to (iii). True, a subset of a linearly independent set will be linearly independent. $\square$

Solution to (iv). True, if a subspace W of a vector space V has the same dimension as V, then W must span V, meaning $W = V$. This is because a subspace of a vector space cannot have a greater dimension than the vector space itself. $\square$