

Mathematical Image Modeling: Homework 3

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Problem 1. Show that $|e^{i\theta} - 1| \leq \theta$ for $\theta \in \mathbb{R}$. (*Hint:* You can use the fundamental theorem of calculus.)

Solution. We have

$$e^{i\theta} - 1 = \int_0^\theta \frac{d}{dt} e^{it} dt = \int_0^\theta i e^{it} dt.$$

Thus,

$$|e^{i\theta} - 1| = \left| \int_0^\theta i e^{it} dt \right| \leq \int_0^\theta |i e^{it}| dt \leq \int_0^\theta 1 dt = \theta. \quad \square$$

Problem 2. Prove that

$$(f * g)'(x) = (f' * g)(x),$$

where ' denotes derivative and * denotes convolution. Just treat this as a formal identity, i.e. assume everything converges nicely and you can pass derivatives through the integral sign.

This identity has an important consequence, namely: “the convolution of f and g is as smooth as the smoother of f and g ”.

Solution. We have

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy.$$

Computing the derivative, we have

$$(f * g)'(x) = \frac{d}{dx} \int_{\mathbb{R}} f(x - y)g(y) dy = \int_{\mathbb{R}} \frac{d}{dx} f(x - y)g(y) dy = \int_{\mathbb{R}} f'(x - y)g(y) dy = (f' * g)(x). \quad \square$$

Problem 3. Prove that if $f(x) = 0$ for $|x| > R$ and $g(x) = 0$ for $|x| > T$, then $(f * g)(x) = 0$ for $|x| > R + T$.

Solution. We have

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy.$$

If $|x| > R + T$, then for any y such that $g(y) \neq 0$, we have $|y| \leq T$. Thus,

$$|x - y| \geq |x| - |y| > (R + T) - T = R.$$

Since $f(x - y) = 0$ for $|x - y| > R$, it follows that $f(x - y) = 0$ whenever $g(y) \neq 0$. Therefore, the integrand $f(x - y)g(y)$ is zero for all y , and hence

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy = 0. \quad \square$$

Problem 4. Show that if $g(x) = f(x + y)$ for some $y \in \mathbb{R}$, then $\hat{g}(\xi) = e^{iy\xi} \hat{f}(\xi)$.

Solution. We have

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-ix\xi} dx = \int_{\mathbb{R}} f(x + y) e^{-ix\xi} dx.$$

Making the substitution $u = x + y$, we have $dx = du$ and thus

$$\hat{g}(\xi) = \int_{\mathbb{R}} f(u) e^{-i(u-y)\xi} du = e^{iy\xi} \int_{\mathbb{R}} f(u) e^{-iu\xi} du = e^{iy\xi} \hat{f}(\xi). \quad \square$$

Problem 5. Show that limits are unique in a metric space. That is, if (X, d) is a metric space and $\{x_n\}$ is a sequence satisfying $x_n \rightarrow x \in X$ and $x_n \rightarrow y \in X$, then $x = y$.

Solution. Since $\{x_n\} \rightarrow x$, we have for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $d(x_n, x) < \varepsilon/2$. Similarly, since $\{x_n\} \rightarrow y$, we have for every $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(x_n, y) < \varepsilon/2$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $d(x, y) = 0$. By the properties of a metric, this implies that $x = y$. \square

Problem 6. (optional for 410, required for 510). Taking the following fact for granted, complete the proof of the Riemann–Lebesgue lemma (i.e. $f \in L^1 \Rightarrow \hat{f} \in C_0$):

Fact: For any $f \in L^1$ and any $\varepsilon > 0$, there exists a function $g \in L^1$ satisfying (i) $xg \in L^1$ and $g' \in L^1$ and (ii) $\|f - g\|_{L^1} < \varepsilon$.

Recall that in class we proved that for $g \in L^1$ satisfying (i), we have $\hat{g} \in C_0$.

Solution. Let $f \in L^1$ and let $\varepsilon > 0$. By the given fact, there exists $g \in L^1$ such that $xg \in L^1$, $g' \in L^1$, and $\|f - g\|_{L^1} < \varepsilon$. Since we have already established that for such a function g , $\hat{g} \in C_0$, it follows that \hat{g} is continuous and vanishes at infinity.

Now, we need to show that $\hat{f} \in C_0$. We can write

$$\hat{f}(\xi) = \hat{g}(\xi) + (\hat{f}(\xi) - \hat{g}(\xi)).$$

Using the properties of the Fourier transform, we have

$$|\hat{f}(\xi) - \hat{g}(\xi)| = \left| \int_{\mathbb{R}} (f(x) - g(x)) e^{-ix\xi} dx \right| \leq \int_{\mathbb{R}} |f(x) - g(x)| dx = \|f - g\|_{L^1} < \varepsilon.$$

Since $\hat{g}(\xi)$ is continuous and vanishes at infinity, for any $\delta > 0$, there exists $M > 0$ such that for all $|\xi| > M$, $|\hat{g}(\xi)| < \delta$. Therefore, for all $|\xi| > M$, we have

$$|\hat{f}(\xi)| \leq |\hat{g}(\xi)| + |\hat{f}(\xi) - \hat{g}(\xi)| < \delta + \varepsilon.$$

Since ε and δ were arbitrary, this shows that $\hat{f}(\xi)$ also vanishes at infinity. Thus, we conclude that $\hat{f} \in C_0$. \square