Several-Variab Calc II: Homework 1

Due on January 14, 2025 at 9:00

Jennifer Thorenson
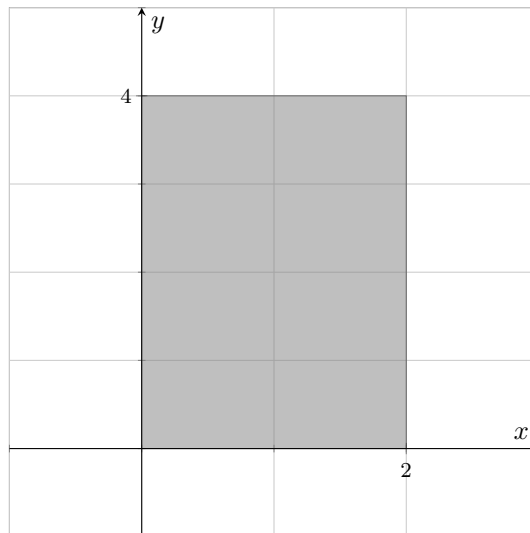
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Problem 1. Evaluate the following double integrals over the given regions.

(i) $\iint_R ye^{xy} \, dA$ where $R = [0, 2] \times [0, 4]$.

(ii) $\iint_D y\sqrt{x^2 - y^2} \, dA$ where D is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 2)$.

(iii) $\iint_D 2xy^3 \, dA$ where D is the region bounded by $y = x$, $y = \frac{1}{x}$, and $y = 2$.

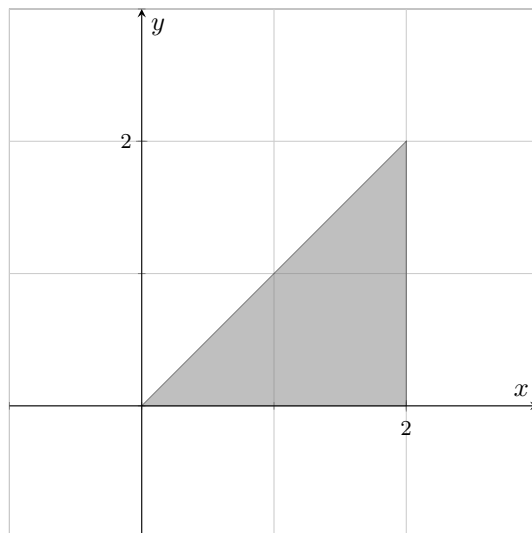*Solution to (i).* Graphing the bounds gives us the following figure.

Therefore, we get the bounds $0 \leq x \leq 2$ and $0 \leq y \leq 4$. Expanding the double integral into an iterated integral and evaluating gives us

$$
\begin{aligned}
\iint_R ye^{xy} \, dA &= \int_0^4 \int_0^2 ye^{xy} \, dx \, dy \\
&= \int_0^4 [e^{xy}]_0^2 \, dy \\
&= \int_0^4 e^{2y} - 1 \, dy \\
&= \left[\frac{e^{2y}}{2} - y \right]_0^4 \\
&= \frac{1}{2}(e^8 - 9). \qquad \square
\end{aligned}
$$

Solution to (ii). Graphing the bounds gives us the following figure.

Therefore, we get the bounds $0 \leq x \leq 2$ and $0 \leq y \leq x$. Expanding the double integral into an iterated integral gives us
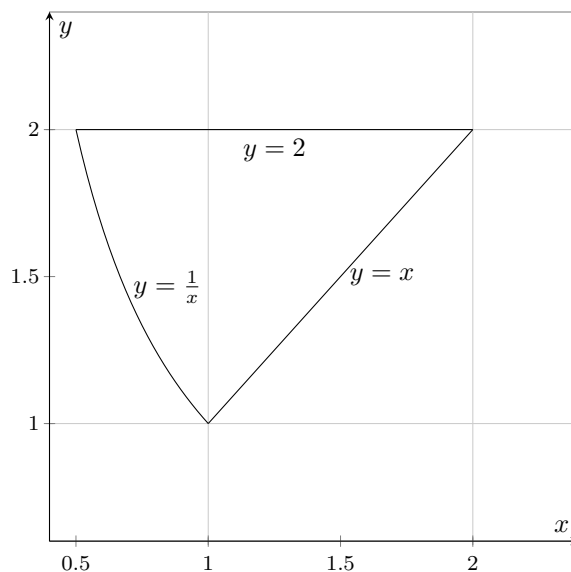
$$\iint_D y\sqrt{x^2 - y^2}\, \mathrm{d}A = \int_0^2 \int_0^x y\sqrt{x^2 - y^2}\, \mathrm{d}y\, \mathrm{d}x.$$

Using a u-substitution of $u = y^2$, we have $\mathrm{d}u = 2y\, \mathrm{d}y$. The bounds go from $0 \mapsto 0$ and $x \mapsto x^2$. This simplifies the integral to

$$\int_0^2 \int_0^x y\sqrt{x^2 - y^2}\, \mathrm{d}y\, \mathrm{d}x = \int_0^2 \int_0^{x^2} \frac{1}{2}\sqrt{x^2 - u}\, \mathrm{d}u\, \mathrm{d}x$$

$$= \int_0^2 \int_0^{x^2} \frac{1}{2}(x^2 - u)^{1/2}\, \mathrm{d}u\, \mathrm{d}x$$

$$= \int_0^2 \frac{1}{2}\left[-\frac{(x^2 - u)^{3/2}}{3/2} \right]_0^{x^2} \mathrm{d}x$$

$$= \int_0^2 \frac{1}{\not{2}} \cdot \frac{\not{2}}{3}\left[-\frac{(x^2 - u)^{3/2}}{3/2} \right]_0^{x^2} \mathrm{d}x$$

$$= \int_0^2 -\frac{1}{3}\left[(x^2 - 0)^{3/2} \right] - \left[(x^2 - 0)^{3/2} \right]\, \mathrm{d}x$$

$$= \int_0^2 \frac{1}{3}\left(x^2 \right)^{3/2} \mathrm{d}x$$

$$= \int_0^2 \frac{1}{3}x^3\, \mathrm{d}x$$

$$= \frac{1}{3} \cdot \left.\frac{x^4}{4}\right|_0^2$$

$$= \frac{1}{3} \cdot \left[\frac{2^4}{4} - \frac{0^4}{4} \right]$$

$$= \frac{1}{3} \cdot \frac{16}{4}$$

$$= \frac{4}{3}. \qquad \qquad \square$$

Solution to (iii). Graphing the bounds gives us the following figure.

Therefore, we get the bounds $1 \le y \le 2$ and $y \le x \le \frac{1}{y}$. Expanding the double integral into an iterated integral and evaluating gives us
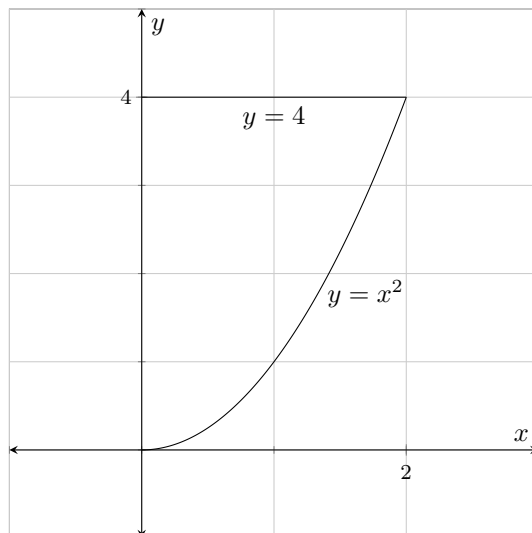
$$
\iint_D 2xy^3 \, \mathrm{d}A = \int_1^2 \int_{1/y}^y 2xy^3 \, \mathrm{d}x \, \mathrm{d}y
$$
$$
= \int_1^2 y^2 y_3 - \frac{y^3}{y^2} \, \mathrm{d}y
$$
$$
= \int_1^2 y^5 - y \, \mathrm{d}y
$$
$$
= \left. \frac{y^6}{6} - \frac{y^2}{2} \right|_1^2
$$
$$
= \frac{2^6}{6} - 2 - \frac{1}{6} + \frac{1}{2}
$$
$$
= 9. \qquad\qquad \square
$$

Problem 2. By changing the order of integration, evaluate the integral

$$
\int_0^2 \int_{x^2}^4 \sqrt{y} \sin(y) \, \mathrm{d}y \, \mathrm{d}x.
$$

Solution. Graphing the bounds gives us the following figure.

Solving the equation $y = x^2$ gives us $x = \pm\sqrt{y}$. But we only want the positive root. Therefore, we get the bounds $0 \le y \le 4$ and $0 \le x \le \sqrt{y}$. Expanding the double integral into an iterated integral gives us

$$\iint_R \sqrt{y}\sin(y)\,\mathrm{d}A = \int_0^4 \int_0^{\sqrt{y}} \sqrt{y}\sin(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^4 \sqrt{y}\sin(y) \cdot x\big|_0^{\sqrt{y}}\,\mathrm{d}y$$

$$= \int_0^4 y\sin(y)\,\mathrm{d}y.$$

Using integration by parts gives us

$$\left.\begin{array}{ll} u = y & \mathrm{d}v = \sin(y)\,\mathrm{d}y \\ \mathrm{d}u = \mathrm{d}y & v = -\cos(y) \end{array}\right\} \Rightarrow \int_0^4 y\sin(y)\,\mathrm{d}y = \left[-y\cos(y) - \int -\cos(y)\,\mathrm{d}y\right]_0^4$$

$$= \left[\sin(y) - 4\cos(y)\right]_0^4$$

$$= 4\sin(4) - 4\cos(4). \qquad \square$$

Problem 3. Use a double integral to find the volume of the following solids.

(i) The solid that is bounded by the coordinate planes and the plane $2x + 3y + z = 6$.

Note: This solid is a tetrahedron. It can be easily drawn by using the lines of intersection between the slant plane and the coordinate planes.

(ii) The solid enclosed by the parabolic cylinder $y = 2 - x^2$ and the planes $z = 2 - y$, $y = x$, and $z = 0$

Note : Surfaces that do not depend on z are vertical surfaces.

Solution to (i). The points of intersection are

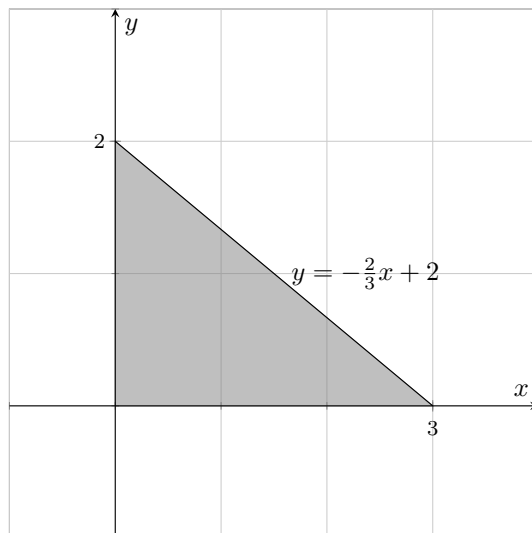$$\text{At } x = 0 = y \Rightarrow z = 6$$

$$\text{At } y = 0 = z \Rightarrow x = 3$$

$$\text{At } z = 0 = x \Rightarrow y = 2.$$

The projection of the plane onto the xy-plane forms a triangle with vertices $(0,0)$, $(3,0)$, and $(0,2)$. Solving for the equation of the line gives us

$$y = -\frac{2}{3}x + 2.$$

Graphing the bounds gives us the following figure.

The equation of the plane can be written as $f(x, y) = 6 - 2x - 3y$. Therefore, we get the bounds for x to be $0 \le x \le 3$ and the bounds for y to be $0 \le y \le -\frac{2}{3}x + 2$. The volume is then and evaluating using Wolfram Alpha to get

$$V = \int_0^3 \int_0^{-\frac{2}{3}x+2} 6 - 2x - 3y \, \mathrm{d}y \, \mathrm{d}x = \int_0^3 6\left[-\frac{2}{3}x + 2\right] - 2\left[-\frac{2}{3}x + 2\right] - \frac{3}{2}\left[-\frac{2}{3} + 2\right]^2 \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^3 12 - 4x - 4x + \frac{4x^2}{3} - \frac{3}{2}\left[4 - \frac{4}{3}x - \frac{4}{3}x + \frac{4}{9}x^2\right] \mathrm{d}x$$

$$= \int_0^3 12 - 4x - 4x + \frac{4x^2}{3} - 6 + 2x + 2x - \frac{72x^2}{78} \, \mathrm{d}x$$

$$= \int_0^3 6 - 4x + \frac{2x^2}{3} \, \mathrm{d}x$$

$$= 6x - \frac{4x^2}{2} + \frac{2x^3}{9} \Big|_0^3 = 18 - 18 + 6 = 6. \qquad \square$$

Solution to (ii). Since $x = y = 2 - x^2$, we get $x^2 + x - 2 = 0$. Solving for x gives us $x = -2$ and $x = 1$. Therefore, the bounds for x are $-2 \le x \le 1$. The bounds for y are $x \le y \le 2 - x^2$. Graphing the bounds gives us the following figure.
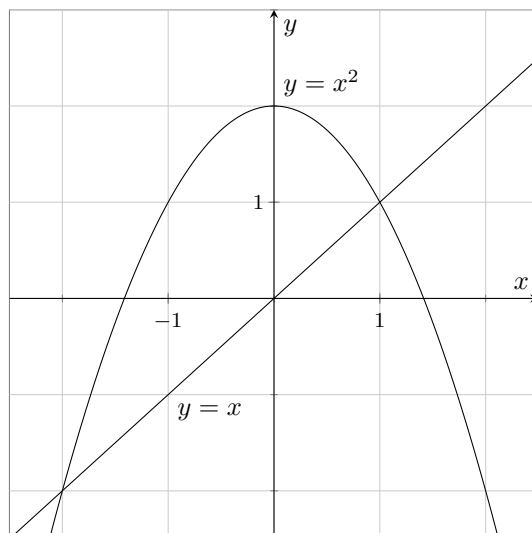
Therefore, expanding the double integral into an iterated integral gives us

$$
V = \int_{-2}^{1} \int_{x}^{2-x^2} 2 - y \, \mathrm{d}y \, \mathrm{d}x = \int_{-2}^{1} 2y - \frac{y^2}{2} \Big|_{x}^{2-x^2} \, \mathrm{d}x
$$

$$
= \int_{-2}^{1} \left[2(2 - x^2) - \frac{(2 - x^2)^2}{2} \right] - \left[2x - \frac{x^2}{2} \right] \, \mathrm{d}x
$$

$$
= \int_{-2}^{1} \left[4 - 2x^2 - \frac{4 - 4x^2 + x^4}{2} \right] - \left[2x - \frac{x^2}{2} \right] \, \mathrm{d}x
$$

$$
= \int_{-2}^{1} 4 - 2x^2 - 2 + 2x^2 - \frac{x^4}{2} - 2x + \frac{x^2}{2} \, \mathrm{d}x
$$

$$
= \int_{-2}^{1} 2 - 2x + \frac{x^2}{2} - \frac{x^4}{2} \, \mathrm{d}x
$$

$$
= 2x - x^2 + \frac{x^3}{6} - \frac{x^5}{10} \Big|_{-2}^{1}
$$

$$
= \left[2 - 1 + \frac{1}{6} - \frac{1}{10} \right] - \left[-4 - 4 - \frac{8}{6} + \frac{32}{10} \right]
$$

$$
= \frac{36}{5}. \qquad \square
$$