1. *This problem will provide another of Cauchy-Schwarz inequality.*

Let V be an inner produce space over $\mathbb{C}$. For any $\mathbf{x}, \mathbf{y} \in V$, define $G = \begin{bmatrix} (\mathbf{x}, \mathbf{x}) & (\mathbf{x}, \mathbf{y}) \\ (\mathbf{y}, \mathbf{x}) & (\mathbf{y}, \mathbf{y}) \end{bmatrix} \in \mathbb{C}^{2\times 2}$.

1). Prove that G is a (Hermitian) positive semi-definite matrix.

2). Prove that eigenvalues of a Hermitian positive semi-definite matrix are all non-negative.

3). Prove the Cauchy-Schwarz inequality, i.e $|(\mathbf{x}, \mathbf{y})| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|$. (Hint: What is the determinant of G? How do we relate determinant of a matrix with its eigenvalues?)

2. *Note this problem gives the formula for the orthogonal projection when a general (not necessarily orthogonal) basis of the subspace is given.*

Consider the standard inner product on $\mathbb{C}^n$. Let $W \subseteq \mathbb{C}^n$ be a subspace. Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ is a basis for W. Denote $B = [\mathbf{w}_1\ \mathbf{w}_2\ \cdots\ \mathbf{w}_m] \in \mathbb{C}^{n\times m}$.

1). Prove that B^*B is (Hermitian) positive definite. (Note B^*B is often referred as the Gramian matrix related to $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$

2). Prove that eigenvalues of a Hermitian positive definite matrix are all positive.

3). Prove that B^*B is invertible.

4). Let $\mathbf{x} \in \mathbb{C}^n$ and let $\mathbf{x}_w$ be the orthogonal projection of $\mathbf{x}$ onto W. Prove that $\mathbf{x}_W = B(B^*B)^{-1}B^*\mathbf{x}$

5). Let $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. By using the formula in Part 4), find the orthogonal projection of $\mathbf{x}_3$ onto the subspace spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$.

3. Find the QR-decomposition for the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$.

4. Let $V = \mathbb{C}^{n\times n}$ with the inner product $(A, B) = \mathrm{Tr}(B^*A)$. Find the orthogonal complement of the subspace of diagonal matrices.

5. Let $A \in \mathbb{C}^{m\times n}$. Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be equipped with the standard inner product. Prove the following statements.

1). $\mathrm{null}(A) = (\mathrm{Range}(A^*))^\perp$.

2). $\mathrm{null}(A^*A) = \mathrm{null}(A)$.

3). $\mathrm{Rank}(A^*A) = \mathrm{Rank}(A) = \mathrm{Rank}(A^*)$.

4). $\mathrm{Range}(A^*A) = \mathrm{Range}(A^*)$.

6. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Let $(\lambda, \mathbf{v})$ be an eigenvalue/eigenvector pair of A. Prove that $(\overline{\lambda}, \mathbf{v})$ is an eigenvalue/eigenvector pair of A^*.

7. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Prove that eigenvectors of A associated with distinct eigenvalues are orthogonal.

8. True or False. (No explanation is needed)

1). Suppose $A \in \mathbb{C}^{n\times n}$. Then $\text{Range}(A^*A) = \text{Range}(A^*) = \text{Range}(A)$.

2). A set of orthonormal vectors must be linearly independent.

3). A set of orthogonal vectors must be linearly independent.

In the statement (4)-(9), V is a finite-dimensional inner product space.

4). Every linear transformation on V has a unique adjoint.

5). For every linear transformation $T : V \to V$ and any given ordered basis B for V, we have $[T^*]_B = ([T]_B)^*$.

6). For any linear transformation T and U on V and scalars a and b, we have

$$(aT + bU)^* = aT^* + bU^*.$$

7). Every self-adjoint linear transformation on V is normal.

8). Linear transformations and their adjoints on V have the same eigenvalues.

9). Linear transformations and their adjoints on V have the same eigenvectors.