

Overview

- Vector space, subspaces, bases and dimensions
- Linear transformation:
 - null space and nullity, range space and rank, dimension theorem
 - One-to-one, onto, isomorphism
 - Matrix representation, composition of linear transformation, change of basis
 - Linear functionals and dual space.
- Inner product space
 - Orthogonal/orthonormal vectors, Gramian matrix
 - Cauchy-Schwarz inequality
 - Gram-Schmidt and QR factorization
 - Orthogonal projections, orthogonal complement
 - Least squares solutions
- Eigenvalues, eigenvectors and eigenspaces of linear transformation.
 - Characteristic polynomials,
 - Diagonalization equivalent conditions
- Spectral Theorem
 - Normal matrix, normal linear transformation.
 - Spectral theorem for normal matrix and normal linear transformation
 - Spectral theorem for Hermitian matrices, self-adjoint linear transformations
 - Singular value decomposition

- Jordan canonical form
 - Jordan canonical form
 - Cayley-Hamilton theorem
 - Generalised eigenvectors, T -invariant spaces.
 - Minimal polynomials
- Application: Exponential of a matrix.

Review on Vectors and matrices

\mathbb{R} = the set of all real numbers, \mathbb{C} = the set of all complex numbers.

Vectors: Notation: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ or \mathbb{C}^n , (i.e. $x_i \in \mathbb{R}$ (or \mathbb{C}))

Algebraic operations: 1). Scalar multiplication: $\forall c \in \mathbb{R}$ (or \mathbb{C}) and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ (or \mathbb{C}^n)

$$c\vec{x} = c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix} \in \mathbb{R}^n \text{ (or } \mathbb{C}^n \text{)}.$$

2). Vector addition: $\forall \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ in \mathbb{R}^n (or \mathbb{C}^n)

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \in \mathbb{R}^n \text{ (or } \mathbb{C}^n \text{)}.$$

Remark: row vectors: $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^{1 \times n}$ (or $\mathbb{C}^{1 \times n}$).

When we say "vectors" here, we mean column vectors.

Matrices: Notation: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$ (or $\mathbb{C}^{m \times n}$)

$= (a_{ij})_{m \times n}$ where $a_{ij} \in \mathbb{R}$ (or \mathbb{C}) is the (i,j) -th entry of A .

Other notations: $A = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}$ (or $\mathbb{C}^{m \times n}$), where $\vec{a}_i \in \mathbb{R}^m$ (or \mathbb{C}^m)

$$= \begin{pmatrix} -\vec{\alpha_1} - \\ -\vec{\alpha_2} - \\ \vdots \\ -\vec{\alpha_m} - \end{pmatrix} \in \mathbb{R}^{m \times n} \text{ (or } \mathbb{C}^{m \times n}), \text{ where } \vec{\alpha_i} \in \mathbb{R}^{1 \times n} \text{ (or } \mathbb{C}^{1 \times n})$$

Block matrices: $A = (A_{ij})$ where A_{ij} is the (i,j) -th block matrix.

Example $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix} \in \mathbb{R}^{3 \times 5}$ is a 2 blocks by 3 blocks matrix

with. $A_{11} = \begin{pmatrix} 1 & 2 \end{pmatrix}$ $A_{12} = \begin{pmatrix} 3 & 4 \end{pmatrix}$ $A_{13} = \begin{pmatrix} 5 \end{pmatrix}$
 $A_{21} = \begin{pmatrix} 6 & 7 \\ 11 & 12 \end{pmatrix}$ $A_{22} = \begin{pmatrix} 8 & 9 \\ 13 & 14 \end{pmatrix}$ $A_{23} = \begin{pmatrix} 10 \\ 15 \end{pmatrix}$

Example $J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{5 \times 5} \rightarrow 2 \text{ blocks} \times 2 \text{ blocks}$

$$J_{11} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad J_{12} = 0 \in \mathbb{R}^{2 \times 3}$$

$$J_{21} = 0 \in \mathbb{R}^{3 \times 2} \quad J_{22} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \in \mathbb{R}^{3 \times 3}$$

$\Rightarrow J$ is a block diagonal matrix.

Examples:

1). Square matrix: $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$)

2). Diagonal matrix: $A = (a_{ij})_{n \times n}$ with $a_{ij} = 0$ for all $i \neq j$.

3). Upper triangular matrix $A = (a_{ij})_{n \times n}$ with $a_{ij} = 0$ for all $i > j$

Algebraic Operations of matrices:

1). Matrix addition: $\forall A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n} \in \mathbb{R}^{m \times n}$ (or $\mathbb{C}^{m \times n}$)

$$A + B = (c_{ij})_{m \times n} \quad \text{where } c_{ij} = a_{ij} + b_{ij}.$$

2). Scalar multiplication: $\forall c \in \mathbb{R}$ (or \mathbb{C}) and $A = (a_{ij})$

$$cA = (ca_{ij})_{m \times n}.$$

3). Matrix Multiplication: $\forall A \in \mathbb{R}^{m \times n}$ (or $\mathbb{C}^{m \times n}$) and $B \in \mathbb{R}^{n \times p}$ (or $\mathbb{C}^{n \times p}$)

$$\Rightarrow AB = (c_{ij})_{m \times p} \in \mathbb{R}^{m \times p} \text{ (or } \mathbb{C}^{m \times p})$$

$$\text{where } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

$$\text{then } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} \quad \text{for all } 1 \leq i \leq m, 1 \leq p \leq j$$

$$= \sum_{k=1}^n a_{ik} b_{kj}$$

using the i -th row of A and j -th column of B .

4). Transpose of a matrix in $\mathbb{R}^{m \times n}$:

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}, \text{ then } A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

Example: $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, then

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

If $A^T A = (C_{ij}) \in \mathbb{R}^{n \times n}$, then $C_{ij} = \underbrace{\text{"the } i\text{-th row of } A^T"}_{A^T} \cdot \underbrace{\text{"the } j\text{-th column of } A"}_A$

$$\begin{aligned} \text{the } i\text{th row of } A^T &\rightarrow \begin{pmatrix} a_{1i} & a_{2i} & \dots & a_{ni} \end{pmatrix} \cdot \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \\ &\quad \uparrow \text{the } j\text{-th column of } A \end{aligned}$$

$$= a_{1i} a_{1j} + a_{2i} a_{2j} + \dots + a_{mi} a_{mj}$$

$$= \sum_{k=1}^m a_{ki} a_{kj}$$

In particular: $C_{ii} = a_{i1} a_{i1} + a_{i2} a_{i2} + \dots + a_{in} a_{in}$

$$= \sum_{k=1}^m a_{ki} a_{ki}$$

$$= \sum_{k=1}^3 a_{ki}^2$$

(This would be helpful for Homework 1, problem 3)

5). Conjugate transpose of a matrix in $\mathbb{C}^{m \times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m \times n}, \text{ then } A^* = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \dots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \dots & \bar{a}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \dots & \bar{a}_{mn} \end{pmatrix} \in \mathbb{C}^{n \times m}$$

(Recall let $z = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$, then $\bar{z} = a - bi$ is the conjugate of z ..

Note. $z \cdot \bar{z} = \bar{z} \cdot z = a^2 + b^2$.)