Funds of Anal I: Homework 7

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Exercise 3.2.4

Let A be a nonempty and bounded above so that $s = \sup(A)$ exists.

(i) Show that $s \in \bar{A}$.

(ii) Can an open set contain its supremum.

Solution 3.2.4

(i) *Proof.* Since every $s - \varepsilon$ has an $a \in A$ with $a > s - \varepsilon$, we can find $a \in V_\varepsilon(s)$, for any $\varepsilon > 0$. This implies that s is a limit point of A. Therefore, $s \in \bar{A}$. □

(ii) No, there doesn't exist any ε-neighborhood of s such that $V_\varepsilon(s) \not\subseteq (s, s+\varepsilon)$.

Exercise 3.2.5

Prove Theorem 3.2.8.

Theorem. *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Solution 3.2.5

Proof. Let $F \subseteq \mathbb{R}$ be closed. Given an arbitrary Cauchy sequence (a_n) contained in F, we know that (a_n) converges to some $a \in \mathbb{R}$. Since F is closed and (a_n) is contained in F, $a \in F$.

Suppose that every Cauchy sequence contained in F converges to a point, l, in F. Since l is a limit point of F, there exists a sequence (a_n) contained in F with $\lim_{n\to\infty}(a_n) = l$. Since (a_n) converges, it must be a Cauchy sequence (by the Cauchy Criterion). Since every Cauchy sequence converges to a limit inside F, we have $l \in F$, meaning that F is closed. $\square$

Exercise 3.2.7

Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(i) Show that the set L is closed.

Solution 3.2.7

(i) *Proof.* Let L' be the set of all limit points of L. Assume $\ell \in L'$. Then, by definition, for every $\varepsilon > 0$, we have

$$L_1 = V_\varepsilon(\ell) \cap (L - \{\ell\}) \neq \emptyset \subseteq L.$$

Then,

$$(\forall y \in L_1)(\forall \varepsilon > 0)[L_2 = V_\delta(y) \cap (A - \{y\}) \neq \emptyset].$$

Now, since ℓ is a limit point of L, for any $\varepsilon > 0$, there exists $y \in L - \{\ell\}$ such that $y \in V_\varepsilon(l)$. By the definition of $y \in L$, there exists $z \in A$, distinct from y, such that $z \in V_\delta(y)$.

Choose $0 < \delta < \varepsilon$ so $V_\delta(x) \subseteq V_\varepsilon(l)$. Since $z \in V_\delta(y) \cap (A - \{y\})$, then $z \in V_\varepsilon(\ell)$ and $z \in A$. So, ℓ is a limit point of A. Therefore, $\ell \in L$ and L is closed. $\square$

Exercise 3.2.11

(i) Prove that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(ii) Does this result about closures extend to infinite unions of sets?

Solution 3.2.11

(i) *Proof.* Let L_a and L_b be the set of limit points for A and B respectively. Let ℓ be a limit point for $A \cup B$. Then, $\ell \in L_a$ or $\ell \in L_b$. So, $\ell \in L_a \cup L_b$. Then, by definition $\overline{A \cup B} = (A \cup B) \cup (L_a \cup L_b) = (A \cup L_a) \cup (B \cup L_b) = \bar{A} \cup \bar{B}$. ☐

(ii) No, the result does not extend to infinite unions of sets. Specifically,

$$\overline{\bigcup_{i \in I} A_i} \neq \bigcup_{i \in I} \bar{A}_i,$$

where $\{A_i\}_{i \in I}$ is a collection of sets, indexed by I.

For a finite union of sets $\overline{A \cup B} = \bar{A} \cup \bar{B}$ holds because any limit point of $A \cup B$ must lie in $\bar{A} \cup \bar{B}$, and the closure distributes over a finite union.

For an infinite union, however, the equality may fail because the closure of an infinite union may include points that are limit points of the entire union, but not limit points of any single set in the collection.

Let $A_n = \{1/n \mid n \in \mathbb{N}\}$ as a counterexample. Then,

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} \cup \{0\} \neq \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} = \bigcup_{n=1}^{\infty} \bar{A}_n.$$

Exercise 3.3.1

Show that if K is compact and nonempty, then $\sup(K)$ and $\inf(K)$ both exist and are elements of K.

Solution 3.3.1

Proof. By Theorem 3.3.4, K is compact if and only if it is closed and bounded. Since $K \neq \emptyset$, it follows that K is bounded above and bounded below. By the Axiom of Completeness, $\sup(K)$ exists because K is bounded above, and $\inf(K)$ exists because K is bounded below. To show $\sup(K) \in K$, note that $\sup(K)$ is a limit point of K (by definition of the least upper bound). Since K is closed, it contains all its limit points, so $\sup(K) \in K$. Similarly, $\inf(K)$ is a limit point of K, and by the closedness of K, $\inf(K) \in K$. $\square$

Exercise 3.3.2

Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(i) $\mathbb{N}$.

(ii) $Q \cap [0, 1]$.

(iii) $S = \{1, {}^1\!/_2, {}^2\!/_3, {}^3\!/_4, {}^4\!/_5, \dots\}$.

Solution 3.3.2

(i) No, since the sequence $x_n = n$ doesn't diverges.

(ii) No, since the sequence $x_n = {}^1\!/\!\sqrt{2} + \frac{1}{n}$ converges to ${}^1\!/\!\sqrt{2} \notin Q \cap [0, 1]$.

(iii) Compact, since, it's bounded, closed, and every sequence converges to the limit value of $1 \in S$.

Exercise 3.3.3

Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Solution 3.3.3

Theorem (Characterization of Compactness in $\mathbb{R}$)**.** *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof.

$\Rightarrow$) Suppose $K \subseteq \mathbb{R}$ is compact. Suppose K isn't bounded. Let (a_n) be a sequence in K. Since K is unbounded, then given any $n \in \mathbb{N}$, we can produce $a_n \in K$ such that $|a_n| > n$. Now, since K is compact, there must exist a subsequence (a_{n_k}) that converges to a limit $a \in K$. But the elements of (a_{n_k}) must satisfy $|a_{n_k}| > n_k$, which implies that (a_{n_k}) is unbounded, meaning it doesn't converge. This gives us a contradiction since every sequence must contain a converging subsequence to be called compact. This implies that K must be bounded.

Now that we know K is bounded, then suppose we are given a sequence (a_n) that's contained in K. By the BWT, there exists a converging subsequence (a_{n_k}) that converges to a limit a. By definition of compactness, $a \in K$. Suppose that (a_n) converges to a limit point. Then, by Theorem 2.5.2, all subsequences converge to the same limit as the original sequence. So, $(a_n) \to x$ and $a \in K$, making a a limit point. Therefore, K is closed.

$\Leftarrow$) Suppose $K \subseteq \mathbb{R}$ is closed and bounded. Let (a_n) be a sequence contained in K. Since K is bounded, by the BWT, there exists a converging subsequence (a_{n_k}) that converges to a limit a. Since K is closed, $a \in K$. Therefore, K is compact. $\square$