Multi-Variable Calculus I: Homework 7

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Problem 1

Use the chain rule to find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ for $u(x, y, z) = x^2 \ln(y^2 + z^2) + e^{-xy^3}$ where $x(s, y) = \frac{t^2}{1 - 3s}$, $y(s, t) = t^2 \cos(3s)$, and $z(s, t) = t^5 s^{-4}$.

Note: Please leave your answer in terms of x, y, z, s, and t.

Solution 1

Let $x_1 = x$, $x_2 = y$, and $x_3 = z$. Then, we have

$$\begin{aligned} \frac{\partial u}{\partial s} &= \sum_{k=3}^{3} \left[\frac{\partial u}{\partial x_k} \cdot \frac{\partial x_k}{\partial s} \right] & \text{and} \quad \frac{\partial u}{\partial t} &= \sum_{k=3}^{3} \left[\frac{\partial u}{\partial x_k} \cdot \frac{\partial x_k}{\partial t} \right] \\ &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s} & \text{and} \quad &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t}. \end{aligned}$$

Performing each one gives us

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x \ln(y^2 + z^2) - y^3 e^{-xy^3} & \frac{\partial u}{\partial y} &= \frac{2x^2 y}{y^2 + z^2} - 3xy^2 e^{-xy^3} & \frac{\partial u}{\partial z} &= \frac{2x^2 z}{y^2 + z^2} \\ \frac{\partial x}{\partial s} &= \frac{3t^2}{(1 - 3s)^2} & \frac{\partial y}{\partial s} &= -3t^2 \sin(3s) & \frac{\partial z}{\partial s} &= -\frac{4t^5}{s^5} \\ \frac{\partial x}{\partial t} &= \frac{2t}{1 - 3s} & \frac{\partial y}{\partial t} &= 2t \cos(3s) & \frac{\partial z}{\partial t} &= 5t^4 s^{-4}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{3t^2(x \ln(y^2 + z^2)^2) - y^3 e^{-xy^3}}{(1 - 3s)^2} + \left[\frac{2x^2 y}{y^2 + z^2} - 3xy^2 e^{-xy^3} \right] \cdot \left[-3t^2 \sin(3s) \right] - \frac{8t^5 x^2 z}{s^5 (y^2 + z^2)} \\ \frac{\partial u}{\partial t} &= \frac{2t(x \ln(y^2 + z^2)^2) - y^3 e^{-xy^3}}{1 - 3s} + \frac{2xyt \cos(3s)(2x - 3y^3 e^{-xy^3} - 3yz^2 e^{-xy^3})}{y^2 + z^2} + \frac{10x^2 z t^4}{s^4 (y^2 + z^2)}. \end{aligned}$$

Problem 2

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ assuming $z = f(x, y)$ for the implicitly defined surface.

$$\tan(x^2 - 3yz) = xz^2 - y^2.$$

Solution 2

For $\dfrac{\partial z}{\partial x}$:

$$\begin{aligned}
& \tan(x^2 - 3yz) && = xz^2 - y^2 \\
\Rightarrow \quad & \frac{\partial}{\partial x}\left[\tan(x^2 - 3yz)\right] && = \frac{\partial}{\partial x}\left[xz^2 - y^2\right] \\
\Rightarrow \quad & \sec^2(x^2 - 3yz) \cdot \frac{\partial}{\partial x}\left[x^2 - 3yz\right] && = z^2 + 2xz\frac{\partial z}{\partial x} \\
\Rightarrow \quad & \sec^2(x^2 - 3yz) \cdot \left[2x - 3y\frac{\partial z}{\partial x}\right] - 2xz\frac{\partial z}{\partial x} && = z^2 \\
\Rightarrow \quad & \frac{\partial z}{\partial x}\left[-2xz - 3y\sec^2(x^2 - 3yz)\right] && = z^2 - 2x\sec^2(x^2 - 3yz) \\
\Rightarrow \quad & \frac{\partial z}{\partial x} && = \frac{z^2 - 2x\sec^2(x^2 - 3yz)}{-2xz - 3y\sec^2(x^2 - 3yz)}.
\end{aligned}$$

For $\dfrac{\partial z}{\partial y}$:

$$\begin{aligned}
& \tan(x^2 - 3yz) && = xz^2 - y^2 \\
\Rightarrow \quad & \frac{\partial}{\partial y}\left[\tan(x^2 - 3yz)\right] && = \frac{\partial}{\partial y}\left[xz^2 - y^2\right] \\
\Rightarrow \quad & \sec^2(x^2 - 3yz) \cdot \frac{\partial}{\partial y}\left[x^2 - 3yz\right] && = -2y \\
\Rightarrow \quad & \sec^2(x^2 - 3yz) \cdot \left[-3y - 3y\frac{\partial z}{\partial y}\right] && = -2y \\
\Rightarrow \quad & \frac{\partial z}{\partial y}\left(-3y\sec^2(x^2 - 3yz)\right) && = -2y + 3y\sec^2(x^2 - 3yz) \\
\Rightarrow \quad & \frac{\partial z}{\partial y} && = \frac{-2y + 3y\sec^2(x^2 - 3yz)}{-3y\sec^2(x^2 - 3yz)}.
\end{aligned}$$

Problem 3

Let $f(x, y)$ be an arbitrary function of (x, y) where $x(r, \varepsilon) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$. Assume that $f(x, y)$ and all of its derivatives are continuous.

(i) Use the chain rule to find f_r, f_θ, f_{rr}, and $f_{\theta\theta}$.

(ii) Evaluate $f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r$ to show that

$$f_{xx} + f_{yy} = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r.$$

Solution 3

(i) Using the chain rule, we get the following

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

Next, we need to compute the partial derivatives, giving us

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta) \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r\cos(\theta).$$

Substituting these into f_r and f_θ

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta.$$

To find f_{rr}, differentiate f_r with respect to r

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right).$$

Expanding using the chain rule

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\cos^2(\theta) + 2\frac{\partial^2 f}{\partial x^2} y\cos(\theta)\sin(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta).$$

Similarly, for $f_{\theta\theta}$

$$\begin{aligned}
\frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left[-\frac{\partial f}{\partial x} r\sin(\theta) + \frac{\partial f}{\partial y} r\cos(\theta)\right] \\
&= \frac{\partial^2 f}{\partial x^2}\left(-r\sin(\theta)\right)^2 + 2\frac{\partial^2 f}{\partial x^2} y\left(-r\sin(\theta)\right)\left(r\cos(\theta)\right) + \frac{\partial^2 f}{\partial y^2}\left(r\cos(\theta)\right)^2 \\
&= r^2\left(\frac{\partial^2 f}{\partial x^2}\sin^2(\theta) - 2\frac{\partial^2 f}{\partial x^2} y\sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\cos^2(\theta)\right).
\end{aligned}$$

(ii) Using the chain rule in Cartesian coordinates

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

Problem 3 continued on next page ...

Therefore, we have shown that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}.$$

Problem 4

Consider the function $f(x,y) = 2xy^3 + \ln(3y - x)$.

(i) Find the directional derivative of f in the direction of $\mathbf{u} = \langle 3, 4 \rangle$ at the point $(2,1)$.

(ii) What is the maximum rate of change of f at $(2,1)$? In what direction does the maximum rate of change occur?

(iii) Find all directions in which the directional derivative of f at $(2,1)$ has the value 1.

Solution 4

(i) The directional derivative is given by multiplying the gradient by the unit vector $\mathbf{u}$

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \frac{\mathbf{u}}{|\mathbf{u}|}.$$

So first, find the gradient of f

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \Rightarrow \nabla f(x,y) = \left\langle 2y^3 - \frac{1}{3y - x}, 6xy^2 + \frac{3}{3y - x} \right\rangle.$$

Then, find the unit vector $\mathbf{u}$

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 3, 4 \rangle}{\sqrt{3^2 + 4^2}} = \frac{\langle 3, 4 \rangle}{5} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

Therefore, the general directional derivative is given by

$$D_{\mathbf{u}} f(x,y) = \left\langle 2y^3 - \frac{1}{3y - x}, 6xy^2 + \frac{3}{3y - x} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

So, at $(2,1)$, we have

$$D_{\mathbf{u}} f(2,1) = \langle 1, 15 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = (1) \cdot \left(\frac{3}{5}\right) + (15) \cdot \left(\frac{4}{5}\right) = \frac{63}{5}.$$

(ii) The maximum rate of change of f at $(2,1)$ is the magnitude of the gradient

$$|\nabla f(2,1)| = \sqrt{(1)^2 + (15)^2} = \sqrt{1 + 225} = \sqrt{226}.$$

The direction of the maximum rate of change is in the direction of the gradient, which is $\langle 1, 15 \rangle$.

(iii) To find all directions in which the directional derivative of f at $(2,1)$ is 1, use the formula (where $\mathbf{v} = \langle v_1, v_2 \rangle$ is a unit vector)

$$D_{\mathbf{v}} f(2,1) = \nabla f(2,1) \cdot \mathbf{v} = 1,$$

which simplifies to $\langle 1, 15 \rangle \cdot \langle v_1, v_2 \rangle = v_1 + 15v_2 = 1$. Since $\mathbf{v}$ is a unit vector, we have $v_1^2 + v_2^2 = 1$. Using Wolfram, we get

$$\mathbf{v} = \langle 1, 0 \rangle \quad \text{or} \quad \mathbf{v} = \left\langle -\frac{112}{113}, \frac{15}{113} \right\rangle.$$