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4.3.3	4.3.4	4.3.6	4.3.8	4.3.9
5	4	5	5	5
4.5.3	4.5.6	4.5.7		
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Fundamentals of Analysis II: Homework 1

Due on January 15, 2025 at 23:59

Yuan Xu 13:00

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Exercise 4.3.3. Supply a proof for Theorem 4.3.9 (Composition of continuous functions) using $\varepsilon - \delta$ characterization of continuity.

Solution. Let f be continuous at c, and let g be continuous at $f(c)$. Let $\varepsilon > 0$ be arbitrary. Since g is continuous at $f(c)$, there exists $\delta_1 > 0$ such that $|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$. Since f is continuous at c, there exists $\delta_2 > 0$ such that $|x - c| < \delta_2 \Rightarrow |f(x) - f(c)| < \delta_1$. Now, consider $\delta = \delta_2$. If $|x - c| < \delta$, then by the continuity of f, we have $|f(x) - f(c)| < \delta_1$. Using the continuity of g, this implies $|g(f(x)) - g(f(c))| < \varepsilon$. Thus, $|x - c| < \delta$ implies $|g(f(x)) - g(f(c))| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $g \circ f$ is continuous at c. ☐

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Exercise 4.3.4. Assume f and g are defined on all of $\mathbb{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$.

(i) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r.$$

(ii) Show that the result in (i) does follow if we assume f and g are continuous.

(iii) Does the result in (i) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Solution to (i). Define $f(x)$ and $g(x)$ as follows

need $q \neq 0$.

$$f(x) = q \quad \text{and} \quad g(x) = \begin{cases} \frac{xr}{q} & \text{if } x \neq q \\ 0 & \text{if } x = q \end{cases}.$$

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Observe that $\lim_{x \to p} f(x) = q$ holds trivially, since $f(x) = q$ for all x. For $g(x)$,

$$\lim_{x \to q} g(x) = \lim_{x \to q} \frac{xr}{q} = \frac{qr}{q} = r.$$

Thus, $\lim_{x \to q} g(x) = r$. However, for the composition

$$\lim_{x \to p} g(f(x)) = g(f(p)) = g(q) = 0.$$

Therefore, $\lim_{x \to p} g(f(x)) = 0 \neq r$, even though $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$. ☐

Solution to (ii). If f and g are both continuous, then we use Theorem 4.3.9, which is proven in Exercise 4.3.3. ☐

Solution to (iii). Not if f is continuous, as in the example I provided, f is continuous. But if g is continuous, then yes. Prove this ☐

Exercise 4.3.6. Provide an example of each or explain why the request is impossible.

(i) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0.

(ii) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

(iii) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

Solution to (i). Define $f(x)$ and $g(x)$ as follows

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}.$$

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Notice that $f(x)$ and $g(x)$ are both not continuous at 0. Then, we get the constant function $f(x)+g(x) = 0$ no matter the x-value, which is continuous at 0. Same thing for their product, we get a constant function $f(x) \cdot g(x) = -1$ no matter the x-value, making it continuous at 0. □

Solution to (ii). Impossible, because the sum of a continuous function and a discontinuous function is always discontinuous. □

Solution to (iii). Let $f(x)$ be a constant function at 0. Then, as long as $g(x)$ is bounded, regardless if it's continuous or not, then by exercise 4.2.7, $f(x)g(x) = 0$. □

Exercise 4.3.8. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

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(i) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(ii) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

(iii) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution to (i). True, using the Sequential Definition for Functional Limits, letting $x_n \to 1$, we have $g(x_n) \geq 0$ and $g(x_n) \to g(1)$. Then, by the Order Limit Theorem, $g(1) \geq 0$. □

Solution to (ii). True. Assume $(\exists c \in \mathbb{R} - \mathbb{Q})[g(c) \neq 0]$. That would cause g to not be continuous at x because we can't make ε smaller than $|g(x)|$ because we can find a rational number r such that $g(r) = 0$ inside any δ-neighborhood. * explain further. □

Solution to (iii). True, since g is continuous on all of $\mathbb{R}$, the positivity of $g(x_0)$ implies that there exists an open interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ around x_0, where $g(x) > 0$ for all $x \in I$. Then, using the fact that any non-empty interval $I \subset \mathbb{R}$ contains uncountably many points. Since $g(x) > 0$ for all $x \in I$, this means $g(x)$ is strictly positive for uncountably many points. □

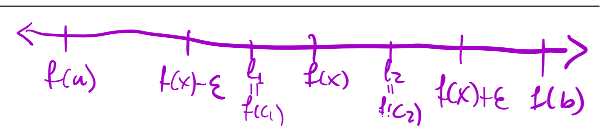
Exercise 4.3.9. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x \mid h(x) = 0\}$. Show that K is a closed set.

Solution. If $K = \emptyset$, then K is closed since $\emptyset^c = \mathbb{R}$, which is open. Let K' be the collection of limit points of K. Let $a \in K'$. Then, there exists a sequence in K such that $x_n \to a$. Since h is continuous on $\mathbb{R}$, then $f(a_n) \to f(a)$. But $f(x_n) = 0$, for all n. So, $h(a) = 0$, implying $a \in K'$. Therefore, $K' \subseteq K$, meaning K is closed. □ 5

Exercise 4.5.3. A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Solution. Let $x \in [a, b]$ and choose $\varepsilon > 0$. Let $\ell_1 \in (f(a), f(x)) \cap (f(x) - \varepsilon, f(x))$, as this ensures that ℓ_1 is both within range of $[a, x]$ and close to $f(x)$. Since f satisfies the intermediate value property, there must exist a $c_1 \in (a, x)$ such that $f(c_1) = \ell_1$. Since ℓ_1 was chosen to satisfy $|f(x) - \ell_1| < \varepsilon$, we get $|f(x) - f(c_1)| < \varepsilon$. Similarly, choose an $\ell_2 \in (f(x), f(b)) \cap (f(b) - \varepsilon, f(b))$. Like before, there must exist a $c_2 \in (x, b)$ such that $f(c_2) = \ell_2$. Then, we get $|f(x) - f(c_2)| < \varepsilon$.

making a graph helps (not always needed)



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Since f is increasing, for all $y \in [c_1, c_2]$, we have $f(c_1) \leq f(y) \leq f(c_2)$. Thus,

$$|f(x) - f(y)| \leq \max\{|f(x) - f(c_1)|, |f(x) - f(c_2)|\} < \varepsilon.$$

From earlier, we know that $|f(x) - f(c_1)| < \varepsilon$ and $|f(x) - f(c_2)| < \varepsilon$, so this implies $|f(x) - f(y)| < \varepsilon$ for all $y \in [c_1, c_2]$. To ensure $y \in [c_1, c_2]$, set $\delta = \min\{x - c_1, c_2 - x\}$. Then, for all $y \in [a, b]$ with $|x - y| < \delta$, it follows that $y \in [c_1, c_2]$, and hence $|f(x) - f(y)| < \varepsilon$. ✓ □

Exercise 4.5.6a. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

Solution. Using the hint, this reduces to finding $x \in [0, 1/2]$ such that $F(x) = 0$, where

$$F(x) = f\left(x + \frac{1}{2}\right) - f(x).$$

Since f is continuous on $[0, 1]$, $F(x)$ is also continuous on the interval where it is defined, which is $[0, 1/2]$. because the argument $x + 1/2$ remains within $[0, 1]$ when $x \in [0, 1/2]$. Evaluating $F(x)$ at certain points gives us

$$\text{At } x = 0 \Rightarrow F(0) = f\left(0 + \frac{1}{2}\right) - f(0) = f(1/2) - f(0)$$
$$\text{At } x = \frac{1}{2} \Rightarrow F(0) = f(1) - f(0). = f(0) - f(1/2)$$

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Since $f(0) = f(1)$, it follows that

$$F\left(\frac{1}{2}\right) = f(0) - f\left(\frac{1}{2}\right).$$

Therefore,

$$F(0) = f\left(\frac{1}{2}\right) - f(0) \quad \text{and} \quad F\left(\frac{1}{2}\right) = f(0) - f\left(\frac{1}{2}\right).$$

From the previous expression, notice that $F(0) = -F(1/2)$, meaning $F(0)$ and $F(1/2)$ have opposite signs or at least one of them is zero. Since $F(x)$ is continuous on $[0, 1/2]$, by the Intermediate Value Theorem, there exists $c \in [0, 1/2]$ such that $F(c) = 0$. Therefore, we have (only if $F(0) \neq 0$)

$$F(c) = f\left(c + \frac{1}{2}\right) - f(c) = 0 \Rightarrow f\left(c + \frac{1}{2}\right) = f(c).$$

Setting $x = c$ and $y = c + 1/2$, we found $|x - y| = 1/2$ and $f(x) = f(y)$. □

Exercise 4.5.7. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

Solution. Define $g(x) = f(x) - x$, which is continuous on $[0, 1]$, since it's the sum of two continuous functions. At $x = 0$, $g(0) = f(0)$. Since the range of f is contained in $[0, 1]$, we know $f(0) \geq 0$, so $g(0) \geq 0$. At $x = 1$, $g(1) = f(1) \leq 1$, so $g(1) \leq 0$. ~~Then,~~ Since $g(x)$ is continuous on $[0, 1]$, $g(0) \geq 0$, and $g(1) \leq 0$, the intermediate value theorem grantees us that there exist a $c \in [0, 1]$ such that $g(c) = 0$. Notice that $f(c) - c = 0 \Rightarrow f(c) = c$, meaning at we have a fixed point at $x = c$. □

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