SOLUTIONS TO HOMEWORK 1

Warning: Very little proofreading has been done.

1. Section 1.2

Exercise 1.2.1.

(1) Prove that $\sqrt{3}$ is irrational. Does a similar argument (to the proof that $\sqrt{3}$ is irrational) work to prove that $\sqrt{6}$ is irrational?
(2) Where does the proof of Theorem 1.1.1 of the book break down if we try to use it to prove that $\sqrt{4}$ is irrational?

Solution. (1) We first do the part about $\sqrt{3}$.

We first claim (this will be used twice below) that if $n \in \mathbb{Z}$ and n^2 is divisible by 3, then n is divisible by 3.

We prove this by contradiction. So suppose n is not divisible by 3. Then there is $m \in \mathbb{Z}$ such that $n = 3m + 1$ or there is $m \in \mathbb{Z}$ such that $n = 3m + 2$ (depending on the remainder when n is divided by 3).

If $n = 3m + 1$, then

$$n^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1.$$

Since $3(3m^2 + 2m)$ is divisible by 3, and 1 is not divisible by 3, it follows that n^2 is not divisible by 3.

If $n = 3m + 2$, then

$$n^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1.$$

Since $3(3m^2 + 4m + 1)$ is divisible by 3, and 1 is not divisible by 3, it follows that n^2 is not divisible by 3.

The claim is proved.

We now prove the statement in the problem by contradiction. So suppose

$$x \in \mathbb{Q} \quad \text{and} \quad x^2 = 3.$$

Write x as a fraction in lowest terms:

$$x = \frac{p}{q}$$

with $p, q \in \mathbb{Z}$, $q \neq 0$, and such that p and q have no common factor. Then

$$3 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}.$$

Multiply through by q^2:

$$3q^2 = p^2.$$

Therefore p^2 is divisible by 3. By the claim at the beginning of the proof, p is divisible by 3. Therefore there is $r \in \mathbb{Z}$ such that

$$p = 3r.$$

Substitute in the previous equation:

$$3q^2 = (3r)^2 = 9r^2.$$

Therefore

$$q^2 = 3r^2.$$

So q^2 is divisible by 3. By the claim at the beginning of the proof, q is divisible by 3. So both p and q are divisible by 3. This contradicts the fact that p and q have no common factor. Therefore x does not exist.

For the part about $\sqrt{6}$: Yes.

Method 1: Proceed as in the main part of the proof above. One gets eventually

$$6q^2 = p^2.$$

Therefore p^2 is divisible by both 2 and 3. Using the results already known, it follows that p is divisible by both 2 and 3. Therefore p is divisible by 6.

Proceed as before, eventually getting

$$q^2 = 6r^2.$$

So, by the same reasoning, q is also divisible by 6. A contradiction is now obtained as before.

Method 2: As before, one gets eventually

$$6q^2 = p^2.$$

Therefore p^2 is divisible by 2. So p is divisible by 2. Therefore there is $r \in \mathbb{Z}$ such that

$$p = 2r.$$

Substitute in the previous equation:

$$6q^2 = (2r)^2 = 4r^2.$$

Therefore

$$3q^2 = 2r^2.$$

So $3q^2$ is even. Since 3 is odd, this can only happen if q^2 is even. Therefore q is even, and p and q have 2 as a common factor.

Method 3: This is the same as Method 2, except working with 3 instead of 2.

(2) One gets eventually

$$4q^2 = p^2$$

(in place of $2q^2 = p^2$). So p^2 is even, whence p is even. Write $p = 2r$. Then

$$4q^2 = p^2 = (2r)^2 = 4r^2.$$

This implies that $q^2 = r^2$, but it certainly does not imply that q^2 is even. □

Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\subseteq \mathbb{R}$.

(1) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
(2) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
(3) Show that $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution. (1) More steps are shown than is really necessary, because in the past there has been confusion over the logic.

Let

$$x \in (A \cap B)^c.$$

Then

$$x \notin A \cap B.$$

Therefore

$$x \notin A \quad \text{or} \quad x \notin B.$$

Hence

$$x \in A^c \quad \text{or} \quad x \in B^c.$$

Thus

$$x \in A^c \cup B^c.$$

This is the desired conclusion.

(2) Again, more steps are shown than is really necessary. Let

$$x \in A^c \cup B^c.$$

Then

$$x \in A^c \quad \text{or} \quad x \in B^c.$$

So

$$x \notin A \quad \text{or} \quad x \notin B.$$

Thus

$$x \notin A \cap B.$$

Hence

$$x \in (A \cap B)^c.$$

This is the desired conclusion.

(3) We first show that that $(A \cup B)^c \subseteq A^c \cap B^c$.
Let
$$x \in (A \cup B)^c.$$
This means that
$$x \notin A \cup B,$$
whence
$$x \notin A \quad \text{and} \quad x \notin B.$$
So
$$x \in A^c \quad \text{and} \quad x \in B^c,$$
whence
$$x \in A^c \cap B^c,$$
as desired.

Now we prove the reverse inclusion.
Let
$$x \in A^c \cap B^c.$$
Then
$$x \in A^c \quad \text{and} \quad x \in B^c,$$
so
$$x \notin A \quad \text{and} \quad x \notin B.$$
It follows that
$$x \notin A \cup B,$$
whence
$$x \in (A \cup B)^c,$$
as desired. $\square$

Exercise 1.2.6.

(b) Find an efficient proof [of triangle inequality] for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.
(d) $\big||a|-|b|\big| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution. (b) Let $a, b \in \mathbb{R}$. Since $ab \leq |ab| = |a| \cdot |b|$ and $a^2 = |a|^2$, we see
$$(a+b)^2 = a^2 + 2ab + b^2 \leq |a|^2 + 2|a| \cdot |b| + |b|^2 = (|a|+|b|)^2.$$
Taking square root proves $|a+b| \leq |a| + |b|$.

(d) Let $a, b \in \mathbb{R}$. Since $(a-b)+b = a$, the triangle inequality gives
$$|a| \leq |a-b| + |b|.$$
Therefore
$$|a| - |b| \leq |a-b|.$$
Similarly
$$|b| - |a| \leq |a-b|.$$
Combining the last two inequalities gives
$$\big||a|-|b|\big| \leq |a-b|.$$
This completes the proof. $\square$

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is , $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
(c) Show that, for an arbitrary function $g \colon \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution. (a) We have

$$f([1,4]) = [1,16] \quad \text{and} \quad f([0,2]) = [0,4].$$

So

$$f([1,4]) \cap f([0,2]) = [1,4] \quad \text{and} \quad f([1,4]) \cup f([0,2]) = [0,16].$$

Also

$$f([1,4] \cap [0,2]) = f([1,2]) = [1,4] = f([1,4]) \cap f([0,2])$$

and

$$f([1,4] \cup [0,2]) = f([0,4]) = [0,16] = f([1,4]) \cup f([0,2]).$$

(b) Example: $A = [-2, -1]$ and $B = [1,2]$. Then

$$f(A \cap B) = f(\varnothing) = \varnothing \quad \text{and} \quad f(A) \cap f(B) = [1,4] \cap [1,4] = [1,4].$$

A second example: $A = \{-1\}$ and $B = \{1\}$. Then

$$f(A \cap B) = f(\varnothing) = \varnothing \quad \text{and} \quad f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}.$$

Many other examples are possible, including ones in which $A \cap B \neq \varnothing$, but there is no example which doesn't use both positive and negative numbers.

(c) Let $y \in f(A \cap B)$. Then there exists $x \in A \cap B$ such that $f(x) = y$. We have $f(x) \in f(A)$ since $x \in A$, and $f(x) \in f(B)$ since $x \in B$. Therefore $y = f(x) \in f(A) \cap f(B)$.

(d) We have $g(A \cup B) = g(A) \cup g(B)$ for all subsets A and B of the domain of g.

First, let $y \in g(A \cup B)$. Then $y = g(x)$ for some $x \in A \cup B$. If $x \in A$, then $y \in g(A) \subseteq g(A) \cup g(B)$, while if $x \in B$, then $y \in g(B) \subseteq g(A) \cup g(B)$. This shows that $g(A \cup B) \subseteq g(A) \cup g(B)$.

For the reverse, let $y \in g(A) \cup g(B)$. If $y \in g(A)$, then there is $x \in A$ such that $g(x) = y$. Then $x \in A \cup B$, so $y = g(x) \in g(A \cup B)$. Similarly, if $y \in g(B)$, then there is $x \in B$ such that $g(x) = y$, and again $y = g(x) \in g(A \cup B)$. This proves $g(A) \cup g(B) \subseteq g(A \cup B)$. □

Exercise 1.2.11. Form the logical negation of each claim. One way to do this is to simply add "It is not the case that ..." in front of each assertion, but for each statement, try to embed the word "not" as deeply into the resulting sentence as possible (or avoid using it altogether). In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + \frac{1}{n} < b$.
(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.
(c) Between every two distinct real numbers there is a rational number.

Solution. (a) There are real numbers a and b such that $a < b$ and such that for every $n \in \mathbb{N}$, we have $a + \frac{1}{n} \geq b$.

(b) Given any real number $x > 0$, there is an $n \in \mathbb{N}$ such that $x \geq 1/n$.

(c) There are distinct real numbers a and b such that every real number between a and b is irrational. □

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence $(y_n)_{n \in \mathbb{N}}$ satisfies $y_n > -6$ for every $n \in \mathbb{N}$.
(b) Use another induction argument to show that the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution. (a) We have $y_1 = 6 > -6$. This is the base case.

Now suppose $y_n > -6$ for some $n \in \mathbb{N}$. Then

$$y_{n+1} = \frac{2y_n - 6}{3} = \frac{2}{3}y_n - 2 > \frac{2}{3}(-6) - 2 = -6.$$

This is the induction step.

(b) We prove by induction on n that $y_n > y_{n+1}$ for all $n \in \mathbb{N}$.

We have $y_1 = 6$ and $y_2 = 2$. This does the base case.

Now assume that $y_n > y_{n+1}$ for some $n \in \mathbb{N}$. Then

$$2y_n > 2y_{n+1},$$

so

$$2y_n - 6 > 2y_{n+1} - 6.$$

whence

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3} = y_{n+2}.$$

This is the induction step.

□

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 of the book has been successfully completed.

(1) Show how induction can be used to prove that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$ and subsets $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}$.

(2) Explain why induction *cannot* be used to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\cap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in N$, but $\cap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(3) Is the statement in Part (b) valid? If so, write a proof which does not use induction.

Solution. (1) The base case ($n = 1$) says $A_1^c = A_1^c$. This is a tautology.

We also need to do the case $n = 2$ separately. This is Exercise 1.2.5(c).

Now we do the induction step. Suppose the statement is true for some $n \in \mathbb{N}$. Then, using the case $n = 2$ at the second step and the induction hypothesis at the third step, we have:

$$\begin{aligned}(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})^c &= \big([A_1 \cup A_2 \cup \cdots \cup A_n] \cup A_{n+1}\big)^c \\ &= \big[(A_1 \cup A_2 \cup \cdots \cup A_n)^c\big] \cap A_{n+1}^c \\ &= A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c.\end{aligned}$$

This is the case $n + 1$, and the induction is complete.

(2) There are many such examples. Here is one: for $i = 1, 2, 3, \ldots$, let

$$B_i = \{1/k : k \geq i,\ k \in \mathbb{N}\}$$

(in particular, $B_1 = \{1/k : k \geq 1\} = \{1, 1/2, 1/3, 1/4 \ldots\}$, $B_2 = \{1/k : k \geq 2\} = \{1/2, 1/3, 1/4 \ldots\}$ etc.). We see that $B_1 \supset B_2 \supset B_3 \supset \ldots$, so that

$$\bigcap_{i=1}^{n} B_i = B_n \neq \emptyset \qquad \text{for every } i \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset.$$

(3) We prove

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq \bigcap_{n=1}^{\infty} A_n^c \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n^c \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right)^c.$$

For the first, let $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$. Then $x \notin \bigcup_{n=1}^{\infty} A_n$. This means that for all $n \in \mathbb{N}$, we have $x \notin A_n$. Thus, for all $n \in \mathbb{N}$, we have $x \in A_n^c$. This says $x \in \bigcap_{n=1}^{\infty} A_n^c$, as desired.

For the reverse inclusion, suppose $x \in \bigcap_{n=1}^{\infty} A_n^c$. Then for all $n \in \mathbb{N}$, we have $x \in A_n^c$, so $x \notin A_n$. This says $x \notin \bigcup_{n=1}^{\infty} A_n$, so $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$, as desired. □

2. Section 1.3

Exercise 1.3.3.

(a) Let $A \subseteq \mathbb{R}$ be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup(B) = \inf(A)$.
(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution. (a) Note: In this problem, we do not get to assume ahead of time that $\inf(A)$ exists. Therefore, we must prove that $\sup(B)$ satisfies the definition of $\inf(A)$.

We had also better check that B is nonempty and bounded above. It is nonempty because, by hypothesis, A has a lower bound. Also, any element of A is clearly an upper bound for B. Since $A \neq \varnothing$, this implies B is bounded above. So $\sup(B)$ exists.

We first show that $\sup(B)$ is a lower bound for A. Let $a \in A$. Then $a \geq b$ for all $b \in B$ (because b is a lower bound for A). Therefore a is an upper bound for B. By the definition of $\sup(B)$, this means that $a \geq \sup(B)$. We have shown that $\sup(B) \leq a$ for all $a \in A$, which is exactly what it means for $\sup(B)$ to be a lower bound for A.

Now let b be an arbitrary lower bound for A. We must show that $b \leq \sup(B)$. But this is immediate because, by definition, $b \in B$.

(b) The proof given for (a) shows, using just the Axiom of Completeness as stated, than whenever $A \subseteq \mathbb{R}$ is nonempty and bounded below, then A has a greatest lower bound in $\mathbb{R}$ (namely, $\sup(B)$ with B as defined in (a)).

□

Exercise 1.3.8abc. Compute, without proofs, the supremums and infimums (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.
(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.
(c) $\{n/(3n+1) : n \in \mathbb{N}\}$.

Solution. (a) The infimum is 0 and the supremum is 1. (For infimum, take $m = 1$ and choose n to be very large. For supremum, take $n = m + 1$ and choose m to be very large).

(b) The infimum is -1 and the supremum is 1. (The set is equal to $\{-1/n : n \in \mathbb{N}\} \cup \{1/n : n \in \mathbb{N}\}$)

(c) The infimum is $1/4$ and the supremum is $1/3$.
(To see this, write

$$\frac{n}{3n+1} = \frac{1}{3}\left(1 - \frac{1}{3n+1}\right).$$

Obviously $\frac{1}{3}$ is an upper bound and one sees by taking n large that it is the least upper bound. The numbers increase with n, so the greatest lower bound is what one gets for $n = 1$, which is $\frac{1}{4}$.) □