MATH 410 - WINTER 2026 - HOMEWORK 1

1. Let $\langle \cdot, \cdot \rangle$ be a real inner product on a vector space V, with induced norm $\| \cdot \|$. Prove the following (called the Cauchy–Schwarz inequality):

$$|\langle v, w \rangle| \leq \|v\| \, \|w\| \quad \text{for any} \quad v, w \in V.$$

Hint. Consider the quadratic polynomial $p(t) = \langle v + tw, v + tw \rangle$, where $t \in \mathbb{R}$.

2. Show that a line segment in $\mathbb{R}^2$ has two-dimensional Lebesgue measure equal to zero. *Optional*: Prove that the same result is true even if the line has infinite length.

3. Find a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that

$$\lim_{n \to \infty} f_n(x) = 0 \quad \text{for all} \quad x \in [0, 1]$$

but

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 1.$$

Why does this not contradict the Lebesgue Dominated Convergence Theorem?

4. Let $V = L^1(\mathbb{R}^n)$ and $W = L^\infty(\mathbb{R}^n)$. Show that if $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ then the mapping T defined by

$$[Tf](x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy$$

is a bounded linear transformation from V to W, with

$$\|T\|_{L^1 \to L^\infty} \leq \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.$$

5. Let W be a closed subset of a normed space V. Show that if $w_k \in W$ and $\lim_{k \to \infty} w_k = w$, then $w \in W$. Recall that by definition W is closed if its complement is open.