Introduction to Proof: Homework 7

Due on November 20, 2024 at 11:59 PM

Victor Ostrik 13:00

Hashem A. Damrah
UO ID: 952102243

Problem 1

Given $Q \Rightarrow R$, prove $[P \Rightarrow T] \Rightarrow [(Q \vee \neg T) \Rightarrow (\neg P \vee R)]$.

Solution 1

1.	$Q \Rightarrow R$	Hypothesis
2.	Assume $P \Rightarrow T$	Dischargeable hypothesis
3.	Assume $Q \vee \neg T$	Dischargeable hypothesis
4.	Assume $P \wedge \neg R$	Dischargeable hypothesis
5.	P	LCS, for 4
6.	$\neg R$	RCS, for 4
7.	T	MP, for 5, for 2
8.	$\neg T$	DI, for 3, for 7
9.	$T \wedge \neg T$	CI, for 7, for 8
10.	$\neg[P \wedge \neg R]$	II, discharge for 4 [4 - 9 unusable]
11.	$\neg[P \wedge \neg R] \Leftrightarrow \neg P \vee R$	Tautology
12.	$\neg P \vee R$	MPB, for 10, for 11
13.	$[Q \vee \neg T] \Rightarrow (\neg P \vee R)$	DT, discharge for 3 [3 - 12 unusable]
14.	$[P \Rightarrow T] \Rightarrow [(Q \vee \neg T) \Rightarrow (\neg P \vee R)]$	DT, discharge for 2 [2 - 13 unusable]

Problem 2

(i) If $C \subseteq A$ and $D \subseteq B$, then prove $D - A \subseteq B - C$.

(ii) Prove $A = X \cap A$ if and only if $A \subseteq X$.

(iii) Prove $A = X \cup A$ if and only if $X \subseteq A$.

Solution 2

(i) Here's the line proof for $D - A \subseteq B - C$.

1. Assume $C \subseteq A$ and $D \subseteq B$.
2. Assume $x \in D - A$.
3. Then, $x \in D$ and $x \notin A$.
4. Since $C \subseteq A$ and $D \subseteq B$, then $x \in B$ and $x \notin C$.
5. By the definition of set difference, $x \in B - C$.
6. Hence, $D - A \subseteq B - C$.
7. Therefore, $(C \subseteq A \wedge D \subseteq B) \Rightarrow (D - A \subseteq B - C)$.

(ii) Here's the line prove for $A = X \cap A$.

1. Assume $A = X \cap A$.
2. Assume $x \in A$.
3. Then $x \in X$ and $x \in A$.
4. Hence, $x \in X$.
5. Therefore, $A \subseteq X$.
6. Assume $A \subseteq X$.
7. Then, by the definition of a subset, $x \in A \Rightarrow x \in X$.
8. Thus, $x \in A \Rightarrow x \in X \cap A$.
9. Similarly, if $x \in X \cap A$, then $x \in A$.
10. Therefore, $A = X \cap A$.
11. Therefore, $A = X \cap A \Leftrightarrow A \subseteq X$.

(iii) Here's the line proof for $A = X \cup A$.

Problem 2 continued on next page . . .

1. Assume $A = X \cup A$.
2. Assume $x \in X \cup A = A$.
3. If $x \in X$, then $x \in A$.
4. Hence, $X \subseteq A$.
5. Assume $X \subseteq A$.
6. Then, by the definition of a subset, $x \in X \Rightarrow x \in A$.
7. Then, $x \in X \cup A$.
8. Therefore, $A = X \cup A$.

Problem 3

Let $f : S \to T$ be a function. Prove that if $X \subseteq T$ and $Y \subseteq T$, then $f^{-1}(X) - f^{-1}(Y) = f^{-1}(X - Y)$.

Solution 3

1. Assume $X \subseteq T$ and $Y \subseteq T$
2. Assume $x \in f^{-1}(X) - f^{-1}(Y)$
3. Then, $x \in f^{-1}(X)$ and $x \notin f^{-1}(Y)$
4. Then, $f(x) \in X$ and $f(x) \notin Y$
5. Then, $f(x) \in X - Y$
6. Then, $x \in f^{-1}(X - Y)$
7. Therefore, $f^{-1}(X) - f^{-1}(Y) \subseteq f^{-1}(X - Y)$
8. Assume $x \in f^{-1}(X - Y)$
9. Then, $f(x) \in X - Y$
10. Then, $f(x) \in X$ and $f(x) \notin Y$
11. Then, $x \in f^{-1}(X)$ and $x \notin f^{-1}(Y)$
12. Then, $x \in f^{-1}(X) - f^{-1}(Y)$
13. Therefore, $f^{-1}(X - Y) \subseteq f^{-1}(X) - f^{-1}(Y)$
14. Therefore, $f^{-1}(X) - f^{-1}(Y) = f^{-1}(X - Y)$

Problem 4

Let $f : S \to T$ be a function, let $A \subseteq S$ and $B \subseteq S$.

(i) Prove $f(A) - f(B) \subseteq f(A - B)$.

(ii) If f is one-to-one, prove $f(A - B) \subseteq f(A) - f(B)$.

(iii) Create an example of an S, T, f, A, and B such that $f(A) - f(B) \neq f(A - B)$.

Solution 4

(i) Here's the line proof for $f(A) - f(B) \subseteq f(A - B)$.

1. Let $f : S \to T$ be a function.
2. Assume $A \subseteq S$ and $B \subseteq S$.
3. Assume $y \in f(A) - f(B)$.
4. Then, $y \in f(A)$ and $y \notin f(B)$.
5. Then, there exists $x \in A$ such that $f(x) = y$ and there does not exist $x \in B$ such that $f(x) = y$.
6. Then, $x \in A$ and $x \notin B$.
7. Then, $x \in A - B$.
8. Then, $y \in f(A - B)$.
9. Therefore, $f(A) - f(B) \subseteq f(A - B)$.

(ii) Here's the line proof for $f(A - B) \subseteq f(A) - f(B)$.

1. Let $f : S \to T$ be a one-to-one function.
2. Assume $A \subseteq S$ and $B \subseteq S$.
3. Assume $y \in f(A - B)$.
4. Then, there exists $x \in A - B$ such that $f(x) = y$.
5. Then, $x \in A$ and $x \notin B$.
6. Then, $f(x) \in f(A)$ and $f(x) \notin f(B)$.
7. Then, $y \in f(A) - f(B)$.
8. Therefore, $f(A - B) \subseteq f(A) - f(B)$.

(iii) Let $S = \{1, 2, 3, 4, 5\}$ and $T = \mathbb{R}$. Let

$$f(X) = \text{Card}(X) + 1.$$

Let $A = \{1, 2, 3\}$ and $B = \{2, 3\}$. Then, $A - B = \{1\}$. Then, we get

$$f(A) = 4, \quad f(B) = 3, \quad f(A) - f(B) = 1, \quad \text{and} \quad f(A - B) = 2.$$

Therefore, $f(A) - f(B) \neq f(A - B)$.

Problem 5

Suppose $f : A \to B$, $X \subseteq A$, $W \subseteq B$, $f(X) \cap W = \emptyset$, and $f(X) \cup W = B$.

(i) Prove that $X \cap f^{-1}(W) = \emptyset$.

(ii) If f is one-to-one, prove that $A = X \cup f^{-1}(W)$.

(iii) If $f(A - X) = W$, prove that f is onto.

Solution 5

(i) Here's the line proof for $X \cap f^{-1}(W) = \emptyset$.

1. Assume $f : A \to B$, $X \subseteq A$, $W \subseteq B$, $f(X) \cap W = \emptyset$, and $f(X) \cup W = B$.
2. Assume $x \in X \cap f^{-1}(W)$.
3. Then, $x \in X$ and $x \in f^{-1}(W)$.
4. Then, $f(x) \in f(X)$ and $f(x) \in W$.
5. That implies that $f(x) \in f(X) \cap W$.
6. But that's a contradiction from the fact that $(X) \cap W = \emptyset$.
7. Therefore, $X \cup f^{-1}(W) = \emptyset$.

(ii) Here's the line proof for $A = X \cup f^{-1}(W)$.

1. Assume $f : A \to B$, $X \subseteq A$, $W \subseteq B$, $f(X) \cap W = \emptyset$, and $f(X) \cup W = B$.
2. Take any $a \in A$.
3. Then $f(a) \in B$. By $f(X) \cup W = B$, we have $f(a) \in f(X)$ or $f(a) \in W$.
4. If $f(a) \in f(X)$, then $a \in X$ (since f is one-to-one).
5. If $f(a) \in W$ then, $a \in f^{-1}(W)$.
6. Thus, $a \in X \cup f^{-1}(W)$.
7. Therefore, $A \subseteq X \cup f^{-1}(W)$.
8. Conversely, observe that $X \subseteq A$ and $f^{-1}(W) \subseteq A$.
9. Thus, $X \cup f^{-1}(W) \subseteq A$.
10. Combining both inclusions, $A = X \cup f^{-1}(W)$.

(iii) Here's the line proof for f is onto.

1. Assume $f : A \to B$, $X \subseteq A$, $W \subseteq B$, $f(X) \cap W = \emptyset$, and $f(X) \cup W = B$.
2. Assume $f(A - X) = W$.
3. Definition of onto is $(\forall b \in B)(\exists a \in A)[f(a) = b]$.
4. Take any $b \in B$.
5. Since $f(X) \cup W = B$, $b \in f(X)$ or $b \in W$.
6. Case 1: If $b \in f(X)$, then there exists $a \in X$ such that $f(a) = b$ (definition of $f(X)$).
7. Case 2: If $b \in W$, then there exists $a \in A - X$ such that $f(a) = b$ (since $f(A - X) = W$).
8. In both cases, there exists $a \in A$ such that $f(a) = b$.
9. Therefore, f is onto.

Problem 6

Prove the statement $1 + 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2$, for all $n \geq 0$.

Solution 6

Proof. Let $P(n) : 1 + 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2$.

Base Case: $P(0) : 1 = (0 + 1)^2 = 1$.

Induction Step: Assume $P(n)$ up to $n = k$. Then, adding the next term $2n + 3$ to both sides gives us $1 + 3 + 5 + 7 + \cdots + (2n + 1) + (2n + 3) = (n + 1)^2 + (2n + 3)$. Simplifying the right hand side gives us

$$(n + 1)^2 + (2n + 3) = n^2 + 2n + 1 + 2n + 3 = n^2 + 4n + 4 = (n + 2)^2 = ((n + 1) + 1),$$

which is equivalent to $P(n + 1)$. Hence, $(\forall n \in \mathbb{N} \cup \{0\})[P(n) \Rightarrow P(n + 1)]$.

By PMI, we proved that $1 + 3 + 5 + 7 + \cdots + (2n + 1) = (n + 1)^2$ for all $n \geq 0$. $\square$

Problem 7

Prove the statement $1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$, for all $n \geq 1$.

Solution 7

Proof. Let $P(n) : 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Base Case: $P(1) : 1^3 = \left[\frac{1(1+1)}{2}\right]^2 = 1$.

Induction Step: Assume $P(n)$ up to $n = k$. Then, adding the next term $(n+1)^3$ to both sides gives us $1^3 + \cdots + n^3 + (n+1)^3 = \left[\dfrac{n(n+1)}{2}\right]^2 + (n+1)^3$. If the statement is true, then we get the following

$$\left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2.$$

Performing some algebra gives us

$$\begin{aligned}
\frac{n^2(n+1)^2}{4} + (n+1)^3 &= \frac{(n+1)^2(n+2)^2}{4} \\
\frac{n^2(n+1)^2 + 4(n+1)^3}{4} &= \frac{(n+1)^2(n+2)^2}{4} \\
n^2(n+1)^2 + 4(n+1)^3 &= (n+1)^2(n+2)^2 \\
(n+1)^2(n^2 + 4(n+1)) &= (n+1)^2(n+2)^2 \\
n^2 + 4(n+1) &= (n+2)^2 \\
n^2 + 4n + 4 &= n^2 + 4n + 4.
\end{aligned}$$

Therefore, we've shown that

$$\left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2.$$

By EPMI, we proved that $1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$ for all $n \geq 1$. $\square$

Problem 8

Prove the statement $\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$, for all $n \geq 1$.

Solution 8

Proof. Let $P(n) : \sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$.

Base Case: $P(1) : (-1)^1 1^2 = (-1)^1 \frac{1(1+1)}{2} = -1$.

Induction Step: Assume $P(n)$ up to n. Then, adding the next term $(-1)^{n+1}(n+1)^2$ to both sides gives us

$$\left(\sum_{k=1}^{n}(-1)^k k^2\right) + (-1)^{n+1}(n+1)^2 = (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2.$$

If the statement is true, then we get the following

$$(-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2 = (-1)^{n+1} \frac{(n+1)(n+2)}{2}.$$

Performing some algebra gives us

$$\begin{aligned}
(-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2 &= (-1)^{n+1} \frac{(n+1)(n+2)}{2} \\
(n+1)\left((-1)^n \cdot \frac{n}{2} + (-1)^{n+1} \cdot (n+1)\right) &= (n+1) \cdot \left(\frac{(-1)^{n+1} \cdot (n+2)}{2}\right) \\
(-1)^n \cdot \frac{n}{2} + (-1)^{n+1} &= (-1)^{n+1} \cdot \frac{(n+2)}{2} \\
\frac{(-1)^n \cdot n + 2(-1)^{n+1} \cdot (n+1)}{2} &= \frac{(-1)^{n+1} \cdot (n+2)}{2} \\
(-1)^n \cdot n + (-1)^{n+1} \cdot 2(n+1) &= (-1)^{n+1} \cdot (n+2) \\
(-1)^n \left(n - 2(n+1)\right) &= (-1)^n \left(-(n+2)\right) \\
n - 2n - 2 &= -n - 2 \\
-n - 2 &= -n - 2.
\end{aligned}$$

Therefore, we've shown that

$$(-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2 = (-1)^{n+1} \frac{(n+1)(n+2)}{2}.$$

By EPMI, we proved that $\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$ for all $n \geq 1$. $\square$

Problem 9

Prove the statement $\frac{(2n)!}{n! \cdot 2^n}$ is an odd number, for every $n \in \mathbb{N}$.

Solution 9

Proof. Let $P(n) : \frac{(2n)!}{n! \cdot 2^n}$ is an odd number.

Base Case: $P(1)$. For $n = 1$,

$$\frac{(2 \cdot 1)!}{1! \cdot 2^1} = \frac{2!}{1 \cdot 2} = \frac{2}{2} = 1.$$

Since 1 is an odd number, $P(1)$ holds.

Induction Step: Assume $P(n)$ holds for some $n \in \mathbb{N}$, i.e.,

$$\frac{(2n)!}{n! \cdot 2^n} \text{ is odd.}$$

We need to show that $P(n+1)$, i.e., $\frac{(2(n+1))!}{(n+1)! \cdot 2^{n+1}}$, is also odd. Starting with

$$\frac{(2(n+1))!}{(n+1)! \cdot 2^{n+1}} = \frac{(2n+2)!}{(n+1)! \cdot 2^{n+1}}.$$

Using the factorial property $(2n+2)! = (2n+2)(2n+1)(2n)!$, this becomes

$$\frac{(2n+2)(2n+1)(2n)!}{(n+1)! \cdot 2^{n+1}}.$$

Rearranging, we write it as

$$\frac{(2n+2)(2n+1)}{(n+1) \cdot 2} \cdot \frac{(2n)!}{n! \cdot 2^n}.$$

By the induction hypothesis, $\frac{(2n)!}{n! \cdot 2^n}$ is odd. Now consider the term

$$\frac{(2n+2)(2n+1)}{(n+1) \cdot 2}.$$

Factor 2 from $(2n+2)$, giving

$$\frac{2(n+1)(2n+1)}{(n+1) \cdot 2} = 2n+1.$$

Since $2n+1$ is odd, the product

$$\frac{(2n+2)(2n+1)}{(n+1) \cdot 2} \cdot \frac{(2n)!}{n! \cdot 2^n}$$

is odd because the product of odd numbers is odd.

Therefore, $P(n+1)$ holds, and $\frac{(2n)!}{n! \cdot 2^n}$ is odd for all $n \in \mathbb{N}$ by the principle of mathematical induction. $\square$

Problem 10

Prove the statement $2^n > n^2$, for all $n > 4$.

Solution 10

Proof. Let $P(n) : 2^n > n^2$.

Base Case: $P(5) : 2^5 = 32 > 25 = 5^2$.

Induction Step: Assume $P(n)$ holds for some $n > 4$, i.e., $2^n > n^2$. We need to show that $P(n+1)$, i.e., $2^{n+1} > (n+1)^2$, is true. Starting with the left-hand side

$$2^{n+1} = 2 \cdot 2^n.$$

By the induction hypothesis, $2^n > n^2$, so

$$2^{n+1} = 2 \cdot 2^n > 2 \cdot n^2.$$

Now compare $2 \cdot n^2$ to $(n+1)^2$

$$2 \cdot n^2 > (n+1)^2 \quad \Leftrightarrow \quad 2 \cdot n^2 > n^2 + 2n + 1.$$

Simplifying gives u

$$2 \cdot n^2 - n^2 > 2n + 1 \quad \Leftrightarrow \quad n^2 > 2n + 1.$$

Since $n > 4$, this inequality holds because

$$n^2 - 2n - 1 = (n-1)^2 - 2 > 0 \quad \text{for all } n > 4.$$

Therefore, $2^{n+1} > (n+1)^2$.

By EPMI, we have proven that $2^n > n^2$ for all $n > 4$. $\square$

Problem 11

Consider the sequence given recursively by $a_0 = 0$ and $a_n = \sqrt{2 + a_{n-1}}$ for all $n \geq 1$. So $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, $a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, and so forth. Then, prove that $a_n \leq 2$ for all $n \geq 0$.

Solution 11

Proof. Let $P(n) : a_n \leq 2$.

Base Case: $P(0) : a_0 = 0 \leq 2$.

Induction Step: Assume $P(n)$ holds for some $n \geq 0$, i.e., assume $a_n \leq 2$. We need to show that $a_{n+1} \leq 2$. By the recursive definition of the sequence,

$$a_{n+1} = \sqrt{2 + a_n}.$$

Starting with the inductive hypothesis $a_n \leq 2$, we add 2 to both sides

$$2 + a_n \leq 2 + 2 = 4.$$

Taking the square root of both sides (noting that square roots preserve inequalities for non-negative numbers), we get

$$\sqrt{2 + a_n} \leq \sqrt{4}.$$

Since $\sqrt{4} = 2$, it follows that

$$a_{n+1} = \sqrt{2 + a_n} \leq 2.$$

Therefore, $P(n + 1)$ holds.

By the principle of mathematical induction, $a_n \leq 2$ for all $n \geq 0$. □

Problem 12

Prove the statement $\left(1+\frac{1}{2}\right)^n > 1+\frac{n}{2}$ for all $n \geq 2$.

Solution 12

Proof. Let $P(n)$ denote the statement $\left(1+\frac{1}{2}\right)^n > 1+\frac{n}{2}$.

Base Case: $P(2)$ states $\left(1+\frac{1}{2}\right)^2 > 1+\frac{2}{2}$. Compute the both sides gives us

$$\left(1+\frac{1}{2}\right)^2 = \frac{3}{2}\cdot\frac{3}{2} = \frac{9}{4} \quad \text{and} \quad 1+\frac{2}{2} = 2.$$

Clearly, $\frac{9}{4} = 2.25 > 2$, so $P(2)$ holds.

Induction Step: Assume $P(n)$ is true for some $k \geq 2$, i.e., assume

$$\left(1+\frac{1}{2}\right)^n > 1+\frac{n}{2}.$$

We need to show that $P(n+1)$ is true, i.e.,

$$\left(1+\frac{1}{2}\right)^{n+1} > 1+\frac{n+1}{2}.$$

Starting with the left-hand side of $P(n+1)$

$$\left(1+\frac{1}{2}\right)^{n+1} = \left(1+\frac{1}{2}\right)^n \cdot \left(1+\frac{1}{2}\right).$$

By the induction hypothesis, $\left(1+\frac{1}{2}\right)^n > 1+\frac{n}{2}$. Substituting this

$$\left(1+\frac{1}{2}\right)^{n+1} > \left(1+\frac{n}{2}\right)\cdot\left(1+\frac{1}{2}\right).$$

Expanding the product on the right-hand side

$$\begin{aligned}\left(1+\frac{n}{2}\right)\cdot\left(1+\frac{1}{2}\right) &= \left(1+\frac{n}{2}\right)+\frac{1}{2}\left(1+\frac{n}{2}\right)\\ &= 1+\frac{n}{2}+\frac{1}{2}+\frac{n}{4} = 1+\frac{1}{2}+\frac{n}{2}+\frac{n}{4}\\ &= 1+\frac{n+2}{2}+\frac{n}{4}.\end{aligned}$$

Since $\frac{n}{4} > 0$ for $n \geq 2$, it follows that

$$\left(1+\frac{n}{2}\right)\cdot\left(1+\frac{1}{2}\right) > 1+\frac{n+2}{2} = 1+\frac{n+1}{2}.$$

Therefore, $\left(1+\frac{1}{2}\right)^{n+1} > 1+\frac{n+1}{2}$.

By the EPMI, $\left(1+\frac{1}{2}\right)^n > 1+\frac{n}{2}$ for all $n \geq 2$. $\square$

Problem 13

Prove the statement $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2} \leq 2 - \frac{1}{n}$, for all $n \in \mathbb{Z}_{\geq 1}$.

Solution 13

Proof. Let $P(n)$ denote the statement $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$.

Base Case: $P(1)$ states $\dfrac{1}{1^2} \leq 2 - \dfrac{1}{1}$. Compute both sides

$$\frac{1}{1^2} = 1 \quad \text{and} \quad 2 - \frac{1}{1} = 1.$$

Clearly, $1 \leq 1$, so $P(1)$ holds.

Induction Step: Assume $P(n)$ holds for some $n \geq 1$, i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

We need to show that $P(n+1)$ is true, i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

Starting with the left-hand side of $P(n+1)$

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}.$$

By the induction hypothesis, we know

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Adding $\frac{1}{(n+1)^2}$ to both sides gives

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$

Performing some algebra gives us

$$\begin{aligned}
2 - \frac{1}{n} + \frac{1}{(n+1)^2} &\leq 2 - \frac{1}{n+1} \\
-\frac{1}{n} + \frac{1}{(n+1)^2} &\leq -\frac{1}{n+1} \\
\frac{1}{(n+1)^2} &\leq \frac{1}{n} - \frac{1}{n+1} \\
\frac{1}{n} - \frac{1}{n+1} &= \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \\
\frac{1}{(n+1)^2} &\leq \frac{1}{n(n+1)}.
\end{aligned}$$

Problem 13 continued on next page ...

Since $n+1>n$, we have $(n+1)^2>n(n+1)$, which implies

$$\frac{1}{(n+1)^2}<\frac{1}{n(n+1)}.$$

Hence, the inequality holds.

By EPMI, we have proven that $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\leq 2-\frac{1}{n}$ for all $n\geq 1$. $\square$

Problem 14

Fill in each box below with a mathematical proposition that makes the biconditional true, and is not a tautology (for example, I don't want you to write "$A \subseteq B$" in the first box, even though this makes the biconditional true). Copy the complete biconditional statements into your homework; do not actually write in the boxes on this worksheet.

(i) $A \subseteq B \Leftrightarrow$

(ii) $A = B \Leftrightarrow$

(iii) $x \in f(A) \Leftrightarrow$

(iv) $y \in f^{-1}(B) \Leftrightarrow$

(v) $x \in A \cup B \Leftrightarrow$

(vi) $x \in A \cap B \Leftrightarrow$

(vii) $x \in A - B \Leftrightarrow$

(viii) $f : S \to T$ is onto $\Leftrightarrow$

(ix) $f : S \to T$ is one-to-one $\Leftrightarrow$

(x) $x \in A \cup (B - C) \Leftrightarrow$

(xi) $X = \emptyset \Leftrightarrow$

Solution 14

(i) $A \subseteq B \Leftrightarrow (\forall x \in A)[x \notin B^c]$.

(ii) $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.

(iii) $x \in f(A) \Leftrightarrow (\exists a \in A)[f(a) = x]$.

(iv) $y \in f^{-1}(B) \Leftrightarrow f(y) \in B$.

(v) $x \in A \cup B \Leftrightarrow x \in A \vee x \in B$.

(vi) $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$.

(vii) $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$.

(viii) $f : S \to T$ is onto $\Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$.

(ix) $f : S \to T$ is one-to-one $\Leftrightarrow (\forall s_1, s_2 \in S)[(f(s_1) = f(s_2)) \Rightarrow (s_1 = s_2)]$.

(x) $x \in A \cup (B - C) \Leftrightarrow x \in A \vee (x \in B \wedge x \notin C$.

(xi) $X = \emptyset \Leftrightarrow (\forall x)[x \notin X]$.

Problem 15

Write definitions for the following sets, using set-builder notation. The first one is done for you.

(i) $X - A$.

(ii) $f^{-1}(B)$.

(iii) $A \cup B$.

(iv) $f(A)$.

(v) $C \cap f^{-1}(B)$.

Solution 15

(i) $X - A = \{x \in X \mid x \notin A\}$.

(ii) $f^{-1}(B) = \{x \in A \mid f(x) \in B\}$.

(iii) $A \cup B = \{x \mid x \in A \vee x \in B\}$.

(iv) $f(A) = \{y \in B \mid (\exists x \in A)[f(x) = y]\}$.

(v) $C \cap f^{-1}(B) = \{x \in C \mid f(x) \in B\}$.