1. Let $T : \mathbb{R}^{1\times 3} \to \mathbb{R}^{1\times 2}$ be a linear transformation defined by

$$T(x,y,z) = (x+z, 2x-z).$$

If $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$, $\mathcal{B}' = \{\beta_1, \beta_2\}$, where

$$\alpha_1 = (1,0,-1), \quad \alpha_2 = (1,1,1), \quad \alpha_3 = (1,0,0), \quad \beta_1 = (0,1), \quad \beta_2 = (1,0).$$

Find the matrix $[T]_{\mathcal{B}}^{\mathcal{B}'}$.

2. Let D be the differentiation operator on $P^3(\mathbb{R})$, i.e.

$$D(g(x)) = g'(x), \quad \text{for} \quad g(x) \in P^3(\mathbb{R}).$$

(Note: D is a linear transformation on $\mathbb{P}^3(\mathbb{R})$.

1). Let $\mathcal{B} = \{1, x, x^2, x^3\}$ be the standard ordered basis for $P^3(\mathbb{R})$. Find the matrix representation $[D]_{\mathcal{B}}$.

2). Let $\mathcal{B}' = \{x^3, x^2, x, 1\}$ be an ordered basis for $P^3(\mathbb{R})$. Find the matrix representation $[D]_{\mathcal{B}'}$.

3. Let T be a linear transformation on the vector space $V = \mathbb{R}^{2\times 2}$ defined by

$$T(A) = 2A + A^T.$$

Let $\mathcal{B} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$. Find the matrix representation $[T]_{\mathcal{B}}$.

4. Let V be a two-dimensional vector space over F, and let $\mathcal{B}$ be an ordered basis for V. If T is a linear transformation on V and $[T]_{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that $T^2 - (a+d)T + (ad-bc)I = 0$.

5. Suppose that T is a linear transformation on a two-dimensional vector space such that T is neither the zero nor the identity linear transformation. Prove that if $T^2 = T$, there is an ordered basis $\mathcal{B}$ for V such that $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(*Hint: Construct a basis* $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ *such that* $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ *and* $T(\boldsymbol{v}_2) = \boldsymbol{0}$.)

6. Let V be a n-dimensional vector space over $\mathbb{R}$. Let T be a linear transformation on V. If $T^n = 0$, and $T^{n-1} \neq 0$, prove that there exists a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$.

(*Hint: Construct a set of the form* $\{\boldsymbol{x}, T(\boldsymbol{x}), T^2(\boldsymbol{x}), \cdots, T^{n-1}(\boldsymbol{x})\}$ *and show that this set is a basis of* V.)

7. Suppose $A \in \mathbb{R}^{n\times n}$ is an invertible matrix. Prove that AB and BA are similar matrices for any $B \in \mathbb{R}^{n\times n}$.

8. Let $A, B \in \mathbb{R}^{n\times n}$. Prove the following statements.

1). If A and B are similar, then $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

2). $AB - BA = I$ is impossible.

Hint: You may use the result from Homework 2 Problem 1.

9. True or False. (No explanation needed.)

In the following statements 1)-3): Let V and W be finite-dimensional vector spaces. Let $T, U : V \to W$ be linear transformations. Let β and γ be ordered bases for V and W, respectively.

1). $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies $T = U$.

2). If $\dim V = n$ and $\dim W = m$, then $[T]_\beta^\gamma \in \mathbb{R}^{n\times m}$.

3). $[T(\mathbf{v})]_\gamma = [T]_\beta^\gamma [\mathbf{v}]_\beta$ for all $\mathbf{v} \in V$.

4). Let $A \in \mathbb{R}^{n\times n}$. If $A^2 = I$, then $A = I$ or $A = -I$.

5). Let $A \in \mathbb{R}^{m\times n}$. Suppose $L_A : R^n \to R^m$ is defined by $L_A(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$. Then $[L_A]_\beta = A$, where β is the standard basis for $\mathbb{R}^n$.