Several-Variab Calc II: Homework 5

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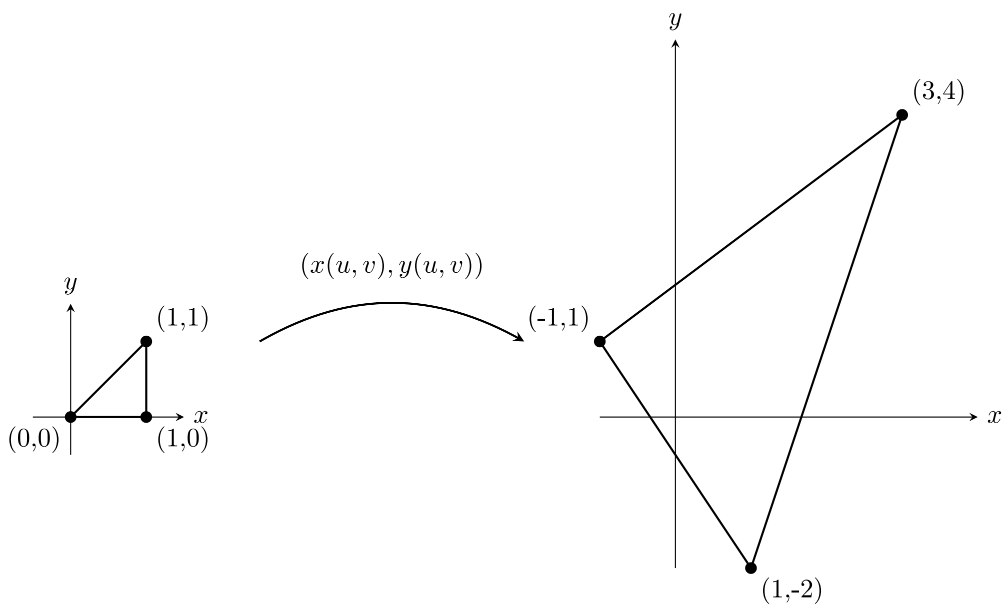
Problem 1. Let D be the triangle with vertices $(-1,1)$, $(1,-2)$, and $(3,4)$.

(i) Let S be the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. Find a change of variables that maps S to D.

Note: Since the change of variable will preserve the boundary as lines, the transformation is affine. That is, $x = g(u,v)$ and $y = h(u,v)$ where $g(u,v) = a_1u + a_2v + a_3$ and $h(u,v) = b_1u + b_2v + b_3$ for constants a_n and b_n, $n = 1,2,3$.

(ii) Use the change of variable to evaluate $\displaystyle\iint_D x + y \,\mathrm{d}A$.

Solution to (i). Graphing the two triangles we get



Therefore, we must get the following transformations

$$(0,0) \mapsto (-1,1), \quad (1,0) \mapsto (1,-2), \quad \text{and} \quad (1,1) \mapsto (3,4).$$

Let $g(u,v) = a_1u + a_2v + a_3$ and $h(u,v) = b_1u + b_2v + b_3$. Then, solving for each variable gives us

$$\begin{aligned}
(0,0) &\mapsto (-1,1) \Rightarrow 0 + 0 + a_3 &&= -1 \Rightarrow a_3 = -1 &&\text{and} \quad 0 + 0 + b_3 &&= 1 \quad \Rightarrow b_3 = 1 \\
(1,0) &\mapsto (1,-2) \Rightarrow a_1 + 0 + a_3 &&= 1 \quad \Rightarrow a_1 = 2 &&\text{and} \quad b_1 + 0 + b_3 &&= -2 \Rightarrow b_1 = -3 \\
(1,1) &\mapsto (3,4) \quad \Rightarrow a_1 + a_2 + a_3 &&= 3 \quad \Rightarrow a_2 = 2 &&\text{and} \quad b_1 + b_2 + b_3 &&= 4 \quad \Rightarrow b_2 = 6.
\end{aligned}$$

Therefore, the change of variable is

$$x = g(u,v) = 2u + 2v - 1 \quad \text{and} \quad y = h(u,v) = -3u + 6v + 1. \qquad \square$$

Solution to (ii). Finding the Jacobian of the transformation we get

$$\left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ -3 & 6 \end{vmatrix} = |18| = 18.$$

Therefore, we have the following double integral

$$\iint_D x + y \,\mathrm{d}A = \iint_S u + v \cdot \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| \mathrm{d}A = 18 \int_0^1 \int_0^u -u + 8v \,\mathrm{d}v \,\mathrm{d}u = 18 \int_0^1 3u^2 \mathrm{d}u = 18. \qquad \square$$

Problem 2. Use an appropriate change of variables to evaluate $\iint_D (2x-y)^2(2y-x)\,\mathrm{d}A$ where D is the triangular region with vertices $(0,1)$, $(2,2)$, and $(1,0)$.

Solution. Let $u = 2x - y$ and $v = 2y - x$. Adding them and solving for x and y gives us

$$x = \frac{2u+v}{3} \quad \text{and} \quad y = \frac{u+2v}{3}.$$

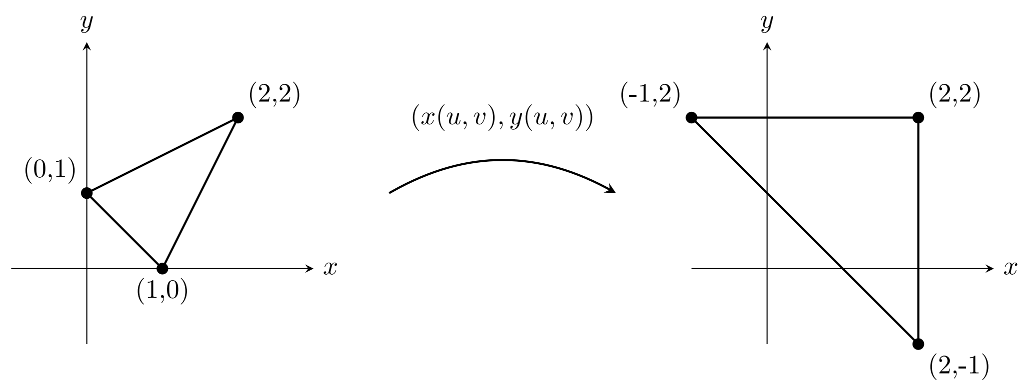
Finding the Jacobian of the transformation we get

$$\left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \left| \frac{4}{9} - \frac{1}{9} \right| = \frac{1}{3}.$$

The new bounds are

$$\begin{aligned} (0,1) &\mapsto (2(0)-1, 2(1)-0) = (-1,2) \\ (2,2) &\mapsto (2(2)-2, 2(2)-2) = (2,2) \\ (1,0) &\mapsto (2(1)-0, 2(0)-1) = (2,-1). \end{aligned}$$

Graphing the triangle we get



Therefore, the double integral becomes

$$\begin{aligned} \iint_D (2x-y)^2(2y-x)\,\mathrm{d}A &= \iint_S uv \cdot \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| \mathrm{d}A \\ &= \frac{1}{3} \int_{-1}^{2} \int_{1-u}^{2} u^2 v \,\mathrm{d}v \,\mathrm{d}u \\ &= \frac{1}{3} \int_{-1}^{2} \left[\frac{1}{2} u^2 v^2 \right]_{1-u}^{2} \mathrm{d}u \\ &= \frac{1}{6} \int_{-1}^{2} 3u^2 + 2u^3 - u^4 \,\mathrm{d}u \\ &= \frac{1}{6} \left[u^3 + \frac{1}{2}u^4 - \frac{1}{5}u^5 \right]_{-1}^{2} = \frac{33}{20}. \end{aligned}$$

□

Problem 3. Use a change of variable to evaluate $\iiint_E z\,\mathrm{d}V$ where E is the solid above $z = 0$ and inside $4x^2 + 16y^2 + 8z^2 = 64$.

Solution. Let $x = 4u$, $y = 2v$, and $z = \sqrt{8}w$. Then, we get

$$\frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{8} = 1 \Rightarrow u^2 + v^2 + w^2 = 1.$$

The Jacobian of the transformation is

$$\left\| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right\| = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{8} \end{vmatrix} = 8\sqrt{8} = 16\sqrt{2}.$$

The bounds are $-1 \leq u \leq 1$, $-\sqrt{1-u^2} \leq v \leq \sqrt{1-u^2}$, and $0 \leq w \leq \sqrt{1-u^2-v^2}$. Converting to spherical coordinates gives us the bounds $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi/2$, and $0 \leq 7 \leq \rho$. Expanding and evaluating the triple integral gives us

$$\begin{aligned} \iiint_E z \, \mathrm{d}V &= \iiint_S (\sqrt{8}w) \cdot \left\| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right\| \mathrm{d}V \\ &= \sqrt{8} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho \cos(\varphi)) \cdot (16\sqrt{2}) \cdot (\rho^2 \sin(\varphi)) \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}\theta \\ &= 64 \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^{\pi/2} \cos(\varphi)\sin(\varphi) \mathrm{d}\varphi \cdot \int_0^1 \rho^3 \, \mathrm{d}\rho \\ &= 64 \cdot 2\pi \cdot \frac{1}{2} \cdot \frac{1}{4} = 16\pi. \end{aligned}$$

□

Problem 4. Use the change of variable $x = u^2$, $y = v^2$, and $z = w^2$ to express the volume of the solid bounded by the surfaces $x = 0$, $y = 0$, and $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ as an iterated integral in the new variables, u, v, and w. It is not required to evaluate the integral.

Solution. Finding the Jacobian of the transformation we get

$$\left\| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right\| = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = |8uvw|.$$

The bounds are $0 \leq u \leq 1$, $0 \leq v \leq 1-u$, and $0 \leq w \leq 1-u-v$. Expanding the triple integral gives us

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} |8uvw| \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u.$$

□

Problem 5. Evaluate $\displaystyle\int_C 12x \, \mathrm{d}s$ where C is the union of the path $y = x^2$ from $(-1,1)$ to $(2,4)$ and the line segment between $(2,4)$ and $(3,2)$.

Solution. We integrate over the following two paths, C_1 and C_2.

C_1: $y = x^2$ from $(-1,1)$ to $(2,4)$. This gives us $\mathbf{r}(t) = \langle r, r^2 \rangle$ from $t = -1$ to $t = 2$. Therefore, the derivative of the position vector is

$$\mathbf{r}'(t) = \langle 1, 2r \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1+4t^2}.$$

Substituting the parameterization into the line integral gives us

$$I_1 = \int_{-1}^2 12t\sqrt{1+4t^2} \, \mathrm{d}t = (4t^2+1)^{3/2}\Big|_{-1}^2 = 17^{3/2} - 5^{3/2} = 17\sqrt{17} - 5\sqrt{5}.$$

C_2: The line segment between $(2,4)$ and $(3,2)$. This gives us $\mathbf{r}(t) = \langle 2+t, 4-2t\rangle$ from $t=0$ to $t=1$. Therefore, the derivative of the position vector is

$$\mathbf{r}'(t) = \langle 1, -2\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1+4} = \sqrt{5}.$$

Substituting the parameterization into the line integral gives us

$$I_2 = \int_0^1 12(2+t)\sqrt{5}\,\mathrm{d}t = 12\sqrt{5}\int_0^1 2+t\,\mathrm{d}t = 12\sqrt{5}\left[2t + \frac{1}{2}t^2\right]_0^1 = 12\sqrt{5}\left(2+\frac{1}{2}\right) = 30\sqrt{5}.$$

Therefore, the total line integral is

$$I = \int_C 12x\,\mathrm{d}s = I_1 + I_2 = 17\sqrt{17} + 25\sqrt{5}.$$

□

Problem 6. Evaluate $\displaystyle\int_C xy\,\mathrm{d}s$ where C is the elliptic helix $x = 2\cos(t)$, $y = 3\sin(t)$, and $z = t$ for $0 \leq t \leq {}^{\pi}/_{2}$.

Solution. We're given the parameterization of the elliptic helix $\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t), t\rangle$. Therefore, the derivative of the position vector is

$$\mathbf{r}'(t) = \langle -2\sin(t), 3\cos(t), 1\rangle \Rightarrow |r'(t)| = \sqrt{4\sin^2(t) + 9\cos^2(t) + 1} = \sqrt{5}\sqrt{\cos^2(t)+1}.$$

Substituting the parameterization into the line integral gives us

$$\int_C xy\,\mathrm{d}s = \int_0^{\pi/2} 6\cos(t)\sin(t)\sqrt{5}\sqrt{\cos^2(t)+1}\,\mathrm{d}t = 6\sqrt{5}\int_0^{\pi/2}\cos(t)\sin(t)\sqrt{\cos^2(t)+1}\,\mathrm{d}t.$$

Using u-substitution with $u = \cos(t)$ gives us

$$6\sqrt{5}\int_0^1 u\sqrt{u^2+1}\,\mathrm{d}u = 6\sqrt{5}\left[\frac{1}{3}(u^2+1)^{3/2}\right]_0^1 = 6\sqrt{5}\left[\frac{2\sqrt{2}}{3} - \frac{1}{3}\right] = 4\sqrt{10} - 2\sqrt{5}.$$

□