Differential Geometry I: Homework 2

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Exercise. Suppose that M^n is a compact smooth n-manifold. Show that there exists a smooth map $F : M \to \mathbb{R}^k$, for some k such that

(i) F is injective at each point,

(ii) and
$$\mathrm{d}F_p : T_pM \to T_{F(p)}\mathbb{R}^k,$$
is injective at each point.

Solution. Since M is compact, it emits a finite atlas, say $\{(U_i, \varphi_i)\}_{i=1}^r$, where each $\varphi_i : U_i \to \mathbb{R}^n$ is a smooth chart.

Let $\{\rho\}_i^r$ be a smooth partition of unity subordinate to this cover. Define the map $G : M \to \mathbb{R}^N$, where

$$p \mapsto (\rho_1(p)\varphi_1(p), \rho_1(p), \rho_2(p)\varphi_2(p), \rho_2(p), \cdots, \rho_r(p)\varphi_r(p), \rho_r(p)),$$

where $N = r(n + 1)$. This map is an immersion and is locally injective on M.

Now, view $G(M) \subseteq \mathbb{R}^N$. By the Whitney Embedding Theorem, for a generic linear map $L : \mathbb{R}^N \to \mathbb{R}^{2n}$, the composition $F = L \circ G : M \to \mathbb{R}^{2n}$ is an embedding. Thus, F is injective and $\mathrm{d}F_p$ is injective at each point. $\square$

Exercise. Let $X \subset L^2[0, 1]$ be defined by

$$X = \left\{ f \in L^2[0, 1] \,\middle|\, \int_0^1 f^2 \,\mathrm{d}x = 1 \right\}.$$

What makes most sense as to define as T_hX, for $h = \sqrt{3}x$?

Solution. Notice that X is the unit sphere in the Hilbert space $L^2[0, 1]$. Define F such that $F : L^2[0, 1] \to \mathbb{R}$ where
$$F(f) = \int_0^1 f(x)^2 \,\mathrm{d}x \Rightarrow X = F^{-1}(1).$$
Then, the differential of F at h is given by

$$\mathrm{d}F_h(g) = 2 \int_0^1 g(x)h(x) \,\mathrm{d}x = 2 \langle g, h\rangle_{L^2}.$$

We want the differential to be tangent to $F = 1$, which gives us the condition that $\mathrm{d}F_h(g) = 0$. We first must check that $\sqrt{3}x$ is in X
$$\int_0^1 (\sqrt{3}x)^2 \,\mathrm{d}x = 3 \int_0^1 x^2 \,\mathrm{d}x = 1.$$
Then, T_hX is given by

$$T_hX = \left\{ g \in L^2[0, 1] \,\middle|\, \int_0^1 xg(x) \,\mathrm{d}x = 0 \right\} \text{ or } \left\{ g \in L^2[0, 1] \,|\, \langle x, g\rangle_{L^2} = 0 \right\}. \qquad \square$$

Exercise 3.1. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $\mathrm{d}F_p : T_pM \to F_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Solution. Assume $\mathrm{d}F_p$ is the zero map for each $p \in M$. Let $p, q \in M$ be in the same component, and let $\gamma : [0, 1] \to M$ be a smooth curve such that $\gamma(0) = p$ and $\gamma(1) = q$. Then, the composition $F \circ \gamma : [0, 1] \to N$ is a smooth curve in N, and its derivative at any $t \in [0, 1]$ is given by

$$(F \circ \gamma)'(t) = \mathrm{d}F_{\gamma(t)}(\gamma'(t)) = 0.$$

Thus, $F \circ \gamma$ is constant, and in particular,

$$F(p) = (F \circ \gamma)(0) = (F \circ \gamma)(1) = F(q).$$

Therefore, F is constant on each component of M.

Conversely, assume the converse. Let F be constant in each component of M. Then, for any vector $v = v^j \frac{\partial}{\partial x^j} \,|_p \in T_p M$, we have

$$\mathrm{d}F_p(v) = v^j \left. \frac{\partial F}{\partial x^j} \right|_p \left. \frac{\partial}{\partial y^i} \right|_{F(p)} = 0.$$

Thus, $\mathrm{d}F_p$ is the zero map for each $p \in M$.

Therefore, $\mathrm{d}F_p : T_p M \to T_{F(p)} N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M. $\square$

Exercise 3.4. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Solution. Define the map $\Phi : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$ by

$$\Phi\left((x, y), t\right) = \left((x, y), t u_{(x,y)}\right) = \left((x, y), t(-y, x)\right).$$

This is well defined since $u_{(x,y)} = (-y, x)$ is a unit tangent vector at the point $(x, y) \in \mathbb{S}^1$. For each fixed p, the mapping $t \mapsto t u_p$ is a linear bijection $\mathbb{R} \to T_p \mathbb{S}^1$.

Let $(p, v) \in T\mathbb{S}^1$. Write $p = (x, y)$. Because v is tangent, $v \cdot p = 0$, so v is a scalar multiple of the tangent direction u_p. Thus there exists $t \in \mathbb{R}$ with $v = t u_p$. Then $\Phi((p, t)) = (p, v)$.

If $\Phi((p, t)) = \Phi((p', t'))$ then the basepoints must match $p = p'$ and $t u_p = t' u_p$. Since $u_p \neq 0$, this forces $t = t'$. So Φ is injective.

Thus Φ is a bijection. Its inverse $\Psi : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ is explicitly

$$\Psi((x, y), v) = ((x, y), t), \qquad t = v \cdot u_{(x,y)}.$$

Indeed v lies in the span of u_p so taking the inner product with u_p recovers the scalar t.

Both Φ and Ψ are given by polynomial (hence smooth) formulas in the coordinates (x, y, t) or (x, y, v_1, v_2). Therefore Φ and Ψ are smooth and are inverses of each other, so Φ is a diffeomorphism. Thus, $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. $\square$

Exercise 3.5. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) \mid \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no *diffeomorphism* with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $\mathrm{d}F(\gamma'(t))$ on the coordinate functions x and y.] *(Used on p. 123.)*

Solution. Define a map $k : \mathbb{S}^1 \to K$ by

$$k(u_1, u_2) = \frac{(u_1, u_2)}{\max(|u_1|, |u_2|)}.$$

This is well-defined, continuous, bijective, and has a continuous inverse $k^{-1} : K \to \mathbb{S}^1$ given by $k^{-1}(y) = y / \|y\|$. Extend this map radially to all of $\mathbb{R}^2$ by defining $F : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F(0) = 0, \qquad F(r, u) = r k(u) \text{ for } r > 0 \text{ and } u \in \mathbb{S}^1.$$

Clearly, F is continuous because k is continuous and the radial extension is continuous in polar coordinates and is bijective with inverse

$$F^{-1}(0) = 0, \qquad F^{-1}(y) = \|y\|\, k^{-1}\!\left(\frac{y}{\|y\|}\right) \text{ for } y \neq 0,$$

which is continuous. It's also on the circle, $F(u) = k(u) \in K$, so $F(\mathbb{S}^1) = K$.

Hence F is a homeomorphism of $\mathbb{R}^2$ sending $\mathbb{S}^1$ to K.

Now, suppose for contradiction, that there exists a diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ with $G(\mathbb{S}^1) = K$. Since $\mathbb{S}^1$ is a smooth embedded 1-manifold in $\mathbb{R}^2$, the image $G(\mathbb{S}^1)$ would also be a smooth embedded 1-manifold in $\mathbb{R}^2$. However, K has corner points (e.g., $(1,1)$) where no well-defined tangent line exists. A smooth embedded 1-manifold cannot have such singular points. Therefore, no such diffeomorphism G can exist. $\square$

Exercise 3.8. Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1\ \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. *(Used on p. 72.)*

Solution. Let γ_1 and γ_2 be two smooth curves starting at p such that $\gamma_1 \sim \gamma_2$. By definition, this means that for every smooth real-valued function f defined in a neighborhood of p, we have

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0).$$

The derivative of the composition can be expressed using the chain rule:

$$(f \circ \gamma_i)'(0) = \mathrm{d}f_p(\gamma_i'(0)) \quad \text{for } i = 1, 2.$$

Therefore, we have

$$\mathrm{d}f_p(\gamma_1'(0)) = \mathrm{d}f_p(\gamma_2'(0)).$$

Since this holds for all smooth functions f, it follows that $\gamma_1'(0) = \gamma_2'(0)$. Thus, the map Ψ is well defined.

To show that Ψ is bijective, we first prove injectivity. Suppose $\Psi[\gamma_1] = \Psi[\gamma_2]$. This means that $\gamma_1'(0) = \gamma_2'(0)$. For any smooth function f defined in a neighborhood of p, we have

$$(f \circ \gamma_1)'(0) = \mathrm{d}f_p(\gamma_1'(0)) = \mathrm{d}f_p(\gamma_2'(0)) = (f \circ \gamma_2)'(0).$$

Thus, $\gamma_1 \sim \gamma_2$, and hence $[\gamma_1] = [\gamma_2]$. Therefore, Ψ is injective.

Next, we prove surjectivity. Let $v \in T_pM$. We need to find a smooth curve γ starting at p such that $\gamma'(0) = v$. Choose a coordinate chart (U, φ) around p such that $\varphi(p) = 0$. In these coordinates, we can define a smooth curve $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ by

$$\tilde{\gamma}(t) = tv.$$

Then, we can define $\gamma : (-\varepsilon, \varepsilon) \to M$ by

$$\gamma(t) = \varphi^{-1}(\tilde{\gamma}(t)).$$

This curve is smooth, starts at p, and its derivative at 0 is

$$\gamma'(0) = \mathrm{d}\varphi^{-1}{}_0(v) = v.$$

Thus, $\Psi[\gamma] = v$, proving that Ψ is surjective.

Since Ψ is both injective and surjective, it is a bijection. Therefore, the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. $\square$