

SOLUTIONS TO HOMEWORK 4

Warning: Little proofreading has been done.

1. SECTION 2.3

Exercise 2.3.5. Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be sequences in \mathbb{R} . Let $(z_n)_{n \in \mathbb{N}}$ be the “shuffled” sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots)$. Prove that $(z_n)_{n \in \mathbb{N}}$ converges if and only if $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are both convergent with $\lim x_n = \lim y_n$.

Solution. We can't use statements about subsequences yet, since they are in Section 2.5 of the book.

Suppose $(z_n)_{n \in \mathbb{N}}$ converges. Define $l = \lim_{n \rightarrow \infty} z_n$. We show that $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = l$. We do both at once. We need the formulas

$$z_{2n-1} = x_n \quad \text{and} \quad z_{2n} = y_n$$

for $n \in \mathbb{N}$.

Let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|z_n - l| < \varepsilon$.

Let $n \in \mathbb{N}$ with $n \geq N$. Then $2n-1 \geq 2N-1 \geq N$, so

$$|x_n - l| = |z_{2n-1} - l| < \varepsilon.$$

Also, $2n \geq 2N \geq N$, so

$$|y_n - l| = |z_{2n} - l| < \varepsilon.$$

This completes the proof that $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = l$.

Now suppose $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$. Define l to be the common value. We use the formula

$$z_n = \begin{cases} x_{(n+1)/2} & n \text{ is odd} \\ y_{n/2} & n \text{ is even.} \end{cases}$$

Let $\varepsilon > 0$.

Choose $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, we have $|x_n - l| < \varepsilon$. Choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$, we have $|y_n - l| < \varepsilon$. Define $N = 2 \max(N_1, N_2)$.

Let $n \in \mathbb{N}$ with $n \geq N$. If n is odd, then

$$\frac{n+1}{2} \geq \frac{N+1}{2} \geq \frac{N}{2} \geq \max(N_1, N_2) \geq N_1.$$

Therefore

$$|z_n - l| = |x_{(n+1)/2} - l| < \varepsilon.$$

If n is even, then

$$\frac{n}{2} \geq \frac{N}{2} \geq \max(N_1, N_2) \geq N_2.$$

Therefore

$$|z_n - l| = |y_{n/2} - l| < \varepsilon.$$

This completes the proof that $\lim_{n \rightarrow \infty} z_n = l$. \square

Exercise 2.3.9.

- (a) Let (a_n) be a bounded (but not necessarily convergent) sequence in \mathbb{R} , and assume $\lim b_n = 0$. Show that $\lim_{n \rightarrow \infty} a_n b_n = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?
- (c) Use part (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution. (a) By definition, there is $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Now let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|b_n| < \frac{\varepsilon}{M+1}.$$

(Comment: We use $M+1$ in case $M=0$.)

Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$|a_n b_n| = |a_n| \cdot |b_n| \leq M |b_n| \leq \frac{M\varepsilon}{M+1} < \varepsilon.$$

This completes the proof that $\lim_{n \rightarrow \infty} a_n b_n = 0$.

The Algebraic Limit Theorem can't be applied because its hypotheses require that $\lim_{n \rightarrow \infty} a_n$ exists.

(c) We are assuming that (a_n) and (b_n) are sequences in \mathbb{R} such that $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} b_n = b$. We have to prove that $\lim_{n \rightarrow \infty} a_n b_n = 0$.

The sequence (b_n) is bounded, by Theorem 2.2.2 of the book. Therefore we may apply part (a), with (a_n) in place of (b_n) and (b_n) in place of (a_n) , to conclude $\lim_{n \rightarrow \infty} a_n b_n = 0$. \square

Exercise 2.3.10 Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

- (a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.
- (b) If $(b_n) \rightarrow b$, then $|b_n| \rightarrow |b|$
- (c) If $(a_n) \rightarrow a$ and $(b_n - a_n) \rightarrow 0$, then $b_n \rightarrow a$
- (d) If $(a_n) \rightarrow 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \rightarrow b$.

Solution. (a) This is false since (a_n) and (b_n) may not converge. For example, choose $a_n = b_n = (-1)^n$. Then $(a_n - b_n) \rightarrow 0$ is trivial, but neither $\lim a_n$ nor $\lim b_n$ exist.

This is true if both (a_n) and (b_n) are both convergence sequences.

(b) Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ satisfying $n \geq N$, we have $|b_n - b| < \varepsilon$.

Let $n \in \mathbb{N}$ satisfy $n \geq N$. Then (see Exercise 1.2.6(d))

$$||b_n| - |b|| \leq |b_n - b| < \varepsilon.$$

This shows $\lim_{n \rightarrow \infty} |b_n| = |b|$.

(c) Let $\varepsilon > 0$. Since $(a_n) \rightarrow a$, there is $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ we have $|a_n - a| < \varepsilon/2$. Since $(b_n - a_n) \rightarrow 0$, there is $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$ we have $|b_n - a_n| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$|b_n - a| \leq |b_n - a_n| + |a_n - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that $\lim_{n \rightarrow \infty} b_n = a$.

(d) The assumption $|b_n - b| \leq a_n$ implies that $a_n \geq 0$.

Let $\varepsilon > 0$. Since $\lim a_n = 0$, there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n| < \varepsilon$.

Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$|b_n - b| \leq a_n = |a_n| < \varepsilon.$$

This shows that $\lim_{n \rightarrow \infty} b_n = b$. \square

2. SECTION 2.4

Exercise 2.4.1.

- (a) Define a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R} recursively by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$. Prove that $(x_n)_{n \in \mathbb{N}}$ converges.

- (b) Now that we know that $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

- (c) Take the limit of each side of the recursive equation in part (a) of this exercise to explicitly compute $\lim x_n$.

Solution. (a) We prove by induction on n that $0 < x_{n+1} < x_n < 4$ for all $n \in \mathbb{N}$.

This is true for $n = 1$, since by definition $x_1 = 3$, while the recursive relation gives $x_2 = 1$.

For the induction step, suppose it is known that $0 < x_{n+1} < x_n < 4$. Then

$$4 - x_{n+1} > 4 - x_n > 0.$$

Therefore we can take inverses and get

$$0 < \frac{1}{4 - x_{n+1}} < \frac{1}{4 - x_n}.$$

That is,

$$0 < x_{n+2} < x_{n+1}.$$

Since we already know $x_{n+1} < x_n < 4$, this gives

$$0 < x_{n+2} < x_{n+1} < 4,$$

as desired. The induction is complete.

It follows that (x_n) is a bounded strictly decreasing sequence. Therefore $\lim x_n$ exists.

- (b) Define $x = \lim_{n \rightarrow \infty} x_n$.

Let $\varepsilon > 0$. Since $\lim x_n = x$, there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|x_n - x| < \varepsilon$.

Let $n \in \mathbb{N}$ with $n \geq N$. Then $n + 1 > n \geq N$. Therefore $|x_{n+1} - x| < \varepsilon$. This shows that $\lim x_{n+1} = x$.

(The fast method, available after the next section, is to observe that (x_{n+1}) is a subsequence of (x_n) .)

- (c) Define $x = \lim_{n \rightarrow \infty} x_n$. Then also $\lim_{n \rightarrow \infty} x_{n+1} = x$. Take limits as $n \rightarrow \infty$ on both sides of the equation

$$x_{n+1} = \frac{1}{4 - x_n},$$

and use the Algebraic Limit Theorem several times, to get

$$x = \frac{1}{4 - x}.$$

Rearrange to get

$$x^2 - 4x + 1 = 0.$$

This has the solutions

$$x = \frac{4 + \sqrt{12}}{2} \quad \text{or} \quad x = \frac{4 - \sqrt{12}}{2}.$$

Simplify to get

$$x = 2 + \sqrt{3} \quad \text{or} \quad x = 2 - \sqrt{3}.$$

Since $(x_n)_{n \in \mathbb{N}}$ is nonincreasing, we have $x \leq x_2 = 1 < 2 + \sqrt{3}$, so $x = 2 - \sqrt{3}$. \square

Exercise 2.4.2 (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y , we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

- (b) This time set $y_1 = 1$ and $y_{n+1} = 3 - 1/y_n$. Can the strategy in (a) be applied to compute the limit of the this sequence?

Solution. (a) The limit of y_n does not exist. Indeed, the sequence is $(1, 2, 1, 2, 1, 2, 1, 2, \dots)$ and it does not converge.

- (b) We first show that the limit exists in this case.

We prove by induction on n that $1 \leq y_n < y_{n+1} < 3$ for all $n \in \mathbb{N}$.

This is true for $n = 1$, since by definition $y_1 = 1$, while the recursive relation gives $y_2 = 2$.

For the induction step, suppose it is known that $1 \leq y_n < y_{n+1} < 3$. Then

$$1 \geq \frac{1}{y_n} > \frac{1}{y_{n+1}} > \frac{1}{3}.$$

Therefore

$$2 \leq 3 - \frac{1}{y_n} < 3 - \frac{1}{y_{n+1}} < 3 - \frac{1}{3}.$$

That is,

$$2 \leq y_{n+1} < y_{n+2} < 3 - \frac{1}{3}.$$

In particular,

$$1 \leq y_{n+1} < y_{n+2} < 3,$$

as desired. The induction is complete.

It follows that $(y_n)_{n \in \mathbb{N}}$ is a bounded strictly increasing sequence. Therefore $y = \lim y_n$ exists. As before, we then also have $\lim y_{n+1} = y$.

Take limits as $n \rightarrow \infty$ on both sides of the equation

$$y_{n+1} = 3 - \frac{1}{y_n},$$

and use the Algebraic Limit Theorem several times, to get

$$y = 3 - \frac{1}{y}.$$

Rearrange to get

$$y^2 - 3y + 1 = 0.$$

This has the solutions

$$y = \frac{3 + \sqrt{5}}{2} \quad \text{or} \quad y = \frac{3 - \sqrt{5}}{2}.$$

Since $(y_n)_{n \in \mathbb{N}}$ is nondecreasing, we have $y \geq y_1 = 1 > (3 - \sqrt{5})/2$, so $y = (3 + \sqrt{5})/2$. \square

Exercise 2.4.6 (Arithmetic-Geometric Mean).

- (a) Explain why $\sqrt{xy} \leq (x + y)/2$ for any two positive numbers x and y . (The geometric mean is always less than the arithmetic mean)
- (b) Now let $0 \leq x_1 \leq y_1$ and defined

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution. (a) For $x, y > 0$, $x - 2\sqrt{xy} + y = (\sqrt{x} - \sqrt{y})^2 \geq 0$. Hence, $x + y \geq 2\sqrt{xy}$ or $(x + y)/2 \geq \sqrt{xy}$.

(b) If $x_1 = 0$ then $x_n = 0$ for all n , and $y_{n+1} = y_n/2 = \dots = y_1/2^n$. Then both x_n and y_n converge to 0. Now, let $0 < x_1 < y_1$. We first prove $0 < x_n \leq y_n$ by induction. Assume we have proved $0 < x_n \leq y_n$. Then, by induction, $x_{n+1} = \sqrt{x_n y_n} > 0$ and $y_{n+1} = (x_n + y_n)/2 > 0$. Hence, by the inequality in (a),

$$0 < x_{n+1} = \sqrt{x_n y_n} \leq \frac{x_n + y_n}{2} = y_{n+1}.$$

Using the inequality $0 < x_n \leq y_n$, we obtain

$$x_{n+1} = \sqrt{x_n y_n} \geq \sqrt{x_n^2} = x_n, \quad n \geq 1,$$

so that $\{x_n\}$ is an increasing sequence and it is also bounded since y_1 is an upper bound. Thus, $\lim x_n$ converges by the Monotone Convergence Theorem. Moreover, using $0 < x_n \leq y_n$, we obtain

$$y_{n+1} = \frac{x_n + y_n}{2} \leq \frac{y_n + y_n}{2} = y_n, \quad n \geq 1,$$

so that y_n is a decreasing sequence and it is bounded from below since x_1 is a lower bound. Thus, $\lim y_n$ converges by the Monotone Convergence Theorem. Furthermore, let $x = \lim x_n$ and $y = \lim y_n$; taking limit $n \rightarrow \infty$ in $x_{n+1} = \sqrt{x_n y_n}$, we obtain $x = \sqrt{xy}$, or $\sqrt{x} = \sqrt{y}$, so that $x = y$. \square