Introduction to Abstract Algebra I: Homework 8

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Exercise 10.14. Find the index of $12\mathbb{Z}$ in $3\mathbb{Z}$.

Solution. The left cosets of $12\mathbb{Z}$ in $3\mathbb{Z}$ are given by

$$m + 12\mathbb{Z} = \{m, m + 12, m - 12, m + 24, m - 24, \ldots\}.$$

Taking m to be an element that isn't in $12\mathbb{Z}$, we find the distinct left cosets

$$0 + 12\mathbb{Z} = \{0, \pm 12, \pm 24, \ldots\}$$
$$3 + 12\mathbb{Z} = \{3, 15, -9, 27, -21, \ldots\}$$
$$6 + 12\mathbb{Z} = \{6, 18, -6, 30, -18, \ldots\}$$
$$9 + 12\mathbb{Z} = \{9, 21, -3, 33, -15, \ldots\}.$$

Thus, there are 4 distinct left cosets of $12\mathbb{Z}$ in $3\mathbb{Z}$, so we have $(3\mathbb{Z} : 12\mathbb{Z}) = 4$. □

Exercise 10.16. Let $\mu = (1, 2, 4, 5)(3, 6)$ in S_6. Find the index of $\langle \mu \rangle$ in S_6.

Solution. The order of μ is given by the least common multiple of the lengths of its disjoint cycles, which is $\mathrm{lcm}(4, 2) = 4$. Thus, the subgroup $\langle \mu \rangle$ has order 4. Since S_6 has order $6! = 720$, the index of $\langle \mu \rangle$ in S_6 is given by

$$(S_6 : \langle \mu \rangle) = |S_6|/|\langle \mu \rangle| = \frac{720}{4} = 180.$$

Thus, there are 180 distinct left cosets of $\langle \mu \rangle$ in S_6, so we have $(S_6 : \langle \mu \rangle) = 180$. □

Exercise 10.32. Let H be a subgroup of a group G and let $a, b \in G$. Prove or provide a counterexample: if $aH = bH$, then $Ha = Hb$.

Solution. The statement is false in general. Equality of right cosets need not imply equality of left cosets. For a counterexample, consider the group $G = D_4$, $H = \{\iota, \mu\}$, $a = \rho$, and $b = \mu\varphi^3$. Then, we have

$$aH = \{\rho, \rho\mu\} = \{\rho, \mu\rho^3\} = bH.$$

However, the left cosets are

$$Ha = \{\iota\rho, \mu\rho\} = \{\rho, \rho^3\mu\} \neq \{\mu\rho^3, \iota\mu\rho^3\} = Hb.$$

□

Exercise 10.34. Let H be a subgroup of a group G and let $a, b \in G$. Prove or provide a counterexample: if $Ha = Hb$, then $Ha^{-1} = Hb^{-1}$.

Solution. Suppose $Ha = Hb$. Then for any $h \in H$, there exists some $h' \in H$ such that $ha = h'b$. Rearranging this equation gives $hb^{-1} = h'a^{-1}$. Since $h, h' \in H$ and H is a subgroup, we have $hb^{-1} \in Ha^{-1}$. Therefore, for any $h \in H$, we have $hb^{-1} \in Ha^{-1}$. This implies that $Hb^{-1} \subseteq Ha^{-1}$.

By a similar argument, we can show that $Ha^{-1} \subseteq Hb^{-1}$. Therefore, we conclude that $Ha^{-1} = Hb^{-1}$. □

Exercise 12.10. Give the order of the element in the factor group $26 + \langle 12 \rangle$ in $\mathbb{Z}_{60}/\langle 12 \rangle$.

Solution. Suppose $Ha = Hb$. Then in particular $a \in Hb$, so there exists some $h \in H$ such that $a = hb$. Taking inverses of both sides gives

$$a^{-1} = b^{-1}h^{-1}.$$

Since $h^{-1} \in H$, this shows that $a^{-1} \in Hb^{-1}$, so $Ha^{-1} \subseteq Hb^{-1}$.

The reverse inclusion follows by a similar argument, so we conclude that $Ha^{-1} = Hb^{-1}$. □

Exercise 12.16. Compute $i_\rho[H]$ for the subgroup $H = \{\iota, \mu\}$ of the dihedral group D_3.

Solution. The map $i_\rho : D_3 \to D_3$ is defined by $i_\rho(x) = \rho x \rho^{-1}$. Since $H = \{\iota, \mu\}$, we compute

$$i_\rho(\iota) = \rho \iota \rho^{-1} = \iota.$$

For the reflection μ, using the relation $\mu\rho = \rho^{-1}\mu$, we have

$$i_\rho(\mu) = \rho\mu\rho^{-1} = \rho\mu\rho^2 = (\rho\mu)\rho^2 = (\rho^{-1}\mu)\rho^2 = \mu\rho.$$

Thus the image of H under i_ρ is

$$i_\rho[H] = \{\iota, \mu\rho\}. \qquad \square$$

Exercise 12.24. Let G_1 and G_2 be groups and $\pi_1 : G_1 \times G_2 \to G_1$ be the function defined by $\pi_1(a, b) = a$. Prove that π_1 is a homomorphism, find $\mathrm{Ker}(\pi_1)$, and prove $(G_1 \times G_2)/\mathrm{Ker}(\pi_1)$ is isomorphic to G_1.

Solution. Let $(a_1, b_1), (a_2, b_2) \in G_1 \times G_2$. Then we have

$$\pi_1((a_1, b_1)(a_2, b_2)) = \pi_1(a_1 a_2, b_1 b_2) = a_1 a_2 = \pi_1(a_1, b_1)\pi_1(a_2, b_2).$$

Thus, π_1 is a homomorphism.

The kernel of π_1 is the set of all elements in $G_1 \times G_2$ that look like (e_1, b) where e_1 is the identity in G_1 and b is any arbitrary element in G_2. Thus, we have

$$\mathrm{Ker}(\pi_1) = \{(e_1, b) \mid b \in G_2\} \cong \{e_1\} \times G_2.$$

By the Fundamental Homomorphism Theorem, we have

$$(G_1 \times G_2)/\mathrm{Ker}(\pi_1) \cong \pi_1[G_1 \times G_2] = G_1.$$

Therefore, we conclude that $(G_1 \times G_2)/\mathrm{Ker}(\pi_1)$ is homomorphic to G_1. It's also clear that this is an isomorphism since the mapping is bijective. $\qquad \square$

Exercise 12.27. Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G, and that G/T is torsion free. (See Exercise 22.)

Solution. The torsion subgroup T of an abelian group G is defined as the set of all elements in G that have finite order. To show that T is a normal subgroup of G, we need to verify that for any $g \in G$ and any $t \in T$, the element gtg^{-1} is also in T. Since G is abelian, we have

$$gtg^{-1} = t.$$

Since t has finite order, it follows that gtg^{-1} also has finite order, and thus $gtg^{-1} \in T$. Therefore, T is a normal subgroup of G.

Now we show that G/T is torsion free. Suppose that $gT \in G/T$ is a torsion element. Then $(gT)^n = T$ for some positive integer n. This means

$$g^n T = T,,$$

so $g^n \in T$, and hence g^n has finite order. Thus $(g^n)^k = e$ for some k, and so

$$g^{nk} = e.$$

Therefore g has finite order, which implies $g \in T$, and hence $gT = T$. This shows that the only torsion element in G/T is the identity element T itself.

Therefore G/T is torsion free. $\qquad \square$

Exercise 12.28. A subgroup H is *conjugate to a subgroup* K of a group G if there exists an inner automorphism i_g of G such that $i_g[H] = K$. Show that conjugacy is an equivalence relation on the collection of subgroups of G.

Solution. To show that conjugacy is an equivalence relation on the collection of subgroups of G, we need to verify that it satisfies the properties of reflexivity, symmetry, and transitivity.

For reflexivity, let H be a subgroup of G. The identity inner automorphism i_e (where e is the identity element of G) satisfies $i_e[H] = H$. Thus, H is conjugate to itself.

For symmetry, let H and K be subgroups of G such that H is conjugate to K. This means there exists an inner automorphism i_g such that $i_g[H] = K$. The inverse inner automorphism $i_{g^{-1}}$ satisfies $i_{g^{-1}}[K] = H$. Therefore, if H is conjugate to K, then K is conjugate to H.

And for transitivity, let H, K, and L be subgroups of G such that H is conjugate to K and K is conjugate to L. This means there exist inner automorphisms i_g and i_h such that $i_g[H] = K$ and $i_h[K] = L$. The composition of these inner automorphisms, i_{hg}, satisfies

$$i_{hg}[H] = i_h[i_g[H]] = i_h[K] = L.$$

Therefore, H is conjugate to L.

Since conjugacy satisfies reflexivity, symmetry, and transitivity, we conclude that it is an equivalence relation on the collection of subgroups of G. $\square$

Exercise 12.30. Find all subgroups of D_3 that are conjugate to $H = \{\iota, \mu\}$. (See Exercise 28.)

Solution. Recall
$$D_3 = \langle r, s \mid r^3 = e,\ s^2 = e,\ srs = r^{-1} \rangle = \{e, r, r^2, s, rs, r^2 s\}.$$

Let $H = \{e, s\}$. Conjugation by powers of r permutes the reflections, since

$$rsr^{-1} = rs \quad \text{and} \quad r^2 s r^{-2} = r^2 s.$$

Thus the conjugates of H are the subgroups generated by the three reflections

$$\{e, s\}, \qquad \{e, rs\}, \quad \text{and} \quad \{e, r^2 s\}.$$

These are distinct order-2 subgroups, and they form the conjugacy class of H. $\square$