Orthogonal Complements and orthogonal Projection.

Definition. Let $W \subseteq V$ be a subspace of V. Then the orthogonal complement of W, denoted by $W^\perp$, is the set of vectors in V that are orthogonal to every vector in W.

i.e. $W^\perp = \{\vec{x} \in V : (\vec{x}, \vec{w}) = 0 \text{ for all } \vec{w} \in W\}$.

Proposition: 1) $W^\perp$ is a subspace of V.

2). Let $B \subseteq W$ be a basis. Then $\vec{x} \in W^\perp$ iff $\vec{x}$ is orthogonal to every vector in B.

3). $W \cap W^\perp = \{\vec{0}\}$, thus the sum $W + W^\perp$ is a direct sum.

4). $V = W \oplus W^\perp$. In particular, $\dim V = \dim W + \dim W^\perp$.

<u>Proof</u>: 1). ① $\vec{0} \in W^\perp \Rightarrow W^\perp \neq \phi$.

② $\forall \vec{x}, \vec{y} \in W^\perp$ and $c \in F$.

$\forall \vec{w} \in W$: $(c\vec{x} + \vec{y}, \vec{w}) = c(\vec{x}, \vec{w}) + (\vec{y}, \vec{w}) = 0 \Rightarrow c\vec{x} + \vec{y} \in W^\perp$.

$\Rightarrow W^\perp$ is a subspace.

2) "$\Leftarrow$" Let $B = \{\vec{y}_1, \cdots, \vec{y}_k\}$ be a basis of W.

Let $\vec{x} \in V$ such that $(\vec{x}, \vec{y}_i) = 0$ for all $i = 1, \cdots, n$

$\forall \vec{y} \in W$: $\vec{y} = \sum_{i=1}^{k} a_i \vec{y}_i$.

Then $(\vec{x}, \vec{y}) = (\vec{x}, \sum_{i=1}^{k} a_i \vec{y}_i) = \sum_{i=1}^{k} \overline{a_i} (\vec{x}, \vec{y}_i) = 0$

$\Rightarrow \vec{x} \in W^\perp$.

"$\Rightarrow$" obvious by the definition of $W^\perp$.

3). $\forall \vec{x} \in W \cap W^{\perp}$, Viewing $\vec{x} \in W$ and $\vec{x} \in W^{\perp}$: $(\vec{x}, \vec{x}) = 0 \Rightarrow \vec{x} = \vec{0}$.

4) Suppose $\{\vec{y}_1, \cdots, \vec{y}_k\}$ is an orthogonal basis of W.

$\forall \vec{x} \in V$: let $\vec{x}_w = \sum_{i=1}^{k} \frac{(\vec{x}, \vec{y}_i)}{(\vec{y}_i, \vec{y}_i)} \vec{y}_i \in W$.

Then by the previous lemma $(\vec{x} - \vec{x}_w, \vec{y}_i) = 0$, for all $i=1, \cdots, k$

$\Rightarrow x - \vec{x}_w \in W^{\perp}$ (by 2))

$\Rightarrow \vec{x} = \vec{x}_w + (x - \vec{x}_w)$

$\Rightarrow V \subseteq W \oplus W^{\perp}$.

Definition Let $\vec{x} \in V$. By the above proposition there exists a unique pair of vectors $\vec{x}_w \in W$ and $\vec{z} \in W^{\perp}$ such that $\vec{x} = \vec{x}_w + \vec{z}$. Then $\vec{x}_w$ is called the orthogonal projection of $\vec{x}$ onto W.

Remarks: By the proof of 4) in the above proposition, by fixing an orthonormal basis of W, one may explicitly compute $\vec{x}_w$.

- The adjoint of a linear transformation

Theorem. Let V be a finite dimensional inner product space. Let $g: V \to F$ be a linear functional. Then there exists a unique $\vec{y} \in V$ such that $g(\vec{x}) = (\vec{x}, \vec{y})$ for any $\vec{x} \in V$.

Proof: Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an orthonormal basis of V.

[Idea: Let $\vec{y} = \sum_{i=1}^{n} a_i \vec{v}_i$ we need to compute a_i.

Then $\vec{y}$ satisfies: $g(\vec{v}_k) = (\vec{v}_k, \vec{y})$ for all $k = 1, \cdots, n$

$$= (\vec{v}_k, \sum_{i=1}^{n} a_i \vec{v}_i)$$

$$= (\vec{v}_k, a_k \vec{v}_k) = \bar{a}_k (\vec{v}_k, \vec{v}_k) = \bar{a}_k$$

$\Rightarrow \overline{g(\vec{v}_k)} = a_k$]

Existence: Let $\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)}\, \vec{v}_i$. Then for any $\vec{x} = \sum_{j=1}^{n} c_j \vec{v}_j \in V$

$$(\vec{x}, \vec{y}) = \left(\sum_{j=1}^{n} c_j \vec{v}_i, \sum_{j=1}^{n} \overline{g(\vec{v}_i)}\, \vec{v}_i \right)$$

$$= \sum_{i=1}^{n} c_i\, g(\vec{v}_i) (\vec{v}_i, \vec{v}_i)$$

$$= \sum_{i=1}^{n} g(c_i \vec{v}_i) = g\left(\sum_{i=1}^{n} c_i \vec{v}_i \right) = g(\vec{x}).$$

Uniqueness: If there exists y and y' such that $g(\vec{x}) = (\vec{x}, \vec{y}) = (\vec{x}, \vec{y}')$ for all $\vec{x}$

$$\Rightarrow \quad (\vec{x}, \vec{y}-\vec{y}\,') = 0 \quad \text{for all } \vec{x} \in V.$$

In particular, take $\vec{x} = \vec{y} - \vec{y}\,'$ then $(\vec{y}-\vec{y}\,', \vec{y}-\vec{y}\,') = 0$

$$\Rightarrow \vec{y} - \vec{y}\,' = \vec{0}$$

$$\Rightarrow \vec{y} = \vec{y}\,'.$$

Theorem: Let V be a finite dimensional inner product space. Let T be a linear transformation on V. Then there exists a unique function $T^*: V \to V$ such that

$$(T(\vec{x}), \vec{y}) = (\vec{x}, T^*(y)) \text{ for all } \vec{x}, \vec{y} \in V.$$

Furthermore, T^* is linear.

Def'n: The linear transformation $T^*: V \to V$ satisfying $(T(x), y) = (x, T^*(y))$ for all $\vec{x}, \vec{y} \in V$ is called the adjoint of linear transformation T

Examples: 1). Let $A \in \mathbb{R}^{n\times n}$. Consider standard inner product on $\mathbb{R}^n$

and $L_A: \mathbb{R}^n \to \mathbb{R}^n$ w/ $L_A(\vec{x}) = A\vec{x}$.

Then $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$: $(L_A(\vec{x}), \vec{y}) = (A\vec{x}, \vec{y}) = (A\vec{x})^T \vec{y}$

$$= \vec{x}^T A^T \vec{y} = (\vec{x}, \underbrace{A^T \vec{y}}_{(L_A)^*(\vec{y})}).$$

$$\Rightarrow (L_A)^* = L_{A^T}.$$

2). Let $A \in \mathbb{C}^{n\times n}$. Consider standard inner product on $\mathbb{C}^n$

and $L_A : \mathbb{C}^n \to \mathbb{C}^n$ w/ $L_A(\vec{x}) = A\vec{x}$.

Then $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$: $(L_A(\vec{x}), \vec{y}) = (A\vec{x}, \vec{y}) = y^* A\vec{x}$

$$= (A^*\vec{y})^* \vec{x}$$

$$= (\vec{x}, \underbrace{A^*\vec{y}}_{(L_A)^*(\vec{y})})$$

$$\Rightarrow (L_A)^* = L_{A^*}.$$

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Proof of the theorem.

For each $\vec{y} \in V$, consider the function $g_{\vec{y}} : \vec{V} \to F$ defined

by $g_y(\vec{x}) = (T(\vec{x}), \vec{y})$. for any $\vec{x} \in V$.

Then for any $\vec{x}_1, \vec{x}_2 \in V$ and $c \in F$:

$$g_{\vec{y}}(c\vec{x}_1 + \vec{x}_2) = (T(c\vec{x}_1 + \vec{x}_2), \vec{y})$$

$$= (cT(\vec{x}_1) + T(\vec{x}_2), \vec{y})$$

$$= c(T(\vec{x}_1), \vec{y}) + (T(\vec{x}_2), \vec{y})$$