Functional Complex Variables I: Homework 8

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Exercise 5.59.3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 + (z^4/9)}.$$

Solution. Using expression (6) in Sec. 59, we have that

$$\frac{1}{1 + (z^4/9)} = \sum_{n=0}^{\infty} \left(-\frac{z^4}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} z^{4n}.$$

Therefore, $f(z)$ can be expressed as

$$f(z) = \frac{z}{9} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} z^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} z^{4n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1}.$$

This only holds for $\left|z^4\right| < 9$, or equivalently, $|z| < \sqrt[4]{9} = \sqrt{3}$. □

Exercise 5.59.4. Show that if $f(z) = \sin(z)$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \qquad (n = 0, 1, 2, \dots).$$

Thus, give an alternative derivation of the Maclaurin series (2) for $\sin(z)$ in Sec. 59.

Solution. Let $f(z) = \sin(z)$. We compute the first few derivatives of $\sin(z)$ and evaluate them at $z = 0$ to get

$$\begin{aligned}
f(z) &= \sin(z) \Rightarrow f(0) &&= 0 \\
f'(z) &= \cos(z) \Rightarrow f'(0) &&= 1 \\
f''(z) &= -\sin(z) \Rightarrow f''(0) &&= 0 \\
f^{(3)}(z) &= -\cos(z) \Rightarrow f^{(3)}(0) &&= -1 \\
f^{(4)}(z) &= \sin(z) \Rightarrow f^{(4)}(0) &&= 0 \\
&\vdots
\end{aligned}$$

Notice, we have the following pattern

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n, \quad n = 0, 1, 2, \dots$$

Therefore, the Maclaurin series for $\sin(z)$ is

$$\sin(z) = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

which matches the known Maclaurin series for $\sin(z)$. □

Exercise 5.59.11. Show that when $z \neq 0$,

(i) $\displaystyle \frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots;$

(ii) $\displaystyle \frac{\sin\left(z^2\right)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots.$

Solution to (i). We can express e^z as its Maclaurin series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Dividing this series by z^2, we get

$$\frac{e^z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots.$$

This series converges for all $z \neq 0$. $\square$

Solution to (ii). The Maclaurin series for $\sin(z)$ is given by

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Substituting z^2 for z, we have

$$\sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}.$$

Dividing this series by z^4, we get

$$\frac{\sin(z^2)}{z^4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n-2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2(2n-1)} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots.$$

This series converges for all $z \neq 0$. $\square$

Exercise 5.59.13. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Solution. We can factor the denominator as follows:

$$4z - z^2 = z(4 - z).$$

Thus, we can rewrite the expression as

$$\frac{1}{4z - z^2} = \frac{1}{z(4-z)} = \frac{1}{4z} \cdot \frac{1}{1 - z/4}.$$

The series expansion for $\frac{1}{1-x}$ is given by

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

which converges for $|x| < 1$. In our case, we have $x = z/4$, so the series converges for $|z/4| < 1$, or equivalently, $|z| < 4$.

Therefore, we can write

$$\frac{1}{4z - z^2} = \frac{1}{4z} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$ $\square$

Exercise 6.74.3. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,\mathrm{d}z,$$

taken counterclockwise around the circle (i) $|z-2|=2$; (ii) $|z|=4$.

Solution to (i). The poles of the integrand are at $z=1$ and $z=\pm 3i$. The circle $|z-2|=2$ contains the pole at $z=1$ but not the poles at $z=3i$ and $z=-3i$. We can use the residue theorem to evaluate the integral. The function $f(z)$ can be written as

$$f(z)=\frac{\Phi(z)}{z-1} \quad \text{where} \quad \Phi(z)=\frac{3z^3+2}{z^2+9}.$$

Since $\Phi(z)$ is analytic on the contour $|z-2|=2$, we can find the residue at the pole $z=1$. The residue at $z=1$ is given by

$$\operatorname*{Res}_{z=1} f(z)=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\frac{3z^3+2}{(z-1)(z^2+9)}=\lim_{z\to 1}\frac{3z^3+2}{z^2+9}=\frac{1}{2}.$$

By the residue theorem, the value of the integral is given by

$$I=2\pi i\cdot\operatorname*{Res}_{z=1}\left(\frac{3z^3+2}{(z-1)(z^2+9)}\right)=2\pi i\cdot\frac{1}{2}=\pi i. \qquad \square$$

Solution to (ii). The circle $|z|=4$ contains all three poles: $z=1$, $z=3i$, and $z=-3i$. We can find the residues at each of these poles.

For the pole at $z=1$, we already calculated the residue, specifically,

$$\operatorname*{Res}_{z=1} f(z)=\frac{1}{2}.$$

Now, we calculate the residues at the poles $z=3i$ and $z=-3i$.

The residue at $z=\pm 3i$ are given by

$$\begin{aligned}
\operatorname*{Res}_{z=3i} f(z)=\lim_{z\to 3i}(z-3i)f(z) &= \lim_{z\to 3i}\frac{3z^3+2}{(z-1)(z+3i)} &&= \frac{3(3i)^3+2}{(3i-1)(3i+3i)} &&= \frac{15+49i}{12} \\
\operatorname*{Res}_{z=-3i} f(z)=\lim_{z\to -3i}(z+3i)f(z) &= \lim_{z\to -3i}\frac{3z^3+2}{(z-1)(z-3i)} &&= \frac{3(-3i)^3+2}{(-3i-1)(-3i-3i)} &&= \frac{15-49i}{12}.
\end{aligned}$$

Now, we can sum the residues

$$\operatorname*{Res}_{z=1} f(z)+\operatorname*{Res}_{z=3i} f(z)+\operatorname*{Res}_{z=-3i} f(z)=\frac{1}{2}+\frac{15+49i}{12}+\frac{15-49i}{12}=3.$$

By the residue theorem, the value of the integral is given by

$$I=2\pi i\cdot\left(\operatorname*{Res}_{z=1} f(z)+\operatorname*{Res}_{z=3i} f(z)+\operatorname*{Res}_{z=-3i} f(z)\right)=2\pi i\cdot 3=6\pi i. \qquad \square$$