Introduction to Toplogy I: Homework 1

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Exercise 1.1.

(i) For each of the three metrics in Example 1.4, sketch the open ball of some radius $r > 0$ around the origin in $\mathbb{R}^2$:

$$B_r(0) = \{(x, y) \in \mathbb{R}^2 \mid \mathrm{d}((x, y), 0) < r\}.$$

(ii) For one of the three metrics (your choice), prove or give a counterexample to the following statement: a sequence of points $(x_1, y_1), (x_2, y_2), \cdots \in \mathbb{R}^2$ converges to a limit (x, y) if and only if $x_n \to x$ and $y_n \to y$ separately, as sequences in $\mathbb{R}$ with the usual metric.

(iii) Why is

$$\mathrm{d}(\mathbf{x}, \mathbf{y}) = \min(\{|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|\}),$$

not a metric on $\mathbb{R}^n$?

Solution to i. For the Euclidean metric, we have $\mathrm{d}_2((x, y), (0, 0))$, we have

$$B_r^{(2)}(0) = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} < r\} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}.$$

This is an open disk of radius r centered ta the origin.

For the taxicab metric, we have $\mathrm{d}_1((x, y), (0, 0))$, we have

$$B_r^{(1)}(0) = \{(x, y) \in \mathbb{R}^2 \mid |x|+|y|< r\}.$$

This is the interior of a diamond (a square rotated by 45°) with vertices at $(r, 0)$, $(0, r)$, $(-r, 0)$, and $(0, -r)$.

For the supremum metric, we have $\mathrm{d}_\infty((x, y), (0, 0))$, we have

$$B_r^{(\infty)}(0) = \{(x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) < r\} = \{(x, y) \in \mathbb{R}^2 \mid |x|< r, |y|< r\}.$$

This is the interior of an axis-aligned square with vertices at (r, r), $(-r, r)$, $(-r, -r)$, and $(r, -r)$.

Graphing these three shapes, we get Figure 1. □

Solution to ii. Take the Euclidean metric d_2. Assume that $(x_n, y_n) \to (x, y)$ in the Euclidean metric. Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathrm{d}_2((x_n, y_n), (x, y)) < \varepsilon$. Notice that for all $n \geq N$, we have

$$|x_n - x|\leq \sqrt{(x_n - x)^2 + (y_n - y)^2} = \mathrm{d}_2((x_n, y_n), (x, y)) < \varepsilon,$$

and similarly $|y_n - y|< \varepsilon$. Hence, $x_n \to x$ and $y_n \to y$ separately.

For the converse, assume that $x_n \to x$ and $y_n \to y$. Let $\varepsilon > 0$ be arbitrary. Choose $N = \max(\{N_1, N_2\})$, where $N_1, N_2 \in \mathbb{N}$ are such that for all $n \geq N_1$, $|x_n - x|< \varepsilon/\sqrt{2}$ and for all $n \geq N_2$, $|y_n - y|< \varepsilon/\sqrt{2}$. Then, for all $n \geq N$,

$$\mathrm{d}_2((x_n, y_n), (x, y)) = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon.$$

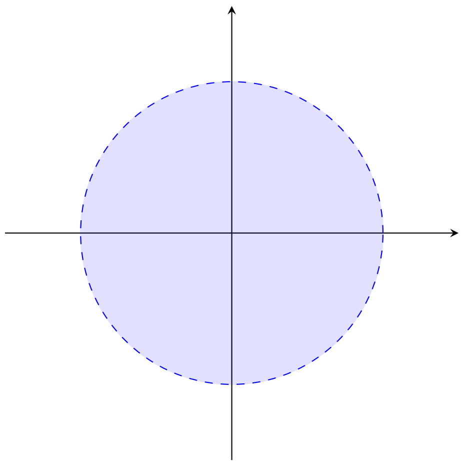
Hence, $(x_n, y_n) \to (x, y)$ in the Euclidean metric.

Therefore, a sequence (x_n, y_n) converges to (x, y) in the Euclidean metric if and only if $x_n \to x$ and $y_n \to y$ separately. □

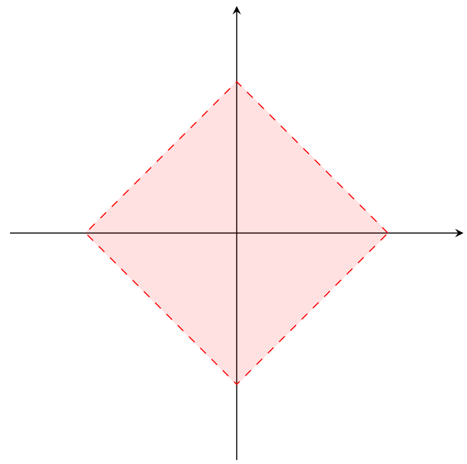
Solution to iii. Clearly, $\mathrm{d}(\mathbf{x}, \mathbf{y})$ satisfies the first property of a metric. However, it fails the second identity, since if two points agree in at least one coordinate, then $\mathrm{d}(\mathbf{x}, \mathbf{y}) = 0$ even if $\mathbf{x} \neq \mathbf{y}$. For example,

$$\mathrm{d}((0, 0), (0, 1)) = \min\{|0 - 0|, |0 - 1|\} = 0,$$

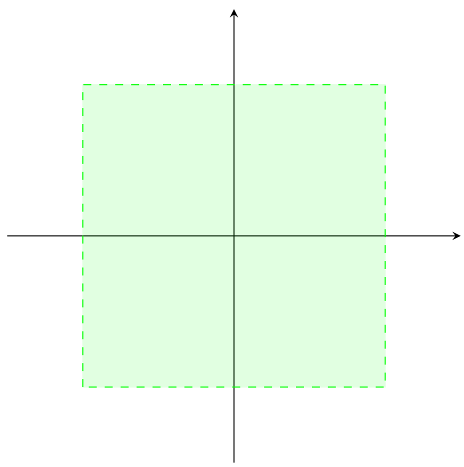
although $(0, 0) \neq (0, 1)$. Thus d is not a metric on $\mathbb{R}^n$. □



Euclidean (d_2): circle of radius r



Taxicab (d_1): diamond (rotated square)



Supremum (d_∞): axis-aligned open square

Figure 1: Open balls of radius r around the origin in $\mathbb{R}^2$ for the three metrics.

Exercise 1.3. Consider the following silly metric on $\mathbb{R}^2$:

$$\mathrm{d}((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1 - y_2| + 1 & \text{if } x_1 \neq x_2 \end{cases}.$$

(i) Prove that d is a metric, that is, it has the three properties listed in Definition 1.2.

(ii) Sketch the open balls of radius 1/2, 1, and 2 around the origin in this metric.

(iii) Give an example of a sequence that converges in the Euclidean metric d_2 but not in our silly metric d.

(iv) Prove that every sequence that converges in d also converges d_2.

Solution to i. Clearly, $\mathrm{d}((x_1, y_1), (x_2, y_2)) = \mathrm{d}((x_2, y_2), (x_1, y_1))$. Also $\mathrm{d}((x_1, y_1), (x_2, y_2)) = 0$ if and only if $(x_1, y_1) = (x_2, y_2)$.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$. For the triangle inequality, observe that we may write

$$\mathrm{d}((x_i, y_i), (x_j, y_j)) = |y_i - y_j| + \delta_{x_i}^{x_j},$$

where δ_a^b is the indicator function that is 0 if $a = b$ and 1 if $a \neq b$. The usual triangle inequality in $\mathbb{R}$ gives $|y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3|$, and the indicator satisfies

$$\delta_{x_1}^{x_3} \leq \delta_{x_1}^{x_2} + \delta_{x_2}^{x_3}.$$

Adding these inequalities yields

$$\mathrm{d}((x_1, y_1), (x_3, y_3)) \leq \mathrm{d}((x_1, y_1), (x_2, y_2)) + \mathrm{d}((x_2, y_2), (x_3, y_3)),$$

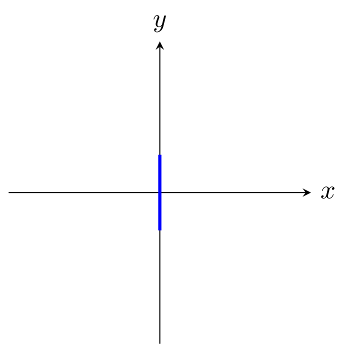
so the triangle inequality holds. Therefore d is a metric on $\mathbb{R}^2$. □

Solution to ii. In this metric, points with the same x-coordinate have distance $|y_1 - y_2|$, while points with different x-coordinates have distance $|y_1 - y_2| + 1$. Hence, for any $r > 0$,

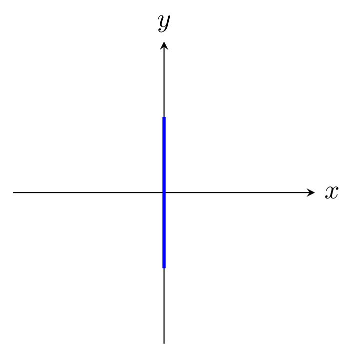
$$B_{\mathrm{d}}((0,0), r) = \{(0, y) \mid x = 0 \text{ and } |y| < r\} \cup \{(x, y) \mid x \neq 0 \text{ and } |y| < r - 1\}.$$

The second set is empty if $r \leq 1$. Thus:

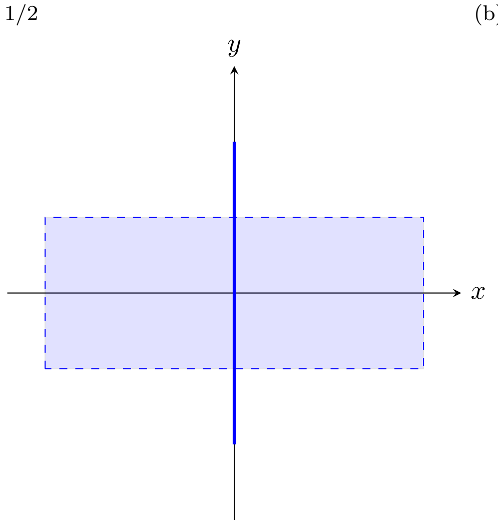
$$\begin{aligned} B_{\mathrm{d}}((0,0), 1/2) &= \{(0, y) \mid x = 0 \text{ and } |y| < 1/2\} \\ B_{\mathrm{d}}((0,0), 1) &= \{(0, y) \mid x = 0 \text{ and } |y| < 1\} \\ B_{\mathrm{d}}((0,0), 2) &= \{(0, y) \mid x = 0 \text{ and } |y| < 2\} \cup \{(x, y) \mid x \neq 0 \text{ and } |y| < 1\}. \end{aligned}$$



(a) Radius 1/2



(b) Radius 1



(c) Radius 2

Figure 2

Graphing these three shapes, we get Figure 2. □

Solution to iii. Take $(a_n) = (1/n, 0)$. Then (a_n) converges to $(0,0)$ in the Euclidean metric d_2 since

$$\mathrm{d}_2(a_n, (0,0)) = \sqrt{(1/n - 0)^2 + (0-0)^2} = 1/n \to 0.$$

But in the silly metric d, we have

$$\mathrm{d}(a_n, (0,0)) = |0-0|+1 = 1 \not\to 0.$$

Hence, (a_n) does not converge to $(0,0)$ in the silly metric. □

Solution to iv. Let $(a_n) = (x_n, y_n)$ converge to $a = (x, y)$ in the metric d, so $\mathrm{d}(a_n, a) \to 0$. Choose $\varepsilon = 1/2$. Then, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\mathrm{d}(a_n, a) < 1/2$. But if $x_n \neq x$ then $\mathrm{d}(a_n, a) = |y_n - y|+1 \geq 1$, contradiction. Hence $x_n = x$ for all $n \geq N_1$.

Now, let $\varepsilon > 0$ be arbitrary. Since $\mathrm{d}(a_n, a) \to 0$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $\mathrm{d}(a_n, a) < \varepsilon$. For $n \geq N := \max(\{N_1, N_2\})$, we have $x_n = x$ and so $\mathrm{d}(a_n, a) = |y_n - y| < e$. Hence, for $n \geq N$,

$$\mathrm{d}_2(a_n, a) = \sqrt{(x_n - x)^2 + (y_n - y)^2} = |y_n - y| < \varepsilon.$$

Therefore $\mathrm{d}_2(a_n, a) \to 0$, and (a_n) converges to a in the Euclidean metric. □

Exercise 1.4. Let X be any set, and let d_X be the *discrete metric*

$$\mathrm{d}_X(p,q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q \end{cases}.$$

(i) Prove that d_X is a metric

(ii) Let (Y, d_Y) be another metric space (not necessarily discrete). Prove that every map $f : X \to Y$ is continuous.

(iii) Prove that a sequence $p_1, p_2, p_3, \dots \in X$ converges in the discrete metric if and only if it is eventually constant.

Solution to i. Clearly, $\mathrm{d}_X(p,q) = \mathrm{d}_X(q,p)$ for all $p, q \in X$. Also, $\mathrm{d}_X(p,q) = 0$ if and only if $p = q$. We now prove that the metric satisfies the triangle inequality. Let $p, q, r \in X$. Assume any two of the points are equal, say $p = q$. Then,

$$\mathrm{d}_X(p,r) = \mathrm{d}_X(q,r) \leq \mathrm{d}_X(p,q) + \mathrm{d}_X(q,r) = 0 + \mathrm{d}_X(q,r) = \mathrm{d}_X(p,r).$$

The cases $p = r$ and $q = r$ are similar. So, assume that p, q, r are all distinct. Then,

$$\mathrm{d}_X(p,r) = 1 \leq \mathrm{d}_X(p,q) + \mathrm{d}_X(q,r) = 1 + 1 = 2.$$

Thus, d_X is a metric on X. □

Solution to ii. Let $p \in X$ and $\varepsilon > 0$. Choose $\delta = 1/2$. Then $B(p, \delta) = \{p\}$. Thus, if $q \in B(p, \delta)$, we must have $q = p$, and so

$$d_Y(f(p), f(q)) = d_Y(f(p), f(p)) = 0 < \varepsilon.$$

Hence f is continuous at p. Since $p \in X$ was arbitrary, f is continuous on X. □

Solution to iii. Assume that the sequence $(p_n) = (p_1, p_2, p_3, \dots)$ converges to $p \in X$. By definition, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathrm{d}_X(p_n, p) < \varepsilon$. Choose $\varepsilon = 1/2$. Then, for all $n \geq N$, $\mathrm{d}_X(p_n, p) < 1/2$. Since the distance between any two distinct points in X is 1, this implies that $p_n = p$ for all $n \geq N$. Thus, the sequence is eventually constant.

Conversely, assume that the sequence (p_n) is eventually constant. Then there exists $N \in \mathbb{N}$ and $p \in X$ such that $p_n = p$ for all $n \geq N$. Let $\varepsilon > 0$ be arbitrary. For this N, we have $\mathrm{d}_X(p_n, p) = 0 < \varepsilon$ whenever $n \geq N$. Hence, by definition, (p_n) converges to p. □

Exercise 1.11. Let (X, d_X) and (Y, d_Y) be metric spaces, let $(p_n) = (p_1, p_2, p_3, \cdots)$ be a sequence that converges to a point ℓ in X, and let $f : X \to Y$ be continuous at ℓ. Prove that the sequence $f(p_n) = f(p_1), f(p_2), f(p_3), \cdots$ converges to $f(\ell)$ in Y.

Solution. Since f is continuous at ℓ (since $\ell \in X$), for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$ with $\mathrm{d}_X(x, \ell) < \delta$, we have $\mathrm{d}_Y(f(x), f(\ell)) < \varepsilon$. Since (p_n) converges to ℓ, for this $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathrm{d}_X(p_n, \ell) < \delta$. Therefore, for all $n \geq N$, we have $\mathrm{d}_Y(f(p_n), f(\ell)) < \varepsilon$. This shows that the sequence $(f(p_n))$ converges to $f(\ell)$ in Y. $\square$