Multi-Variable Calculus I: Homework 8

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Problem 1

Find the tangent plane to the surface

$$x^3 y \cos(z^2 + x - y) = 3,$$

at the point $P = (-1, 3, 2)$.

Solution 1

Let $f(x, y, z) = x^3 y \cos(z^2 + x - y) - 3$. The surface is defined implicitly by $f(x, y, z) = 0$.

The gradient of f, $\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$, is normal to the tangent plane. Thus, we compute the partial derivatives of f

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2 y \cos(z^2 + x - y) - x^3 y \sin(z^2 + x - y) \\
\frac{\partial f}{\partial y} &= x^3 \cos(z^2 + x - y) + x^3 y \sin(z^2 + x - y) \\
\frac{\partial f}{\partial z} &= -2zx^3 y \sin(z^2 + x - y) \\
.
\end{aligned}$$

At P, substitute $x = -1$, $y = 3$, and $z = 2$ into $\nabla f(x, y, z)$.

$$\nabla f(-1, 3, 2) = \langle 3(1)(3)(1) - (-1)(3)(0), -1(1) + (-1)(3)(0), -2(2)(-1)(3)(0) \rangle = \langle 9, -1, 0 \rangle.$$

The equation of the tangent plane at $P = (-1, 3, 2)$ is given by

$$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0.$$

Substituting $\nabla f(P) = (9, -1, 0)$ and $P = (-1, 3, 2)$ will give us the equation of the tangent plane. Thus, we get

$$9x - y + 12 = 0.$$

Problem 2

Find and classify (local maximum, local minimum, or saddle) the critical points of the functions.

(i) $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$.

(ii) $f(x,y) = e^y(y^2 - x^2)$.

Solution 2

(i) Let $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$. To find the critical points, we compute the first partial derivatives of f

$$\frac{\partial f}{\partial x} = 6x^2 + y^2 + 10x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy + 2y.$$

Set $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ to solve for x and y.

$$\begin{aligned} \frac{\partial f}{\partial y} &= 0 \Rightarrow 2y(x+1) \ = 0 \Rightarrow y &&= 0 &&\text{or} \quad x = -1 \\ \frac{\partial f}{\partial x} &= 0 \Rightarrow 6x^2 + 10x = 0 \Rightarrow x(6x+10) &&= 0 \Rightarrow x = 0 &&\text{or} \quad x = -\frac{5}{3}. \end{aligned}$$

Thus, the critical points are $(0,0)$ and $(-5/3, 0)$. Substituting $x = -1$ into $\frac{\partial f}{\partial x} = 0$ gives us

$$6(-1)^2 + y^2 + 10(-1) = 0 \Rightarrow 6 + y^2 - 10 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$$

Thus, the additional critical points are $(-1, 2)$ and $(-1, -2)$.

To classify the critical points, we compute the second partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = 12x + 10, \quad \frac{\partial^2 f}{\partial y^2} = 2x + 2, \quad \text{and} \quad \frac{\partial f}{\partial x} y = 2y.$$

The Hessian determinant is

$$H = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial x} y\right)^2 = (12x+10)(2x+2) - (2y)^2.$$

(a) At $(0,0)$, we get the following values

$$\frac{\partial^2 f}{\partial x^2} = 10, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 f}{\partial x\, \partial y} = 0.$$

Plugging these values into the Hessian determinant gives us

$$H = (10)(2) - (0)^2 = 20 > 0, \quad \frac{\partial^2 f}{\partial x^2} > 0.$$

Thus, $(0,0)$ is a local minimum.

(b) At $(-5/3, 0)$, we get the following values

$$\frac{\partial^2 f}{\partial x^2} = 12(-\frac{5}{3}) + 10 = -10, \quad \frac{\partial^2 f}{\partial y^2} = 2(-\frac{5}{3}) + 2 = -\frac{4}{3}, \quad \text{and} \quad \frac{\partial f}{\partial x} y = 0.$$

Plugging these values into the Hessian determinant gives us

$$H = (-10)(-\frac{4}{3}) - (0)^2 = \frac{40}{3} > 0, \quad \frac{\partial^2 f}{\partial x^2} < 0.$$

Thus, $(-5/3, 0)$ is a local maximum.

(c) At $(-1, 2)$ and $(-1, -2)$, we get the following values

$$\frac{\partial^2 f}{\partial x^2} = 12(-1) + 10 = -2, \quad \frac{\partial^2 f}{\partial y^2} = 2(-1) + 2 = 0, \quad \text{and} \quad \frac{\partial f}{\partial x}y = 4 \quad \text{or} \quad -4.$$

Plugging these values into the Hessian determinant gives us

$$H = (-2)(0) - (4)^2 = -16 < 0.$$

Thus, $(-1, 2)$ and $(-1, -2)$ are saddle points.

(ii) Let $f(x, y) = e^y(y^2 - x^2)$. To find the critical points, compute the first partial derivatives

$$\frac{\partial f}{\partial x} = -2xe^y \quad \text{and} \quad \frac{\partial f}{\partial y} = e^y(2y + y^2 - x^2).$$

Set $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ to get

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0 \Rightarrow -2xe^y &&= 0 \Rightarrow x &&= 0 \\ \frac{\partial f}{\partial y} &= 0 \Rightarrow e^y(2y + y^2) &&= 0 \Rightarrow 2y + y^2 &&= 0 \Rightarrow y(y+2) = 0 \Rightarrow y = 0 \quad \text{or} \quad y = -2. \end{aligned}$$

Thus, the critical points are $(0, 0)$ and $(0, -2)$.

To classify the critical points, compute the second partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = -2e^y, \quad \frac{\partial^2 f}{\partial y^2} = e^y(2 + 2y + y^2 - x^2), \quad \text{and} \quad \frac{\partial f}{\partial x}y = -2xe^y.$$

The Hessian determinant is

$$H = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2.$$

(a) At $(0, 0)$, we get the following values

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial f}{\partial x}y = 0.$$

Plugging these values into the Hessian determinant gives us

$$H = (-2)(2) - (0)^2 = -4 < 0.$$

Thus, $(0, 0)$ is a saddle point.

(b) At $(0, -2)$, we get the following values

$$\frac{\partial^2 f}{\partial x^2} = -2e^{-2}, \quad \frac{\partial^2 f}{\partial y^2} = e^{-2}(2 - 4 + 4) = 2e^{-2}, \quad \text{and} \quad \frac{\partial f}{\partial x}y = 0.$$

Plugging these values into the Hessian determinant gives us

$$H = (-2e^{-2})(2e^{-2}) - (0)^2 = -4e^{-4} > 0, \quad \frac{\partial^2 f}{\partial x^2} < 0.$$

Since $H > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, the critical point $(0, -2)$ is a local maximum.