Introduction to Toplogy I: Homework 9

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Nicolas Addington

Hashem A. Damrah
UO ID: 952102243

Exercise 11.2. Prove that the subset $[0, 1] \subset \mathbb{R}$ is not compact in the lower limit topology from Example 7.3(d).

Solution. Let $U = [0, 1]$. Take the open cover of U given by $V_x = [x, x + \varepsilon)$, for some $\varepsilon > 0$. Clearly, each V_x is open in the lower limit topology, and

$$U \subset \bigcup_{x \in [0,1]} V_x.$$

Take any finite subcover $\{V_{x_1}, \cdots, V_{x_n}\}$. Pick any two points, x_i and x_j such that $x_i < x_j$. Make $\varepsilon > 0$ small enough such that $x_i + \varepsilon < x_j$. Then, the there exists a point $p \in (x_i + \varepsilon, x_j)$ that is not covered by any of the V_{x_k}'s. Thus, no finite subcover exists, and $[0, 1]$ is not compact in the lower limit topology. □

Exercise 11.4. Let X be a topological space, and let Φ be a set of continuous functions $X \to [0, \infty)$ such that for every $x \in X$ there is some $f \in \Phi$ with $f(x) > 0$.

(i) Prove that if X is compact then there are $f_1, \cdots, f_n \in \Phi$ such that $f_1(x) + \cdots + f_n(x) > 0$ for all $x \in X$.

(ii) Give a counterexample when X is not compact.

(If you know something about rings and ideals, you can consider the ring of continuous functions on a topological space X; then for any $p \in X$, the functions that vanish at p turn out to form a maximal ideal. This problem can be used to prove that if X is compact, then every maximal ideal comes from some point p in this way. But if X is not compact then there are more maximal ideals.)

Solution to (i). Assume X is compact. For each $x \in X$, there exists some $f_x \in \Phi$ such that $f_x(x) > 0$. Since f_x is continuous, there exists an open neighborhood U_x of x such that for all $y \in U_x$, $f_x(y) > 0$. The collection $\{U_x \mid x \in X\}$ forms an open cover of X. By compactness, there exists a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$. Correspondingly, we have functions $f_{x_1}, f_{x_2}, \ldots, f_{x_n} \in \Phi$. Now, for any $y \in X$, there exists some U_{x_i} in the finite subcover such that $y \in U_{x_i}$. Therefore, $f_{x_i}(y) > 0$. Then, we have

$$f_{x_1}(y) + f_{x_2}(y) + \cdots + f_{x_n}(y) \geq f_{x_i}(y) > 0.$$ □

Solution to (ii). □

Exercise 11.6.

(i) Let X be a Hausdorff space. By definition, distinct points of X have disjoint neighborhoods. Proposition 11.6 proved that a compact subset $K \subset X$ and a point $p \notin K$ have disjoint neighborhoods. Now prove that two disjoint compact sets C, $K \subset X$ have disjoint neighborhoods: that is, if $C \cap K = \emptyset$ then there are open sets $U, V \subset X$ with $C \subset X$, $K \subset V$, and $U \cap V = \emptyset$.

(ii) Give an example of a non-Hausdorff space X, a compact subset $K \subset X$, and a point $p \in X \setminus K$ such that every neighborhood of p meets every neighborhood of K.

Solution to (i). Let C and K be two disjoint compact subsets of a Hausdorff space X. For each point $c \in C$, since X is Hausdorff, for each point $k \in K$, there exist disjoint open neighborhoods $U_{c,k}$ of c and $V_{c,k}$ of k. The collection $\{V_{c,k} \mid k \in K\}$ forms an open cover of K. By compactness of K, there exists a finite subcover $\{V_{c,k_1}, V_{c,k_2}, \ldots, V_{c,k_m}\}$. Correspondingly, we have open neighborhoods $U_{c,k_1}, U_{c,k_2}, \ldots, U_{c,k_m}$ of c. Define

$$U_c = \bigcap_{i=1}^{m} U_{c,k_i} \quad \text{and} \quad V_c = \bigcup_{i=1}^{m} V_{c,k_i}.$$

Then, U_c is an open neighborhood of c and V_c is an open neighborhood of K, with $U_c \cap V_c = \emptyset$. The collection $\{U_c \mid c \in C\}$ forms an open cover of C. By compactness of C, there exists a finite subcover $\{U_{c_1}, U_{c_2}, \ldots, U_{c_n}\}$. Correspondingly, we have open neighborhoods $V_{c_1}, V_{c_2}, \ldots, V_{c_n}$ of K. Define

$$U = \bigcup_{j=1}^{n} U_{c_j} \quad \text{and} \quad V = \bigcap_{j=1}^{n} V_{c_j}.$$

Then, U is an open neighborhood of C and V is an open neighborhood of K, with $U \cap V = \emptyset$. □

Solution to (ii). □

Exercise 11.7.

(i) Prove that a non-empty subset $A \subset \mathbb{R}$ is compact in the lower semi-continuous topology from Example 7.3(c) if and only if A is bounded below and contains its infimum.

(ii) Let X be a compact space, and let $f : X \to \mathbb{R}$ be lower semicontinuous, that is, continuous with respect to the lower semicontinuous topology on $\mathbb{R}$. Prove that f attains its minimum: there is a point $p \in X$ such that $f(p) \leq f(x)$ for all $x \in X$.

Solution to (i). Assume A is compact in the lower semi-continuous topology. Since A is non-empty, let $m = \inf A$. Assume for contradiction that $m \notin A$. Consider the open cover of A given by

$$U_a = \left(\frac{m+a}{2}, \infty\right).$$

Clearly, the collection $\{U_a \mid a \in A\}$ is an open cover for A. Now, since A is compact, there exists a finite subcover $\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\}$ that covers A. Let

$$b = \min_{i \in [1,n]} \frac{m + a_i}{2}.$$

Clearly, $m < b$. However, pick any point $c \in (m, b)$. Then, $c \notin U_{a_i}$ for all $i \in [1, n]$, which contradicts the fact that $\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\}$ is a cover for A. Thus, $m \in A$. Also, since A is non-empty and compact, it must be bounded below.

Now, conversely, assume A is bounded below and contains its infimum m. Since $m \in A$, it's contained in any open cover, U_m, of A, which has the form (a, ∞), where $a < m$. But then, $A \subset (a, \infty) = U_m$, since every $x \in A$ satisfies $x \geq m > a$. Thus, any open cover of A has a finite subcover (in fact, just one set suffices), and A is compact. □

Solution to (ii). Assume X is compact and $f : X \to \mathbb{R}$ is lower semicontinuous. Let $m = \inf_{x \in X} f(x)$. Consider the open cover of X given by

$$U_x = (f(x), \infty).$$

Clearly, the collection $\{U_x \mid x \in X\}$ is an open cover for X. Since X is compact, there exists a finite subcover $\{U_{x_1}, U_{x_2}, \cdots, U_{x_n}\}$ that covers X. Let

$$b = \min_{i \in [1,n]} f(x_i).$$

Clearly, $b \geq m$. Now, for any $x \in X$, there exists some U_{x_i} in the finite subcover such that $x \in U_{x_i}$. Therefore, $f(x) > f(x_i) \geq b$. Thus, for all $x \in X$, $f(x) \geq b$. Since b is a lower bound for $f(X)$ and m is the greatest lower bound, we have $m \geq b$. Combining this with the earlier inequality, we get $m \leq b \leq m$, which implies $b = m$. Therefore, there exists some x_j such that $f(x_j) = m$, or equivalently, $f(x_j) = m$. Thus, f attains its minimum at the point $x_j \in X$. □

Exercise 11.8. Let (X, d) be a metric space. A map $f : X \to X$ is called a *contraction mapping* if there is an $r \in [0, 1)$ such that

$$\mathrm{d}(f(p), f(q)) \leq r \cdot \mathrm{d}(p, q),$$

for all $p, q \in X$, while it is called a *weak contraction mapping* if we just have

$$\mathrm{d}(f(p), f(q)) < \mathrm{d}(p, q),$$

whenever $p \neq q$. If X is complete then the Banach fixed-point theorem (Theorem 4.13) stated that a contraction mapping has a fixed point, while Exercise 4.5(a) asked you to show that a weak contraction mapping need not have a fixed point.

(i) Prove that if X is compact and $f : X \to X$ is a weak contraction mapping, then f has fixed point.

Hint: Prove that the function $g : X \to \mathbb{R}$ given by $g(p) = \mathrm{d}(f(p), p)$ is continuous; thus it achieves its minimum by the extreme value theorem, and if this minimum is not zero then you get a contradiction.

(ii) Let $X = [0, 1/2]$ with the usual metric, which is compact. Prove that the map $f : X \to X$ given by $f(x) = x^2$ is a weak contraction mapping, but not a contraction mapping.

(iii) Give an example of a compact metric space (X, d) and a map $f : X \to X$ that satisfies $\mathrm{d}(f(p), f(q)) \leq \mathrm{d}(p, q)$ but has no fixed point.

Solution to (i). Assume X is compact and $f : X \to X$ is a weak contraction mapping. Define the mapping $g : X \to \mathbb{R}$ by $g(p) = \mathrm{d}(f(p), p)$. Pick two points $p, q \in X$. By the triangle inequality, we have

$$\mathrm{d}(f(p), p) \leq \mathrm{d}(f(p), f(q)) + \mathrm{d}(f(q), q) + \mathrm{d}(q, p).$$

But since f is a weak contraction mapping, we have

$$\mathrm{d}(f(p), p) < \mathrm{d}(p, q) + \mathrm{d}(f(q), q) + \mathrm{d}(q, p) = 2\mathrm{d}(p, q) + \mathrm{d}(f(q), q).$$

Therefore, we have

$$g(p) - g(q) = \mathrm{d}(f(p), p) - \mathrm{d}(f(q), q) < 2\mathrm{d}(p, q).$$

By symmetry, we also have $g(q) - g(p) < 2\mathrm{d}(p, q)$. Therefore, we have $|g(p) - g(q)| < 2\mathrm{d}(p, q)$. This shows that g is continuous. Since X is compact, g attains a minimum at some $p_0 \in X$. If $g(p_0) = 0$, then $f(p_0) = p_0$, and we have found a fixed point. Now, assume for contradiction that $g(p_0) > 0$. Then, we have

$$g(f(p_0)) = \mathrm{d}(f(f(p_0)), f(p_0)) < \mathrm{d}(f(p_0), p_0) = g(p_0).$$

This contradicts the fact that $g(p_0)$ is the minimum value of g. Thus, we must have $g(p_0) = 0$, and f has a fixed point at p_0. □

Solution to (ii). Let $X = [0, 1/2]$ with the usual metric $\mathrm{d}(x, y) = |x - y|$. Consider the map $f : X \to X$ given by $f(x) = x^2$. For any two points $x, y \in X$, without loss of generality, assume $x < y$. Then, we have

$$\mathrm{d}(f(x), f(y)) = |x^2 - y^2| = |x - y||x + y| \leq |x - y| \cdot 1 = \mathrm{d}(x, y).$$

Since $x + y \leq 1$ for all $x, y \in [0, 1/2]$, we have $\mathrm{d}(f(x), f(y)) < \mathrm{d}(x, y)$ whenever $x \neq y$. Thus, f is a weak contraction mapping. However, to show that f is not a contraction mapping, assume for contradiction that there exists some $r \in [0, 1)$ such that

$$\mathrm{d}(f(x), f(y)) \leq r \cdot \mathrm{d}(x, y),$$

for all $x, y \in X$. Pick $x = 0$ and $y = 1/2$. Then, we have

$$\mathrm{d}(f(0), f(1/2)) = |0 - (1/2)^2| = 1/4 \leq r \cdot |0 - 1/2| = r/2.$$

This implies that $r \geq 1/2$. Now, pick $x = 1/4$ and $y = 1/2$. Then, we have

$$\mathrm{d}(f(1/4), f(1/2)) = |(1/4)^2 - (1/2)^2| = 3/16 \leq r \cdot |1/4 - 1/2| = r/4.$$

This implies that $r \geq 3/4$. Continuing this process, we can see that for any $\varepsilon > 0$, we can find points $x, y \in X$ □

Solution to (iii). □

Exercise 11.10. A continuous map $f : X \to Y$ is called proper if the preimage of any compact set $K \subset Y$ is compact.

(i) Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is proper.

(ii) Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$ is not proper.

(iii) Prove that if f is proper then the preimage of every point is compact.

(iv) Give an example of a continuous map $f : X \to Y$ for which the preimage of every point is compact, but nonetheless f is not proper.

(v) Prove that if X is compact and Y is Hausdorff then any continuous map $f : X \to Y$ is proper.

(vi) Let X and Y be topological spaces. Prove that the projection $p : X \times Y \to X$ is proper if and only if Y is compact.

Solution to (i). The pre-image of some number a under the map f is given by

$$f^{-1}(a) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = a\}.$$

If $a < 0$, then $f^{-1}(a) = \emptyset$, which is compact. If $a = 0$, then $f^{-1}(a) = \{(0, 0)\}$, which is also compact. If $a > 0$, then $f^{-1}(a)$ is the circle of radius $\sqrt{a}$ centered at the origin, which is closed and bounded in $\mathbb{R}^2$, and hence compact by the Heine-Borel theorem. Now, consider any compact set $K \subset \mathbb{R}$. Since K is compact, it is closed and bounded, so there exist real numbers m and M such that

$$K \subset [m, M].$$

Therefore, we have

$$f^{-1}(K) \subset f^{-1}([m, M]) = \bigcup_{a \in [m,M]} f^{-1}(a).$$

Since $[m, M]$ is closed and bounded, it is compact. The union of the pre-images of all points in a compact set is also compact, as each pre-image is compact. Thus, $f^{-1}(K)$ is compact, and f is a proper map. □

Solution to (ii). The equation $x^2 - y^2 = 0$ is true if and only if $x = y$. Thus, the pre-image of 0 under the map f is given by

$$f^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \mid x = y\} = \{(t, t) \mid t \in \mathbb{R}\},$$

which is the line $y = x$ in $\mathbb{R}^2$. This set is not bounded, and hence not compact. Therefore, f is not a proper map. □

Solution to (iii). Let $f : X \to Y$ be a proper map. Consider any point $y \in Y$. The set $\{y\}$ is compact in Y since we can take the finite subcover consisting of just the set $\{y\}$ itself. Since f is proper, the pre-image $f^{-1}(\{y\})$ is compact in X. Thus, the pre-image of every point under a proper map is compact. □

Solution to (iv). □

Solution to (v). Assume that X is compact, Y is Hausdorff, and $f : X \to Y$ is continuous. Consider any compact set $K \subset Y$. Since Y is Hausdorff, K is closed in Y. Since f is continuous, the pre-image $f^{-1}(K)$ is closed in X. Since X is compact, any closed subset of X is also compact. Therefore, $f^{-1}(K)$ is compact in X, and f is a proper map. □

Solution to (vi). Assume that the projection $p : X \times Y \to X$ is proper. Consider any compact set $K \subset X$. Since p is proper, the pre-image

$$p^{-1}(K) = K \times Y.$$

is compact in $X \times Y$. By the Heine-Borel theorem, this implies that Y is compact.

Now, conversely, assume that Y is compact. Consider any compact set $K \subset X$. The pre-image

$$p^{-1}(K) = K \times Y.$$

Since both K and Y are compact, their product $K \times Y$ is also compact by the Tychonoff theorem. Thus, p is a proper map. □