

Several-Variable Calc II: Homework 6

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Problem 1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = (x+z)\mathbf{i} + 2y\mathbf{j} + (y+2x)\mathbf{k}$ where C is

- (i) the line segment from $(-1, 0, 1)$ and $(0, -1, 2)$.
- (ii) the curve $\mathbf{r}(t) = \langle t-1, t^2-2t, t^3+1 \rangle$ from $0 \leq t \leq 1$.

Solution to (i). Let C be the line segment from $P(-1, 0, 1)$ to $Q(0, -1, 2)$. Then, we have

$$\begin{aligned}\mathbf{r}(t) &= (1-t)P + tQ = (1-t)(-1, 0, 1) + t(0, -1, 2) = \langle -1+t, -t, 1+t \rangle \\ \Rightarrow \mathbf{r}'(t) &= \langle 1, -1, 1 \rangle.\end{aligned}$$

Converting \mathbf{F} from a vector function of x and y to a vector function of t , we have

$$\mathbf{F}(x, y) = \langle x+z, 2y, y+2x \rangle = \langle (-1+t) + (-1+t), -2t, -t + (-1+t) \rangle = \langle 2t, -2t, -2+t \rangle = \mathbf{F}(t).$$

Therefore, the line integral over the vector field \mathbf{F} along the line segment C is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(t) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (1) \cdot (2t) + (-1) \cdot (-2t) + (1) \cdot (-2+t) dt \\ &= \int_0^1 2t + 2t - 2 + t dt \\ &= \int_0^1 5t - 2 dt = \left(\frac{5}{2}t - 2t \right) \Big|_0^1 = \frac{1}{2}.\end{aligned}\quad \square$$

Solution to (ii). Let C be the curve $\mathbf{r}(t) = \langle t-1, t^2-2t, t^3+1 \rangle$ where $0 \leq t \leq 1$. Then, we have

$$\mathbf{r}'(t) = \langle 1, 2t-2, 3t^2 \rangle.$$

Converting \mathbf{F} from a vector function of x and y to a vector function of t , we have

$$\begin{aligned}\mathbf{F}(x, y) &= \langle x+z, 2y, y+2x \rangle = \langle (t-1) + (t^3+1), 2(t^2-2t), (t^2-2t) + 2(t-1) \rangle \\ &= \langle t^3+t, 2t^2-4t, t^2-2 \rangle.\end{aligned}$$

Evaluating the dot product of \mathbf{F} and \mathbf{r}' , we have

$$\begin{aligned}\mathbf{F}(t) \cdot \mathbf{r}'(t) &= \langle t^3+t, 2t^2-4t, t^2-2 \rangle \cdot \langle 1, 2t-2, 3t^2 \rangle \\ &= (t^3+t) \cdot (1) + (2t^2-4t) \cdot (2t-2) + (t^2-2) \cdot (3t^2) \\ &= t^3+t+4t^3-4t^2-8t^2+8t+3t^4-6t^2 \\ &= 3t^4+5t^3-18t^2+9t.\end{aligned}$$

Therefore, the line integral over the vector field \mathbf{F} along the curve C is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(t) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 3t^4 + 5t^3 - 18t^2 + 9t dt \\ &= \frac{3}{5} + \frac{5}{4} - \frac{18}{3} + \frac{9}{2} = \frac{7}{20}.\end{aligned}\quad \square$$

Problem 2. Compute the amount of work done by the vector field $\mathbf{F} = \langle -y, x, x-z \rangle$ moving a particle along the curve of intersection of the surfaces $x^2+y^2=9$ and $y-x+z=1$ oriented in the counterclockwise direction about the cylinder.

Note: Do not use a method we haven't discussed yet. The use of anything other than directly evaluating the line integral will not be accepted.

Solution. Let C be the curve of intersection of the surfaces $x^2 + y^2 + z^2 = 9$ and $y - x + z = 1$. Converting to cylindrical coordinates, we have

$$\left. \begin{array}{l} x^2 + y^2 = 9 \\ y - x + z = 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} r^2 = 9 \\ z = 1 - r(\sin(\theta) - \cos(\theta)) \end{array} \right\} \Rightarrow \left. \begin{array}{l} r = 3 \\ z = 1 - 3(\sin(\theta) - \cos(\theta)) \end{array} \right\} \\ \Rightarrow z = 1 - 3(\sin(\theta) - \cos(\theta)).$$

Parametrizing the curve C as

$$\begin{aligned} \mathbf{r}(t) &= \langle 3 \cos(\theta), 3 \sin(\theta), 1 - 3(\sin(\theta) - \cos(\theta)) \rangle \\ \Rightarrow \mathbf{r}'(t) &= \langle -3 \sin(\theta), 3 \cos(\theta), 3(\cos(\theta) + \sin(\theta)) \rangle, \end{aligned}$$

where $0 \leq t \leq 2\pi$, we have Converting \mathbf{F} from a vector function of x and y to a vector function of t , we have

$$\begin{aligned} \mathbf{F}(t) &= \langle -y(t), x(t), x(t) - z(t) \rangle = \langle -3 \sin(\theta), 3 \cos(\theta), 3 \cos(\theta) - (1 - 3(\sin(\theta) - \cos(\theta))) \rangle \\ &= \langle -3 \sin(\theta), 3 \cos(\theta), 3 \sin(\theta) - 1 \rangle. \end{aligned}$$

Evaluating the dot product of \mathbf{F} and \mathbf{r}' , we have

$$\begin{aligned} \mathbf{F}(t) \cdot \mathbf{r}'(t) &= \langle -3 \sin(\theta), 3 \cos(\theta), 3 \sin(\theta) - 1 \rangle \cdot \langle -3 \sin(\theta), 3 \cos(\theta), 3(\cos(\theta) + \sin(\theta)) \rangle \\ &= (-3 \sin(\theta)) \cdot (-3 \sin(\theta)) + (3 \cos(\theta)) \cdot (3 \cos(\theta)) + (3 \sin(\theta) - 1) \cdot (3(\cos(\theta) + \sin(\theta))) \\ &= 9(\sin^2(\theta) + \cos^2(\theta)) + (-9 \sin^2(\theta) - 9 \sin(\theta) \cos(\theta) + 3 \cos(\theta) + 3 \sin(\theta)) \\ &= 9 - 9 \sin^2(\theta) - 9 \sin(\theta) \cos(\theta) + 3 \cos(\theta) + 3 \sin(\theta). \end{aligned}$$

Computing the work integral over the vector field \mathbf{F} along the curve C , we have

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} 9 - 9 \sin^2(\theta) - 9 \sin(\theta) \cos(\theta) + 3 \cos(\theta) + 3 \sin(\theta) dt \\ &= \int_0^{2\pi} 9 d\theta - 9 \int_0^{2\pi} \sin^2(\theta) d\theta - 9 \int_0^{2\pi} \sin(\theta) \cos(\theta) d\theta + 3 \int_0^{2\pi} \cos(\theta) d\theta + 3 \int_0^{2\pi} \sin(\theta) d\theta \Big|_0^{2\pi} \\ &= 9\theta - \frac{9}{2}(\theta - \sin(\theta) \cos(\theta)) - \frac{9}{2} \cos^2(\theta) + 3 \sin(\theta) - 3 \cos(\theta) \Big|_0^{2\pi} \\ &= \left[9(2\pi) - \frac{9}{2}(2\pi - \sin(2\pi) \cos(2\pi)) - \frac{9}{2} \cos^2(2\pi) + 3 \sin(2\pi) - 3 \cos(2\pi) \right] \\ &\quad - \left[9(0) - \frac{9}{2}(0 - \sin(0) \cos(0)) - \frac{9}{2} \cos^2(0) + 3 \sin(0) - 3 \cos(0) \right] \\ &= \left[18\pi - \frac{9}{2}(2\pi - (0)(-1)) - \frac{9}{2}(1) + 3(0) - 3(1) \right] \\ &\quad - \left[0 - \frac{9}{2}(0 - (0)(1)) - \frac{9}{2}(1) + 3(0) - 3(1) \right] \\ &= 18\pi - 9\pi - \frac{9}{2} - 3 + \frac{9}{2} + 3 = 9\pi. \end{aligned} \quad \square$$

Problem 3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Use the fundamental theorem for line integrals whenever it applies.

- (i) $\mathbf{F} = \langle 2xe^{xy} + x^2ye^{xy} + 3x^2, x^3e^{xy} + 2\sin(y) \rangle$, C is the line segment from $(-1, 0)$ to $(0, 3)$.
- (ii) $\mathbf{F} = \langle y^3 - 2x, 3xy^2 + \sin(\pi y) \rangle$, C is the path $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.

(iii) $\mathbf{F} = \langle 6xy - z^2, 3x^2 + 6y^2, 1 - 2xz \rangle$, C is the circular helix $\mathbf{r}(t) = \langle t, 2\cos(t), 2\sin(t) \rangle$, $0 \leq t \leq \pi$.

(iv) $\mathbf{F} = \langle y + z, x - 2z, x + 2y \rangle$, C is the intersection of sphere $x^2 + y^2 + z^2 = 5$ and the plane $x = 1$ in the first octant oriented upward.

Solution to (i). A vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative if there exists a potential function $f(x, y)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Evaluating each partial derivative, we have

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} [2xe^{xy} + x^2ye^{xy} + 3x^2] \\ &= 2x \cdot \frac{\partial}{\partial y} [e^{xy}] + x^2e^{xy} \cdot \frac{\partial}{\partial y} [y] + x^2y \cdot \frac{\partial}{\partial y} [e^{xy}] + \frac{\partial}{\partial y} [3x^2] \\ &= 2x^2e^{xy} + x^2e^{xy} + x^3ye^{xy} + 0 \\ &= 3x^2e^{xy} + x^3ye^{xy} \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} [x^3e^{xy} + 2\sin(y)] \\ &= e^{xy} \cdot \frac{\partial}{\partial x} [x^3] + x^3 \cdot \frac{\partial}{\partial x} [e^{xy}] + \frac{\partial}{\partial x} [2\sin(y)] \\ &= 3x^2e^{xy} + x^3ye^{xy} + 0 \\ &= 3x^2e^{xy} + x^3ye^{xy}. \end{aligned}$$

Clearly, they are the same. Also, note that \mathbf{F} is simply connected, as there are no singularities. Therefore, \mathbf{F} is conservative.

Therefore, we have the following system of equations

$$f_x = 2xe^{xy} + x^2ye^{xy} + 3x^2 \tag{1}$$

$$f_y = x^3e^{xy} + 2\sin(y) \tag{2}$$

Integrating equation 2 with respect to y , we have

$$\begin{aligned} f &= \int f_y \, dy = \int x^3e^{xy} + 2\sin(y) \, dy \\ &= x^2e^{xy} + 2\cos(y) + g(x). \end{aligned}$$

Plugging this into equation 1, we have

$$f_x = 2xe^{xy} + x^2ye^{xy} + 3x^2 = P \Rightarrow g'(x) = 3x^2 \Rightarrow g(x) = x^3.$$

Therefore, the potential function is

$$f(x, y) = x^2e^{xy} + 2\cos(y) + x^3.$$

Therefore, the line integral over the vector field \mathbf{F} along the line segment C is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(0, 3) - f(-1, 0) \\ &= [-2\cos(3)] - [1 - 1 - 2] \\ &= -2\cos(3) + 2. \end{aligned}$$

□

Solution to (ii). A vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative if there exists a potential function $f(x, y)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Evaluating each partial derivative, we have

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} [y^3 - 2x] \\ &= 3y^2 - 0 \\ &= 3y^2 \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} [3xy^2 + \sin(\pi y)] \\ &= 3y^2 + 0 \\ &= 3y^2.\end{aligned}$$

Clearly, they are the same. Also, note that \mathbf{F} is simply connected, as there are no singularities. Therefore, \mathbf{F} is conservative.

Therefore, we have the following system of equations

$$f_x = y^3 - 2x \tag{3}$$

$$f_y = 3xy^2 + \sin(\pi y) \tag{4}$$

Integrating equation 3 with respect to x , we have

$$\begin{aligned}f &= \int f_x \, dx = \int y^3 - 2x \, dx \\ &= y^3 x - x^2 + g(y).\end{aligned}$$

Plugging this into equation 4, we have

$$f_y = 3xy^2 + h'(y) = Q \Rightarrow h'(y) = \sin(\pi y) \Rightarrow h(y) = -\frac{1}{\pi} \cos(\pi y).$$

Therefore, the potential function is

$$f(x, y) = xy^3 - x^2 - \frac{1}{\pi} \cos(\pi y).$$

Therefore, the line integral over the vector field \mathbf{F} along the path C is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(4, 2) - f(1, 1) \\ &= \left[32 - 16 - \frac{1}{\pi} \cos(2\pi) \right] - \left[1 - 1 - \frac{1}{\pi} \cos(\pi) \right] \\ &= 16 - \frac{2}{\pi}.\end{aligned} \quad \square$$

Solution to (iii). A vector field $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if there exists a potential function $f(x, y, z)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Evaluating each partial derivative, we have

$$\frac{\partial P}{\partial y} = 6x = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial z} = -2z = \frac{\partial P}{\partial x}.$$

Clearly, they are the same. Also, note that \mathbf{F} is simply connected, as there are no singularities. Therefore, \mathbf{F} is conservative.

Therefore, we have the following system of equations

$$f_x = 6xy - z^2 \quad (5)$$

$$f_y = 3x^2 + 6y^2 \quad (6)$$

$$f_z = 1 - 2xz. \quad (7)$$

Integrating equation 5 with respect to x , we have

$$f = \int 6xy - z^2 dx = 3x^2y - xz^2 + g(y, z).$$

Plugging this into equation 6, we have

$$f_y = 3x^2 + g_y(y, z) = Q \Rightarrow g_y(y, z) = 6y^2 \Rightarrow g(y, z) = 2y^3 + h(z).$$

Plugging this into equation 7, we have

$$f_z = 1 - 2xz + h'(z) = R \Rightarrow h'(z) = 1 \Rightarrow h(z) = z.$$

Therefore, the potential function is

$$f(x, y, z) = 3x^2y - xz^2 + 2y^3 + z.$$

The starting and ending points are

$$A = \mathbf{r}(0) = \langle 0, 2, 0 \rangle \quad \text{and} \quad B = \langle \pi, -2, 0 \rangle.$$

Therefore, the line integral over the vector field \mathbf{F} along the circular helix C is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(\pi, -2, 0) - f(0, 2, 0) \\ &= [3\pi^2(-2) - \pi(0)^2 + 2(-2)^3 + 0] - [3(0)^2(2) - 0(0)^2 + 2(2)^3 + 0] \\ &= -6\pi^2 - 32. \end{aligned} \quad \square$$

Solution to (iv). A vector field $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if there exists a potential function $f(x, y, z)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Evaluating each partial derivative, we have

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = -2 \neq 2 = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = 1 = \frac{\partial P}{\partial z}.$$

Clearly, they are not the same. Therefore, \mathbf{F} is not conservative.

Since C is the intersection of the sphere $x^2 + y^2 + z^2 = 5$ and the plane $x = 1$ in the first octant, we get $x = 1$, $y^2 + z^2 = 4$, and $y, z \geq 0$. Therefore, we have

$$\mathbf{r}(t) = \langle 1, 2 \cos(t), 2 \sin(t) \rangle \Rightarrow \mathbf{r}'(t) = \langle 0, -2 \sin(t), 2 \cos(t) \rangle.$$

The bounds for t are $0 \leq t \leq \frac{\pi}{2}$. Converting \mathbf{F} from a vector function of x and y to a vector function of t , we have

$$\mathbf{F}(x, y) = \langle y + z, x - 2z, x + 2y \rangle = \langle 2 \cos(t) + 2 \sin(t), 1 - 8 \sin(t), 1 + 8 \cos(t) \rangle = \mathbf{F}(t).$$

Evaluating the dot product of \mathbf{F} and \mathbf{r}' , we have

$$\begin{aligned}\mathbf{F}(t) \cdot \mathbf{r}'(t) &= \langle 2 \cos(t) + 2 \sin(t), 1 - 8 \sin(t), 1 + 8 \cos(t) \rangle \cdot \langle 0, -2 \sin(t), 2 \cos(t) \rangle \\ &= (2 \cos(t) + 2 \sin(t)) \cdot (0) + (1 - 8 \sin(t)) \cdot (-2 \sin(t)) + (1 + 8 \cos(t)) \cdot (2 \cos(t)) \\ &= -2 \sin(t) + 16 \sin^2(t) + 2 \cos(t) + 16 \cos^2(t) \\ &= 2 \cos(t) - 2 \sin(t) + 16.\end{aligned}$$

Therefore, the line integral over the vector field \mathbf{F} along the curve C is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} \mathbf{F}(t) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{\frac{\pi}{2}} 2 \cos(t) - 2 \sin(t) + 16 dt \\ &= [2 \sin(t) - 2 \cos(t) + 16t] \Big|_0^{\frac{\pi}{2}} \\ &= [2 + 8\pi] - [-2] = 4 + 8\pi.\end{aligned}$$

□

Problem 4. Consider $\mathbf{F} = \langle P, Q \rangle$ where $P(x, y) = \frac{-y}{x^2 + y^2}$ and $Q(x, y) = \frac{x}{x^2 + y^2}$.

(i) Show $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the domain of \mathbf{F} .

(ii) Use the definition of the line integral to show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ where C is the circle $x^2 + y^2 = a^2$, counterclockwise orientation, for any constant $a > 0$. Is \mathbf{F} conservative?

Solution to (i). We have

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{-y}{x^2 + y^2} \right) = \frac{-(x^2 + y^2) - (-y)(2y)}{(x^2 + y^2)^2} = \frac{-x^2 - y^2 + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.\end{aligned}$$

Therefore, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the domain of \mathbf{F} .

□

Solution to (ii). We parameterize the circle C

$$\mathbf{r}(t) = \langle a \cos(t), a \sin(t) \rangle.$$

Evaluating $\mathbf{r}'(t)$ gives us

$$\mathbf{r}'(t) = \langle -a \sin(t), a \cos(t) \rangle,$$

where $0 \leq t \leq 2\pi$. Converting \mathbf{F} from a vector function of x and y to a vector function of t , we have

$$\mathbf{F}(t) = \left\langle \frac{-a \sin(t)}{a^2}, \frac{a \cos(t)}{a^2} \right\rangle = \left\langle \frac{-\sin(t)}{a}, \frac{\cos(t)}{a} \right\rangle.$$

Evaluating the dot product of \mathbf{F} and \mathbf{r}' , we have

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = \left\langle \frac{-\sin(t)}{a}, \frac{\cos(t)}{a} \right\rangle \cdot \langle -a \sin(t), a \cos(t) \rangle = \frac{-\sin(t)}{a} \cdot (-a \sin(t)) + \frac{\cos(t)}{a} \cdot (a \cos(t)) = 1.$$

Thus, the line integral evaluates to

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 dt = 2\pi.$$

A vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if there exists a function $f(x, y)$ such that $\nabla f = \langle P, Q \rangle$. A necessary condition for \mathbf{F} to be conservative is that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, which we verified in part (i).

However, \mathbf{F} is not defined at $(0, 0)$, and the domain $\mathbb{R}^2 - \{(0, 0)\}$ is not simply connected, since any loop enclosing the origin cannot be continuously shrunk to a point without leaving the domain. A conservative vector field must be path-independent, meaning that the line integral around any closed curve should be zero.

Since we computed

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi \neq 0,$$

we conclude that \mathbf{F} is not conservative. □