

SOLUTIONS TO HOMEWORK 5

Warning: Little proofreading has been done.

1. SECTION 2.4

Exercise 2.4.7 (Limit Superior). Let (a_n) be a bounded sequence.

- (a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.
- (b) The *limit superior* of (a_n) , or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

- (c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.
- (d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution. (a) Since the sequence (a_n) is bounded, there is an $M > 0$ such that $|a_n| \leq M$, or $-M \leq a_n \leq M$. As a consequence, we see that $-M \leq y_n \leq M$, or $|y_n| \leq M$, so that y_n is bounded. By its definition,

$$y_n = \sup\{a_k : k \geq n\} = \sup\{a_k : k \geq n+1\} \cup \{y_n\} \geq \sup\sup\{a_k : k \geq n+1\} = y_{n+1},$$

so that y_n is a decreasing sequence. Hence, by the Monotone convergence theory, (y_n) converges.

(b) Let $z_n = \inf\{a_k : k \geq n\}$. The same argument shows that z_n is an increasing bounded sequence, so that (z_n) converges. We can then define

$$\liminf a_n = \lim z_n.$$

- (c) Since $z_n \leq y_n$ for every $n \in \mathbb{N}$, $\liminf a_n \leq \limsup a_n$ follows from order limit theorem.

There are many such examples. One example is $a_n = (-1)^n$. Then $\liminf a_n = -1$ and $\limsup a_n = 1$ so that $\liminf a_n < \limsup a_n$.

- (d) Since $z_n \leq a_n \leq y_n$, if $\liminf a_n = \limsup a_n = a$, then by the squeeze theorem, $\lim a_n = a$.

Assume $\lim a_n = a$. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that, for all $n > N$, $|a_n - a| < \varepsilon$ or $a - \varepsilon < a_n < a + \varepsilon$. Since $y_n = \sup\{a_k : k \geq n\}$ and $z_n = \inf\{a_k : k \geq n\}$, we see that if $n \geq N$, then $a - \varepsilon \leq z_n \leq y_n \leq a + \varepsilon$, so that $|z_n - a| < \varepsilon$ and $|y_n - a| < \varepsilon$. Thus, we conclude that $\lim y_n = a$ and $\lim z_n = a$. \square

2. SECTION 2.5

Exercise 2.5.1. Give an example of each of the following, or prove that such a request is impossible.

- (a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
- (b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
- (c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.
- (d) An unbounded sequence with a convergent subsequence.

Solution. (a) This is not possible. A subsequence of a subsequence is a subsequence of the original sequence, and the Bolzano-Weierstrass Theorem implies that the bounded subsequence has in turn a convergent subsequence.

(b) Define

$$a_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ 1 + \frac{1}{n} & n \text{ odd} \end{cases}$$

Then $(a_{2n})_{n \in \mathbb{N}} = \left(\frac{1}{2n}\right)_{n \in \mathbb{N}}$ is a subsequence of $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$, and we already know $\frac{1}{n} \rightarrow 0$, so

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} \frac{1}{2n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Also, $(a_{2n-1})_{n \in \mathbb{N}} = \left(1 + \frac{1}{2n-1}\right)_{n \in \mathbb{N}}$ is a subsequence of $\left(1 + \frac{1}{n}\right)_{n \in \mathbb{N}}$, and we can combine the Algebraic Limit Theorem with limits, we already know to get $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$, so

$$\lim_{n \rightarrow \infty} a_{2n-1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n-1}\right) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1.$$

Finally, it is obvious that $a_n \neq 0$ and $a_n \neq 1$ for all $n \in \mathbb{N}$.

(c) One example is taking the sequence to be the sequence of rational numbers. Here is another example. Take the sequence to be

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 1, \dots\right).$$

For every $n \in \mathbb{N}$, there is a subsequence which has the form

$$\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots\right).$$

We already know that this sequence converges to $\frac{1}{n}$.

□

Exercise 2.5.3.

- (a) Prove that if an infinite series converges, then the associative property holds. That is, assume $a_1 + a_2 + a_3 + a_4 + \dots$ converges to a limit L (that is, the sequence of partial sums $(s_n)_{n \in \mathbb{N}}$ converges to L). Show that any regrouping of the terms

$$(a_1 + a_2 + \dots + a_{n_1}) + (a_{n_1+1} + \dots + a_{n_2}) + (a_{n_2+1} + \dots + a_{n_3}) + \dots$$

leads to a series which also converges to L .

- (b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in part a apply to this example?

Solution. (a) The partial sums of the series $\sum_{n=1}^{\infty}$ are given by

$$s_n = a_1 + a_2 + \dots + a_n$$

for $n \in \mathbb{N}$.

Define $n_0 = 0$, and define

$$b_k = a_{n_{k-1}+1} + \dots + a_{n_k}$$

for $k \in \mathbb{N}$. Thus the new series is $\sum_{k=1}^{\infty} b_k$. Its partial sums are given by

$$t_k = b_1 + b_2 + \dots + b_k.$$

It is immediate that

$$t_k = a_1 + a_2 + \dots + a_{n_k} = s_{n_k}.$$

By hypothesis, $\lim_{n \rightarrow \infty} s_n = L$. Therefore, by Theorem 2.5.2 of the book,

$$\sum_{k=1}^{\infty} (a_{n_{k-1}+1} + \dots + a_{n_k}) = \sum_{k=1}^{\infty} b_k = \lim_{k \rightarrow \infty} t_k = \lim_{k \rightarrow \infty} s_{n_k} = L.$$

This completes the proof.

- (b) The series in the example discussed at the end of Section 2.1 does not converge. Indeed, the sequence of partial sums is

$$(-1, 0, -1, 0, -1, 0, -1, 0, \dots),$$

which has (constant) subsequences converging to the distinct limits -1 and 0 .

□

3. SECTION 2.6

Exercise 2.6.2 Give an example of each of the following, or argue that such a request is impossible.

- (a) A Cauchy sequence that is not monotone.
- (b) A Cauchy sequence with an unbounded subsequence.
- (c) A divergence monotone sequence with a Cauchy subsequence.
- (d) An unbounded sequence containing a subsequence that is Cauchy.

Solution. (a) Define $a_n = (-1)^n/n$ for $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} a_n = 0$. Therefore (a_n) is Cauchy.

The sequence $(a_n)_{n \in \mathbb{N}}$ is not monotone because $a_1 < a_2$ but $a_2 > a_3$.

There are many other examples. Here is a rather trivial one:

$$(-1, 1, -1, 0, 0, 0, 0, 0, 0, \dots).$$

(b) No such thing exists. Every Cauchy sequence converges, so every subsequence converges to the same limit, and a convergence sequence is bounded.

(c) No such thing exists. Let (a_n) be an increasing sequence. If (a_{n_k}) is a Cauchy subsequence of (a_n) , then it must converge, hence bounded. That is, there is an $M > 0$ such that $|a_{n_k}| \leq M$. However, for every $n \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $n_k \leq n \leq n_{k+1}$. By the monotonicity, $a_{n_k} \leq a_n \leq a_{n_{k+1}}$. Hence, we must have $|a_n| \leq M$. This shows that (a_n) is bounded, hence (a_n) converges.

(d) There are many such sequences. Here is an example. Define

$$a_n = \begin{cases} n & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$$

Then $(a_n)_{n \in \mathbb{N}}$ is clearly unbounded, but the subsequence $(a_{2n})_{n \in \mathbb{N}}$ is constant, therefore convergent, therefore Cauchy. \square

Exercise 2.6.3 If $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$

is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

- (a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.
- (b) Do the same for the product $(x_n y_n)$.

Solution. (a) Let $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N_1$, we have $|x_m - x_n| < \frac{1}{2}\varepsilon$. Choose $N_2 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N_2$, we have $|y_m - y_n| < \frac{1}{2}\varepsilon$. Define $N = \max(N_1, N_2)$. Let $m, n \in \mathbb{N}$ with $m, n \geq N$. Then

$$|(x_m + y_m) - (x_n + y_n)| = |(x_m - x_n) + (y_m - y_n)| \leq |x_m - x_n| + |y_m - y_n| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

(b) We need a little preparation. By Lemma 2.6.3 of the book, (x_n) and (y_n) are bounded. Thus, there are $M_1, M_2 \in [0, \infty)$ such that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N_1$, we have

$$|x_m - x_n| < \frac{\varepsilon}{2M_2 + 1}.$$

Choose $N_2 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N_2$, we have

$$|y_m - y_n| < \frac{\varepsilon}{2M_1 + 1}.$$

Define $N = \max(N_1, N_2)$. Let $m, n \in \mathbb{N}$ with $m, n \geq N$. Then

$$\begin{aligned} |x_m y_m - x_n y_n| &= |(x_m y_m - x_n y_m) + (x_n y_m - x_n y_n)| \\ &\leq |x_m - x_n| \cdot |y_m| + |x_n| \cdot |y_m - y_n| \\ &\leq \left(\frac{\varepsilon}{2M_2 + 1} \right) M_2 + M_1 \left(\frac{\varepsilon}{2M_1 + 1} \right) \\ &< \varepsilon. \end{aligned}$$

□

Remark 1: It is legitimate to use Lemma 2.6.3 of the book, since it doesn't depend on completeness. It *isn't* legitimate to use the fact that Cauchy sequences converge, and then the fact that convergent sequences are bounded, since this requires completeness.

Remark 2: We divide by $2M_1 + 1$ and $2M_2 + 1$ in case $M_1 = 0$ or $M_2 = 0$.