

Several-Variable Calc II: Homework 8

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Problem 1. Evaluate $\iint_S x \, dS$ where S is the portion of the plane $4x + 2y + z = 8$ in octant one.

Solution. Solving for z , we get $z = 8 - 4x - 2y$. The region lies in the first octant, meaning $x \geq 0$, $y \geq 0$, and $z \geq 0$. Finding the intersection points with each axis, we get

$$x = 0 = y \Rightarrow z = 8$$

$$x = 0 = z \Rightarrow y = 4$$

$$y = 0 = z \Rightarrow x = 2.$$

Thus, the region is a triangle with vertices at $(0, 0, 8)$, $(0, 4, 0)$, and $(2, 0, 0)$. The normal vector to the plane is $\mathbf{N} = \nabla f = \langle 4, 2, 1 \rangle$. Therefore, the surface element is given by

$$\left| \frac{\partial z}{\partial(x, y)} \right| dx \, dy = \sqrt{1^2 + 2^2 + 4^2} \, dx \, dy = \sqrt{21} \, dx \, dy.$$

Therefore, we get

$$dS = \sqrt{21} \, dx \, dy.$$

The equation of the line connecting $(2, 0)$ and $(0, 4)$ is $y = -2x + 4$. Therefore, the bounds of integration are $0 \leq x \leq 2$ and $0 \leq y \leq -2x + 4$. Thus, we get

$$\begin{aligned} \iint_S x \, dS &= \int_0^2 \int_0^{-2x+4} x \sqrt{21} \, dy \, dx \\ &= \sqrt{21} \cdot \int_0^2 xy \Big|_0^{-2x+4} dx \\ &= \sqrt{21} \cdot \int_0^2 -2x^2 + 4x \, dx \\ &= \sqrt{21} \cdot \left(-\frac{2}{3}x^3 + 2x^2 \right) \Big|_0^2 \\ &= \sqrt{21} \left(-\frac{16}{3} + 8 \right) = \frac{8\sqrt{21}}{3}. \end{aligned}$$

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Problem 2. Evaluate $\iint_S x^2 + y^2 + z^2 \, dS$ where S is the boundary surface of the solid above the plane $z = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

Solution. In spherical coordinates, we have $x = 2 \sin(\theta) \cos(\varphi)$, $y = 2 \sin(\theta) \sin(\varphi)$, and $z = 2 \cos(\theta)$. The cap is bounded by $z \geq 1$, or $2 \cos(\theta) \geq 1 \Rightarrow \cos(\theta) \geq 1/2$. This gives us the bounds $0 \leq \theta \leq \pi/3$. The surface element is given by

$$dS = 4 \sin(\theta) \, d\theta \, d\varphi.$$

On the spherical cap, we have

$$\begin{aligned} I_1 &= \iint x^2 + y^2 + z^2 \, dS = \int_0^{2\pi} \int_0^{\pi/3} 4 \cdot 4 \sin(\theta) \, d\theta \, d\varphi \\ &= 16 \cdot \int_0^{2\pi} d\varphi \cdot \int_0^{\pi/3} \sin(\theta) \, d\theta \\ &= 16 \cdot 2\pi \cdot \cos(\theta) \Big|_{\pi/3}^0 \\ &= 32\pi \cdot \left(\cos(0) - \cos\left(\frac{\pi}{3}\right) \right) \end{aligned}$$

$$= 32\pi \cdot \left(-\frac{1}{2} + 1\right) = 16\pi.$$

On the disk, $x^2 + y^2 + z^2 = 4$ and $z = 1$, we have $x^2 + y^2 = 3$. Using polar coordinates, $x = \sqrt{3}\cos(\varphi)$ and $y = \sqrt{3}\sin(\varphi)$, we get the bounds $0 \leq \varphi \leq 2\pi$ and $0 \leq r \leq \sqrt{3}$. The surface

$$dS = r \, dr \, d\varphi.$$

Therefore, we get

$$\begin{aligned} I_2 &= \iint x^2 + y^2 + z^2 \, dS = \int_0^{2\pi} \int_0^{\sqrt{3}} (r^2 + 1) \cdot r \, dr \, d\varphi \\ &= \int_0^{2\pi} d\varphi \cdot \int_0^{\sqrt{3}} r^3 + r \, dr \\ &= 2\pi \cdot \left(\frac{r^4}{4} + \frac{r^2}{2}\right)_0^{\sqrt{3}} \\ &= 2\pi \cdot \left(\frac{9}{4} + \frac{3}{2}\right) = \frac{15\pi}{2}. \end{aligned}$$

Therefore, the total integral is

$$I = I_1 + I_2 = 16\pi + \frac{15\pi}{2} = \frac{47\pi}{2}. \quad \square$$

Problem 3. A funnel has the shape of the cone $z = \sqrt{x^2 + y^2}$ for $1 \leq z \leq 3$. The mass density of the funnel is $\rho(x, y, z) = 5 - z$. Find the mass of the funnel.

Solution. Rewriting in cylindrical coordinates, we have $x = r \cos(\theta)$, $y = r \sin(\theta)$, and $z = r$. For $1 \leq z \leq 3$, we have $1 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. The normal vector is given by the gradient of the function $F(x, y, z) = z - \sqrt{x^2 + y^2}$,

$$\nabla F = \left\langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}, 1 \right\rangle.$$

It's magnitude is given by

$$\begin{aligned} |\nabla F| &= \sqrt{\left(-\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(-\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1^2} \\ &= \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + 1} = \sqrt{2}. \end{aligned}$$

Therefore, the surface element is given by

$$dS = |\nabla F| \, dr \, d\theta \sqrt{2} \, dr \, d\theta.$$

The mass is given by

$$\begin{aligned} M &= \iiint_S \rho \, dS = \int_0^{2\pi} \int_1^3 (5 - r) \cdot r \sqrt{2} \, dr \, d\theta \\ &= \sqrt{2} \cdot 2\pi \cdot \int_1^3 5r - r^2 \, dr \\ &= 2\sqrt{2}\pi \cdot \left(\frac{5r^2}{2} - \frac{r^3}{3}\right)_1^3 \end{aligned}$$

$$\begin{aligned}
&= 2\sqrt{2}\pi \cdot \left(\frac{27}{2} - \frac{13}{6}\right) \\
&= 2\sqrt{2}\pi \cdot \frac{34\sqrt{2}}{3} = \frac{68\sqrt{2}\pi}{3}. \quad \square
\end{aligned}$$

Problem 4. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x - 2y, 2z + 8x, y + z \rangle$ downward across the portion of the plane $4x + y + z = 12$ that lies in the first octant.

Solution. The normal vector to the plane is $\mathbf{N} = \nabla f = \langle 4, 1, 1 \rangle$. The region lies in the first octant, meaning $x \geq 0$, $y \geq 0$, and $z \geq 0$. Finding the intersection points with each axis, we get

$$\begin{aligned}
x = 0 = y &\Rightarrow z = 12 \\
x = 0 = z &\Rightarrow y = 12 \\
y = 0 = z &\Rightarrow x = 3.
\end{aligned}$$

Rewriting the plane equation, we get $z = 12 - 4x - y$. The normal vector to the plane is $\mathbf{N} = \nabla f = \langle 4, 1, 1 \rangle$. Since we want a downward orientation, we check the z -component of ∇F . Since it is positive (+1), the normal vector is already pointing upward. To orient it downward, we take the negative of the normal vector, $\mathbf{N} = \langle -4, -1, -1 \rangle$. Therefore, the surface element is given by

$$d\mathbf{S} = \langle -4, -1, -1 \rangle dA.$$

Computing the dot product, we get

$$\mathbf{F} \cdot d\mathbf{S} = (x - 2y)(-4) + (2z + 8x)(-1) + (y + z)(-1) = 10y - 36.$$

The equation of the line connecting $(3, 0)$ and $(0, 12)$ is $y = -4x + 12$. Therefore, the bounds of integration are $0 \leq x \leq 3$ and $0 \leq y \leq -4x + 12$. Thus, we get

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^{-4x+12} 10y - 36 \, dy \, dx \\
&= \int_0^3 5y^2 - 36y \Big|_0^{-4x+12} dx \\
&= \int_0^3 5(-4x + 12)^2 - 36(-4x + 12) \, dx \\
&= \int_0^3 5(16x^2 - 96x + 144) - 432 + 144x \, dx \\
&= \int_0^3 80x^2 - 336x + 288 \, dx \\
&= \frac{80}{3}x^3 - 168x^2 + 288x \Big|_0^3 = 72. \quad \square
\end{aligned}$$

Problem 5. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle xz, x, y \rangle$ and S is the hemisphere $x^2 + y^2 + z^2 = 9$ with $y \geq 0$ oriented in the direction of the positive y -axis.

Solution. In spherical coordinates, we have $x = 3 \sin(\theta) \cos(\varphi)$, $y = 3 \sin(\theta) \sin(\varphi)$, and $z = 3 \cos(\theta)$. Since we're considering the positive half of y , we have $0 \leq \theta \leq \pi$ and $-2/\pi \leq \varphi \leq 2/\pi$ for the upper hemisphere. Parametrizing, we get

$$\mathbf{r} = \langle 3 \sin(\theta) \cos(\varphi), 3 \sin(\theta) \sin(\varphi), 3 \cos(\theta) \rangle.$$

We know that

$$\begin{aligned}\mathbf{r}_\theta &= \langle 3 \cos(\theta) \cos(\varphi), 3 \cos(\theta) \sin(\varphi), -3 \sin(\theta) \rangle \\ \mathbf{r}_\varphi &= \langle -3 \sin(\theta) \sin(\varphi), 3 \sin(\theta) \cos(\varphi), 0 \rangle.\end{aligned}$$

Therefore, the normal vector is given by

$$\begin{aligned}\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_\varphi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 \cos(\theta) \cos(\varphi) & 3 \cos(\theta) \sin(\varphi) & -3 \sin(\theta) \\ -3 \sin(\theta) \sin(\varphi) & 3 \sin(\theta) \cos(\varphi) & 0 \end{vmatrix} \\ &= \langle 9 \sin^2(\theta) \sin(\varphi), 9 \sin^2(\theta) \sin(\varphi), 9 \cos(\theta) \cos(\varphi) \sin(\theta) \cos(\varphi) + 9 \cos(\theta) \sin(\varphi) \sin(\theta) \sin(\varphi) \rangle \\ &= \langle 9 \sin^2(\theta) \sin(\varphi), 9 \sin^2(\theta) \sin(\varphi), 9 \cos(\theta) \sin(\theta) \cos^2(\varphi) + 9 \cos(\theta) \sin(\theta) \sin^2(\varphi) \rangle \\ &= \langle 9 \sin^2(\theta) \cos(\varphi), 9 \sin^2(\theta) \sin(\varphi), 9 \cos(\theta) \sin(\theta) \rangle.\end{aligned}$$

Since the surface is oriented in the positive y -axis, we take the negative of the normal vector, $\mathbf{N} = \langle -9 \sin^2(\theta) \cos(\varphi), -9 \sin^2(\theta) \sin(\varphi), -9 \cos(\theta) \sin(\theta) \rangle$. Therefore, the surface element is given by

$$\begin{aligned}dS &= |\mathbf{N}| d\theta d\varphi \\ &= 9 \sqrt{\sin^4(\theta) \cos^2(\varphi) + \sin^4(\theta) \sin^2(\varphi) + \cos^2(\theta) \sin^2(\theta)} d\theta d\varphi \\ &= 9 \sin(\theta) \sqrt{\sin^2(\theta) \cos^2(\varphi) + \sin^2(\theta) \sin^2(\varphi) + \cos^2(\theta)} d\theta d\varphi \\ &= 9 \sin(\theta) \sqrt{\sin^2(\theta) (\cos^2(\varphi) + \sin^2(\varphi)) + \cos^2(\theta)} d\theta d\varphi \\ &= 9 \sin(\theta) \sqrt{\sin^2(\theta) + \cos^2(\theta)} d\theta d\varphi \\ &= 9 \sin(\theta) d\theta d\varphi.\end{aligned}$$

Computing the dot product, we get

$$\mathbf{F} \cdot \mathbf{N} = 81 \cos(\theta) \sin^3(\theta) \cos^2(\varphi) + 27 \sin^3(\theta) \cos(\varphi) \sin(\varphi) + 27 \cos(\theta) \sin^2(\theta) \sin(\varphi).$$

Therefore, we get

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^\pi \int_{-\pi/2}^{\pi/2} 81 \cos(\theta) \sin^3(\theta) \cos^2(\varphi) + 27 \sin^3(\theta) \cos(\varphi) \sin(\varphi) + 27 \cos(\theta) \sin^2(\theta) \sin(\varphi) d\varphi d\theta \\ &= \text{BEAUTIFUL MATH STUFF} \\ &= 0.\end{aligned}$$

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