In this problem set, the questions marked with a $(\star)$ are required for students enrolled in 583 and bonus for students enrolled in 483.

1. Differentiation and Transformation of Four-Vectors

   Two inertial frames S and S' are related by a Lorentz boost with speed v in the $+x$-direction. Units are chosen so that $c = 1$. The transformation is

$$t' = \gamma(t - vx), \tag{1}$$
$$x' = \gamma(x - vt), \tag{2}$$
$$y' = y, \tag{3}$$
$$z' = z, \tag{4}$$

   where $\gamma = (1 - v^2)^{-1/2}$.

   A particle moves along the x-axis in frame S with worldline

$$x(t) = at^2, \qquad y = z = 0,$$

   where a is a constant.

   The four-position is $x^\mu = (t, x, y, z)$, and the four-velocity is defined as

$$u^\mu = \frac{dx^\mu}{d\tau},$$

   where τ is the particle's proper time.

   (a) (5 points) Compute u^μ in frame S, expressing your answer in terms of t and a. (Hint: first compute dx^μ/dt, then use $d\tau = dt\sqrt{1 - v^2}$, where $v = dx/dt$.)

   (b) (2 points) Show explicitly that

$$u^\mu u_\mu = -1.$$

   (c) (5 points) Using the Lorentz transformation above, compute the components of u'^μ in frame S'. Verify that

$$u'^\mu u'_\mu = -1,$$

   and explain briefly why this result was expected.

   (d) (5 points) The four-acceleration is defined as

$$a^\mu = \frac{du^\mu}{d\tau}.$$

   Compute a^μ for the particle in frame S and show that

$$u^\mu a_\mu = 0.$$

(e) (2 points) Explain why differentiation with respect to proper time preserves the four-vector character of a quantity, whereas differentiation with respect to coordinate time t generally does not.

2. **Tachyons**

(a) (2 points) Argue that a kind of particle that always moves *faster* than the velocity of light would be consistent with Lorentz invariance in the sense that if its speed is greater than light in one frame, it will be greater than light in all frames. (Such a hypothetical particle is called a tachyon).

(b) (2 points) Show that the tangent vector to the trajectory of a tachyon is space-like and can be written $u^\alpha = dx^\alpha/ds$, where s is the spacelike interval along the trajectory. Show that $\mathbf{u} \cdot \mathbf{u} = 1$.

(c) (2 points) Evaluate the components of a tachyon's four-velocity $\mathbf{u}$ in terms of the three velocity $\vec{V} = d\vec{x}/dt$.

(d) (2 points) Define the four-momentum by $\mathbf{p} = m\mathbf{u}$ and find the relation between energy and momentum for a tachyon.

(e) (2 points) Show that there is an inertial frame where the energy of any tachyon is negative.

(f) (2 points) Show that if tachyons interact with normal particles, a normal particle could emit a tachyon with total energy and three-momentum being conserved. This result suggests a world containing tachyons would be unstable and there is no evidence for tachyons in nature.

3. **Relativistic charged particle** Consider the action of a (relativistic) charge particle coupled to a background gauge field, A_μ, living in Minkowski space. It is governed by the action

$$S = -m \int d\tau + q \int A_\mu(\boldsymbol{x}) dx^\mu \,, \tag{5}$$

where $\boldsymbol{A}$ is a 1-form and m, q are the mass and charge of the particle, respectively.

(a) (3 points) Suppose that the particle travels along the worldline $x^\mu(\lambda)$. Write the action in terms of x^μ and $dx^\mu/d\lambda$.

(b) (5 points) Find the canonical momenta p_μ and show that

$$p_\mu = mu_\mu + qA_\mu \,. \tag{8}$$

In other words, show that the canonical momentum, which is defined by the Lagrangian procedure, is different than the mechanical momentum $p_\mu = mu_\mu$.

(c) (3 points) Complete the Euler-Lagrange equations to show that

$$\frac{dp_\gamma}{d\lambda} = q \frac{\partial A_\mu}{\partial x^\gamma} \frac{dx^\mu}{d\lambda} \,. \tag{12}$$

(d) (6 points) ($\star$) Show that your answer in the previous part can be written as

$$m\frac{du_\gamma}{d\tau} = qF_{\gamma\mu}u^\mu \, , \tag{14}$$

where

$$F_{\gamma\mu} = \partial_\gamma A_\mu - \partial_\mu A_\gamma \tag{15}$$

is the field strength tensor.

(e) (5 points) Show that $F_{\gamma\mu}$ is antisymmetric and has the correct transformation properties to be a tensor.

(f) (6 points) ($\star$) By considering the acceleration of slow-moving particles, show that the components of $F_{\gamma\mu}$ agree with

$$F_{\gamma\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} . \tag{20}$$

(g) (4 points) ($\star$) Write the independent components of Eq. 15 corresponding to E_x and B_x. Does this match with what you've learned in E&M of how fields are derived from a vector potential? What is the physical interpretation of the component A_0?