Chapter 1

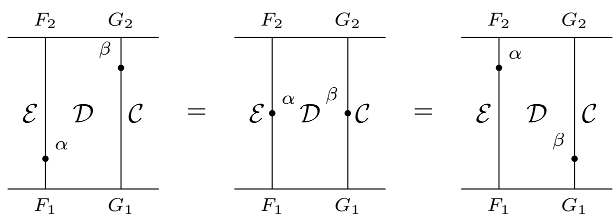
Introduction to Lie Algebras and Representation Theory

Chapter 2

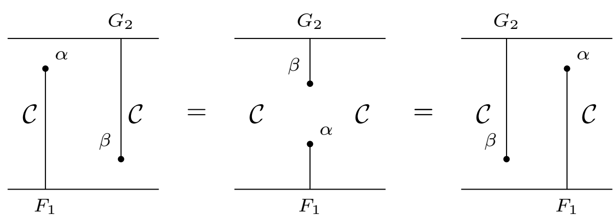
Introduction to Soergel Bimodules

2.7 How to Draw Monoidal Categories

Exercise 7.8. Show that the axioms of a 2-category imply the following equalities.



(2.1)



(2.2)

Solution to 7.8. The diagram 2.1 follows from the axioms of a 2-category. In particular, the interchange law states that horizontal and vertical compositions of 2-morphisms are compatible. More precisely, given 2-morphisms $\alpha : F_1 \implies F_2$ and $\beta : G_1 \implies G_2$, the horizontal composition $\beta * \alpha : G_1 \circ F_1 \implies G_2 \circ F_2$ is defined, and satisfies:

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha).$$

In 2.1, each equality reflects this interchange identity. That is, we may either first compose vertically and then horizontally, or compose horizontally first and then vertically, and the result is the same. This coherence condition is one of the core structural axioms of any strict 2-category.

For diagram 2.2, this identity expresses the fact that the horizontal composition of 2-morphisms with identity 1-morphisms on either side is strictly associative, and that such composition is natural. Specifically, suppose:

$$\alpha : F_1 \implies G_2 \quad \text{and} \quad \beta : \mathrm{id}_{\mathcal{C}} \implies \mathrm{id}_{\mathcal{C}},$$

are 2-morphisms in the hom-category $\mathcal{C}(\mathcal{C}, \mathcal{C})$. Then, under horizontal composition, we form:

$$(\mathrm{id}_{\mathcal{C}} * \alpha) \circ (\beta * \mathrm{id}_{F_1}) = (\beta \circ \mathrm{id}_{\mathcal{C}}) * (\mathrm{id}_{\mathcal{C}} \circ \alpha) = \beta * \alpha.$$

This follows from the interchange law and the unit laws of 2-categories. In particular, composing with identity 2-morphisms has no effect, and the strict associativity of composition allows us to freely rebracket the diagram. This justifies the equality of all three diagrams above. $\square$

Exercise 7.16. We can view the algebra $A = \mathbb{R}[x]/(x^2)$ as an object in the monoidal category of $\mathbb{R}$-vector spaces. Let $\cap : A \otimes A \to \mathbb{R}$ denote the linear map which sends $f \otimes g$ to the coefficient of x in fg. Let $\cup : \mathbb{R} \to A \otimes A$ denote the map which sends 1 to $x \otimes 1 + 1 \otimes x$.

(i) We wish to encode these maps diagrammatically, drawing $\cap$ as a cap and $\cup$ as a cup. Justify this diagrammatic notation, by checking the isotopy relations.

(ii) Draw a sequence of nested circles, as in an archery target. Evaluate this morphism.

Solution to 7.16(i). Let $v = ax + b \in A$. To check isotopy relations, we need to check the following compositions: $(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax + b)$ and $(\mathrm{id}_A \otimes \cup) \circ (\cap \otimes \mathrm{id}_A)(ax+b)$. To keep notation simple, recall that $\mathbb{R} \otimes A \cong A \cong A \otimes \mathbb{R}$.

We deal with the first composition first. We know that $\mathrm{id}_A \otimes \cup : A \to A \otimes A \otimes A$, we get

$$(\mathrm{id}_A \otimes \cup)(ax + b) = (ax + b) \otimes (x \otimes 1 + 1 \otimes x) = (ax + b) \otimes x \otimes 1 + (ax + b) \otimes 1 \otimes x.$$

Now, we know that $\cap \otimes \mathrm{id}_A : A \otimes A \otimes A \to A$, applying this to the above, we get

$$(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax + b) = (\cap \otimes \mathrm{id}_A)((ax + b) \otimes x \otimes 1 + (ax + b) \otimes 1 \otimes x).$$

Computing $\cap((ax + b) \otimes x)$, we compute fg to get $(ax + b) \cdot x = ax^2 + bx$. Therefore, $\cap((ax + b) \otimes x) = b$. Similarly, we get $\cap((ax + b) \otimes 1) = a$. Therefore, we have

$$(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax + b) = b + ax = ax + b.$$

Therefore, we have shown that the first composition is the identity.

We now deal with the second composition, $(\mathrm{id}_A \otimes \cup) \circ (\cap \otimes \mathrm{id}_A)(ax+b)$. Computing the inner composition, $\cup \otimes \mathrm{id}_A : A \otimes A \to A \otimes A \otimes A$, we get

$$(\cup \otimes \mathrm{id}_A)(ax + b) = (x \otimes 1 + 1 \otimes x) \otimes (ax + b) = x \otimes 1 \otimes (ax + b) + 1 \otimes x \otimes (ax + b).$$

Now, we know that $\mathrm{id}_A \otimes \cap : A \otimes A \otimes A \to A$, so applying this to the above, we get

$$(\mathrm{id}_A \otimes \cap)(x \otimes 1 \otimes (ax + b) + 1 \otimes x \otimes (ax + b)) = ax + b.$$

Therefore, $(\mathrm{id}_A \otimes \cap) \circ (\cup \otimes \mathrm{id}_A)(ax + b) = ax + b$, as desired.

Thus, we have shown that the isotopy relations hold in vector spaces. $\square$

Solution to 7.16(ii). Let f be the entire morphism. For simplicity, I use the notation, $\cup^{\otimes k} : \mathbb{R} \to A^{\otimes k}$, $\cap^{\otimes k} : A^{\otimes k} \to \mathbb{R}$, and $f^k = \cap^{\otimes k} \circ \cup^{\otimes k} : \mathbb{R} \to \mathbb{R}$, to denote k-circles.

For the 1-circle case, we have the following morphism

$$f(1) = (\cap \circ \cup)(1) = \cap(\cup(1)) = \cap(x \otimes 1 + 1 \otimes x) = 1 + 1 = 2.$$

For the 2-circle case, we have the following morphism

$$f^2(1) = \left(\cap^{\otimes 2} \circ \cup^{\otimes 2}\right)(1) = \cap^{\otimes 2}\left(\cup^{\otimes 2}(1)\right) = \cap^{\otimes 2}(x \otimes 1 + 1 \otimes x \otimes x \otimes 1 + 1 \otimes x) = 4.$$

For the k-circle case, we have the following morphism

$$f^k(1) = \left(\cap^{\otimes k} \circ \cup^{\otimes k}\right)(1).$$

Since $\cup(1) = x \otimes 1 + 1 \otimes x$, then $\cup^{\otimes k}(1)$ is the sum of 2^k simple tensors, each term of the form $v_1 \otimes w_1 \otimes \cdots \otimes v_k \otimes w_k$, where each pair $(v_i \otimes w_i)$ is either $x \otimes 1$ or $1 \otimes x$. Applying $\cap^{\otimes k}$ to each such term gives 1 for every pair, since $\cap(x \otimes 1) = \cap(1 \otimes x) = 1$. Therefore, we get $f^k(1) = 2^k$. $\square$

Exercise 7.17. This question is about the Temperley–Lieb category.

(i) Finish the proof that the isotopy relation holds in vector spaces.

(ii) There is a map $V \otimes V \to V \otimes V$ which sends $x \otimes y \mapsto y \otimes x$. Draw this as an element of the Temperley–Lieb category (a linear combination of diagrams).

(iii) Find an endomorphism of 2 strands which is killed by placing a cap on top. Can you find one which is an idempotent? Also find an endomorphism killed by putting a cup on bottom.

(iv) (Harder) Find an idempotent endomorphism of 3 strands which is killed by a cap on top (for either of the two placements of the cap).

Solution to 7.17(i). Define the following swap function:

$$\tau^k \left(\bigotimes_{n=1}^{k} v_n \right) = \bigotimes_{n=1}^{k} v_{k-n+1}.$$

I'll be using the notation $\lfloor f^k(x) \rfloor$ to denote the coefficient of x^k in the polynomial $f(x)$.

To prove isotopy relations hold in vector spaces, we need to prove the following: the following identities,

$$(\cap \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \cup) = \mathrm{id}_V$$
$$(\mathrm{id}_V \otimes \cap) \circ (\cup \otimes \mathrm{id}_V) = \mathrm{id}_V,$$

loop evaluation, symmetry of cups and caps, and sliding. I've already shown the snake identities in 7.16(i) and the loop evaluation in 7.16(ii). $\square$

Solution to 7.17(ii). $\square$

Solution to 7.17(iii). $\square$

Exercise 7.19. One can think about the right mate and the left mate as "twisting" or "rotating" α by 180° to the right or to the left. Visualize what it would mean to twist α by 360° to the right, yielding another 2-morphism $\alpha^{\vee\vee} : E \to F$. Verify that ${}^{\vee}\alpha = \alpha^{\vee}$, if and only if $\alpha = \alpha^{\vee\vee}$. Thus cyclicity is the same as "360 degree rotation invariance," which one might expect from any planar picture.

Solution to 7.19. $\square$

Exercise 7.20. Suppose that B is an object in a monoidal category with biadjoints, and $\Phi : B \otimes B \otimes B \to \mathbb{1}$ is a cyclic morphism. What should it mean to "rotate" Φ by 120°? Suppose that $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{C}$. What can you say about the 120° rotation of Φ, vis a vis Φ? What if $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{R}$?

Solution to 7.20. $\square$

2.9 The Dihedral Cathedral

Exercise 9.25. Let our base ring be some specialization of $\mathbb{Z}[\delta]$. Inside $\mathrm{TL}_{n,\delta}$ let T be the vector space of elements which are killed by all the $(n-1)$ caps on top, and let B be the space killed by cups on the bottom. For an element $x \in \mathrm{TL}_{n,\delta}$ let $\bar{x}$ denote the same element with each diagram flipped upside down. Thus, for example, $x \in T$ if and only if $\bar{x} \in B$.

(i) Show that any crossingless matching is either the identity diagram, or has both a cap on bottom and a cup on top.

(ii) We now make the following assumption:

$$\text{There exist some } f \in T \text{ for which the coefficients of the identity diagram is invertible.} \tag{2.3}$$

Why is this equivalent to the analogous assumption for B?

(iii) Let $f \in T$, with invertible coefficient c for the identity diagram. Let $g \in B$, with invertible coefficient d for the identity diagram. Compute the composition fg in two ways and deduce that f and g are colinear.

(iv) Assuming 2.3 deduce that $T = B$, that this space is one-dimensional, and that $f = \bar{f}$ for $f \in T$.

(v) Thus, assuming 2.3, there is a unique element $\mathrm{JW}_n \in T$ whose identity coefficient is 1. Prove that JW_n is idempotent. (If we construct JW_n in some other way, this proves 2.3.)

Solution to 9.25(i). □

Solution to 9.25(ii). □

Solution to 9.25(iii). □

Solution to 9.25(iv). □

Solution to 9.25(v). □

Exercise 9.26. Let TL_n be the Temperley–Lieb algebra with n-strands where the bubble evaluates to $-[2] = q + q^{-1} \in \mathbb{Q}(q)$. Clearly, JW_1 is just the identity element, where the condition of being killed by caps and cups is vacuous. Verify the following recursive formula:

(2.4)

In this last diagram, the cup on top matches the a-th and $(a+1)$-st boundary points, counting from the left.

Solution to 9.26. □

Exercise 9.27. Prove the following recursive formula.

Solution to 9.27. □

Exercise 9.28. The trace of an element $a \in \mathrm{TL}_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the following closed diagram:

(2.5)

(i) Calculate the trace of JW_n.

(ii) Suppose that q is a primitive $2m$-th root of unity. What is the trace of JW_{m-1}? What do you get when you rotate JW_{m-1} by one strand?

Solution to 9.28(i). □

Solution to 9.28(ii). □

Exercise 9.34.

(i) Write down the two-color relations when $m = 2$. Prove that $B_s B_t \simeq B_t B_s$ by constructing inverse isomorphisms.

(ii) Write down the two-color relations when $m = 3$. Prove that $B_s B_t B_S \simeq X \oplus B_s$, where X is the image of an idempotent constructed using two 6-valent vertices, by following the rubric of Exercise 8.39.

(iii) (Still for m = 3) Similarly, one has $B_t B_s B_t \simeq Y \oplus B_t$. Prove that X is isomorphic to Y. Extend the rubric of Exercise 8.39 to a rubric which describes when two summands of different objects are isomorphic.

Solution to 9.34(i). □

Solution to 9.34(ii). □

Solution to 9.34(iii). □

Exercise 9.35. Prove that there is an autoequivalence of $\mathcal{H}_{\mathrm{BS}}$ which flips each diagram vertically (resp. horizontally). See Exercise 8.10 for inspiration.

Solution to 9.35. □

Exercise 9.36. Show that the diagram obtained by attaching a "handle" to the left or right of a Jones–Wenzl projector equals 0. For example,

(2.6)

(Hint: use (9.16).)

Solution to 9.36. □

Exercise 9.37.

(i) A *pitchfork* is a diagram of the form

(2.7)

(or its color swap). The death by pitchfork relation states that the diagram obtained by placing a pitchfork anywhere on top or bottom of a Jones–Wenzl projector equals 0. For example:

(2.8)

Why is death by pitchfork implied by the defining property of the Jones–Wenzl projector?

(ii) Use (9.28) and (9.31) to prove that the diagram obtained by placing a pitchfork anywhere on top or bottom of the $2m$-valent vertex equals 0. We also call this *death by pitchfork*.

Solution to 9.37(i). □

Solution to 9.37(ii). □

Exercise 9.38.

(i) Prove (9.29) and (9.30) using the relations in (9.27). (Hint: each relation follows from two careful applications of (9.27c). Alternatively, (9.29) can be proved by repeatedly applying (9.30).

(ii) Prove (9.27b) using (9.28) and the other relations in (9.27). (Hint: first use (8.12) to create a dot and a trivalent vertex on the left hand side, and then dispose of the trivalent vertex with two-color associativity.)

The following exercise is harder, but very worthwhile.

Solution to 9.38(i). □

Solution to 9.38(ii). □

Solution to 9.38(iii). □

Exercise 9.39.

(i) Prove (9.28) using the relations in (9.27).

(ii) Prove that the Elias–Jones–Wenzl relation (9.27b) follows from two-color associativity (9.27c) and two-color dot contraction (9.28).

Solution to 9.39(i). □

Solution to 9.39(ii). □

Chapter 3

Research Papers

Exercise 1. Compute the value of a bigon at $q = 1$ or at general q.

Solution to 1. □

Exercise 2. Look at (2.9). Can you find associativity and coassociativity inside? Use only these relations and (2.4) to prove (2.9).

Solution to 2. □

Exercise 3. Write down what (2.10) means explicitly for some small values of k, l, r, s, until you get a feeling for how it works. You'll definitely want an exampel where k-l+r-s is at least 2 eventually. Then try to verify it using vectors for small values.

Solution to 3. □

Exercise 4. Try to prove Lemma 2.9 from Light Ladders and Clasp Conjectures

Solution to 4. □

Exercise 5. Remember how for the Temperley-Lieb algebra you described the "Crossing" $v \otimes w \mapsto w \otimes v$ as a linear combination of other maps. Let's do this again, but with webs this time. You're going to have to use $q = 1$ do this exercise, so forget about the q-deformation.

Consider the map $\Lambda^1 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^1 V$ which just swaps the tensor factors. This is a linear combination of:

(i) The web which merges $1, 2$ into 3 and then splits 3 into $2, 1$.

(ii) The web which splits $1, 2$ into $1, 1, 1$ and then merges $1, 1, 1$ into $2, 1$.

Find the linear combo.

Solution to 5(i). □

Solution to 5(ii). □

Exercise 6. Consider the map $\Lambda^2 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^2 V$ which just swaps the tensor factors. This is a linear combination of:

(i) the web which merges $2, 2$ into 4 and then splits 4 into $2, 2$.

(ii) the web which splits $2, 2$ into $2, 1, 1$ and then merges $2, 1, 1$ into $3, 1$ and then splits back to $2, 1, 1$ and marges back to $2, 2$.

(iii) the identity of $2, 2$.

Find the linear combo.

Solution to 6(i). □

Solution to 6(ii). $\square$

Solution to 6(iii). $\square$

Chapter 4

Misc