Math 307, Homework #5
Due Wednesday, November 6

For the logic proofs we do from now on, try to do them without using any non-obvious tautologies—that is, without using tautologies other than those appearing in Theorem 2.8 on page 48 of the book.

1. Show that $Q \Rightarrow S,\ R \Rightarrow T \vdash (Q \vee R) \Rightarrow (S \vee T)$.
2. Show that $(Q \wedge \sim T) \Rightarrow (Y \vee \sim P),\ Y \Rightarrow (V \vee \sim X) \vdash (P \wedge X) \Rightarrow [Q \Rightarrow (T \vee V)]$.
3. Give a line proof that $(A \subseteq X \wedge B \subseteq X) \Rightarrow (A \cup B \subseteq X)$.
4. Give a line proof that $(X \subseteq A \wedge X \subseteq B) \Rightarrow (X \subseteq A \cap B)$.

For any $n \in \mathbb{Z}$, recall that $M_n = \{x \in \mathbb{Z} \mid x \equiv_n 0\}$; that is, M_n is the set of all multiples of n.

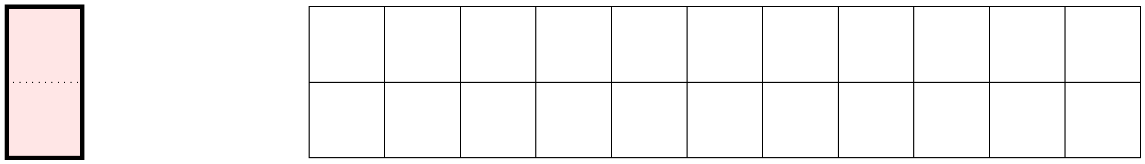
In each of questions 5–7, if you choose to prove the claim, give a line proof. To disprove an equality, explain which subset direction you are disproving and give a specific element which disproves it.

5. Prove or disprove: If $a, b \in \mathbb{N}$ and $a \leq b$ then $M_a \subseteq M_b$.
6. Prove or disprove: $M_4 \cap M_6 = M_{24}$.
7. Prove or disprove: $M_4 \cap M_9 = M_{36}$.
8. Give a line proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

For sets A and X, recall that $X - A = \{x \mid x \in X \wedge x \notin A\}$.

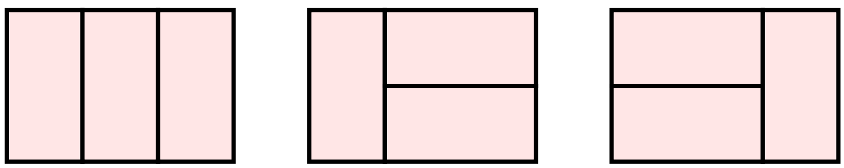
9. Give a line proof showing that $X - (A \cup B) = (X - A) \cap (X - B)$, for all sets A, B, and X.
10. Give a line proof showing that $X - (A \cap B) = (X - A) \cup (X - B)$, for all sets A, B, and X.
11. If f is a function from S to T and $A \subseteq S$, define $f(A) = \{x \mid (\exists y \in A)[x = f(y)]\}$. This is called the **image of A under f**.
    (a) Give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ where $f(M_3 \cap \mathbb{N}) = \mathbb{N}$.
    (b) Suppose $f\colon S \to T$ and $A \subseteq S$, $B \subseteq S$.
        (i) Prove that $f(A \cup B) = f(A) \cup f(B)$
        (ii) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$
        (iii) Give an example of sets S, T, A, B, and a function f for which $f(A) \cap f(B) \not\subseteq f(A \cap B)$. [Hint: Start by trying to prove that $f(A) \cap f(B) \subseteq f(A \cap B)$, and see where you get stuck.]
12. Yoda has a bunch of eggs, and you have to figure out how many.
    (a) He tells you two facts: When you separate the eggs into groups of 11, there are 3 left over. When you separate the eggs into groups of 8, there are 4 left over.
    Given these facts, what is the least number of eggs that Yoda could have?
    (b) Suppose Yoda also tells you that he has between 100 and 200 eggs. Given this additional piece of information, you can determine exactly how many eggs Yoda has. How many?
    (c) The next day Yoda comes back with a lot more eggs. This time he says: When I separate the eggs into groups of 11, there are 3 left over. When I separate them into groups of 300, there are 51 left over. What is the least number of eggs that Yoda could have?

13. You have a huge collection of 1×2 dominoes (or tiles), and a 2×11 checkerboard:



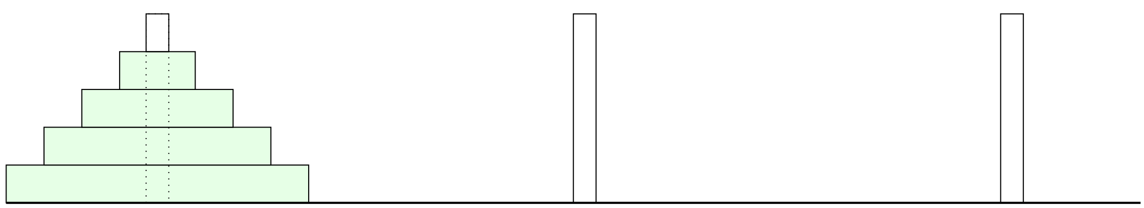
Your goal in this problem is to determine how many different ways there are to tile the checkerboard using your dominoes.

To solve this problem, it is best to solve some smaller versions first. Let S_n denote the number of different ways to tile a $2 \times n$ checkerboard. For example, $S_3 = 3$ as we see below:



Make a table showing the values of S_n for $1 \leq n \leq 11$. As you are doing this, try to find a systematic way of finding all the tilings. Note any patterns that you find, and see if you can explain them. As a check, you should get $S_6 = 13$.

14. A certain puzzle has three pegs, the leftmost peg starting out with a tower of n disks. The object of the puzzle is to move this tower to the rightmost peg. The rules are that you can only move one disk at a time, the disk has to be moved from one peg to another, and you can never place a larger disk on top of a smaller disk. The following picture shows the starting position of the puzzle when $n = 4$:



Your goal in this problem is to determine the minimum number of moves needed to solve the puzzle when there are 11 disks. Approach this in a similar way to what we did in problem #13, by making a table showing the minimum number of moves to solve the n-disk game for $1 \leq n \leq 11$.