

## SOLUTIONS TO HOMEWORK 5

**Warning:** Little proofreading has been done.

### 1. SECTION 2.4

**Exercise 2.4.7 (Limit Superior).** Let  $(a_n)$  be a bounded sequence.

- (a) Prove that the sequence defined by  $y_n = \sup\{a_k : k \geq n\}$  converges.
- (b) The *limit superior* of  $(a_n)$ , or  $\limsup a_n$ , is defined by

$$\limsup a_n = \lim y_n,$$

where  $y_n$  is the sequence from part (a) of this exercise. Provide a reasonable definition for  $\liminf a_n$  and briefly explain why it always exists for any bounded sequence.

- (c) Prove that  $\liminf a_n \leq \limsup a_n$  for every bounded sequence, and give an example of a sequence for which the inequality is strict.
- (d) Show that  $\liminf a_n = \limsup a_n$  if and only if  $\lim a_n$  exists. In this case, all three share the same value.

*Solution.* (a) Since the sequence  $(a_n)$  is bounded, there is an  $M > 0$  such that  $|a_n| \leq M$ . or  $-M \leq a_n \leq M$ . As a consequence, we see that  $-M \leq y_n \leq M$ , or  $|y| \leq M$ , so that  $y_n$  is bounded. By its definition,

$$y_n = \sup\{a_k : k \geq n\} = \sup\{a_k : k \geq n+1\} \cup \{y_n\} \geq \sup \sup\{a_k : k \geq n+1\} = y_{n+1},$$

so that  $y_n$  is a decreasing sequence. Hence, by the Monotone convergence theory,  $(y_n)$  converges.

(b) Let  $z_n = \inf\{a_k : k \geq n\}$ . The same argument shows that  $z_n$  is an increasing bounded sequence, so that  $(z_n)$  converges. We can then define

$$\liminf a_n = \lim z_n.$$

(c) Since  $z_n \leq y_n$  for every  $n \in \mathbb{N}$ ,  $\liminf a_n \leq \limsup a_n$  follows from order limit theorem.

There are many such examples. One example is  $a_n = (-1)^n$ . Then  $\liminf a_n = -1$  and  $\limsup a_n = 1$  so that  $\liminf a_n < \limsup a_n$ .

(d) Since  $z_n \leq a_n \leq y_n$ , if  $\liminf a_n = \limsup a_n = a$ , then by the squeeze theorem,  $\lim a_n = a$ .

Assume  $\lim a_n = a$ . Let  $\varepsilon > 0$ . There is  $N \in \mathbb{N}$  such that, for all  $n > N$ ,  $|a_n - a| < \varepsilon$  or  $a - \varepsilon < a_n < a + \varepsilon$ . Since  $y_n = \sup\{a_k : k \geq n\}$  and  $z_n = \inf\{a_k : k \geq n\}$ , we see that if  $n \geq N$ , then  $a - \varepsilon \leq z_n \leq y_n \leq a + \varepsilon$ , so that  $|z_n - a| < \varepsilon$  and  $|y_n - a| < \varepsilon$ . Thus, we conclude that  $\lim y_n = a$  and  $\lim z_n = a$ .  $\square$

### 2. SECTION 2.5

**Exercise 2.5.1.** Give an example of each of the following, or prove that such a request is impossible.

- (a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
- (b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
- (c) A sequence that contains subsequences converging to every point in the infinite set  $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$ .
- (d) An unbounded sequence with a convergent subsequence.

*Solution.* (a) This is not possible. A subsequence of a subsequence is a subsequence of the original sequence, and the Bolzano-Weierstrass Theorem implies that the bounded subsequence has in turn a convergent subsequence.

(b) Define

$$a_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ 1 + \frac{1}{n} & n \text{ odd} \end{cases}$$

Then  $(a_{2n})_{n \in \mathbb{N}} = (\frac{1}{2n})_{n \in \mathbb{N}}$  is a subsequence of  $(\frac{1}{n})_{n \in \mathbb{N}}$ , and we already know  $\frac{1}{n} \rightarrow 0$ , so

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} \frac{1}{2n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Also,  $(a_{2n-1})_{n \in \mathbb{N}} = (1 + \frac{1}{2n-1})_{n \in \mathbb{N}}$  is a subsequence of  $(1 + \frac{1}{n})_{n \in \mathbb{N}}$ , and we can combine the Algebraic Limit Theorem with limits, we already know to get  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n}) = 1$ , so

$$\lim_{n \rightarrow \infty} a_{2n-1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n-1}\right) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1.$$

Finally, it is obvious that  $a_n \neq 0$  and  $a_n \neq 1$  for all  $n \in \mathbb{N}$ .

(c) One example is taking the sequence to be the sequence of rational numbers. Here is another example. Take the sequence to be

$$(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 1, \dots).$$

For every  $n \in \mathbb{N}$ , there is a subsequence which has the form

$$(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots).$$

We already know that this sequence converges to  $\frac{1}{n}$ . □

### Exercise 2.5.3.

- (a) Prove that if an infinite series converges, then the associative property holds. That is, assume  $a_1 + a_2 + a_3 + a_4 + \dots$  converges to a limit  $L$  (that is, the sequence of partial sums  $(s_n)_{n \in \mathbb{N}}$  converges to  $L$ ). Show that any regrouping of the terms

$$(a_1 + a_2 + \dots + a_{n_1}) + (a_{n_1+1} + \dots + a_{n_2}) + (a_{n_2+1} + \dots + a_{n_3}) + \dots$$

leads to a series which also converges to  $L$ .

- (b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in part a apply to this example?

*Solution.* (a) The partial sums of the series  $\sum_{n=1}^{\infty}$  are given by

$$s_n = a_1 + a_2 + \dots + a_n$$

for  $n \in \mathbb{N}$ .

Define  $n_0 = 0$ , and define

$$b_k = a_{n_{k-1}+1} + \dots + a_{n_k}$$

for  $k \in \mathbb{N}$ . Thus the new series is  $\sum_{k=1}^{\infty} b_k$ . Its partial sums are given by

$$t_k = b_1 + b_2 + \dots + b_k.$$

It is immediate that

$$t_k = a_1 + a_2 + \dots + a_{n_k} = s_{n_k}.$$

By hypothesis,  $\lim_{n \rightarrow \infty} s_n = L$ . Therefore, by Theorem 2.5.2 of the book,

$$\sum_{k=1}^{\infty} (a_{n_{k-1}+1} + \dots + a_{n_k}) = \sum_{k=1}^{\infty} b_k = \lim_{k \rightarrow \infty} t_k = \lim_{k \rightarrow \infty} s_{n_k} = L.$$

This completes the proof.

- (b) The series in the example discussed at the end of Section 2.1 does not converge. Indeed, the sequence of partial sums is

$$(-1, 0, -1, 0, -1, 0, -1, 0, \dots),$$

which has (constant) subsequences converging to the distinct limits  $-1$  and  $0$ . □

## 3. SECTION 2.6

**Exercise 2.6.2** Give an example of each of the following, or argue that such a request is impossible.

- (a) A Cauchy sequence that is not monotone.
- (b) A Cauchy sequence with an unbounded subsequence.
- (c) A divergence monotone sequence with a Cauchy subsequence.
- (d) An unbounded sequence containing a subsequence that is Cauchy.

*Solution.* (a) Define  $a_n = (-1)^n/n$  for  $n \in \mathbb{N}$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$ . Therefore  $(a_n)$  is Cauchy.

The sequence  $(a_n)_{n \in \mathbb{N}}$  is not monotone because  $a_1 < a_2$  but  $a_2 > a_3$ .

There are many other examples. Here is a rather trivial one:

$$(-1, 1, -1, 0, 0, 0, 0, 0, 0, \dots).$$

(b) No such thing exists. Every Cauchy sequence converges, so every subsequence converges to the same limit, and a convergence sequence is bounded.

(c) No such thing exists. Let  $(a_n)$  be an increasing sequence. If  $(a_{n_k})$  is a Cauchy subsequence of  $(a_n)$ , then it must converge, hence bounded. That is, there is an  $M > 0$  such that  $|a_n| \leq M$ . However, for every  $n \in \mathbb{N}$ , there is  $n_k \in \mathbb{N}$  such that  $n_k \leq n \leq n_{k+1}$ . By the monotonicity,  $a_{n_k} \leq a_n \leq a_{n_{k+1}}$ . Hence, we much have  $|a_n| \leq M$ . This shows that  $(a_n)$  is bounded, hence  $(a_n)$  converges.

(d) There are many such sequences. Here is an example. Define

$$a_n = \begin{cases} n & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$$

Then  $(a_n)_{n \in \mathbb{N}}$  is clearly unbounded, but the subsequence  $(a_{2n})_{n \in \mathbb{N}}$  is constant, therefore convergent, therefore Cauchy.  $\square$

**Exercise 2.6.3** If  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  are Cauchy sequences, then one easy way to prove that  $(x_n + y_n)$  is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4,  $(x_n)$  and  $(y_n)$  must be convergent, and the Algebraic Limit Theorem then implies  $(x_n + y_n)$  is convergent and hence Cauchy.

- (a) Give a direct argument that  $(x_n + y_n)$  is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.
- (b) Do the same for the product  $(x_n y_n)$ .

*Solution.* (a) Let  $\varepsilon > 0$ . Choose  $N_1 \in \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  with  $m, n \geq N_1$ , we have  $|x_m - x_n| < \frac{1}{2}\varepsilon$ . Choose  $N_2 \in \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  with  $m, n \geq N_2$ , we have  $|y_m - y_n| < \frac{1}{2}\varepsilon$ . Define  $N = \max(N_1, N_2)$ . Let  $m, n \in \mathbb{N}$  with  $m, n \geq N$ . Then

$$|(x_m + y_m) - (x_n + y_n)| = |(x_m - x_n) + (y_m - y_n)| \leq |x_m - x_n| + |y_m - y_n| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

(b) We need a little preparation. By Lemma 2.6.3 of the book,  $(x_n)$  and  $(y_n)$  are bounded. Thus, there are  $M_1, M_2 \in [0, \infty)$  such that  $|x_n| \leq M_1$  and  $|y_n| \leq M_2$  for all  $n \in \mathbb{N}$ .

Let  $\varepsilon > 0$ . Choose  $N_1 \in \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  with  $m, n \geq N_1$ , we have

$$|x_m - x_n| < \frac{\varepsilon}{2M_2 + 1}.$$

Choose  $N_2 \in \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  with  $m, n \geq N_2$ , we have

$$|y_m - y_n| < \frac{\varepsilon}{2M_1 + 1}.$$

Define  $N = \max(N_1, N_2)$ . Let  $m, n \in \mathbb{N}$  with  $m, n \geq N$ . Then

$$\begin{aligned} |x_m y_m - x_n y_n| &= |(x_m y_m - x_n y_m) + (x_n y_m - x_n y_n)| \\ &\leq |x_m - x_n| \cdot |y_m| + |x_n| \cdot |y_m - y_n| \\ &\leq \left( \frac{\varepsilon}{2M_2 + 1} \right) M_2 + M_1 \left( \frac{\varepsilon}{2M_1 + 1} \right) \\ &< \varepsilon. \end{aligned}$$

□

Remark 1: It is legitimate to use Lemma 2.6.3 of the book, since it doesn't depend on completeness. It *isn't* legitimate to use the fact that Cauchy sequences converge, and then the fact that convergent sequences are bounded, since this requires completeness.

Remark 2: We divide by  $2M_1 + 1$  and  $2M_2 + 1$  in case  $M_1 = 0$  or  $M_2 = 0$ .