Implicit Differentiation

Consider an implicitly defined surface $F(x,y,z)=C$

Assume that x and y are independent and $z=f(x,y)$

so that as x or y changes, z must change to

remain on surface. What is $\frac{\partial z}{\partial x}$? What is $\frac{\partial z}{\partial y}$?

Ex! $x^2+y^2+z^2=4$ defines a sphere of radius 2.

$$\left.\begin{aligned} z &= \sqrt{4-x^2-y^2} \\ z &= -\sqrt{4-x^2-y^2} \end{aligned}\right\} \text{ Explicit functions of } (x,y) \text{ that satisfy implicit relation.}$$

$F(x,y,z)=C$ where $z=f(x,y)$

$$\frac{\partial}{\partial x}\left(F(x,y,z)\right)=\frac{\partial}{\partial x}(C)$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{but } \frac{dx}{dx}=1 \text{ and } \frac{dy}{dx}=0$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=\frac{-F_x}{F_z}$$

$$\frac{\partial}{\partial y}\left(F(x,y,z)\right)=\frac{\partial}{\partial y}(C)$$

$$\frac{\partial F}{\partial x}\frac{dx}{dy}+\frac{\partial F}{\partial y}\frac{dy}{dy}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0 \quad \text{but } \frac{dx}{dy}=0 \text{ and } \frac{dy}{dy}=1$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$$

Ex: Given $xyz = \cos(xz^2 - y)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$xyz - \cos(xz^2 - y) = 0$$

Define $F(x,y,z) = xyz - \cos(xz^2 - y)$

Then the surface is $F = 0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(yz + z^2\sin(xz^2 - y))}{xy + 2xz\sin(xz^2 - y)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(xz - \sin(xz^2 - y))}{xy + 2xz\sin(xz^2 - y)}$$

§14.6: Directional Derivative

Recall, for a function $f(x,y)$

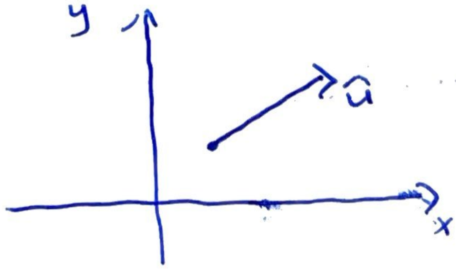
$$f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

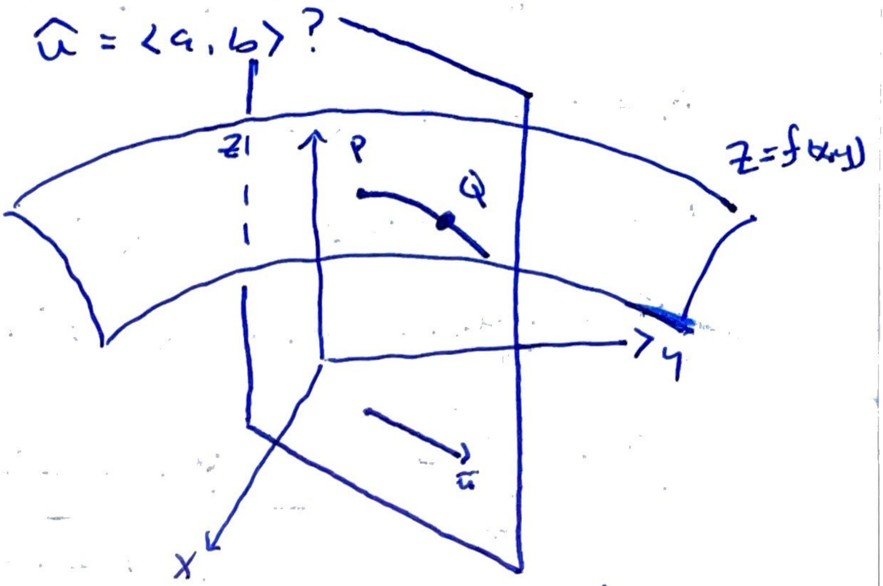
Rate of change of f with respect to x: rate of change of f in direction of $\hat{i} = \langle 1, 0 \rangle$

$$f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Rate of change of f with respect to y; rate of change of f in direction of $\hat{j} = \langle 0, 1 \rangle$

What is the rate of change of f in the direction of the unit vector $\hat{u} = \langle a, b \rangle$?





Intersect the surface $z=f(x,y)$ with a vertical plane in direction of $\hat{u}$ forming a curve of intersection

Let $P(x_0, y_0, f(x_0, y_0))$ be a point on the curve.

Let $Q(x, y, f(x,y))$ be any other point on the curve.

Project P and Q onto xy-plane

$$P'(x_0, y_0, 0)$$

$$Q'(x, y, 0)$$

In the plane, $\overrightarrow{P'Q'} = \langle x-x_0, y-y_0 \rangle$ is parallel to $\hat{u}$. There exists a scalar, h, so that

$$\langle x-x_0, y-y_0 \rangle = h\langle a, b \rangle$$

Since $\hat{u}$ is a unit vector, h is the distance traveled in direction of $\hat{u}$ in the domain of f.

$x = x_0 + ah$

$y = y_0 + bh$

Difference Quotient of f with respect to distance is

$$\frac{\Delta f}{h} = \frac{f(x,y) - f(x_0,y_0)}{h} = \frac{f(x_0+ah,\, y_0+bh) - f(x_0,y_0)}{h}$$

As $h \to 0$, $Q \to P$ and the average rate of change of f with respect to distance approaches the instantaneous rate of change.

Defn: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of the unit vector $\hat{u} = \langle a, b \rangle$ is

$$D_{\hat{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ah,\, y_0+bh) - f(x_0,y_0)}{h}$$

if the limit exists,

Theorem: If $f(x,y)$ is differentiable at (x_0, y_0), then it has a directional derivative in the direction of any unit vector $\hat{u} = \langle a, b \rangle$ and

$$D_{\hat{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

Proof: Let $g(h) = f(x_0 + ah, y_0 + bh)$

Then $g'(0) = \lim_{h \to 0} \dfrac{g(h) - g(0)}{h} = \lim_{h \to 0} \dfrac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$

$= D_{\hat{u}} f(x_0, y_0)$

Also, by chain rule,

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

$$g'(0) = a f_x(x_0, y_0) + b f_y(x_0, y_0) = D_{\hat{u}} f(x_0, y_0)$$

Ex: Find the derivative of $f(x,y) = x^2 + x\sqrt{y}$ in direction of $\vec{u} = \langle 2, -3 \rangle$

Unit vector in direction of $\vec{u}$ is $\hat{u} = \frac{1}{\sqrt{13}} \langle 2, -3 \rangle$

$$f_x = 2x + \sqrt{y}$$

$$f_y = \frac{x}{2\sqrt{y}}$$

$$D_{\hat{u}} f = \frac{2}{\sqrt{13}}(2x + \sqrt{y}) - \frac{3}{\sqrt{13}}\left(\frac{x}{2\sqrt{y}}\right)$$