

# Fundamentals of Analysis II: Homework 6

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**Exercise 7.4.2.**

(i) Let  $g(x) = x^3$ , and classify each of the following as positive, negative, or zero

$$(i) \int_0^{-1} g + \int_0^1 g \quad (ii) \int_1^0 g + \int_0^1 g \quad (iii) \int_1^{-2} g + \int_0^1 g.$$

(ii) Show that if  $b \leq a \leq c$  and  $f$  is integrable on  $[b, c]$ , then it is still the case that  $\int_a^b f = \int_a^c f + \int_c^b f$ .

*Solution to (i).* Converting the integrals to their respective values, we have

$$\int_0^{-1} g + \int_0^1 g = -\int_{-1}^0 x^3 dx + \int_0^1 x^3 dx = -\left(-\frac{1}{4}\right) + \frac{1}{4} = \frac{1}{2}.$$

Therefore, the integral  $\int_0^{-1} g + \int_0^1 g$  is positive.

Converting the integrals to their respective values, we have

$$\int_1^0 g + \int_0^1 g = -\int_0^1 x^3 dx + \int_0^1 x^3 dx = 0.$$

Therefore, the integral  $\int_1^0 g + \int_0^1 g$  is zero.

Converting the integrals to their respective values, we have

$$\int_1^{-2} g + \int_0^1 g = -\int_{-2}^1 x^3 dx + \int_0^1 x^3 dx = -\left(-\frac{15}{4}\right) + \frac{1}{4} = 4.$$

Therefore, the integral  $\int_1^{-2} g + \int_0^1 g$  is positive. □

*Solution to (ii).* Assume  $b \leq a \leq c$  and  $f$  is integrable on  $[b, c]$ . By the additivity of the integral, we have

$$\int_b^c f = \int_b^a f + \int_a^c f.$$

Rearranging the terms, we have

$$\begin{aligned} \int_b^a f &= \int_b^c f - \int_a^c f \\ \Rightarrow -\int_a^b f &= \int_b^c f - \int_a^c f \\ \Rightarrow \int_a^b f &= -\int_b^c f + \int_a^c f \\ \Rightarrow \int_a^b f &= \int_a^c f + \int_c^b f. \end{aligned}$$

□

**Exercise 7.4.3.** Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(i) If  $|f|$  is integrable on  $[a, b]$ , the  $f$  is also integrable on this set.

(ii) Assume  $g$  is integrable and  $g(x) \geq 0$  on  $[a, b]$ . If  $g(x) > 0$  for an infinite number of points  $x \in [a, b]$ , then  $\int_a^b g > 0$ .

(iii) If  $g$  is continuous on  $[a, b]$  and  $g(x) \geq 0$  with  $g(y_0) > 0$  for at least one point  $y_0 \in [a, b]$ , then  $\int_a^b g > 0$ .

*Solution to (i).* True. Assume  $|f|$  is integrable on  $[a, b]$ . By the integrability criterion, for any  $\varepsilon > 0$ , there is a partition  $P_\varepsilon$  such that

$$U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \varepsilon.$$

Since  $-|f| \leq f \leq |f|$ , we have

$$L(|f|, P_\varepsilon) \leq L(f, P_\varepsilon), \quad U(f, P_\varepsilon) \leq U(|f|, P_\varepsilon).$$

This implies that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \varepsilon.$$

By the integrability criterion,  $f$  is integrable on  $[a, b]$ . □

*Solution to (ii).* False. Consider the function

$$g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then  $g$  is non-negative and positive on an infinite set of points. However, it is zero almost everywhere, so the integral  $\int_a^b g = 0$ . □

*Solution to (iii).* True. Since  $g$  is continuous and  $g(y_0) > 0$ , there is an interval around  $y_0$  where  $g(x) > 0$ . Since this is a nonzero interval and  $g(x) \geq 0$ , the integral over this interval is positive. Hence,  $\int_a^b g > 0$ . □

**Exercise 7.4.6.** Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact:

(i) If  $f$  satisfies  $|f(x)| \leq M$  on  $[a, b]$ , show

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

(ii) Prove that if  $f$  is integrable on  $[a, b]$ , then so is  $f^2$ .

(iii) Now show that if  $f$  and  $g$  are integrable, then  $fg$  is integrable. (Consider  $(f + g)^2$ .)

*Solution to (i).* Using the difference of squares factorization

$$f(x)^2 - f(y)^2 = (f(x) - f(y))(f(x) + f(y)).$$

Taking absolute values,

$$|f(x)^2 - f(y)^2| = |f(x) - f(y)| \cdot |f(x) + f(y)|.$$

Since  $|f(x)| \leq M$  and  $|f(y)| \leq M$ , we get

$$|f(x) + f(y)| \leq |f(x)| + |f(y)| \leq M + M = 2M.$$

Thus,

$$|f(x)^2 - f(y)^2| \leq 2M|f(x) - f(y)|. \quad \square$$

*Solution to (ii).* Since  $f$  is integrable, it satisfies the definition of integrability: for every  $\varepsilon > 0$ , there exists a partition  $P$  such that the difference between the upper sum and lower sum is less than  $\varepsilon$ . That is,

$$U(f, P) - L(f, P) < \varepsilon.$$

Now, we analyze  $f^2$ . From Step 1, we have the inequality:

$$|f(x)^2 - f(y)^2| \leq 2M|f(x) - f(y)|.$$

This means that the function  $f^2$  has variations that are controlled by  $f$ . Since  $f$  is integrable, the variation  $|f(x) - f(y)|$  can be made arbitrarily small by refining the partition. Using this bound, we can show that  $U(f^2, P) - L(f^2, P)$  also becomes arbitrarily small for sufficiently fine partitions. Therefore,  $f^2$  is integrable. □

*Solution to (iii).* Rearranging  $(f + g)^2 = f^2 + 2fg + g^2$ , we get

$$fg = \frac{(f + g)^2 - f^2 - g^2}{2}.$$

Since we have already established that if  $f$  is integrable, then  $f^2$  is integrable, and since  $f$  and  $g$  are both given to be integrable, we conclude that  $f^2$ ,  $g^2$ , and  $(f + g)^2$  are all integrable.

Since the set of integrable functions is closed under addition and scalar multiplication, it follows that  $fg$  is also integrable.  $\square$

**Exercise 7.5.1.**

(i) Let  $f(x) = |x|$  and define  $F(x) = \int_{-1}^x f$ . Find a piecewise algebraic formula for  $F(x)$  for all  $x$ . Where is  $F$  continuous? Where is  $F$  differentiable? Where does  $F'(x) = f(x)$ ?

(ii) Repeat part (i) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

*Solution to (i).* Define  $f(x)$  as

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Evaluating  $F(x)$ , we get

$$F(x) = \int -1^x |t| dt.$$

We then have two cases to consider. If  $x < 0$ , then from  $-1$  to  $x$ , we are in the negative region

$$F(x) = \int_{-1}^x -t dt = \frac{1}{2} - \frac{x^2}{2}.$$

If  $x \geq 0$ , then from  $-1$  to  $x$ , we are in the positive region

$$F(x) = \int_{-1}^0 -t dt + \int_0^x t dt = \frac{1}{2} + \frac{x^2}{2}.$$

Therefore, the piecewise formula for  $F(x)$  is

$$F(x) = \begin{cases} \frac{1}{2} - \frac{x^2}{2} & \text{if } x < 0 \\ \frac{1}{2} + \frac{x^2}{2} & \text{if } x \geq 0. \end{cases}$$

Each piece is a polynomial and polynomials are continuous, so we only need to check continuity at  $x = 0$ . Evaluate the left and right limits

$$\begin{aligned} \lim_{x \rightarrow 0^-} F(x) &= \lim_{x \rightarrow 0^-} \left( \frac{1}{2} - \frac{x^2}{2} \right) = \frac{1}{2} \\ \lim_{x \rightarrow 0^+} F(x) &= \lim_{x \rightarrow 0^+} \left( \frac{1}{2} + \frac{x^2}{2} \right) = \frac{1}{2} \\ F(0) &= \frac{1}{2}. \end{aligned}$$

Since the left and right limits are equal to the value of the function at  $x = 0$ ,  $F(x)$  is continuous at  $x = 0$ , making  $F(x)$  continuous everywhere.

Again, since each piece is a polynomial,  $F(x)$  is differentiable everywhere, except possibly at  $x = 0$ . Evaluate the left and right derivatives at  $x = 0$ ,

$$F'(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Evaluating the left and right derivatives at  $x = 0$ ,

$$\begin{aligned} \lim_{x \rightarrow 0^-} F'(x) &= \lim_{x \rightarrow 0^-} -x = 0 \\ \lim_{x \rightarrow 0^+} F'(x) &= \lim_{x \rightarrow 0^+} x = 0. \end{aligned}$$

Since the left and right derivatives are equal at  $x = 0$ ,  $F(x)$  is differentiable at  $x = 0$ , making  $F(x)$  differentiable everywhere.

Therefore, we have

$$F'(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Clearly,  $F'(x) = f(x)$  for all  $x$ . □

*Solution to (ii).* Define  $f(x)$  as

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Evaluating  $F(x)$ , we get

$$F(x) = \int_{-1}^x f \, dt.$$

We then have two cases to consider. If  $x < 0$ , then from  $-1$  to  $x$ , we are in the region where  $f(x) = 1$

$$F(x) = \int_{-1}^x 1 \, dt = x + 1.$$

If  $x \geq 0$ , then from  $-1$  to  $x$ , we are in the region where  $f(x) = 2$

$$F(x) = \int_{-1}^0 1 \, dt + \int_0^x 2 \, dt = 1 + 2x.$$

Therefore, the piecewise formula for  $F(x)$  is

$$F(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 1 + 2x & \text{if } x \geq 0. \end{cases}$$

Each piece is a polynomial and polynomials are continuous everywhere, so we only need to check continuity at  $x = 0$ . Evaluate the left and right limits

$$\begin{aligned} \lim_{x \rightarrow 0^-} F(x) &= \lim_{x \rightarrow 0^-} (x + 1) = 1 \\ \lim_{x \rightarrow 0^+} F(x) &= \lim_{x \rightarrow 0^+} (1 + 2x) = 1 \\ F(0) &= 1. \end{aligned}$$

Since the left and right limits are equal to the value of the function at  $x = 0$ ,  $F(x)$  is continuous at  $x = 0$ , making  $F(x)$  continuous everywhere.

Differentiating each piece, we get

$$F'(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Evaluating the left and right derivatives at  $x = 0$ ,

$$\begin{aligned} \lim_{x \rightarrow 0^-} F'(x) &= \lim_{x \rightarrow 0^-} 1 = 1 \\ \lim_{x \rightarrow 0^+} F'(x) &= \lim_{x \rightarrow 0^+} 2 = 2. \end{aligned}$$

Since the left and right derivatives are not equal at  $x = 0$ ,  $F(x)$  is not differentiable at  $x = 0$ , making  $F(x)$  differentiable everywhere except at  $x = 0$ .

Therefore, we have

$$F'(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Clearly,  $F'(x) \neq f(x)$  for all  $x \neq 0$ . □

**Exercise 7.5.2.** Decide whether each statement is true or false, providing a short justification for each conclusion.

- (i) If  $g = h'$  for some  $h$  on  $[a, b]$ , then  $g$  is continuous on  $[a, b]$ .
- (ii) If  $g$  is continuous on  $[a, b]$ , then  $g = h'$  for some  $h$  on  $[a, b]$ .
- (iii) If  $H(x) = \int_a^x h$  is differentiable at  $c \in [a, b]$ , then  $h$  is continuous at  $c$ .

*Solution to (i).* False. Consider the function  $h : [-1, 1] \rightarrow \mathbb{R}$  defined as

$$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then,  $g : [-1, 1] \rightarrow \mathbb{R}$  defined as

$$g(x) = \begin{cases} h'(x) = 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0, but  $g = h'$ . □

*Solution to (ii).* True. By the Fundamental Theorem of Calculus, if  $g$  is continuous on  $[a, b]$ , then  $g$  is the derivative of some function  $h$  on  $[a, b]$ . □

*Solution to (iii).* False. Consider the function

$$H(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then,  $H(x) = 0$  and differentiable at 0, but  $h$  is not continuous at 0. □

**Exercise 7.5.4.** Show that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $\int_a^x f = 0$  for all  $x \in [a, b]$ , then  $f(x) = 0$  everywhere on  $[a, b]$ . Provide an example to show that this conclusion does not follow if  $f$  is not continuous.

*Solution.* Since  $f$  is continuous, by Theorem 7.5.1, part (ii), letting  $F(x) = \int_a^x f = 0$  for all  $x \in [a, b]$ , we have  $F'(x) = f(x) = 0$  for all  $x \in [a, b]$ . Therefore,  $f(x) = 0$  everywhere on  $[a, b]$ . If  $f$  is not continuous, then this does not hold. □