Matrix Representation of linear transformations

Let V and W be finite-dimensional vector spaces. Let $T: V \to W$ be a linear transformation.

Review: Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an ordered basis for V. Then for any $\vec{x} \in V$, there exists a unique set of n-tuples $x_1, x_2, \cdots, x_n$ such that $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$

Then $[\vec{x}]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ (or $\mathbb{C}^n$) is called the coordinates of $\vec{x}$ related to the ordered basis B.

Example. 1) Let $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \in \mathbb{R}^3$ be an ordered basis of $\mathbb{R}^3$

Then $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \left[\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2). Let $B = \{x^2, x, 1\} \in P_2(\mathbb{R})$ be an ordered basis

Then $f(x) = 2 - x + 3x^2 = 3 \cdot (x^2) + (-1) \cdot (x) + 2 \cdot (1)$

$\Rightarrow [f(x)]_B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Definition. Let $T: V \to W$ be a linear transformation. Let $\dim V = n$ and $\dim W = m$.

Let $B_V = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \subseteq V$ be an ordered basis for V

$B_W = \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\} \subseteq W$ be an ordered basis for W.

Denote: $\vec{t}_i = [T(\vec{v}_i)]_{B_W} \in \mathbb{R}^m$ (or $\mathbb{C}^m$).

i.e. $\vec{t}_i$ is the coordinates of $T(\vec{v}_i)$ releated to B_W.

Then $[T]_{B_V}^{B_W} = [\vec{t}_1 \ \vec{t}_2 \ \cdots \ \vec{t}_n] \in \mathbb{R}^{m\times n}$ (or $\mathbb{C}^{m\times n}$)

is called the matrix representation of T relative to B_V and B_W.

Remarks: Denote: $[T]_{B_V}^{B_W} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$

Then $\vec{t}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$ which is from: $T\vec{v}_i = a_{1i}\vec{w}_1 + a_{2i}\vec{w}_2 + \cdots + a_{mi}\vec{w}_m$

$= (\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_m) \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$

informal notation. similar to matrix multiplication

for all $i = 1, 2, \cdots, n$.

$\Rightarrow (T\vec{v}_1 \ T\vec{v}_2 \ \cdots \ T\vec{v}_n) = (\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_m) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

$$\boxed{\Leftrightarrow T(\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n) = (\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_m)[T]_{B_V}^{B_W}}$$

Examples: Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$. Recall $L_A: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$. is the "left-multiplication by A" linear transformation.

Let $B_1 = \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\} \subseteq \mathbb{R}^n$ be the standard basis for $\mathbb{R}^n$.

Let $B_2 = \{\vec{E}_1, \vec{E}_2, \cdots, \vec{E}_m\} \subseteq \mathbb{R}^m$ be the standard

$$L_A(e_i) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \longleftarrow \text{the ith column of } A$$

$$= a_{1i}\vec{E}_1 + a_{21}\vec{E}_2 + \cdots + a_{mi}\vec{E}_m$$

$$\Rightarrow [L_A(\vec{e}_i)]_{B_2} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \rightarrow \text{still the i-th column of } A$$

$$\Rightarrow [L_A]_{B_1}^{B_2} = \left(\underset{\substack{\uparrow \\ \text{first column} \\ \text{of } A}}{[L_A(\vec{e}_1)]_{B_2}} \quad \underset{\substack{\uparrow \\ \text{2nd column of } A}}{[L_A(\vec{e}_2)]_{B_2}} \quad \cdots \quad \underset{\substack{\uparrow \\ \text{n-th column of } A.}}{[L_A(\vec{e}_n)]_{B_2}} \right)$$

$$= A$$

Notation: let V be a finite-dimensional vector space. let $T: V \to V$ be a linear transformation on V. Let B be an ordered basis for V. Then $[T]_B^B$ is also denoted by $[T]_B$.

2). Let $M = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. Define $T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$

$$A \longmapsto AM - MA$$

let $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the standard basis

Find $[T]_B$.

Idea: Denote $\vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Need to compute (1) $T\vec{v}_i$, for $i = 1, 2, 3, 4$.

(2). Find the coordinates $[T\vec{v}_i]_B$ for $i = 1, 2, 3, 4$.

(3). Put $[T\vec{v}_i]_B$ in each columns of $[T]_B$.

Solution: $T\vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= 0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [T\vec{v}_1]_B = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, $T\vec{v}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow [T\vec{v}_2]_B = \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$T\vec{v}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow [T\vec{v}_3]_B = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$T\vec{v}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow [T\vec{v}_4]_B = \begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow [T]_B = \begin{pmatrix} 0 & -1 & -2 & 0 \\ 2 & -1 & 0 & -2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

Proposition: Let $T: V \to W$ be a linear transformation. Let B_1 and B_2 be ordered basis for V and W, respectively. Then $\forall\, \vec{x} \in V$: $[T\vec{x}]_{B_2} = [T]_{B_1}^{B_2}[\vec{x}]_{B_1}$

<u>Proof</u>: Denote $B_1 = \{\vec{v}_1, \dots, \vec{v}_n\}$ $\quad B_2 = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$

$$[\vec{x}]_{B_1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Then: $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$T\vec{x} = T(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n)$$

$$= x_1(T\vec{v}_1) + x_2(T\vec{v}_2) + \cdots + x_n(T\vec{v}_n)$$

$$= \left(T\vec{v}_1 \quad T\vec{v}_2 \quad \cdots \quad T\vec{v}_n\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

From the previous remark

$$= \left(\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m\right) \underbrace{[T]_{B_1}^{B_2} \, [\vec{x}]_{B_1}}_{\text{each entry is the coefficient of } \vec{w}_i \text{ when multiplying out}}$$

$$\Rightarrow \quad [T\vec{x}]_{B_2} = [T]_{B_1}^{B_2} [\vec{x}]_{B_1}$$

Definition: let $T: V \to W$ and $U: W \to Z$ be linear transformations.

$\forall \vec{x} \in V$: define the composition $UT: V \to Z$ by

$$(UT)(\vec{x}) = U(T(\vec{x})).$$

Proposition: The above composition map $UT: V \to Z$ is a linear transformation

Proof: $\forall \vec{x}, \vec{y} \in V$ and $c \in \mathbb{R}$ (or $\mathbb{C}$).

$$(UT)(a\vec{x} + \vec{y}) = U(T(a\vec{x} + \vec{y}))$$

$$= U(aT(\vec{x}) + T(\vec{y}))$$

$$= a\,U(T(\vec{x})) + U(T(\vec{y}))$$

$$= a(UT)(\vec{x}) + (UT)(\vec{y}) \quad .$$

Proposition: let $T: V \to W$ and $U: W \to Z$ be linear transformations

let $B_1 = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$, $B_2 = \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$, and $B_3 = \{\vec{z}_1, \vec{z}_2, \cdots, \vec{z}_k\}$ be ordered bases for V, W, and W. Then $[UT]_{B_1}^{B_3} = [U]_{B_2}^{B_3} [T]_{B_1}^{B_2}$.

// end of Jan 24