Differential Geometry: Homework 6

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Exercise 3.3.1. Show that at the origin $(0,0,0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

Solution. Let $h(x,y) = z = axy$. We compute the necessary partial derivatives to evaluate the Gaussian curvature K and the mean curvature H at the origin. Computing the first-order partial derivatives, we have

$$h_x = ay \quad \text{and} \quad h_y = ax.$$

Evaluated at the origin $(0,0)$, we have

$$h_x(0,0) = 0 \quad \text{and} \quad h_y(0,0) = 0.$$

Next, we compute the second-order partial derivatives

$$h_{xx} = 0, \quad h_{yy} = 0, \quad \text{and} \quad h_{xy} = a.$$

Now, we can compute the Gaussian curvature K at the origin, to get

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2} = \frac{0 \cdot 0 - a^2}{(1+0+0)^2} = \frac{-a^2}{1} = -a^2.$$

Lastly, we compute the mean curvature H at the origin, to get

$$\begin{aligned} H &= \frac{\left(1 + h_y^2\right) h_{xx} - 2h_x h_y h_{xy} + \left(1 + h_x^2\right) h_{yy}}{2\left(1 + h_x^2 + h_y^2\right)^{3/2}} \\ &= \frac{(1+0) \cdot 0 - 2 \cdot 0 \cdot 0 \cdot a + (1+0) \cdot 0}{2 \cdot (1+0+0)^{3/2}} \\ &= \frac{0}{2} = 0. \end{aligned}$$

□

Exercise 3.3.3. Determine the asymptotic curves of the catenoid

$$\mathbf{x}(u,v) = (\cosh(v)\cos(u), \cosh(v)\sin(u), v).$$

Solution. To find the asymptotic curves, we must identify the directions in which the second fundamental form vanishes. We begin by computing the first and second derivatives of $\mathbf{x}(u,v)$

$$\begin{aligned} \mathbf{x}_u &= (-\cosh(v)\sin(u), \cosh(v)\cos(u), 0) \\ \mathbf{x}_v &= (\sinh(v)\cos(u), \sinh(v)\sin(u), 1) \\ \mathbf{x}_{uu} &= (-\cosh(v)\cos(u), -\cosh(v)\sin(u), 0) \\ \mathbf{x}_{uv} &= (-\sinh(v)\sin(u), \sinh(v)\cos(u), 0) \\ \mathbf{x}_{vv} &= (\cosh(v)\cos(u), \cosh(v)\sin(u), 0). \end{aligned}$$

Next, we compute the unit normal vector

$$\begin{aligned} \mathbf{x}_u \times \mathbf{x}_v &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -\cosh(v)\sin(u) & \cosh(v)\cos(u) & 0 \\ \sinh(v)\cos(u) & \sinh(v)\sin(u) & 1 \end{vmatrix} \\ &= (\cosh(v)\cos(u), \cosh(v)\sin(u), -\cosh(v)\sinh(v)) \\ \text{and} \quad \|\mathbf{x}_u \times \mathbf{x}_v\| &= \sqrt{\cosh^2(v) + \cosh^2(v)\sinh^2(v)} = \cosh(v)\sqrt{1 + \sinh^2(v)} = \cosh^2(v). \end{aligned}$$

Therefore, the unit normal vector is

$$\mathbf{N} = \left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, -\tanh(v) \right).$$

Now compute the coefficients of the second fundamental form

$$\begin{aligned}
e &= \langle \mathbf{x}_{uu}, \mathbf{N} \rangle = (-\cosh(v)\cos(u), -\cosh(v)\sin(u), 0) \cdot \left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, -\tanh(v) \right) \\
&= -\cos(u)^2 - \sin(u)^2 = -1 \\
f &= \langle \mathbf{x}_{uv}, \mathbf{N} \rangle = (-\sinh(v)\sin(u), \sinh(v)\cos(u), 0) \cdot \left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, -\tanh(v) \right) \\
&= \frac{-\sinh(v)\sin(u)\cos(u) + \sinh(v)\cos(u)\sin(u)}{\cosh(v)} = 0 \\
g &= \langle \mathbf{x}_{vv}, \mathbf{N} \rangle = (\cosh(v)\cos(u), \cosh(v)\sin(u), 0) \cdot \left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, -\tanh(v) \right) \\
&= \cos(u)^2 + \sin(u)^2 = 1.
\end{aligned}$$

Therefore, the second fundamental form is $\mathrm{II} = -\,\mathrm{d}u^2 + \mathrm{d}v^2$. Setting $\mathrm{II} = 0$, we find $-\,\mathrm{d}u^2 + \mathrm{d}v^2 = 0 \Rightarrow \mathrm{d}u = \pm\,\mathrm{d}v$. Integrating, the asymptotic curves are

$$u + v = \text{const.}, \quad u - v = \text{const.}$$

□

Exercise 3.3.5. Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right),$$

and show that

(i) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

(ii) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

(iii) The principal curvatures are

$$\kappa_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad \kappa_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

(iv) The lines of the curvature are the coordinate curves.

(v) The asymptotic curves are $u + v = \text{const.}$, $u - v = \text{const.}$

Solution to (i). We begin by computing the tangent vectors:

$$\mathbf{x}_u = \left(1 - u^2 + v^2, 2uv, 2u\right) \quad \text{and} \quad \mathbf{x}_v = \left(2uv, 1 - v^2 + u^2, -2v\right).$$

Then, the coefficients of the first fundamental form are given by:

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = (1 - u^2 + v^2)(2uv) + (2uv)(1 - v^2 + u^2) - 4uv = 0
\end{aligned}$$

$$G = \langle \mathbf{x}_v, x_v \rangle = (2uv)^2 + (1 - v^2 + u^2)^2 + 4v^2.$$

Expanding both E and G, we find:

$$\begin{aligned} E &= (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \\ &= 1 - 2u^2 + 2v^2 + u^4 - 2u^2v^2 + v^4 + 4u^2v^2 + 4u^2 \\ &= 1 + u^4 + 2u^2 + v^4 + 2v^2 + 2u^2v^2 \\ &= (1 + u^2 + v^2)^2. \end{aligned}$$

Similarly,

$$\begin{aligned} G &= (1 - v^2 + u^2)^2 + 4u^2v^2 + 4v^2 \\ &= 1 - 2v^2 + 2u^2 + u^4 - 2u^2v^2 + v^4 + 4u^2v^2 + 4v^2 \\ &= 1 + u^4 + 2u^2 + v^4 + 2v^2 + 2u^2v^2 \\ &= (1 + u^2 + v^2)^2. \end{aligned}$$

Hence, we conclude:

$$E = (1 + u^2 + v^2)^2 = G \quad \text{and} \quad F = 0.$$ □

Solution to (ii). To compute the coefficients of the second fundamental form, we first compute the unit normal vector. The cross product is given by

$$\mathbf{x}_u \times \mathbf{x}_v = \left(-4u, -4v, (1 + u^2 + v^2)^2\right) \quad \text{and} \quad \|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{16u^2 + 16v^2 + (1 + u^2 + v^2)^4}.$$

Next, we compute the second derivatives

$$\begin{aligned} \mathbf{x}_{uu} &= (-2u, 2v, 2)\,, \\ \mathbf{x}_{uv} &= (2v, 2u, 0)\,, \\ \mathbf{x}_{vv} &= (2u, -2v, -2)\,. \end{aligned}$$

Then, dotting with $\mathbf{x}_u \times \mathbf{x}_v$, we obtain

$$\begin{aligned} e &= \frac{\langle \mathbf{x}_{uu}, \mathbf{x}_u \times \mathbf{x}_v \rangle}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{8u^2 + 8v^2 + 2(1 + u^2 + v^2)^2}{\|\mathbf{x}_u \times \mathbf{x}_v\|} &= 2 \\ f &= \frac{\langle \mathbf{x}_{uv}, \mathbf{x}_u \times \mathbf{x}_v \rangle}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{-8uv + 0}{\|\mathbf{x}_u \times \mathbf{x}_v\|} &= 0 \\ g &= \frac{\langle \mathbf{x}_{vv}, \mathbf{x}_u \times \mathbf{x}_v \rangle}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{-8u^2 - 8v^2 - 2(1 + u^2 + v^2)^2}{\|\mathbf{x}_u \times \mathbf{x}_v\|} &= -2. \end{aligned}$$

So the coefficients are

$$e = 2, \quad f = 0, \quad \text{and} \quad g = -2.$$ □

Solution to (iii). Recall that the principal curvatures κ_1, κ_2 are the eigenvalues of the shape operator and are given by

$$\kappa_{1,2} = \frac{eG - 2fF + gE \pm \sqrt{(eG - gE)^2 + 4(fE - eF)^2}}{2(EG - F^2)}.$$

Since $F = f = 0$, the formula simplifies

$$\kappa_{1,2} = \frac{eG + gE \pm \sqrt{(eG - gE)^2}}{2EG}.$$

Using $E = G = (1 + u^2 + v^2)^2$, and $e = 2$, $g = -2$, we compute

$$\begin{aligned} \kappa_1 &= \frac{2E + (-2)E + \sqrt{(2E - (-2)E)^2}}{2E^2} = \frac{0 + \sqrt{(4E)^2}}{2E^2} = \frac{4E}{2E^2} = \frac{2}{E} \\ \kappa_2 &= \frac{0 - \sqrt{(4E)^2}}{2E^2} = -\frac{2}{E} \end{aligned}$$

Hence,

$$\kappa_1 = \frac{2}{(1 + u^2 + v^2)^2} \quad \text{and} \quad \kappa_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

□

Solution to (iv). The lines of curvature are the integral curves of the principal directions. Since $F = f = 0$, the shape operator diagonalizes in the coordinate directions, and the coordinate curves are orthogonal and aligned with the principal directions. Therefore, the coordinate curves $u =$ const., $v =$ const. are the lines of curvature. □

Solution to (v). Asymptotic curves are the curves along which the normal curvature vanishes. On a surface where the principal curvatures κ_1, κ_2 have opposite signs, the asymptotic directions correspond to directions in which the second fundamental form vanishes. In our case, the second fundamental form is

$$\mathrm{II} = e \, \mathrm{d}u^2 + 2f \, \mathrm{d}u \, \mathrm{d}v + g \, \mathrm{d}v^2 = 2 \, \mathrm{d}u^2 - 2 \, \mathrm{d}v^2.$$

Set $\mathrm{II} = 0$, we find

$$2 \, \mathrm{d}u^2 - 2 \, \mathrm{d}v^2 = 0 \Rightarrow \mathrm{d}u^2 = \mathrm{d}v^2 \Rightarrow \mathrm{d}u = \pm \, \mathrm{d}v.$$

Integrating, we obtain

$$u + v = \text{const.} \quad \text{and} \quad u - v = \text{const.}$$

Hence, the asymptotic curves are the families $u + v = c$, $u - v = c$. □

Exercise 3.3.7. $(\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v))$, $\varphi \neq 0$ is given as a surface of revolution with constant Gaussian curvature K. To determine the functions φ and ψ, choose the parameter v in such a way that $(\varphi')^2 + (\psi')^2 = 1$ (geometrically, this means that v is the arc length of the generating curve $(\varphi(v), \psi(v))$). Show that

(i) φ satisfies $\varphi'' + K\varphi = 0$ and ψ is given by $\psi = \int \sqrt{1 - (\varphi')^2} \, \mathrm{d}v$; thus, $0 < u < 2\pi$, and the domain of v is such that the last integral makes sense.

(ii) All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane xOy are given by

$$\varphi(v) = C\cos(v), \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2(v)} \, \mathrm{d}v,$$

where C is a constant ($C = \varphi(0)$). Determine the domain of v and draw a rough sketch of the profile of the surface in the xz-plane for the cases $C = 1$, $C > 1$, $C < 1$. Observe that $C = 1$ gives a sphere.

(iii) All surfaces of revolution with constant curvature $K = -1$ may be given by one of the following types:

1. $\varphi(v) = C\cosh(v)$,
   $\psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2(v)} \, \mathrm{d}v$.
2. $\varphi(v) = C\sinh(v)$,
   $\psi(v) = \int_0^v \sqrt{1 - C^2 \cosh^2(v)} \, \mathrm{d}v$.
3. $\varphi(v) = e^v$,
   $\psi(v) = \int_0^v \sqrt{1 - e^{2v}} \, \mathrm{d}v$.

Determine the domain of v and draw a rough sketch of the profile of the surface in the xz-plane.

(iv) The surface of type 3 in part (iii) is the pseudosphere of Exercise 6.

(v) The only surfaces of revolution with $K \equiv 0$ are the right circular cylinder, the right circular cone, and the plane.

Solution to (i). Computing the partial derivatives of $\mathbf{x}(u,v) = (\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v))$, we have

$$\begin{gathered}
\mathbf{x}_u = (-\varphi(v)\sin(u), \varphi(v)\cos(u), 0), \quad \mathbf{x}_v = (\varphi'(v)\cos(u), \varphi'(v)\sin(u), \psi'(v)) \\
\mathbf{x}_{uu} = (-\varphi(v)\cos(u), -\varphi(v)\sin(u), 0), \quad \mathbf{x}_{uu} = (\varphi''(v)\cos(u), \varphi''(v)\sin(u), \psi''(v)) \\
\mathbf{x}_{uv} = (-\varphi'(v)\sin(u), \varphi'(v)\cos(u), 0)
\end{gathered}$$

Computing the normal vector $\mathbf{N}$ to the surface, we have

$$\tilde{\mathbf{N}} = \mathbf{x}_u \wedge \mathbf{x}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -\varphi(v)\sin(u) & \varphi(v)\cos(u) & 0 \\ \varphi'(v)\cos(u) & \varphi'(v)\sin(u) & \psi'(v) \end{vmatrix} = (\varphi(v)\psi'(v)\cos(u), \varphi(v)\psi'(v)\sin(u), -\varphi(v)\varphi'(v)).$$

Normalizing it, we have

$$\begin{aligned}
\mathbf{N} = \frac{\tilde{\mathbf{N}}}{\|\tilde{\mathbf{N}}\|} &= \frac{(\varphi(v)\psi'(v)\cos(u), \varphi(v)\psi'(v)\sin(u), -\varphi(v)\varphi'(v))}{\varphi(v)\sqrt{(\psi'(v))^2 + (\varphi'(v))^2}} \\
&= (\psi'(v)\cos(u), \psi'(v)\sin(u), -\varphi'(v)).
\end{aligned}$$

Computing the coefficients for the first and second fundamental form, we have

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \varphi^2(v) \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (\varphi'(v))^2 + (\psi'(v))^2 = 1 \\
e &= \langle \mathbf{x}_{uu}, \mathbf{N} \rangle = -\varphi(v)\psi'(v) \\
f &= \langle \mathbf{x}_{uv}, \mathbf{N} \rangle = 0 \\
g &= \langle \mathbf{x}_{vv}, \mathbf{N} \rangle = \varphi''(v)\psi'(v) - \varphi'(v)\psi''(v).
\end{aligned}$$

Plugging in the values of the coefficients into the formulas for the Gaussian curvature, we have

$$\begin{aligned}
K = \frac{eg - f^2}{EG - F^2} &= \frac{(-\varphi\psi')(\varphi''\psi' - \varphi'\psi'')}{\varphi^2} \\
&= \frac{(-\varphi\psi')\left(\varphi''\psi' + \varphi' \cdot \left(\frac{\varphi'\varphi''}{\psi'}\right)\right)}{\varphi^2} \\
&= \frac{-\varphi\varphi''\psi'\left(\psi' + \frac{(\varphi')^2}{\psi'}\right)}{\varphi^2} \\
&= \frac{\varphi''}{\varphi} \cdot \left((\varphi')^2 + (\psi')^2\right) \\
\Rightarrow \quad 0 &= \varphi'' + K\varphi.
\end{aligned}$$

Solving for ψ, we have

$$(\varphi')^2 + (\psi')^2 = 1 \Rightarrow \psi' = \sqrt{1 - (\varphi')^2} \Rightarrow \psi = \int \sqrt{1 - (\varphi')^2}\, \mathrm{d}v.$$

The domain of v is any open interval $I \subset \mathbb{R}$ on which φ is differentiable and the integrand is real, which happens when $(\varphi')^2 \leq 1$. $\square$

Solution to (ii). Taking the first and second derivatives of $\varphi(v) = C\cos(v)$, we have

$$\varphi'(v) = -C\sin(v) \quad \text{and} \quad \varphi''(v) = -C\cos(v).$$

Plugging these into the equation $\varphi'' + K\varphi = 0$, we have

$$-C\cos(v) + KC\cos(v) = 0 \Rightarrow K = 1.$$

Plugging $\varphi = C\cos(v)$ into the equation for ψ, we have

$$\psi(v) = \int_0^v \sqrt{1 - C^2\sin^2(v)}\,\mathrm{d}v.$$

For the integrand to be real, we require

$$C^2\sin^2(v) \leq 1 \Leftrightarrow |\sin(v)| \leq \frac{1}{C}.$$

So the domain depends on the value of C, giving us three cases: $C < 1$, $C = 1$, and $C > 1$.

If $C < 1$, then $1/C > 1$, and since $|\sin(v)| \leq 1$ for all $v \in \mathbb{R}$, making the domain $\mathbb{R}$.

If $C = 1$, then $|\sin(v)| \leq 1$ still holds for all v, so again the domain is $\mathbb{R}$. Notice that this case gives us a sphere of radius 1 because

$$\varphi(v) = \cos(v) \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - \sin^2(v)}\,\mathrm{d}v = \int_0^v \cos(v)\,\mathrm{d}v = \sin(v).$$

If $C > 1$, then $1/C < 1$, so $|\sin(v)| \leq 1/C$ only holds for v in the interval

$$v \in \left(-\arcsin\left(\frac{1}{C}\right), \arcsin\left(\frac{1}{C}\right)\right).$$

The graphs for each case are given in figure 1. □

Solution to (iii). For all three cases, we have

$$\begin{aligned}
\varphi_1(v) &= C\cosh(v), & \varphi_1'(v) &= C\sinh(v), & \varphi_1''(v) &= C\cosh(v) \\
\varphi_2(v) &= C\sinh(v), & \varphi_2'(v) &= C\cosh(v), & \varphi_2''(v) &= C\sinh(v) \\
\varphi_3(v) &= e^v, & \varphi_3'(v) &= e^v, & \varphi_3''(v) &= e^v.
\end{aligned}$$

Clearly, φ_1, φ_2, and φ_3 are all solutions to the differential equation $\varphi'' + K\varphi = 0$ when $K = -1$. Plugging these into the equation for ψ, we have

$$\begin{aligned}
\psi_1(v) &= \int_0^v \sqrt{1 - C^2\sinh^2(v)}\,\mathrm{d}v \\
\psi_2(v) &= \int_0^v \sqrt{1 - C^2\cosh^2(v)}\,\mathrm{d}v \\
\psi_3(v) &= \int_0^v \sqrt{1 - e^{2v}}\,\mathrm{d}v.
\end{aligned}$$

Now, we deal with the domains of v for each case.

For the first case, just like before, we have

$$1 - C^2\sinh^2(v) \geq 0 \Leftrightarrow C^2\sinh^2(v) \leq 1 \Leftrightarrow |\sinh(v)| \leq \frac{1}{C}.$$

Since $|\sinh(v)|$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$, the inequality holds when

$$v \in \left(-\sinh^{-1}\left(\frac{1}{C}\right), \sinh^{-1}\left(\frac{1}{C}\right)\right).$$

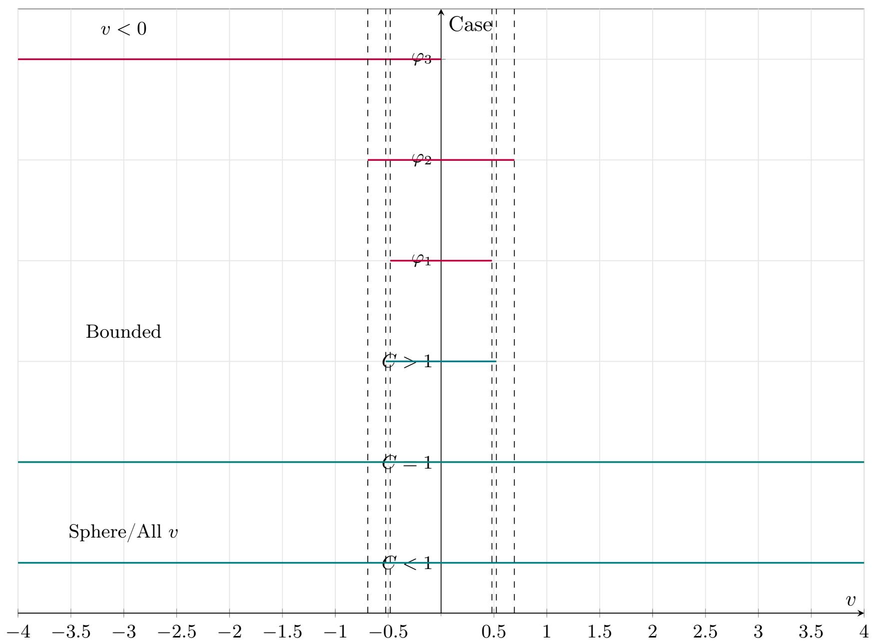

Figure 1: Sketch of the profile of the surface in the xz-plane for the cases $C = 1$, $C > 1$, and $C < 1$.

For the second case, we have

$$1 - C^2\cosh^2(v) \geq 0 \Leftrightarrow C^2\cosh^2(v) \leq 1.$$

But $\cosh(v) \geq 1$ for all v, so $C^2 \leq 1$ is required. If $C^2 < 1$, then

$$v \in \left(-\cosh^{-1}\left(\frac{1}{C}\right), \cosh^{-1}\left(\frac{1}{C}\right)\right).$$

If $C^2 = 1$, then $\cosh^2(v) \leq 1$ only when $v = 0$, so the domain is $\{0\}$. If $C^2 > 1$, then the integrand is imaginary for all v, so there is no valid domain.

For the third case, we have

$$1 - e^{2v} \geq 0 \Leftrightarrow e^{2v} \leq 1 \Leftrightarrow v \leq 0.$$

So the domain is $(-\infty, 0]$. $\square$

Solution to (iv). From Exercise 6, the pseudosphere is the surface of revolution generated by rotating the tractrix about the z-axis. A classical parametrization of the tractrix is

$$\varphi(v) = \operatorname{sech}(v) \quad \text{and} \quad \psi(v) = v - \tanh(v),$$

where φ is the radial function and ψ is the height function. However, we now compare this to the type (3) surface from part (iii), where

$$\varphi(v) = e^v \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - e^{2v}}\, dv.$$

For this integral to be real-valued, we must restrict to the domain where $1 - e^{2v} > 0$, i.e., $v < \log(1) = 0$, so $v \in (-\infty, 0)$. Observe that this parametrization arises from solving the differential equation $\varphi''(v) + K\varphi(v) = 0$, with $K = -1$, satisfied by $\varphi(v) = e^v$. The associated profile curve generates a surface of revolution with constant Gaussian curvature $K = -1$, and the form of φ and ψ matches the construction of the pseudosphere.

In Exercise 6, we also saw that the pseudosphere is a surface of revolution with constant negative Gaussian curvature $K = -1$, and that it can be parametrized as

$$\mathbf{x}(u, v) = (\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v)),$$

where φ satisfies $\varphi'' + K\varphi = 0$. Therefore, the surface of type (3) in part (iii) is indeed a particular parametrization of the pseudosphere. □

Solution to (v). As found in part (i), the Gaussian curvature K is given by the equation

$$K = -\frac{\varphi''(v)}{\varphi(v)}.$$

If $K \equiv 0$, then this gives

$$-\frac{\varphi''(v)}{\varphi(v)} = 0 \Leftrightarrow \varphi''(v) = 0.$$

Solving this second-order linear ODE, we find $\varphi(v) = Av + B$, for some constants A and B.

The form of $\psi(v)$ is then determined by the condition that the parametrization is regular. We recall that the arc length condition for a surface of revolution requires

$$\psi'(v) = \sqrt{1 - (\varphi'(v))^2} = \sqrt{1 - A^2}.$$

So $\psi(v) = \sqrt{1 - A^2}v + C$, where $C \in \mathbb{R}$. This makes sense only when $|A| \leq 1$; otherwise the metric would be degenerate or complex. This gives us three cases: $A = 0$, $A \neq 0$, and $A = 0$ and $B = 0$.

If $A = 0$, then $\varphi(v) = B$, a constant. The profile curve is a horizontal line, and the surface of revolution is a right circular cylinder.

If $A \neq 0$, then $\varphi(v) = Av + B$ is linear. The profile curve is a straight line not parallel to the axis of revolution. Rotating it generates a right circular cone, as long as $B \neq 0$.

If $A = 0$ and $B = 0$, then $\varphi(v) \equiv 0$, which is not allowed since it degenerates the surface. However, if we parametrize the surface directly as a horizontal plane (e.g. $\mathbf{x}(u, v) = (u, v, 0)$), then $K = 0$, and it is a surface of revolution in the trivial sense (with arbitrary axis).

Thus, the only surfaces of revolution with $K \equiv 0$ are the right circular cylinder, the right circular cone, and the plane. □

Exercise 3.3.22. Let $h : S \to \mathbb{R}$ be a differentiable function on a surface S, and let $p \in S$ be a critical point of h (i.e., $\mathrm{d}h_p = 0$). Let $w \in T_p(S)$ and let

$$\alpha : (-\varepsilon, \varepsilon) \to S,$$

be a parametrized curve with $\alpha(0) = p$, $\alpha'(0) = \mathbf{w}$. Set

$$H_p h(\mathbf{w}) = \left.\frac{\mathrm{d}^2(h \circ \alpha)}{\mathrm{d}t^2}\right|_{t=0}.$$

(i) Let $\mathbf{x} : U \to S$ be a parametrization of S at p, and show that (the fact that p is a critical point of h is essential here)

$$H_p h(u'\mathbf{x}_u + v'\mathbf{x}_v) = h_{uu}(p)(u')^2 + 2h_{uv}(p)u'v' + h_{vv}(p)(v')^2.$$

Conclude that $H_p h : T_p(S) \to \mathbb{R}$ is a well-defined (i.e., it does not depend on the choice of $\mathbf{x}$) quadratic form on $T_p(S)$. $H_p h$ is called the *Hessian* of h at p.

(ii) Let $h : S \to \mathbb{R}$ be the height function of S relative to $T_p(S)$; that is, $h(q) = \langle q - p, \mathbf{N}(p)\rangle$, $q \in S$. Verify that p is a critical point of h and thus that the Hessian $H_p h$ is well defined. Show that if $\mathbf{w} \in T_p(S)$, $|\mathbf{w}| = 1$, then

$$H_p h(\mathbf{w}) = \text{normal curvature at } p \text{ in the direction of } \mathbf{w}.$$

Conclude that the Hessian at p of the height function relative to $T_p(S)$ is the second fundamental form of S at p.

Solution to (i). Let $\tilde{h} = h \circ \alpha$. Then,

$$H_p h(\mathbf{w}) = \left.\frac{\mathrm{d}^2 \tilde{h}}{\mathrm{d}t^2}\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\frac{\partial \tilde{h}}{\partial u} u'(t) + \frac{\partial \tilde{h}}{\partial v} v'(t)\right] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\tilde{h}_u u'(t) + \tilde{h}_v v'(t)\right].$$

By the product rule and chain rule, we compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{h}_u u') = \tilde{h}_{uu} u' u' + \tilde{h}_u u'' = \tilde{h}_{uu}(u')^2 + \tilde{h}_u u'' \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{h}_v v') = \tilde{h}_{vv}(v')^2 + \tilde{h}_v v'' \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{h}_u v') = \tilde{h}_{uv} u' v' \quad \text{(and similarly } \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{h}_v u') = \tilde{h}_{uv} v' u'\text{)}.
\end{aligned}$$

Putting everything together,

$$H_p h(\mathbf{w}) = \left.\frac{\mathrm{d}^2 \tilde{h}}{\mathrm{d}t^2}\right|_{t=0} = \tilde{h}_{uu}(p)(u')^2 + 2\tilde{h}_{uv}(p) u' v' + \tilde{h}_{vv}(p)(v')^2 + \tilde{h}_u(p) u''(0) + \tilde{h}_v(p) v''(0).$$

Since p is a critical point of h, we have $\mathrm{d}h_p = 0$, which implies $\tilde{h}_u(p) = h_u(p) = 0$ and $\tilde{h}_v(p) = h_v(p) = 0$. Thus, the last two terms vanish, and we are left with

$$H_p h(\mathbf{w}) = \tilde{h}_{uu}(p)(u')^2 + 2\tilde{h}_{uv}(p) u' v' + \tilde{h}_{vv}(p)(v')^2.$$

This shows that $H_p h : T_p(S) \to \mathbb{R}$ is a homogeneous degree-2 polynomial in the components of $\mathbf{w}$, and hence a quadratic form. Since the expression depends only on second-order partials and the velocity vector $\mathbf{w} = u' \mathbf{x}_u + v' \mathbf{x}_v$, it is independent of the chosen parametrization $\mathbf{x}$, and therefore $H_p h$ is a well-defined quadratic form on $T_p(S)$. □

Solution to (ii). Let $h(q) = \langle q - p, \mathbf{N}(p)\rangle$ be the height function of S relative to the tangent plane $T_p(S)$, where $\mathbf{N}(p)$ is the unit normal vector to S at p. Again, let $\tilde{h}(t) = (h \circ \alpha)(t)$. To verify that p is a critical point of h, let $\alpha : (-\varepsilon, \varepsilon) \to S$ be any smooth curve on S with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{w} \in T_p(S)$. Then,

$$\begin{aligned}
&\tilde{h}(t) = \langle \alpha(t) - p, \mathbf{N}(p)\rangle \\
\Rightarrow &\left.\frac{\mathrm{d}\tilde{h}}{\mathrm{d}t}\right|_{t=0} = \langle \alpha'(0), \mathbf{N}(p)\rangle = \langle \mathbf{w}, \mathbf{N}(p)\rangle = 0,
\end{aligned}$$

since $\mathbf{w} \in T_p(S)$ and $\mathbf{N}(p)$ is normal to $T_p(S)$. Thus, $\mathrm{d}h_p(\mathbf{w}) = 0$ for all $\mathbf{w} \in T_p(S)$, so p is a critical point of h, and hence $H_p h$ is well-defined.

Now, differentiating again

$$H_p h(\mathbf{w}) = \left.\frac{\mathrm{d}^2 \tilde{h}}{\mathrm{d}t^2}\right|_{t=0} = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle \alpha(t) - p, \mathbf{N}(p)\rangle\right|_{t=0} = \langle \alpha''(0), \mathbf{N}(p)\rangle.$$

Since α lies on the surface S, and $\alpha''(0)$ is the acceleration vector of the curve at p, its normal component measures the curvature of S in the direction of $\mathbf{w}$. Therefore,

$$H_p h(\mathbf{w}) = \langle \alpha''(0), \mathbf{N}(p)\rangle = \text{normal curvature of } S \text{ at } p \text{ in the direction of} \mathbf{w}.$$

This is precisely the definition of the second fundamental form $\mathrm{II}_p(\mathbf{w}, \mathbf{w})$, and hence,

$$H_p h(\mathbf{w}) = \mathrm{II}_p(\mathbf{w}, \mathbf{w}).$$

Therefore, the Hessian at p of the height function relative to $T_p(S)$ is equal to the second fundamental form at p. □