Math 441/541 - Linear Algebra - Midterm Exam

CRN: 26334/26337,   Feburary 7, 2025

Last Name: ________________________,   **First Name:** ________________________

1. There are 5 problems. The total credit is 50 points with opportunities of 10 bonus points. The minimum of 50 and your score will be used as your exam score (out of 50).
2. No collaboration is allowed for this exam. The work on this exam is to be yours and yours alone. Failure to adhere to this policy will result in a zero on this exam. Any instances of suspected cheating will also be reported to the Dean of Students' Office.
3. No calculators, books or other material are permitted.
4. In order to receive full credit you need to show your work, use complete sentences to explain your answers, and present your answers as clearly as possible.
5. Failure to follow directions specific to a problem will result in loss of points.
6. Put away and **silence** your cell phones, tablets, laptops, etc. There is no reason to look at your phone during this test, as the current time is displayed on the clock in the classroom.
7. Once this exam begins, you will have 50 minutes to complete your solutions.

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1. [8 pts] Answer the following questions.

1). Give the definition of a **subspace** of a vector space.

Let V be a vector space. Let $W \subseteq V$ be an non-empty set. Then W is called a subspace of V if W itself is a vector space

2). State the **replacement theorem**.

Let V be a vector space. Let $G \subseteq V$ be a subset of V such that $\operatorname{Span} G = V$ and $|G| = n$. Let L be an linearly independent set in V. with $|L| = m$. Then

1) $m \leq n$

2) there exists a subset $H \subseteq G$ such that $|H| = n-m$ and $\operatorname{Span}(L \cup H) = V$.

2. [8 pts] Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For any $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 b_1,\ a_2 + b_2), \quad \text{and} \quad c(a_1, a_2) = (ca_1,\ ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

No. Let (x_1, x_2) be the zero vector

$$(a_1, a_2) + (x_1, x_2) = (a_1 x_1,\ a_2 + x_2) = (a_1, a_2)$$

$$\Rightarrow \begin{matrix} a_1 x_1 = a_1 \\ a_2 + x_2 = a_2 \end{matrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix}$$

$\Rightarrow$ $(1, 0)$ is the zero vector.

Then $\forall (0, a_2) \in V$ does not have its inverse

b/c: $(0, a_2) + (b_1, b_2) = (0, a_2 + b_2) \neq (1, 0)$

for any $(b_1, b_2) \in V$.

Or:

$$c(a_1, a_2) + c(b_1, b_2) = (ca_1, ca_2) + (cb_1, cb_2)$$

$$= (c^2 a_1 a_2,\ c(a_2 + b_2))$$

$$c\left[(a_1, a_2) + (b_1, b_2)\right] = c(a_1 a_2,\ b_1 + b_2)$$

$$= (ca_1 a_2,\ c(b_1 + b_2))$$

For any $a_1 a_2 \neq 0$ and $c^2 \neq c$

$$(c^2 a_1 a_2,\ c(a_2 + b_2)) \neq (ca_1 a_2,\ c(b_1 + b_2)).$$

Or

$$(c+d)(a_1, a_2) = ((c+d)a_1,\ (c+d)a_2)$$

$$c(a_1, a_2) + d(a_1, a_2) = (ca_1, ca_2) + (da_1, da_2)$$

$$= (cda_1,\ (c+d)a_2)$$

In general $c+d \neq cd$ for $c, d \in \mathbb{R}$.

3. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & a \\ d & c+d \end{bmatrix}.$$

1). [6 pts] Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^{2\times 2}$. Find the matrix representation $[T]_{\mathcal{B}}$.

2). [4 pts] Let f be a linear functional on $\mathbb{R}^{2\times 2}$ defined by $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a - b + c - d$. Let $T^t : (\mathbb{R}^{2\times 2})^* \to (\mathbb{R}^{2\times 2})^*$ be the transpose (or dual map) of the linear transformation T defined above. Find the formula for the linear transformation $T^t(f) : \mathbb{R}^{2\times 2} \to \mathbb{R}$.

3). [2 pts] Let $\mathcal{B}^*$ be an order basis for V^*, which is the dual basis of $\mathcal{B}$. Find the matrix representation $[T^t]_{\mathcal{B}^*}$.

Coordinates

1)

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow [T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2)

$$T^t(f)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = f \circ T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = f\left(\begin{bmatrix} a+b & a \\ d & c+d \end{bmatrix}\right) = (a+b) - a + d - (c+d)$$

$$= b - c$$

3)

$$[T^t]_{\mathcal{B}^*} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

4. Let $V = \mathbb{P}^1(\mathbb{R})$, i.e. the vector space of all polynomials with degree less or equal to one. Let the linear functionals $f_1, f_2 \in V^*$ be defined as follows,

$$f_1(p(x)) = \int_0^1 p(x)\,dx, \quad f_2(p(x)) = \int_0^2 p(x)\,dx, \text{ for any } p(x) \in V.$$

1). [10 pts] Prove that $\{f_1, f_2\}$ is a basis for V^*.

Note: You may directly use the result about the dimensions of V and V^*.

2). [4 pts] (*Bonus Problem*) Find the basis of V that is the dual basis to $\{f_1, f_2\}$.

1). Suppose $a_1 f_1 + a_2 f_2 = 0$ Then $a_1 f_1(p(x)) + a_2 f_2(p(x)) = 0$ for all $p(x) \in V$.

then take $p(x) = 1$: $f_1(p(x)) = \int_0^1 1\, dx = 1$

$f_2(p(x)) = \int_0^2 1\, dx = 2$

$\Rightarrow a_1 f_1(p(x)) + a_2 f_2(p(x)) = a_1 + 2a_2 = 0$

Take $p(x) = 2x$: $f_1(p(x)) = \int_0^1 2x\, dx = x^2 \Big|_0^1 = 1$

$f_2(p(x)) = \int_0^2 2x\, dx = x^2 \Big|_0^2 = 4$

You can also take $p(x) = mx + b$ for any $m, b \in \mathbb{R}$ (see next page)

$\Rightarrow a_1 f_1(p(x)) + a_2 f_2(p(x)) = a_1 + 4a_2 = 0$

As $\begin{cases} a_1 + 2a_2 = 0 \\ a_1 + 4a_2 = 0 \end{cases}$ has a unique solution $a_1 = a_2 = 0$

the set $\{f_1, f_2\}$ is linearly indepedent

As $\dim V^* = \dim V = 2$

$\Rightarrow \{f_1, f_2\}$ is a basis of V^*.

2). Let $p_1(x) = ax + b$ and $p_2(x) = cx + d$ be dual space of $\{f_1, f_2\}$

Then $f_1(p_1) = \int_0^1 (ax+b)dx = (\frac{1}{2}ax^2+bx)\Big|_0^1 = \frac{1}{2}a+b = 1$

$f_1(p_2) = \cdots = \frac{1}{2}c+d = 0$

$f_2(p_1) = \int_0^2 (ax+b)dx = (\frac{1}{2}ax^2+bx)\Big|_0^2 = 2a+2b = 1$

$f_2(p_2) = \cdots = 2c+2d = 0$

Then $\begin{cases} \frac{1}{2}a+b=1 \\ 2a+2b=0 \end{cases} \Leftrightarrow \begin{cases} 2a+4b=4 \\ 2a+2b=0 \end{cases} \Rightarrow \begin{cases} a=-2 \\ b=2 \end{cases}$

$\begin{cases} \frac{1}{2}c+d=0 \\ 2c+2d=1 \end{cases} \Leftrightarrow \begin{cases} 2c+4d=0 \\ 2c+2d=1 \end{cases} \Rightarrow \begin{cases} c=1 \\ d=-\frac{1}{2} \end{cases}$

$\Rightarrow \{-2x+2,\ x-\frac{1}{2}\}$ is a dual basés to $\{f_1, f_2\}$.

Part 1). take $p(x) = mx+b$ $\forall m, b \in \mathbb{R}$

$f_1(p(x)) = \int_0^1 (mx+b)dx = (\frac{1}{2}mx^2+bx)\Big|_0^1 = \frac{1}{2}m+b$

$f_2(p(x)) = \int_0^2 (mx+b)dx = (\frac{1}{2}mx^2+bx)\Big|_0^2 = 2m+2b$

$(a_1f_1+a_2f_2)(p(x)) = a_1(\frac{1}{2}m+b)+a_2(2m+2b) = 0$

$\Leftrightarrow (\frac{1}{2}a_1+2a_2)m+(a_1+2a_2)b = 0$ (*)

(*) is true for all $m, b \in \mathbb{R}$

$\Rightarrow \begin{cases} \frac{1}{2}a_1+2a_2=0 \\ a_1+2a_2=0 \end{cases} \Leftrightarrow \begin{cases} a_1=0 \\ a_2=0 \end{cases}$

5. Let V be the vector space of all polynomials over $\mathbb{R}$. Let $T : V \to V$ be a function defined by

$$T(p(x)) = xp(x) \quad \text{for any} \quad p(x) \in V.$$

1). [4 pts] Prove that T is a linear transformation from V to V.

2). [8 pts] Let D be the differentiation operator on V, i.e. $D(p(x)) = p'(x)$ for any $p(x) \in V$. Prove that $DT - TD = I$, where $I : V \to V$ is the identity map.

3). [6 pts] (*Bonus Problem*) Prove that $DT^n - T^nD = nT^{n-1}$ for any positive integer n. (Note that T^0 is the identity map).

Note: $n = 1$ case is proved in Part 2).

1) $\forall\ f(x), p(x) \in V$: and $\forall\ c \in \mathbb{R}$.

$$T(cf(x) + p(x)) = x(cf(x) + p(x))$$
$$= cxf(x) + xp(x)$$
$$= cT(f(x)) + T(p(x)).$$

2). $\forall\ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

$$D(p(x)) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

$$T \cdot D(p(x)) = x\,(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1})$$
$$= a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots + n a_n x^n$$
$$= \sum_{i=1}^{n} i\, a_i x^i$$

$$T(p(x)) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1}$$

$$D(T(p(x))) = a_0 + 2a_1 x + 3a_2 x^2 + \cdots + (n+1) a_n x^{n+1}$$

$$= \sum_{i=0}^{n} (i+1)\, a_i x^i$$

$$\Rightarrow (DT - TD)(p(x)) = \sum_{i=0}^{n} (i+1) a_i x^i - \sum_{i=1}^{n} i a_i x^i$$

Extra page for computation

$$= a_0 + \sum_{i=1}^{n} \left[(i+1) a_i x^i - i a_i x^i \right]$$

$$= a_0 + \sum_{i=1}^{n} a_i x^i$$

$$= p(x)$$

$$\Rightarrow DT - TD = I.$$

Shorter Proof: $\forall p(x) \in V$: $(DT - TD)(p(x))$

$$= D(T(p(x))) - TD(p(x))$$

$$= D(x p(x)) - T(p'(x))$$

↓ product rule for derivatives

$$= p(x) + x p'(x) - x p'(x)$$

$$= p(x)$$

$$\Rightarrow DT - TD = I.$$

3) Induction on n.

Base case: $n=1$, proved in (2).

Inductive steps: Assume that $DT^{k} - T^{k}D = kT^{k-1}$.

Then:
$$DT^{k+1} - T^{k+1}D = DT^{k+1} - T^{k}DT + T^{k}DT - T^{k+1}D$$
$$= (DT^{k} - T^{k}D)T + T^{k}(DT - TD)$$
$$= kT^{k-1}\cdot T + T^{k}\cdot I$$
$$= kT^{k} + T^{k}$$
$$= (k+1)T^{k}.$$

$\blacksquare$

(More explicit proof): Note that
$$T^{n}(p(x)) = T^{n-1}(T(p(x))) = T^{n-1}(x\,p(x))$$
$$= T^{n-2}(T(x\,p(x))) = T^{n-2}(x^{2}p(x))$$
$$= \cdots = x^{n}p(x), \quad \forall p(x) \in V.$$

$$(DT^{n} - T^{n}D)(p(x)) = D[T^{n}(p(x))] - T^{n}[D(p(x))]$$
$$= D(x^{n}p(x)) - T^{n}(p'(x))$$
$$= nx^{n-1}p(x) + x^{n}p'(x) - x^{n}p'(x)$$
$$= nx^{n-1}p(x)$$
$$= nT^{n-1}(p(x))$$

$\blacksquare$