1. Let T be the linear transformation on $\mathbb{R}^4$ which is represented in standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what condition on a, b and c is T diagonalizable?

2. Let T be a linear transformation on the n-dimensional vector space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.

3. Let T be an invertible linear transformation on a vector space V. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

4. Let $A, B \in \mathbb{R}^{n\times n}$. This problem is to conclude that AB and BA have exactly the same set of eigenvalues.

1). Assume that $\lambda I - AB$ is invertible. Prove that

$$(\lambda I - BA)\Big[I + B(\lambda I - AB)^{-1}A\Big] = \lambda I.$$

2). Use Part 1) to prove that AB and BA have the same eigenvalues. (Note: The algebraic multiplicity of the same eigenvalues may not be the same.)

5. Let $A \in \mathbb{C}^{n\times n}$. Let g be a polynomial over $\mathbb{C}$. Prove that c is an eigenvalue of $g(A)$ if and only if $c = g(\lambda)$ for some eigenvalue λ of A.

6. Suppose $V = W_1 \oplus W_2$. Prove that for any $\mathbf{v} \in V$, there exists a unique pair of vectors $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.

7. True or False. (No explanation needed.)

1). Let $A \in \mathbb{C}^{n\times n}$, then A has exactly n eigenvalues (counting the multiplicities).

2). Let $T : V \to V$ be a linear transformation, where $\dim V = n$. Then T is diagonalizable if and only if T has n distinct eigenvalues.

3). Similar matrices always have the same eigenvalues.

4). Similar matrices always have the same eigenvectors.

5). The sum of two eigenvectors of a linear transformation T is always an eigenvector of T.

6). If λ_1 and λ_2 are distinct eigenvalues of a linear transformation, then $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$.

7). A linear transformation T on a finite-dimensional vector space is diagonalizable if and only if the algebraic multiplicity of each eigenvalue λ equals to its geometric multiplicity.

8). Suppose $W_1, W_2, \cdots W_m \subset V$ are subspaces. Then $W_1 + W_2 + \cdots + W_m$ is a direct sum if $W_i \cap W_j = \{\mathbf{0}\}$ for any $i \neq j$.