Differential Geometry: Homework 8

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Exercise 4.3.1. Show that if $\mathbf{x}$ is an orthogonal parametrization, that is, $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{GE}}\right)_u\right].$$

Solution. Using the given equations in Do Carmo, we see that

$$\Gamma_{11}^1 = \frac{E_u}{2E}, \quad \Gamma_{11}^2 = -\frac{E_v}{2G}, \quad \Gamma_{12}^1 = \frac{E_v}{2E}, \quad \Gamma_{12}^2 = \frac{G_u}{2G}, \quad \Gamma_{22}^1 = -\frac{G_u}{2E}, \quad \text{and} \quad \Gamma_{22}^2 = \frac{G_v}{2G}.$$

We already know that

$$\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2 = -EK.$$

Therefore, computing the partials of Γ_{12}^2 and Γ_{11}^2, we have

$$\left(\Gamma_{12}^2\right)_u = \left(\frac{G_u}{2G}\right)_u = \frac{GG_{uu} - G_u^2}{2G^2} \quad \text{and} \quad \left(\Gamma_{11}^2\right)_v = \left(\frac{E_v}{2G}\right)_v = \frac{GE_{vv} - G_vE_v}{2G^2}.$$

Now we combine like terms to get

$$\begin{aligned} K &= \frac{G_u^2 - GG_{uu}}{2EG^2} + \frac{G_vE_v - GE_{vv}}{2EG^2} + \frac{E_v^2}{4E^2G} - \frac{G_u^2}{4EG^2} - \frac{E_vG_v}{4EG^2} + \frac{E_uG_u}{4E^2G} \\ &= \frac{-GG_{uu} + G_vE_v - GE_{vv}}{2EG^2} + \frac{G_u^2}{2EG^2} - \frac{G_u^2}{4EG^2} - \frac{E_vG_v}{4EG^2} + \frac{E_v^2}{4E^2G} + \frac{E_uG_u}{4E^2G} \\ &= \frac{-GG_{uu} - GE_{vv}}{2EG^2} + \frac{G_vE_v}{2EG^2} - \frac{E_vG_v}{4EG^2} + \frac{G_u^2}{4EG^2} + \frac{E_v^2}{4E^2G} + \frac{E_uG_u}{4E^2G}. \end{aligned}$$

Now notice that

$$\left(\frac{G_u}{\sqrt{EG}}\right)_u = \frac{G_{uu}}{\sqrt{EG}} - \frac{G_u}{2(EG)^{3/2}}(E_uG + EG_u) \quad \text{and} \quad \left(\frac{E_v}{\sqrt{EG}}\right)_v = \frac{E_{vv}}{\sqrt{EG}} - \frac{E_v}{2(EG)^{3/2}}(E_vG + EG_v).$$

Then combining these results, we have

$$\begin{aligned} \left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u &= \frac{E_{vv} + G_{uu}}{\sqrt{EG}} - \frac{1}{2(EG)^{3/2}}\left[E_v(E_vG + EG_v) + G_u(E_uG + EG_u)\right] \\ &= \frac{E_{vv} + G_{uu}}{\sqrt{EG}} - \frac{1}{2(EG)^{3/2}}\left(E_v^2G + E_vEG_v + G_uE_uG + G_uEG_u\right). \end{aligned}$$

Therefore,

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]. \qquad \square$$

Exercise 4.3.2. Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\Delta(\log(\lambda)),$$

where $\Delta\varphi$ denotes the Laplacian $\left(\partial^2\varphi/\partial u^2\right) + \left(\partial^2\varphi/\partial v^2\right)$ of the function φ. Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $K = \text{const.} = 4c$.

Solution. Using the equation proven in exercise 4.3.1, we have

$$\begin{aligned} K &= -\frac{1}{2\sqrt{\lambda\lambda}}\left(\left(\frac{\lambda_v}{\sqrt{\lambda\lambda}}\right)_v + \left(\frac{\lambda_u}{\sqrt{\lambda\lambda}}\right)_u\right) \\ &= -\frac{1}{2\lambda}\left(\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2\lambda}\left((\ln_v(\lambda))_v + (\ln_u(\lambda))_u\right) \\
&= -\frac{1}{2\lambda}\left(\frac{\partial^2 \ln(\lambda)}{\partial u^2} + \frac{\partial^2 \ln(\lambda)}{\partial v^2}\right) \\
&= -\frac{1}{2\lambda}\Delta(\log(\lambda)).
\end{aligned}
$$
□

Exercise 4.3.4. Show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

Solution. Parameterizing a sphere by spherical coordinates, we have

$$\mathbf{x}(u, v) = (\rho \sin(u) \cos(v), \rho \sin(u) \sin(v), \rho \cos(u)),$$

where ρ is the radius of the sphere, $u \in [0, \pi]$ is the polar angle, and $v \in [0, 2\pi)$ is the azimuthal angle. Computing the partial derivatives, we have

$$\mathbf{x}_u = (\rho \cos(u) \cos(v), \rho \cos(u) \sin(v), -\rho \sin(u)) \quad \text{and} \quad \mathbf{x}_v = (-\rho \sin(u) \sin(v), \rho \sin(u) \cos(v), 0).$$

The first fundamental form coefficients are

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \rho^2, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad \text{and} \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \rho^2 \sin^2(u).$$

Therefore, using the Gauss formula from exercise 4.3.1, we have

$$
\begin{aligned}
K &= -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{GE}}\right)_u\right] \\
&= -\frac{1}{2\rho^2 \sin(u)}\left[\left(\frac{0}{\rho \sin(u)}\right)_v + \left(\frac{\rho^2 \cos(u)}{\rho^2}\right)_u\right] \\
&= -\frac{-\sin(u)}{2\rho^2 \sin(u)} \\
&= \frac{1}{\rho^2}.
\end{aligned}
$$

Parametrizing a plane by Cartesian coordinates, we have

$$\bar{\mathbf{x}}(u, v) = (u, v, au + bv + c).$$

The partial derivatives are

$$\bar{\mathbf{x}}_u = (1, 0, a) \quad \text{and} \quad \bar{\mathbf{x}}_v = (0, 1, b).$$

The first fundamental form coefficients are

$$\bar{E} = \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = 1, \quad \bar{F} = \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle = 0, \quad \text{and} \quad \bar{G} = \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle = 1.$$

Therefore, using the Gauss formula from exercise 4.3.1, we have

$$\bar{K} = -\frac{1}{2\sqrt{\bar{E}\bar{G}}}\left[\left(\frac{\bar{E}_v}{\sqrt{\bar{E}\bar{G}}}\right)_v + \left(\frac{\bar{G}_u}{\sqrt{\bar{G}\bar{E}}}\right)_u\right] = -\frac{1}{2}\left[\left(\frac{0}{1}\right)_v + \left(\frac{0}{1}\right)_u\right] = 0.$$

Since the Gaussian is invariant under isometries and $1/\rho^2 \neq 0$, $\forall \rho > 0$, we conclude that no neighborhood of a point in a sphere may be isometrically mapped into a plane. □

Exercise 4.3.8. Compute the Christoffel symbols an open set of the plane

(i) In Cartesian coordinates.

(ii) In polar coordinates.

Use the Gauss formula to compute K in both cases.

Solution to (i). Parametrizing the plane by Cartesian coordinates, we have

$$\mathbf{x}(u,v) = (u, v, au + bv + c).$$

From exercise 4.3.4, the first fundamental form coefficients are

$$E = 1, \quad F = 0, \quad \text{and} \quad G = 1$$
$$E_u = 0 = E_v \quad \text{and} \quad G_u = 0 = G_v$$

The Christoffel symbols are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Clearly, all the Christoffel symbols are zero, which imply that $K = 0$. □

Solution to (ii). Parametrizing the plane by polar coordinates, we have

$$\mathbf{x}(u,v) = (u\cos(v), u\sin(v), au\cos(v) + bu\sin(v) + c).$$

The first fundamental form coefficients are

$$E = u^2, \quad F = 0, \quad \text{and} \quad G = 1$$
$$E_u = 2u, \quad E_v = 0, \quad \text{and} \quad G_u = 0 = G_v.$$

The Christoffel symbols are given by

$$\begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, we get

$$\Gamma_{11}^1 = \frac{1}{u}, \quad \Gamma_{11}^2 = 0, \quad \Gamma_{12}^1 = 0, \quad \Gamma_{12}^2 = 0, \quad \Gamma_{22}^1 = 0, \quad \Gamma_{22}^2 = 0.$$

Using the Gauss formula from exercise 4.3.1, we have

$$\begin{aligned} K &= -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{GE}} \right)_u \right] = -\frac{1}{2u} \left[\left(\frac{0}{u} \right)_v + \left(\frac{0}{u} \right)_u \right] \\ &= -\frac{1}{2u} \left[\left(\frac{0}{u} \right)_v + \left(\frac{0}{u} \right)_u \right] = 0. \end{aligned}$$ □

Exercise 4.4.1.

(i) Show that if a curve $C \subset S$ is both a line of curvature and a geodesic, then C is a plane curve.

(ii) Show that if a (nonrectilinear) geodesic is a plane curve, then it is a line of curvature.

(iii) Give an example of a line of curvature which is a plane curve and not a geodesic.

Solution to (i). Let α be the curve parametrized by arc length. Since α is geodesic, we have $k_g = 0$. Therefore,

$$k^2 = k_g^2 + k_n^2 \Rightarrow k^2 = k_n^2 \Rightarrow k = k_n = k \langle n, \mathbf{N} \rangle \Rightarrow \langle n, \mathbf{N} \rangle = 1 \Rightarrow n = \mathbf{N},$$

where n is the principal normal to the curve in $\mathbb{R}^3$ and $\mathbf{N}$ is the unit normal to the surface.

Since α is also a line of curvature, we have $\mathrm{d}\mathbf{N}(T) = \lambda T$ for some principal curvature λ. But we also have $n = \mathbf{N}$, so

$$\frac{\mathrm{d}n}{\mathrm{d}s} = \lambda T.$$

This implies that the derivative of the principal normal vector lies in the direction of the tangent vector. Hence, the binormal vector $B = T \times n$ is constant. Since the osculating plane is spanned by T and n, it remains fixed. Therefore, α lies entirely in a fixed plane, and is a plane curve. □

Solution to (ii). Let α be a nonrectilinear geodesic parametrized by arc length. Since α is a geodesic, we have $k_g = 0$, and therefore the total curvature satisfies $n = \mathbf{N}$, as we showed in the previous problem. Thus, the principal normal vector n of the curve agrees with the surface normal vector $\mathbf{N}$ along α.

Now suppose further that α lies in a plane $P \subset \mathbb{R}^3$. Since the binormal vector $B = T \times n$ is orthogonal to both T and $n = \mathbf{N}$, we have that B is constant and perpendicular to the plane P.

Because $\mathbf{N}$ coincides with the principal normal n, which is perpendicular to the fixed binormal B, it follows that the surface normal vector $\mathbf{N}$ stays in the same direction as the curve moves — i.e., the surface bends uniformly in the direction of the curve's tangent vector T.

Differentiating $\mathbf{N}$ along α,

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = \frac{\mathrm{d}n}{\mathrm{d}s} = -kT + \tau B.$$

But since B is constant and the curve is planar, we must have $\tau = 0$. So

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -kT.$$

This implies that the shape operator S satisfies

$$\mathrm{d}\mathbf{N}(T) = -kT,$$

i.e., T is an eigenvector of the shape operator. Hence, the curve is a line of curvature. □

Solution. (iii) Consider the parallel curves on a surface of revolution, such as a circle of latitude on a sphere (excluding the equator). Let S be the unit sphere in $\mathbb{R}^3$, and let $\alpha(s)$ be the circle of latitude defined by

$$\alpha(s) = (\cos(s)\sin(\theta_0), \sin(s)\sin(\theta_0), \cos(\theta_0)),$$

where $\theta_0 \in (0, \pi) \setminus \{\pi/2\}$ is fixed. Then α lies in the plane $z = \cos(\theta_0)$ and is clearly a plane curve.

On a surface of revolution, parallels are always lines of curvature. However, α is not a geodesic unless $\theta_0 = \pi/2$, in which case the parallel is a great circle (i.e., a geodesic). Since $\theta_0 \neq \pi/2$, α is not a geodesic – the geodesic curvature k_g is nonzero.

Therefore, α is a line of curvature, is planar, but not a geodesic. □

Exercise 4.4.2. Prove that a curve $C \subset S$ is both an asymptotic curve and a geodesic if and only if C is a (segment of a) straight line.

Solution. Let $\alpha(s)$ be a regular curve on S parametrized by arc length. Suppose α is both a geodesic and an asymptotic curve. Since it is a geodesic, we have

$$k_g = 0 \Rightarrow k^2 = k_n^2,$$

and since it is asymptotic, we have

$$k_n = \langle n, \mathbf{N} \rangle = 0 \Rightarrow k = 0.$$

Thus, the total curvature $k = 0$, which implies that $\alpha''(s) = 0$, so α is a straight line in $\mathbb{R}^3$.

Conversely, suppose that $\alpha(s)$ is a straight line in $\mathbb{R}^3$ lying on the surface S. Then $\alpha''(s) = 0 \Rightarrow k = 0$. In particular, both the normal curvature $k_n = k \langle n, \mathbf{N} \rangle = 0$ and the geodesic curvature $k_g = 0$ vanish. Therefore, α is both an asymptotic curve and a geodesic, and hence a straight line.

Thus, a curve $C \subset S$ is both an asymptotic curve and a geodesic if and only if C is a (segment of a) straight line. □

Exercise 4.4.3. Show, without using Prop. 5, that the straight lines are the only geodesics of a plane.

Solution. Since the surface is a plane, its normal vector $\mathbf{N}$ is constant and perpendicular to the plane. For any curve α lying in the plane, the principal normal vector n lies in the plane itself. Therefore,

$$\langle n, \mathbf{N} \rangle = 0,$$

for all points on α. Consequently, the normal curvature of any curve on the plane satisfies

$$k_n = k \langle n, \mathbf{N} \rangle = 0,$$

so every curve in the plane is an asymptotic curve.

Recall that the curvature k of the curve satisfies

$$k^2 = k_g^2 + k_n^2 = k_g^2,$$

since $k_n = 0$. Thus, the geodesic curvature k_g equals the total curvature k of the curve. Geodesics have zero geodesic curvature, so $k_g = 0$, which implies $k = 0$. Hence, the geodesics on the plane are precisely the curves with zero curvature – that is, straight lines. Therefore, the straight lines are the only geodesics in a plane. □

Exercise 4.4.4. Let u and w be vector fields along a curve $\alpha : I \to S$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle v(t), w(t) \rangle = \left\langle \frac{\mathrm{D}v}{\mathrm{d}t}, w(t) \right\rangle + \left\langle v(t), \frac{\mathrm{D}w}{\mathrm{d}t} \right\rangle.$$

Solution. The derivative of the inner product satisfies the product rule, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle v, w \rangle = \left\langle \frac{\mathrm{d}v}{\mathrm{d}t}, w \right\rangle + \left\langle v, \frac{\mathrm{d}w}{\mathrm{d}t} \right\rangle.$$

However, the ordinary derivatives $\mathrm{d}v/\mathrm{d}t$ and $\mathrm{d}w/\mathrm{d}t$ may not lie in the tangent plane. We can decompose them as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{D}v}{\mathrm{d}t} + v_n\mathbf{N} \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{D}w}{\mathrm{d}t} + w_n\mathbf{N},$$

where v_n and w_n are scalar functions and $\mathbf{N}$ is the unit normal vector to the surface.

Since v and w are tangent vector fields, we have $\langle \mathbf{N}, v \rangle = \langle \mathbf{N}, w \rangle = 0$. Using the bilinearity of the inner product,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle v, w \rangle &= \left\langle \frac{\mathrm{D}v}{\mathrm{d}t} + v_n\mathbf{N}, w \right\rangle + \left\langle v, \frac{\mathrm{D}w}{\mathrm{d}t} + w_n\mathbf{N} \right\rangle \\ &= \left\langle \frac{\mathrm{D}v}{\mathrm{d}t}, w \right\rangle + v_n \langle \mathbf{N}, w \rangle + \left\langle v, \frac{\mathrm{D}w}{\mathrm{d}t} \right\rangle + w_n \langle v, \mathbf{N} \rangle \\ &= \left\langle \frac{\mathrm{D}v}{\mathrm{d}t}, w \right\rangle + \left\langle v, \frac{\mathrm{D}w}{\mathrm{d}t} \right\rangle. \end{aligned}$$

□