$\Rightarrow \quad c\vec{w}_1 + \vec{w}_2 = cT(\vec{v}_1) + T(\vec{v}_2) = T(c\vec{v}_1 + \vec{v}_2)$

$\Rightarrow \quad c\vec{w}_1 + \vec{w}_2 \in \text{Range}(T)$

$\Rightarrow$ Range$(T) \subseteq W$ is a subspace. $\boxtimes$

Definition: The dimension of Ker(T) is called the nullity of T.
The dimension of Range(T) is called the rank of T.

Theorem: (Rank-nullity Theorem or Dimension Theorem). Let V be a finite-dimensional vector space. Let $T: V \to W$ be a linear transformation. Then

$$\dim V = \text{nullity}(T) + \text{rank}(T).$$

<u>Proof</u>: Suppose dimension $V = n$, and nullity$(T) = k$. We need to prove that rank$(T) = n-k$.

Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\} \subseteq \text{Ker}(T)$ be a basis.

Extend $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ to a basis of V: $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$
(Note this extension can be carried out by Replacement Theorem.).

Claim: $S = \{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\} \subseteq \text{Range}(T)$ is a basis of Range(T).

Followed by the claim: Rank$(T) = n-k$. Therefore

Thus $\dim V = n$, nullity$(T) = k$, rank$(T) = n-k$

$$\Rightarrow \dim V = \text{nullity}(T) + \text{rank}(T)$$

end of Jan 17

Proof of Claim: Prove S is linearly independent.

Suppose $c_{k+1}, \cdots, c_n$ satisfies $c_{k+1}T(\vec{v}_{k+1}) + \cdots + c_n T(\vec{v}_n) = \vec{0}$

$\Rightarrow \quad T(c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n) = \vec{0}$

$\Rightarrow \quad c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n \in \text{Ker}(T) = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_k\}$

$\exists\, c_1, c_2, \cdots, c_k$ such that:

$$c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

$\Rightarrow -c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n = \vec{0}$

Since $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$ is a basis of V, thus linearly independent

$\Rightarrow \quad -c_1 = -c_2 = \cdots = -c_k = c_{k+1} = \cdots = c_n = 0$

$\Rightarrow \quad c_{k+1}T(\vec{v}_{k+1}) + \cdots + c_n T(\vec{v}_n) = \vec{0}$ implies $c_{k+1} = \cdots = c_n = 0$

$\Rightarrow \{T(v_{k+1}), T(v_{k+2}), \cdots, T(v_n)\}$ is linearly independent.

② Prove that $\text{Span}\, S = \text{Range}(T)$.

$\forall\, \vec{w} \in \text{Range}(T) \quad \exists\, \vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis of V, $\exists\, a_1, \cdots, a_k, a_{k+1}, \cdots, a_n$

such that $\vec{v} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} + \cdots + a_n\vec{v}_n$

$$\Rightarrow \vec{w} = T(\vec{v}) = T(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} + \cdots + a_n\vec{v}_n)$$

$$= \underbrace{a_1T(\vec{v}_1) + \cdots + a_kT(\vec{v}_k)}_{\vec{v}_1, \cdots, \vec{v}_k \in \text{Ker}(T) \Rightarrow T(\vec{v}_1) = \cdots = T(\vec{v}_k) = \vec{0}} + a_{k+1}T(\vec{v}_{k+1}) + \cdots + a_nT(\vec{v}_n)$$

$$= a_{k+1}T(\vec{v}_{k+1}) + \cdots + a_nT(\vec{v}_n)$$

$\Rightarrow \quad \vec{w} \in \text{Span}\{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\} \Rightarrow \text{Range}\,T \subseteq \text{Span}\{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\}$

As $\text{Span}\{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\} \subseteq \text{Range}(T)$,

$\text{Span}\{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\} = \text{Range}(T)$

By ① and ②, $\{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\}$ is a basis of Range(T)

$\Rightarrow$ $\text{rank}(T) = n-k$. ∎

Definition: Let $T: V \to W$ be a linear transformation.

1) T is called one-to-one if " $\forall \vec{x}, \vec{y} \in V$, $\vec{x} \neq \vec{y}$ implies $T(\vec{x}) \neq T(\vec{y})$

or equivalently " $T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$"

2) T is called onto if $\forall \vec{w} \in W$: $\exists \vec{v} \in V$ such that $\vec{w} = T(\vec{v})$.

or equivalently $W = \text{Range}(T)$.

3). T is called an isomphism if T is both one-to-one and onto.

Definition: Two vector spaces V and W are called isomophic, if there exists an isomorphism $T: V \to W$.

Proposition: A linear transformation $T: V \to W$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

<u>Proof</u>: "$\Rightarrow$" Suppose T is one-to-one.

Since $T(\vec{0}) = \vec{0}$ $\Rightarrow$ $\vec{0} \in \ker(T)$

$\forall \vec{x} \in \ker(T)$ $\Rightarrow$ $T(\vec{x}) = \vec{0} = T(\vec{0})$ $\Rightarrow$ $\vec{x} = \vec{0}$ as T is one-to-one

$\Rightarrow$ $\ker(T) = \{\vec{0}\}$

"$\Leftarrow$" Suppose $\ker(T)=\{\vec{0}\}$

If $T(\vec{x})=T(\vec{y}) \Leftrightarrow T(\vec{x})-T(\vec{y})=\vec{0}$

$\Leftrightarrow T(\vec{x}-\vec{y})=\vec{0}$

$\Rightarrow \vec{x}-\vec{y} \in \ker(T)=\{\vec{0}\}$

$\Leftrightarrow \vec{x}-\vec{y}=\vec{0} \quad \Rightarrow \vec{x}=\vec{y}$

$\Rightarrow$ T is one-to-one.

Notation: If T is a linear transformation from V to V, we may say "T is a linear transformation on V".

Proposition. Let T be a linear transformation on a finite dimensional space V. Then the following statements are equivalent.

1). T is one-to-one

2). T is onto

3). T is an isomorphism.

<u>Proof</u>: Homework. (Hint: use rank-nullity theorem).

<u>Remarks</u>: The above statements are not equivalent for infinite-dimensional vector spaces. See homework problems #   in Homework 3

End of Jan 22.