§14.3: Partial Derivatives

Recall, if $y = f(x)$, then $f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

is the instantaneous rate of change of with respect to x at $x = a$.

Now, for $f(x,y)$, the partial derivatives computes the rate of change with respect to each variable separately.

If $y = b$ and x is allowed to vary, then the partial derivative of $f(x,y)$ at (a,b) with respect to x is $f_x = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$ if the limit exists.

Similarly, if $x = a$ and y is allowed to vary, then the partial derivative of $f(x,y)$ at (a,b) with respect to y is

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$ if the limit exists.

Notation: $z = f(x,y)$

$$z_x = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

$$z_y = f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

Ex! Consider $f(x,y) = x^2y^5 + \sin(x^3 - y^{-2}) + e^{x^2/y}$

To compute, f_x, treat y as a constant and differentiate with respect to x.

$$f_x = 2xy^5 + \cos(x^3 - y^{-2})(3x^2) + e^{x^2/y}(2xy^{-1})$$

$$f_y = 5x^2y^4 + \cos(x^3 - y^{-2})(2y^{-3}) + e^{x^2/y}(-x^2y^{-2})$$

Ex: A model for the surface area of a human body is given by the function $S(w,h) = 0.1091 w^{0.425} h^{0.725}$ where w is weight (lb) and h is height (inches). Surface area is measured in ft^2

Calculate $\frac{\partial S}{\partial w}(160,70)$: Rate of change of surface area with respect to weight when $w = 160$ lb and $h = 70$ in

$$\frac{\partial S}{\partial w} = 0.1091(0.425) w^{-0.575} h^{0.725}$$

$$\frac{\partial S}{\partial w}(160,70) \approx 0.05452 \ \frac{ft^2}{lb}$$

When the person weighs 160 lb and has height 70 in, their surface area is increasing at a rate of 0.05452 ft^2 per lb.

Therefore if the weight increased from 160 to 161, surface area would approximately increase by 0.05452

$$S(161,70) \approx S(160,70) + \frac{\partial S}{\partial w}(160,70) = 20.579$$

$$S(161,70) = 20.5789 \quad \text{(evaluating function)}$$

Ex: $f(x,y) = \arctan\left(\frac{x}{y}\right)$

$$f_x = \frac{1}{\left(\frac{x}{y}\right)^2+1}\left(\frac{1}{y}\right) = \frac{1}{\frac{x^2}{y}+y} = \frac{y}{x^2+y^2}$$

$$f_y = \frac{1}{\left(\frac{x}{y}\right)^2+1}\left(\frac{-x}{y^2}\right) = \frac{-x}{x^2+y^2}$$

Second Order Derivatives: Consider $z = f(x,y)$

$$z_x = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

$$z_y = f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

Differentiate again to find second order derivatives

$$z_{xx} = \frac{\partial}{\partial x}(z_x) = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$z_{yy} = \frac{\partial}{\partial y}(z_y) = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$z_{xy} = \frac{\partial}{\partial y}(z_x) = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$z_{yx} = \frac{\partial}{\partial x}(z_y) = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

Mixed Partial Derivatives

Ex! Find second order derivatives of

$f(x,y) = \arctan\left(\frac{x}{y}\right)$

$$f_x = \frac{y}{x^2+y^2} = y(x^2+y^2)^{-1}$$

$$f_{xx} = -y(x^2+y^2)^{-2}(2x) = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_{xy} = (x^2+y^2)^{-1} - y(x^2+y^2)^{-2}(2y)$$

$$= \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f_y = \frac{-x}{x^2+y^2} = -x(x^2+y^2)^{-1}$$

$$f_{yy} = x(x^2+y^2)^{-2}(2y) = \frac{2xy}{(x^2+y^2)^2}$$

$$f_{yx} = -(x^2+y^2)^{-1} + x(x^2+y^2)^{-2}(2x)$$

$$= \frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Note! $f_{xy} = f_{yx}$