Ex: $f(x,y) = \sqrt{x^2+y^2}$

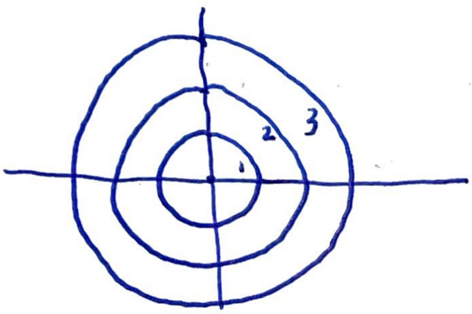
Domain $\mathbb{R}^2$

Range $[0, \infty)$

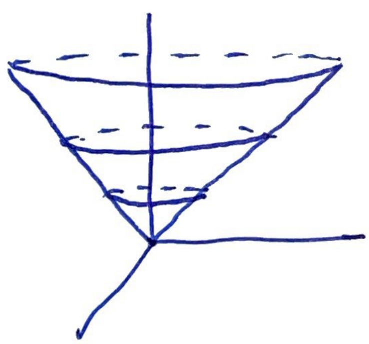
Level curves

$\sqrt{x^2+y^2} = K$ for $K \geq 0$

$x^2+y^2 = K^2$ Circle with radius K.



Given unit change in f, there is a unit increase in radius of circular contour.



$z = \sqrt{x^2+y^2}$

Cone.

Functions of Three Variables

Defn: A function, $f(x,y,z)$, assigns to each (x,y,z) in the domain of f a unique number $f(x,y,z)$. The domain is a subset of $\mathbb{R}^3$ and the range is a subset of $\mathbb{R}$.

The graph of $f(x,y,z)$ is the collection of all ordered 4-tuples $(x,y,z, f(x,y,z))$ in $\mathbb{R}^4$.

Level Surface

Given a function, $f(x,y,z)$, the level surfaces are all equations $f(x,y,z) = K$ for K in range of f.

The equation $f(x,y,z) = K$ is a surface in $\mathbb{R}^3$ containing all points (x,y,z) that produces the range value K for f.

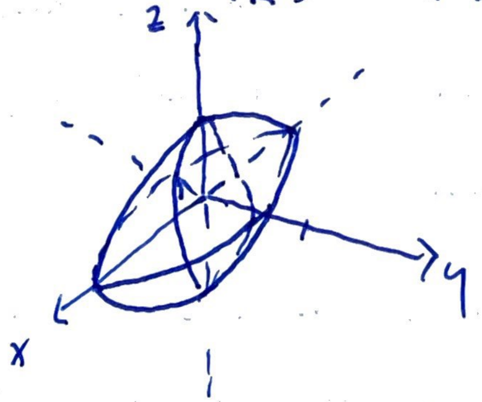
Ex: $f(x,y,z) = x^2 + 4y^2 + 9z^2$

Domain: $\mathbb{R}^3$

Range: $[0,\infty)$

For each $K \geq 0$, $x^2 + 4y^2 + 9z^2 = K$ defines an ellipsoid.

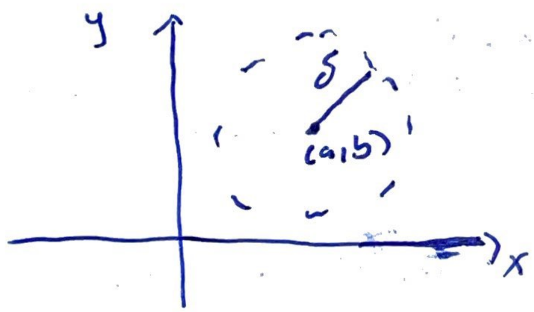
$$\left(\frac{x}{\sqrt{K}}\right)^2 + \left(\frac{2y}{\sqrt{K}}\right)^2 + \left(\frac{3z}{\sqrt{K}}\right)^2 = 1$$

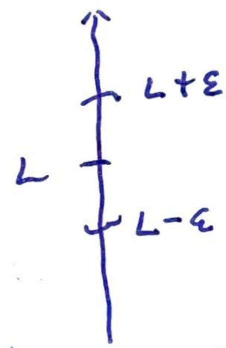


As K increases, the ellipsoid expands

§14.2: Limits and Continuity

Defn: Let $f(x,y)$ be a function whose domain D contains all points arbitrarily close to (a,b). We say the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $(x,y) \in D$ with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.



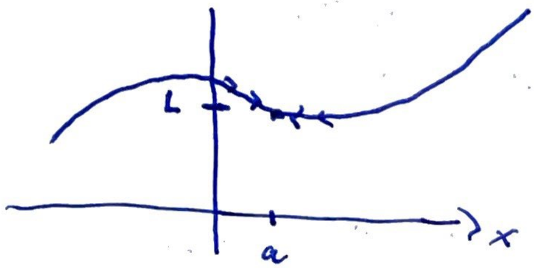


For every distance ε in the range, there is a distance δ in the domain so that if (x,y) is within δ of (a,b), then f is within ε of L.

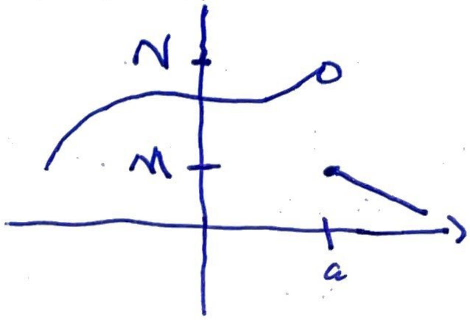
If this ε-δ relation can be found for every possible ε, then the limit exists and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

Recall, $\lim_{x \to a} f(x) = L$ if and only if

$\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$



$\lim_{x \to a} f(x) = L$



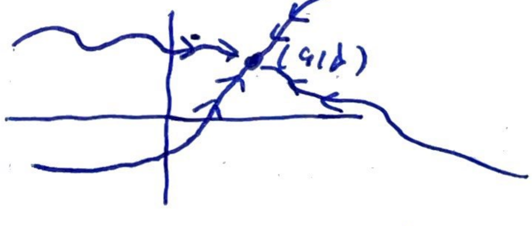
$\lim_{x \to a^-} f(x) = N$

$\lim_{x \to a^+} f(x) = M$

$N \neq M$ $\lim_{x \to a} f(x)$ DNE

Paths: If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along the path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along the path C_2, then

$\lim_{(x,y) \to (a,b)} f(x,y)$ DNE



Note: Paths cannot be used to prove a limit exists since we cannot check the infinitely many curves that pass through (a,b).

Ex: Determine if the limit exists.

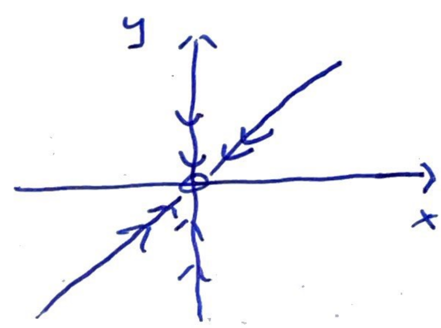
a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{6x^3y}{2x^4+y^4}$ $\qquad f(x,y) = \dfrac{6x^3y}{2x^4+y^4}$

Domain: $D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

Path 1: $x=0$ (y-axis)

$$f(0,y) = \frac{6(0)y}{0+y^4} = \frac{0}{y^4} = 0 \text{ if } y \neq 0$$

$$\lim_{(0,y)\to(0,0)} f(0,y) = \lim_{(0,y)\to(0,0)} 0 = 0$$



Path 2: $y = x$

$$f(x,x) = \frac{6x^3x}{2x^4+x^4} = \frac{6x^4}{3x^4} = 2 \quad \text{if } x \neq 0$$

$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{(x,x)\to(0,0)} 2 = 2$$

Since $\lim\limits_{(0,y)\to(0,0)} f(0,y) \neq \lim\limits_{(x,x)\to(0,0)} f(x,x)$, then

$\lim\limits_{(x,y)\to(0,0)} f$ DNE

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

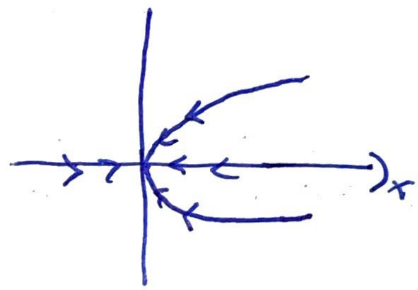
$f(x,y) = \dfrac{xy^4}{x^2+y^8}$

Domain: $D = \{(x,y) \in \mathbb{R}^2 ; (x,y) \neq (0,0)\}$

Path 1: $y=0$ (x-axis)

$$f(x,0) = \frac{x(0)}{x^2+0} = \frac{0}{x^2} = 0 \quad \text{if } x \neq 0$$

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0$$



Path 2: $x = y^4$

$$f(y^4, y) = \frac{y^4 y^4}{y^8+y^8} = \frac{y^8}{2y^8} = \frac{1}{2} \quad \text{if } y \neq 0$$

$$\lim_{y\to 0} f(y^4, y) = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

Since $\displaystyle\lim_{x\to 0} f(x,0) \neq \lim_{y\to 0} f(y^4, y)$, then

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ DNE