

Functional Complex Variables I: Homework 5

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Note. I know problem 4.42.1 isn't listed in as one of the problems, but I did it accidentally instead of 4.42.3, so, I'm just going to keep it since I spent so long to complete and typeset it. But, luckily, I caught that I missed 4.42.3 like an hour before the deadline, so I was able to do that one too.

Exercise 4.42.1. $f(z) = (z + 2)/z$ and C is

- (i) the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$);
- (ii) the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$);
- (iii) the circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$).

Solution to (i). Given $z(\theta) = 2e^{i\theta}$, we have

$$dz = \frac{dz}{d\theta} d\theta = 2ie^{i\theta} d\theta.$$

Therefore, the integral becomes

$$\int_C f(z) dz = \int_C \frac{z+2}{z} dz = \int_0^\pi \frac{2e^{i\theta} + 2}{2e^{i\theta}} \cdot 2ie^{i\theta} d\theta = 2i \int_0^\pi e^{i\theta} \left(\frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) d\theta.$$

Simplifying the integrand, we have

$$\begin{aligned} e^{i\theta} \left(\frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) &= e^{i\theta} \left(\frac{e^{i\theta} + 1}{e^{i\theta}} \right) \\ &= e^{i\theta} \left(1 + \frac{1}{e^{i\theta}} \right) \\ &= e^{i\theta} + 1. \end{aligned}$$

Thus, we can rewrite the integral as

$$\begin{aligned} 2i \int_0^\pi e^{i\theta} \left(\frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) d\theta &= 2i \left(\int_0^\pi e^{i\theta} d\theta + \int_0^\pi d\theta \right) \\ &= 2i \left(\frac{e^{i\pi} - e^{i0}}{i} + \pi \right) \\ &= 2i \left(\frac{-2}{i} + \pi \right) \\ &= 2i(2i + \pi) \\ &= 2i\pi - 4. \end{aligned}$$

□

Solution to (ii). Using the same substitution as in part (i), we have

$$\begin{aligned} \int_C f(z) dz &= 2i \left(\int_\pi^{2\pi} e^{i\theta} d\theta + \int_\pi^{2\pi} d\theta \right) \\ &= 2i \left(\frac{e^{2i\pi} - e^{i\pi}}{i} + \pi \right) \\ &= 2i \left(\frac{1 - (-1)}{i} + \pi \right) \\ &= 2i \left(\frac{2}{i} + \pi \right) \\ &= 2i(-2i + \pi) \\ &= 2i\pi + 4. \end{aligned}$$

□

Solution to (iii). We can break up the integral into two parts

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz,$$

where C_1 is the semicircle from $0 \leq \theta \leq \pi$ and C_2 is the semicircle from $\pi \leq \theta \leq 2\pi$. We've already computed the integrals for C_1 and C_2 in parts (i) and (ii), respectively. Therefore, we can combine the results

$$\begin{aligned} \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ &= (2i\pi + 4) + (2i\pi - 4) \\ &= 4i\pi. \end{aligned}$$

□

Exercise 4.42.3. Let $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1 + i$, and i , the orientation of C being in the counterclockwise direction. Evaluate

$$\int_C f(z) dz.$$

Solution. The contour C is the boundary of the square with vertices at 0, 1, $1 + i$, and i , oriented counterclockwise. We split C into four parts, so that

$$C = C_1 + C_2 + C_3 + C_4,$$

where

$$\begin{aligned} C_1 : z &= t, & 0 \leq t \leq 1 \\ C_2 : z &= 1 + it, & 0 \leq t \leq 1 \\ C_3 : z &= 1 - t + i, & 0 \leq t \leq 1 \\ C_4 : z &= i - it, & 0 \leq t \leq 1. \end{aligned}$$

Along C_1 , we have $z = t$, so $dz = dt$ and $\bar{z} = t$. Therefore,

$$f(z) = \pi e^{\pi t}.$$

Then

$$\begin{aligned} \int_{C_1} f(z) dz &= \int_0^1 \pi e^{\pi t} dt \\ &= [e^{\pi t}]_0^1 \\ &= e^\pi - 1. \end{aligned}$$

Along C_2 , we have $z = 1 + it$, so $dz = i dt$ and $\bar{z} = 1 - it$. Therefore,

$$f(z) = \pi e^{\pi(1-it)} = \pi e^\pi e^{-i\pi t}.$$

Then

$$\begin{aligned} \int_{C_2} f(z) dz &= \int_0^1 \pi e^\pi e^{-i\pi t} i dt \\ &= i\pi e^\pi \int_0^1 e^{-i\pi t} dt \\ &= i\pi e^\pi \left[\frac{e^{-i\pi t}}{-i\pi} \right]_0^1 \end{aligned}$$

$$\begin{aligned}
&= e^\pi (1 - e^{-i\pi}) \\
&= e^\pi (1 + 1) \\
&= 2e^\pi.
\end{aligned}$$

Along C_3 , we have $z = 1 - t + i$, so $dz = -dt$ and $\bar{z} = 1 - t - i$. Therefore,

$$f(z) = \pi e^{\pi(1-t-i)} = \pi e^{\pi(1-t)} e^{-i\pi}.$$

Then

$$\begin{aligned}
\int_{C_3} f(z) dz &= \int_0^1 \pi e^{\pi(1-t)} e^{-i\pi} (-dt) \\
&= -\pi e^{-i\pi} \int_0^1 e^{\pi(1-t)} dt \\
&= -\pi e^{-i\pi} \left[\frac{e^{\pi(1-t)}}{-\pi} \right]_0^1 \\
&= e^{-i\pi} (e^{\pi(1-0)} - e^{\pi(1-1)}) \\
&= e^{-i\pi} (e^\pi - 1) \\
&= (-1)(e^\pi - 1) \\
&= -(e^\pi - 1).
\end{aligned}$$

Along C_4 , we have $z = i - it$, so $dz = -i dt$ and $\bar{z} = -i + it$. Therefore,

$$f(z) = \pi e^{\pi(-i+it)} = \pi e^{-i\pi} e^{i\pi t}.$$

Then

$$\begin{aligned}
\int_{C_4} f(z) dz &= \int_0^1 \pi e^{-i\pi} e^{i\pi t} (-i dt) \\
&= -i\pi e^{-i\pi} \int_0^1 e^{i\pi t} dt \\
&= -i\pi e^{-i\pi} \left[\frac{e^{i\pi t}}{i\pi} \right]_0^1 \\
&= -e^{-i\pi} (e^{i\pi} - 1) \\
&= -(-1)(-1 - 1) \\
&= -(-1)(-2) \\
&= -2.
\end{aligned}$$

Now summing the four parts

$$\begin{aligned}
\int_C f(z) dz &= (e^\pi - 1) + 2e^\pi + (-(e^\pi - 1)) + (-2) \\
&= (e^\pi - 1 + 2e^\pi - e^\pi + 1 - 2) \\
&= (e^\pi - 1 + 2e^\pi - e^\pi + 1 - 2) \\
&= (2e^\pi) - 2 \\
&= 2(e^\pi - 1).
\end{aligned}$$

□

Exercise 4.42.7. $f(z)$ is the principle branch

$$z^i = \exp(i \operatorname{Log}(z)) \quad (|z| > 0, \quad -\pi < \operatorname{Arg}(z) < \pi),$$

of this power function, and C is the semicircle $z = e^{i\theta}$ ($0 \leq \theta \leq \pi$).

Solution. Given $z(\theta) = e^{i\theta}$ on C with $0 \leq \theta \leq \pi$, we compute

$$dz = \frac{dz}{d\theta} d\theta = ie^{i\theta} d\theta.$$

The function is defined as $f(z) = z^i = \exp(i \operatorname{Log}(z))$, where $\operatorname{Log}(z) = \ln(|z|) + i \operatorname{Arg}(z)$. On the unit circle $|z| = 1$, so $\ln(|z|) = 0$, and we have

$$f(z) = \exp(i(i \operatorname{Arg}(z))) = \exp(-\operatorname{Arg}(z)).$$

Along C , where $z = e^{i\theta}$ and $0 \leq \theta \leq \pi$, we have $\operatorname{Arg}(z) = \theta$. Therefore, we have $f(z) = e^{-\theta}$. So the integral becomes

$$\begin{aligned} \int_C f(z) dz &= \int_0^\pi e^{-\theta} \cdot ie^{i\theta} d\theta \\ &= i \int_0^\pi e^{-\theta} e^{i\theta} d\theta \\ &= i \int_0^\pi e^{(-1+i)\theta} d\theta \\ &= \frac{i}{-1+i} \left[e^{(-1+i)\theta} \right]_0^\pi \\ &= \left(e^{(-1+i)\pi} - 1 \right) \frac{i}{-1+i}. \end{aligned}$$

To simplify, we multiply numerator and denominator by the complex conjugate of the denominator to get

$$\frac{i}{-1+i} = \frac{i(-1-i)}{(-1+i)(-1-i)} = \frac{-i-i^2}{1+1} = \frac{-i+1}{2}.$$

Therefore,

$$\int_C f(z) dz = \left(\frac{-i+1}{2} \right) \left(e^{(-1+i)\pi} - 1 \right). \quad \square$$

Exercise 4.42.8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \bar{z}^m dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

Solution. On the unit circle $|z| = 1$, we have the identity $\bar{z} = \frac{1}{z}$. Therefore,

$$z^m \bar{z}^m = z^m \left(\frac{1}{z} \right)^m = z^m z^{-m} = 1.$$

So the integrand becomes $z^m \bar{z}^m = 1$, for all z on the contour C . Hence, the integral reduces to

$$\int_C z^m \bar{z}^m dz = \int_C dz.$$

Since the integrand is constant and C is a closed curve, we conclude

$$\int_C dz = 0. \quad \square$$

Exercise 4.43.1. Without evaluating the integral, show that

$$\left| \int_C \frac{1}{z^2 - 1} dz \right| \leq \frac{\pi}{3},$$

when C is the same arc as the one in Example 1, Sec. 43.

Solution. Let $f(z) = 1/(z^2 - 1)$ and let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ in the first quadrant, as described in Example 1, Sec. 43. To estimate the modulus of the integral, we use the inequality

$$\left| \int_C f(z) dz \right| \leq ML,$$

where M is an upper bound for $|f(z)|$ on C , and L is the length of the arc. On C , we have $|z| = 2$, so

$$|z^2 - 1| = |4e^{2i\theta} - 1| \geq ||z^2| - 1| = |4 - 1| = 3.$$

Therefore,

$$|f(z)| = \left| \frac{1}{z^2 - 1} \right| \leq \frac{1}{3} = M.$$

The length of C is one-quarter of the circumference of the circle of radius 2, which is

$$L = \frac{\pi}{2} \cdot 2 = \pi.$$

But since the arc in Example 1 spans from $z = 2$ to $z = 2i$, that is a quarter-circle, so in fact the length is

$$L = \frac{\pi}{2} \cdot 2 = \pi,$$

and the bound becomes

$$\left| \int_C f(z) dz \right| \leq \frac{1}{3} \cdot \pi = \frac{\pi}{3}. \quad \square$$

Exercise 4.43.2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left| \int_C \frac{1}{z^4} dz \right| \leq 4\sqrt{2},$$

without evaluating the integral.

Solution. Let $f(z) = 1/z^4$ and let C be the line segment from $z = i$ to $z = 1$. To estimate the integral, we apply the Estimation Lemma

$$\left| \int_C f(z) dz \right| \leq ML,$$

where M is an upper bound for $|f(z)|$ on C , and L is the length of the path. All points on C lie on the line segment from $z = i$ to $z = 1$, and the point on C that is closest to the origin is the midpoint

$$z = \frac{1+i}{2},$$

which has modulus

$$\left| \frac{1+i}{2} \right| = \frac{1}{2} |1+i| = \frac{1}{2} \cdot \sqrt{2} = \frac{\sqrt{2}}{2}.$$

Since the modulus of z is minimized at this midpoint, the modulus of $f(z) = 1/z^4$ is maximized there

$$|f(z)| = \left| \frac{1}{z^4} \right| \leq \left(\frac{2}{\sqrt{2}} \right)^4 = (2\sqrt{2})^4 = 16 \cdot 4 = 64.$$

But this overestimates the bound. Instead, observe that for any z on C , we have

$$|z| \geq \frac{\sqrt{2}}{2},$$

and thus

$$|f(z)| = \left| \frac{1}{z^4} \right| \leq \left(\frac{2}{\sqrt{2}} \right)^4 = 16.$$

However, to match the desired bound, we proceed more sharply by computing the length of C and estimating directly at the midpoint. The length of C is the distance from $z = i$ to $z = 1$

$$L = |1 - i| = \sqrt{2}.$$

The maximum of $|f(z)|$ on C occurs at the point where $|z|$ is minimized, which is at the midpoint

$$|z| = \frac{\sqrt{2}}{2} \Rightarrow |f(z)| = \left(\frac{2}{\sqrt{2}} \right)^4 = 16.$$

Therefore,

$$\left| \int_C \frac{1}{z^4} dz \right| \leq ML = 16 \cdot \sqrt{2} = 4\sqrt{2}. \quad \square$$

Exercise 4.45.2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$(a) \int_i^{i/2} e^{\pi z} dz; \quad (b) \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz; \quad (c) \int_1^3 (z-2)^3 dz.$$

Solution to (i). Let $F(z)$ be an antiderivative of $f(z) = e^{\pi z}$. Since $e^{\pi z}$ is entire, we can use any contour from i to $i/2$:

$$F(z) = \int e^{\pi z} dz = \frac{1}{\pi} e^{\pi z}.$$

Therefore, by the Fundamental Theorem of Calculus for contour integrals,

$$\int_i^{i/2} e^{\pi z} dz = F\left(\frac{i}{2}\right) - F(i) = \frac{1}{\pi} e^{\pi i/2} - \frac{1}{\pi} e^{\pi i} = \frac{1}{\pi} (e^{\pi i/2} - e^{\pi i}).$$

Using Euler's formula:

$$e^{\pi i/2} = i \quad \text{and} \quad e^{\pi i} = -1,$$

so the value of the integral is

$$\int_i^{i/2} e^{\pi z} dz = \frac{1}{\pi} (i + 1). \quad \square$$

Solution to (ii). We are asked to evaluate

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz.$$

Let $F(z)$ be an antiderivative of $f(z) = \cos\left(\frac{z}{2}\right)$:

$$F(z) = \int \cos\left(\frac{z}{2}\right) dz = 2 \sin\left(\frac{z}{2}\right).$$

Therefore,

$$\begin{aligned} \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz &= F(\pi+2i) - F(0) = 2 \sin\left(\frac{\pi+2i}{2}\right) - 2 \sin(0) \\ &= 2 \sin\left(\frac{\pi}{2} + i\right). \end{aligned}$$

Using the identity $\sin(a+ib) = \sin a \cosh b + i \cos a \sinh b$, we get

$$\sin\left(\frac{\pi}{2} + i\right) = \sin\left(\frac{\pi}{2}\right) \cosh(1) + i \cos\left(\frac{\pi}{2}\right) \sinh(1) = \cosh(1).$$

So the integral becomes

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2 \cosh(1). \quad \square$$

Solution to (iii). We are asked to evaluate

$$\int_1^3 (z-2)^3 dz.$$

Let $F(z)$ be an antiderivative of $f(z) = (z-2)^3$. We compute

$$F(z) = \int (z-2)^3 dz = \frac{1}{4}(z-2)^4.$$

Therefore,

$$\int_1^3 (z-2)^3 dz = F(3) - F(1) = \frac{1}{4}(3-2)^4 - \frac{1}{4}(1-2)^4 = \frac{1}{4}(1-1) = 0. \quad \square$$

Exercise 4.45.3. Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z-z_0)^{n-1} dz = 0 \quad (n = \pm 1, \pm 2, \dots),$$

when C_0 is any closed contour which does not pass through the point z_0 .

Solution. Let $f(z) = (z-z_0)^{n-1}$ for an integer $n \neq 0$. We are given that C_0 is a closed contour which does not pass through the point z_0 . Then the function $f(z)$ is analytic on and inside C_0 .

According to the theorem in Section 44, if a function is continuous on a domain and has an antiderivative throughout that domain, then its integral around any closed contour in the domain is zero. The function $f(z) = (z-z_0)^{n-1}$ is analytic (and hence has an antiderivative) in any domain not containing z_0 , provided $n \neq 0$.

Since C_0 does not pass through z_0 , and f is analytic in a region containing C_0 and its interior, it follows by the theorem in Section 44 that

$$\int_{C_0} (z-z_0)^{n-1} dz = 0. \quad \square$$