Differential Geometry: Homework 7

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Exercise 4.2.1. Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$F(u,v) = (u\sin(\alpha)\cos(v), u\sin(\alpha)\sin(v), u\cos(\alpha))$$
$$(u,v) \in U = \{(u,v) \in \mathbb{R}^2 \mid u > 0\}, \quad \alpha = \text{const.}$$

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(i) Prove that F is a local diffeomorphism of U onto a cone C with the vertex at the origin and 2α as the angle of the vertex.

(ii) Is F a local isometry?

Solution to (i). Let $F : U \to \mathbb{R}^3$ be defined by

$$F(u,v) = (u\sin(\alpha)\cos(v), u\sin(\alpha)\sin(v), u\cos(\alpha)), \quad u > 0, v \in \mathbb{R}.$$

We claim that $F(U)$ is the cone

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\tan^2(\alpha), \quad z \geq 0\},$$

with vertex at the origin and vertex angle 2α.

First, note the vertex is at the origin since $F(0,v) = (0,0,0)$ for all v.

Next, for any $(u,v) \in U$,

$$x^2 + y^2 = (u\sin(\alpha)\cos(v))^2 + (u\sin(\alpha)\sin(v))^2 = u^2\sin^2(\alpha)(\cos^2(v) + \sin^2(v)) = u^2\sin^2(\alpha),$$

and

$$z^2 = (u\cos(\alpha))^2 = u^2\cos^2(\alpha).$$

Hence,

$$x^2 + y^2 = z^2 \cdot \frac{\sin^2(\alpha)}{\cos^2(\alpha)} = z^2\tan^2(\alpha),$$

confirming $F(U) \subseteq C$.

Finally, to show F is a local diffeomorphism, observe the Jacobian matrix of F has rank 2 for all $(u,v) \in U$ (since the partial derivatives with respect to u and v are linearly independent), so F is an immersion and a local homeomorphism onto its image.

Therefore, F parametrizes the cone C locally diffeomorphically.

The vertex angle of the cone is 2α because the generating lines of the cone form an angle α with the z-axis. □

Solution to (ii). Computing the partial derivatives, we have

$$F_u = (\sin(\alpha)\cos(v), \sin(\alpha)\sin(v), \cos(\alpha)) \quad \text{and} \quad F_v = (-u\sin(\alpha)\sin(v), u\sin(\alpha)\cos(v), 0).$$

Then, the coefficients for the first fundamental form are

$$\begin{aligned} E &= \langle F_u, F_u \rangle = \sin^2\alpha + \cos^2\alpha = 1 \\ F &= \langle F_u, F_v \rangle = 0 \\ G &= \langle F_v, F_v \rangle = u^2\sin^2\alpha. \end{aligned}$$

So the first fundamental form is

$$I = \mathrm{d}u^2 + u^2\sin^2(\alpha)\,\mathrm{d}v^2.$$

This is the same as the first fundamental form of a cone with vertex angle 2α parametrized by

$$G(u,v) = u(\sin(\alpha)\cos(v), \sin(\alpha)\sin(v), \cos(\alpha)).$$

Therefore, by proposition 1, F is a local isometry. □

Exercise 4.2.2. Prove the following "converse" of Prop. 1: Let $\varphi : S \to \bar{S}$ be an isometry and $\mathbf{x} : U \to S$ a parametrization at $p \in S$; then $\bar{\mathbf{x}} = \varphi \circ \mathbf{x}$ is a parametrization at $\varphi(p)$ and $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$.

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Solution. Let $\varphi : S \to \bar{S}$ be an isometry, and let $\mathbf{x} : U \to S$ be a local parametrization at $p \in S$. Define $\bar{\mathbf{x}} = \varphi \circ \mathbf{x} : U \to \bar{S}$. Since φ is a diffeomorphism and $\mathbf{x}$ is a parametrization, it follows that $\bar{\mathbf{x}}$ is also a parametrization at $\varphi(p)$.

Since φ is an isometry, it preserves the inner product of tangent vectors. That is, for any $q = \mathbf{x}(u, v)$ and any tangent vectors $w_1, w_2 \in T_q(S)$,

$$\langle w_1, w_2 \rangle_q = \langle d\varphi_q(w_1), d\varphi_q(w_2) \rangle_{\varphi(q)}.$$

In particular, we consider the tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$, and compute the coefficients of the first fundamental form

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle. \end{aligned}$$

Now, since $\bar{\mathbf{x}} = \varphi \circ \mathbf{x}$, the chain rule gives

$$\bar{\mathbf{x}}_u = d\varphi(\mathbf{x}_u) \quad \text{and} \quad \bar{\mathbf{x}}_v = d\varphi(\mathbf{x}_v).$$

Using the fact that φ is an isometry, we have

$$\begin{aligned} \bar{E} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = \langle d\varphi(\mathbf{x}_u), d\varphi(\mathbf{x}_u) \rangle = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = E \\ \bar{F} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle = \langle d\varphi(\mathbf{x}_u), d\varphi(\mathbf{x}_v) \rangle = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = F \\ \bar{G} &= \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle = \langle d\varphi(\mathbf{x}_v), d\varphi(\mathbf{x}_v) \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = G. \end{aligned}$$

Therefore, the first fundamental form of $\bar{\mathbf{x}}$ is equal to that of $\mathbf{x}$, and so $E = \bar{E}$, $F = \bar{F}$, and $G = \bar{G}$. ☐

Exercise 4.2.3. Show that a diffeomorphism $\varphi : S \to \bar{S}$ is an isometry if and only if the arc length of any parametrized curve in S is equal to the arc length of the image curve by φ.

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Solution. Assume φ is an isometry. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a smooth parametrized curve, and let $\bar{\alpha} = \varphi \circ \alpha : (-\varepsilon, \varepsilon) \to \bar{S}$ be the image of α under φ. The arc length of α is

$$s = \int_{-\varepsilon}^{\varepsilon} \|\alpha'(t)\| \, \mathrm{d}t.$$

Since φ is an isometry, it preserves the inner product of tangent vectors. In particular, it preserves their lengths

$$\|\bar{\alpha}'(t)\| = \|\mathrm{d}\varphi(\alpha'(t))\| = \|\alpha'(t)\|.$$

Therefore,

$$\bar{s} = \int_{-\varepsilon}^{\varepsilon} \|\bar{\alpha}'(t)\| \, \mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} \|\alpha'(t)\| \, \mathrm{d}t = s.$$

So the arc length of α is equal to the arc length of $\bar{\alpha}$.

Assume that for any parametrized smooth curve α in S, the arc length of α is equal to the arc length of $\bar{\alpha} = \varphi \circ \alpha$ in $\bar{S}$. Let $\mathbf{x} : U \to S$ be a local parametrization around $p \in S$, and define $\bar{\mathbf{x}} = \varphi \circ \mathbf{x} : U \to \bar{S}$, a local parametrization around $\bar{p} = \varphi(p)$. Let $\mathbf{v} = (v_1 \ v_2)$ be a tangent vector at a point $u \in U$, and consider a curve $\gamma(t) = \mathbf{x}(u + t\mathbf{v})$ in S. Then its arc length is given by

$$s = \int_{-\varepsilon}^{\varepsilon} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{x}(u + t\mathbf{v}) \right\| \mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} \sqrt{\mathbf{v}^T \cdot \mathrm{I}(u) \cdot \mathbf{v}} \, \mathrm{d}t = 2\varepsilon \cdot \sqrt{\mathbf{v}^T \mathrm{I}(u) v},$$

where $\mathrm{I}(u)$ is the first fundamental form matrix for $\mathbf{x}$ at u. The image curve $\bar{\gamma}(t) = \bar{\mathbf{x}}(u + t\mathbf{v})$ has arc length

$$\bar{s} = \int_{-\varepsilon}^{\varepsilon} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathbf{x}}(u + t\mathbf{v}) \right\| \mathrm{d}t = 2\varepsilon \cdot \sqrt{\mathbf{v}^T \bar{\mathrm{I}}(u) \mathbf{v}},$$

where $\bar{\mathrm{I}}(u)$ is the first fundamental form matrix for $\bar{\mathbf{x}}$ at u. Since arc length is preserved, $s = \bar{s}$ for all $\mathbf{v}$, so

$$\mathbf{v}^T \mathrm{I}(u) \mathbf{v} = \mathbf{v}^T \bar{\mathrm{I}}(u) \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbb{R}^2.$$

This implies $\mathrm{I}(u) = \bar{\mathrm{I}}(u)$, so the first fundamental forms of $\mathbf{x}$ and $\bar{\mathbf{x}}$ agree at u. Since the first fundamental form determines the metric, this means that $\mathrm{d}\varphi_p$ preserves inner products between tangent vectors, i.e., φ is an isometry.

Therefore, φ is an isometry if and only if the arc length of any parametrized curve in S is equal to the arc length of the image curve by φ. □

Exercise 4.2.4. Use the stereographic projection (cf. Exercise 16, Sec. 2-2) to show that the sphere is locally conformal to a plane.

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Solution. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin. The stereographic projection from the north pole $N = (0, 0, 1)$ onto the plane $z = 0$ is given by

$$\pi(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right).$$

This map is smooth on $S^2 \setminus \{N\}$ and its inverse is

$$\pi^{-1}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Let $\mathbf{x}(u, v) = \pi^{-1}(u, v)$ be a parametrization of the sphere minus the north pole. Computing the first fundamental form of this parametrization yields

$$E = G = \frac{4}{(u^2 + v^2 + 1)^2}, \qquad F = 0.$$

Thus, the metric in (u, v) coordinates is conformal to the Euclidean metric

$$\mathrm{I} = \lambda(u, v)^2 (\mathrm{d}u^2 + \mathrm{d}v^2), \quad \text{with } \lambda(u, v) = \frac{2}{u^2 + v^2 + 1}.$$

Therefore, the stereographic projection induces a conformal (angle-preserving) correspondence between the sphere (minus a point) and the plane. This proves that the sphere is locally conformal to a plane. □

Exercise 4.2.8. Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that

$$|G(p) - G(q)| = |p - q| \text{ for all } p, q \in \mathbb{R}^3.$$

(that is, G is a *distance-preserving* map). Prove that there exists $p_0 \in \mathbb{R}^3$ and a linear isometry (cf. Exercise 7) F of the vector space $\mathbb{R}^3$ such that

$$G(p) = F(p) + p_0 \text{ for all } p \in \mathbb{R}^3.$$

Solution. Define $p_0 := G(0)$ and $F(p) = G(p) - p_0$, for all $p \in \mathbb{R}^3$. Then, for all $p, q \in \mathbb{R}^3$, we have

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$$|F(p) - F(q)| = |(F(p) - p_0) - (F(q) - p_0)| = |G(p) - G(q)| = |p - q|.$$

So, F is a distance-preserving map. It also preserves the norms, via

$$\|F(p)\| = \|F(p) - F(0)\| = \|G(p) - G(0)\| = \|p - 0\| = \|p\|,$$

since

$$F(0) = G(0) - p_0 = G(0) - G(0) = 0.$$

We now show that F is a linear map. Since F is distance-preserving and satisfies $F(0) = 0$, it also preserves inner products. To see this, for all $p, q \in \mathbb{R}^3$, we use the polarization identity

$$\langle p, q \rangle = \frac{1}{2}(\|p+q\|^2 - \|p\|^2 - \|q\|^2),$$

and since F preserves norms, we get

$$\langle F(p), F(q) \rangle = \frac{1}{2}(\|F(p) + F(q)\|^2 - \|F(p)\|^2 - \|F(q)\|^2) = \frac{1}{2}(\|p+q\|^2 - \|p\|^2 - \|q\|^2) = \langle p, q \rangle .$$

Hence, F preserves inner products.

Now let $p, q \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then,

$$\|F(p+q) - (F(p) + F(q))\|^2 = \|F(p+q)\|^2 + \|F(p) + F(q)\|^2 - 2\langle F(p+q), F(p) + F(q) \rangle .$$

Because F preserves norms and inner products, this equals

$$\|p+q\|^2 + \|p+q\|^2 - 2\langle p+q, p+q \rangle = 0,$$

so $F(p+q) = F(p) + F(q)$. A similar argument shows $F(\lambda p) = \lambda F(p)$. Hence, F is linear.

Finally, since F is a linear map that preserves inner products, it is a linear isometry of $\mathbb{R}^3$. Therefore, $G(p) = F(p) + p_0$, with F a linear isometry and $p_0 = G(0)$, as desired. □

Exercise 4.2.10. Let S be a surface of revolution. Prove that the rotations about its axis are isometries of S.

Solution. Let $\mathbf{x}(u, v) = (f(v)\cos(u), f(v)\sin(u), g(v))$ be a surface of revolution of the generating curve $\alpha(u, v) = (f(v), g(v))$ along some axis of rotation. Computing th first fundamental form, we have

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = f^2(v) \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (f'(v))^2 + (g'(v))^2, \end{aligned}$$

giving us

$$\mathrm{I}_p(\mathbf{w}) = f^2(v)\,\mathrm{d}u^2 + ((f'(v))^2 + (g'(v))^2)\,\mathrm{d}v^2,$$

for some $p \in S$ and $\mathbf{w} \in T_p(S)$.

+5 Now consider the map $\varphi_\theta(u, v) = (u + \theta, v)$, which corresponds to a rotation about the axis of revolution by a fixed angle θ. This induces a new parametrization of the surface,

$$\bar{\mathbf{x}}(u, v) = \mathbf{x}(\varphi_\theta(u, v)) = \mathbf{x}(u + \theta, v).$$

We compute the first fundamental form of $\bar{\mathbf{x}}$ to get

$$\begin{aligned} \bar{\mathbf{x}}_u(u, v) &= \frac{\partial}{\partial u}\mathbf{x}(u + \theta, v) = \mathbf{x}_u(u + \theta, v) \\ \bar{\mathbf{x}}_v(u, v) &= \frac{\partial}{\partial v}\mathbf{x}(u + \theta, v) = \mathbf{x}_v(u + \theta, v). \end{aligned}$$

Then the coefficients of the first fundamental form for $\bar{\mathbf{x}}$ are

$$\begin{aligned} \bar{E} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = \langle \mathbf{x}_u(u + \theta, v), \mathbf{x}_u(u + \theta, v) \rangle = f^2(v) \\ \bar{F} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle = \langle \mathbf{x}_u(u + \theta, v), \mathbf{x}_v(u + \theta, v) \rangle = 0 \end{aligned}$$

$$\bar{G} = \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle = \langle \mathbf{x}_v(u+\theta, v), \mathbf{x}_v(u+\theta, v) \rangle = (f'(v))^2 + (g'(v))^2.$$

These are identical to the original coefficients E, F, G.

Since the map $\varphi_\theta(u, v) = (u + \theta, v)$ is smooth, has a smooth inverse $\varphi_{-\theta}$, and preserves the first fundamental form everywhere on the domain of the parametrization, it defines a global isometry of the surface. In particular, φ_θ acts globally on S because the surface of revolution is entirely covered by the parametrization $\mathbf{x}(u, v)$ with $u \in [0, 2\pi)$ (mod 2π) and v in an interval I. Thus, the rotations about the axis of revolution define smooth diffeomorphisms from S to itself that preserve the Riemannian metric. Therefore, they are global isometries. □

Exercise 4.2.14. We say that a differentiable map $\varphi : S_1 \to S_2$ *preserves angles* when for every $p \in S_1$ and every pair $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S_1)$ we have

$$\cos(\mathbf{v}_1, \mathbf{v}_2) = \cos(\mathrm{d}\varphi_p(\mathbf{v}_1), \mathrm{d}\varphi_p(\mathbf{v}_2)).$$

Prove that φ is locally conformal if and only if it preserves angles.

Solution. Assume $\varphi : S_1 \to S_2$ is a locally conformal map, i.e., for all $p \in S_1$ and $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S_1)$, we have

$$\langle \mathrm{d}\varphi_p(\mathbf{v}_1), \mathrm{d}\varphi_p(\mathbf{v}_2) \rangle = \lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \tag{1}$$

where $\lambda(p)$ is a nowhere-zero differentiable function on S. Notice that $\|\mathrm{d}\varphi_p(\mathbf{v}_1)\| = \sqrt{\langle \mathrm{d}\varphi_p(\mathbf{v}_1), \mathrm{d}\varphi_p(\mathbf{v}_1) \rangle} = \sqrt{\lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \lambda(p) \|\mathbf{v}_1\|$, and similarly for $\mathbf{v}_2$. Dividing both sides of equation 1 by $\|\mathrm{d}\varphi_p(\mathbf{v}_1)\| \|\mathrm{d}\varphi_p(\mathbf{v}_2)\|$, we obtain

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$$\begin{aligned} \cos(\mathrm{d}\varphi_p(v_1), \mathrm{d}\varphi_p(v_2)) &= \frac{\langle \mathrm{d}\varphi_p(\mathbf{v}_1), \mathrm{d}\varphi_p(\mathbf{v}_2) \rangle}{\|\mathrm{d}\varphi_p(\mathbf{v}_1)\| \|\mathrm{d}\varphi_p(\mathbf{v}_2)\|} \\ &= \frac{\lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\lambda(p) \|\mathbf{v}_1\| \lambda(p) \|\mathbf{v}_2\|} \\ &= \frac{\lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\lambda^2(p) \|\mathbf{v}_1\| \|\mathbf{v}_2\|} \\ &= \frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|} \\ &= \cos(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Therefore, φ preserves angles if φ is locally conformal.

Assume φ preserves angles. Then for all $p \in S_1$ and all $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S_1)$, we have

$$\frac{\langle \mathrm{d}\varphi_p(\mathbf{v}_1), \mathrm{d}\varphi_p(\mathbf{v}_2) \rangle}{\|\mathrm{d}\varphi_p(\mathbf{v}_1)\| \|\mathrm{d}\varphi_p(\mathbf{v}_2)\|} = \frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|}.$$

Let $\mathbf{v} \in T_p(S_1)$ be a nonzero vector, and define

$$\lambda(p) \coloneqq \frac{\|\mathrm{d}\varphi_p(\mathbf{v})\|}{\|\mathbf{v}\|} > 0.$$

We claim that for all $\mathbf{w} \in T_p(S_1)$, we have

$$\|\mathrm{d}\varphi_p(\mathbf{w})\| = \lambda(p) \|\mathbf{w}\| \quad \text{and} \quad \langle \mathrm{d}\varphi_p(\mathbf{v}_1)\, \mathrm{d}\varphi_p(\mathbf{v}_2) \rangle = \lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle.$$

We claim that for all $\mathbf{w} \in T_p(S_1)$, we have

$$\|\mathrm{d}\varphi_p(\mathbf{w})\| = \lambda(p) \|\mathbf{w}\| \quad \text{and} \quad \langle \mathrm{d}\varphi_p(\mathbf{v}_1), \mathrm{d}\varphi_p(\mathbf{v}_2) \rangle = \lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle.$$

To see this, fix any $\mathbf{w} \in T_p(S_1)$ and apply angle preservation to the pair $\mathbf{v}, \mathbf{w}$. Then

$$\cos(\mathbf{v}, \mathbf{w}) = \cos(\mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w})) = \frac{\langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle}{\|\mathrm{d}\varphi_p(\mathbf{v})\| \, \|\mathrm{d}\varphi_p(\mathbf{w})\|}.$$

Substituting $\lambda(p) = \|\mathrm{d}\varphi_p(\mathbf{v})\| / \|\mathbf{v}\|$ and rearranging gives

$$\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \, \|\mathbf{w}\|} = \frac{\langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle}{\lambda(p) \, \|\mathbf{v}\| \, \|\mathrm{d}\varphi_p(\mathbf{w})\|} \Rightarrow \langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle = \lambda(p) \langle \mathbf{v}, \mathbf{w} \rangle \cdot \frac{\|\mathrm{d}\varphi_p(\mathbf{w})\|}{\|\mathbf{w}\|}.$$

So

$$\frac{\|\mathrm{d}\varphi_p(\mathbf{w})\|}{\|\mathbf{w}\|} = \lambda(p),$$

and hence

$$\|\mathrm{d}\varphi_p(\mathbf{w})\| = \lambda(p) \, \|\mathbf{w}\|.$$

Now that both $\mathbf{v}$ and $\mathbf{w}$ scale by $\lambda(p)$, it follows that

$$\langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle = \lambda^2(p) \langle \mathbf{v}, \mathbf{w} \rangle.$$

Since $\mathbf{v}, \mathbf{w}$ were arbitrary, this proves that φ is locally conformal at p.

Therefore, φ is locally conformal if and only if it preserves angles. □

Exercise 4.2.15. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(x, y) = (u(x, y), v(x, y))$, where u and v are differentiable functions that satisfy the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Show that φ is a local conformal map from $\mathbb{R}^2 - Q$ into $\mathbb{R}^2$, where $Q = \{(x, y) \in \mathbb{R}^2 \mid u_x^2 + u_y^2 = 0\}$

Solution. Let $\varphi(x, y) = (u(x, y), v(x, y))$ be a differentiable map satisfying the Cauchy-Riemann equations. Then, the Jacobian matrix $D\varphi$ at a point (x, y) is

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$$D\varphi = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix},$$

since φ satisfies the Cauchy-Riemann equations.

This matrix is of the form

$$D\varphi = u_x \begin{bmatrix} 1 & \frac{u_y}{u_x} \\ -\frac{u_y}{u_x} & 1 \end{bmatrix} \quad (\text{when } u_x \neq 0).$$

More importantly, for all points not in $Q = \{(x, y) \mid u_x^2 + u_y^2 = 0\}$, the Jacobian is nonzero and we can interpret $D\varphi$ as a similarity transformation: a composition of a rotation and a scaling.

Indeed, the Jacobian satisfies

$$D\varphi^\top D\varphi = \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} = \begin{bmatrix} u_x^2 + u_y^2 & 0 \\ 0 & u_x^2 + u_y^2 \end{bmatrix} = (u_x^2 + u_y^2)\mathrm{I}.$$

This shows that the differential of φ preserves angles and scales all vectors by the same factor locally – that is, it is conformal.

Therefore, φ is a local conformal map on $\mathbb{R}^2 \setminus Q$. □