Proposition: If $T: V \to W$ is a linear transformation, then $T(\vec{0}_v) = \vec{0}_w$

<u>Proof</u>: Denote $\vec{\omega} = T(\vec{0}_v) \in W$.

$$\vec{\omega} = T(\vec{0}_v + \vec{0}_v) = T(\vec{0}_v) + T(\vec{0}_v) = 2\vec{\omega}$$

$$\Rightarrow \vec{\omega} = 2\vec{\omega} \Rightarrow 2\vec{\omega} - \vec{\omega} = \vec{0} \Rightarrow \vec{\omega} = \vec{0} \in W.$$

Example: 1) Translation. Let $\vec{a} \in \mathbb{R}^n$ be a non-zero vector. Define $T_{\vec{a}}: \mathbb{R}^n \to \mathbb{R}^n$ by $T_{\vec{a}}(\vec{x}) = \vec{x} + \vec{a}$. Then $T_{\vec{a}}(\vec{0}) = \vec{a} \neq \vec{0}$

$\Rightarrow T_{\vec{a}}$ is NOT a linear transformation.

2) Let $\vec{a} \in \mathbb{R}^m$ be a non-zero vector. Define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = \vec{a}$. Then T is not a linear transformation.

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Proposition: Let V be a finite-dimensional vector space with a basis $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$. If $U, T: V \to W$ are linear transformations such that $U(\vec{v}_i) = T(\vec{v}_i)$ for all $i = 1, \cdots, n$. Then $U = T$.

Remarks: This proposition implies that a linear transformation is uniquely determined by its actions on a basis of the domain space.

Proof: We need to prove $U(\vec{x}) = T(\vec{x})$ for any $\vec{x} \in V$.

$\forall \vec{x} \in V$. As $B = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, there exists $a_1, \cdots, a_n$ such that: $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$

Then $T(\vec{x}) = T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)$

$$= a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n) \quad \text{T is a linear transformation}$$

$$= a_1 U(\vec{v}_1) + \cdots + a_n U(\vec{v}_n) \quad T(\vec{v}_i) = U(\vec{v}_i) \text{ for all } i=1,\cdots,n.$$

$$= U(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \quad \text{U is a linear transformation}$$

$$= U(\vec{x}).$$

$\boxtimes$

Definition: Let V and W be vector spaces.

Let $\mathcal{L}(V, W)$ be the set of all linear transformations from V to W. For any $U, T \in \mathcal{L}(V, W)$, and any $c \in \mathbb{R}$ (or $\mathbb{C}$), define

$$(U+T)(\vec{x}) = U(\vec{x}) + T(\vec{x}) \quad \forall \vec{x} \in V. \rightarrow \text{Linear transformation addition}$$

$$(cT)(\vec{x}) = c\,T(\vec{x}) \quad \forall \vec{x} \in V. \rightarrow \text{Linear transformation scalar multiplication}$$

Proposition: $\mathcal{L}(V, W)$ along with "+" and scalar multiplication defined above forms a vector space.

(Proof: Homework. Need to check Axiom 0 – VIII are all satisfied)

Definition: Let V, W be vector spaces. Let $T: V \to W$ be a linear transformation

1). The null space or kernel of T, denoted by $Null(T)$ or $Ker(T)$ is defined to be the set of vectors $\vec{v} \in V$ s.t. $T(\vec{v}) = \vec{0}$

i.e. $Ker(T) = \{ \vec{v} \in V : T(\vec{v}) = \vec{0} \}$

2). The range of T, denoted by $Range(T)$, is defined to be the subset of W consisting of all images of T, i.e.

$$Range\, T = \{ \vec{y} \in W : \exists \vec{v} \in V \text{ s.t. } \vec{y} = T(\vec{v}) \}$$

Example: 1). $Ker(Id) = \{\vec{0}\}$, $Range(Id) = V$

$Ker(T_{\vec{0}}) = V$, $Range(T_{\vec{0}}) = \{\vec{0}\}$

2). Let $A \in \mathbb{R}^{m \times n}$ and $L_A: \mathbb{R}^n \to \mathbb{R}^m$ be the "left-multiplication by A" linear transformation.

$Ker(L_A) = Null(A)$, $Range(L_A) = Range(A) =$ the column space of A.

Theorem: Let $T: V \to W$ be a linear transformation. Then

1): $Ker(T)$ is a subspace of V.

2). $Range(T)$ is a subspace of W.

<u>Proof</u>: 1) $\forall \vec{x}, \vec{y} \in Ker(T)$ and $c \in \mathbb{R}$ (or $\mathbb{C}$) $T(\vec{x}) = \vec{0}$ and $T(\vec{y}) = \vec{0}$

Then $T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y}) = c\vec{0} + \vec{0} = \vec{0}$

$\Rightarrow c\vec{x} + \vec{y} \in Ker(T)$. $\Rightarrow Ker(T) \subseteq V$ is a subspace

2) $\forall \vec{w}_1, \vec{w}_2 \in Range(T)$ and $c \in \mathbb{R}$ (or $\mathbb{C}$).

$\exists \vec{v}_1, \vec{v}_2 \in V$ such that: $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$

$\Rightarrow \quad c\vec{w}_1 + \vec{w}_2 = cT(\vec{v}_1) + T(\vec{v}_2) = T(c\vec{v}_1 + \vec{v}_2)$

$\Rightarrow \quad c\vec{w}_1 + \vec{w}_2 \in \text{Range}(T)$

$\Rightarrow$ Range$(T) \subseteq W$ is a subspace. $\boxtimes$

Definition: The dimension of Ker(T) is called the nullity of T.
The dimension of Range(T) is called the rank of T.

Theorem: (Rank-nullity Theorem or Dimension Theorem). Let V be a finite-dimensional vector space. Let $T: V \to W$ be a linear transformation. Then

$$\dim V = \text{nullity}(T) + \text{rank}(T).$$

<u>Proof</u>: Suppose dimension $V = n$, and nullity$(T) = k$. We need to prove that rank$(T) = n-k$.

Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\} \subseteq \text{Ker}(T)$ be a basis.

Extend $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ to a basis of V: $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$

(Note this extension can be carried out by Replacement Theorem.).

Claim: $S = \{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\} \subseteq \text{Range}(T)$ is a basis of Range(T).

Followed by the claim: Rank$(T) = n-k$. Therefore

Thus $\dim V = n$, nullity$(T) = k$, rank$(T) = n-k$

$\Rightarrow \quad \dim V = \text{nullity}(T) + \text{rank}(T)$

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