Introduction to Toplogy I: Homework 3

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Nicolas Addington

Hashem A. Damrah
UO ID: 952102243

Exercise 2.2. Let (X, d) be a metric space and $A \subset X$. Use Proposition 2.6 to prove that

$$\partial A = \partial(X \setminus A) = \overline{A} \cap \overline{X \setminus A}.$$

Solution. Assume $x \in \partial A$. By definition, that means $x \in \overline{A} \setminus \operatorname{int} A$. Then, $x \in \overline{A}$ and $x \notin \operatorname{int} A$. By Proposition 2.6, we have $x \notin \operatorname{int} A$ if and only if $x \in \overline{X \setminus A}$. Therefore, $x \in \overline{A}$ and $x \in \overline{X \setminus A}$, which implies

$$x \in \overline{A} \cap \overline{X \setminus A}.$$

Hence, $\partial A \subset \overline{A} \cap \overline{X \setminus A}$.

Now, for the converse. Assume $x \in \overline{A} \cap \overline{X \setminus A}$. Then, $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. By Proposition 2.6 again, $x \in \overline{X \setminus A}$ if and only if $x \notin \operatorname{int} A$. Therefore, $x \in \overline{A}$ and $x \notin \operatorname{int} A$, which means $x \in \partial A$. Hence,

$$\overline{A} \cap \overline{X \setminus A} \subset \partial A.$$

Combining both inclusions, we conclude that

$$\partial A = \overline{A} \cap \overline{X \setminus A}.$$

Finally, since the expression is symmetric in A and $X \setminus A$, it follows that

$$\partial A = \partial(X \setminus A). \qquad \square$$

Exercise 2.3. Prove the analogue of Proposition 2.7 for closures, without appealing to Proposition 2.6.

(i) If $A \subset B$, then $\bar{A} \subset \bar{B}$.

(ii) $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

(iii) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

Give an example to show that the inclusion can be strict.

(iv) $\bar{\bar{A}} = \bar{A}$.

Solution to (i). Assume $A \subset B$. Let $p \in \bar{A}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects A. Since $A \subset B$, it follows that $B_r(p)$ also intersects B. Therefore, for every $r > 0$, the open ball $B_r(p)$ intersects B, which means $p \in \bar{B}$. Hence, $\bar{A} \subset \bar{B}$. $\square$

Solution to (ii). Assume $p \in \bar{A} \cup \bar{B}$. Without loss of generality, assume $p \in \bar{A}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects A. Since $A \subset A \cup B$, it follows that $B_r(p)$ also intersects $A \cup B$. Therefore, for every $r > 0$, the open ball $B_r(p)$ intersects $A \cup B$, which means $p \in \overline{A \cup B}$. Hence, $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

Now, let $p \in \overline{A \cup B}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects $A \cup B$. This means that for each $r > 0$, there exists a point in either A or B that lies within the ball. If there are infinitely many such points in A, then p is a limit point of A and thus belongs to $\bar{A}$. Similarly, if there are infinitely many such points in B, then p is a limit point of B and thus belongs to $\bar{B}$. In either case, we have $p \in \bar{A} \cup \bar{B}$. Hence, $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

Combining both inclusions, we conclude that $\bar{A} \cup \bar{B} = \overline{A \cup B}$. $\square$

Solution to (iii). Assume $p \in \overline{A \cap B}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects $A \cap B$. This means that for each $r > 0$, there exists a point in both A and B that lies within the ball. Therefore, for every $r > 0$, the open ball $B_r(p)$ intersects A and also intersects B. This implies that $p \in \bar{A}$ and $p \in \bar{B}$. Hence, $p \in \bar{A} \cap \bar{B}$, and we conclude that

$$\overline{A \cap B} \subset \bar{A} \cap \bar{B}.$$

To show that the inclusion can be strict, consider the metric space $(\mathbb{R}, \mathrm{d})$ with the usual metric. Let

$$A = (0,1) \quad \text{and} \quad B = (1,2).$$

Then,

$$A \cap B = (0,1) \cap (1,2) = \emptyset,$$

so

$$\overline{A \cap B} = \bar{\emptyset} = \emptyset.$$

However,

$$\bar{A} = [0,1] \quad \text{and} \quad \bar{B} = [1,2],$$

so

$$\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}.$$

Thus, $\overline{A \cap B} = \emptyset \subsetneq \{1\} = \bar{A} \cap \bar{B}$. □

Solution to (iv). Assume $p \in \bar{\bar{A}}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects $\bar{A}$. This means that for each $r > 0$, there exists a point in $\bar{A}$ that lies within the ball. Since $\bar{A}$ is the closure of A, it follows that for every $r > 0$, the open ball $B_r(p)$ also intersects A. Therefore, $p \in \bar{A}$. Hence, $\bar{\bar{A}} \subset \bar{A}$.

Now, let $p \in \bar{A}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects A. Since $A \subset \bar{A}$, it follows that for every $r > 0$, the open ball $B_r(p)$ also intersects $\bar{A}$. Therefore, $p \in \bar{\bar{A}}$. Hence, $\bar{A} \subset \bar{\bar{A}}$.

Combining both inclusions, we conclude that $\bar{\bar{A}} = \bar{A}$. □

Exercise 2.4. Define the closed ball

$$\bar{B}_r(p) = \{q \in X \mid \mathrm{d}(p,q) \leq r\}.$$

(i) Prove that $\bar{B}_r(p)$ equals its own closure.

(ii) Prove that $\overline{B_r(p)} \subset \bar{B}_r(p)$: that is, the closure of the open ball is contained in the closed ball. But give an example to show that the inclusion can be strict.

Hint: For the proof, you may quote Exercise 2.3(a). For the example, you might take $X = \mathbb{Z}$ with the usual metric inherited from $\mathbb{R}$, or any set with a discrete metric (Exercise 1.4).

Solution to (i). If $q \in \overline{\bar{B}_r(p)}$ then for every $s > 0$ there exists $x \in \bar{B}_r(p)$ with $\mathrm{d}(q,x) < s$. Thus $\mathrm{d}(p,q) \leq \mathrm{d}(p,x) + \mathrm{d}(x,q) \leq r + s$ for every $s > 0$, so $\mathrm{d}(p,q) \leq r$ and $q \in \bar{B}_r(p)$. Conversely, if $q \in \bar{B}_r(p)$ then for every $s > 0$ we have $q \in B_s(q) \cap \bar{B}_r(p)$, so $q \in \overline{\bar{B}_r(p)}$. Hence $\overline{\bar{B}_r(p)} = \bar{B}_r(p)$. □

Solution to (ii). Let $q \in \overline{B_r(p)}$. By definition of closure (or Exercise 2.3(a)), every open neighbourhood of q meets $B_r(p)$. Hence for each $s > 0$ there exists a point $x_s \in B_r(p)$ with

$$\mathrm{d}(q, x_s) < s \quad \text{and} \quad \mathrm{d}(p, x_s) < r.$$

By the triangle inequality,

$$\mathrm{d}(p,q) \leq \mathrm{d}(p, x_s) + \mathrm{d}(x_s, q) < r + s.$$

This inequality holds for every $s > 0$. If $d(p,q) > r$ then choosing $s < \mathrm{d}(p,q) - r$ would contradict $\mathrm{d}(p,q) < r + s$. Thus we must have $\mathrm{d}(p+q) \leq r$, so $q \in \bar{B}_r(p)$. Since q was arbitrary in $\overline{B_r(p)}$, we obtain $\overline{B_r(p)} \subset \bar{B}_r(p)$.

As for the example where the inclusion is strict, take $X = \mathbb{Z}$ with the metric inherited from $\mathbb{R}$ let $p = 0$ and $r = 1$.

$$B_1(0) = \{n \in \mathbb{Z} \mid |n| < 1\} = \{0\},$$

so $\overline{B_1(0)} = \{0\}$ (closure in X is still $\{0\}$). But

$$\bar{B}_1(0) = \{n \in \mathbb{Z} \mid |n| \leq 1\} = \{-1, 0, 1\}.$$

Hence $\overline{B_1(0)} = \{0\} \subsetneq \{-1, 0, 1\} = \bar{B}_1(0)$, so the inclusion can be strict. □

Exercise 3.1. In Example 1.4 we saw three different metrics on $\mathbb{R}^2$. Prove one of the following:

(i) A subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the taxicab metric.

(ii) A subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the square metric.

(iii) A subset $A \subset \mathbb{R}^2$ is open in the taxicab metric if and only if it is open in the square metric.

Solution. We show that a subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the taxicab metric. Notice that we have

$$\mathrm{d}_2(x, y) \leq \mathrm{d}_1(x, y) \leq \sqrt{2}\,\mathrm{d}_2(x, y),$$

which holds for all $x, y \in \mathbb{R}^2$. We will use these inequalities to show that openness in one metric implies openness in the other.

Suppose that A is open in the Euclidean metric. Let $x \in A$. Then, by definition of openness, there exists $r > 0$ such that the Euclidean open ball

$$B_2(x, r) = \{y \in \mathbb{R}^2 : \mathrm{d}_2(x, y) < r\},$$

is contained in A. Now consider any point $y \in B_1(x, r)$, where

$$B_1(x, r) = \{y \in \mathbb{R}^2 : \mathrm{d}_1(x, y) < r\}.$$

Using the inequality $\mathrm{d}_2(x, y) \leq \mathrm{d}_1(x, y)$, we have $\mathrm{d}_2(x, y) < r$ whenever $\mathrm{d}_1(x, y) < r$. Hence $B_1(x, r) \subseteq B_2(x, r) \subseteq A$. This shows that for every $x \in A$, there exists an $r > 0$ such that $B_1(x, r) \subset A$. Therefore, A is open in the taxicab metric.

Conversely, suppose that A is open in the taxicab metric. Let $x \in A$. Then there exists $r > 0$ such that

$$B_1(x, r) = \{y \in \mathbb{R}^2 : \mathrm{d}_1(x, y) < r\} \subseteq A.$$

Using the inequality $\mathrm{d}_1(x, y) \leq \sqrt{2}\,\mathrm{d}_2(x, y)$, we see that if $\mathrm{d}_2(x, y) < r/\sqrt{2}$, then $\mathrm{d}_1(x, y) < r$, and hence $y \in B_1(x, r) \subseteq A$. Therefore,

$$B_2\left(x, \frac{r}{\sqrt{2}}\right) \subseteq B_1(x, r) \subseteq A.$$

This shows that each point $x \in A$ has a Euclidean neighborhood contained in A, so A is open in the Euclidean metric.

Thus a subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the taxicab metric. □

Exercise 3.2. Let $X = \mathbb{Q}$ with the metric induced from the usual one on $\mathbb{R}$: that is, $\mathrm{d}(x, y) = |x - y|$ for all $x, y \in \mathbb{Q}$, but we're thinking about $\mathbb{Q}$ in itself and forgetting about the rest of $\mathbb{R}$.

(i) Prove that the subset

$$\{x \in \mathbb{Q} \mid x^2 < 1\},$$

is open but not closed.

(ii) Prove that the subset

$$\{x \in \mathbb{Q} \mid x^2 \leq 2\},$$

is both open and closed

(You may use the fact that $\sqrt{2}$ is irrational without proving it.)

Solution to (i). Let $A = \{x \in \mathbb{Q} \mid x^2 < 1\}$. We will show that A is open in $\mathbb{Q}$.

Clearly, $A = (-1, 1) \cap \mathbb{Q}$. Let $p \in A$. Then, $p \in (-1, 1)$, so there exists an $\varepsilon > 0$ such that the open interval $(p - \varepsilon, p + \varepsilon)$ is contained in $(-1, 1)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose ε small enough so that $(p - \varepsilon, p + \varepsilon) \cap \mathbb{Q} \subset A$. Therefore, for every $p \in A$, there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(p) \cap \mathbb{Q} \subset A$. This shows that A is open in $\mathbb{Q}$.

Next, we will show that A is not closed in $\mathbb{Q}$. The closure of A in $\mathbb{Q}$ is given by

$$\overline{A} = \{x \in \mathbb{Q} \mid x^2 \leq 1\}.$$

This is because the points -1 and 1 are limit points of A in $\mathbb{R}$, but they are not in $\mathbb{Q}$. Since $-1, 1 \notin A$, we have $\overline{A} \neq A$. Therefore, A is not closed in $\mathbb{Q}$.

Hence, we conclude that A is open but not closed in $\mathbb{Q}$. □

Solution to (ii). Let $B = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$. We will show that B is both open and closed in $\mathbb{Q}$.

First, we show that B is closed in $\mathbb{Q}$. The closure of B in $\mathbb{Q}$ is given by

$$\overline{B} = \{x \in \mathbb{Q} \mid x^2 \leq 2\}.$$

This is because the points $-\sqrt{2}$ and $\sqrt{2}$ are limit points of B in $\mathbb{R}$, but they are not in $\mathbb{Q}$. Since $-\sqrt{2}, \sqrt{2} \notin B$, we have $\overline{B} = B$. Therefore, B is closed in $\mathbb{Q}$.

Next, we show that B is open in $\mathbb{Q}$. Let $p \in B$. Then, $p^2 \leq 2$, so there exists an $\varepsilon > 0$ such that the open interval $(p - \varepsilon, p + \varepsilon)$ is contained in the interval $(-\sqrt{2}, \sqrt{2})$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose ε small enough so that $(p - \varepsilon, p + \varepsilon) \cap \mathbb{Q} \subset B$. Therefore, for every $p \in B$, there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(p) \cap \mathbb{Q} \subset B$. This shows that B is open in $\mathbb{Q}$.

Hence, we conclude that B is both open and closed in $\mathbb{Q}$. □

Exercise 3.3. Let $A \subset C^1([0, 1])$ be the set of functions with simple roots as in Example 3.3. Prove that A is open in the C^1 metric.

Hint: For a given $f \in A$, take the ball of radius

$$r = \inf_{x \in [0,1]} (|f(x)| + |f'(x)|)).$$

Solution. We show that the set $A \subset C^1([0, 1])$ consisting of all functions with simple roots is open in the C^1 metric. Recall that for $f, g \in C^1([0, 1])$, the C^1 metric is given by

$$\mathrm{d}(f, g) = \max_{x \in [0,1]} |f(x) - g(x)| + \max_{x \in [0,1]} |f'(x) - g'(x)|.$$

A function $f \in C^1([0, 1])$ has a *simple root* at x_0 if $f(x_0) = 0$ and $f'(x_0) \neq 0$. The set A is defined as

$$A = \{f \in C^1([0, 1]) : f(x) = 0 \Rightarrow f'(x) \neq 0\}.$$

Let $f \in A$. We must show that there exists $r > 0$ such that if $g \in C^1([0, 1])$ satisfies $\mathrm{d}(f, g) < r$, then $g \in A$. Taking

$$r = \inf_{x \in [0,1]} (|f(x)| + |f'(x)|),$$

notice that $r > 0$ must hold. This is because f is continuously differentiable, implying both $|f(x)|$ and $|f'(x)|$ are continuous on $[0, 1]$, meaning their sum is continuous and attains a minimum. If this minimum were zero, then there would exist $x_0 \in [0, 1]$ such that $f(x_0) = 0$ and $f'(x_0) = 0$, contradicting the assumption that all roots of f are simple. Hence $r > 0$.

Now suppose $g \in C^1([0, 1])$ satisfies $\mathrm{d}(f, g) < r$. This means that for all $x \in [0, 1]$,

$$|f(x) - g(x)| < r \quad \text{and} \quad |f'(x) - g'(x)| < r.$$

We claim that g also has only simple roots. Suppose, for contradiction, that there exists $x_0 \in [0,1]$ such that $g(x_0) = 0$ and $g'(x_0) = 0$. Then

$$|f(x_0)|+|f'(x_0)|= |f(x_0) - g(x_0)|+|f'(x_0) - g'(x_0)|,$$

Since both $|f(x_0) - g(x_0)|$ and $|f'(x_0) - g'(x_0)|$ are strictly less than r, we obtain

$$|f(x_0)|+|f'(x_0)|< 2r.$$

However, by the definition of r as the infimum of $|f(x)|+|f'(x)|$, we must have $|f(x_0)|+|f'(x_0)|\geq r$, giving us

$$r \leq |f(x_0)|+|f'(x_0)|< 2r,$$

which is not a contradiction in itself. But notice that if $\mathrm{d}(f,g) < r/2$, then both $|f(x_0) - g(x_0)|$ and $|f'(x_0) - g'(x_0)|$ are less than $r/2$, giving

$$|f(x_0)|+|f'(x_0)|\leq |f(x_0) - g(x_0)|+|f'(x_0) - g'(x_0)|< r,$$

contradicting the definition of r as the infimum of $|f(x)|+|f'(x)|$. Therefore, for all g with $\mathrm{d}(f,g) < r/2$, no such x_0 can exist, and g must have only simple roots.

Hence, for each $f \in A$, the C^1 ball

$$B\left(f, \frac{r}{2}\right) = \{g \in C^1([0,1]) : \mathrm{d}(f,g) < r/2\},$$

is contained in A. Therefore, A is open in the C^1 metric. □

Exercise 3.5. Without using Proposition 3.9,

(i) Prove that if $U, V \subset X$ are open, then the intersection is again open.

(ii) Give an example of a metric space (X, d) and countably many open sets $U_1, U_2, U_3, \dots \subset X$ such that their intersection $U_1 \cap U_2 \cap U_3 \cap \cdots$ is not open.

(iii) Let I be a set, and suppose that for each $i \in I$, we have an open set $U_i \subset X$. Prove that the union $\bigcup_{i\in I} U_i$ is again open.

(Don't assume that the index set I is countable!)

Solution to (i). Assume $U, V \subset X$ are open sets. Let $p \in U \cap V$. Since U is open, there exists an $\varepsilon_1 > 0$ such that the open ball $B_{\varepsilon_1}(p) \subset U$. Similarly, since V is open, there exists an $\varepsilon_2 > 0$ such that the open ball $B_{\varepsilon_2}(p) \subset V$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then, the open ball $B_\varepsilon(p)$ is contained in both U and V, i.e., $B_\varepsilon(p) \subset U \cap V$. Therefore, for every point $p \in U \cap V$, there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(p) \subset U \cap V$. This shows that $U \cap V$ is open. □

Solution to (ii). Consider the metric space $(\mathbb{R}, \mathrm{d})$, where d is the standard Euclidean metric. For each $n \in \mathbb{N}$, define

$$U_n = \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Each U_n is open in $\mathbb{R}$ because it is an open interval. Observe that the sequence of sets is nested, i.e.,

$$U_1 \supset U_2 \supset U_3 \supset \cdots.$$

Now consider their intersection:

$$\bigcap_{n=1}^{\infty} U_n = \{0\}.$$

The set $\{0\}$ is not open in $\mathbb{R}$, since for any $\varepsilon > 0$, the open ball $B_\varepsilon(0) = (-\varepsilon, \varepsilon)$ contains points other than 0. Therefore, no open ball centered at 0 is contained in $\{0\}$. □

Solution to (iii). Let I be an index set, and for each $i \in I$, let $U_i \subset X$ be an open set. Let $p \in \bigcup_{i \in I} U_i$. Then, there exists some index $j \in I$ such that $p \in U_j$. Since U_j is open, there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(p) \subset U_j$. Since $U_j \subset \bigcup_{i \in I} U_i$, it follows that $B_\varepsilon(p) \subset \bigcup_{i \in I} U_i$. Therefore, for every point $p \in \bigcup_{i \in I} U_i$, there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(p) \subset \bigcup_{i \in I} U_i$. This shows that $\bigcup_{i \in I} U_i$ is open. $\square$