1. Use the following conclusion to solve the given problems.

Let $A \in \mathbb{R}^{n\times m}$ *and* $B \in \mathbb{R}^{m\times n}$ *with* $n \geq m$*. Then* $\det(\lambda I_n - AB) = \lambda^{n-m}\det(\lambda I_m - BA)$.

1). Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}^T\mathbf{x} = 1$. Find the eigenvalues for $I_n - 2\mathbf{x}\mathbf{x}^T$.

$$\begin{aligned}\det(\lambda I_n - (I_n - 2\mathbf{x}\mathbf{x}^T)) &= \det((\lambda - 1)I_n + 2\mathbf{x}\mathbf{x}^T)\\ &= (\lambda-1)^{n-1}\det[(\lambda-1)I + 2\mathbf{x}^T\mathbf{x}]\\ &= (\lambda-1)^{n-1}(\lambda+1).\end{aligned}$$

Therefore $I_n - 2\mathbf{x}\mathbf{x}^T$ has two eigenvalues: $\lambda = 1$ with multiplicity $n-1$, and $\lambda = -1$ with multiplicity 1.

2). Let $\mathbf{x} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mathbb{R}^n$, and $\mathbf{y} = \begin{pmatrix} b_1 \\ \cdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$. Find the eigenvalues for $I_n - \mathbf{x}\mathbf{y}^T$.

$$\begin{aligned}\det(\lambda I_n - (I_n - \mathbf{x}\mathbf{y}^T)) &= \det[(\lambda-1)I_n + \mathbf{x}\mathbf{y}^T]\\ &= (\lambda-1)^{n-1}\det(\lambda - 1 + \mathbf{y}^T\mathbf{x})\end{aligned}$$

Thus $I_n - \mathbf{x}\mathbf{y}^T$ has two eigenvalues: $\lambda = 1$ with multiplicity $n-1$, and $\lambda = 1 - a_1b_1 - a_2b_2 - \cdots - a_nb_n$ with multiplicity 1

2. Prove that an upper triangular matrix with zeros in all the diagonal entries is nilpotent. (Note: A matrix A is nilpotent if and only if there exists a positive integer k such that A^k=0.)

Assume that A is an $n \times n$ upper triangular matrix with zero in all the diagonal entries. Then the characteristic polynomial of A is $f(\lambda) = \det(A - \lambda I) = (-1)^n \lambda^n$. By Cayley-Hamilton theorem $f(A) = (-1)^n A^n = 0$, or $A^n = 0$. Therefore A is nilpotent.

3. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Prove that $A^{-1} = g(A)$, for some polynomial $g(x)$ with $\deg(g(x)) = n - 1$.

Suppose the characteristic polynomial of A is

$$f(\lambda) = \det(A - \lambda I) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

Then on one hand $f(0) = \det(A) = a_0$. As A is invertible, $\det(A) =\neq 0$. Thus $a_0 \neq 0$. On the other hand, by Cayley-Hamilton theorem $f(A) = 0$, i.e.

$$f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

As $a_0 \neq 0$, the above equation is equivalent to

$$-\frac{1}{a_0}[(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A] = I$$

$$-\frac{1}{a_0} \cdot A \cdot [(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I] = I$$

Thus $A^{-1} = -\frac{1}{a_0}[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I]$.

4. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Define

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2 \qquad (*).$$

Prove that $(*)$ defines an inner product on $\mathbb{R}^2$.

We will check all the four axioms of an inner production for the above operation $*$.

1). Let $\mathbf{x} = (x_1, x_2)$, then $(\mathbf{x}, \mathbf{x}) = x_1x_1 - x_2x_1 - x_1x_2 + 4x_2x_2 = (x_1 - x_2)^2 + 3x_2^2 \geq 0$ for any $x_1, x_2 \in \mathbb{R}$. $(x_1 - x_2)^2 + 3x_2^2 = 0$ if and only if $x_1 - x_2 = 0$ and $x_2 = 0$, i.e. $\mathbf{x} = (x_1, x_2) = \mathbf{0}$.

2). Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Then

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2 \quad \text{and} \quad (\mathbf{y}, \mathbf{x}) = y_1x_1 - y_2x_1 - y_1x_2 + 4y_2x_2$$

Compare the above terms, one notice that $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$.

3). Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Then $c\mathbf{x} = (cx_1, cx_2)$.

$$(c\mathbf{x}, \mathbf{y}) = (cx_1)y_1 - (cx_2)y_1 - (cx_1)y - 2 + 4(cx_2)y_2 = c(x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2) = c(\mathbf{x}, \mathbf{y}).$$

4). Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$. Then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)$. Then

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}, \mathbf{z}) &= (x_1 + y_1)z_1 - (x_2 + y_2)z_1 - (x_1 + y_1)z_2 + 4(x_2 + y_2)z_2 \\
&= (x_1z_1 - x_2z_1 - x_1z_2 + 4x_2z_2) + (y_1z_1 - y_2z_1 - y_1z_2 + 4y_2z_2) \\
&= (\mathbf{x}, \mathbf{z}) + (\mathbf{y}, \mathbf{z}).
\end{aligned}$$

By 1), 2), 3) and 4), $*$ defines an inner product on $\mathbb{R}^2$.

5. Let $A \in \mathbb{C}^{n\times n}$ and assume A is Hermitian positive-definite. Prove that $(\mathbf{x},\mathbf{y}) = \mathbf{y}^* A\mathbf{x}$ defines an inner product on $\mathbb{C}^n$.

We will check all the four axioms of an inner production for the above operation $(\mathbf{x},\mathbf{y}) = \mathbf{y}^* A\mathbf{x}$

1). Since A is positive definite, $(\mathbf{x},\mathbf{x}) = \mathbf{x}^* A\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$ and thus $(\mathbf{x},\mathbf{x}) = \mathbf{x}^* A\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

2). Since $A^* = A$, and the property that the conjugate transpose of a product of matrices is equal to product of conjugate transpose of the matrices in the product in reverse order, we have

$$(\mathbf{y},\mathbf{x}) = \mathbf{x}^* A\mathbf{y} = (\mathbf{y}^* A^*\mathbf{x})^* = (\mathbf{y}^* Ax)^* = [(\mathbf{x},\mathbf{y})]^*$$

As $(\mathbf{x},\mathbf{y})$ is a number (i.e. 1×1 matrix, $[(\mathbf{x},\mathbf{y})]^* = \overline{(\mathbf{x},\mathbf{y})}$. Thus$(\mathbf{y},\mathbf{x}) = \overline{(\mathbf{x},\mathbf{y})}$.

3). For any $c \in \mathbb{C}$: $(c\mathbf{x},\mathbf{y}) = \mathbf{y}^* A(c\mathbf{x}) = c(\mathbf{y}^* A\mathbf{x}) = c(\mathbf{x},\mathbf{y})$.

4). For any $\mathbf{x},\mathbf{y}$ and $\mathbf{z}$:

$$(\mathbf{x}+\mathbf{y},\mathbf{z}) = \mathbf{z}^* A(\mathbf{x}+\mathbf{y}) = \mathbf{z}^* A\mathbf{x} + \mathbf{z}^* A\mathbf{y} = (\mathbf{x},\mathbf{z}) + (\mathbf{y},\mathbf{z}).$$

By 1), 2), 3) and 4), $(\mathbf{x},\mathbf{y}) = \mathbf{y}^* A\mathbf{x}$ defines an inner product on $\mathbb{C}^n$.

6. Verify the following polarization identity for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$(\mathbf{x}, \mathbf{y}) = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2.$$

Since $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$.

$$\begin{aligned}
&\frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2 \\
&= \frac{1}{4}(\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) - \frac{1}{4}(\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}) \\
&= \frac{1}{4}[(\mathbf{x}, \mathbf{x}) + (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) - [(\mathbf{x}, \mathbf{x}) - (\mathbf{x}, \mathbf{y}) - (\mathbf{y}, \mathbf{x}) + (\mathbf{y}, \mathbf{y})] \\
&= \frac{1}{4}[2(\mathbf{x}, \mathbf{y}) + 2(\mathbf{y}, \mathbf{x})] \\
&= \frac{1}{4} \cdot 4(\mathbf{x}, \mathbf{y}) \\
&= (\mathbf{x}, \mathbf{y})
\end{aligned}$$

7. Let V be an inner product space. Prove the following triangular inequality for any $\mathbf{x}, \mathbf{y} \in V$:

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||.$$

It is equivalent to prove that

$$(||\mathbf{x} + \mathbf{y}||)^2 \leq (||\mathbf{x}|| + ||\mathbf{y}||)^{\mathbf{2}} = ||\mathbf{x}||^{\mathbf{2}} + \mathbf{2}||\mathbf{x}||||\mathbf{y}|| + ||\mathbf{y}||^{\mathbf{2}}.$$

First notice that

$$\mathrm{Re}(\mathbf{x}, \mathbf{y}) \leq |\mathrm{Re}(\mathbf{x}, \mathbf{y})| \leq |(\mathbf{x}, \mathbf{y})| \leq ||\mathbf{x}||||\mathbf{y}||.$$

where $|\mathrm{Re}(\mathbf{x}, \mathbf{y})| \leq |(\mathbf{x}, \mathbf{y})|$ is because

$$|(\mathbf{x}, \mathbf{y})| = \sqrt{[\mathrm{Re}(\mathbf{x}, \mathbf{y})]^2 + [\mathrm{Im}(\mathbf{x}, \mathbf{y})]^2} \geq \sqrt{[\mathrm{Re}(\mathbf{x}, \mathbf{y})]^2} = |\mathrm{Re}(\mathbf{x}, \mathbf{y})|,$$

and $|(\mathbf{x}, \mathbf{y})| \leq ||\mathbf{x}||||\mathbf{y}||$ is by Cauchy-Schwartz Inequality. Therefore

$$\begin{aligned}(||\mathbf{x} + \mathbf{y}||)^2 &= (\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) \\ &= (\mathbf{x}, \mathbf{x}) + (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) \\ &= ||\mathbf{x}||^2 + 2\mathrm{Re}(\mathbf{x}, \mathbf{y}) + ||\mathbf{y}||^2 \\ &\leq ||\mathbf{x}||^2 + 2||\mathbf{x}||||\mathbf{y}|| + ||\mathbf{y}||^{\mathbf{2}} \\ &= (||\mathbf{x}|| + ||\mathbf{y}||)^2.\end{aligned}$$

8. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be orthonormal vectors in $\mathbb{R}^n$. Show that $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ are also orthonormal if and only if $A \in \mathbb{R}^{n\times n}$ is orthogonal.

"$\Leftarrow$": If A is orthogonal, then $A^T A = I_n$.

$$(A\mathbf{x}_i, A\mathbf{x}_j) = (A\mathbf{x}_i)^T (A\mathbf{x}_j) = \mathbf{x}_i^T A^T A\mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = \delta_{ij}$$

the last equality in the above computation is because $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are orthonormal vectors. Thus $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ are also orthonormal.

"$\Rightarrow$ ": $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ are also orthonormal, Let $X = \begin{pmatrix} | & | & & | \\ \mathbf{x_1} & \mathbf{x_2} & \cdots & \mathbf{x_n} \\ | & | & & | \end{pmatrix}$. Since $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are orthonormal vectors.

$$X^T X = XX^T = I_n$$

On the other hand

$$AX = \begin{pmatrix} | & | & & | \\ A\mathbf{x_1} & A\mathbf{x_2} & \cdots & A\mathbf{x_n} \\ | & | & & | \end{pmatrix}.$$

Since $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ are also orthonormal,

$$(AX)^T AX = X^T A^T AX = I_n$$

Now left multiply X and right multiply X^T on the last two terms in the above equation, we get

$$XX^TA^TAXX^T = XI_nX^T, \quad \Leftrightarrow \quad A^TA = I.$$

Therefore A is orthogonal.

9. True or False. (No explanation needed.)

1). An inner product is a scalar-valued function on the set of ordered pairs of vectors. T

2). An inner product is linear in both components. F

3). If $(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{v}$ in an inner product space, then $\mathbf{w} = \mathbf{0}$. T

4). A set of orthonormal vectors must be linearly independent. T

5). A set of orthogonal vectors must be linearly independent. F

6). A matrix in $\mathbb{R}^{n\times n}$ is orthogonal if and only if its column vectors are orthogonal. F