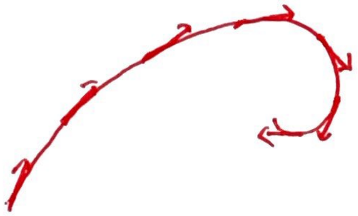
Curvature

A curve has a **smooth** parametrization, $\vec{r}(t)$, on an interval I if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$ on I.

A curve is **smooth** if it has a smooth parametrization.

Suppose the curve, C, is smooth with parametrization, $\vec{r}(t)$. Then the unit tangent vector is $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



At each point along C, $\hat{T}(t)$ exists and determines the direction of the curve.

The curvature of a curve is a measure of how quickly the curve changes direction.

Defn: Let C be a smooth curve with parametrization, $\vec{r}(t)$. Then the **curvature** is $\kappa = \left|\dfrac{d\hat{T}}{ds}\right|$ where s is arc length.

Note: Suppose $\hat{T} = \hat{T}(s)$ where $s = s(t) = \int_a^t |\vec{r}\,'(u)|\,du$

Then by chain rule, $\dfrac{d\hat{T}}{dt} = \dfrac{d\hat{T}}{ds}\dfrac{ds}{dt} = \dfrac{d\hat{T}}{ds}|\vec{r}\,'(t)|$

$$\frac{d\hat{T}}{ds} = \frac{\frac{d\hat{T}}{dt}}{|\vec{r}\,'(t)|}$$

$$K = \left|\frac{d\hat{T}}{ds}\right| = \frac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|}$$

Ex: Find the curvature of a circle of radius a.

$$x^2 + y^2 = a^2$$

$$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$$

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2(\sin^2 t + \cos^2 t)} = \sqrt{a^2} = a$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t\rangle$$

$$\hat{T}'(t) = \langle -\cos t, -\sin t\rangle$$

$$|\hat{T}'(t)| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$$

$$K = \frac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{a}$$

The curvature decreases as the radius increases.

Theorem: Let C be a smooth curve with parametrization $\vec{r}(t)$. Then the curvature is $\kappa = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

Proof: Given the parametrization, $\vec{r}(t)$, the unit tangent vector is $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{r}\,'(t)}{s'(t)}$

where $s(t)$ is the arc length function.

$$\vec{r}\,'(t) = s'(t)\,\hat{T}(t)$$

$$\vec{r}\,''(t) = s''(t)\,\hat{T}(t) + s'(t)\,\hat{T}'(t)$$

$$\begin{aligned}
\vec{r}\,' \times \vec{r}\,'' &= (s'(t)\,\hat{T}(t)) \times (s''(t)\,\hat{T}(t) + s'(t)\,\hat{T}'(t)) \\
&= s'(t)s''(t)(\hat{T} \times \hat{T}) + (s'(t))^2(\hat{T} \times \hat{T}') \\
&= (s'(t))^2(\hat{T} \times \hat{T}') \qquad \text{since } \hat{T} \times \hat{T} = \vec{0}
\end{aligned}$$

Note: Since $|\hat{T}| = 1$ for all t, then $\hat{T}$ and $\hat{T}'$ are orthogonal.

$$\begin{aligned}
|\hat{T}|^2 &= 1 \\
\hat{T} \cdot \hat{T} &= 1 \\
\frac{d}{dt}(\hat{T} \cdot \hat{T}) &= \frac{d}{dt}(1) \\
\hat{T}' \cdot \hat{T} + \hat{T} \cdot \hat{T}' &= 0 \\
2\hat{T} \cdot \hat{T}' &= 0 \implies T \,\&\, \hat{T} \text{ are orthogonal}
\end{aligned}$$

Therefore,

$$|\vec{r}' \times \vec{r}''| = |(s')^2 (\hat{T} \times \hat{T}')|$$

$$= (s')^2 |\hat{T}| |\hat{T}'| \sin(\tfrac{\pi}{2})$$

$$= (s')^2 |\hat{T}'| \quad \text{since } |\hat{T}| = 1$$

$$|\hat{T}'| = \frac{|\vec{r}' \times \vec{r}''|}{(s')^2} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^2}$$

Curvature is

$$K = \frac{|\hat{T}'|}{|\vec{r}'|} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

Ex! Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$$\vec{r}' = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{r}'' = \langle 0, 2, 6t \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle 6t^2, -6t, 2 \rangle = 2\langle 3t^2, -3t, 1 \rangle$$

$$|\vec{r}' \times \vec{r}''| = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$|\vec{r}'| = \sqrt{1 + 4t^2 + 9t^4}$$

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(9t^4 + 4t^2 + 1)^{3/2}}$$

Unit tangent, unit normal and unit binormal vectors

Given a smooth curve with parametrization, $\vec{r}(t)$,

Unit tangent vector is $\hat{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Since $|\hat{T}(t)| = 1$ for all t, $\hat{T}(t)$ and $\hat{T}'(t)$ are orthogonal

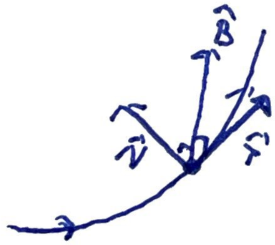
Unit normal vector is $\hat{N}(t) = \dfrac{\hat{T}'(t)}{|\hat{T}'(t)|}$

Orthogonal to $\hat{T}(t)$ and points in direction curve is bending.

$\hat{T}(t) \times \hat{N}(t)$ is orthogonal to $\hat{T}(t)$ and $\hat{N}(t)$.

$|\hat{T}(t) \times \hat{N}(t)| = |\hat{T}(t)|\,|\hat{N}(t)| \sin(\frac{\pi}{2}) = (1)(1)(1) = 1$

Unit Binormal vector is $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$



Plane containing $\hat{B}$ and $\hat{N}$ contains all vectors orthogonal to $\hat{T}$ and is the <u>normal plane</u>.

Plane containing $\hat{T}$ and $\hat{N}$ is osculating plane. It contains tangent direction and turning direction. It is the plane that best fits the curve at the point.

Ex, Find the unit tangent, unit normal, and binormal vectors of $\vec{r} = \langle t^2, \frac{2}{3}t^3, t \rangle$

$\vec{r}' = \langle 2t, 2t^2, 1 \rangle$

$|\vec{r}'| = \sqrt{4t^2 + 4t^4 + 1} = \sqrt{(2t^2+1)^2} = 2t^2+1$

$T(t) = \dfrac{\vec{r}'}{|\vec{r}'|} = \dfrac{1}{2t^2+1} \langle 2t, 2t^2, 1 \rangle$

$T'(t) = \dfrac{-4t}{(2t^2+1)^2} \langle 2t, 2t^2, 1 \rangle + \dfrac{1}{2t^2+1} \langle 2, 4t, 0 \rangle$

$= \dfrac{1}{(2t^2+1)^2} \left[\langle -8t^2, -8t^3, -4t \rangle + \langle 4t^2+2, 8t^3+4t, 0 \rangle \right]$

$= \dfrac{1}{(2t^2+1)^2} \langle 2-4t^2, 4t, -4t \rangle$

$|\vec{T}'(t)| = \dfrac{1}{(2t^2+1)^2} |\langle 2-4t^2, 4t, -4t \rangle| = \dfrac{1}{(2t^2+1)^2} \sqrt{4-16t^2+16t^4+16t^2+16t^2}$

$= \dfrac{1}{(2t^2+1)^2} \sqrt{16t^4+16t^2+4} = \dfrac{1}{(2t^2+1)^2} \sqrt{(4t^2+2)^2} = \dfrac{2(2t^2+1)}{(2t^2+1)^2}$

$= \dfrac{2}{2t^2+1}$

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = \dfrac{1}{2t^2+1} \langle 1-2t^2, 2t, -2t \rangle$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \dfrac{1}{2t^2+1} \langle 2t, 2t^2, 1 \rangle \times \dfrac{1}{2t^2+1} \langle 1-2t^2, 2t, -2t \rangle$

$= \dfrac{1}{(2t^2+1)^2} \left[\langle 2t, 2t^2, 1 \rangle \times \langle 1-2t^2, 2t, -2t \rangle \right]$

$= \dfrac{1}{(2t^2+1)^2} \langle -4t^3 - 2t, 1+2t^2, 2t^2+4t^4 \rangle = \dfrac{1}{2t^2+1} \langle -2t, 1, 2t^2 \rangle$