**1.** *This problem will provide another of Cauchy-Schwarz inequality.*

Let $V$ be an inner produce space over $\mathbb{C}$. For any $\mathbf{x}, \mathbf{y} \in V$, define $G = \begin{bmatrix} (\mathbf{x}, \mathbf{x}) & (\mathbf{x}, \mathbf{y}) \\ (\mathbf{y}, \mathbf{x}) & (\mathbf{y}, \mathbf{y}) \end{bmatrix} \in \mathbb{C}^{2\times 2}$.

1). Prove that $G$ is a (Hermitian) positive semi-definite matrix.

2). Prove that eigenvalues of a Hermitian positive semi-definite matrix are all non-negative.

3). Prove the Cauchy-Schwarz inequality, i.e $|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$. (Hint: What is the determinant of $G$? How do we relate determinant of a matrix with its eigenvalues?)

1). $G^* = G$ as $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ and $(\mathbf{x}, \mathbf{x}), (\mathbf{y}, \mathbf{y}) \in \mathbb{R}$. Consider any $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2$.

$$
\begin{aligned}
\mathbf{c}^* G \mathbf{c} &= [\overline{c}_1 \quad \overline{c}_2] \begin{bmatrix} (\mathbf{x}, \mathbf{x}) & (\mathbf{x}, \mathbf{y}) \\ (\mathbf{y}, \mathbf{x}) & (\mathbf{y}, \mathbf{y}) \end{bmatrix} \begin{bmatrix} ( & c_1 \\ & c_2 \end{bmatrix} \\
&= [\overline{c}_1(\mathbf{x}, \mathbf{x}) + \overline{c}_2(\mathbf{y}, \mathbf{x}) \quad \overline{c}_1(\mathbf{x}, \mathbf{y}) + \overline{c}_2(\mathbf{y}, \mathbf{y})] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= [(\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \mathbf{x}) \quad (\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \mathbf{y})] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= c_1(\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \mathbf{x}) + c_2(\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \mathbf{y})] \\
&= (\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \overline{c}_1\mathbf{x}) + (\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \overline{c}_2\mathbf{y}) \\
&= (\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}, \overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}) \\
&= \|\overline{c}_1\mathbf{x} + \overline{c}_2\mathbf{y}\|^2 \geq 0
\end{aligned}
$$

2). Suppose $G\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq 0$. As $G$ is positive semi-definite, then

$$0 \leq \mathbf{v}^* G \mathbf{v} = \mathbf{v}^* \lambda \mathbf{v} = \lambda \mathbf{v}^* \mathbf{v}.$$

As $\mathbf{v}^*\mathbf{v} > 0$, $\lambda \geq 0$.

3). Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $G$. Then $\det G = \lambda 1 \cdot \lambda_2 \geq 0$. On the other hand

$$0 \leq \det G = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) - (\mathbf{x}, \mathbf{y})(\mathbf{y}, \mathbf{x}),$$

which is equivalent to

$$|(\mathbf{x}, \mathbf{y})|^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$$

**2.** *Note this problem gives the formula for the orthogonal projection when a general (not necessarily orthogonal) basis of the subspace is given.*

Consider the standard inner product on $\mathbb{C}^n$. Let $W \subseteq \mathbb{C}^n$ be a subspace. Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ is a basis for $W$. Denote $B = [\mathbf{w}_1\ \mathbf{w}_2\ \cdots\ \mathbf{w}_m] \in \mathbb{C}^{n\times m}$.

1). Prove that $B^*B$ is (Hermitian) positive definite. (Note $B^*B$ is often referred as the Gramian matrix related to $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$

2). Prove that eigenvalues of a Hermitian positive definite matrix are all positive.

3). Prove that $B^*B$ is invertible.

4). Let $\mathbf{x} \in \mathbb{C}^n$ and let $\mathbf{x}_w$ be the orthogonal projection of $\mathbf{x}$ onto $W$. Prove that $\mathbf{x}_W = B(B^*B)^{-1}B^*\mathbf{x}$

5). Let $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. By using the formula in Part 4), find the orthogonal projection of $\mathbf{x}_3$ onto the subspace spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$.

1). $(B^*B)^* = B^*B$. Thus $B$ is Hermitian. For any $\mathbf{x} \in \mathbb{C}^n$,

$$\mathbf{x}^*(B^*B)\mathbf{x} = (B\mathbf{x})^*(B\mathbf{x}) = ||B\mathbf{x}||^2 \geq 0.$$

Denote $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, then $B\mathbf{x} = \sum_{i=1}^{m} x_i\mathbf{w}_i$. Then $||B\mathbf{x}||^2 = 0$ if and only if $B\mathbf{x} = \sum_{i=1}^{m} x_i\mathbf{w}_i = \mathbf{0}$.

Since $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ is linearly independent, $\sum_{i=1}^{m} x_i\mathbf{w}_i = \mathbf{0}$ if and only if $x_1 = \cdots = x_m = 0$.

Thus we proved that $\mathbf{x}^*(B^*B)\mathbf{x} = (B\mathbf{x})^*(B\mathbf{x}) = ||B\mathbf{x}||^2 \geq 0$ and $\mathbf{x}^*(B^*B)\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$, i.e. $B^*B$ is positive definite.

2). Suppose $B^*B\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq 0$. As $B$ is positive definite, then

$$0 < \mathbf{v}^*B^B\mathbf{v} = \mathbf{v}^*\lambda\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v}.$$

As $\mathbf{v}^*\mathbf{v} > 0$, $\lambda > 0$.

3). Let $\lambda_1, \cdots \lambda_n$ be all the eigenvalues of $B^*B$. Then $\det(B^*B) = \lambda_1 \cdots \lambda_n > 0$. Thus $B^*B$ is invertible.

4). Suppose $\vec{x}_w = c_1\vec{w}_1 + \cdots + c_m\vec{w}_m = (\vec{w}_1\ \vec{w}_2 \cdots \vec{w}_m)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$

denote $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$. Then $\vec{x}_w = B\vec{c}$

$\vec{x} - \vec{x}_w \in W^\perp \iff (\vec{x}-\vec{x}_w, \vec{w}_i) = 0$ for all $i=1,\cdots,m$

$\iff (\vec{x}, \vec{w}_i) = (\vec{x}_w, \vec{w}_i)$

$= \left(\sum_{k=1}^{m} c_k\vec{w}_k, \vec{w}_i\right)$

$= \sum_{k=1}^{m} c_k(\vec{w}_k, \vec{w}_i)$

$$(\vec{x}, \vec{w}_i) = \begin{pmatrix} (\vec{w}_1, \vec{w}_i) & (\vec{w}_2, \vec{w}_i) & \cdots & (\vec{w}_m, \vec{w}_i) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} (\vec{x}, \vec{w}_1) \\ (\vec{x}, \vec{w}_2) \\ \vdots \\ (\vec{x}, \vec{w}_m) \end{pmatrix} = \begin{pmatrix} (\vec{w}_1, \vec{w}_1) & (\vec{w}_2, \vec{w}_1) & \cdots & (\vec{w}_m, \vec{w}_1) \\ (\vec{w}_1, \vec{w}_2) & (\vec{w}_2, \vec{w}_2) & \cdots & (\vec{w}_m, \vec{w}_2) \\ \vdots & & & \\ (\vec{w}_1, \vec{w}_m) & (\vec{w}_2, \vec{w}_m) & \cdots & (\vec{w}_m, \vec{w}_m) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \vec{w}_1^* \vec{x} \\ \vec{w}_2^* \vec{x} \\ \vdots \\ \vec{w}_m^* \vec{x} \end{pmatrix} = \begin{pmatrix} w_1^* w_1 & w_1^* w_2 & \cdots & w_1^* w_m \\ w_2^* w_1 & w_2^* w_2 & \cdots & w_2^* w_m \\ \vdots & & & \\ w_m^* w_1 & w_m^* w_2 & \cdots & w_m^* w_m \end{pmatrix} \vec{c}$$

$$\begin{pmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_m^* \end{pmatrix} \vec{x} = \begin{pmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_m^* \end{pmatrix} (\vec{w}_1 \;\; w_2 \;\; \cdots \;\; w_m) \vec{c}$$

$$B^* \vec{x} = B^* B \vec{c}$$

$$\Rightarrow \vec{c} = (B^* B)^{-1} B^* \vec{x}$$

$$\Rightarrow \vec{x}_w = B\vec{c} = B(B^*B)^{-1} B^* \vec{x}.$$

5) $B = (\vec{x}_1 \;\; \vec{x}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \vec{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$B^*B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\Rightarrow (B^*B)^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$\vec{x}_w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 2/3 \\ 5/3 \end{pmatrix}$$

**3.** Find the $QR$-decomposition for the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$.

Denote $\vec{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

<u>Step 1</u>: $\vec{y}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \|\vec{y}_1\| = \sqrt{2} \Rightarrow \vec{u}_1 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

<u>Step 2</u>: $\dfrac{(\vec{x}_2, \vec{y}_1)}{(\vec{y}_1, \vec{y}_1)} = \dfrac{2}{2} = 1 \Rightarrow \vec{y}_2 = \vec{x}_2 - \dfrac{(\vec{x}_2, \vec{y}_1)}{(\vec{y}_1, \vec{y}_1)} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$(\Rightarrow \vec{x}_2 = \vec{y}_1 + \vec{y}_2)$

$\Rightarrow \|\vec{y}_2\| = \sqrt{2} \Rightarrow \vec{u}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Step 3: $\dfrac{(\vec{x}_3, \vec{y}_1)}{(\vec{y}_1, \vec{y}_1)} = \dfrac{2}{2} = 1$, $\dfrac{(\vec{x}_3, \vec{y}_2)}{(\vec{y}_2, \vec{y}_2)} = \dfrac{2}{2} = 1$

$\Rightarrow \vec{y}_3 = \vec{x}_3 - \dfrac{(\vec{x}_3, \vec{y}_1)}{(\vec{y}_1, \vec{y}_1)} \vec{y}_1 - \dfrac{(\vec{x}_3, \vec{y}_2)}{(\vec{y}_2, \vec{y}_2)} \vec{y}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$(\Rightarrow \vec{x}_3 = \vec{y}_1 + \vec{y}_2 + \vec{y}_3$

$\Rightarrow \|\vec{y}_3\| = 1 \qquad \|\vec{u}_3\| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$A = (\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3) = (\vec{y}_1 \ \vec{y}_2 \ \vec{y}_3) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3) \begin{pmatrix} \|\vec{y}_1\| & & \\ & \|\vec{y}_2\| & \\ & & \|\vec{y}_3\| \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3) \begin{pmatrix} \sqrt{2} & & \\ & \sqrt{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}}_{R}$$

**4.** Let $V = \mathbb{C}^{n\times n}$ with the inner product $(A, B) = \mathrm{Tr}(\cancel{A^*B}\ B^*A)$. Find the orthogonal complement of the subspace of diagonal matrices.

Let $S$ = the subspace of all diagonal matrices.

Let $E^{i,i}$ be the matrix with 1 at the $(i,i)$-th entry, and zero everywhere else

Then a basis of $S$ is given by: $B = \{E^{i,i} : i = 1, 2, \cdots, n\}$.

$\forall A \in S^{\perp}$ if and only if $(A, E^{i,i}) = 0$ for all $i = 1, \cdots, n$.

Denote $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $(E^{i,i})^* A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ → the i-th row

$$\Rightarrow (A, E^{i,i}) = \mathrm{Tr}\left((E^{i,i})^* A\right) = a_{ii} = 0$$

$$\Rightarrow A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in S^{\perp} \Leftrightarrow a_{11} = a_{22} = \cdots = a_{nn} = 0$$

$$\Rightarrow S^{\perp} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} : a_{11} = a_{22} = \cdots = a_{nn} = 0 \right\}.$$

**5.** Let $A \in \mathbb{C}^{m\times n}$. Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be equipped with the standard inner product. Prove the following statements.

1). $\text{null}(A) = (\text{Range}(A^*))^\perp$.

For any $\mathbf{y} \in \text{Range}(A^*)$, there exists $\mathbf{z} \in \mathbb{C}^m$ such that $\mathbf{y} = A^*\mathbf{z}$.

Take any $\mathbf{x} \in \text{Null}(A)$, then $A\mathbf{x} = \mathbf{0}$. And we have

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{z}) = (A^*\mathbf{z})^*\mathbf{x} = \mathbf{z}^*A\mathbf{x} = (A\mathbf{x}, \mathbf{z}) = (\mathbf{0}, \mathbf{z}) = 0.$$

Thus $\mathbf{x} \in (\text{Range}(A^*))^\perp$. Therefore, $\text{Null}(A) \subseteq (\text{Range}(A^*))^\perp$.

Conversely, For any $\mathbf{z} \in (\text{Range}(A^*))^\perp$, and any $\mathbf{x} \in \mathbb{C}^m$:

$$(A\mathbf{z}, \mathbf{x}) = \mathbf{x}^*A\mathbf{z} = (A^*\mathbf{x})^*\mathbf{z} = (\mathbf{z}, \mathbf{A}^*\mathbf{x}) = 0.$$

The above last equality is because $A^*\mathbf{x} \in \text{Range}(A^*)$ and $\mathbf{z} \in (\text{Range}(A^*))^\perp$. As $(A\mathbf{z}, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{C}^m$, $A\mathbf{z} = \mathbf{0}$, i.e. $\mathbf{z} \in \text{Null}(A)$. Therefore $\text{Range}(A^*))^\perp \subseteq \text{Null}(A)$.

Thus, we proved $\text{Range}(A^*))^\perp = \text{Null}(A)$.

2). $\text{null}(A^*A) = \text{null}(A)$.

For any $\mathbf{x} \in \text{Null}(A)$, $A\mathbf{x} = \mathbf{0}$. Thus $A^*A\mathbf{x} = A^*\mathbf{0} = \mathbf{0}$. Therefore $\text{Null}(A) \subseteq \text{Null}(A^*A)$. Conversely, for any $\mathbf{x} \in \text{Null}A^*A$,$A^*A\mathbf{x} = \mathbf{0}$. Therefore

$$0 = \mathbf{x}^*A^*A\mathbf{x} = (A\mathbf{x})^*A\mathbf{x} = (A\mathbf{x}, A\mathbf{x}) = ||A\mathbf{x}||^2$$

Therefore $A\mathbf{x} = \mathbf{0}$. Thus $\mathbf{x} \in \text{Null}(A)$. Thus $\text{Null}(A^*A) \subseteq \text{Null}(A)$.

Therefore, we proved $\text{Null}(A^*A) = \text{Null}(A)$.

3). $\text{Rank}(A^*A) = \text{Rank}(A) = \text{Rank}(A^*)$.

By 1) and 2): $\dim(\text{Range}(A^*A)^\perp) = \dim(\text{Null}(A)) = \dim(\text{Null}(A^*A))$

$\text{Range}(A^*) \oplus (\text{Range}(A^*))^\perp = \mathbb{C}^n \Rightarrow \dim(\text{Range}(A^*)^\perp) = n - \text{Rank}(A^*)$.

By dimension theorem: $n = \dim(\text{Null}(A)) + \text{Rank}(A) \Rightarrow \dim(\text{Null}(A)) = n - \text{rank}(A)$

By dimension theorem: $\dim(\text{null}(A^*A)) + \text{rank}(A^*A) = n \Rightarrow \dim(\text{null}(A^*A)) = n - \text{rank}(A^*A)$

$\Rightarrow n - \text{Rank}(A^*) = n - \text{rank}(A) = n - \text{rank}(A^*A)$

$\Rightarrow \text{rank}(A^*) = \text{rank}(A) = \text{rank}(A^*A)$.

4). $\text{Range}(A^*A) = \text{Range}(A^*)$.

$\forall \vec{y} \in \text{Range}(A^*A)$ : $\exists \vec{x}$ s.t. $\vec{y} = A^*A\vec{x} = A^*(A\vec{x}) \in \text{Range}(A^*)$

$\Rightarrow \text{Range}(A^*A) \subseteq \text{Range}(A^*)$.

As $\text{Rank}(A^*A) = \text{Rank}(A^*)$

$\Rightarrow \text{Range}(A^*A) = \text{Range}(A^*)$.

**6.** Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Let $(\lambda, \mathbf{v})$ be an eigenvalue/eigenvector pair of $A$. Prove that $(\overline{\lambda}, \mathbf{v})$ is an eigenvalue/eigenvector pair of $A$.

Proof:

$$(A-\lambda I)^*(A-\lambda I) = (A^* - \overline{\lambda} I)(A-\lambda I) = A^*A - A^*\lambda I - A\overline{\lambda} I + \overline{\lambda}\lambda I$$

$$\overset{\downarrow A \text{ normal}}{=} AA^* - \lambda I A^* - A\overline{\lambda} I + \overline{\lambda}\lambda I$$

$$= (A-\lambda I)A^* - (A-\lambda I)\overline{\lambda} I$$

$$= (A-\lambda I)(A^* - \overline{\lambda} I)$$

Note: $(A\vec{x}, \vec{y}) = \vec{y}^* A\vec{x} = (A^*\vec{y})^*\vec{x} = (\vec{x}, A^*\vec{y})$.

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0} \iff (A-\lambda I)\vec{v} = \vec{0}$$

$$\iff ((A-\lambda I)\vec{v}, (A-\lambda I)\vec{v}) = 0$$

$$\iff (\vec{v}, (A-\lambda I)^*(A-\lambda I)\vec{v}) = 0$$

$$\iff (\vec{v}, (A-\lambda I)(A-\lambda I)^*\vec{v}) = 0$$

$$\iff (\vec{v}, ((A-\lambda I)^*)^*(A-\lambda I)^*\vec{v}) = 0$$

$$\iff ((A-\lambda I)^*\vec{v}, (A-\lambda I)^*\vec{v}) = 0$$

$$\iff \|(A-\lambda I)^*\vec{v}\|^2 = 0$$

$$\iff (A-\lambda I)^*\vec{v} = \vec{0}$$

$$\iff (A^* - \overline{\lambda} I)\vec{v} = \vec{0}$$

$$\iff A^*\vec{v} = \overline{\lambda}\vec{v}.$$

**7.** Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Prove that eigenvectors of $A$ associated with distinct eigenvalues are orthogonal.

Suppose $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$ for $\vec{v}_1 \neq \vec{0}$ and $\vec{v}_2 \neq \vec{0}$ and $\lambda_1 \neq \lambda_2$.

Since $A$ is normal, $A^*\vec{v}_1 = \bar{\lambda}_1 \vec{v}_1$ and $A^*\vec{v}_2 = \bar{\lambda}_2 \vec{v}_2$

$$\lambda_1(\vec{v}_1, \vec{v}_2) = (\lambda_1\vec{v}_1, \vec{v}_2) = (A\vec{v}_1, \vec{v}_2) = (\vec{v}_1, A^*\vec{v}_2)$$

$$= (\vec{v}_1, \bar{\lambda}_2\vec{v}_2)$$

$$= \lambda_2(\vec{v}_1, \vec{v}_2)$$

$\Rightarrow \quad (\lambda_1 - \lambda_2)(\vec{v}_1, \vec{v}_2) = 0$

Since $\lambda_1 - \lambda_2 \neq 0$, $(\vec{v}_1, \vec{v}_2) = 0 \Rightarrow \vec{v}_1$ and $\vec{v}_2$ are orthogonal.

**8.** True or False. (No explanation is needed)

F 1). Suppose $A \in \mathbb{C}^{n\times n}$. Then $\text{Range}(A^*A) = \text{Range}(A^*) = \text{Range}(A)$.

T 2). A set of orthonormal vectors must be linearly independent.

F 3). A set of orthogonal vectors must be linearly independent.

*In the statement (4)-(9), V is a finite-dimensional inner product space.*

T 4). Every linear transformation on $V$ has a unique adjoint.

F 5). For every linear transformation $T : V \to V$ and any given ordered basis $B$ for $V$, we have $[T^*]_B = ([T]_B)^*$.

F 6). For any linear transformation $T$ and $U$ on $V$ and scalars $a$ and $b$, we have

$$(aT + bU)^* = aT^* + bU^*.$$

T 7). Every self-adjoint linear transformation on $V$ is normal.

F 8). Linear transformations and their adjoints on $V$ have the same eigenvalues.

F ~~T~~ 9). Linear transformations and their adjoints on $V$ have the same eigenvectors.

(The correct statement should be: "<u>Normal</u> linear transformation - - - - ")