The cross product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$= \langle a_2 b_3 - a_3 b_2, \; a_3 b_1 - a_1 b_3, \; a_1 b_2 - a_2 b_2 \rangle$$

$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$$
$$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$$

The cross product is not a commutative operation. Due to the differences in the components, $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$.

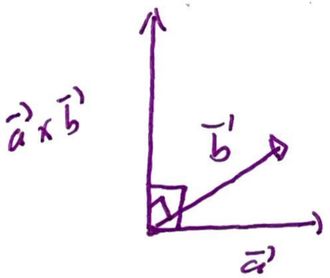
Ex: Let $\vec{a} = \langle 1, 5, 2 \rangle$ and $\vec{b} = \langle 3, -1, 1 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 5 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix}$$

$$= \langle 5 - (-2), -(1-6), -1-15 \rangle$$

$$= \langle 7, 5, -16 \rangle$$

$\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$ and its direction follows right hand rule.



Right Hand Rule: Using right hand, curl fingers from $\vec{a}$ to $\vec{b}$, then thumb points orthogonally in direction of $\vec{a} \times \vec{b}$.

Theorem: Let θ be angle between $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$. Then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

Proof:

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$|\vec{a} \times \vec{b}|^2 = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$$

$$= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3$$
$$+ a_1^2 b_3^2 + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2$$

$$= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2$$
$$+ a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2$$
$$- \big(a_1^2 b_1^2 + 2a_1 a_2 b_1 b_2 + 2a_1 a_3 b_1 b_3$$
$$+ 2a_2 a_3 b_2 b_3 + a_2^2 b_2^2 + a_3^2 b_3^2 \big)$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$$

$$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ since $\sin\theta \ge 0$ when $0 \le \theta \le \pi$

Theorem:

Two nonzero vectors are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$

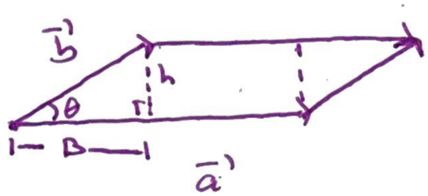
Proof: Nonzero vectors are parallel if and only if angle between them is either $\theta = 0$ or $\theta = \pi$.

Since $\sin\theta = \sin(0) = \sin(\pi) = 0$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$
$$= 0$$

If $|\vec{a} \times \vec{b}| = 0$, then $\vec{a} \times \vec{b} = \vec{0}$.

Consider the parallelogram determined by $\vec{a}$ and $\vec{b}$. Find its area.



$$A = \tfrac{1}{2}Bh + h(|\vec{a}| - B) + \tfrac{1}{2}Bh$$
$$= Bh + |\vec{a}|h - Bh$$
$$= |\vec{a}|h$$

From right triangle, $\sin\theta = \dfrac{h}{|\vec{b}|} \Rightarrow h = |\vec{b}|\sin\theta$

Area of parallelogram is $A = |\vec{a}|h$

$$= |\vec{a}||\vec{b}|\sin\theta$$
$$= |\vec{a} \times \vec{b}|$$

Therefore $\vec{a} \times \vec{b}$ is a vector orthogonal to both $\vec{a}$ and $\vec{b}$ and its magnitude is the area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$.