SOLUTIONS TO HOMEWORK 2

Warning: Very little proofreading has been done.

1. Section 1.3

Exercise 1.3.1.

(a) Write a formal definition in the style of Definition 1.3.2 of the book for the *infimum* or *greatest lower bound* of a subset of $\mathbb{R}$.
(b) Now, state and prove a version of Lemma 1.3.7 of the book for greatest lower bounds.

Solution. (a) Let $A \subseteq \mathbb{R}$ and let $s \in \mathbb{R}$. Then $s = \inf(A)$ if, first, s is a lower bound for A, and, second, whenever c is a lower bound for A then $s \geq c$.

(b) The statement is: Let $A \subseteq \mathbb{R}$, and let s be an upper bound for A. Then $s = \inf(A)$ if and only if for every $\varepsilon > 0$ there is $a \in A$ such that $a < s + \varepsilon$.

For the proof, first suppose $s = \inf(A)$. Let $\varepsilon > 0$. Then $s + \varepsilon$ is not a lower bound for A, so there is $a \in A$ such that $s + \varepsilon > a$.

Now suppose that for every $\varepsilon > 0$ there is $a \in A$ such that $a < s + \varepsilon$. Since we are given that s is a lower bound for A, we need only show that if $t > s$ then t is not a lower bound for A. Set $\varepsilon = t - s > 0$. By hypothesis, there exists $a \in A$ such that $a < s + \varepsilon = t$. So t is not a lower bound for A, as desired. $\square$

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution. In one hand, since a is an upper bound of A and $\sup A$ is the least user bound, we must have $\sup A \leq a$. On the other hand, $a \in A$, so that $a \leq \sup A$. Consequently, $a \leq \sup A \leq a$. Hence, $a = \sup A$. $\square$

Exercise 1.3.9. (a) If $\sup(A) < \sup(B)$, show that there exists an element $b \in B$ that is an upper bound for A.

Solution. The hypotheses imply that $\sup(A)$ is not an upper bound for B. Therefore there exists a $b \in B$ such that $b > \sup(A)$.

We finish the proof by showing that b is an upper bound for A. So let $x \in A$. Then $x \leq \sup(A) < b$, so $x \leq b$. $\square$

Alternate solution. Let $\varepsilon = \sup(B) - \sup(A)$. Then $\varepsilon > 0$. By Lemma 1.3.7, there is a $b \in B$ such that $b > \sup(B) - \varepsilon$. Thus

$$b > \sup(B) - \varepsilon = \sup(A).$$

Therefore b is an upper bound for A. $\square$

Exercise 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup(A) \leq \sup(B)$.
(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution. (a) This is true. It suffices to show that $\sup(B)$ is an upper bound for A. Let $x \in A$. Then $x \in B$, so $x \leq \sup(B)$.

(b) This is true. Since $\sup A < \inf B$, the number

$$c = \frac{\sup A + \inf B}{2}$$

satisfies $\sup A < c < \inf B$. Consequently, $a \leq \sup A < c < \inf B \leq b$ for all $a \in A$ and $b \in B$.

(c) This is false. Let $c = 1$,

$$A = \{1 - \tfrac{1}{n} : n \in \mathbb{N}\} \qquad \text{and} \qquad B = \{1 + \tfrac{1}{n} : n \in \mathbb{N}\}.$$

Then $a < c < b$ for all $a \in A$ and $b \in B$, but $\sup A = 1 = \inf B$. $\square$

2. Section 1.4

Exercise 1.4.1. Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.
(b) Shown that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in I$ as long as $a \neq 0$.
(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can you say about $s + t$ and st?

Solution. (a) Write $a = p_1/q_1$ and $b = p_2/q_2$ with $p_i, q_i \in \mathbb{N}$. Then

$$ab = \frac{p_1 p_2}{q_1 q_2} \qquad \text{and} \qquad a + b = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

so that $ab \in \mathbb{Q}$ and $a + b \in \mathbb{Q}$.

(b) If $a + t \in \mathbb{Q}$, then $t = (a + t) - a$ would be in $\mathbb{Q}$ by (1), contradiction to $a \in \mathbb{I}$. Similarly, if $at \in \mathbb{Q}$, then $t = at/a$ would be in $\mathbb{Q}$.

(c) $\mathbb{I}$ is not closed under addition, nor is it closed under multiplication. Consider, for example, $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then $a + b = 0 \in \mathbb{Q}$ and $ab = -1 \in \mathbb{Q}$.

Given two irrational numbers s and t, there is not much one can say about $s + t$ and st. They could be either rational or irrational numbers. $\square$

Exercise1.4.3. Prove that $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution. Set $I_n = \left(0, \frac{1}{n}\right)$ and

$$E = \bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} \left(0, \tfrac{1}{n}\right)$$

If $x \leq 0$, then $x \notin I_1$, so $x \notin E$. If $x > 0$, then there is $n \in \mathbb{N}$ such that $\frac{1}{n} < x$, so $x \notin I_n$, whence $x \notin E$. So E has no elements.

For the last part: All the hypotheses of the Nested Interval Property, except closedness of the intervals, are satisfied. $\square$

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. Obviously b is an upper bound of T. For any $\varepsilon > 0$, there is a rational number t between $b - \varepsilon$ and b by the density of $\mathbb{Q}$ in $\mathbb{R}$. Then $t \in T$ and $t > b - \varepsilon$. By the definition of supreme, $b = \sup T$. $\square$