1. Let T be the linear transformation on $\mathbb{R}^4$ which is represented in standard ordered basis by the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what condition on a, b and c is T diagonalizable?

Answer. $\det(T - \lambda I) = \lambda^4 = 0$

$\Rightarrow$ T has only $\lambda = 0$ as its eigenvalue, and the algebraic multiplicity of $\lambda = 0$ is 4.

$E_0 = \text{Null}(T) \cong \text{Null}(A) \subseteq \mathbb{R}^4$.

Note $\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0} \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : ax_1 = bx_2 = cx_3 = 0, x_4 \text{ free} \right\}$.

If $a \neq 0$ or $b \neq 0$ or $c \neq 0$ then $\dim \text{Null}(A) < 4$.

Since T is diagonalizable if and only if $\dim E_0 = 4$.

$\Rightarrow$ T is diagonalizable if and only if $\text{Null}(A) = \mathbb{R}^4$.

$\Rightarrow$ T is diagonalizable if and only if $a = b = c = 0$ or $T = 0$ (the zero transformation)

2. Let T be a linear transformation on the n-dimensional vector space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.

Proof : Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be n- distinct eigenvalues of T

Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be eigenvectors corresponding to $\lambda_1, \cdots, \lambda_n$, respectively

Then $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent and $T(\vec{v}_i) = \lambda(\vec{v}_i)$, $i = 1, \cdots, n$.

Let $B = \{\vec{v}_1, \cdots, \vec{v}_n\}$. Then B is a basis for V.

Then $[T(\vec{v}_i)]_B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in F^n$, where λ_i is at the i-th entry, and all all other entries are zeros

3. Let T be an invertible linear transformation on a vector space V. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Proof. Since T is invertible, all the eigenvalues of T are nonzero.
(proved in HW 5).

Let λ be an eigenvalue of T with eigenvector $\vec{x}$. Then $T(\vec{x}) = \lambda \vec{x}$.

Then $T^{-1}(T(\vec{x})) = \lambda T^{-1}(\vec{x})$

$\Leftrightarrow \quad \vec{x} = \lambda T^{-1}(\vec{x})$

$\Leftrightarrow \quad T^{-1}(\vec{x}) = \frac{1}{\lambda} \vec{x}$

$\Rightarrow$ λ^{-1} is an eigenvalue of T^{-1} with eigenvector $\vec{x}$.

$\square$

4. Let $A, B \in \mathbb{R}^{n\times n}$. This problem is to conclude that AB and BA have exactly the same set of eigenvalues.

1). Assume that $\lambda I - AB$ is invertible. Prove that

$$(\lambda I - BA)\left[I + B(\lambda I - AB)^{-1}A\right] = \lambda I.$$

2). Use Part 1) to prove that AB and BA have the same eigenvalues. (Note: The algebraic multiplicity of the same eigenvalues may not be the same.)

1). $(\lambda I - BA)\left[I + B(\lambda I - AB)^{-1}A\right]$

$= (\lambda I - BA) + \lambda B(\lambda I - AB)^{-1}A - BAB(\lambda I - AB)^{-1}A$

$= (\lambda I - BA) + B(\lambda I)(\lambda I - AB)^{-1}A - B[AB](\lambda I - AB)^{-1}A$

$= (\lambda I - BA) + B[(\lambda I - AB)](\lambda I - AB)^{-1}A$ ↓ Factor B on the left, factor $(\lambda I - AB)^{-1}A$ on the right.

$= (\lambda I - BA) + BA = \lambda I.$

2). Let us consider the invertibility of $\lambda I - AB$ and $\lambda I - BA$.

· <u>Case 1</u>: $\lambda \neq 0$: By part 1), if $\lambda I - AB$ is invertible, then $(\lambda I - BA)$ is also invertible since $(\lambda I - BA)^{-1} = \frac{1}{\lambda}\left(I + B(\lambda I - AB)^{-1}A\right)$.

Similarly one may show that if $\lambda I - BA$ is invertible then $(\lambda I - AB)$ is invertible since

$$(\lambda I - AB)^{-1} = \frac{1}{\lambda}\left(I + A(\lambda I - BA)^{-1}B\right)$$

Case 2: $\lambda = 0$: $0I - AB$ is invertible $\Leftrightarrow$ $\det(AB) \neq 0$

$\Leftrightarrow$ $\det(BA) = \det(AB) \neq 0$

$\Leftrightarrow$ $-BA$ is invertible.

By Case 1 and Case 2, one may conclude

$\forall \lambda \in \mathbb{R}$: $\lambda I - AB$ is invertible if and only if $\lambda I - BA$ is invertible

$\Leftrightarrow$ $\forall \lambda \in \mathbb{R}$: $\lambda I - AB$ is not invertible if and only if $\lambda I - BA$ is invertible.

Therefore, λ is an eigenvalue of AB $\Leftrightarrow$ $\lambda I - AB$ is not invertible

$\Leftrightarrow$ $\lambda - BA$ is not invertible

$\Leftrightarrow$ λ is an eigenvalue of AB

5. Let $A \in \mathbb{C}^{n\times n}$. Let g be a polynomial over $\mathbb{C}$. Prove that c is an eigenvalue of $g(A)$ if and only if $c = g(\lambda)$ for some eigenvalue λ of A.

Proof: "$\Leftarrow$": If λ is an eigenvalue of A, then there exists $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda\vec{v}$.

Claim: $A^k\vec{v} = \lambda^k\vec{v}$ for any $k \in \mathbb{Z}^+$

Proof of the claim by induction on k.

Base case: $k=1$: $A\vec{v} = \lambda\vec{v}$.

Inductively, suppose $A^{k-1}(\vec{v}) = \lambda^{k-1}\vec{v}$.

Then $A^k\vec{v} = A^{k-1}(A\vec{v}) = A^{k-1}(\lambda\vec{v}) = \lambda A^{k-1}\vec{v} = \lambda^k\vec{v}$

Suppose $g(x) = a_n x^n + \cdots + a_1 x + a_0$

Then $g(A)\vec{v} = (a_n A^n + \cdots + a_1 A + a_0 I)\vec{v}$

$= a_n(A^n\vec{v}) + \cdots + a_1(A\vec{v}) + a_0\vec{v}$

$= a_n(\lambda^n\vec{v}) + \cdots + a_1(\lambda\vec{v}) + a_0\vec{v}$

$= (a_n\lambda^n + \cdots + a_1\lambda + a_0)\vec{v}$

$= g(\lambda)\vec{v}$.

$\Rightarrow g(\lambda)$ is an eigenvalue of $g(A)$.

" $\Rightarrow$ ": let g be a polynomial over $\mathbb{C}$.

Then $c-g(x)=a(a_1-x)(a_2-x)\cdots(a_k-x)$, for some $a,a_1,\cdots,a_k\in\mathbb{C}$

plug A into the above polynomial:

$$cI-g(A)=a(a_1I-A)(a_2I-A)\cdots(a_kI-A) \text{ where } a\neq 0.$$

Since c is an eigenvalue of $g(A)$, $\det(cI-g(A))=0$

$$\Rightarrow\ 0=\det(cI-g(A))=\det\left[a(a_1I-A)(a_2I-A)\cdots(a_kI-A)\right]$$

$$=a^n\det(a_1I-A)\det(a_2I-A)\cdots\det(a_kI-A)$$

$\Rightarrow$ At least one of the terms among $\det(a_1I-A),\cdots,\det(a_kI-A)$ is 0.

Without loss of generality, we may assume $\det(a_1I-A)=0$.

Then a_1 is an eigenvalue of A.

Denote $\lambda=a_1$, then:

$$c-g(x)=a(\lambda-x)(a_2-x)\cdots(a_k-x).$$

plug in $x=\lambda$: $c-g(\lambda)=a(\lambda-\lambda)\cdots(a_k-x)=0$

$\Rightarrow$ $c=g(\lambda)$, where λ is an eigenvalue of A.

6. Suppose $V = W_1 \oplus W_2$. Prove that for any $\mathbf{v} \in V$, there exists a unique pair of vectors $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.

Proof. Suppose $\forall \vec{v} \in W_1 \oplus W_2$: $\exists\, w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$

such that $\vec{v} = \vec{w}_1 + \vec{w}_2 = \vec{w}_1' + \vec{w}_2'$

$\Rightarrow \vec{w}_1 - \vec{w}_1' = \vec{w}_2' - \vec{w}_2$

As $\vec{w}_1 - \vec{w}_1' \in W_1$ and $\vec{w}_2' - \vec{w}_2 \in W_2$ (b/c W_1 and W_2 are subspaces)

$\Rightarrow \vec{w}_1 - \vec{w}_1' = \vec{w}_2' - \vec{w}_2 \in W_1 \cap W_2 = \{\vec{0}\}$ (b/c $W_1 + W_2$ is a direct sum)

$\Rightarrow \vec{w}_1 = \vec{w}_1'$, $\vec{w}_2 = \vec{w}_2'$.

$\Rightarrow \forall \vec{v} \in W_1 \oplus W_2$, there exists a unique pair $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$. $\blacksquare$

7. True or False. (No explanation needed.)

T 1). Let $A \in \mathbb{C}^{n\times n}$, then A has exactly n eigenvalues (counting the multiplicities).

F 2). Let $T : V \to V$ be a linear transformation, where $\dim V = n$. Then T is diagonalizable if and only if T has n distinct eigenvalues.

T 3). Similar matrices always have the same eigenvalues.

F 4). Similar matrices always have the same eigenvectors.

F 5). The sum of two eigenvectors of a linear transformation T is always an eigenvector of T.

T 6). If λ_1 and λ_2 are distinct eigenvalues of a linear transformation, then $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$.

F 7). A linear transformation T on a finite-dimensional vector space is diagonalizable if and only if the algebraic multiplicity of each eigenvalue λ equals to its geometric multiplicity.

F 8). Suppose $W_1, W_2, \cdots W_m \subset V$ are subspaces. Then $W_1 + W_2 + \cdots + W_m$ is a direct sum if $W_i \cap W_j = \{\mathbf{0}\}$ for any $i \neq j$.

For 7): The characteristic polynomial also needs to be factored completely into all linear factors over the given field.