Math 307, Homework #4
Due Wednesday, October 30
SOLUTIONS TO SELECTED PROBLEMS

1. Identify each of the following statements as true or false. Where you can, prove the statement by giving an example or disprove it by giving a counterexample.
   (a) $(\exists x)[x \in \mathbb{Z}_9 \wedge x^2 \in \{5,7\}]$
   True: $5 \in \mathbb{Z}_9$ and $5^2 = 7$, so $5^2 \in \{5,7\}$.
   (b) $(\forall a)[(a \in \mathbb{Z} \wedge a^2 \equiv_{11} 16) \Rightarrow a \equiv_{11} 4]$
   False: $7 \in \mathbb{Z}$ and $7^2 \equiv_{11} 16$ (since $11|(49-16)$). But $7 \not\equiv_{11} 4$.
   (c) $(\exists n)[n \in \mathbb{N} \wedge \left(\frac{1}{2}(n^2+n)+2 \text{ is prime}\right)]$
   True: $1 \in \mathbb{N}$ and $\frac{1}{2}(1^2+1)+2 = 3$, which is prime.
   (d) $(\forall n)[n \in \mathbb{N} \Rightarrow \left(\frac{1}{2}(n^2+n)+2 \text{ is prime}\right)]$
   False: $3 \in \mathbb{Z}$ and $\frac{1}{2}(3^2+3)+2 = 8$, which is not prime.
   (e) $(\exists a,b)(a,b \in \mathbb{Z} \wedge 12a+20b = 4)$
   True: $12 \cdot (-3) + 20 \cdot (2) = 4$ and $-3, 2 \in \mathbb{Z}$.
   (f) $\{x \,|\, x \in \mathbb{Z}_8 \wedge 4 \cdot_8 x = 0\} \cap \{4 \cdot_8 x \,|\, x \in \mathbb{Z}_8\} = \emptyset$
   False: 0 is in the given intersection, so the set is nonempty.
   (g) $(\forall n)[(n \in \mathbb{N} \wedge n \equiv_2 1 \wedge n > 3) \Rightarrow 3|n^2-1]$
   False: $9 \in \mathbb{N}$ and $9 \equiv_2 1$ and $9 > 3$, but $9^2 - 1 = 80$ and $3 \nmid 80$.
   (h) $(\forall a,b)[(a,b \in \mathbb{Z} \wedge 12|ab) \Rightarrow (12|a \vee 12|b)]$
   False: $12|3 \cdot 4$ but $12 \nmid 3$ and $12 \nmid 4$.

2. Give a line proof that $(\forall n)[n \in \mathbb{N} \Rightarrow n^2+(n+1)^2+(n+2)^2 \equiv_3 2]$.
   Proof:
   1. Assume $n \in \mathbb{N}$.
   2. Then $n^2+(n+1)^2+(n+2)^2 = n^2+(n^2+2n+1)+(n^2+4n+4) = 3n^2+6n+5$.
   3. So $n^2+(n+1)^2+(n+2)^2-2 = 3n^2+6n+5-2 = 3n^2+6n+3 = 3(n^2+2n+1)$.
   4. Hence, $3|n^2+(n+1)^2+(n+2)^2-2$.
   5. Therefore $n^2+(n+1)^2+(n+2)^2 \equiv_3 2$.
   6. So $n \in \mathbb{N} \Rightarrow n^2+(n+1)^2+(n+2)^2 \equiv_3 2$ (DT)
   7. $(\forall n)[n \in \mathbb{N} \Rightarrow n^2+(n+1)^2+(n+2)^2 \equiv_3 2]$ (IU)

3. Decide if the following statement is true or false. If it is true, give a line proof. If it is false, give a counterexample.

$$(\forall a,b,c)\big[(a,b,c \in \mathbb{Z} \wedge a > 0 \wedge a|(b-1) \wedge a|(c-1)\,) \Rightarrow a|(bc-1)\big].$$

   Proof:
   1. Assume $a,b,c \in \mathbb{Z}$ and $a > 0$ and $a|b-1$ and $a|c-1$.
   2. Then $b-1 = ax$ for some $x \in \mathbb{Z}$.
   3. And $c-1 = ay$ for some $y \in \mathbb{Z}$.
   4. So $b = ax+1$ and $c = ay+1$.
   5. Then $bc-1 = (ax+1)(ay+1)-1 = (a^2xy+ax+ay+1)-1 = a(axy+x+y)$.
   6. So $a|bc-1$.
   7. $(a,b,c \in \mathbb{Z} \wedge a > 0 \wedge a|b-1 \wedge a|c-1) \Rightarrow a|bc-1$ (DT)
   8. $(\forall a,b,c)[(a,b,c \in \mathbb{Z} \wedge a > 0 \wedge a|b-1 \wedge a|c-1) \Rightarrow a|bc-1]$ (IU)

4. Theorem: $(\forall y)[(y \in \mathbb{Z} \land 4y^2 \equiv_7 0) \Rightarrow y \equiv_7 0]$

   Proof:

   1. Assume $y \in \mathbb{Z}$ and $4y^2 \equiv_7 0$.
   2. Then $7|4y^2$.
   3. 7 is prime, so by property (P) we know $7|4$ or $7|y^2$.
   4. But $7 \nmid 4$, so $7|y^2$.
   5. Using Property (P) again, either $7|y$ or $7|y$.
   6. Therefore $7|y$ (by using the tautology $P \lor P \Rightarrow P$).
   7. So $y \equiv_7 0$.
   8. $(y \in \mathbb{Z} \land 4y^2 \equiv_7 0) \Rightarrow y \equiv_7 0$ (DT)
   9. $(\forall y)[(y \in \mathbb{Z} \land 4y^2 \equiv_7 0) \Rightarrow y \equiv_7 0]$ (IU)

5. Give a line proof that $(\forall n)[(n \in \mathbb{N} \land 3|n \land n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12]$.

   Proof:

   1. Assume $n \in \mathbb{N}$ and $3|n$ and $n \equiv_5 3$.
   2. Then $n = 3u$ for some $u \in \mathbb{Z}$.
   3. And $5|n-3$, so $n - 3 = 5r$ for some $r \in \mathbb{Z}$.
   4. So $3u = n = 5r + 3$.
   5. Rearranging, we get $3(u-1) = 5r$.
   6. So $3|5r$.
   7. By Property (P), $3|5$ or $3|r$.
   8. But $3 \nmid 5$, so $3|r$.
   9. Therefore $r = 3s$ for some $s \in \mathbb{Z}$.
   10. And so $n = 5r + 3 = 5(3s) + 3 = 15s + 3$.
   11. $n^2 + n - 12 = (15s+3)^2 + (15s+3) - 12 = 225s^2 + 90s + 9 + 15s + 3 - 12 = 15(15s^2 + 7s)$
   12. So $15|n^2 + n - 12$
   13. $n^2 + n \equiv_{15} 12$.
   14. $(n \in \mathbb{N} \land 3|n \land n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12$ (DT)
   15. $(\forall n)[(n \in \mathbb{N} \land 3|n \land n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12]$ (IU)

6. A *rational number* is a number of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. As you learned in elementary school, a rational number can always be written in the form where $\gcd(a,b) = 1$. Fill in the blanks below to give a proof of the following theorem; feel free to insert extra steps if you think it will help clarify the proof.

   Theorem: $\sqrt{2}$ is not a rational number.

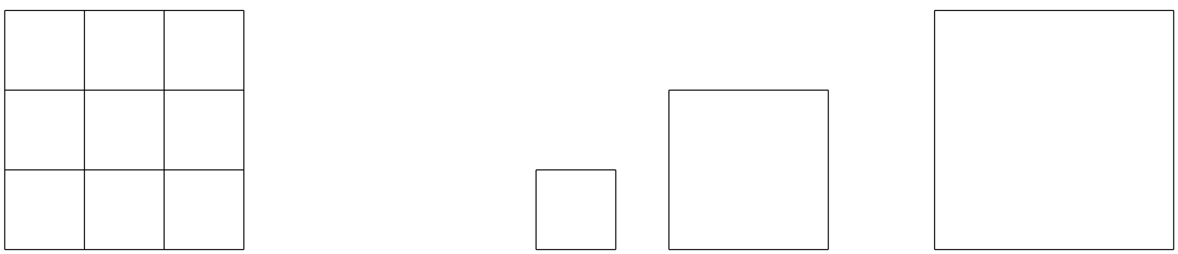
   1 Assume that $\sqrt{2}$ is a rational number.
   2. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $\gcd(a,b) = 1$.
   3. So $2 = \frac{a^2}{b^2}$, and therefore $a^2 = 2b^2$.
   4. So $2|a^2$.
   5. Then by Property (P), $2|a$.
   6. $a = 2r$ for some $r \in \mathbb{Z}$.
   7. $2b^2 = 4r^2$
   8. $b^2 = 2r^2$
   9. $2|b^2$
   10. So using Property (P), $2|b$.
   11. $b = 2s$ for some $s \in \mathbb{Z}$.
   12. This is a contradiction, because $2|a$ and $2|b$, but also $\gcd(a,b) = 1$.
   13. So $\sqrt{2}$ is not a rational number. (II)

7. Give a line proof showing that $\sqrt{10}$ is not a rational number.

Proof:

1 Assume that $\sqrt{10}$ is a rational number.
2. Then $\sqrt{10} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $\gcd(a, b) = 1$.
3. So $10 = \frac{a^2}{b^2}$, and therefore $a^2 = 10b^2$.
4. Then $a^2 = 2(5b^2)$, so $2|a^2$.
5. Then by Property (P), $2|a$.
6. $a = 2r$ for some $r \in \mathbb{Z}$.
7. $10b^2 = a^2 = 4r^2$
8. So $5b^2 = 2r^2$.
9. $2|5b^2$
10. So using Property (P), $2|5$ or $2|b^2$.
11. But $2 \nmid 5$, so $2|b^2$.
12. Using Property (P) again, $2|b$.
13. This is a contradiction, because $2|a$ and $2|b$, but also $\gcd(a, b) = 1$.
14. So $\sqrt{10}$ is not a rational number. (II)

8. A 3×3 grid has 14 squares in it:



There are 1×1 squares, 2×2 squares, and 3×3 squares, and if you count all the squares that you see in the above grid you should get 14.

Figure out how many squares there are in a 10×10 grid, and explain your answer. Give an exact number, not just a formula for computing it.

Hints for doing this: Get a sense of the problem by tackling smaller versions. Try a 2×2 grid, you already did the 3×3 grid, maybe look at 4×4 and 5×5 grids. Analyze these smaller problems and try to find some underlying patterns.