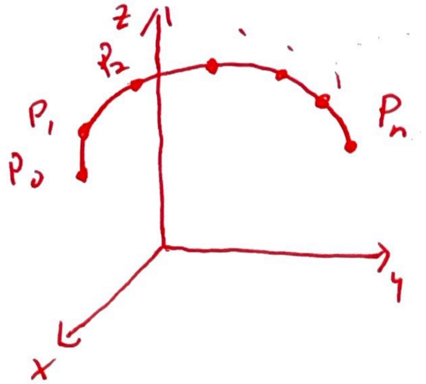
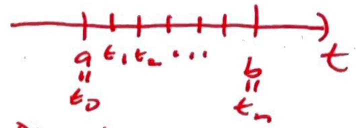
§13.3'. Arc Length and Curvature

Arc Length

Suppose a space curve is defined by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$, where f, g, and h are differentiable.

If the curve is traversed one time, find its arc length.





Divide $[a,b]$ into n subintervals of width Δt

$\vec{r}(t_i) = \overrightarrow{OP_i} = \langle x_i, y_i, z_i \rangle = \langle f(t_i), g(t_i), h(t_i) \rangle$

Approximate the arc length, $\overset{\frown}{P_{i-1}P_i}$, by the distance from P_{i-1} to P_i.

$$\overset{\frown}{P_{i-1}P_i} \approx |P_{i-1}P_i|$$

$$= \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$$

$\Delta x_i = f(t_i) - f(t_{i-1})$

$\Delta y_i = g(t_i) - g(t_{i-1})$

$\Delta z_i = h(t_i) - h(t_{i-1})$

$$= \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2}\ \Delta t$$

$$= \sqrt{(x'(t_i^*))^2 + (y'(t_i^{**}))^2 + (z'(t_i^{***}))^2}\ \Delta t$$

for $t_i^*, t_i^{**}, t_i^{***} \in [t_{i-1}, t_i]$

Therefore the arc length is

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(x'(t_i^*))^2 + (y'(t_i^{**}))^2 + (z'(t_i^{***}))^2}\, \Delta t$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

Arc Length is the integral of the magnitude of tangent vector given a position, $\vec{r}(t)$.

Ex: Find the arc length of $\vec{r}(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 12, 12t^{1/2}, 6t \rangle = 6\langle 2, 2t^{1/2}, t \rangle$

$|\vec{r}\,'(t)| = 6\sqrt{4 + 4t + t^2} = 6\sqrt{(2+t)^2}$

$= 6|t+2| = 6(t+2)$ since $0 \le t \le 1$

$$L = \int_0^1 |\vec{r}\,'(t)|\, dt$$

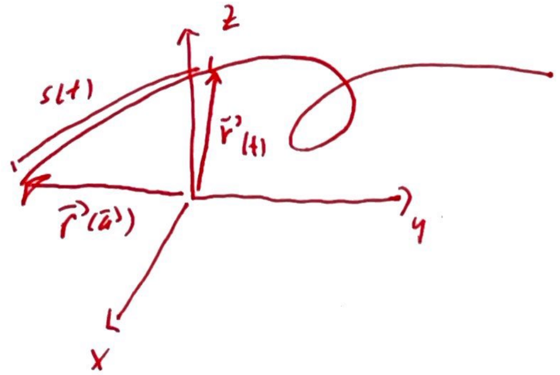
$$= \int_0^1 6(t+2)\, dt$$

$$= 3t^2 + 12t \Big|_0^1$$

$$= 15$$

Arc Length Parametrization

Consider a space curve, C, defined by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$.



Arc Length Function

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

measures distances along curve for $a \leq u \leq t$ for each $t \in [a,b]$.

We can reparametrize C with respect to arc length by using $s(t)$ to find $t = t(s)$.

Ex: Parametrize $\vec{r}(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle$ from $P(0,0,0)$ in the direction of increasing t with respect to arc length.

$|\vec{r}\,'(t)| = 6(t+2)$

Note $\vec{r}(0) = \langle 0,0,0 \rangle$ so parameter starts at $t=0$

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

$$= \int_0^t (6u+12)\,du$$

$$= 3u^2 + 12u \Big|_0^t = 3t^2 + 12t$$

$s = 3t^2 + 12t$

$\Rightarrow t = \dfrac{-12 \pm \sqrt{144+12s}}{6} = -2 \pm \frac{1}{3}\sqrt{36+3s}$

$s=0$ (zero length) must correspond to $t=0$ (minimum parameter)

$s=0 \Rightarrow t = -2 \pm \frac{\sqrt{36}}{3} = -2 \pm 2$

Therefore $t = -2 + \frac{1}{3}\sqrt{36+3s}$

$$\vec{r}(s) = \left\langle 12\left(-2+\tfrac{1}{3}\sqrt{36+3s}\right), 8\left(-2+\tfrac{1}{3}\sqrt{36+3s}\right)^{3/2}, 3\left(-2+\tfrac{1}{3}\sqrt{36+3s}\right)^{3/2} \right\rangle \quad 0 \leq s \leq 15$$