Introduction to Topology II: Homework 1

Due on January 14, 2026 at 23:59

Nick Proudfoot 13:00

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Problem 1. Show that, if X has finitely many connected components, then each component is both open and closed. On the other hand, find an example of a space X none of whose connected components are open sets.

Solution. Assume that X has finitely many connected components, say

$$X = \bigcup_{i=1}^{n} C_i.$$

Each component C_i is closed in X since connected components are always closed. To see that C_i is also open, note that

$$X \setminus C_i = \bigcup_{j \neq i} C_j,$$

which is a finite union of closed sets and hence closed. Therefore C_i is open. Thus every connected component of X is both open and closed.

For the second part, let $X = \mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$. Since $\mathbb{Q}$ is totally disconnected, every connected component is a single point. No singleton $\{q\} \subset \mathbb{Q}$ is open, because every open set in $\mathbb{Q}$ contains infinitely many rational numbers. Hence $\mathbb{Q}$ is a space none of whose connected components are open. □

Problem 2. Fix real numbers $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Use connectedness of the interval $[a, b]$ to prove the intermediate value theorem, which says that there exists an element $c \in (a, b)$ with $f(c) = 0$.

Solution. Take the partition $A = \{x \in A \mid f(x) \leq 0\} = [f(a), f(0)]$ and $B = \{x \in A \mid f(x) \geq 0\} = [f(0), f(b)]$. Both A and B are non-empty, as $f(a) < 0$ and $f(b) > 0$. Notice that $A \cup B = [a, b]$. Since $[a, b]$ is connected, then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.

Take a point c in either of these intersections. If $c \in \bar{A} \cap B$, then there exists a sequence (a_n) in A such that $a_n \to c$. By continuity of f, we have $f(a_n) \to f(c)$. Since $f(a_n) \leq 0$ for all n, we have $f(c) \leq 0$. But since $c \in B$, we also have $f(c) \geq 0$. Therefore, $f(c) = 0$.

Likewise, if $c \in A \cap \bar{B}$, then there exists a sequence (b_n) in B such that $b_n \to c$. By continuity of f, we have $f(b_n) \to f(c)$. Since $f(b_n) \geq 0$ for all n, we have $f(c) \geq 0$. But since $c \in A$, we also have $f(c) \leq 0$. Therefore, $f(c) = 0$.

Thus, there must exist a point $c \in (a, b)$ such that $f(c) = 0$. □

Problem 3. Prove that, if $f : X \to Y$ is surjective and X is path-connected, then so is Y.

Solution. Since X is path-connected, for any two points $x_1, x_2 \in X$, there exists a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Since f is surjective, for every point $c \in [f(x_1), f(x_2)]$, there exists a point $d \in [x_1, x_2]$ such that $f(d) = c$. Since f is continuous, the composition $f \circ \gamma : [0, 1] \to Y$ is also continuous. Furthermore, $(f \circ \gamma)(0) = f(x_1)$ and $(f \circ \gamma)(1) = f(x_2)$. Therefore, Y is path-connected. □

Problem 4. Prove that, if X and Y are path-connected, then so is $X \times Y$.

Solution. Since X and Y are path-connected, for any two points $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, there exist continuous maps $\gamma_X : [0, 1] \to X$ and $\gamma_Y : [0, 1] \to Y$ such that $\gamma_X(0) = x_1$, $\gamma_X(1) = x_2$, $\gamma_Y(0) = y_1$, and $\gamma_Y(1) = y_2$. For two points $(x_1, y_1), (x_2, y_2) \in X \times Y$, we can the continuous map $\gamma : [0, 1] \to X \times Y$ defined by

$$\gamma(t) = (\gamma_X(t), \gamma_Y(t)).$$

Since γ is composed of two continuous maps, it is also continuous. Furthermore, $\gamma(0) = (x_1, y_1)$ and $\gamma(1) = (x_2, y_2)$. Therefore, $X \times Y$ is path-connected. □

Problem 5. Suppose that $X = A \cup B$ with A, B, and $A \cap B$ all path-connected. Show that, if $A \cap B$ is nonempty, then X is also path-connected.

Solution. Assume $A \cap B$ is non-empty. Then, since A and B are path-connected, we can find a path between any point in A to $A \cap B$, and a path between any point in B to $A \cap B$. Let $x_1, x_2 \in X$. If both points are in A or both points are in B, then there exists a path between them. If $x_1 \in A$ and $x_2 \in B$, then, we can define the path $\gamma : [0,1] \to X$, where $\gamma(0) = x_1$, $\gamma(0.5) = c$, and $\gamma(1) = x_2$. Since $A \cap B$ is path-connected, there exists a continuous map $\delta : [0,1] \to A \cap B$ such that $\delta(0) = x_1$ and $\delta(1) = x_2$. Therefore, X is path-connected. □

Problem 6. For each description below, name a familiar space that is homeomorphic to the corresponding identification space (no proofs required).

(i) The cylinder $S^1 \times [0,1]$ with each of its boundary circles collapsed to a point. (That is, $(x,s) \sim (y,t)$ if and only if $s = t \in \{0,1\}$.)

(ii) The torus $S^1 \times S^1$ with both a longitude $(1,0) \times S^1$ and a meridian $S^1 \times (0,1)$ collapsed to a point.

(iii) The Möbius strip M with its boundary circle collapsed to a point.

Solution to (i). Collapsing each boundary circle of the cylinder $S^1 \times [0,1]$ to a point identifies the two ends of the cylinder to two distinct points. The resulting space is homeomorphic to the 2-sphere S^2. □

Solution to (ii). Collapsing both a longitude and a meridian of the torus $S^1 \times S^1$ to a point yields a space homeomorphic to the wedge sum of two 2-spheres, $S^2 \vee S^2$. □

Solution to (iii). Collapsing the boundary circle of the Möbius strip M to a point produces a space homeomorphic to the real projective plane $\mathbb{R}P^2$. □

Problem 7. Give an example of an identification map $f : X \to Y$ and a subspace $A \subset X$ such that the surjection $f : A \to f(A)$ is not an identification map.

Solution. Let $X = [0,1]$ and define an equivalence relation by identifying the endpoints $0 \sim 1$. Let $Y = X/\sim$, which is homeomorphic to the circle S^1, and let $f : X \to Y$ be the quotient map. Then f is an identification map.

Now let $A = \{0\} \cup (1/2, 1] \subset X$. The restriction $f|_A \colon A \to f(A)$ is a continuous surjection. However, it is not an identification map. Indeed, the subset $(1/2, 1]$ is open in A, but its image under f is not open in the subspace $f(A)$, since neighborhoods of $f(0) = f(1)$ necessarily intersect the image of $(1/2, 1]$. Thus the quotient topology on $f(A)$ does not agree with the subspace topology induced from Y.

Therefore, $f : X \to Y$ is an identification map, but the restricted map $f : A \to f(A)$ is not. □

Problem 8. Define $f : S^2 \to \mathbb{R}^4$ by the formula $f(x,y,z) = (x^2 - y^2, xy, xz, yz)$. You can convince yourself (and you may assume) that $f(x,y,z) = f(a,b,c)$ only if $(a,b,c) = (x,y,z)$ or $(a,b,c) = (-x,-y,-z)$. Show that f descends to a map $g : \mathbb{R}P^2 \to \mathbb{R}^4$, and that g is a homeomorphism from $\mathbb{R}P^2$ onto its image.

Solution. Let $\pi : S^2 \to \mathbb{R}P^2$ denote the quotient map identifying antipodal points. By assumption, if

$$f(x,y,z) = f(a,b,c),$$

then $(a,b,c) = (x,y,z)$ or $(a,b,c) = (-x,-y,-z)$. Hence f is constant on the equivalence classes of π, and therefore f descends to a well-defined map $g : \mathbb{R}P^2 \to \mathbb{R}^4$, satisfying $g \circ \pi = f$.

Since f is continuous and π is a quotient map, the induced map g is continuous. Moreover, g is injective: if $g([\mathbf{x}]) = g([\mathbf{y}])$, then $f(\mathbf{x}) = f(\mathbf{y})$, which implies $\mathbf{y} = \mathbf{x}$ or $\mathbf{y} = -\mathbf{x}$, so $[\mathbf{x}] = [\mathbf{y}]$ in $\mathbb{R}P^2$.

The space $\mathbb{R}P^2$ is compact, being the continuous image of the compact space S^2, and $\mathbb{R}^4$ is Hausdorff. Therefore, a continuous injective map from $\mathbb{R}P^2$ into $\mathbb{R}^4$ is a homeomorphism onto its image. Hence g is a homeomorphism from $\mathbb{R}P^2$ onto $g(\mathbb{R}P^2) \subset \mathbb{R}^4$. □