Fundamentals of Analysis I: Homework 9

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SECTION 4.2

Exercise 4.2.1

(i) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

(ii) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

Solution to (i). Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$. By the Sequential Criterion for Functional Limits, since $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, all sequences $(a_n) \to c$, we have $f(x_n) \to L$ and $g(x_n) \to M$. By the Algebraic Limit Theorem for sequences, we have $\lim_{n\to\infty}[f(x_n)+g(x_n)] = L+M$. Therefore, $\lim_{x\to c}[f(x)+g(x)] = L+M$. □

Solution to (ii). Let $\varepsilon > 0$. Let δ_1 such that $0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon/2$. Let δ_2 such that $0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Let x be arbitrary. Suppose $0 < |x-c| < \delta$. Then, we get the following

$$0 < |x-c| < \delta \Rightarrow |[f(x)+g(x)] - [M+L]| < |f(x) - M| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $0 < |x-c| < \delta \Rightarrow |[f(x)+g(x)] - [M+L]| < \varepsilon$.
It follows that $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in A)[0 < |x-c| < \delta \Rightarrow |[f(x)+g(x)] - [M+L]| < \varepsilon]$. □

Exercise 4.2.3 Review the definition of Thomae's function $t(x)$ from Section 4.1.

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(i) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

(ii) Now, compute $\lim_{n\to\infty} t(x_n)$, $\lim_{n\to\infty} t(y_n)$, and $\lim_{n\to\infty} t(z_n)$.

(iii) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1 B to verify the claim (Given $\varepsilon > 0$, consider the set of points $\{x \in \mathbb{R} \mid t(x) \geq \varepsilon\}$. Argue that all the points in this set are isolated).

Solution to (i). Let $(a_n) = (0, 1/2, 2/3, 3/4, \ldots)$, $(b_n) = (0, 1/3, 2/4, 3/5, \ldots)$, and $(c_n) = (0, 1/4, 2/5, 3/6, \ldots)$. □

Solution to (ii). We have $\lim_{n\to\infty} t(a_n) = 1$, $\lim_{n\to\infty} t(b_n) = 1$, and $\lim_{n\to\infty} t(c_n) = 1$. □

Solution to (iii). I conjecture that $\lim_{x\to 1} t(x) = 0$.

Let $\varepsilon > 0$. From the definition of $t(x)$, the condition $t(x) \geq \varepsilon$ implies that $x = \frac{m}{n}$ in lowest terms with $n \leq \frac{1}{\varepsilon}$. Since m is bounded by $|m| \leq n$ and n is bounded by $\frac{1}{\varepsilon}$, there are finitely many such rational numbers.

These rational points $\frac{m}{n}$, where $n \leq \frac{1}{\varepsilon}$, form a discrete set. Hence, outside a small neighborhood of these points, $t(x) < \varepsilon$. Define $\delta > 0$ as the minimum distance between 1 and any of these finitely many rational points

$$\delta = \min\left(\left\{\text{abs}(1-x) \,\middle|\, x = \frac{m}{n}, n \leq \frac{1}{\varepsilon}\right\}\right).$$

Then, for any $x \in \mathbb{R}$ such that $0 < |x-1| < \delta$, it follows that $t(x) < \varepsilon$.
Therefore, $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[0 < |x-1| < \delta \Rightarrow t(x) < \varepsilon]$.
It follows that $\lim_{x\to 1} t(x) = 0$. □

Exercise 4.2.7 Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A in the sense that that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$.

Show that $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

Solution. Let $g : A \to \mathbb{R}$ and let f be bounded on A. Then, there exists an $M > 0$ such that $(\forall x \in A)[|f(x)| \leq M]$. Then, $(\forall x \in A)[|f(x) \cdot g(x)| \leq M \cdot |g(x)|]$. Set δ small enough such that $|g(x)| < \frac{\varepsilon}{M}$. Then, we get $|f(x) \cdot g(x)| \leq M \cdot |g(x)| < M \cdot \frac{\varepsilon}{M} = \varepsilon$. ☐

Exercise 4.2.8 Compute each limit or state that it does not exist. Use the tools developed in this section to justify your answer.

(i) $\lim_{x\to 2} \frac{|x-2|}{x-2}$.

(ii) $\lim_{x\to \frac{7}{4}} \frac{|x-2|}{x-2}$.

(iii) $\lim_{x\to 0} (-1)^{\lfloor 1/x \rfloor}$.

(iv) $\lim_{x\to 0} \sqrt[3]{x}(-1)^{\lfloor 1/x \rfloor}$.

Solution to (i). The limit does not exist.

Let $(x_n) = 2 + \frac{1}{n}$ and $(y_n) = 2 - \frac{1}{n}$. Since $\lim_{n\to\infty} a_n = 2 = \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} f(x_n) = 1 \neq -1 = \lim_{n\to\infty} f(y_n)$, then by the Divergence Criterion for Functional Limits, the limit does not exist. ☐

Solution to (ii). The limit is -1.

The function $\frac{|x-2|}{x-2}$ can be simplified as

$$\frac{|x-2|}{x-2} = \begin{cases} 1, & \text{if } x-2 > 0 \\ -1, & \text{if } x-2 < 0 \end{cases}.$$

Notice that when $x = \frac{7}{4}$, we have $x - 2 = \frac{7}{4} - 2 = -\frac{1}{4} < 0$. Thus, as $x \to \frac{7}{4}$, $x - 2 < 0$.

Let $\varepsilon > 0$. Suppose $0 < |x - \frac{7}{4}| < \delta$, for some $\delta > 0$. Thus, for x sufficiently close to $\frac{7}{4}$, $\frac{|x-2|}{x-2} = -1$, and we find

$$\left| \frac{|x-2|}{x-2} + 1 \right| = |-1+1| = 0 < \varepsilon.$$

We can choose any $\delta > 0$, such as $\delta = 1$, to ensure this inequality holds because the function is constant at -1 near $x = \frac{7}{4}$ when $x - 2 < 0$.

Therefore, $\lim_{x\to \frac{7}{4}} \frac{|x-2|}{x-2} = -1$. ☐

Solution to (iii). The limit does not exist.

Let $(x_n) = \frac{1}{2n}$ and $(y_n) = \frac{1}{2n+1}$. Since $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} f(x_n) = 1 \neq -1 = \lim_{n\to\infty} f(y_n)$, then by the Divergence Criterion for Functional Limits, the limit does not exist. ☐

Solution to (iv). The limit is 0.

Let $g(x) = (-1)^{\lfloor 1/x \rfloor}$ and $g(x) = \sqrt[3]{x}$. Then $\lim_{x\to 0} g(x) = 0$ and $f(x)$ is bounded by $M > 0$, then, by Exercise 4.2.7, $\lim_{x\to 0} f(x) \cdot g(x) = 0$. ☐

I don't know if you want me to prove $\lim_{x\to 0} f(x)$, but I will anyway.

Proof of $\lim_{x \to 0} f(x) = 0$. Let $\varepsilon > 0$. Let $\delta = \varepsilon^3$. Let x be arbitrary. Suppose $0 < |x| < \delta$. Then, we get

$$0 < |x| < \delta \Rightarrow |\sqrt[3]{x}| < \delta^{1/3} = \varepsilon.$$

Therefore, $0 < |x| < \delta \Rightarrow |\sqrt[3]{x}| < \varepsilon$.
It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[0 < |x| < \delta \Rightarrow |\sqrt[3]{x}| < \varepsilon]$. □

Exercise 4.2.11 Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = L$ at some point $c \in A$, show $\lim_{x \to c} g(x) = L$ as well.

Solution. Let $\varepsilon > 0$. Assume $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$. By definition of a limit, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad 0 < |x - c| < \delta_2 \Rightarrow |h(x) - L| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - c| < \delta$, we have

$$L - \frac{\varepsilon}{2} < f(x) < L + \frac{\varepsilon}{2} \quad \text{and} \quad L - \frac{\varepsilon}{2} < h(x) < L + \frac{\varepsilon}{2}.$$

Suppose $0 < |x - c| < \delta$. Then, we get

$$\begin{aligned} |g(x) - L| &\leq |g(x) - f(x)| + |f(x) - L| \\ &\leq |h(x) - f(x)| + |f(x) - L| \\ &\leq |h(x) - L| + |L - f(x)| + |f(x) - L| \\ &\leq |h(x) - L| + 2|f(x) - L| \\ &< \frac{\varepsilon}{2} + 2\frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, $0 < |x - c| < \delta \Rightarrow |g(x) - L| < \varepsilon$.
It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[0 < |x - c| < \delta \Rightarrow |g(x) - L| < \varepsilon]$. □

SECTION 4.3

Exercise 4.3.1 Let $g(x) = \sqrt[3]{x}$

(i) Prove that g is continuous at $c = 0$.

(ii) Prove that g is continuous at a point $c \neq 0$ (The identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ will be helpful).

Solution to (i). Let $\varepsilon > 0$. Let $\delta = \varepsilon^3$. Let x be arbitrary. For $c = 0$, suppose $|x - 0| < \delta$. Then, we get

$$|x| < \delta \Rightarrow |\sqrt[3]{x}| < \delta^{1/3} = \varepsilon.$$

Therefore, $|x| < \delta \Rightarrow |\sqrt[3]{x}| < \varepsilon$.
It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)[|x| < \delta \Rightarrow |\sqrt[3]{x}| < \varepsilon]$.
Therefore, by the definition of continuity, g is continuous at $c = 0$. □

Solution to (ii). Let $\varepsilon > 0$. Choose $\delta < |c|$. Let x be arbitrary. For $c \neq 0$, suppose $|x - c| < \delta$. Using the given inequality, we get

$$|x^{1/3} - c^{1/3}| = |x - c| \cdot |x^{2/3} + x^{1/3} \cdot c^{1/3} + c^{2/3}|.$$

Since $\delta < |c|$, we get $|x - c| < \delta \Rightarrow |x| < \delta < \delta + |c| < 2|c|$. Finding an upper bound for the right-hand side of the first inequality gives us

$$\begin{aligned} |x - c| \cdot |x^{2/3} + x^{1/3} \cdot c^{1/3} + c^{2/3}| &\leq |x^{2/3}| + |x^{1/3} \cdot c^{1/3}| + |c^{2/3}| \\ &\leq 2^{2/3} \cdot |c^{2/3}| + 2^{1/3} \cdot |c^{2/3}| + |c^{2/3}| = M. \end{aligned}$$

Then, we get $|x^{1/3} - c^{1/3}| \leq |x - c| \cdot M$. Let $\delta = \frac{\varepsilon}{M}$. Then, we get $|x - c| \cdot M < \delta \cdot M = \varepsilon$

Therefore, $|x - c| < \delta \Rightarrow |x^{1/3} - c^{1/3}| < \varepsilon$.

It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[|x - c| < \delta \Rightarrow |x^{1/3} - c^{1/3}| < \varepsilon]$.

Therefore, by the definition of continuity, g is continuous at $c \neq 0$. Since g is continuous at $c = 0$ and $c \neq 0$, g is continuous on A (its domain). □

Exercise 4.3.5 Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Solution. Let $\varepsilon > 0$. Since c is an isolated point, then we can choose a $\delta > 0$ small enough such that $x \in V_\delta(c)$ if and only if $x = c$. Then, we get $|x - c| = 0 < \delta$. Then, clearly $|f(x) - f(c)| = 0 < \varepsilon$.

Therefore, $(\forall \varepsilon > 0)(\exists \delta > 0)[|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon]$.

Therefore, by the definition of continuity, f is continuous at c. □