

Multi-Variable Calculus I: Homework 1

Due on October 8, 2024 at 8:00 AM

Jennifer Thorenson 08:00

Hashem A. Damrah

Problem 1

Consider the following regions of \mathbb{R}^3 described by an inequality. Describe each region in words and sketch it to the best of your ability. Note: Inequalities define a solid region in space rather than a surface.

① $4 < x^2 + y^2 + z^2$.

② $x^2 + z^2 \leq 9$.

Solution 1

- ① The inequality represents the points (x, y, z) whose distance from the origin is more than 2. This creates a sphere of radius 2 centered at the origin (see ??). All the points are outside the sphere.
- ② Since $y = 0$ in this inequality, that means the inequality is being drawn on the xz -plane. This is just a circle of radius 3 centered at the origin (see ??)



Problem 2

Consider the vectors $\mathbf{a} = \langle -2, 5, 4 \rangle$ and $\mathbf{b} = \langle 4, 8, 1 \rangle$.

- ① Find $2\mathbf{a} - 3\mathbf{b}$.
- ② Find the length of the vectors \mathbf{a} and \mathbf{b} .
- ③ Find the angle between the vectors \mathbf{a} and \mathbf{b} .
- ④ Find a unit vector in the direction of \mathbf{a} .
- ⑤ Find $\text{comp}_{\mathbf{a}}(\mathbf{b})$ and $\text{proj}_{\mathbf{a}}(\mathbf{b})$.

Solution 2

- ① $2\langle -2, 5, 4 \rangle - 3\langle 4, 8, 1 \rangle = \langle -4, 10, 8 \rangle - \langle 12, 24, 3 \rangle = \langle -16, -14, 5 \rangle$.
- ② $|\mathbf{a}| = \sqrt{(-2)^2 + 5^2 + 4^2} = \sqrt{45}$.
 $|\mathbf{b}| = \sqrt{4^2 + 8^2 + 1^2} = \sqrt{81} = 9$.
- ③ $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{36}{\sqrt{45} \cdot 9} = \frac{4\sqrt{5}}{15}$. This gives us $\arccos\left(\frac{4\sqrt{5}}{15}\right) \approx 54^\circ$.
- ④ The unit vector for \mathbf{a} is $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\langle -2, 5, 4 \rangle}{\sqrt{45}} = \left\langle -\frac{2}{\sqrt{45}}, \frac{5}{\sqrt{45}}, \frac{4}{\sqrt{45}} \right\rangle$.
- ⑤ $\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{36}{\sqrt{45}}$.
 $\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \frac{36}{45} \langle -2, 5, 4 \rangle = \left\langle -\frac{8}{5}, 4, \frac{16}{5} \right\rangle$.



Problem 3

Find the values of x such that the vectors $\langle 6, 3x, x \rangle$ and $\langle -3, 1, x \rangle$ are orthogonal.

Solution 3

Both vectors are orthogonal if and only if their dot product is zero. Thus, solving for x gives us

$$\begin{aligned}\langle 6, 3x, x \rangle \cdot \langle -3, 1, x \rangle = 0 &\implies 6(-3) + 3x(1) + x(x) = 0 \\ &\implies -18 + 3x + x^2 = 0 \\ &\implies (x - 3)(x + 6) = 0.\end{aligned}$$

This gives us $x = 3$ and $x = -6$. Therefore, the following vector pairs are orthogonal to one another

$$\begin{aligned}&\langle 6, 9, 3 \rangle \quad \text{and} \quad \langle -3, 1, 3 \rangle \\ &\langle 6, -18, -6 \rangle \quad \text{and} \quad \langle -3, 1, -6 \rangle.\end{aligned}$$



Problem 4

If $\mathbf{r} = \langle x, y, z \rangle$ is an arbitrary position vector and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are constant position vectors, show that $(\mathbf{r} - \mathbf{u}) \cdot (\mathbf{r} - \mathbf{v}) = 0$ defines a sphere. Find the center and radius of the sphere.

Solution 4

Expanding the expression gives us

$$\begin{aligned}
 (\mathbf{r} - \mathbf{u}) \cdot (\mathbf{r} - \mathbf{v}) &= 0 \\
 \implies \langle x - u_1, y - u_2, z - u_3 \rangle \cdot \langle x - v_1, y - v_2, z - v_3 \rangle &= 0 \\
 \implies (x - u_1)(x - v_1) + (y - u_2)(y - v_2) + (z - u_3)(z - v_3) &= 0 \\
 \implies x^2 - (u_1 + v_1)x + u_1v_1 + y^2 - (u_2 + v_2)y + u_2v_2 + z^2 - (u_3 + v_3)z + u_3v_3 &= 0 \\
 \implies x^2 - (u_1 + v_1)x + y^2 - (u_2 + v_2)y + z^2 - (u_3 + v_3)z &= D_1,
 \end{aligned}$$

where D_n is the sum of all the constant terms. Lastly, we need to factor everything by completing the square, giving us

$$\begin{aligned}
 x^2 - (u_1 + v_1)x + y^2 - (u_2 + v_2)y + z^2 - (u_3 + v_3)z &= D_1 \\
 \implies x^2 - (u_1 + v_1)x + \left(\frac{u_1 + v_1}{2}\right)^2 + y^2 - (u_2 + v_2)y + \left(\frac{u_2 + v_2}{2}\right)^2 + z^2 - (u_3 + v_3)z + \left(\frac{u_3 + v_3}{2}\right)^2 &= D_2 \\
 \implies \left(x - \frac{u_1 + v_1}{2}\right)^2 + \left(y - \frac{u_2 + v_2}{2}\right)^2 + \left(z - \frac{u_3 + v_3}{2}\right)^2 &= D_3,
 \end{aligned}$$

where $\frac{u_1 + v_1}{2} = h$, $\frac{u_2 + v_2}{2} = k$, $\frac{u_3 + v_3}{2} = l$, and $\sqrt{D_3} = r$. This gives us the following center and radius for the sphere:

$$\text{Center} = (h, k, l) = \left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}, \frac{u_3 + v_3}{2}\right)$$

$$\text{Radius} = r = \sqrt{D_3}.$$



Problem 5

Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then \mathbf{u} and \mathbf{v} have the same length.

Solution 5