Abstract Linear Algebra: Homework 2

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Problem 1. The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is defined as the sum of the diagonal elements of A, i.e., $\text{Tr}(A) = \sum_{j=1}^{n} a_{jj}$. Prove that $\text{Tr}(AB) = \text{Tr}(BA)$ for any $n \times n$ matrices A and B.

Solution. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. The (i,i)-th element of AB and BA are given as follows

$$(AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki} \quad \text{and} \quad (BA)_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki}.$$

Substituting both of these into the definition of the trace gives us

$$\text{Tr}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \quad \text{and} \quad \text{Tr}(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}.$$

Notice that both expressions involve summing over all pairs (i,k). Since addition is commutative, the order of summation does not matter, and thus $\text{Tr}(AB) = \text{Tr}(BA)$. □

Problem 2. State the replacement theorem.

Solution. Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a linearly independent set of vectors in V. If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a spanning set of V, then

(i) $n \leq m$, and

(ii) It is possible to replace n vectors in $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ with $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ such that the resulting set still spans V. □

Problem 3. Let V be a vector space. Prove that the zero vector in V is unique.

Solution. Let $\mathbf{e}$ and $\mathbf{f}$ be zero vectors in V. Since $(\forall \mathbf{a} \in V)[\mathbf{e} \cdot \mathbf{a} = \mathbf{a}]$, then, we have $\mathbf{e} \cdot \mathbf{f} = \mathbf{f}$. Since $(\forall \mathbf{b} \in V)[\mathbf{b} \cdot \mathbf{f} = \mathbf{b}]$, then, we have $\mathbf{e} \cdot \mathbf{f} = \mathbf{e}$. Therefore, $\mathbf{e} = \mathbf{f}$. □

Problem 4. Let $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{C}\}$. Define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \quad \text{and} \quad c(a_1, a_2) = (a_1, 0).$$

Determine whether or not V is a vector space over $\mathbb{C}$ with these operations. Justify your answer.

Solution. It isn't a vector space over $\mathbb{C}$ as it doesn't have an identity scalar element, as $(\forall c \in \mathbb{C})[c(a_1, a_2) = (a_1, 0)]$, but in order for V to be a vector space over $\mathbb{C}$, $(\exists c \in \mathbb{C})[c(a_1, a_2) = (a_1, a_2)]$. □

Problem 5. Let $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{C}\}$. Define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Determine whether or not V is a vector space over $\mathbb{C}$ with these operations. Justify your answer.

Solution. It isn't a vector space over $\mathbb{C}$ since it isn't commutative, as $(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$. □

Problem 6. If W_1 and W_2 are subspaces of a vector space V, prove that $W_1 \cap W_2$ is a subspace of V.

Solution. Let $\mathbf{x}, \mathbf{y} \in W_1 \cap W_2$. This means that $\mathbf{x}, \mathbf{y} \in W_1$ and $\mathbf{x}, \mathbf{y} \in W_2$. Since they both are subspaces of the vector space V, then $(\forall c \in \mathbb{F})[c\mathbf{x} + \mathbf{y} \in W_1]$ and $(\forall c \in \mathbb{F})[c\mathbf{x} + \mathbf{y} \in W_2]$. Then, by definition, we get, $(\forall c \in \mathbb{F})[c\mathbf{x} + \mathbf{y} \in W_1 \cap W_2]$. Therefore, $W_1 \cap W_2$ must be a subspace of the vector space V. □

Problem 7. Consider the following subsets in $\mathbb{C}^n$:

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n \mid a_1 + \cdots + a_n = 0 \right\}, W_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n \mid a_1 + \cdots + a_n = c \text{ where } c \neq 0 \right\}.$$

Prove that W_1 is a subspace of $\mathbb{C}^n$, but W_2 is not a subspace of $\mathbb{C}^n$.

Solution. The set W_1 is a subspace of $\mathbb{C}^n$ as $(\forall c \in \mathbb{F})(\forall \mathbf{x}, \mathbf{y} \in W_1)[c\mathbf{x} + \mathbf{y} \in W_1]$, since $c(a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = c\mathbf{0} + \mathbf{0} = \mathbf{0} \in W_1$.

The set W_2 is not a subspace of $\mathbb{C}^n$ as $(\nexists \mathbf{e} \in W_2)(\forall \mathbf{v} \in W_2)[\mathbf{v} + \mathbf{e} = \mathbf{v} = \mathbf{e} + \mathbf{v}]$, since $a_1 + \cdots + a_n \neq 0$. □

Problem 8. Let S be the subset of all symmetric matrices in $\mathbb{R}^{n\times n}$, i.e., $S = \left\{ A \in \mathbb{R}^{n\times n} \middle| A^T = A \right\}$. Prove that S is a subspace of $\mathbb{R}^{n\times n}$.

Solution. The set S is a subspace of $\mathbb{R}^{n\times n}$ since $(\forall c \in \mathbb{F})(\forall A, B \in S)[cA + B \in S]$, since $cA + B = (cA + B)^T = (cA)^T + (B)^T = cA^T + B^T = cA + B$. □

Problem 9. Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ be an ordered basis for V. Prove that for any $\mathbf{x} \in V$, there exists a unique set of scalars $\{a_1, a_2, \cdots, a_n\}$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n.$$

Solution. Let $\mathbf{x} \in V$. Since $\mathcal{B}$ is a basis for V, then $\mathbf{x}$ can be written as a linear combination of the vectors in $\mathcal{B}$. By the spanning property, there exist scalars $\{a_1, \cdots, a_n\}$ such that $\mathbf{x} = \sum_{i=1}^{n} a_i\mathbf{x}_i$. Suppose there's another set of scalars $\{b_1, \cdots, b_n\}$ such that $\mathbf{x} = \sum_{i=1}^{n} b_i\mathbf{x}_i$. Subtracting the two equations gives us $0 = \sum_{i=1}^{n} (a_i - b_i)\mathbf{x}_i$. Let $c_i = a_i - b_i$. Since $\mathcal{B}$ is a basis, the vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ are linearly independent. Therefore, $c_i = 0$ for all i. Thus, $a_i = b_i$ for all i, and the set of scalars is unique. □

Problem 10. True or False. (No explanation needed).

(i) A vector space may have more than one zero vector.

(ii) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

(iii) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace.

(iv) If W and U are subspaces of V, then $W \cup U$ is a subspace of V.

Solution to (i). False, since by problem 3, I've shown that the zero vector in a vector space is unique. □

Solution to (ii). No, take $f(x) = x^2 + x$ and $g(x) = -x^2 + x$. Then, $(f + g)(x) = 2x$. □

Solution to (iii). Yes, by definition, a subset W of a vector space V if it's not empty and closed under addition and scalar multiplication. But since W is already a vector space, that means it's not empty and closed under addition and scalar multiplication, so it's a subspace of V. □

Solution to (iv). No, as it might not be closed under addition or scalar multiplication. For example, let $W = \{(x, 0) \mid x \in \mathbb{R}\}$ and $U = \{(0, y) \mid y \in \mathbb{R}\}$. Then, $(1, 0) + (0, 1) = (1, 1) \notin W \cup U$. □