Mathematical Image Modeling: Homework 3

Due on January 28, 2026 at 23:59

Jason Murphy 10:00

Hashem A. Damrah
UO ID: 952102243

Problem 1. Show that $|e^{i\theta} - 1| \leq \theta$ for $\theta \in \mathbb{R}$. (*Hint:* You can use the fundamental theorem of calculus.)

Solution. We have

$$e^{i\theta} - 1 = \int_0^\theta \frac{\mathrm{d}}{\mathrm{d}t} e^{it} \, \mathrm{d}t = \int_0^\theta ie^{it} \, \mathrm{d}t.$$

Thus,

$$|e^{i\theta} - 1| = \left| \int_0^\theta ie^{it} \, \mathrm{d}t \right| \leq \int_0^\theta |ie^{it}| \mathrm{d}t \leq \int_0^\theta 1 \, \mathrm{d}t = \theta.$$

□

Problem 2. Prove that

$$(f * g)'(x) = (f' * g)(x),$$

where $'$ denotes derivative and $*$ denotes convolution. Just treat this as a formal identity, i.e. assume everything converges nicely and you can pass derivatives through the integral sign.

This identity has an important consequence, namely: "the convolution of f and g is as smooth as the smoother of f and g".

Solution. We have

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) \, \mathrm{d}y.$$

Computing the derivative, we have

$$(f * g)'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} f(x - y) g(y) \, \mathrm{d}y = \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x} f(x - y) g(y) \, \mathrm{d}y = \int_{\mathbb{R}} f'(x - y) g(y) \, \mathrm{d}y = (f' * g)(x).$$

□

Problem 3. Prove that if $f(x) = 0$ for $|x| > R$ and $g(x) = 0$ for $|x| > T$, then $(f * g)(x) = 0$ for $|x| > R + T$.

Solution. We have

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) \, \mathrm{d}y.$$

If $|x| > R + T$, then for any y such that $g(y) \neq 0$, we have $|y| \leq T$. Thus,

$$|x - y| \geq |x| - |y| > (R + T) - T = R.$$

Since $f(x - y) = 0$ for $|x - y| > R$, it follows that $f(x - y) = 0$ whenever $g(y) \neq 0$. Therefore, the integrand $f(x - y) g(y)$ is zero for all y, and hence

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) \, \mathrm{d}y = 0.$$

□

Problem 4. Show that if $g(x) = f(x + y)$ for some $y \in \mathbb{R}$, then $\hat{g}(\xi) = e^{iy\xi} \hat{f}(\xi)$.

Solution. We have

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-ix\xi} \, \mathrm{d}x = \int_{\mathbb{R}} f(x + y) e^{-ix\xi} \, \mathrm{d}x.$$

Making the substitution $u = x + y$, we have $\mathrm{d}x = \mathrm{d}u$ and thus

$$\hat{g}(\xi) = \int_{\mathbb{R}} f(u) e^{-i(u-y)\xi} \, \mathrm{d}u = e^{iy\xi} \int_{\mathbb{R}} f(u) e^{-iu\xi} \, \mathrm{d}u = e^{iy\xi} \hat{f}(\xi).$$

□

Problem 5. Show that limits are unique in a metric space. That is, if (X, d) is a metric space and $\{x_n\}$ is a sequence satisfying $x_n \to x \in X$ and $x_n \to y \in X$, then $x = y$.

Solution. Since $\{x_n\} \to x$, we have for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\mathrm{d}(x_n, x) < \varepsilon/2$. Similarly, since $\{x_n\} \to y$, we have for every $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $\mathrm{d}(x_n, y) < \varepsilon/2$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have

$$\mathrm{d}(x, y) \leq \mathrm{d}(x, x_n) + \mathrm{d}(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mathrm{d}(x, y) = 0$. By the properties of a metric, this implies that $x = y$. $\square$

Problem 6. (optional for 410, required for 510). Taking the following fact for granted, complete the proof of the Riemann–Lebesgue lemma (i.e. $f \in L^1 \Rightarrow \hat{f} \in C_0$):

Fact: For any $f \in L^1$ and any $\varepsilon > 0$, there exists a function $g \in L^1$ satisfying (i) $xg \in L^1$ and $g' \in L^1$ and (ii) $\|f - g\|_{L^1} < \varepsilon$.

Recall that in class we proved that for $g \in L^1$ satisfying (i), we have $\hat{g} \in C_0$.

Solution. Let $f \in L^1$ and let $\varepsilon > 0$. By the given fact, there exists $g \in L^1$ such that $xg \in L^1$, $g' \in L^1$, and $\|f - g\|_{L^1} < \varepsilon$. Since we have already established that for such a function g, $\hat{g} \in C_0$, it follows that $\hat{g}$ is continuous and vanishes at infinity.

Now, we need to show that $\hat{f} \in C_0$. We can write

$$\hat{f}(\xi) = \hat{g}(\xi) + (\hat{f}(\xi) - \hat{g}(\xi)).$$

Using the properties of the Fourier transform, we have

$$|\hat{f}(\xi) - \hat{g}(\xi)| = \left| \int_{\mathbb{R}} (f(x) - g(x)) e^{-ix\xi} \, \mathrm{d}x \right| \leq \int_{\mathbb{R}} |f(x) - g(x)| \mathrm{d}x = \|f - g\|_{L^1} < \varepsilon.$$

Since $\hat{g}(\xi)$ is continuous and vanishes at infinity, for any $\delta > 0$, there exists $M > 0$ such that for all $|\xi| > M$, $|\hat{g}(\xi)| < \delta$. Therefore, for all $|\xi| > M$, we have

$$|\hat{f}(\xi)| \leq |\hat{g}(\xi)| + |\hat{f}(\xi) - \hat{g}(\xi)| < \delta + \varepsilon.$$

Since ε and δ were arbitrary, this shows that $\hat{f}(\xi)$ also vanishes at infinity. Thus, we conclude that $\hat{f} \in C_0$. $\square$