

# Differential Geometry: Homework 7

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**Exercise 4.2.1.** Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$\begin{aligned} F(u, v) &= (u \sin(\alpha) \cos(v), u \sin(\alpha) \sin(v), u \cos(\alpha)) \\ (u, v) \in U &= \{(u, v) \in \mathbb{R}^2 \mid u > 0\}, \quad \alpha = \text{const.} \end{aligned}$$

- (i) Prove that  $F$  is a local diffeomorphism of  $U$  onto a cone  $C$  with the vertex at the origin and  $2\alpha$  as the angle of the vertex.
- (ii) Is  $F$  a local isometry?

*Solution to (i).* Let  $F : U \rightarrow \mathbb{R}^3$  be defined by

$$F(u, v) = (u \sin(\alpha) \cos(v), u \sin(\alpha) \sin(v), u \cos(\alpha)), \quad u > 0, v \in \mathbb{R}.$$

We claim that  $F(U)$  is the cone

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \tan^2(\alpha), \quad z \geq 0\},$$

with vertex at the origin and vertex angle  $2\alpha$ .

First, note the vertex is at the origin since  $F(0, v) = (0, 0, 0)$  for all  $v$ .

Next, for any  $(u, v) \in U$ ,

$$x^2 + y^2 = (u \sin(\alpha) \cos(v))^2 + (u \sin(\alpha) \sin(v))^2 = u^2 \sin^2(\alpha) (\cos^2(v) + \sin^2(v)) = u^2 \sin^2(\alpha),$$

and

$$z^2 = (u \cos(\alpha))^2 = u^2 \cos^2(\alpha).$$

Hence,

$$x^2 + y^2 = z^2 \cdot \frac{\sin^2(\alpha)}{\cos^2(\alpha)} = z^2 \tan^2(\alpha),$$

confirming  $F(U) \subseteq C$ .

Finally, to show  $F$  is a local diffeomorphism, observe the Jacobian matrix of  $F$  has rank 2 for all  $(u, v) \in U$  (since the partial derivatives with respect to  $u$  and  $v$  are linearly independent), so  $F$  is an immersion and a local homeomorphism onto its image.

Therefore,  $F$  parametrizes the cone  $C$  locally diffeomorphically.

The vertex angle of the cone is  $2\alpha$  because the generating lines of the cone form an angle  $\alpha$  with the  $z$ -axis.  $\square$

*Solution to (ii).* Computing the partial derivatives, we have

$$F_u = (\sin(\alpha) \cos(v), \sin(\alpha) \sin(v), \cos(\alpha)) \quad \text{and} \quad F_v = (-u \sin(\alpha) \sin(v), u \sin(\alpha) \cos(v), 0).$$

Then, the coefficients for the first fundamental form are

$$\begin{aligned} E &= \langle F_u, F_u \rangle = \sin^2 \alpha + \cos^2 \alpha = 1 \\ F &= \langle F_u, F_v \rangle = 0 \\ G &= \langle F_v, F_v \rangle = u^2 \sin^2 \alpha. \end{aligned}$$

So the first fundamental form is

$$I = du^2 + u^2 \sin^2(\alpha) dv^2.$$

This is the same as the first fundamental form of a cone with vertex angle  $2\alpha$  parametrized by

$$G(u, v) = u(\sin(\alpha) \cos(v), \sin(\alpha) \sin(v), \cos(\alpha)).$$

Therefore, by proposition 1,  $F$  is a local isometry.  $\square$

**Exercise 4.2.2.** Prove the following “converse” of Prop. 1: Let  $\varphi : S \rightarrow \bar{S}$  be an isometry and  $\mathbf{x} : U \rightarrow S$  a parametrization at  $p \in S$ ; then  $\bar{\mathbf{x}} = \varphi \circ \mathbf{x}$  is a parametrization at  $\varphi(p)$  and  $E = \bar{E}$ ,  $F = \bar{F}$ ,  $G = \bar{G}$ .

*Solution.* Let  $\varphi : S \rightarrow \bar{S}$  be an isometry, and let  $\mathbf{x} : U \rightarrow S$  be a local parametrization at  $p \in S$ . Define  $\bar{\mathbf{x}} = \varphi \circ \mathbf{x} : U \rightarrow \bar{S}$ . Since  $\varphi$  is a diffeomorphism and  $\mathbf{x}$  is a parametrization, it follows that  $\bar{\mathbf{x}}$  is also a parametrization at  $\varphi(p)$ .

Since  $\varphi$  is an isometry, it preserves the inner product of tangent vectors. That is, for any  $q = \mathbf{x}(u, v)$  and any tangent vectors  $w_1, w_2 \in T_q(S)$ ,

$$\langle w_1, w_2 \rangle_q = \langle d\varphi_q(w_1), d\varphi_q(w_2) \rangle_{\varphi(q)}.$$

In particular, we consider the tangent vectors  $\mathbf{x}_u$  and  $\mathbf{x}_v$ , and compute the coefficients of the first fundamental form

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

Now, since  $\bar{\mathbf{x}} = \varphi \circ \mathbf{x}$ , the chain rule gives

$$\bar{\mathbf{x}}_u = d\varphi(\mathbf{x}_u) \quad \text{and} \quad \bar{\mathbf{x}}_v = d\varphi(\mathbf{x}_v).$$

Using the fact that  $\varphi$  is an isometry, we have

$$\bar{E} = \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = \langle d\varphi(\mathbf{x}_u), d\varphi(\mathbf{x}_u) \rangle = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = E$$

$$\bar{F} = \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle = \langle d\varphi(\mathbf{x}_u), d\varphi(\mathbf{x}_v) \rangle = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = F$$

$$\bar{G} = \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle = \langle d\varphi(\mathbf{x}_v), d\varphi(\mathbf{x}_v) \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = G.$$

Therefore, the first fundamental form of  $\bar{\mathbf{x}}$  is equal to that of  $\mathbf{x}$ , and so  $E = \bar{E}$ ,  $F = \bar{F}$ , and  $G = \bar{G}$ .  $\square$

**Exercise 4.2.3.** Show that a diffeomorphism  $\varphi : S \rightarrow \bar{S}$  is an isometry if and only if the arc length of any parametrized curve in  $S$  is equal to the arc length of the image curve by  $\varphi$ .

*Solution.* Assume  $\varphi$  is an isometry. Let  $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$  be a smooth parametrized curve, and let  $\bar{\alpha} = \varphi \circ \alpha : (-\varepsilon, \varepsilon) \rightarrow \bar{S}$  be the image of  $\alpha$  under  $\varphi$ . The arc length of  $\alpha$  is

$$s = \int_{-\varepsilon}^{\varepsilon} \|\alpha'(t)\| dt.$$

Since  $\varphi$  is an isometry, it preserves the inner product of tangent vectors. In particular, it preserves their lengths

$$\|\bar{\alpha}'(t)\| = \|d\varphi(\alpha'(t))\| = \|\alpha'(t)\|.$$

Therefore,

$$\bar{s} = \int_{-\varepsilon}^{\varepsilon} \|\bar{\alpha}'(t)\| dt = \int_{-\varepsilon}^{\varepsilon} \|\alpha'(t)\| dt = s.$$

So the arc length of  $\alpha$  is equal to the arc length of  $\bar{\alpha}$ .

Assume that for any parametrized smooth curve  $\alpha$  in  $S$ , the arc length of  $\alpha$  is equal to the arc length of  $\bar{\alpha} = \varphi \circ \alpha$  in  $\bar{S}$ . Let  $\mathbf{x} : U \rightarrow S$  be a local parametrization around  $p \in S$ , and define  $\bar{\mathbf{x}} = \varphi \circ \mathbf{x} : U \rightarrow \bar{S}$ , a local parametrization around  $\bar{p} = \varphi(p)$ . Let  $\mathbf{v} = (v_1, v_2)$  be a tangent vector at a point  $u \in U$ , and consider a curve  $\gamma(t) = \mathbf{x}(u + t\mathbf{v})$  in  $S$ . Then its arc length is given by

$$s = \int_{-\varepsilon}^{\varepsilon} \left\| \frac{d}{dt} \mathbf{x}(u + t\mathbf{v}) \right\| dt = \int_{-\varepsilon}^{\varepsilon} \sqrt{\mathbf{v}^T \cdot \mathbf{I}(u) \cdot \mathbf{v}} dt = 2\varepsilon \cdot \sqrt{\mathbf{v}^T \mathbf{I}(u) \mathbf{v}},$$

where  $I(u)$  is the first fundamental form matrix for  $\mathbf{x}$  at  $u$ . The image curve  $\bar{\gamma}(t) = \bar{\mathbf{x}}(u + t\mathbf{v})$  has arc length

$$\bar{s} = \int_{-\varepsilon}^{\varepsilon} \left\| \frac{d}{dt} \bar{\mathbf{x}}(u + t\mathbf{v}) \right\| dt = 2\varepsilon \cdot \sqrt{\mathbf{v}^T \bar{I}(u) \mathbf{v}},$$

where  $\bar{I}(u)$  is the first fundamental form matrix for  $\bar{\mathbf{x}}$  at  $u$ . Since arc length is preserved,  $s = \bar{s}$  for all  $\mathbf{v}$ , so

$$\mathbf{v}^T I(u) \mathbf{v} = \mathbf{v}^T \bar{I}(u) \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbb{R}^2.$$

This implies  $I(u) = \bar{I}(u)$ , so the first fundamental forms of  $\mathbf{x}$  and  $\bar{\mathbf{x}}$  agree at  $u$ . Since the first fundamental form determines the metric, this means that  $d\varphi_p$  preserves inner products between tangent vectors, i.e.,  $\varphi$  is an isometry.

Therefore,  $\varphi$  is an isometry if and only if the arc length of any parametrized curve in  $S$  is equal to the arc length of the image curve by  $\varphi$ .  $\square$

**Exercise 4.2.4.** Use the stereographic projection (cf. Exercise 16, Sec. 2-2) to show that the sphere is locally conformal to a plane.

*Solution.* Let  $S^2 \subset \mathbb{R}^3$  be the unit sphere centered at the origin. The stereographic projection from the north pole  $N = (0, 0, 1)$  onto the plane  $z = 0$  is given by

$$\pi(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right).$$

This map is smooth on  $S^2 \setminus \{N\}$  and its inverse is

$$\pi^{-1}(u, v) = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Let  $\mathbf{x}(u, v) = \pi^{-1}(u, v)$  be a parametrization of the sphere minus the north pole. Computing the first fundamental form of this parametrization yields

$$E = G = \frac{4}{(u^2 + v^2 + 1)^2}, \quad F = 0.$$

Thus, the metric in  $(u, v)$  coordinates is conformal to the Euclidean metric

$$I = \lambda(u, v)^2 (du^2 + dv^2), \quad \text{with } \lambda(u, v) = \frac{2}{u^2 + v^2 + 1}.$$

Therefore, the stereographic projection induces a conformal (angle-preserving) correspondence between the sphere (minus a point) and the plane. This proves that the sphere is locally conformal to a plane.  $\square$

**Exercise 4.2.8.** Let  $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a map such that

$$|G(p) - G(q)| = |p - q| \quad \text{for all } p, q \in \mathbb{R}^3.$$

(that is,  $G$  is a *distance-preserving* map). Prove that there exists  $p_0 \in \mathbb{R}^3$  and a linear isometry (cf. Exercise 7)  $F$  of the vector space  $\mathbb{R}^3$  such that

$$G(p) = F(p) + p_0 \quad \text{for all } p \in \mathbb{R}^3.$$

*Solution.* Define  $p_0 := G(0)$  and  $F(p) = G(p) - p_0$ , for all  $p \in \mathbb{R}^3$ . Then, for all  $p, q \in \mathbb{R}^3$ , we have

$$|F(p) - F(q)| = |(F(p) - p_0) - (F(q) - p_0)| = |G(p) - G(q)| = |p - q|.$$

So,  $F$  is a distance-preserving map. It also preserves the norms, via

$$\|F(p)\| = \|F(p) - F(0)\| = \|G(p) - G(0)\| = \|p - 0\| = \|p\|,$$

since

$$F(0) = G(0) - p_0 = G(0) - G(0) = 0.$$

We now show that  $F$  is a linear map. Since  $F$  is distance-preserving and satisfies  $F(0) = 0$ , it also preserves inner products. To see this, for all  $p, q \in \mathbb{R}^3$ , we use the polarization identity

$$\langle p, q \rangle = \frac{1}{2}(\|p + q\|^2 - \|p\|^2 - \|q\|^2),$$

and since  $F$  preserves norms, we get

$$\langle F(p), F(q) \rangle = \frac{1}{2}(\|F(p) + F(q)\|^2 - \|F(p)\|^2 - \|F(q)\|^2) = \frac{1}{2}(\|p + q\|^2 - \|p\|^2 - \|q\|^2) = \langle p, q \rangle.$$

Hence,  $F$  preserves inner products.

Now let  $p, q \in \mathbb{R}^3$  and  $\lambda \in \mathbb{R}$ . Then,

$$\|F(p + q) - (F(p) + F(q))\|^2 = \|F(p + q)\|^2 + \|F(p) + F(q)\|^2 - 2\langle F(p + q), F(p) + F(q) \rangle.$$

Because  $F$  preserves norms and inner products, this equals

$$\|p + q\|^2 + \|p + q\|^2 - 2\langle p + q, p + q \rangle = 0,$$

so  $F(p + q) = F(p) + F(q)$ . A similar argument shows  $F(\lambda p) = \lambda F(p)$ . Hence,  $F$  is linear.

Finally, since  $F$  is a linear map that preserves inner products, it is a linear isometry of  $\mathbb{R}^3$ . Therefore,  $G(p) = F(p) + p_0$ , with  $F$  a linear isometry and  $p_0 = G(0)$ , as desired.  $\square$

**Exercise 4.2.10.** Let  $S$  be a surface of revolution. Prove that the rotations about its axis are isometries of  $S$ .

*Solution.* Let  $\mathbf{x}(u, v) = (f(v) \cos(u), f(v) \sin(u), g(v))$  be a surface of revolution of the generating curve  $\alpha(u, v) = (f(v), g(v))$  along some axis of rotation. Computing the first fundamental form, we have

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = f^2(v) \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (f'(v))^2 + (g'(v))^2, \end{aligned}$$

giving us

$$I_p(\mathbf{w}) = f^2(v) du^2 + ((f'(v))^2 + (g'(v))^2) dv^2,$$

for some  $p \in S$  and  $\mathbf{w} \in T_p(S)$ .

Now consider the map  $\varphi_\theta(u, v) = (u + \theta, v)$ , which corresponds to a rotation about the axis of revolution by a fixed angle  $\theta$ . This induces a new parametrization of the surface,

$$\bar{\mathbf{x}}(u, v) = \mathbf{x}(\varphi_\theta(u, v)) = \mathbf{x}(u + \theta, v).$$

We compute the first fundamental form of  $\bar{\mathbf{x}}$  to get

$$\begin{aligned} \bar{\mathbf{x}}_u(u, v) &= \frac{\partial}{\partial u} \mathbf{x}(u + \theta, v) = \mathbf{x}_u(u + \theta, v) \\ \bar{\mathbf{x}}_v(u, v) &= \frac{\partial}{\partial v} \mathbf{x}(u + \theta, v) = \mathbf{x}_v(u + \theta, v). \end{aligned}$$

Then the coefficients of the first fundamental form for  $\bar{\mathbf{x}}$  are

$$\begin{aligned} \bar{E} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = \langle \mathbf{x}_u(u + \theta, v), \mathbf{x}_u(u + \theta, v) \rangle = f^2(v) \\ \bar{F} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle = \langle \mathbf{x}_u(u + \theta, v), \mathbf{x}_v(u + \theta, v) \rangle = 0 \end{aligned}$$

$$\bar{G} = \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle = \langle \mathbf{x}_v(u + \theta, v), \mathbf{x}_v(u + \theta, v) \rangle = (f'(v))^2 + (g'(v))^2.$$

These are identical to the original coefficients  $E, F, G$ .

Since the map  $\varphi_\theta(u, v) = (u + \theta, v)$  is smooth, has a smooth inverse  $\varphi_{-\theta}$ , and preserves the first fundamental form everywhere on the domain of the parametrization, it defines a global isometry of the surface. In particular,  $\varphi_\theta$  acts globally on  $S$  because the surface of revolution is entirely covered by the parametrization  $\mathbf{x}(u, v)$  with  $u \in [0, 2\pi) \pmod{2\pi}$  and  $v$  in an interval  $I$ . Thus, the rotations about the axis of revolution define smooth diffeomorphisms from  $S$  to itself that preserve the Riemannian metric. Therefore, they are global isometries.  $\square$

**Exercise 4.2.14.** We say that a differentiable map  $\varphi : S_1 \rightarrow S_2$  *preserves angles* when for every  $p \in S_1$  and every pair  $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S_1)$  we have

$$\cos(\mathbf{v}_1, \mathbf{v}_2) = \cos(d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2)).$$

Prove that  $\varphi$  is locally conformal if and only if it preserves angles.

*Solution.* Assume  $\varphi : S_1 \rightarrow S_2$  is a locally conformal map, i.e., for all  $p \in S_1$  and  $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S_1)$ , we have

$$\langle d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2) \rangle = \lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad (1)$$

where  $\lambda(p)$  is a nowhere-zero differentiable function on  $S$ . Notice that  $\|d\varphi_p(\mathbf{v}_1)\| = \sqrt{\langle d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_1) \rangle} = \sqrt{\lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \lambda(p) \|\mathbf{v}_1\|$ , and similarly for  $\mathbf{v}_2$ . Dividing both sides of equation 1 by  $\|d\varphi_p(\mathbf{v}_1)\| \|d\varphi_p(\mathbf{v}_2)\|$ , we obtain

$$\begin{aligned} \cos(d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2)) &= \frac{\langle d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2) \rangle}{\|d\varphi_p(\mathbf{v}_1)\| \|d\varphi_p(\mathbf{v}_2)\|} \\ &= \frac{\lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\lambda(p) \|\mathbf{v}_1\| \lambda(p) \|\mathbf{v}_2\|} \\ &= \frac{\lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\lambda^2(p) \|\mathbf{v}_1\| \|\mathbf{v}_2\|} \\ &= \frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|} \\ &= \cos(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Therefore,  $\varphi$  preserves angles if  $\varphi$  is locally conformal.

Assume  $\varphi$  preserves angles. Then for all  $p \in S_1$  and all  $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S_1)$ , we have

$$\frac{\langle d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2) \rangle}{\|d\varphi_p(\mathbf{v}_1)\| \|d\varphi_p(\mathbf{v}_2)\|} = \frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|}.$$

Let  $\mathbf{v} \in T_p(S_1)$  be a nonzero vector, and define

$$\lambda(p) := \frac{\|d\varphi_p(\mathbf{v})\|}{\|\mathbf{v}\|} > 0.$$

We claim that for all  $\mathbf{w} \in T_p(S_1)$ , we have

$$\|d\varphi_p(\mathbf{w})\| = \lambda(p) \|\mathbf{w}\| \quad \text{and} \quad \langle d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2) \rangle = \lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle.$$

We claim that for all  $\mathbf{w} \in T_p(S_1)$ , we have

$$\|d\varphi_p(\mathbf{w})\| = \lambda(p) \|\mathbf{w}\| \quad \text{and} \quad \langle d\varphi_p(\mathbf{v}_1), d\varphi_p(\mathbf{v}_2) \rangle = \lambda^2(p) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle.$$

To see this, fix any  $\mathbf{w} \in T_p(S_1)$  and apply angle preservation to the pair  $\mathbf{v}, \mathbf{w}$ . Then

$$\cos(\mathbf{v}, \mathbf{w}) = \cos(\mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w})) = \frac{\langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle}{\|\mathrm{d}\varphi_p(\mathbf{v})\| \|\mathrm{d}\varphi_p(\mathbf{w})\|}.$$

Substituting  $\lambda(p) = \|\mathrm{d}\varphi_p(\mathbf{v})\| / \|\mathbf{v}\|$  and rearranging gives

$$\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{\langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle}{\lambda(p) \|\mathbf{v}\| \|\mathrm{d}\varphi_p(\mathbf{w})\|} \Rightarrow \langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle = \lambda(p) \langle \mathbf{v}, \mathbf{w} \rangle \cdot \frac{\|\mathrm{d}\varphi_p(\mathbf{w})\|}{\|\mathbf{w}\|}.$$

So

$$\frac{\|\mathrm{d}\varphi_p(\mathbf{w})\|}{\|\mathbf{w}\|} = \lambda(p),$$

and hence

$$\|\mathrm{d}\varphi_p(\mathbf{w})\| = \lambda(p) \|\mathbf{w}\|.$$

Now that both  $\mathbf{v}$  and  $\mathbf{w}$  scale by  $\lambda(p)$ , it follows that

$$\langle \mathrm{d}\varphi_p(\mathbf{v}), \mathrm{d}\varphi_p(\mathbf{w}) \rangle = \lambda^2(p) \langle \mathbf{v}, \mathbf{w} \rangle.$$

Since  $\mathbf{v}, \mathbf{w}$  were arbitrary, this proves that  $\varphi$  is locally conformal at  $p$ .

Therefore,  $\varphi$  is locally conformal if and only if it preserves angles.  $\square$

**Exercise 4.2.15.** Let  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $\varphi(x, y) = (u(x, y), v(x, y))$ , where  $u$  and  $v$  are differentiable functions that satisfy the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Show that  $\varphi$  is a local conformal map from  $\mathbb{R}^2 - Q$  into  $\mathbb{R}^2$ , where  $Q = \{(x, y) \in \mathbb{R}^2 \mid u_x^2 + u_y^2 = 0\}$

*Solution.* Let  $\varphi(x, y) = (u(x, y), v(x, y))$  be a differentiable map satisfying the Cauchy-Riemann equations. Then, the Jacobian matrix  $D\varphi$  at a point  $(x, y)$  is

$$D\varphi = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix},$$

since  $\varphi$  satisfies the Cauchy-Riemann equations.

This matrix is of the form

$$D\varphi = u_x \begin{bmatrix} 1 & \frac{u_y}{u_x} \\ -\frac{u_y}{u_x} & 1 \end{bmatrix} \quad (\text{when } u_x \neq 0).$$

More importantly, for all points not in  $Q = \{(x, y) \mid u_x^2 + u_y^2 = 0\}$ , the Jacobian is nonzero and we can interpret  $D\varphi$  as a similarity transformation: a composition of a rotation and a scaling.

Indeed, the Jacobian satisfies

$$D\varphi^\top D\varphi = \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} = \begin{bmatrix} u_x^2 + u_y^2 & 0 \\ 0 & u_x^2 + u_y^2 \end{bmatrix} = (u_x^2 + u_y^2)I.$$

This shows that the differential of  $\varphi$  preserves angles and scales all vectors by the same factor locally – that is, it is conformal.

Therefore,  $\varphi$  is a local conformal map on  $\mathbb{R}^2 \setminus Q$ .  $\square$