

# Functional Complex Variables I: Homework 7

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**Exercise 4.49.7.** Show that if  $C$  is a positively oriented simple closed contour, then the area of the region enclosed by  $C$  can be written

$$\frac{1}{2i} \int_C \bar{z} dz.$$

*Solution.* Let  $z = x + iy$ , where  $x, y \in \mathbb{R}$ . Then, we have

$$\bar{z} = x - iy \quad \text{and} \quad dz = dx + i dy.$$

Now compute the integrand:

$$\bar{z} dz = (x - iy)(dx + i dy) = x dx + xi dy - iy dx - i^2 y dy.$$

Since  $i^2 = -1$ , we get

$$\bar{z} dz = x dx + ix dy - iy dx + y dy = (x dx + y dy) + i(x dy - y dx).$$

So,

$$\frac{1}{2i} \int_C \bar{z} dz = \frac{1}{2i} \int_C [(x dx + y dy) + i(x dy - y dx)] = \frac{1}{2i} \left[ \int_C (x dx + y dy) + i \int_C (x dy - y dx) \right].$$

The first integral  $\int_C (x dx + y dy)$  is zero because it represents the line integral of the gradient of the scalar function  $1/2(x^2 + y^2)$ , and over a closed path the integral of a gradient is zero

$$\int_C (x dx + y dy) = 0.$$

So we're left with just

$$\frac{1}{2i} \int_C \bar{z} dz = \frac{1}{2i} \cdot i \int_C (x dy - y dx) = \frac{1}{2} \int_C (x dy - y dx).$$

Notice that this is an expression for the area of the planar region bounded by a positively oriented curve,

$$A = \frac{1}{2} \int_C (x dy - y dx).$$

Therefore,

$$A = \frac{1}{2i} \int_C \bar{z} dz. \quad \square$$

**Exercise 4.52.1.** Let  $C$  denote the positively oriented boundary of the square whose sides lie along the lines  $x = \pm 2$  and  $y = \pm 2$ . Evaluate each of these integrals:

$$\begin{array}{lll} \text{(i)} \int_C \frac{e^{-z}}{z - (\pi i/2)} dz; & \text{(ii)} \int_C \frac{\cos(z)}{z(z^2 + 8)} dz; & \text{(iii)} \int_C \frac{z}{2z + 1} dz; \\ \text{(iv)} \int_C \frac{\cosh(z)}{z^4} dz; & \text{(v)} \int_C \frac{\tan(z/2)}{(z - x_0)^2} dz. & \end{array}$$

*Solution to (i).* Let  $f(z) = e^{-z}$ , which is entire (analytic everywhere), and note that  $\pi i/2$  lies inside the square since its imaginary part is  $\frac{\pi}{2} < 2$ . By Cauchy's Integral Formula, we have

$$\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}} dz = 2\pi i \cdot e^{-\frac{\pi i}{2}} = 2\pi i \cdot \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) = 2\pi i \cdot (-i) = 2\pi. \quad \square$$

*Solution to (ii).* The singularities are at  $z = 0$  and  $z = \pm 2\sqrt{2}i$ . All of these are within the square (since  $\sqrt{8} \approx 2.828 < 4$ ). Let

$$f(z) = \frac{\cos(z)}{z^2 + 8},$$

which is analytic at  $z = 0$ . Then by Cauchy's Integral Formula, we have

$$\int_C \frac{\cos(z)}{z(z^2 + 8)} dz = 2\pi i \cdot \frac{\cos(0)}{0^2 + 8} = 2\pi i \cdot \frac{1}{8} = \frac{\pi i}{4}. \quad \square$$

*Solution to (iii).* The integrand can be rewritten as

$$\frac{z}{2z + 1} = \frac{1}{2} \cdot \frac{2z}{2z + 1} = \frac{1}{2} \cdot \left(1 - \frac{1}{2z + 1}\right).$$

Now integrate term-by-term, we have

$$\int_C \frac{z}{2z + 1} dz = \frac{1}{2} \int_C \left(1 - \frac{1}{2z + 1}\right) dz.$$

The integral of 1 over a closed contour is 0, and  $1/(2z + 1)$  has a simple pole at  $z = -\frac{1}{2}$ , which lies inside the square. So:

$$\int_C \frac{z}{2z + 1} dz = -\frac{1}{2} \int_C \frac{1}{2z + 1} dz = -\frac{1}{2} \cdot 2\pi i \cdot \frac{1}{2} = -\frac{\pi i}{2}. \quad \square$$

*Solution to (iv).* We use the fact that

$$\int_C \frac{f(z)}{(z - z_0)^n} dz = \frac{2\pi i}{(n - 1)!} f^{(n-1)}(z_0),$$

for  $n = 4$  and  $z_0 = 0$ . Here,  $f(z) = \cosh(z)$  is entire, so, we have

$$\int_C \frac{\cosh(z)}{z^4} dz = \frac{2\pi i}{3!} \cosh^{(3)}(0).$$

Recall the Taylor series expansion for  $\cosh(z)$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!},$$

which imply that  $\cosh^{(3)}(0) = 0$ . Therefore,

$$\int_C \frac{\cosh(z)}{z^4} dz = 0. \quad \square$$

*Solution to (v).* Let  $f(z) = \tan(z/2)$ , which is analytic inside the square (its singularities occur at  $z = (2n + 1)\pi$ , none of which are inside the square since  $\pi > 3$ ).

The integrand has a pole of order 2 at  $z = x_0 \in (-2, 2)$ , and we can use the derivative form of Cauchy's Integral Formula,

$$\int_C \frac{f(z)}{(z - x_0)^2} dz = 2\pi i \cdot f'(x_0).$$

Computing the derivative of  $f(z)$ , we have

$$f'(z) = \frac{1}{2} \sec^2\left(\frac{z}{2}\right) \Rightarrow f'(x_0) = \frac{1}{2} \sec^2\left(\frac{x_0}{2}\right).$$

Therefore, we have

$$\int_C \frac{\tan(z/2)}{(z - x_0)^2} dz = \pi i \cdot \sec^2\left(\frac{x_0}{2}\right). \quad \square$$

**Exercise 4.52.2.** Find the value of the integral of  $g(z)$  around the circle  $|z - i| = 2$  in the positive sense when

$$(i) \ g(z) = \frac{1}{z^2 + 4};$$

$$(ii) \ g(z) = \frac{1}{(z^2 + 4)^2}.$$

*Solution to (i).* Factoring the denominator, we have  $z^2 + 4 = (z + 2i)(z - 2i)$ . The singularities are at  $z = 2i$  and  $z = -2i$ . The contour  $|z - i| = 2$  is centered at  $z = i$  and has radius 2. Therefore,  $|2i - i| = 1 < 2$  and  $|-2i - i| = 3 > 2$ , so the singularity at  $z = 2i$  lies *inside* the contour, while the singularity at  $z = -2i$  lies *outside*.

Since only  $z = 2i$  is inside, we write

$$g(z) = \frac{1}{(z - 2i)(z + 2i)} = \frac{1}{z + 2i} \cdot \frac{1}{z - 2i}.$$

The function  $f(z) = 1/(z + 2i)$  is analytic on and inside the contour (since  $z = -2i$  is outside). So we can use Cauchy's Integral Formula for the simple pole at  $z = 2i$

$$\int_{|z-i|=2} \frac{f(z)}{z - 2i} dz = 2\pi i \cdot f(2i) = 2\pi i \cdot \frac{1}{2i + 2i} = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}. \quad \square$$

*Solution to (ii).* Again, factoring the denominator, we have  $(z^2 + 4)^2 = [(z - 2i)(z + 2i)]^2$ . So the integrand has a pole of order 2 at  $z = 2i$ , which is *inside* the circle, and another at  $z = -2i$ , which is *outside* the circle. We can rewrite the integrand as

$$g(z) = \frac{1}{[(z - 2i)^2(z + 2i)^2]} = \frac{1}{(z + 2i)^2} \cdot \frac{1}{(z - 2i)^2}.$$

Let

$$f(z) = \frac{1}{(z + 2i)^2},$$

which is analytic inside and on the circle (since  $z = -2i$  is outside). We apply the Cauchy Integral Formula for derivatives, to get

$$\int_{|z-i|=2} \frac{f(z)}{(z - 2i)^2} dz = 2\pi i \cdot f'(2i).$$

Computing the derivative of  $f$ , we have  $f'(z) = -2(z + 2i)^{-3}$ . So,

$$f'(2i) = -2(4i)^{-3} = -2 \cdot \frac{1}{64i^3} = -2 \cdot \frac{1}{64(-i)} = \frac{2}{64i} = \frac{1}{32i}.$$

Therefore, we have

$$\int_{|z-i|=2} \frac{1}{(z^2 + 4)^2} dz = 2\pi i \cdot \frac{1}{32i} = \frac{\pi}{16}. \quad \square$$

**Exercise 4.52.3.** Let  $C$  be the circle  $|z| = 3$ , described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds \quad (|z| \neq 3),$$

then  $g(2) = 8\pi i$ . What is the value of  $g(z)$  when  $|z| > 3$ ?

*Solution.* Assume that  $|2| < 3$ . Since  $z = 2$  is strictly *inside* the contour  $C$  (because  $|2| < 3$ ), we apply Cauchy's Integral Formula, to get

$$\int_C \frac{f(s)}{s - z} ds = 2\pi i \cdot f(z).$$

Therefore, we have

$$g(2) = 2\pi i \cdot f(2) = 2\pi i \cdot (2(2)^2 - 2 - 2) = 2\pi i \cdot (8 - 2 - 2) = 2\pi i \cdot 4 = 8\pi i.$$

Assume that  $|z| > 3$ , i.e.,  $z$  is *outside* the contour  $C$ . In this case, the function  $f(s)/(s - z)$  is analytic in  $s$  on and inside the contour  $C$ , because  $z$  is outside the region enclosed by  $C$  and  $f$  is entire.

Since the integrand is analytic inside and on  $C$ , and  $C$  is a closed curve, the Cauchy-Goursat Theorem implies that

$$g(z) = \int_C \frac{f(s)}{s - z} ds = 0. \quad \square$$

**Exercise 4.52.4.** Let  $C$  be any simple closed contour, described in the positive sense in the  $z$ -plane, and write

$$g(z) = \int_C \frac{s^3 + 2z}{(s - z)^3} ds.$$

Show that  $g(z) = 6\pi iz$  when  $z$  is inside  $C$  and that  $g(z) = 0$  when  $z$  is outside.

*Solution.* If  $z$  is *outside* the contour  $C$ , then the integrand is analytic on and inside  $C$  since the denominator never vanishes. Therefore, by Cauchy's Theorem, we have

$$g(z) = \int_C \frac{s^3 + 2z}{(s - z)^3} ds = 0.$$

Now, for when  $z$  is *inside* the contour  $C$ . We can write the integrand as

$$\frac{s^3 + 2z}{(s - z)^3} = \frac{s^3}{(s - z)^3} + \frac{2z}{(s - z)^3}.$$

Let us evaluate each term using Cauchy's Integral Formula for derivatives to get

$$\int_C \frac{f(s)}{(s - z)^n} ds = \frac{2\pi i}{(n - 1)!} f^{(n-1)}(z), \quad n \geq 1.$$

For the first term, we have  $f(s) = s^3$  and we compute the second derivative of  $f$  at  $s = z$  to get

$$f'(s) = 3s^2 \quad \text{and} \quad f''(s) = 6s \Rightarrow f''(z) = 6z.$$

Thus, we have

$$\int_C \frac{s^3}{(s - z)^3} ds = \frac{2\pi i}{2!} \cdot 6z = \pi i \cdot 6z = 6\pi iz.$$

For the second term, we have  $f(s) = 2z$ , which is constant with respect to  $s$ , so, we have

$$\int_C \frac{2z}{(s - z)^3} ds = 2z \cdot \int_C \frac{1}{(s - z)^3} ds.$$

But  $1/(s - z)^3$  is the third derivative kernel for the constant function, which is 0, giving us

$$\int_C \frac{1}{(s - z)^3} ds = 0.$$

So the second term vanishes. Therefore, we have

$$g(z) = \int_C \frac{s^3 + 2z}{(s - z)^3} ds = 6\pi iz + 0 = 6\pi iz. \quad \square$$

**Exercise 4.52.7.** Let  $C$  be the unit circle  $z = e^{i\theta}$  ( $-\pi \leq \theta \leq \pi$ ). First show that for any real constant  $a$ ,

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

Then, write this integral in terms of  $\theta$  to give the integration formula

$$\int_0^\pi e^{a \cos(\theta)} \cos(a \sin(\theta)) d\theta = \pi.$$

*Solution.* Note that the function  $\frac{e^{az}}{z}$  is analytic everywhere inside and on  $C$ , except for a simple pole at  $z = 0$ , which lies inside  $C$ . We apply the Cauchy Integral Formula, to get

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0),$$

for a function  $f$  analytic on and inside  $C$ , and  $z_0$  inside  $C$ . In our case,  $f(z) = e^{az}$ , and  $z_0 = 0$ , so

$$\int_C \frac{e^{az}}{z} dz = 2\pi i e^{a \cdot 0} = 2\pi i.$$

Let  $z = e^{i\theta}$ , with  $-\pi \leq \theta \leq \pi$ , then  $dz = ie^{i\theta} d\theta$ . So, we have

$$\int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} \cdot ie^{i\theta} d\theta = i \int_{-\pi}^{\pi} e^{ae^{i\theta}} d\theta.$$

Now write  $e^{ae^{i\theta}}$  in terms of real and imaginary parts, to get

$$e^{ae^{i\theta}} = e^{a(\cos \theta + i \sin \theta)} = e^{a \cos \theta} \cdot e^{ia \sin \theta} = e^{a \cos \theta} [\cos(a \sin \theta) + i \sin(a \sin \theta)].$$

So,

$$\begin{aligned} i \int_{-\pi}^{\pi} e^{ae^{i\theta}} d\theta &= i \int_{-\pi}^{\pi} e^{a \cos \theta} [\cos(a \sin \theta) + i \sin(a \sin \theta)] d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta - \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) d\theta. \end{aligned}$$

We know this entire expression equals  $2\pi i$ , so

$$i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta - \int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) d\theta = 2\pi i.$$

Equating real and imaginary parts, the imaginary part gives

$$\int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2\pi.$$

Since the integrand is even (both  $\cos$  and  $e^{a \cos \theta}$  are even functions),

$$\int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2 \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta.$$

So,

$$2 \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2\pi \Rightarrow \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi.$$

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