Math 307, Homework # 1
Due Wednesday, October 9
SOLUTIONS TO SELECTED PROBLEMS

1. In English we have lots of different ways of expressing conditional statements. For each of the following, rewrite it using the "If–Then" form in a way that preserves the meaning:
   (a) Passing the test requires getting 70%.
   (b) My cat gets scared when the phone rings.
   (c) I get mad whenever you do that.
   (d) Being first in line guarantees getting a ticket.
   (e) You will fail the class unless you hand in your exam.

2. Recall the tautologies $\sim(P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q)$, $\sim(P \vee Q) \Leftrightarrow (\sim P \wedge \sim Q)$, and $\sim(P \Rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$. Use these to write English language versions for the negation of each sentence below:
   (a) If you wash the dishes, I will pay you \$10.
   (b) I will wash the car or I will mow the lawn.
   (c) I ate steak last night and I ate cereal this morning.
   (d) If I go to the movies, then I will see Star Wars or I will see Harry Potter.
   (e) If it is raining, then I will stay home or I will go to the movies.
   (f) If I am a monkey and you are a baboon then pigs can fly.

3. Give the truth table for each of the following statements. Say which of the statements are tautologies.
   (a) $\sim[\sim P \wedge \sim Q]$
   (b) $(P \vee Q) \Leftrightarrow (\sim P \Rightarrow Q)$
   (c) $(Q \wedge \sim Q) \Rightarrow P$
   (d) $Q \wedge (P \Rightarrow Q)$
   (e) $(P \wedge Q) \Rightarrow (P \vee Q)$
   (f) $(P \wedge Q) \Leftrightarrow (\sim Q \Rightarrow P)$
   (g) $(P \vee Q) \Rightarrow (\sim Q \vee P)$
   (h) $[P \Rightarrow (Q \Rightarrow R)] \Leftrightarrow [(P \wedge Q) \Rightarrow R]$.

For the proofs below, you can use the logic rules MP, LCS/RCS, CI, and MT. See the list of rules on the course webpage. For exercises that call for filling in blanks, you need to copy the whole proof onto a sheet of your homework—don't just fill in the blanks on this exercise sheet.

When you use a tautology, say where you have seen it before: either cite it by name (e.g., DeMorgan's Law), say"in class", or refer to a specific homework problem where you checked it was a tautology. If you can't do any of these, include the truth table that shows it is a tautology.

4. Prove that $P, P \Leftrightarrow Q \vdash Q$ by filling in the blanks below. This rule of logic is called "Modus Ponens for the Biconditional (MPB)". Once you have proven it, you can use it in the other problems.

| | Statement | Explanation |
|---|---|---|
| 1. | $P$ | hypothesis |
| 2. | $P \Leftrightarrow Q$ | hyp. |
| 3. | $P \Rightarrow Q$ | LCS for 2 (remember that $P \Leftrightarrow Q := (P \Rightarrow Q) \wedge (Q \Rightarrow P)$) |
| 4. | $Q$ | MP for 3 and 1 |

5. Fill in the blanks to give a proof of $[P \wedge Q] \Rightarrow R$, $\sim R$, $P \vdash \sim Q$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $[P \wedge Q] \Rightarrow R$ | hyp. |
| 2. | $\sim R$ | hyp. |
| 3. | $\sim [P \wedge Q]$ | MT, For 1, For 2 |
| 4. | $\sim [P \wedge Q] \Leftrightarrow [\sim P \vee \sim Q]$ | taut., De Morgan |
| 5. | $\sim P \vee \sim Q$ | MPB, For 4, For 3 |
| 6. | $[\sim P \vee \sim Q] \Rightarrow [P \Rightarrow \sim Q]$ | taut., compare with $[\sim P \vee \sim Q] \Leftrightarrow [P \Rightarrow \sim Q]$ |
| 7. | $P \Rightarrow \sim Q$ | MP, For 6, For 5 |
| 8. | P | hyp. |
| 9. | $\sim Q$ | MP, For 7, For 8 |

6. Show that $\sim Q \Rightarrow \sim P$, $\sim Q \wedge R \vdash \sim P$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $\sim Q \Rightarrow \sim P$ | hyp. |
| 2. | $\sim Q \wedge R$ | hyp. |
| 3. | $\sim Q$ | LCS, For 2 |
| 4. | $\sim P$ | MP, For 1, For 3. |

7. Show that $Q \vee R$, $\sim R \vdash Q$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $Q \vee R$ | hyp. |
| 2. | $\sim R$ | hyp. |
| 3. | $(Q \vee R) \Leftrightarrow (\sim R \Rightarrow Q)$ | taut. |
| 4. | $\sim R \Rightarrow Q$ | MPB, For 3, For 1 |
| 5. | $Q$ | MP, For 4, For 2 |

8. Show that $R \Rightarrow S$, $R \wedge Q \vdash R \wedge S$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $R \Rightarrow S$ | hyp. |
| 2. | $R \wedge Q$ | hyp. |
| 3. | $R$ | LCS, For 2 |
| 4. | $S$ | MP, For 1, For 3 |
| 5. | $R \wedge S$ | CI, For 3, For 4 |

9. Show that $\sim [P \wedge R] \Rightarrow \sim S$, $\sim S \Rightarrow Q$, $\sim [P \wedge R] \vdash \sim S \wedge Q$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $\sim [P \wedge R] \Rightarrow \sim S$ | hyp. |
| 2. | $\sim S \Rightarrow Q$ | hyp. |
| 3. | $\sim [P \wedge R]$ | hyp. |
| 4. | $\sim S$ | MP For 1, For 3 |
| 5. | $Q$ | MP For 2, For 4 |
| 6. | $\sim S \wedge Q$ | CI, For 4, For 5 |

10. Show that $[P \vee R] \Rightarrow Q$, $P \vdash P \wedge Q$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $[P \vee R] \Rightarrow Q$ | hyp. |
| 2. | $P$ | hyp. |
| 3. | $P \Rightarrow [P \vee R]$ | taut. |
| 4. | $P \vee R$ | MP For 3, For 2 |
| 5. | $Q$ | MP For 1, For 4 |
| 6. | $P \wedge Q$ | CI, For 2, For 5 |

11. Show that $R \wedge \sim Q$, $P \Rightarrow Q \vdash R \wedge \sim P$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $R \wedge \sim Q$ | hyp. |
| 2. | $P \Rightarrow Q$ | hyp. |
| 3. | $R$ | LCS, For 1 |
| 4. | $\sim Q$ | RCS, For 1 |
| 5. | $\sim P$ | MT, For 2, For 4 |
| 6. | $R \wedge \sim P$ | CI, For 3, For 5. |

12. Show that $Q$, $[P \Rightarrow Q] \Rightarrow [R \wedge S] \vdash S$. (This can be done in 6 lines if you find the right tautology. Try working backwards.)

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $Q$ | hyp. |
| 2. | $[P \Rightarrow Q] \Rightarrow [R \wedge S]$ | hyp. |
| 3. | $Q \Rightarrow [P \Rightarrow Q]$ | taut. |
| 4. | $P \Rightarrow Q$ | MP For 3, For 1 |
| 5. | $R \wedge S$ | MP, For 2, For 4 |
| 6. | $S$ | RCS, For 5. |

13. Show that $Q \Rightarrow S$, $\sim S \vee R$, $\sim R \vdash \sim Q$.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $Q \Rightarrow S$ | hyp. |
| 2. | $\sim S \vee R$ | hyp. |
| 3. | $\sim R$ | hyp. |
| 4. | $(\sim S \vee R) \Leftrightarrow (S \Rightarrow R)$ | taut. |
| 5. | $S \Rightarrow R$ | MPB, For 4, For 2 |
| 6. | $\sim S$ | MT, For 5, For 3. |
| 7. | $\sim Q$ | MT, For 1, For 6. |

14. There are five people considering going to the beach: Amanda, Bill, Chris, Debbie, and Eeyore. The following are true statements:

- If Amanda goes to the beach, then Eeyore will not go.
- If Debbie goes (to the beach), then Chris will go too.
- If Debbie does not go, then Eeyore will go.
- If Eeyore goes to the beach, then either Amanda will go or Bill will not go.

This is all too complicated for Chris, who gets fed up and decides not to go to the beach. Determine who goes to the beach, and who stays home.

Convert this into a problem in symbolic logic. I suggest you use "A" to stand for the proposition "Amanda goes to the beach". Give a symbolic logic proof, as we have been doing in the above exercises, where you use the rules of logic to deduce who does and does not go to the beach. You should have five hypotheses.

Proof.

| | Statement | Explanation |
|---|---|---|
| 1. | $A \Rightarrow \sim E$ | hyp. |
| 2. | $D \Rightarrow C$ | hyp. |
| 3. | $\sim D \Rightarrow E$ | hyp. |
| 4. | $E \Rightarrow (A \vee \sim B)$ | hyp. |
| 5. | $\sim C$ | hyp. |
| 6. | $\sim D$ | MT, For 2, For 5. |
| 7. | $E$ | MP, For 3, For 6. |
| 8. | $\sim A$ | MT, For 1, For 7 |
| 9. | $A \vee \sim B$ | MP For 4, For 7 |
| 10. | $(A \vee \sim B) \Leftrightarrow (\sim A \Rightarrow \sim B)$ | taut. |
| 11. | $\sim A \Rightarrow \sim B$ | MPB For 10, For 9 |
| 12. | $\sim B$ | MP, For 11, For 8 |

The above demonstration has shown $\sim A$, $\sim B$, $\sim C$, $\sim D$, and $E$. So Eeyore is the only one who goes to the beach (and he is very sad that everyone deserted him).

15. I have a penny, a dime, and a quarter. If you make a true statement, I will give you one of the coins. If you make a false statement, I will give you nothing. What statement can you make that will *guarantee* that you get the quarter?

    Answer: The easiest solution is to say "You will not give me the penny AND you will not give me the dime". Another possibility is "I will get the quarter OR I will get nothing."

16. Write out the operation tables for $(\mathbb{Z}_5, \cdot_5)$ and $(\mathbb{Z}_8, +_8)$.

17. Solve the equation $(7 \cdot_{11} x) +_{11} 10 = 4$ in $\mathbb{Z}_{11}$. (As always in this course, you should explain your methods. Just writing down the answer with no explanation will not earn many points.)

    Solution.

    Let us add 1 (additive inverse of 10) to both sides and use associativity of addition:

$$((7 \cdot_{11} x) +_{11} 10) +_{11} 1 = 4 +_{11} 1,$$
$$(7 \cdot_{11} x) +_{11} (10 +_{11} 1) = 5,$$
$$(7 \cdot_{11} x) +_{11} 0 = 5,$$
$$7 \cdot_{11} x = 5.$$

Now let us multiply the last equation by 8 (multiplicative inverse of 7) and use associativity of multiplication:

$$8 \cdot_{11} (7 \cdot_{11} x) = 8 \cdot_{11} 5,$$
$$(8 \cdot_{11} 7) \cdot_{11} x = 7,$$
$$1 \cdot_{11} x = 7,$$
$$x = 7.$$

Let us check that $x = 7$ is a solution:

$$(7 \cdot_{11} x) +_{11} 10 = (7 \cdot_{11} 7) +_{11} 10 = 5 +_{11} 10 = 4.$$

Answer: $x = 7$ as element of $\mathbb{Z}_{11}$.

18. Make a table showing all the perfect squares in $\mathbb{Z}_{11}$ (I mean $0^2$, $1^2$, $2^2$, and so on). Use this to help you find all solutions to the equation

$$x^2 + 4x + 10 = 0$$

in $\mathbb{Z}_{11}$. (Hint: Complete the square.) I strongly suggest that you check your answers at the end, by plugging them back into the quadratic equation.

Solution.

Let us complete the square:

$$x^2 + 4x + 10 = 0$$

$$x^2 + 4x + 4 + 6 = 0$$

$$(x+2)^2 + 6 = 0$$

$$(x+2)^2 + 6 = 0.$$

Now let us add 5 (additive inverse of 6) to the equation:

$$(x+2)^2 + 6 + 5 = 5$$

$$(x+2)^2 = 5.$$

Thus we are interested in values of $x$ such that the square of $x+2$ is 5. Let us make a table of squares modulo 11 (we use letter $m$ since $x$ has a specific different meaning above):

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m^2 \pmod{11}$ | 0 | 1 | 4 | 9 | 5 | 3 | 3 | 5 | 9 | 4 | 1 |

Thus 5 is indeed a square and we have 2 options:

$$x + 2 = 4 \text{ or } x + 2 = 7.$$

By adding 9 (additive inverse of 2) we get

$$x = 4 + 9 \equiv 2 \pmod{11} \text{ or } x = 7 + 9 \equiv 5 \pmod{11}.$$

Let us check these solutions:

$$x = 2 : x^2 + 4x + 10 = 2^2 + 4 \cdot 2 + 10 = 22 \equiv 0 \pmod{11},$$

$$x = 5 : x^2 + 4x + 10 = 5^2 + 4 \cdot 5 + 10 = 55 \equiv 0 \pmod{11}.$$

Thus both solutions work.

Answer: there are two solutions $x = 2$ and $x = 5$ (as elements of $\mathbb{Z}_{11}$).