Math 307, Homework #4
Due Wednesday, October 30

The most common way to prove a statement of the form $(\exists x)[P(x)]$ is to produce an explicit example of an x for which $P(x)$ is true. For example, to prove

$$(\exists n)[n \in \mathbb{N} \wedge 3|(n^2+2n)]$$

we can simply say "4 is a natural number, $4^2 + 2 \cdot 4 = 24$, and $3|24$." This is called *giving an example*.

The negation of $(\forall x)[P(x)]$ is $(\exists x)[\sim P(x)]$. Disproving the former statement is the same as proving the latter statement. So applying the above principle, if we want to disprove a "for all" statement then we should try to produce an explicit x such that $\sim P(x)$ is true. This is called *giving a counterexample*. For instance: we can disprove the statement

$$(\forall n)[n \in \mathbb{Z} \Rightarrow 3|n^2+2n]$$

simply by saying "5 is in $\mathbb{Z}$, $5^2 + 2 \cdot 5 = 35$, and $3 \nmid 35$". (Note that we are disproving $P \Rightarrow Q$ by proving $P \wedge \sim Q$.)

1. Identify each of the following statements as true or false. Where you can, prove the statement by giving an example or disprove it by giving a counterexample.
   (a) $(\exists x)[x \in \mathbb{Z}_9 \wedge x^2 \in \{5, 7\}]$
   (b) $(\forall a)[(a \in \mathbb{Z} \wedge a^2 \equiv_{11} 16) \Rightarrow a \equiv_{11} 4]$
   (c) $(\exists n)[n \in \mathbb{N} \wedge \big(\frac{1}{2}(n^2+n)+2 \text{ is prime}\big)]$
   (d) $(\forall n)[n \in \mathbb{N} \Rightarrow \big(\frac{1}{2}(n^2+n)+2 \text{ is prime}\big)]$
   (e) $(\exists a, b)(a, b \in \mathbb{Z} \wedge 12a + 20b = 4)$
   (f) $\{x \,|\, x \in \mathbb{Z}_8 \wedge 4 \cdot_8 x = 0\} \cap \{4 \cdot_8 x \,|\, x \in \mathbb{Z}_8\} = \emptyset$
   (g) $(\forall n)[(n \in \mathbb{N} \wedge n \equiv_2 1 \wedge n > 3) \Rightarrow 3|n^2 - 1]$
   (h) $(\forall a, b)[(a, b \in \mathbb{Z} \wedge 12|ab) \Rightarrow (12|a \vee 12|b)]$
2. Give a line proof that $(\forall n)[n \in \mathbb{N} \Rightarrow n^2 + (n+1)^2 + (n+2)^2 \equiv_3 2]$.
3. Decide if the following statement is true or false. If it is true, give a line proof. If it is false, give a counterexample.

$$(\forall a, b, c)\big[(a, b, c \in \mathbb{Z} \wedge a > 0 \wedge a|(b-1) \wedge a|(c-1)\,) \Rightarrow a|(bc-1)\big].$$

4. Here is an important property of prime numbers. If p is prime, then

$$\text{Property (P)}: \qquad (\forall x, y \in \mathbb{Z})[p|xy \Rightarrow (p|x \vee p|y)].$$

   Using this, fill in the blanks below to give a proof of the following theorem:
   Theorem: $(\forall y)[(y \in \mathbb{Z} \wedge 4y^2 \equiv_7 0) \Rightarrow y \equiv_7 0]$

Proof:

1. Assume $y \in \mathbb{Z}$ and $4y^2 \equiv_7 0$.
2. Then $\boxed{\qquad\qquad}$.
3. 7 is prime, so by property (P) we know $7|4$ or $\boxed{\qquad\qquad}$.
4. But $7 \nmid 4$, so $\boxed{\qquad\qquad}$.
5. Using Property (P) again, either $7|y$ or $\boxed{\qquad\qquad}$.
6. Therefore $7|y$ (by using the tautology $\boxed{\qquad\qquad}$).
7. $\boxed{\qquad\qquad}$.
8. $\boxed{\qquad\qquad}$.

5. Give a line proof that $(\forall n)[(n \in \mathbb{N} \wedge 3|n \wedge n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12]$.

   Hint: you will need Property (P) at some point of your proof.

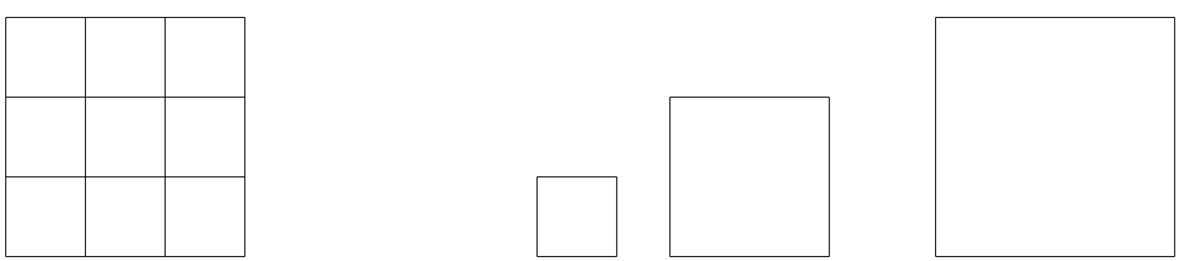
6. A *rational number* is a number of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. As you learned in elementary school, a rational number can always be written in the form where $\gcd(a, b) = 1$. Fill in the blanks below to give a proof of the following theorem; feel free to insert extra steps if you think it will help clarify the proof.

   Theorem: $\sqrt{2}$ is not a rational number.

1 Assume that $\sqrt{2}$ is a rational number.
2. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $\gcd(a, b) = 1$.
3. So $2 = \frac{a^2}{b^2}$, and therefore $a^2 = \boxed{\qquad\qquad}$.
4. So $2|a^2$.
5. Then by Property (P), $\boxed{\qquad\qquad}$.
6. $a = 2r$ for some $r \in \mathbb{Z}$.
7. $2b^2 = \boxed{\qquad\qquad}$.
8. $b^2 = \boxed{\qquad\qquad}$.
9. $2|b^2$
10. So using Property (P), $\boxed{\qquad\qquad}$.
11. $b = 2s$ for some $s \in \mathbb{Z}$.
12. This is a contradiction, because $\boxed{\qquad\qquad}$.
13. $\boxed{\qquad\qquad}$.

7. Give a line proof showing that $\sqrt{10}$ is not a rational number.

8. A 3×3 grid has 14 squares in it:



There are 1×1 squares, 2×2 squares, and 3×3 squares, and if you count all the squares that you see in the above grid you should get 14.

Figure out how many squares there are in a 10×10 grid, and explain your answer. Give an exact number, not just a formula for computing it.

Hints for doing this: Get a sense of the problem by tackling smaller versions. Try a 2×2 grid, you already did the 3×3 grid, maybe look at 4×4 and 5×5 grids. Analyze these smaller problems and try to find some underlying patterns.