Quiz 1 Solution **Fall 2024**

1. (5 pts) State the Axiom of Completeness.

   Solution. Every nonempty set in $\mathbb{R}$ that is bounded from above has a least upper bound. $\square$

2. (10 pts) (a) Find the least upper bound and the greatest lower bound of the set $\left\{1+\dfrac{1}{3n} : n \in \mathbb{N}\right\}$.

   (b) Give an example of a bounded set S so that its l.u.b is not in S (i.e., $\sup S \notin S$).

   Solution. (a) The least upper bound is $1+\frac{1}{3}=\frac{4}{3}$. The greatest lower bound is 1.

   (b) Here are two examples:
   $A=[0,1)$ (with $\sup A=1$) and $B=\{-\frac{1}{n} : n \in \mathbb{N}\}$ (with $\sup B=0$). $\square$

3. (10 pts) Let a be a real number and $\varepsilon>0$. Use the Theorem *Density of* $\mathbb{Q}$ *in* $\mathbb{R}$ to show that the interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many rational numbers.

   Solution 1. Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$) states that there is a rational number between any two given real numbers.

   There is a rational number in r_1 that satisfies $a<r_1<a+\varepsilon$. And there is a rational number r_2 satisfying $a<r_2<r_1$. The process goes on. Assumed we have found $a<r_n<r_{n-1}<....<r_1<b$. By the density of $\mathbb{Q}$, there is a rational number r_{n+1} satisfying $a<r_{n+1}<r_n$. Thus, by induction, we have infinitely many rationals in $(a, a+\varepsilon)$ and, hence, in $(a-\varepsilon, a+\varepsilon)$. $\square$

   Solution 2. Let r_1 be a rational number in $(a-\varepsilon, a)$ and r_2 be a rational number in $(a, a+\varepsilon)$. Then $r_1<r_2$. The middle point $r_3=(r_1+r_2)/2$ of (r_1, r_2) is a rational number. The middle point of r_2 and r_3 is also a rational number, call it r_4, and $r_4=(r_2+r_3)/2$. More generatlly, we define $r_{n+1}=(r_{n-1}+r_n)/2$. Then all $r_n : n \in \mathbb{N}$ are rational numbers (use induction if needed). Thus, there are infinitely many rational numbers in $(a-\varepsilon, a+\varepsilon)$. $\square$