Problem (Scalar Density Functions).

(i) Find the total mass over the cube $R : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 4$ if the density at (x, y, z) is given by $\rho(x, y, z) = \sqrt{z}$.

(ii) Find the total mass over the cylinder $R : x^2 + y^2 \leq 4,\ 0 \leq z \leq 3$ if the density at (x, y, z) is given by $\rho(x, y, z) = z$.

Solution to (i). Evaluating the integral, we get

$$\int_0^1 \int_0^1 \int_0^4 \sqrt{z}\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \int_0^1 \frac{2}{3}(4)^{3/2}\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \int_0^1 \frac{16}{3}\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \frac{16}{3}\, \mathrm{d}x = \frac{16}{3}. \qquad \square$$

Solution to (ii). Evaluating the integral, we get

$$\int_0^{2\pi} \int_0^2 \int_0^3 z\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int_0^2 \frac{1}{2}(3)^2\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int_0^2 \frac{9}{2}\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \frac{9}{2}(2)\, \mathrm{d}\theta = 9\pi. \qquad \square$$

Problem (Basic Line Integrals). Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, given that

(i) $\mathbf{F} = \langle y, x \rangle$ and C is the line segment from $(0, 0)$ to $(1, 1)$.

(ii) $\mathbf{F} = \langle x^2, y^2 \rangle$ and C is the line segment from $(1, 0)$ to $(3, 2)$.

(iii) $\mathbf{F} = \langle e^x, \sin(y) \rangle$ and C is the line segment from $(0, 0)$ to $(2, \pi)$.

(iv) $\mathbf{F} = \langle x^2 - y^2, 2xy \rangle$ and C is the line segment from $(-1, 2)$ to $(3, 4)$.

Solution to (i). Parametrizing the line segment, we get $\mathbf{r}(t) = \langle t, t \rangle$, where $0 \leq t \leq 1$. The derivative is $\mathbf{r}'(t) = \langle 1, 1 \rangle$. Converting $\mathbf{F}$ to a function of t, we get $\mathbf{F}(\mathbf{r}(t)) = \langle t, t \rangle$. Thus, the line integral is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \int_0^1 \langle t, t \rangle \cdot \langle 1, 1 \rangle\, \mathrm{d}t = \int_0^1 2t\, \mathrm{d}t = 1. \qquad \square$$

Solution to (ii). Parametrizing the line segment, we get $\mathbf{r}(t) = (1 - t)P + tQ = (1 - t)\langle 1, 0 \rangle + t\langle 2, 2 \rangle = \langle 1 + 2t, 2t \rangle$, where $0 \leq t \leq 1$. The derivative is $\mathbf{r}'(t) = \langle 2, 2 \rangle$. Converting $\mathbf{F}$ to a function of t, we get $\mathbf{F}(\mathbf{r}(t)) = \langle (1 + 2t)^2, (2t)^2 \rangle = \langle 1 + 4t + 4t^2, 4t^2 \rangle$. Thus, the line integral is

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t &= \int_0^1 \langle 1 + 4t + 4t^2, 4t^2 \rangle \cdot \langle 2, 2 \rangle\, \mathrm{d}t \\
&= \int_0^1 2(1 + 4t + 4t^2) + 8t^2\, \mathrm{d}t \\
&= \int_0^1 2 + 8t + 8t^2 + 8t^2\, \mathrm{d}t \\
&= \int_0^1 2 + 8t + 16t^2\, \mathrm{d}t \\
&= 2t + 4t^2 + \frac{16}{3}t^3 \bigg|_0^1 = 2 + 4 + \frac{16}{3} = \frac{34}{3}. \qquad \square
\end{aligned}$$

Solution to (iii). Parametrizing the line segment, we get $\mathbf{r}(t) = \langle 2t, \pi t \rangle$, where $0 \leq t \leq 1$. The derivative is $\mathbf{r}'(t) = \langle 2, \pi \rangle$. Converting $\mathbf{F}$ to a function of t, we get $\mathbf{F}(\mathbf{r}(t)) = \langle e^{2t}, \sin(\pi t) \rangle = \langle e^{2t}, 0 \rangle$. Thus, the line integral is

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t &= \int_0^1 \langle e^{2t}, 0 \rangle \cdot \langle 2, \pi \rangle\, \mathrm{d}t \\
&= \int_0^1 2e^{2t}\, \mathrm{d}t = e^{2t} \bigg|_0^1 = e^2 - 1. \qquad \square
\end{aligned}$$

Solution to (iv). Parametrizing the line segment, we get $\mathbf{r}(t) = \langle -1+4t, 2+2t \rangle$, where $0 \le t \le 1$. The derivative is $\mathbf{r}'(t) = \langle 4, 2 \rangle$. Converting $\mathbf{F}$ to a function of t, we get $\mathbf{F}(\mathbf{r}(t)) = \langle -3+12t^2, -4+12t+16t^2 \rangle$. Thus, the line integral is

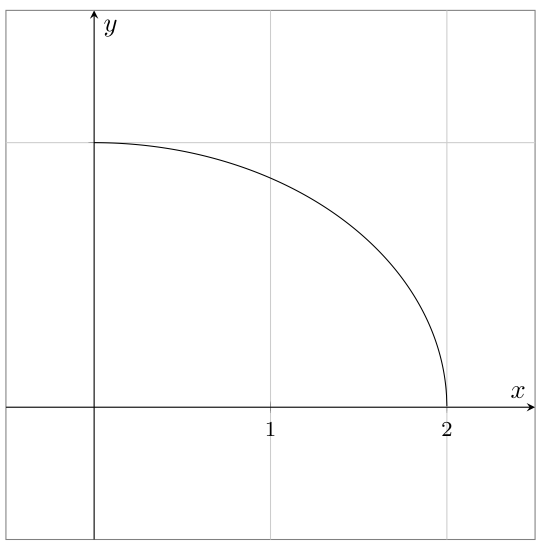
$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,\mathrm{d}t &= \int_0^1 \langle -3+12t^2, -4+12t+16t^2 \rangle \cdot \langle 4, 2\rangle \,\mathrm{d}t \\ &= \int_0^1 4(-3+12t^2) + 2(-4+12t+16t^2)\,\mathrm{d}t \\ &= \int_0^1 -20+80t^2+24t\,\mathrm{d}t \\ &= -20t + \frac{80}{3}t^3 + 12t^2 \Big|_0^1 = -20 + \frac{80}{3} + 12 = \frac{56}{3}. \end{aligned}$$

□

Problem (Change of Variables and Green's Theorem). Evaluate the line integral $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F} = \langle -2y, x \rangle$ and C is from $(0,0)$ to $(2,0)$, then from $(2,0)$ to $(0,1)$ on the curve $4 = x^2 + 4y^2$, then from $(0,1)$ to $(0,0)$. Clearly sketch the curve and the orientation of the curve.

NOTE: This is pretty much the exact same problem given on the midterm except for $\mathbf{F}$, as I don't remember, but the bounds are correct. Jennifer said she wants us to use change of variables and then use Green's Theorem to evaluate.

Solution. Graphing the curve, we get



Dividing both sides of the ellipse by 4, we get

$$\frac{x^2}{4} + y^2 = 1,$$

giving us the change of variables $x = 2r\cos(\theta)$ and $y = r\sin(\theta)$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. The Jacobian is

$$J = \begin{vmatrix} 2\cos(\theta) & -2r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = 2r\cos^2(\theta) + 2r\sin^2(\theta) = 2r.$$

Finding the partials of P and Q with respect to x and y, we have

$$\frac{\partial P}{\partial y} = -2 = \frac{\partial Q}{\partial x} = 1 \Rightarrow \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 3.$$

Thus, we can use Green's Theorem to evaluate the line integral. The line

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \iint_R 3 \, \mathrm{d}A = \int_0^{2\pi} \int_0^1 3 \cdot 2r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} 6 \, \mathrm{d}\theta = 12\pi. \qquad \square$$

Problem (Use the Fundamental Theorem of Line Integrals). Evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, given that

(i) $\mathbf{F} = \langle 2x, 3y^2 \rangle$ and C is the curve from $(1, 2)$ to $(3, 5)$.

(ii) $\mathbf{F} = \langle 2x, 4y \rangle$ and C is the arc of the parabola $x = y^2$ from $(4, -2)$ to $(1, 1)$.

(iii) $\mathbf{F} = \langle \sin(y) + e^x, x\cos(y) \rangle$ and C is the curve from $(0, 0)$ to $(\pi, 0)$ to (π, π) to $(0, \pi)$ to $(0, 0)$.

Solution to (i). Checking if $\mathbf{F}$ is conservative, we have

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} = 0.$$

Since $\mathbf{F}$ is simply connected and the partials are continuous, $\mathbf{F}$ is conservative. Thus, we can use the Fundamental Theorem of Line Integrals to evaluate the line integral. We are given the following system of partial differential equations

$$f_x = 2x \tag{1}$$
$$f_y = 3y^2 \tag{2}$$

Integrating (1) with respect to x, we have

$$f = \int 2x \, \mathrm{d}x = x^2 + g'(y).$$

From (2), we have

$$g'(y) = 3y^2 \Rightarrow g(y) = y^3.$$

Thus, the potential function is $f = x^2 + y^3$. Evaluating the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(3, 5) - f(1, 2) = 3^2 + 5^3 - (1^2 + 2^3) = 9 + 125 - 1 - 8 = 125. \qquad \square$$

Solution to (ii). Checking if $\mathbf{F}$ is conservative, we have

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} = 0.$$

Since $\mathbf{F}$ is simply connected and the partials are continuous, $\mathbf{F}$ is conservative. Thus, we can use the Fundamental Theorem of Line Integrals to evaluate the line integral. We are given the following system of partial differential equations

$$f_x = 2x \tag{3}$$
$$f_y = 4y \tag{4}$$

Integrating (3) with respect to x, we have

$$f = \int 2x \, \mathrm{d}x = x^2 + g'(y).$$

From (4), we have

$$g'(y) = 4y \Rightarrow g(y) = 2y^2.$$

Thus, the potential function is $f = x^2 + 2y^2$. Evaluating the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(1, 1) - f(4, -2) = 1^2 + 2(1)^2 - (4^2 + 2(-2)^2) = 1 + 2 - 16 - 8 = -21. \qquad \square$$

Solution to (iii). Checking if $\mathbf{F}$ is conservative, we have

$$\frac{\partial P}{\partial y} = \cos(y) = \frac{\partial Q}{\partial x} = \cos(y).$$

Since $\mathbf{F}$ is simply connected and the partials are continuous, $\mathbf{F}$ is conservative. Thus, we can use the Fundamental Theorem of Line Integrals to evaluate the line integral. We are given the following system of partial differential equations

$$f_x = \sin(y) + e^x \tag{5}$$
$$f_y = x\cos(y) \tag{6}$$

Integrating (5) with respect to x, we have

$$f = \int \sin(y) + e^x \, \mathrm{d}x = e^x + x\sin(y) + g(y).$$

Differentiating f with respect to y, we have $f_y = x\cos(y) + g'(y)$. Comparing this with (6), we have

$$g'(y) = 0 \Rightarrow g(y) = C,$$

but we can ignore the constant of integration. Thus, the potential function is $f = e^x + x\sin(y)$. Evaluating the closed line integral

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(0,0) - f(0,0) = e^0 + 0\sin(0) - (e^0 + 0\sin(\pi)) = 1 - 1 = 0. \qquad \square$$

Problem (Surface Integrals).

(i) Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the disk $x^2 + y^2 \leq 1$.

(ii) Find the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2+y^2 = 4$and above the xy-plane

(iii) Find the surface area of the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ excluding the circular base in the xy-plane.

NOTE: Don't worry about this one too much, as the problem on the midterm was probably the easiest one on the exam.

Solution to (i). The partials of z with respect to x and y are $z_x = 2x$ and $z_y = 2y$. Thus, we get $1+z_x^2+z_y^2 = 1 + (2x)^2 + (2y)^2 = 1 + 4x^2 + 4y^2$. Since the region of integration is the disk $x^2 + y^2 \leq 1$, we can convert to polar coordinates. Therefore, we get $1 + 4x^2 + 4y^2 = 1 + 4r^2$. Therefore, the surface area is

$$\iint_R \sqrt{1 + z_x^2 + z_y^2}\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Then, this is just a simple u-sub with $u = 1 + 4r^2$ giving us $\mathrm{d}u = 8r\,\mathrm{d}r \Rightarrow r\,\mathrm{d}r = \mathrm{d}u/8$, and you should be able to find the new bounds and evaluate the rest. $\square$

Solution to (ii). The partials of z with respect to x and y are

$$z_x = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad z_y = \frac{y}{\sqrt{x^2+y^2}}.$$

Thus,

$$1 + z_x^2 + z_y^2 = 1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = 2.$$

Since the region of integration is the disk $x^2 + y^2 \leq 4$, we can convert to polar coordinates. Therefore, we get

$$z^2 = 1 + \frac{x^2 + y^2}{x^2 + y^2} = 1 + \frac{r^2}{r^2} = 1 + 1 = 2 \Rightarrow z = \sqrt{2}.$$

Therefore, the surface area is

$$\iint_R \sqrt{1 + z_x^2 + z_y^2}\,\mathrm{d}A = \int_0^{2\pi}\int_0^2 \sqrt{2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^2 \sqrt{2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta. \qquad \square$$

Solution to (iii). For a sphere of radius R, the surface area element in spherical coordinates is given by $\mathrm{d}S = R^2\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi$. For the given sphere, $R = \sqrt{9} = 3$. Thus, we get the bounds for θ as $0 \leq \theta \leq \pi/2$ and for φ as $0 \leq \varphi \leq 2\pi$. The surface area is

$$\begin{aligned}
\iint_R \mathrm{d}S = \int_0^{2\pi}\int_0^{\pi/2} 3^2\sin(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\theta &= 9\int_0^{2\pi}\int_0^{\pi/2}\sin(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= 9\int_0^{2\pi} -\cos(\varphi)\Big|_0^{\pi/2}\,\mathrm{d}\theta \\
&= 9\int_0^{2\pi} -(-1)\,\mathrm{d}\theta = 9\int_0^{2\pi} 1\,\mathrm{d}\theta = 9(2\pi) = 18\pi. \qquad \square
\end{aligned}$$

Problem (Exact Spherical Problem). Setup the triple integral $\iiint \frac{z}{\sqrt{x^2+y^2}}\mathrm{d}V$ using spherical coordinates over the region above the inside $z = 2\sqrt{x^2+y^2}$ and above the sphere $x^2 + y^2 + z^2 = 2z$. Don't evaluate it, just set it up.

Solution. Rewriting the given sphere, $x^2+y^2+(z^2-2z) = 0$, and completing the square $x^2+y^2+(z-1)^2 = 1$. This represents a sphere centered at $(0, 0, 1)$ with radius 1. In spherical coordinates, we get $\rho = 2\cos(\varphi)$.

For the cone, dividing both sides by $\sqrt{x^2+y^2}$ gives $z/\sqrt{x^2+y^2} = 2$. This is the equation of the cone $\varphi = \arctan(1/2)$.

Since the region is symmetric around the z-axis, we integrate over the full circular angle $0 \leq \theta \leq 2\pi$. The region is bounded below by the cone and above by the sphere. So, φ ranges from the cone's angle $\varphi = \arctan(1/2)$, which gives us $\varphi = \pi/2$, giving us the bounds $0 \leq \varphi \leq \pi/2$. The radius ρ extends from the origin to the sphere, $0 \leq \rho \leq 2\cos(\varphi)$, giving us the bounds $0 \leq \rho \leq 2\cos(\varphi)$.

Converting the integrand to spherical coordinates, we get

$$\frac{z}{\sqrt{x^2+y^2}} = \frac{\rho\cos(\varphi)}{\rho\sin(\varphi)} = \cot(\varphi).$$

The volume element in spherical coordinates is $\rho^2\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$. Thus, the triple integral is

$$\iiint \frac{z}{\sqrt{x^2+y^2}}\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos(\varphi)} \cot(\varphi)\rho^2\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta. \qquad \square$$