Several-Variab Calc II: Homework 3

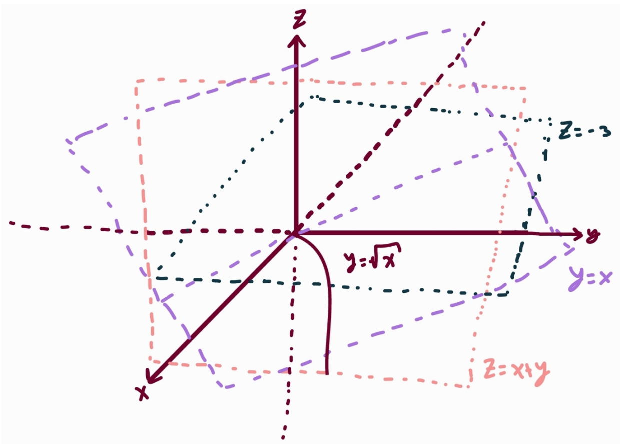
Due on January 28, 2025 at 9:00

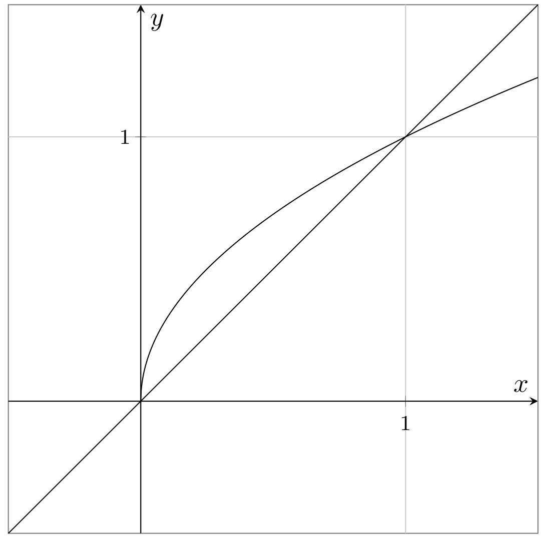
Jennifer Thorenson

Hashem A. Damrah
UO ID: 952102243

Problem 1. Evaluate $\iiint_E 3y\,\mathrm{d}V$ where E is the solid bounded by the cylinder $y = \sqrt{x}$ and the planes $y = x$, $z = -3$, and $z = x + y$.

Solution. Graphing the solid and the bounds on the xy-plane gives us





Clearly, from the graph, the bounds for x are $0 \leq x \leq 2$ and for y are $x \leq y \leq \sqrt{x}$. The bounds for z are $-3 \leq z \leq x + y$. Expanding the triple integral into an iterated integral gives us

$$\begin{aligned}
\iiint_E 3y\,\mathrm{d}V &= \int_0^1 \int_x^{\sqrt{x}} \int_{-3}^{x+y} 3y\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \int_x^{\sqrt{x}} 3yz\Big|_{-3}^{x+y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \int_x^{\sqrt{x}} 3y(x+y+3)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \int_x^{\sqrt{x}} 3y^2 + 3yx + 9y\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 y^3 + \frac{3y^2x}{2} + \frac{9y^2}{2}\Big|_x^{\sqrt{x}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left(x^{3/2} + \frac{3x^2}{2} + \frac{9x}{2}\right) - \left(x^3 + \frac{3x^3}{2} + \frac{9x^2}{2}\right)\mathrm{d}x \\
&= \int_0^1 x^{3/2} - \frac{5x^3}{2} - 3x^2 + \frac{9x}{2}\,\mathrm{d}x \\
&= \frac{2}{5}x^{5/2} - \frac{5x^4}{8} - x^3 + \frac{9x^2}{4}\Big|_0^1 \\
&= \frac{2}{5} - \frac{5}{8} - 1 + \frac{9}{4} = \frac{41}{40}.
\end{aligned}$$

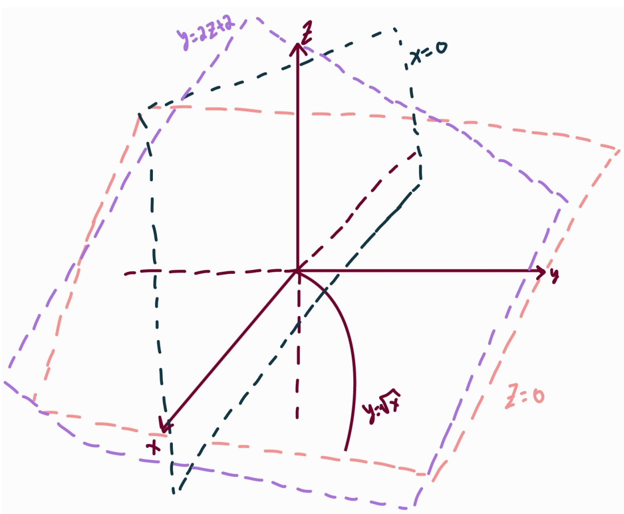
□

Problem 2. For any continuous $f(x, y, z)$, write $\iiint_E f(x, y, z)\,\mathrm{d}V$ as an iterated integral in the six orders of integration where E is a solid bounded by the surfaces $z = 0$, $x = 0$, $y + 2z = 2$, and $y = \sqrt{x}$. Clearly sketch the solid and each of the projections of E onto the coordinate planes.

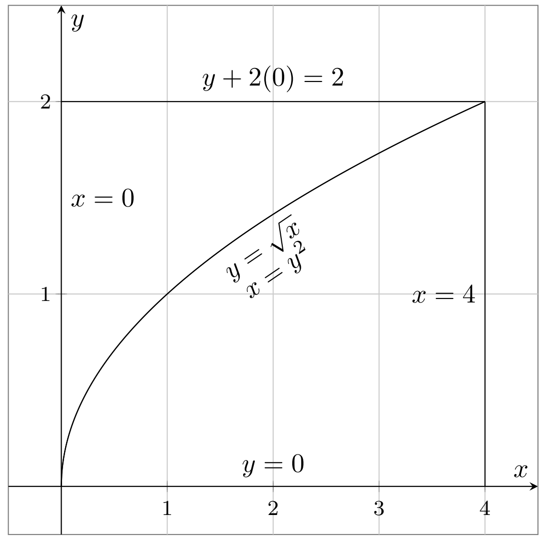
Solution. For the xy-plane, we get the following functions $y = 2$, $x = 0$, and $y = \sqrt{x} \Rightarrow x = y^2$. Finding the intersection point of $y = 2$ and $y = \sqrt{x}$ gives us $x = 4$. They intersect at $(4, 2)$. The graph is shown in figure **??**.

For the yz-plane, we get the following functions $z = 1 - {}^{y}/_{2} \Rightarrow y = 2 - 2z$ and $z = 0$. Solving for their intersections gives us $y = 0$ and $z = 0$. They intersect at $(0, 0)$ and $(2, 0)$. The graph is shown in figure **??**.

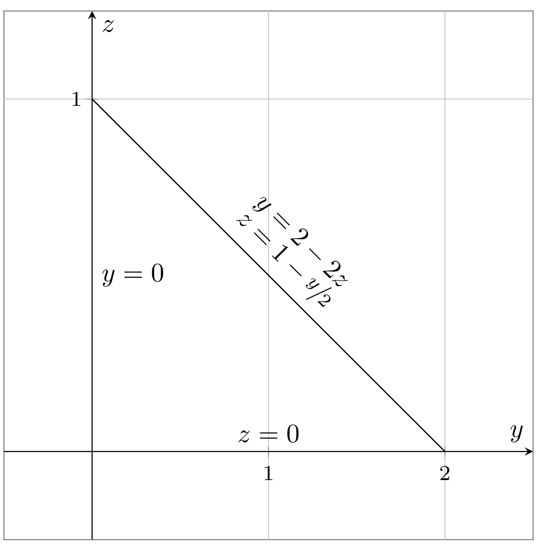
For the xz-plane, we get the following functions $z = -{}^{\sqrt{x}}/_{2} + 1$ (since $y = \sqrt{x}$) and $z = 0$. Solving for their intersections gives us $x = 4$. They intersect at $(4, 0)$. The graph is shown in figure **??**.



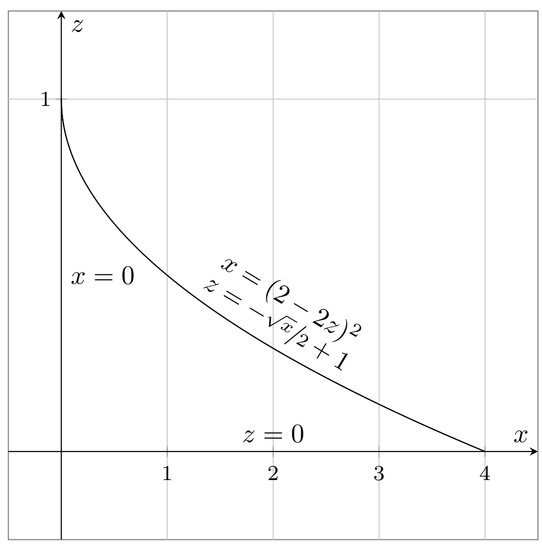
(a) Solid



(b) Boundaries: xy-plane



(c) Boundaries: yz-plane



(d) Boundaries: xz-plane

From figure 2b, we can create two integrals: $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ and $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$. For the first integral, the bounds for x are $0 \leq x \leq 4$ and for y, they are $\sqrt{x} \leq y \leq 2$. For the second integral, the bounds for y are $0 \leq y \leq 2$ and for x, they are $0 \leq x \leq y^2$. The bounds for z are the same for both, $0 \leq z \leq 1 - {}^{y}/_{2}$. Therefore, we get the following two integrals

$$\int_0^4 \int_{\sqrt{x}}^2 \int_0^{1-y/2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \iiint_E f(x,y,z)\,\mathrm{d}V = \int_0^2 \int_0^{y^2} \int_0^{1-y/2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

From figure 2c, we can create two integrals: $\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. For the first integral, the bounds for y are $0 \leq y \leq 2$ and for z, they are $0 \leq z \leq 1 - {}^{y}\!/\!_{2}$. For the second integral, the bounds for z are $0 \leq z \leq 1$ and for y, they are $0 \leq y \leq 2 - 2z$. The bounds for x are the same for both, $0 \leq x \leq y^2$. Therefore, we get the following two integrals

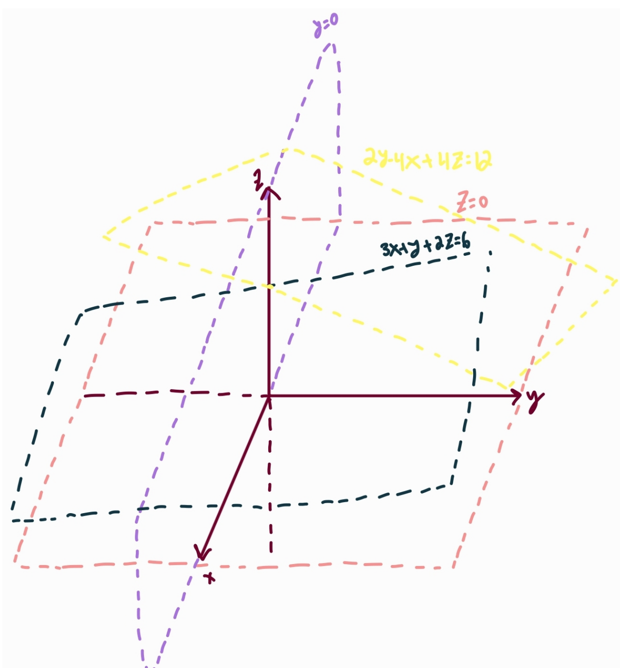
$$\int_0^2 \int_0^{1-y/2} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \iiint_E f(x,y,z)\,\mathrm{d}V = \int_0^1 \int_0^{2-2z} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

From figure 2d, we can create two integrals: $\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$ and $\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$. For the first integral, the bounds for z are $0 \leq z \leq 1$ and for x, they are $0 \leq x \leq (2-2z)^2$. For the second integral, the bounds for x are $0 \leq x \leq 4$ and for z, they are $0 \leq z \leq -\sqrt{x}/2 + 1$. The bounds for y are the same for both, $\sqrt{x} \leq y \leq 2 - 2z$. Therefore, we get the following two integrals

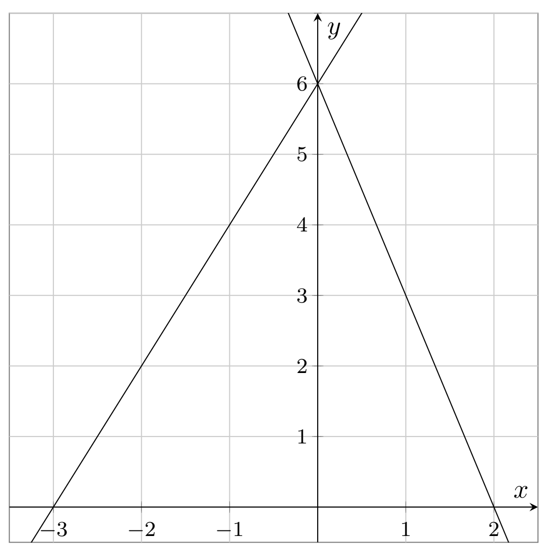
$$\int_0^1 \int_0^{(2-2z)^2} \int_{\sqrt{x}}^{2-2z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z = \iiint_E f(x,y,z)\,\mathrm{d}V = \int_0^4 \int_0^{-\sqrt{x}/2+1} \int_{\sqrt{x}}^{2-2z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x. \qquad \square$$

Problem 3. Use a triple integral to find the volume of the tetrahedron bounded by $z = 0$, $y = 0$, $3x+y+2z = 6$, and $2y - 4x + 4z = 12$.

Solution. Graphing the solid and the bounds on the xy-plane gives us



(a) Solid

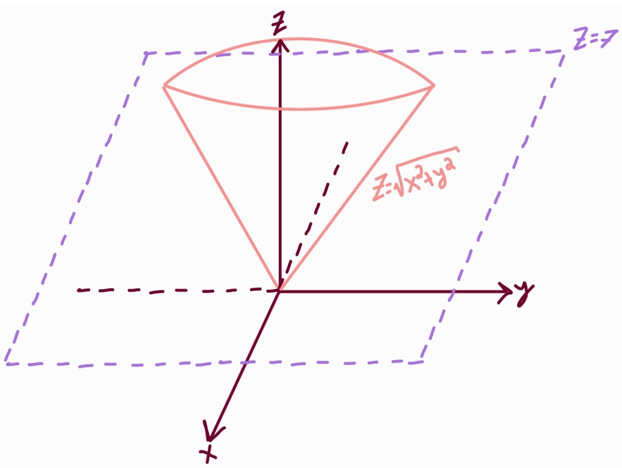


(b) Boundaries of the solid

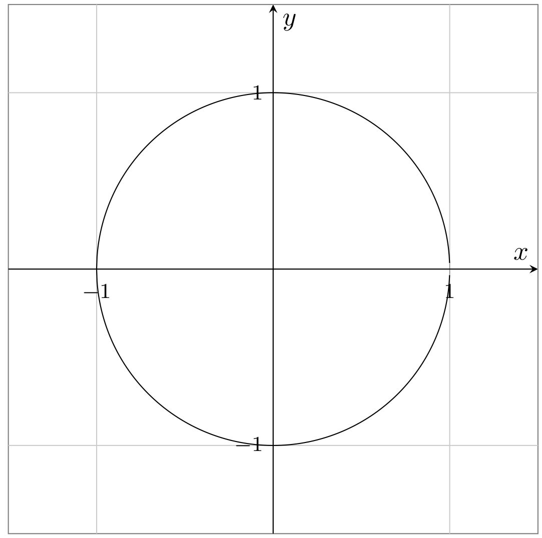
We can find the points of intersection of the planes $3x + y + 2z = 6$ and $2y - 4x + 4z = 12$ to get

$$\begin{aligned} x = 0 = y:&\quad z = 3 \quad \text{and} \quad z = 3 \\ y = 0 = z:&\quad x = 2 \quad \text{and} \quad x = -3 \\ x = 0 = z:&\quad y = 6 \quad \text{and} \quad y = 6. \end{aligned}$$

Therefore, for the plane $3x + y + 2z = 6$, we get the intersection points $(2, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 3)$ and for the plane $2y - 4x + 4z = 12$, we get the intersection points $(-3, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 3)$. Therefore, we get the bounds for x as $-3 \leq x \leq 2$. If we take the y bounds to be $0 \leq y \leq 6$, then we'd have to split the



(a) Solid



(b) Boundaries of the solid

z-integral into two parts, one part going from 0 to one of the top planes and another from the other top plane to the bottom plane. Instead, we can take the z bounds to be $0 \leq z \leq 3$, and y as $6+2x-2z \leq y \leq 6-3x-2z$. Expanding the triple integral into an iterated integral gives us

$$\begin{aligned}
\iiint_E 1 \, \mathrm{d}V &= \int_{-3}^{2} \int_{0}^{3} \int_{6+2x-2z}^{6-3x-2z} 1 \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x \\
&= \int_{-3}^{2} \int_{0}^{3} y \Big|_{6+2x-2z}^{6-3x-2z} \mathrm{d}z \, \mathrm{d}x \\
&= \int_{-3}^{2} \int_{0}^{3} -5x - z \, \mathrm{d}z \, \mathrm{d}x \\
&= \int_{-3}^{2} -5xz - \frac{z^2}{2} \Big|_{0}^{3} \mathrm{d}x \\
&= \int_{-3}^{2} -15x - \frac{9}{2} \, \mathrm{d}x \\
&= -\frac{15x^2}{2} - \frac{9}{2}x \Big|_{-3}^{2} \\
&= \left[-\frac{15(4)}{2} - \frac{9(2)}{2} \right] - \left[-\frac{15(9)}{2} - \frac{9(-3)}{2} \right] = -30 - 9 + 54 = 15.
\end{aligned}$$

□

Problem 4. Consider the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 1$. Find the center of mass of the solid using the density function $\rho(x, y, z) = z(x^2 + y^2)$.

Solution. I'm going to use cylindrical coordinates for this problem. Finding the intersection between the two surfaces gives us

$$z = \sqrt{x^2 + y^2} = r \Rightarrow r = 1.$$

Graphing the solid and the intersection gives us Using polar coordinates, we can write the density function as $\rho(r, \theta, z) = zr^2$ and for z, we get $z = r$. Therefore, the bounds for θ are $0 \leq \theta \leq 2\pi$, for r are $0 \leq r \leq 1$, and for z, we get $r \leq z \leq 1$.

The mass of the solid is given by

$$m = \iiint_E \rho(x, y, z) \, \mathrm{d}V = \int_{0}^{2\pi} \int_{0}^{1} \int_{r}^{1} zr^2 \, r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^1 \left.\frac{r^3 z^2}{2}\right|_r^1 \mathrm{d}r \\
&= \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^1 \frac{r^3}{2} - \frac{r^5}{2}\,\mathrm{d}r \\
&= \int_0^{2\pi} \mathrm{d}\theta \cdot \left.\frac{r^4}{8} - \frac{r^6}{12}\right|_0^1 \\
&= 2\pi \cdot \left(\frac{1}{8} - \frac{1}{12}\right) = 2\pi \cdot \left(\frac{1}{24}\right) = \frac{\pi}{12}.
\end{aligned}
$$

The following moments are given by

$$
\begin{aligned}
M_{yz} &= \iiint_E x\rho(x,y,z)\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_r^1 zr^4\cos(\theta)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = 0 \\
M_{xz} &= \iiint_E y\rho(x,y,z)\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_r^1 zr^4\sin(\theta)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = 0 \\
M_{xy} &= \iiint_E z\rho(x,y,z)\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_r^1 z^2 r^3\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 \left.\frac{z^3 r^3}{3}\right|_r^1 \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{1^3 r^3}{3} - \frac{r^3 r^3}{3}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{r^3}{3}(1-r^3)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{3}\int_0^1 r^3(1-r^3)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{3}\left[\int_0^1 r^3\,\mathrm{d}r - \int_0^1 r^6\,\mathrm{d}r\right]\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{3}\left[\left.\frac{r^4}{4} - \frac{r^7}{7}\right|_0^1\right]\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{3}\cdot\frac{7-4}{28}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{3}\cdot\frac{3}{28}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{28}\,\mathrm{d}\theta \\
&= \frac{1}{28}\cdot 2\pi = \frac{\pi}{14}.
\end{aligned}
$$

Note that the moments M_{yz} and M_{xz} are zero because the symmetry about the x-axis and y-axis respectively. Therefore, we get the following values

$$
\bar{x} = \frac{M_{yz}}{m} = 0, \quad \bar{y} = \frac{M_{xz}}{m} = 0, \quad \text{and} \quad \bar{z} = \frac{M_{xy}}{m} = \frac{\pi/14}{\pi/12} = \frac{6}{7}.
$$

Therefore, the center of mass of the solid is at $(0, 0, 6/7)$. □