SOLUTIONS TO HOMEWORK 8

Warning: Little proofreading has been done.

1. Section 3.3

Exercise 3.3.5 Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) An arbitrary intersection of compact sets is compact.
(b) An arbitrary union of compact sets is compact.
(c) Let A be arbitrary, and let K be compact. Then the intersection $A \cap K$ is compact.
(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq 0$.

Solution to (a). True. Let $(K_i)_{i \in I}$ be a family of compact subsets of $\mathbb{R}$, with $I \neq \varnothing$. Then K_i is closed for every $i \in I$. Therefore $\bigcap_{i \in I} K_i$ is closed.

Choose some $i_0 \in I$. Then K_{i_0} is bounded. Since $\bigcap_{i \in I} K_i \subseteq K_{i_0}$, this set is also bounded. Therefore $\bigcap_{i \in I} K_i$ is compact. □

Solution to (b). False. For $n \in \mathbb{N}$, let $F_n = [n, n+1]$. Then F_n is compact. However, $\bigcup_{n=1}^{\infty} F_n = [1, \infty)$ is unbounded and, hence not compact. □

Solution to (c). False. Set $A = \mathbb{R} \setminus \{0\}$ and $K = [0,1]$. Then K is closed and bounded, hence compact. However, $A \cap K = (0,1]$ is not closed, because 0 is a limit point of $(0,1]$ (proved below) but $0 \notin (0,1]$. So $A \cap K$ is not compact. □

Solution to (d). False. For $n \in \mathbb{N}$, let $F_n = \{m \in N : m \geq n\} = \{n, n+1, n+2, \ldots\}$. Then F_n is closed, since F_n has no limit points, and $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$. However, the intersection $\bigcap_{n=1}^{\infty} F_n = \emptyset$. □

Problem 3.3.9 Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite sub cover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.
(b) Argue that there must exist an $x \in K$ such that $x \in I_n$ for all n.
(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Solution to (a). We bisect I_0 into two intervals, $I_0^{(1)}$ and $I_0^{(2)}$. At least one of $I_0^{(1)} \cap K$ and $I_0^{(2)} \cap K$ cannot be finitely covered. Chose one interval that cannot be, call it I_1. We then bisect I_1 and choose I_2 as one of the half intervals whose intersection with K cannot be finitely covered. Continue this process, we obtain a nested closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$, so that $I_n \cap K$ cannot be finitely covered. By our construction, $|I_n| = |I_0|/2^n$ so that $\lim I_n = 0$. □

Solution to (b). This follows from Nested Compact Set Property, since $K \cap I_n$ is compact (proved in Exercise 3.3.4 (a)). □

Solution to (c). Since x is an element of the open set O_{λ_0}, there is an $\varepsilon > 0$ such that the open interval $V_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ is a subset of O_{λ_0}. Since $\lim |I_n| = 0$, there is an $N \in \mathbb{N}$ such that $|I_n| < \varepsilon/2$ for all $n \geq N$. This shows that I_n is a subset of $(x - \varepsilon, x + \varepsilon)$ and, in particular, a subset of O_{λ_0}. But this means that I_n, for $n \geq N$, can be covered by *one* open set in the original cover. This is a contradiction. □

2. Section 4.2

Exercise 4.2.2 For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ε challenge.

(a) $\lim_{x\to 3}(5x-6)=9$, where $\varepsilon=1$.
(b) $\lim_{x\to 4}\sqrt{x}=2$, where $\varepsilon=1$.

Solution to (a). The largest possible value of δ is $\delta=1/5$, since

$$|(5x-6)-9|=|5x-15|=5|x-3|<\varepsilon=1$$

is equivalent to $|x-3|<1/5$. □

Solution to (b). The largest possible value of δ is $\delta=3$. We need

$$|\sqrt{x}-2|<\varepsilon=1$$

which is equivalent to

$$1=-1+2\sqrt{x}<2+1=3,$$

or, after taking square,

$$1<x<9$$

so that

$$-3<x-4<5 \qquad \text{or} \qquad -5<4-x<3.$$

Thus, the largest interval for $|x-4|<\delta$ is when $\delta=3$. □

Exercise 4.2.5 Use Definition 4.2.1 to supply proofs for the following limit statements.

(a) $\lim_{x\to 2}(3x+4)=10$.
(b) $\lim_{x\to 0}x^3=0$.
(c) $\lim_{x\to 2}(x^2+x-1)=5$.
(d) $\lim_{x\to 3}1/x=1/3$.

Solution to (a). Let $\varepsilon>0$. Define $\delta=\frac{1}{3}\varepsilon$. Then $\delta>0$. Let $x\in\mathbb{R}$ satisfy $0<|x-2|<\delta$. Then

$$|(3x+4)-10|=|3x-6|=3|x-2|<3\delta=\varepsilon.$$

This completes the proof. □

Solution to (b). Let $\varepsilon>0$. Define $\delta=\sqrt[3]{\varepsilon}$. Then $\delta>0$. Let $x\in\mathbb{R}$ satisfy $0<|x|<\delta$. Then

$$|x^3|=|x|^3<\left(\sqrt[3]{\varepsilon}\right)^3=\varepsilon.$$

This completes the proof. □

Solution to (c). Let $\varepsilon>0$. Define

$$\delta=\min\left(1,\,\frac{\varepsilon}{6}\right).$$

Then $\delta>0$. Let $x\in\mathbb{R}$ satisfy $0<|x-2|<\delta$. Then $1<x<3$, so

$$|x+3|\le 3+3\le 6.$$

Now

$$\begin{aligned}|(x^2+x-1)-5|=|x^2+x-6|=|(x+3)(x-2)|&\le|x+3|\cdot|x-2|\\ &\le 6|x-2|<6\delta\le 6\cdot\frac{\varepsilon}{6}=\varepsilon.\end{aligned}$$

This completes the proof. □

Solution to (d). Let $\varepsilon > 0$. Define
$$\delta = \min(1,\, 6\varepsilon).$$
Then $\delta > 0$. Let $x \in \mathbb{R}$ satisfy $0 < |x-3| < \delta$. Then $2 < x < 4$, so
$$\frac{1}{3|x|} = \frac{1}{3x} \leq \frac{1}{3 \cdot 2} = \frac{1}{6}.$$
Now
$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|x-3|}{3x} \leq \frac{1}{6}|x-3| < \frac{1}{6}\delta \leq \frac{1}{6} \cdot 6\varepsilon = \varepsilon.$$
This completes the proof. □

Example 4.2.6 Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller δ will also suffice.
(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.
(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x)-2]^2 = 3(L-2)^2$.
(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

Solution to (a). Let A be the domain of f. Suppose we know that for every $x \in A$ with $0 < |x-c| < \delta$, we have $|f(x) - L| < \varepsilon$. Suppose $0 < \delta_0 < \delta$. Then for every $x \in A$ with $0 < |x-c| < \delta_0$, we have $0 < |x-c| < \delta$, so $|f(x) - L| < \varepsilon$. □

Solution to (b). This is not true. For example, consider the function f defined by
$$f(x) = \begin{cases} 0 & x \neq 0, \\ 1 & x = 0. \end{cases}$$
Then $\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$. □

Solution to (c). This is true. It follows from the algebraic limit theorem (Corollary 4.2.4). □

Solution to (d). This is false. For example, let $f(x) = x$ and
$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$
Then $\lim_{x\to 0} f(x) = 0$, but
$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} \frac{1}{x}$$
does not exist. □