Fundamentals of Analysis II: Homework 8

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Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

Exercise 6.2.9. Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

(i) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.

(ii) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

(iii) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

Solution to (i). Let $\varepsilon > 0$. Since f_n and g_n are uniformly convergent, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|f_n - f_m| < \varepsilon/2$ and $|g_n - g_m| < \varepsilon/2$. Then, for all $n, m \geq N$ and using the triangle inequality, we have

$$\begin{aligned}|(f_n + g_n) - (f_m + g_m)| &= |f_n - f_m + g_n - g_m| \\ &\leq |f_n - f_m| + |g_n - g_m| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore, $(f_n + g_n)$ converges uniformly. □

Solution to (ii). Let (f_n) and (g_n) be sequences of functions on $[0, 1]$ defined by $f_n(x) = x_n = g_n(x)$. First, we show that (f_n) and (g_n) converge uniformly to 0 on $[0, 1]$.

For all $x \in [0, 1]$, we have $x^n \to 0$ as $n \to \infty$. Thus, the pointwise limit is

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \text{for all } x \in [0, 1].$$

To check uniform convergence, we compute

$$\sup_{x\in[0,1]} |f_n(x) - 0| = \sup_{x\in[0,1]} x^n.$$

Since the maximum value of x^n on $[0, 1]$ occurs at $x = 1$, we obtain

$$\sup_{x\in[0,1]} x^n = 1^n = 1 \to 0,$$

which confirms that (f_n) converges uniformly to 0. Similarly, (g_n) also converges uniformly to 0.

Now consider the product sequence (h_n) given by $h_n(x) = f_n(x)g_n(x) = x^{2n}$. The pointwise limit is

$$\lim_{n\to\infty} h_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}.$$

To determine uniform convergence, we compute

$$\sup_{x\in[0,1]} |h_n(x) - 0| = \sup_{x\in[0,1]} x^{2n}.$$

Since x^{2n} attains its maximum at $x = 1$, we obtain

$$\sup_{x\in[0,1]} x^{2n} = 1,$$

which does not tend to 0. Thus, (h_n) does not converge uniformly to 0. □

Solution to (iii). Let $\varepsilon > 0$. Since f_n and g_n are uniformly convergent, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, we get $|f_n - f_m| < \varepsilon/2$ and $|g_n - g_m| < \varepsilon/2$. Since f_n and g_n are bounded by $M > 0$, then for all $n, m \geq N$ and using the triangle inequality, we have

$$|f_n g_n - f_m g_m| \leq |f_n g_n - f_n g_m| + |f_n g_m - f_m g_m|$$

$$
\begin{aligned}
&= |f_n| \cdot |g_n - g_m| + |g_m| \cdot |f_n - f_m| \\
&< M \cdot |g_n - g_m| + M \cdot |f_n - f_m| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

Therefore, $(f_n g_n)$ converges uniformly. □

Exercise 6.3.4. Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbb{R}$.

Solution. First, we show that $h_n \to 0$ uniformly on $\mathbb{R}$.

For all $x \in \mathbb{R}$, we have

$$|h_n(x)| = \left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}}.$$

Since the right-hand side does not depend on x and satisfies

$$\sup_{x\in\mathbb{R}} |h_n(x)| = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty,$$

it follows that $h_n \to 0$ uniformly on $\mathbb{R}$.

Now, we analyze the derivatives. Differentiating $h_n(x)$ gives

$$h_n'(x) = \sqrt{n}\cos(nx).$$

To show that (h_n') diverges for every $x \in \mathbb{R}$, we consider two cases: pointwise divergence and unboundedness in the supremum norm.

For case 1, fix any $x \in \mathbb{R}$. Since $\cos(nx)$ oscillates between -1 and 1, for any fixed x, there exists an increasing sequence $\{n_k\}$ such that $\cos(n_k x)$ accumulates at both 1 and -1 infinitely often. Consequently, the sequence $h_{n_k}'(x) = \sqrt{n_k}\cos(n_k x)$ takes arbitrarily large positive and negative values as $n_k \to \infty$. This implies that $h_n'(x)$ does not converge for any fixed x.

For case 2, since $|\cos(nx)| \leq 1$, we obtain

$$\sup_{x\in\mathbb{R}} |h_n'(x)| = \sup_{x\in\mathbb{R}} \sqrt{n}|\cos(nx)| \leq \sqrt{n}.$$

To achieve equality, we choose $x = 0$ (or any multiple of 2π), for which $\cos(nx) = 1$. Thus, we have

$$\sup_{x\in\mathbb{R}} |h_n'(x)| = \sqrt{n} \to \infty \text{ as } n \to \infty.$$

This shows that (h_n') is unbounded in the supremum norm, confirming its divergence.

Therefore, while $h_n \to 0$ uniformly, its derivatives h_n' diverge for every $x \in \mathbb{R}$. □

Exercise 7.4.7(i). Review the discussion immediately preceding Theorem 7.4.4.

(i) Produce an example of a sequence $f_n \to 0$ pointwise on $[0,1]$ where $\lim_{n\to\infty} \int_0^1 f_n$ does not exist.

Solution to (i). Define f_n as

$$f_n(x) = \begin{cases} n^2 & \text{if } 1/n \leq x < 2/n,\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Then, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. Since $f(0) = 0$ by definition, and for each $x > 0$, there exists an $N \in \mathbb{N}$ such that $x > 2/N$ and $f_n(x) = 0$ for all $n \geq N$, we have

$$\int_0^1 f_n(x)\,\mathrm{d}x = n^2\left(\frac{2}{n} - \frac{1}{n}\right) = n,$$

meaning that $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n$ does not exist. $\square$

Exercise 7.4.8. For each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 1/2^n \end{cases},$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show that H is integrable and compute $\int_0^1 H$.

Proof. We first show that the infinite series defining $H(x)$ converges and is integrable. Observe that for each n,

$$0 \leq h_n(x) \leq \frac{1}{2^n} \text{ for all } x \in [0,1].$$

Since $\sum \frac{1}{2^n}$ is a convergent geometric series, the Weierstras M-test shows that $\sum_n h_n(x)$ converges uniformly on H. By Theorem 7.4.4, $H(x)$ is integrable.

Next, we compute $\int_0^1 H(x)\,\mathrm{d}x$.

$$\int_0^1 H(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^1 h_n(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_{\frac{1}{2^n}}^1 \frac{1}{2^n}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right).$$

Expanding the summand and simplifying, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right) &= \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{4^n} \\ &= \left(\frac{1}{1-\frac{1}{2}}\right) - \left(\frac{1}{1-\frac{1}{4}}\right) \\ &= 2 - \frac{4}{3} = \frac{2}{3}. \end{aligned}$$

Therefore, $H(x)$ is integrable, and

$$\int_0^1 H(x)\,\mathrm{d}x = \frac{2}{3}.$$ $\square$

Exercise 6.4.2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(i) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(ii) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(iii) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exists constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution to (i). True. Let $\varepsilon > 0$. Applying the Cauchy Criterion with $n = m+1$, we have that $|g_n(x)| < \varepsilon$ for all $x \in A$. Therefore, (g_n) converges uniformly to zero. $\square$

Solution to (ii). True. Let $\varepsilon > 0$. Notice that

$$\left|\sum_{k=m+1}^{n} f_k(x)\right| = \sum_{k=m+1}^{n} f_k(x) \leq \sum_{k=m+1}^{n} g_k(x) = \left|\sum_{k=m+1}^{n} g_k(x)\right| < \varepsilon.$$

Therefore, $\sum_{n=1}^{\infty} f_n$ converges uniformly. $\square$

Solution to (iii). False. Consider the following sequence of functions, defined on $[0, 1)$.

$$g_{ij}(x) = \begin{cases} 2^{-i} & 2^{-i}(j-1) \leq x < 2^{-i}j \\ 0 & \text{otherwise} \end{cases},$$

with $i \geq 1$ and $j = 1, \cdots, 2^i$. Define the function sequence $f_n(x) = g_{ij}(x)$, where we enumerate $f_n(x)$ by listing all g_{ij} in increasing order of i, then increasing order of j. That is, first take g_{11}, then g_{21}, g_{22}, then g_{31}, g_{32}, g_{33}, and so on. This ordering ensures that every function in the collection $\{g_{ij}\}$ appears exactly once in the sequence $\{f_n\}$.

The sum

$$S_i(x) = \sum_{j=1}^{2^i} g_{ij}(x) = \sum_{j=1}^{2^i} 2^{-i} = 1.$$

Since the partial sums stabilize at 1, the sequence of partial sums of $f_n(x)$ converges uniformly to the function $H(x) = 1$.

Each function $g_{ij}(x)$ satisfies

$$\max g_{ij}(x) = 2^{-i}.$$

Since there are 2^i such functions at level i, the sum of these maxima is

$$\sum_{i=1}^{\infty} \sum_{j=1}^{2^i} 2^{-i} = \sum_{i=1}^{\infty} 2^i \cdot 2^{-i} = \sum_{i=1}^{\infty} 1 = \infty.$$

This sum diverges, proving that the series $\sum_{n=1}^{\infty} f_n$ does not converge uniformly. □

Exercise 6.4.4. Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1 + x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Solution. Define $h_n(x)$ as

$$h_n(x) = \frac{x^{2n}}{1 + x^{2n}}.$$

For $|x| \geq 1$, $h_n(x)$ does not approach 0 as $n \to \infty$, so the series diverges. For $|x| < 1$, $|h_n(x)| \leq x^{2n}$, which is a geometric series in x^2, which converges. So, $g(x)$ converges by the Order Limit Theorem.

For any $0 \leq a < 1$, $|h(x)| = a^{2n} = M_n$ over the interval $[-a, a]$ for all $n \in \mathbb{N}$. Thus, by the Weiestrass M-test, $g(x)$ uniformly converges over $[-a, a]$ for all $0 \leq a < 1$. Since $g(x)$ converges uniformly on $[-a, a]$ for all $0 \leq a < 1$, it follows that $g(x)$ is continuous on $(-1, 1)$. □