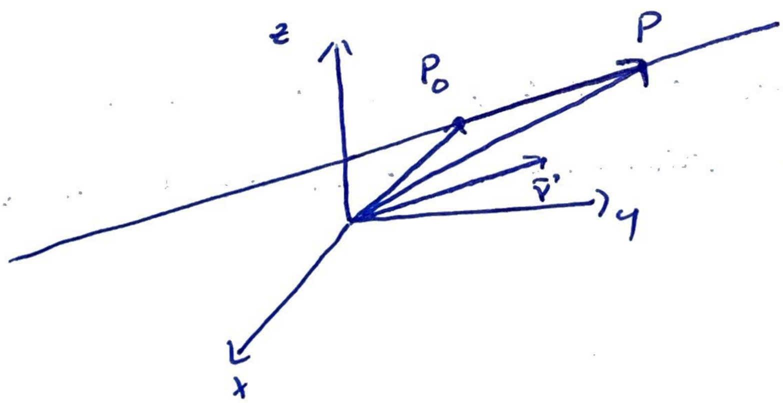
Note: If $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, then $\vec{a}$ and $\vec{b} \times \vec{c}$ are orthogonal. Therefore $\vec{a}, \vec{b}, \vec{c}$ lie in a plane and are called <u>coplanar</u>.

§12.5: Lines and Planes

<u>Lines</u>

A line in $\mathbb{R}^3$ is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\vec{v} = \langle a, b, c \rangle$ parallel to the line.



Let $P(x, y, z)$ be any point on line.

$\vec{OP} = \vec{OP_0} + \vec{P_0P}$

Since $\vec{P_0P}$ and $\vec{v}$ are parallel, there exists a scalar t such that

$\vec{P_0P} = t\vec{v}$

Let $\vec{OP} = \langle x, y, z \rangle = \vec{r}$ and $\vec{OP_0} = \langle x_0, y_0, z_0 \rangle = \vec{r_0}$

Then $\vec{OP} = \vec{OP_0} + t\vec{v}$

$\vec{r}(t) = \vec{r_0} + t\vec{v}$ is vector equation of line.

Using the components,

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$= \langle x_0, y_0, z_0 \rangle + \langle at, bt, ct \rangle$$

$$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$$\left.\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}\right\} \text{Parametric Equations of Line.}$$

Ex: Find line through $P_0(1, 4, -2)$ parallel to $\hat{i} - 3\hat{j} + 5\hat{k}$.

$$\vec{r} = \langle 1, 4, -2 \rangle + t\langle 1, -3, 5 \rangle$$

OR

$$\begin{cases} x(t) = 1 + t \\ y(t) = 4 - 3t \\ z(t) = -2 + 5t \end{cases}$$

The symmetric equations for a line are formed by eliminating the parameter, t, from parametric equations.

$x = x_0 + at \Rightarrow t = \frac{x - x_0}{a}$ $\quad a \neq 0$

$y = y_0 + bt \Rightarrow t = \frac{y - y_0}{b}$ $\quad b \neq 0$

$z = z_0 + ct \Rightarrow t = \frac{z - z_0}{c}$ $\quad c \neq 0$

Symmetric Equations are $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ if $a, b, c \neq 0$

Ex! Find line through $A(1, 5, 3)$ and $B(4, 1, 3)$

A direction vector is $\vec{v} = \vec{AB} = \langle 3, -4, 0 \rangle$

Vector equation is $\vec{r}(t) = \vec{OA} + t\,\vec{AB}$

$= \langle 1, 5, 3 \rangle + t \langle 3, -4, 0 \rangle$

Parametric Equations are $x(t) = 1 + 3t$

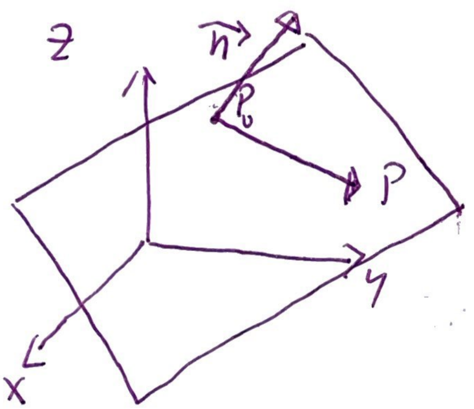
$y(t) = 5 - 4t$

$z(t) = 3$

Symmetric Equations $\frac{x - 1}{3} = \frac{y - 5}{-4}$; $z = 3$

Planes

A plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n} = \langle a, b, c \rangle$ orthogonal to the plane called a normal vector.



Let $P(x, y, z)$ be any point in the plane. Then $\vec{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ lies in the plane and is orthogonal to $\vec{n}$.

By orthogonality,

$\vec{n} \cdot \vec{P_0P} = 0$; Vector equation of plane.

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$; Scalar Equation

Ex: Find plane through $P_0(1, 4, -2)$ with normal $\vec{n} = \langle 2, 1, 5 \rangle$.

$$\langle 2, 1, 5 \rangle \cdot \langle x - 1, y - 4, z + 2 \rangle = 0$$

$$2(x - 1) + y - 4 + 5(z + 2) = 0$$

OR, $2x + y + 5z = -4$

Ex: Find the plane containing points $P(1,2,3)$, $Q(-1,2,4)$, and $R(1,3,5)$.

$\vec{PQ} = \langle -2, 0, 1 \rangle$
$\vec{PR} = \langle 0, 1, 2 \rangle$

$\vec{PQ}$ and $\vec{PR}$ are vectors in the plane.

$\vec{PQ} \times \vec{PR}$ is orthogonal to $\vec{PQ}$ and $\vec{PR}$ and therefore orthogonal to plane.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \langle -1, 4, -2 \rangle$$

Let $S(x, y, z)$ be any point in plane.

$\vec{PS} = \langle x-1, y-2, z-3 \rangle$ lies in plane.

$$(\vec{PQ} \times \vec{PR}) \cdot \vec{PS} = 0$$

$$-(x-1) + 4(y-2) - 2(z-3) = 0$$

$$-x + 4y - 2z = 1$$

Note: The equation of a plane has the form $ax + by + cz = d$ with constants a, b, c, and d.

The components of the normal are the scalar coefficients: $\vec{n} = \langle a, b, c \rangle$.