1. Let V be a n-dimensional vector space. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subset V$ is a spanning set of V, i.e. Span $(S) = V$. Prove that S is a basis of V.

2. Consider $V = \mathbb{R}^{n\times n}$ and let $S = \{A \in V : \text{Tr}(A) = 0\}$.

1). Prove that S is a subspace of V.

2). Find a basis for S. Make sure to justify that the set you give is a basis.

3. Let W_1 and W_2 be subspaces of a vector space V. Let $\dim W_1 = m$ and $\dim W_2 = p$. Define $W_1 + W_2 = \{\mathbf{x_1} + \mathbf{x_2} : \quad \mathbf{x_1} \in W_1,\ \mathbf{x_2} \in W_2\}$. Prove that

1). $W_1 + W_2$ is a subspace of V;

2). $\dim(W_1 + W_2) = m + p - \dim(W_1 \cap W_2)$.

4. Consider the following subspaces of $\mathbb{R}^{2\times 2}$,

$$W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in \mathbb{R}^{2\times 2}, a, b, c \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \in \mathbb{R}^{2\times 2}, a, b \in \mathbb{R} \right\}.$$

Compute the dimension of the subspace $W_1 + W_2$. Explain your answer. (Note: the definition of $W_1 + W_2$ is given in Problem 3).

5. Show that the polynomials $2, 1 + t, t + t^2$ form a basis for $P^2(\mathbb{R})$. Then find the coordinate of $3 + t + 2t^2$ in this basis.

6. Let $V = \mathbb{R}^\infty = \{(a_1, a_2, \cdots) : a_1, a_, \cdots \in \mathbb{R}\}$. Define $T : V \to V$ by

$$T((a_1, a_2, a_3, \cdots)) = (a_2, a_3, \cdots).$$

1). Prove that T is a linear transformation on V.

2). Prove that T is onto, but not one-to-one.

7. Let $V = \mathbb{R}^\infty = \{(a_1, a_2, \cdots) : a_1, a_, \cdots \in \mathbb{R}\}$. Define $T : V \to V$ by

$$T((a_1, a_2, a_3, \cdots)) = (0, a_1, a_2, \cdots).$$

1). Prove that T is a linear transformation on V.

2). Prove that T is one-to-one, but not onto.

8. Let V and W be vector spaces over F. Let $\mathcal{L}(V, W)$ be the set of all linear transformations from V to W. For any $T, U \in \mathcal{L}(V, W)$, define $T + U$ by

$$(T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x}), \text{ for any } \mathbf{x} \in V.$$

For any $T \in \mathcal{L}(V, W)$ and $c \in F$, define cT by

$$(cT)(\mathbf{x}) = cT(\mathbf{x}), \text{ for any } \mathbf{x} \in V.$$

Prove that $\mathcal{L}(V, W)$ with the above addition and scalar multiplication is a vector space over F.

9. True of false. (No explanation needed)

1). If S is a linear dependent set, then each vector in S is a linear combination of other vectors in S.

2). Any set containing the zero vector is a linearly dependent.

3). Subset of linearly independent set is linearly independent.

4). Let V be a vector space. Let $W \subseteq V$ be a subspace with $\dim W = \dim V$. Then $W = V$.