1. Verify that the determinant of $\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{bmatrix}$ is $\prod_{1\le i<j\le 3}(t_j - t_i)$.

$$\det\begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ t_1 & t_2-t_1 & t_3-t_1 \\ t_1^2 & t_2^2-t_1^2 & t_3^2-t_1^2 \end{pmatrix} = (t_2-t_1)(t_3^2-t_1^2)-(t_3-t_1)(t_2^2-t_1^2)$$

$$= (t_2-t_1)(t_3-t_1)(t_3+t_1) - (t_3-t_1)(t_2-t_1)(t_2+t_1)$$

$$= (t_2-t_1)(t_3-t_1)(t_3+t_1-t_2-t_1)$$

$$= (t_2-t_1)(t_3-t_1)(t_3-t_2)$$

2. Use the method introduced in the class to find a polynomial $p(x)$ in $\mathbb{P}^3(\mathbb{R})$ such that $p(1) = 1$, $p(2) = 3$, $p(3) = -1$, and $p(4) = 2$.

$t_1 = 1,\quad t_2 = 2,\quad t_3 = 3,\quad t_4 = 4$

$c_1 = 1,\quad c_2 = 3,\quad c_3 = -1,\quad c_4 = 2.$

$$p(x) = \sum_{i=1}^{4} c_i \prod_{\substack{j=1\\ j\neq i}}^{4} \frac{(x-t_j)}{(t_i-t_j)}$$

$$= 1\cdot\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 3\cdot\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} - \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 2\cdot\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

$$= \frac{1}{6}(13x^3 - 96x^2 + 209x - 120).$$

3. Let f be the linear functional on $\mathbb{R}^2$ defined by $f(x_1, x_2) = 2x_1 - 3x_2$. Let T be a linear transformation defined by $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$. Let T^t be the transpose linear transformation of T on the dual space of $\mathbb{R}^2$. Find the formula for the linear functional $T^t(f) : \mathbb{R}^2 \to \mathbb{R}$.

For any $(x_1, x_2) \in \mathbb{R}^2$:

$$T^t(f)\left(x_1, x_2\right) = f \circ T\,(x_1, x_2) = f\left(x_1 - x_2,\, x_1 + x_2\right)$$

$$= 2\left(x_1 - x_2\right) - 3(x_1 + x_2)$$

$$= -x_1 - 5x_2.$$

4. Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by $f(p) = \displaystyle\int_a^b p(x)\,dx$. Let D be the differentiation operator on V, and $D^t : V^* \to V^*$ be the the transpose linear transformation of D on the dual space V^*. Find the formula for the linear functional $D^t(f) : V \to \mathbb{R}$.

For any $p(x) \in V$:

$$D^t(f)\,(p(x)) = f\cdot\left(D(p(x))\right) = \int_a^b p'(x)\,dx$$

$$= p(x)\Big|_a^b$$

$$= p(b) - p(a).$$

5. Let $V = \mathbb{R}^{n\times n}$ and let $B \in \mathbb{R}^{n\times n}$ be a fixed matrix. Let $T : V \to V$ be the linear transformation defined by $T(A) = AB - BA$, and $f : V \to \mathbb{R}$ be the trace linear functional defined by $f(C) = \mathrm{Tr}(C)$. Let $T^t : V^* \longrightarrow V^*$ be t the transpose linear transformation of T on the dual space V^*. Find the formula for the linear functional $T^t(f) : V \to \mathbb{R}$.

For any $A \in \mathbb{R}^{n\times n}$:

$$T^t(f)(A) = f(T(A)) = f(AB-BA)$$
$$= \mathrm{Tr}(AB-BA)$$
$$= 0.$$

6. Let $\mathbb{R}^\infty$ be a vector space of infinite sequences $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ of real numbers.

1). Define a linear transformation $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$T(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, \alpha_1, \alpha_2, \alpha_3, \cdots).$$

Find the eigenvalue(s) and eigenvectors of T or prove there are no eigenvalues or eigenvectors for T.

Denote $\vec{x} = (\alpha_1, \alpha_2, \alpha_3, \cdots)$, $\forall \lambda \in \mathbb{R}$

$T\vec{x} = (0, \alpha_1, \alpha_2, \cdots)$, $\lambda\vec{x} = \lambda(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots)$

$T\vec{x} = \lambda\vec{x} \iff (0, \alpha_1, \alpha_2, \cdots) = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots)$

$$\iff \begin{cases} 0 = \lambda\alpha_1 & (1) \\ \alpha_1 = \lambda\alpha_2 & (2) \\ \alpha_2 = \lambda\alpha_3 & (3) \\ \vdots \end{cases}$$

Case 1: If $\lambda = 0$, then: $\alpha_1 = 0$, $\alpha_2 = 0$, $\cdots$,

$\Rightarrow$ If $\lambda = 0$, then $T\vec{x} = \lambda\vec{x}$ is only true when $\vec{x} = \vec{0}$

$\Rightarrow$ $\lambda = 0$ is not an eigenvalue

Case 2: If $\lambda \neq 0$, by (1): $\alpha_1 = 0$ then by (2): $\alpha_2 = 0$, $\cdots$.

$\Rightarrow$ If $\lambda \neq 0$, then $T\vec{x} = \lambda\vec{x}$ is only true when $\vec{x} = \vec{0}$

$\Rightarrow$ $\forall \lambda \neq 0$ is not an eigenvalue

By Case 1 and Case 2: T has no eigenvalues.

2). Define a linear transformation $U : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$U(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\alpha_2, \alpha_3, \alpha_4, \cdots).$$

Find the eigenvalue(s) and eigenvectors of U or prove there are no eigenvalues or eigenvectors for U.

Denote $\vec{x} = (\alpha_1, \alpha_2, \alpha_3, \cdots)$, $\forall \lambda \in \mathbb{R}$

$U(\vec{x}) = U(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\alpha_2, \alpha_3, \alpha_4, \cdots)$

$\lambda \vec{x} = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots)$

$U(\vec{x}) = \lambda\vec{x} \iff (\alpha_2, \alpha_3, \alpha_4, \cdots) = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots)$

$$\iff \begin{cases} \alpha_2 = \lambda\alpha_1 \\ \alpha_3 = \lambda\alpha_2 \\ \alpha_4 = \lambda\alpha_4 \\ \vdots \end{cases}$$

$\Rightarrow$ For any $\lambda \in \mathbb{R}$ and $\vec{x} = (\alpha_1, \lambda\alpha_1, \lambda^2\alpha, \cdots)$

$= \alpha(1, \lambda, \lambda^2, \cdots)$ with $\alpha \neq 0$

$\Rightarrow U(\vec{x}) = \lambda\vec{x}$

$\Rightarrow$ Every real number λ is an eigenvalue of U associated with eigenvector $\alpha(1, \lambda, \lambda^2, \lambda^3, \cdots)$ $\forall \alpha \in \mathbb{R}$.

7. Let $A \in \mathbb{C}^{n\times n}$. Let $\lambda_1, \cdots, \lambda_n$ be all the eigenvalues of A.

1). Prove that the determinant of A equals to the product of all eigenvalues of A, i.e.

$$\det(A) = \lambda_1 \cdot \cdots \cdot \lambda_n.$$

2). Use Part 1) to prove that A is invertible if and only if $\lambda_i \neq 0$ for all $i = 1, \cdots, n$.

Proof: 1) As $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all the roots of the characteristic polynomial of A, $\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$

plug in $\lambda=0$: $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

2) A is invertible if and only if $\det A \neq 0$

As $\det A = \lambda_1\cdots\lambda_n$, $\det A \neq 0$ if and only if $\lambda_i \neq 0$ for all $i=1,\cdots,n$.

8. Let λ_1 and λ_2 be distinct eigenvalues of a linear transformation $T : V \to V$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors associated with λ_1 and λ_2 respectively. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Proof: Suppose $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0}$ (*)

Apply T on both sides of (*):

$$T(a_1\vec{v}_1 + a_2\vec{v}_2) = \vec{0}$$

$$\Leftrightarrow \lambda_1 a_1\vec{v}_1 + \lambda_2 a_2\vec{v}_2 = \vec{0} \quad ①$$

multiply λ_2 on both sides of (*):

$$\lambda_2 a_1\vec{v}_1 + \lambda_2 a_2\vec{v}_2 = \vec{0} \quad ②$$

① - ②: $(\lambda_1-\lambda_2)a_1\vec{v}_1 = \vec{0}$

As $\vec{v}_1$ is an eigenvector, $\vec{v}_1 \neq \vec{0}$

As λ_1 and λ_2 are distinct, $\lambda_1-\lambda_2 \neq 0$

$\Rightarrow a_1 = 0$.

plug $a_1=0$ into (*), we get $a_2\vec{v}_2 = \vec{0}$

As $\vec{v}_2$ is an eigenvector, $\vec{v}_2 \neq \vec{0}$, thus $a_2=0$

$\Rightarrow a_1 = a_2 = 0$

$\Rightarrow \vec{v}_1$ and $\vec{v}_2$ are linearly independent.

9. Let T be the linear transformation on $\mathbb{R}^4$ which is represented in standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what condition on a, b and c is T diagonalizable? Explain your answer.

10. Let T be a linear transformation on an n-dimensional vector space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.