

Multi-Variable Calculus I: Homework 5

Due on November 5, 2024 at 8:00 AM

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Problem 1

A ball with mass 0.8 kg is thrown southward into the air with a speed of 20 m/s at an angle of 30° to the ground. A southwest ($S45^\circ W$) wind applies a steady force of 2 N to the ball. Where does the ball land relative to where it was thrown and with what speed?

Solution 1

The initial velocity vector \mathbf{v}_0 can be broken down into horizontal and vertical components

$$\begin{aligned} v_{0_x} &= v_0 \cos(\theta) = 20 \cos(30^\circ) = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} \approx 17.32 \text{ m/s} \\ v_{0_y} &= v_0 \sin(\theta) = 20 \sin(30^\circ) = 20 \times \frac{1}{2} = 10 \text{ m/s}. \end{aligned}$$

So, the horizontal component of the velocity is $10\sqrt{3}$ m/s and the vertical component is 10 m/s.

The wind force is directed southwest, or 45° west of south. Decompose this force into its $x-y$ components

$$\begin{aligned} W_x &= W \cos(45^\circ) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.141 \text{ N} \\ W_y &= W \sin(45^\circ) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.141 \text{ N}. \end{aligned}$$

Using $F = ma$, the acceleration components due to the wind are

$$\begin{aligned} a_x &= \frac{W_x}{m} = \frac{1.141}{0.8} \approx 1.43 \text{ m/s}^2 \\ a_y &= \frac{W_y}{m} = \frac{1.141}{0.8} \approx 1.43 \text{ m/s}^2. \end{aligned}$$

For the vertical direction, we have the initial upward velocity $v_{0_y} = 10$ m/s and the acceleration is downward due to gravity

$$a_{\text{vertical}} = -g = -9.8 \text{ m/s}^2.$$

The time T of flight can be found by setting the vertical displacement to zero (since it returns to the same height from which it was launched)

$$y(t) = v_{0_y}t + \frac{1}{2}(-g)t^2 = 0 \Rightarrow t = \frac{10}{4.9} \approx 2.04 \text{ s}.$$

Therefore, the total flight time is around 2.04 seconds.

The horizontal displacement has two components – one due to the initial southward velocity and one due to the wind's effect. The southward displacement is given by

$$d_s = v_{0_x}T + \frac{1}{2}a_yT^2 \approx 39.$$

The westward displacement is due to the wind's westward force

$$d_w = \frac{1}{2}a_xT^2 = \frac{1}{2} \times 1.43 \times 2.04^2 \approx 2.98.$$

The ball lands approximately 40 meters south and 3 meters west of where it was thrown. The resultant

displacement D from the starting point is

$$D = \sqrt{40^2 + 2.98^2} \approx 40.11 \text{ m.}$$

The final speed can be found by considering the horizontal and vertical velocity components at $t = T$

$$\begin{aligned} v_x &= v_{0x} + a_y T = 17.32 + (1.43)(2.04) \approx 20.24 \text{ m/s} \\ v_y &= v_{0y} - gT = 10 - (9.8)(2.04) \approx -10 \text{ m/s.} \end{aligned}$$

Therefore, the final speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20.24)^2 + (-10)^2} = \sqrt{509.6576} \approx 22.58.$$

Problem 2

Given the position vector $\mathbf{r}(t) = \langle t, 2e^t, e^{2t} \rangle$, find the curvature, velocity, speed, and the tangential and normal components of the acceleration vector.

Solution 2

Velocity is just the first derivative of the position vector

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \langle 1, 2e^t, 2e^{2t} \rangle.$$

The speed is the magnitude of the velocity vector

$$s(t) = |\mathbf{v}(t)| = \sqrt{1 + 4e^{2t} + 4e^{4t}}.$$

Acceleration is the second derivative of the position vector, or the first derivative of the velocity vector

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \langle 0, 2e^t, 4e^{2t} \rangle.$$

We can find the curvature using the following formula

$$\kappa = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}.$$

The cross product of $\mathbf{v}(t)$ and $\mathbf{a}(t)$ is

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2e^t & 2e^{2t} \\ 0 & 2e^t & 4e^{2t} \end{vmatrix} = \langle 4e^{3t}, -4e^{2t}, 2e^t \rangle.$$

The magnitude of their cross product is

$$|\mathbf{v}(t) \times \mathbf{a}(t)| = \sqrt{(4e^{3t})^2 + (-4e^{2t})^2 + (2e^{2t})^2} = 2\sqrt{4e^{6t} + 5e^{4t}}.$$

We already know that $|\mathbf{v}(t)| = s(t)$. So, using that, we get the curvature of the position vector as

$$\kappa = \frac{2\sqrt{4e^{6t} + 5e^{4t}}}{(1 + 4e^{2t} + 4e^{4t})^{3/2}}.$$

The tangential and normal components of the acceleration vector are given as follows

$$\begin{aligned} a_T(t) &= \frac{(1)(0) + (2e^t)(2e^t) + (2e^{2t})(4e^{2t})}{\sqrt{1 + 4e^{2t} + 4e^{4t}}} = \frac{4(e^{2t} + 2e^{4t})}{1 + 2e^{2t}} \\ a_N(t) &= \frac{2e^t \sqrt{4e^{3t} + 4e^{2t} + 1}}{\sqrt{1 + 4e^{2t} + 4e^{4t}}} = \frac{2e^t \sqrt{4e^{3t} + 4e^{2t} + 1}}{1 + 2e^{2t}}. \end{aligned}$$

Problem 3

Find and sketch the domain of the functions

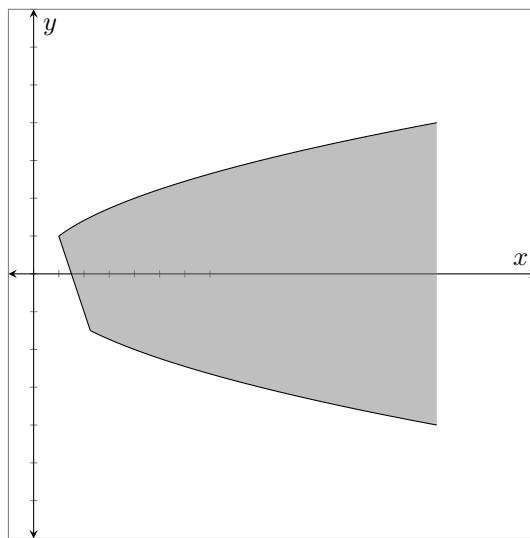
(i) $f(x, y) = \sqrt{y + 2x - 3} \ln(x - y^2)$.

(ii) $f(x, y) = \arccos(x^2 + y^2 - 8)$.

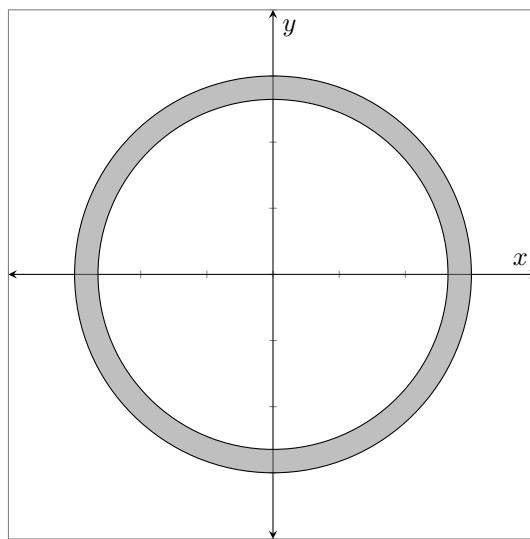
(iii) $f(x, y, z) = \sqrt{36 - 4x^2 - y^2 - 9z^2}$.

Solution 3

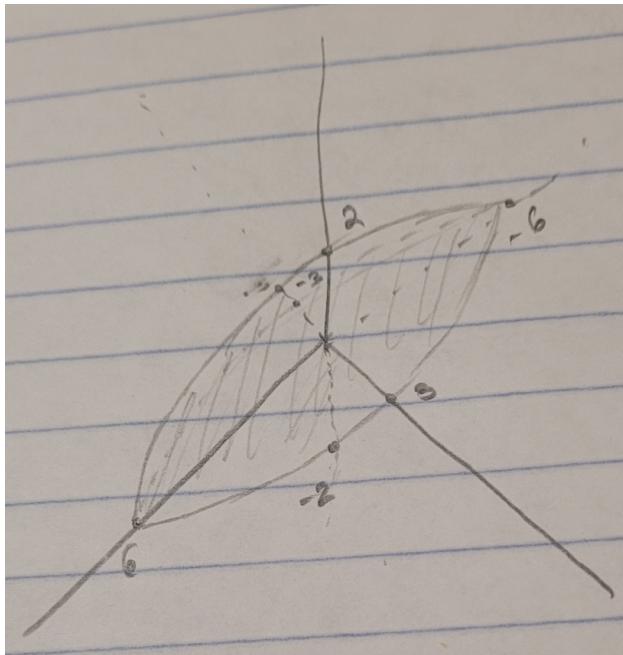
- (i) For the square root to be defined, $y + 2x - 3 \geq 0 \Rightarrow y \geq 3 - 2x$. For the natural logarithm to be defined, $x - y^2 > 0 \Rightarrow x > y^2$. So, the domain of the function is $D = \{(x, y) \in \mathbb{R}^2 \mid y \geq 3 - 2x \wedge x > y^2\}$.



- (ii) The function arccos is defined between the interval $[-1, 1]$, meaning the original function is only defined when $-1 \leq x^2 + y^2 - 8 \leq 1$. For the upper bound, we have $x^2 + y^2 \leq 9$ and for the lower bound, we have $x^2 + y^2 \geq 7$. So, the domain of the function is $D = \{(x, y) \in \mathbb{R}^2 \mid 7 \leq x^2 + y^2 \leq 9\}$.



- (iii) The function is defined when $36 - 4x^2 - y^2 - 9z^2 \geq 0$. This gives us $4x^2 + y^2 + 9z^2 \leq 36$, which is an ellipsoid in 3D space. This means that the function is defined on the surface of the ellipsoid along with all the points inside it. So, the domain of the function is $D = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{4} \leq 1 \right\}$.



Problem 4

Describe and draw several level curves for the functions

- (i) $f(x, y) = xy.$
- (ii) $f(x, y) = \ln(y) - x.$

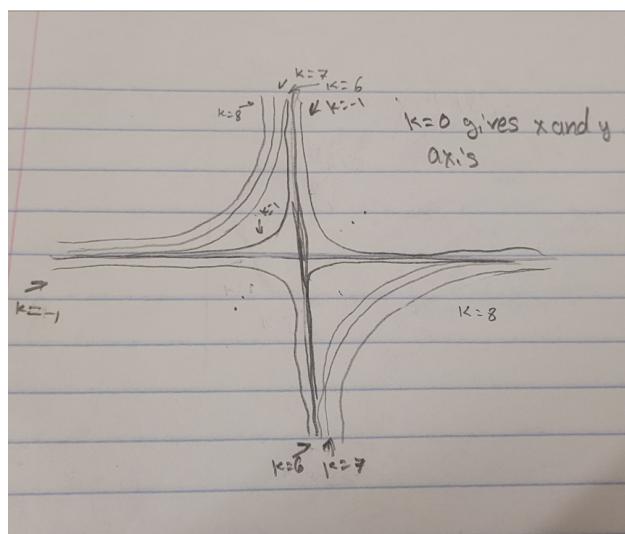
Solution 4

- (i) When $k = 0$, the level curve is along the x -axis and y -axis.

For $k > 0$, we get hyperbolas in the first and third quadrants, curving away from the origin.

For $k < 0$, the hyperbolas lie in the second and fourth quadrants.

As $|k|$ increases, the hyperbolas get farther and farther from the origin.



- (ii) For $k = 0$, the curve $y = e^x$ represents a standard exponential growth function.

For $k > 0$, the exponential curves shift upwards, increasing the y -values for each x .

For $k < 0$, the curves shift downwards.

As $|k|$ increases, the curves shift higher and higher.

