Abstract Linear Algebra: Homework 1

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Problem 1 State the rank-nullity theorem for an $m \times n$ matrix with real entries.

Solution. Let A be an $m \times n$ matrix with real entries. Then the rank-nullity theorem states that

$$\operatorname{rank}(A) + \operatorname{Null}(A) = n \quad \text{and} \quad \operatorname{rank}(A^T) + \operatorname{Null}(A) = m. \qquad \square$$

Problem 2 Given that $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, find a basis for $\operatorname{Null}(A)$ and a basis for $\operatorname{Range}(A)$.

Solution. The null space of A consists of all solutions to the equation $A\mathbf{x} = \mathbf{0}$. We need to convert B into reduced row echelon form to find the basis for the null space of A. The reduced row echelon form of B is

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{aligned} x_1 &= -2x_2 - 4x_4 \\ x_2 &= x_2 \\ x_3 &= \frac{7}{5}x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned} .$$

Parametrizing the solution, we have

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ \frac{7}{5}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, a basis for $\operatorname{Null}(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The basis for the range of A (or $\operatorname{Col}(A)$) is the set of pivot columns of A. The pivot columns of A are the first, third, and fourth columns. Therefore, a basis for Range(A) is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}.$$

□

Problem 3 Let A be an $m \times n$ matrix with real entires. Prove that $A = 0$ if and only if $\mathrm{Tr}(A^T A) = 0$.

Solution.

$\Rightarrow$) Assume $A = 0$, where 0 is the $m \times n$ zero matrix. Then,

$$A^T A = 0^T 0 = 0,$$

and the trace of the zero matrix is

$$\mathrm{Tr}(A^T A) = \mathrm{Tr}(0) = 0.$$

Thus, if $A = 0$, we have $\mathrm{Tr}(A^T A) = 0$.

$\Leftarrow$) Assume $\mathrm{Tr}(A^T A) = 0$. By the definition of $A^T A$,

$$A^T A = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \langle \mathbf{a}_1, \mathbf{a}_2 \rangle & \cdots & \langle \mathbf{a}_1, \mathbf{a}_n \rangle \\ \langle \mathbf{a}_2, \mathbf{a}_1 \rangle & \langle \mathbf{a}_2, \mathbf{a}_2 \rangle & \cdots & \langle \mathbf{a}_2, \mathbf{a}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{a}_n, \mathbf{a}_1 \rangle & \langle \mathbf{a}_n, \mathbf{a}_2 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{a}_n \rangle \end{bmatrix},$$

where $\mathbf{a}_i$ are the column vectors of A. The trace of $A^T A$ is

$$\mathrm{Tr}(A^T A) = \sum_{i=1}^{n} \langle \mathbf{a}_i, \mathbf{a}_i \rangle = \sum_{i=1}^{n} \|\mathbf{a}_i\|^2.$$

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Since $\mathrm{Tr}(A^T A) = 0$, it follows that

$$\sum_{i=1}^{n} \|\mathbf{a}_i\|^2 = 0.$$

Each term $\|\mathbf{a}_i\|^2$ is a sum of squares of the entries of $\mathbf{a}_i$ and is non-negative. Therefore, $\|\mathbf{a}_i\|^2 = 0$ for all i, which implies $\mathbf{a}_i = 0$ for all i.

Thus, $A = 0$. □

Problem 4 True or False. No explanation needed.

(i) If A is a 3×3 matrix, then $\det(3A) = 9 \det(A)$.

(ii) If A, B are invertible $n \times n$ matrices, then $[(AB)^T]^{-1} = (B^T)^{-1}(A^T)^{-1}$

Solution.

(i) False. If A is a 3×3 matrix, then $\det(3A) = 3^3 \det(A) = 27 \det(A)$, not $9 \det(A)$.

(ii) True. The inverse of the transpose of a product of matrices is the product of the transposes of the inverses, i.e., $[(AB)^T]^{-1} = (B^T)^{-1}(A^T)^{-1}$.

$\square$