Multi-Variable Calculus I: Homework 4

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Problem 1

Reparameterize the curve $\mathbf{r}(t) = \langle 1, t^2, t^3 \rangle$ with respect to arc length measured from the point $(1, 0, 0)$ in the direction of t increasing.

Solution 1

To reparameterize the curve $\mathbf{r}(t) = \langle 1, t^2, t^3 \rangle$ with respect to arc length s measured from the point $(1, 0, 0)$ in the direction of increasing t, we need to find s as a function t, then invert it to express t as a function of s.

The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \langle 0, 2t, 3t^2 \rangle$. The speed of the curve at time t is given by the magnitude of $\mathbf{r}'(t)$.

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| = \sqrt{0^2 + (2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}.$$

The arc length s from $t = 0$ to an arbitrary point t is

$$\begin{aligned}
s = \int_0^t \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| \mathrm{d}u &= \int_0^t u\sqrt{4 + 9u^2}\,\mathrm{d}u \\
&= \int_4^{4+9t^2} \frac{\sqrt{v}}{18}\,\mathrm{d}v \\
&= \frac{1}{18}\int_4^{4+9t^2} \sqrt{v}\,\mathrm{d}v \\
&= \frac{1}{18}\cdot\frac{2}{3}[v^{3/2}]_4^{4+9t^2} \\
&= \frac{1}{27}\left[\left(4 + 9t^2\right)^{3/2} - 4^{3/2}\right] \\
&= \frac{1}{27}\left(\left(4 + 9t^2\right)^{3/2} - 8\right).
\end{aligned}$$

To find t as a function of s, we need to solve the equation for t

$$\begin{aligned}
& \frac{1}{27}\left(\left(4 + 9t^2\right)^{3/2} - 8\right) &&= s \\
\Rightarrow\quad & \left(4 + 9t^2\right)^{3/2} - 8 &&= 27s \\
\Rightarrow\quad & \left(4 + 9t^2\right)^{3/2} &&= 27s + 8 \\
\Rightarrow\quad & 4 + 9t^2 &&= (27s + 8)^{2/3} \\
\Rightarrow\quad & 9t^2 &&= (27s + 8)^{2/3} - 4 \\
\Rightarrow\quad & t^2 &&= \frac{(27s + 8)^{2/3} - 4}{9} \\
\Rightarrow\quad & t &&= \pm\sqrt{\frac{(27s + 8)^{2/3} - 4}{9}}.
\end{aligned}$$

But since t is increasing, we choose the positive square root. Therefore,

$$\mathbf{r}(s) = \left\langle \sqrt{\frac{(27s + 8)^{2/3} - 4}{9}}, \left(\sqrt{\frac{(27s + 8)^{2/3} - 4}{9}}\right), \left(\sqrt{\frac{(27s + 8)^{2/3} - 4}{9}}\right)^3 \right\rangle.$$

Problem 2

Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln(t), t\ln(t)\rangle$ at the point $(1, 0, 0)$.

Solution 2

The curvature of a curve $\mathbf{r}(t)$ is given by the formula

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

The first and second derivatives of $\mathbf{r}(t)$ are

$$\mathbf{r}'(t) = \langle 2t, \frac{1}{t}, \ln(t) + 1\rangle \quad \text{and} \quad \mathbf{r}''(t) = \langle 2, -\frac{1}{t^2}, \frac{1}{t}\rangle.$$

The cross product of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ is

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & {}^1/_t & \ln(t) + 1 \\ 2 & -{}^1/_{t^2} & {}^1/_t \end{vmatrix} = \left\langle \frac{\ln(t) + 2}{t^2}, 2\ln(t), -\frac{4}{t} \right\rangle.$$

The magnitude of $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \frac{\sqrt{t^2\left(\left(\frac{\ln(t)+2}{t^2}\right)^2 + 4\ln^2(t)\right) + 16}}{t}.$$

The curvature is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{t^2\left(\left(\frac{\ln(t)+2}{t^2}\right)^2 + 4\ln^2(t)\right) + 16}}{(4t^2 + t^2(\ln(t) + 1)^2 + 1)^{3/2}}.$$

At $t = 1$, we get

$$\kappa(1) = \frac{\sqrt{30}}{18}.$$

Problem 3

Show that for a smooth curve, $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa \mathbf{N}(t)$. Therefore, at each point along the curve, $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ and $\mathbf{N}(t)$ are parallel and $\mathbf{N}$ points in the direction of curvature along the curve.

Solution 3

Proof. For a smooth curve $\mathbf{r}(t)$ parameterized by arc length s, the unit tangent vector is defined a

$$\mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}.$$

By definition, $\mathbf{T}$ has unit length 1.

We now differentiate $\mathbf{T}$ with respect to s to get

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}.$$

Since $\mathbf{T}$ has unit length, $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is orthogonal to $\mathbf{T}$. This is because

$$2\mathbf{T} \cdot \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = 0.$$

Therefore, $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is perpendicular to $\mathbf{T}$.

By definition, the curvature κ of the curve at a point is the magnitude of the rate of change of $\mathbf{T}$ with respect to s

$$\kappa = \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right|.$$

We can write

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa \mathbf{N}(t),$$

where $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$ that points in the direction of $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$. The principal normal vector of the curve is $\mathbf{N}$.

This result tells us that

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \quad \text{and} \quad \mathbf{N}(t),$$

are parallel, with $\mathbf{N}$ pointing in the direction of $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$. Thus, the principal normal vector $\mathbf{N}$ points in the direction of curvature along the curve, and its magnitude is scaled by the curvature κ.

In summary, we have shown that

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa \mathbf{N}(t),$$

where $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ and $\mathbf{N}$ are parallel, and $\mathbf{N}$ points in the direction of curvature along the curve. □

Problem 4

Given a space curve with smooth parametrization, $\mathbf{r}(t)$, the binormal vector is $\hat{B} = \hat{T} \times \hat{N}$. By properties of the cross product, it is a unit vector orthogonal to both $\hat{T}$ and $\hat{N}$.

Note: Given $\hat{B} = \hat{B}(s)$, where $s(t)$ is the arc length, by Chain Rule, $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}t} = \dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}[\mathbf{r}'(t)]$

(i) Compute and simplify $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}$.

(ii) Show that $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is orthogonal to $\hat{B}$.

(iii) Show that $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is orthogonal to $\hat{T}$.

(iv) Explain why $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is therefore parallel to $\hat{N}$.

Note: Since $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is parallel to $\hat{N}$, it has the form $\dfrac{\mathrm{d}B}{\mathrm{d}s} = -\tau\hat{N}$. The scalar function τ is called the torsion of the space curve. It measures the rate at which the curve is twisting out of the osculating plane toward or away from the binormal vector, $\hat{B}$. This is similar to how curvature measures the rate at which the curve is bending towards the unit normal vector.

The dot product can be used to solve for τ.

$$\begin{aligned}
-\tau\hat{N} &= \frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \\
-\tau\hat{N}\cdot\hat{N} &= \frac{\mathrm{d}\hat{B}}{\mathrm{d}s}\cdot\hat{N} \\
\tau &= -\frac{\mathrm{d}\hat{B}}{\mathrm{d}s}\cdot\hat{N} \quad \text{since } \hat{N}\cdot\hat{N} = 1.
\end{aligned}$$

Similar to curvature, this is an unpleasant computation.

Solution 4

To compute $\dfrac{\mathrm{d}\hat{B}}{\mathrm{d}s}$, we differentiate $\hat{B}$ with respect to s

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(\hat{T}\times\hat{N}).$$

Using the product rule for the cross product, we get

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{T}}{\mathrm{d}s}\times\hat{N} + \hat{T}\times\frac{\mathrm{d}\hat{N}}{\mathrm{d}s}.$$

Since $\dfrac{\mathrm{d}\hat{T}}{\mathrm{d}s} = \kappa\hat{N}$, we substitute

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = (\kappa\hat{N})\times\hat{N} + \hat{T}\times\frac{\mathrm{d}\hat{N}}{\mathrm{d}s}.$$

The first term, $(\kappa\hat{N}) \times \hat{N} = 0$, because $\hat{N} \times \hat{N} = 0$. Thus,

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \hat{T} \times \frac{\mathrm{d}\hat{N}}{\mathrm{d}s}.$$

To proceed, we note that $\frac{\mathrm{d}\hat{N}}{\mathrm{d}s}$ is in the direction of $\hat{T}$ and $\hat{B}$ by the Frenet-Serret formulas. In fact, $\frac{\mathrm{d}\hat{N}}{\mathrm{d}s} = -\kappa\hat{T} + \tau\hat{B}$, where τ is the torsion. Substituting, we get

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \hat{T} \times (-\kappa\hat{T} + \tau\hat{B}) = \tau(\hat{T} \times \hat{B}).$$

Since $\hat{T} \times \hat{B} = \hat{N}$, we have

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \tau\hat{N}.$$

We want to show that $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{B} = 0$. Substituting from above,

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \tau\hat{N},$$

we get

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{B} = (\tau\hat{N}) \cdot \hat{B} = \tau(\hat{N} \cdot \hat{B}).$$

Since $\hat{N}$ and $\hat{B}$ are orthogonal unit vectors, $\hat{N} \cdot \hat{B} = 0$. Therefore,

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{B} = 0,$$

showing that $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is orthogonal to $\hat{B}$.

We want to show that $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{T} = 0$. From our previous result,

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = \tau\hat{N}.$$

Thus,

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{T} = (\tau\hat{N}) \cdot \hat{T} = \tau(\hat{N} \cdot \hat{T}).$$

Since $\hat{N}$ and $\hat{T}$ are also orthogonal unit vectors, $\hat{N} \cdot \hat{T} = 0$. Therefore,

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{T} = 0,$$

showing that $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is orthogonal to $\hat{T}$.

We have shown that $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ is orthogonal to both $\hat{T}$ and $\hat{B}$. Since $\hat{T}$, $\hat{N}$, and $\hat{B}$ form an orthonormal basis for the space around the curve, $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ must be parallel to $\hat{N}$ (the only remaining direction). Therefore, we can write

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = -\tau\hat{N},$$

where τ is a scalar function called the torsion of the curve.

To compute the torsion, we can use the equation

$$\tau = -\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{N}.$$

Since $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} = -\tau\hat{N}$, we have

$$\tau = -\frac{\mathrm{d}\hat{B}}{\mathrm{d}s} \cdot \hat{N},$$

which allows us to find the torsion by taking the dot product of $\frac{\mathrm{d}\hat{B}}{\mathrm{d}s}$ with $\hat{N}$.