Chapter 2

Matrix Lie Groups Solutions

Exercise 1. Let a be an irrational real number. Show that the set of numbers of the form $e^{2\pi i n a}$, $n \in \mathbb{Z}$, is dense in S^1. Now, let G be the following subgroup of $\mathrm{GL}(2, \mathbb{C})$

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{iat} \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

Show that

$$\overline{G} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{is} \end{pmatrix} \middle| \, t, s \in \mathbb{R} \right\},$$

where $\overline{G}$ denotes the closure of the set G inside the space of 2×2 matrices.

Note: The group $\overline{G}$ can be thought of as the torus $S^1 \times S^1$, which in turn can be thought of as $[0, 2\pi] \times [0, 2\pi]$, with the ends of the intervals identified. The set $G \subset [0, 2\pi] \times [0, 2\pi]$ is called an **irrational line**. Draw a picture of this set and you should see why G is dense in $[0, 2\pi] \times [0, 2\pi]$.

Solution. We first show that the set $\left\{ e^{2\pi i n a} \right\}$ is dense in S^1. We want to show that

$$\overline{\left\{ e^{2\pi i n a} \right\}} = S^1.$$

Define the sequence $\theta_n = na \pmod 1$, which lives in the unit interval $[0, 1)$. The sequence $\left\{ e^{2\pi i \theta_n} \right\}$ consists of points on the unit circle. Since a is irrational, the sequence θ_n is dense in $[0, 1]$. Since the mapping $\theta \mapsto e^{2\pi i \theta}$ is a continuous function from $[0, 1]$ to S^1, it follows that the sequence $\left\{ e^{2\pi i n a} \right\}$ is dense in S^1. Therefore, for every $\epsilon > 0$, we can find an n such that $e^{2\pi i n a}$ is within ϵ of $e^{2\pi i \theta}$.

Now, we can show that G is dense in $S^1 \times S^1$. The elements of G are determined by pairs of (t, at), which trace out points in the torus $S^1 \times S^1$. That is, we're looking at the set

$$\{ t \pmod{2\pi}, at \pmod{2\pi} | \, t \in \mathbb{R} \} \subset [0, 2\pi] \times [0, 2\pi].$$

Since a is irrational, the set of points $(t \pmod{2\pi}, at \pmod{2\pi})$ is dense in $[0, 2\pi] \times [0, 2\pi]$, since the second coordinate, $at \pmod{2\pi}$ acts are an irrational rotation of the first coordinate. Since the torus $S^1 \times S^1$ is exactly the set of matrices of the form

$$\left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{is} \end{pmatrix} \middle| \, t, s \in \mathbb{R} \right\},$$

and our set is dense in this torus, it follows that the closure of G is precisely

$$\overline{G} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{is} \end{pmatrix} \middle| \, t, s \in \mathbb{R} \right\}. \qquad \square$$

Exercise 2. *Orthogonal Group.* Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^n$, $\langle x, y\rangle = \sum_i x_i y_i$. Show that a matrix A preserves inner products if and only if the column vectors of A are orthonormal.

Show that for any $n \times n$ real matrix B,

$$\langle Bx, y\rangle = \langle x, B^T y\rangle,$$

where $\left(B^T\right)_{ij} = B_{ji}$. Using this fact, show that a matrix A preserves inner products if and only if $A^T A = I$.

Note: a similar analysis applies to the complex orthogonal groups $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$.

Solution. We first show that a matrix A preserves inner products if and only if the column vectors of A are orthonormal.

Assume A preserves inner products. The standard inner product is defined as $\langle x, y\rangle = x^T y$. Since A preserves inner products, that means $\langle Ax, Ay\rangle = (Ax)^T Ay = x^T A^T Ay = x^T y$. This means $A^T = A^{-1}$, since we require $A^T A = I$ to preserve inner products. Let $A = (\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n)$, where $\mathbf{v}_i$ are the column vectors of A. Then, we have

$$A^T A = \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \cdots \\ \mathbf{v}_n^T \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1\rangle & \langle \mathbf{v}_1, \mathbf{v}_2\rangle & \cdots & \langle \mathbf{v}_1, \mathbf{v}_n\rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1\rangle & \langle \mathbf{v}_2, \mathbf{v}_2\rangle & \cdots & \langle \mathbf{v}_2, \mathbf{v}_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1\rangle & \langle \mathbf{v}_n, \mathbf{v}_2\rangle & \cdots & \langle \mathbf{v}_n, \mathbf{v}_n\rangle \end{pmatrix} = I \tag{2.1}$$

Therefore, we must have $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \delta_{ij}$, for $i, j = 1, \cdots, n$. Therefore, the column vectors of A are orthonormal.

Now, assume the columns of A are orthonormal. Same thing as shown in equation 2.1. Therefore, we have

$$\langle Ax, Ay\rangle = (Ax)^T By = x^T A^T Ay = x^T y.$$

Therefore, A preserves inner products.

Thus, we have shown that a matrix A preserves inner products if and only if the column vectors of A are orthonormal.

Now, we show that $\langle Bx, y\rangle = \langle x, B^T y\rangle$. Notice that

$$\langle Bx, y\rangle = (Bx)^T y = x^T B^T y = \langle x, B^T y\rangle.$$

Since we have proven that A preserves inner products if and only if $A^T A = I$ $\square$

Exercise 3. *Unitary groups.* Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{C}^n$, $\langle x, y\rangle = \sum_i \overline{x_i} y_i$. Following Exercise 2, show that $A^* A = I$ if and only if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$. ($(A^*)_{ij} = \overline{A_{ji}}$).

Solution. We first show that $A^* A = I$ if and only if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$.

Assume $A^* A = I$. The standard inner product is defined as $\langle x, y\rangle = \sum_i \overline{x_i} y_i$. Since A preserves inner products, that means $\langle Ax, Ay\rangle = (Ax)^* Ay = x^* A^* Ay = x^* y = \langle x, y\rangle$. Therefore, we have shown that $A^* A = I$ implies $\langle Ax, Ay\rangle = \langle x, y\rangle$.

Now, assume $\langle Ax, Ay\rangle = \langle x, y\rangle$. We have

$$\langle Ax, Ay\rangle = (Ax)^* Ay = x^* A^* Ay = x^* y = \langle x, y\rangle.$$

Therefore, $A^* A = I$.

Thus, we have shown that $A^* A = I$ if and only if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$. $\square$

Exercise 4. *Generalized orthogonal groups.* Let $[x, y]_{n,k}$ be the symmetric bilinear form on $\mathbb{R}^{n+k}$ defined in 2.1. Let gj be the $(n+k) \times (n+k)$ diagonal matrix with first n diagonal entries equal to one, and last k diagonal entries equal to minus one

$$g = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}.$$

Show that for all $x, y \in \mathbb{R}^{n+k}$,

$$[x, y]_{n,k} = \langle x, gy \rangle.$$

Show that a $(n+k) \times (n+k)$ real matrix A is in $\mathrm{O}(n,k)$ if and only if $A^T g A = g$. Show that $\mathrm{O}(n,k)$ and $\mathrm{SO}(n,k)$ are subgroups of $\mathrm{GL}(n+k, \mathbb{R})$, and are matrix Lie groups.

Solution. We first show that $[x, y]_{n,k} = \langle x, gy \rangle$. By definition, the symmetric bilinear form $[x, y]_{n,k}$ is given by

$$[x, y]_{n,k} = x^T g y.$$

We also have the standard Euclidean inner product

$$\langle x, y \rangle = x^T y.$$

Since g is the diagonal matrix

$$g = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix},$$

it follows that $\langle x, gy \rangle = x^T (gy)$. By substituting g, we get $\langle x, gy \rangle = x^T g y$. Since this matches the definition of $[x, y]_{n,k}$, we conclude that

$$[x, y]_{n,k} = \langle x, gy \rangle.$$

Now, we show that a $(n+k) \times (n+k)$ real matrix A is in $\mathrm{O}(n,k)$ if and only if $A^T g A = g$.

Assume $A \in \mathrm{O}(n,k)$. By definition, A preserves the bilinear form, meaning that for all $x, y \in \mathbb{R}^{n+k}$, we have

$$[Ax, Ay]_{n,k} = [x, y]_{n,k}.$$

Expanding this using the definition of the bilinear form,

$$(Ax)^T g (Ay) = x^T g y.$$

Since this holds for all x, y, it follows that

$$A^T g A = g.$$

Conversely, if $A^T g A = g$, then for any $x, y \in \mathbb{R}^{n+k}$,

$$(Ax)^T g (Ay) = x^T A^T g A y = x^T g y = [x, y]_{n,k},$$

which implies that A preserves the bilinear form.

Thus, we have shown that a $(n+k) \times (n+k)$ real matrix A is in $\mathrm{O}(n,k)$ if and only if $A^T g A = g$.

Now, we show that $\mathrm{O}(n,k)$ and $\mathrm{SO}(n,k)$ are subgroups of $\mathrm{GL}(n+k, \mathbb{R})$ and are matrix Lie groups. We check closure, identity, and inverses.

If $A, B \in \mathrm{O}(n,k)$, then

$$(AB)^T g (AB) = B^T (A^T g A) B = B^T g B = g.$$

Hence, $AB \in \mathrm{O}(n,k)$, so $\mathrm{O}(n,k)$ is closed under multiplication.

The identity matrix I_{n+k} satisfies $I^T g I = g$, so $I \in \mathrm{O}(n,k)$.

If $A \in \mathrm{O}(n,k)$, then $A^T g A = g$. Multiplying on the right by A^{-1} and on the left by A^{-T}, we obtain $A^{-T} g = g A^{-1}$ Taking transposes gives $(A^{-1})^T g A^{-1} = g$, so $A^{-1} \in \mathrm{O}(n,k)$.

Thus, $\mathrm{O}(n,k)$ is a subgroup of $\mathrm{GL}(n+k, \mathbb{R})$.

For the special indefinite orthogonal group, $\mathrm{SO}(n,k)$, we impose the additional determinant condition

$$\mathrm{SO}(n,k) = \{A \in \mathrm{O}(n,k) \mid \det(A) = 1\}.$$

Since the determinant of a product satisfies $\det(AB) = \det(A) \cdot \det(B)$, the determinant condition is preserved under multiplication and inverses, ensuring that $\mathrm{SO}(n,k)$ is also a subgroup.

A matrix Lie group is a subgroup of $\mathrm{GL}(n+k,\mathbb{R})$ that is also a smooth submanifold of $\mathbb{R}^{(n+k)^2}$. The defining equation $A^TgA = g$ imposes ${}^{(n+k)(n+k+1)}/_2$ independent constraints on the $(n+k)^2$ entries of A, since $A^TgA - g = 0$ is a symmetric matrix equation. The set of solutions forms a submanifold of $\mathbb{R}^{(n+k)^2}$. The determinant condition $\det(A) = 1$ is a smooth function, defining $\mathrm{SO}(n,k)$ as a submanifold of $\mathrm{O}(n,k)$.

Since both are smooth manifolds and closed under multiplication and inversion, they are matrix Lie groups. □

Exercise 5. *Symplectic groups.* Let $B[x,y]$ be the skew-symmetric bilinear form on $\mathbb{R}^{2^n}$ given by $B[x,y] = \sum_{i=1}^n x_i y_{n+1} - x_{n+i} y_i$. Let J be the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Show that for all $x, y \in \mathbb{R}^{2n}$,

$$B[x,y] = \langle x, Jy \rangle.$$

Show that a $2n \times 2n$ matrix A is in $\mathrm{Sp}(n,\mathbb{R})$ if and only if $A^TJA = J$. Show that $\mathrm{Sp}(n,\mathbb{R})$ is a subgroup of $\mathrm{GL}(2n,\mathbb{R})$, and a matrix Lie group.

Note: a similar analysis applies to $\mathrm{Sp}(n,\mathbb{C})$.

Solution. We first show that $B[x,y] = \langle x, Jy \rangle$. By definition, the skew-symmetric bilinear form $B[x,y]$ is given by

$$B[x,y] = \sum_{i=1}^n x_i y_{n+i} - x_{n+i} y_i.$$

The standard Euclidean inner product is given by $\langle x, y \rangle = x^T y$ Since J is defined as

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we compute $\langle x, Jy \rangle = x^T(Jy)$. Expanding using the structure of J,

$$Jy = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix},$$

so that

$$x^TJy = (x_1^T x_2^T) \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = x_1^T y_2 - x_2^T y_1 = \sum_{i=1}^n x_i y_{n+i} - x_{n+i} y_i.$$

This matches the definition of $B[x,y]$, proving the first statement.

Now, we show that a $2n \times 2n$ matrix A is in $\mathrm{Sp}(n,\mathbb{R})$ if and only if $A^TJA = J$.

Assume $A \in \mathrm{Sp}(n,\mathbb{R})$. By definition, A preserves the bilinear form, meaning that for all $x, y \in \mathbb{R}^{2n}$, $B[Ax, Ay] = B[x,y]$. Expanding this using the definition of the bilinear form, $(Ax)^TJ(Ay) = x^TJy$. Since this holds for all x, y, it follows that $A^TJA = J$. Conversely, if $A^TJA = J$, then for any $x, y \in \mathbb{R}^{2n}$,

$$(Ax)^TJ(Ay) = x^TA^TJAy = x^TJy = B[x,y],$$

which implies that A preserves the bilinear form. Thus, we have shown that $A \in \mathrm{Sp}(n,\mathbb{R})$ if and only if $A^TJA = J$.

Finally, we show that $\mathrm{Sp}(n,\mathbb{R})$ is a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ and a matrix Lie group. Again, we check closure, identity, and inverses.

If $A, B \in \mathrm{Sp}(n,\mathbb{R})$, then

$$(AB)^TJ(AB) = B^T(A^TJA)B = B^TJB = J,$$

so $AB \in \mathrm{Sp}(n, \mathbb{R})$.

The identity matrix I_{2n} satisfies $I^T J I = J$, so $I \in \mathrm{Sp}(n, \mathbb{R})$.

If $A \in \mathrm{Sp}(n, \mathbb{R})$, then $A^T J A = J$. Multiplying on the right by A^{-1} and on the left by A^{-T}, we obtain $A^{-T} J = J A^{-1}$. Taking transposes gives $(A^{-1})^T J A^{-1} = J$, so $A^{-1} \in \mathrm{Sp}(n, \mathbb{R})$.

Thus, $\mathrm{Sp}(n, \mathbb{R})$ is a subgroup of $\mathrm{GL}(2n, \mathbb{R})$.

To show that $\mathrm{Sp}(n, \mathbb{R})$ is a matrix Lie group, we note that the defining equation $A^T J A = J$ imposes $n(2n+1)$ independent constraints on the $4n^2$ entries of A, forming a submanifold of $\mathbb{R}^{4n^2}$. Since it is also closed under multiplication and inversion, $\mathrm{Sp}(n, \mathbb{R})$ is a matrix Lie group. $\square$

Exercise 6. *The groups* $\mathrm{O}(2)$ *and* $\mathrm{SO}(2)$. Show that the matrix

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

is in $\mathrm{SO}(2)$, and that

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}.$$

Show that every element A of $\mathrm{O}(2)$ is of one of the two forms

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

(If A is of the first form, then $\det(A) = 1$; if A is of the second form, then $\det(A) = -1$.)

Hint: Recall that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be in $\mathrm{O}(2)$, the column vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ must be unit vectors, and must be orthogonal.

Solution. We first show that the given matrix is in $\mathrm{SO}(2)$. A matrix is in $\mathrm{SO}(2)$ if it is orthogonal and has determinant 1. Consider

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The determinant is computed as

$$\det(A) = \cos^2(\theta) + \sin^2(\theta) = 1.$$

The transpose of A is

$$A^T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

We verify that $A^T A = I$

$$A^T A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = I.$$

Hence, $A \in \mathrm{SO}(2)$.

Next, we prove the composition rule,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}.$$

Performing matrix multiplication,

$$\begin{pmatrix} \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi) \end{pmatrix}.$$

Using angle sum identities,

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}.$$

Thus, matrix multiplication corresponds to addition of angles.

Now, we classify all elements of $\mathrm{O}(2)$. Suppose $A \in \mathrm{O}(2)$, so its column vectors must be orthonormal. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The orthogonality conditions are

$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + cd &= 0. \end{aligned}$$

Setting $a = \cos(\theta)$ and $c = \sin(\theta)$, we solve for b and d. The two possible solutions are,

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

which has determinant 1, and

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

which has determinant -1. These two cases classify all elements of $\mathrm{O}(2)$. □

Exercise 7. *The groups* $\mathrm{O}(1,1)$ *and* $\mathrm{SO}(1,1)$. Show that

$$A = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix},$$

is in $\mathrm{SO}(1,1)$, and that

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix} = \begin{pmatrix} \cosh(t+s) & \sinh(t+s) \\ \sinh(t+s) & \cosh(t+s) \end{pmatrix}.$$

Show that every element of $\mathrm{O}(1,1)$ can be written in one of the four forms

$$\begin{aligned} A &= \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \\ A &= \begin{pmatrix} -\cosh(t) & \sinh(t) \\ \sinh(t) & -\cosh(t) \end{pmatrix} \\ A &= \begin{pmatrix} \cosh(t) & -\sinh(t) \\ \sinh(t) & -\cosh(t) \end{pmatrix} \\ A &= \begin{pmatrix} -\cosh(t) & -\sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}. \end{aligned}$$

(Since $\cosh(t)$ is always positive, there is no overlap among the four cases. Matrices of the first two forms have determinant one, matrices of the last two forms have determinant minus one.)

Hint: For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be in $\mathrm{O}(1,1)$, we must have $a^2 - c^2 = 1$, $b^2 - d^2 = -1$, and $ad - cd = 0$. The set of points (a, c) in the plane with $a^2 - c^2 = 1$ (i.e., $a = \pm\sqrt{1 + c^2}$) is a hyperbola.

Solution. We first show that the given matrix is in $\mathrm{SO}(1,1)$. A matrix is in $\mathrm{SO}(1,1)$ if it preserves the quadratic form $x^2 - y^2$ and has determinant 1. Consider

$$A = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}.$$

The determinant is computed as

$$\det(A) = \cosh^2(t) - \sinh^2(t) = 1.$$

We verify that A preserves the quadratic form, $A^T J A = J$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Computing,

$$A^T J A = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}.$$

Simplifying, we find that A satisfies the defining property of $\mathrm{SO}(1,1)$.

Next, we prove the composition rule,

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}.$$

Performing matrix multiplication,

$$\begin{pmatrix} \cosh(t)\cosh(s) + \sinh(t)\sinh(s) & \cosh(t)\sinh(s) + \sinh(t)\cosh(s) \\ \sinh(t)\cosh(s) + \cosh(t)\sinh(s) & \sinh(t)\sinh(s) + \cosh(t)\cosh(s) \end{pmatrix}.$$

Using hyperbolic identities,

$$\begin{pmatrix} \cosh(t+s) & \sinh(t+s) \\ \sinh(t+s) & \cosh(t+s) \end{pmatrix}.$$

Thus, matrix multiplication corresponds to addition of parameters.

Now, we classify all elements of $\mathrm{O}(1,1)$. Suppose $A \in \mathrm{O}(1,1)$, so it must satisfy the conditions:

$$\begin{aligned} a^2 - c^2 &= 1 \\ b^2 - d^2 &= -1 \\ ad - bc &= 0. \end{aligned}$$

Solving for valid forms, we obtain four cases

$$A = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

$$A = \begin{pmatrix} -\cosh(t) & \sinh(t) \\ \sinh(t) & -\cosh(t) \end{pmatrix}$$

$$A = \begin{pmatrix} \cosh(t) & -\sinh(t) \\ \sinh(t) & -\cosh(t) \end{pmatrix}$$

$$A = \begin{pmatrix} -\cosh(t) & -\sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}.$$

Since $\cosh(t)$ is always positive, there is no overlap among the four cases. Matrices of the first two forms have determinant 1, while those of the last two forms have determinant -1. These four cases classify all elements of $\mathrm{O}(1,1)$. $\square$

Exercise 8. *The group* SU(2)*.* Show that if α, β are arbitrary complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, then the matrix

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \tag{2.2}$$

is in SU(2). Show that every $A \in \mathrm{SU}(2)$ can be expressed in the form in equation 2.2 for a unique pair (α, β) satisfying $|\alpha|^2 + |\beta|^2 = 1$. (Thus SU(2) can be thought of as the three-dimensional sphere S^3 sitting inside $\mathbb{C}^2 = \mathbb{R}^4$. In particular, this shows that SU(2) is connected and simply connected.)

Solution. We first show that the given matrix is in SU(2). A matrix is in SU(2) if it is unitary and has determinant 1. Consider

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}.$$

The determinant of A is computed as

$$\det(A) = \alpha\overline{\alpha} - (-\overline{\beta})\beta = |\alpha|^2 + |\beta|^2 = 1.$$

The conjugate transpose of A is

$$A^* = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix}.$$

We verify that $A^*A = I$

$$A^*A = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}\alpha + \overline{\beta}\beta & -\overline{\alpha}\overline{\beta} + \overline{\beta}\overline{\alpha} \\ -\beta\alpha + \alpha\beta & -\beta(-\overline{\beta}) + \alpha\overline{\alpha} \end{pmatrix}.$$

Since $|\alpha|^2 + |\beta|^2 = 1$, we obtain $A^*A = I$, proving that $A \in \mathrm{SU}(2)$.

Next, we show that every element of SU(2) can be written in this form. Suppose

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2).$$

Since B is unitary, $B^*B = I$ gives the conditions

$$\begin{aligned} |a|^2 + |c|^2 &= 1 \\ |b|^2 + |d|^2 &= 1 \\ a\overline{b} + c\overline{d} &= 0. \end{aligned}$$

Since $\det(B) = 1$, we also have $ad - bc = 1$ Setting $a = \alpha$, $c = \beta$, $b = -\overline{\beta}$, and $d = \overline{\alpha}$ satisfies all these conditions, with the constraint $|\alpha|^2 + |\beta|^2 = 1$ ensuring that B is unitary. Uniqueness follows from the linear independence of α and β as complex numbers.

This shows that every $A \in \mathrm{SU}(2)$ can be expressed in the given form, proving that SU(2) is topologically equivalent to the 3-sphere S^3 in $\mathbb{C}^2 = \mathbb{R}^4$. $\square$

Exercise 9. *The groups* $\mathrm{Sp}(1,\mathbb{R})$*,* $\mathrm{Sp}(1,\mathbb{C})$*, and* $\mathrm{Sp}(1)$*.* Show that $\mathrm{Sp}(1,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$, $\mathrm{Sp}(1,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$, and $\mathrm{Sp}(1) = \mathrm{SU}(2)$.

Solution. We first show that $\mathrm{Sp}(1,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$. By definition, $\mathrm{Sp}(1,\mathbb{R})$ consists of 2×2 real matrices preserving the symplectic form given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A matrix $A \in \mathrm{Sp}(1,\mathbb{R})$ satisfies $A^TJA = J$. Explicit computation shows that this condition is equivalent to requiring that $\det(A) = 1$, which is precisely the definition of $\mathrm{SL}(2,\mathbb{R})$. Thus, we conclude that $\mathrm{Sp}(1,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$.

Next, we show that $\mathrm{Sp}(1, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$. The complex symplectic group consists of 2×2 complex matrices preserving the complex symplectic form J. Again, the defining relation $A^T J A = J$ reduces to the determinant condition $\det(A) = 1$, which characterizes $\mathrm{SL}(2, \mathbb{R})$. Hence, $\mathrm{Sp}(1, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$.

Finally, we show that $\mathrm{Sp}(1) = \mathrm{SU}(2)$. The quaternionic symplectic group $\mathrm{Sp}(1)$ consists of 2×2 complex matrices preserving both the symplectic form J and the Hermitian form given by $A^* A = I$. Since $\mathrm{SU}(2)$ consists of 2×2 unitary matrices with determinant 1, it follows that the preservation conditions defining $\mathrm{Sp}(1)$ are equivalent to those of $\mathrm{SU}(2)$. Thus, we conclude that $\mathrm{Sp}(1) = \mathrm{SU}(2)$. $\square$

Exercise 10. *The Heisenberg group.* Determine the center $Z(h)$ of the Heisenberg group H. Show that the quotient group $H/Z(H)$ is abelian.

Solution. We first determine the center $Z(H)$ of the Heisenberg group H. The group operation is matrix multiplication. Direct computation shows that the product of two elements is given by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a + a' & c + c' + ab' \\ 0 & 1 & b + b' \\ 0 & 0 & 1 \end{pmatrix}.$$

The center $Z(H)$ consists of all elements that commute with every element of H. Let

$$X = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}.$$

Their commutator is given by

$$XYY^{-1}X^{-1} = \begin{pmatrix} 1 & 0 & ab' - a'b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For this to be the identity matrix, we require $ab' - a'b = 0$ for all $a, b, a', b' \in \mathbb{R}$. This implies that c must be the only free parameter in the center, meaning that $Z(H) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, c \in \mathbb{R} \right\}$.

Now, we show that the quotient group $H/Z(H)$ is abelian. Consider the natural projection map

$$\pi : H \to H/Z(H),$$

given by

$$\pi \left(\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right) = (a, b) \in \mathbb{R}^2.$$

The group operation in $H/Z(H)$ is then given by

$$(a, b) \cdot (a', b') = (a + a', b + b').$$

Since addition in $\mathbb{R}^2$ is commutative, it follows that $H/Z(H)$ is an abelian group. $\square$

Exercise 11. *Connectedness of* $\mathrm{SO}(n)$. Show that $\mathrm{SO}(n)$ is connected, following the outline below.

For the $n = 1$ case, there is not much to show, since a 1×1 matrix with determinant one must be 1. Assume, then, that $n \geq 2$ Let $\mathbf{e}_1$ denote the vector

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

Given any unit vector $\mathbf{v} \in \mathbb{R}^n$, show that there exists a continuous path $R(t)$ in $\mathrm{SO}(n)$ such that $R(0) = I$ and $R(1)\mathbf{e}_1 = \mathbf{v}$. (Thus any unit vector can be "continuously rotated" to $\mathbf{e}_1$.)

Now show that any element R of $\mathrm{SO}(n)$ can be connected to an element of $\mathrm{SO}(n-1)$, and proceed by induction.

Solution. Let $\mathcal{B} = \{\mathbf{v}, \mathbf{e}_1\}$ be a basis for a two-dimensional plane. By the Gram-Schmitt process, we can construct an orthonormal basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2\}$ for the same plane. Let $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2$ be defined as

$$\mathbf{u}_2 = \frac{\mathbf{e}_1 - \langle \mathbf{e}_1, \mathbf{v} \rangle \mathbf{v}}{\|\mathbf{e}_1 - \langle \mathbf{e}_1, \mathbf{v} \rangle \mathbf{v}\|}.$$

Let $\theta(t) : [0,1] \to \mathbb{R}$, defined by

$$\theta(0) = 0 \quad \text{and} \quad \theta(1) = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{e}_1}{\|\mathbf{v}\|\|\mathbf{e}_1\|}\right).$$

Then, we can construct a rotation $R(t) \in \mathrm{SO}(n)$ as a block matrix that acts as a rotation by $\theta(t)$ in the plane spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$, and as the identity on the orthogonal complement. This defines a continuous path with $R(0) = I$ and $R(1)\mathbf{v} = \mathbf{e}_1$.

We'll use induction to show that $\mathrm{SO}(n)$ is connected for all $n \geq 1$. The base case is trivial, which is trivially connected.

Assume $\mathrm{SO}(n)$ is connected for some $n \geq 1$. We need to show that $\mathrm{SO}(n+1)$ is connected. Let $R \in \mathrm{SO}(n+1)$. Consider the first column of R, which is a unit vector $\mathbf{v} \in \mathbb{R}^{n+1}$.

Since $R(1)$ maps t to $\mathbf{e}_1$, we consider the path $t \mapsto R(t)T$. Observe that

$$R(1)T\mathbf{e}_1 = R(1)\mathbf{v} = \mathbf{e}_1.$$

This means that $R(1)T$ is an element of $\mathrm{SO}(n+1)$ whose first column is $\mathbf{e}_1$.

We can write $R(1)T$ as a block matrix,

$$R(1)T = \begin{pmatrix} 1 & 0 \\ 0 & T' \end{pmatrix},$$

where T' is an element of $\mathrm{SO}(n)$. By the induction hypothesis, there exists a continuous path from the identity matrix I_n to T' in $\mathrm{SO}(n)$. We now concatenate the paths:

1. The path $R(t)$ from I_{n+1} to $R(1)$,
2. The path $t \mapsto R(1)T$ from $R(1)$ to $R(1)T$,
3. The path from I_n to T' in $\mathrm{SO}(n)$, which lifts to a path from I_{n+1} to $R(1)T$ in $\mathrm{SO}(n+1)$.

Concatenating these paths gives a continuous path from I_{n+1} to any element of $\mathrm{SO}(n+1)$, proving that $\mathrm{SO}(n+1)$ is connected.

By induction, $\mathrm{SO}(n)$ is connected for all $n \geq 1$. □

Exercise 12. *The polar decomposition of* $\mathrm{SL}(n, \mathbb{R})$. Show that every element A of $\mathrm{SL}(n, \mathbb{R})$ can be written uniquely in the form $A = RH$, where $R \in \mathrm{SO}(n)$, and H is a symmetric, positive-definite matrix with determinant one (That is, $H^T = H$, and $\langle \mathbf{x}, H\mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$).

Hint: If A could be written in this form, then we would have

$$A^T A = H^T R^T R H = H R^{-1} R H = H^2.$$

Thus H would have to be the unique positive-definite symmetric square root of $A^T A$.

Note: A similar argument gives polar decompositions for $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{GL}(n, \mathbb{C})$. For example, every element A of $\mathrm{SL}(n, \mathbb{C})$ can be written uniquely as $A = UH$, with $U \in \mathrm{SU}(n)$, and H is a self-adjoint positive definite matrix with determinant one.

Solution. Consider the matrix $A^T A$, which is symmetric and positive definite because for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, we have

$$(A^T A)^T = (A)^T (A^T)^T = A^T A \quad \text{and} \quad \mathbf{x}^T A^T A \mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 > 0.$$

Since $A \in \mathrm{SL}(n, \mathbb{R})$, we have $\det(A) = 1$, implying $\det(A^T A) = \det(A^T) \cdot \det(A) = 1 \cdot 1 = 1$ since determinant is multiplicative. By the spectral theorem, $A^T A$ has an orthonormal eigenbasis with positive eigenvalues, so it admits a unique positive-definite square root, denoted H, such that

$$H = \sqrt{A^T A} \implies H^2 = A^T A.$$

Define $R = AH^{-1}$. We check that R is orthogonal

$$R^T R = (H^{-1} A^T)(A H^{-1}) = H^{-1} A^T A H^{-1} = H^{-1} H^2 H^{-1} = I.$$

Thus, $R \in \mathrm{SO}(n)$ since $\det(R) = {}^{\det(A)}/_{\det(H)} = {}^1/_1 = 1$, proving existence.

Suppose $A = R_1 H_1 = R_2 H_2$ are two such decompositions. Then,

$$H_1^{-1} R_1^{-1} R_2 H_2 = I.$$

Multiplying on the right by H_2^{-1} and on the left by H_1, we obtain

$$H_1 H_1^{-1} R_1^{-1} R_2 H_2 H_2^{-1} = H_1 H_2^{-1} = I,$$

so $H_1 = H_2$. This implies $R_1 = R_2$, proving uniqueness.

Thus, every element of $\mathrm{SL}(n, \mathbb{R})$ has a unique polar decomposition. □

Exercise 13. *The connectedness of* $\mathrm{SL}(n, \mathbb{R})$. Using the polar decomposition of $\mathrm{SL}(n, \mathbb{R})$ and the connectedness of $\mathrm{SO}(n)$, show that $\mathrm{SL}(n, \mathbb{R})$ is connected.

Hint: Recall that if H is a real, symmetric matrix, then there exists a *real* orthogonal matrix R_1 such that $H = R_1 D R_1^{-1}$, where D is diagonal.

Solution. Since we are dealing with $\mathrm{SL}(n, \mathbb{R})$, we add the restriction that H is of determinant one. By the polar decomposition, we can write $A = RH$, where $R \in \mathrm{SO}(n)$ and H is a symmetric, positive-definite matrix with determinant one.

Also, by the hint, we can write $H = R_1 D R_1^{-1}$, where $R_1 \in \mathrm{O}(n)$ and D is a diagonal matrix. The space of symmetric matrices with determinant 1 that are also positive definite forms a connected space. This follows because the space of positive-definite diagonal matrices with determinant 1 is connected, and conjugation by an orthogonal matrix does not change connectivity.

By exercise 11, we know that $\mathrm{SO}(n)$ is connected. Since each element in $\mathrm{SL}(n, \mathbb{R})$ can be written as RH, where $R \in \mathrm{SO}(n)$ and H belongs to a connected space, and the product of connected spaces is connected, we conclude that $\mathrm{SL}(n, \mathbb{R})$ is connected. □

Exercise 14. *The connectedness of* $\mathrm{GL}(n, \mathbb{R})^+$. Show that $\mathrm{GL}(n, \mathbb{R})^+$ is connected.

Solution. For any $A \in \mathrm{GL}(n, \mathbb{R})$, the polar decomposition expresses A uniquely as $A = U_A P_A$, $U_A \in \mathrm{O}(n)$ (i.e., $U_A U_A^T = I$), and P_A is a symmetric positive-definite matrix (i.e., $P_A = \sqrt{A^\top A}$, and P_A has only positive eigenvalues).

Since $A, B \in \mathrm{GL}(n, \mathbb{R})^+$, we know that $\det(A) > 0$ and $\det(B) > 0$, which implies U_A, U_B have determinant $+1$, so $U_A, U_B \in \mathrm{SO}(n)$.

Given $A, B \in \mathrm{GL}(n, \mathbb{R})^+$ with their polar decompositions $A = U_A P_A$ and $B = U_B P_B$, we can construct a continuous path from A to B as follows. Since $\mathrm{SO}(n)$ is path-connected, there exists a smooth path U_t in $\mathrm{SO}(n)$ such that $U_0 = U_A$ and $U_1 = U_B$. One explicit choice is the geodesic interpolation, $U_t = U_A \exp\left(t \log\left(U_A^\top U_B\right)\right)$, which remains in $\mathrm{SO}(n)$ for all $t \in [0, 1]$. Since the space of symmetric positive-definite matrices is also path-connected, we use the interpolation $P_t = (1 - t)P_A + tP_B$. This remains positive definite for all $t \in [0,1]$ because the sum of two positive-definite matrices with positive weights remains positive definite. Now, we can define the path $A_t = U_t P_t$, for $t \in [0,1]$ Since U_t remains in $\mathrm{SO}(n)$ and P_t remains positive definite, each A_t is invertible with $\det(A_t) > 0$, ensuring $A_t \in \mathrm{GL}(n, \mathbb{R})^+$ for all t.

Verifying continuity, we get:

1. The function $t \mapsto U_t$ is continuous because it is constructed from matrix exponentiation, which is smooth.

2. The function $t \mapsto P_t$ is trivially continuous as it is a convex combination of continuous matrices.

3. Since matrix multiplication is continuous, the final path $t \mapsto A_t = U_t P_t$ is continuous.

Thus, $\mathrm{GL}(n, \mathbb{R})^+$ is connected. □

Exercise 15. Show that the set of translations is a normal subgroup of the Euclidean group, and also of the Poincaré group. Show that $(\mathrm{E}(n)/\text{translations}) \cong \mathrm{O}(n)$.

Solution. We begin by proving that the set of translations forms a normal subgroup of the Euclidean group $\mathrm{E}(n)$ and the Poincaré group.

The Euclidean group $\mathrm{E}(n)$ consists of all isometries of $\mathbb{R}^n$, which can be written as affine transformations

$$g(x) = Ax + b, \quad A \in \mathrm{O}(n), \quad b \in \mathbb{R}^n.$$

The subgroup of translations consists of elements of the form

$$T_b(x) = x + b, \quad b \in \mathbb{R}^n.$$

Given any translation T_b and an isometry $g(x) = Ax + c \in \mathrm{E}(n)$, we compute the conjugation

$$gT_b g^{-1}(x) = g(T_b(g^{-1}(x))).$$

Since $g^{-1}(x) = A^{-1}x - A^{-1}c$, we obtain

$$T_b(g^{-1}(x)) = A^{-1}x - A^{-1}c + b.$$

Applying g gives

$$gT_b g^{-1}(x) = A(A^{-1}x - A^{-1}c + b) + c = x + Ab.$$

Since Ab is still a translation, we conclude that the set of translations is normal in $\mathrm{E}(n)$.

Similarly, in the Poincaré group (which consists of Lorentz transformations and translations in Minkowski space), the same argument applies, replacing $\mathrm{O}(n)$ with the Lorentz group $O(1, n-1)$ and showing that translations remain normal under conjugation.

Next, we show that $\mathrm{E}(n)/\text{translations} \cong \mathrm{O}(n)$. Define a map

$$\varphi : \mathrm{E}(n) \to \mathrm{O}(n), \quad \varphi(A, b) = A.$$

This is a homomorphism with kernel consisting precisely of the translations (i.e., those transformations where $A = I$). Since the first isomorphism theorem states that

$$\mathrm{E}(n)/\ker(\varphi) \cong \mathrm{Im}(\varphi),$$

we conclude that

$$\mathrm{E}(n)/\text{translations} \cong \mathrm{O}(n).$$

□

Exercise 16. *Harder.* Show that every Lie group homomorphism $\phi : \mathbb{R} \to S^1$ is of the form $\phi(x) = e^{iax}$ for some $a \in \mathbb{R}$. In particular, every such homomorphism is smooth.

Solution. A Lie group homomorphism $\phi : \mathbb{R} \to S^1$ is a smooth map that preserves the group structure, meaning

$$\phi(x+y) = \phi(x)\phi(y) \quad \forall x, y \in \mathbb{R}.$$

Since $\mathbb{R}$ is an additive group and $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ is a multiplicative group, we seek to determine the structure of such a homomorphism.

Define $\theta(x) \in \mathbb{R}$ such that $\phi(x) = e^{i\theta(x)}\phi(x) = e^{i\theta(x)}$ The homomorphism property then implies

$$e^{i\theta(x+y)} = e^{i\theta(x)}e^{i\theta(y)},$$

which simplifies to

$$\theta(x+y) = \theta(x) + \theta(y) \pmod{2\pi}.$$

This means θ is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}/2\pi\mathbb{Z}, +)$. Since $\mathbb{R}$ is torsion-free, θ must be of the form $\theta(x) = ax$ for some $a \in \mathbb{R}$. Thus, the general form of ϕ is $\phi(x) = e^{iax}$.

Since $\phi(x)$ is expressed as a smooth function of x, it follows that every such Lie group homomorphism is smooth. □