

Functional Complex Variables I: Homework 8

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Exercise 5.59.3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 + (z^4/9)}.$$

Solution. Using expression (6) in Sec. 59, we have that

$$\frac{1}{1 + (z^4/9)} = \sum_{n=0}^{\infty} \left(-\frac{z^4}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} z^{4n}.$$

Therefore, $f(z)$ can be expressed as

$$f(z) = \frac{z}{9} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} z^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} z^{4n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1}.$$

This only holds for $|z^4| < 9$, or equivalently, $|z| < \sqrt[4]{9} = \sqrt{3}$. \square

Exercise 5.59.4. Show that if $f(z) = \sin(z)$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \quad (n = 0, 1, 2, \dots).$$

Thus, give an alternative derivation of the Maclaurin series (2) for $\sin(z)$ in Sec. 59.

Solution. Let $f(z) = \sin(z)$. We compute the first few derivatives of $\sin(z)$ and evaluate them at $z = 0$ to get

$$\begin{aligned} f(z) &= \sin(z) \Rightarrow f(0) = 0 \\ f'(z) &= \cos(z) \Rightarrow f'(0) = 1 \\ f''(z) &= -\sin(z) \Rightarrow f''(0) = 0 \\ f^{(3)}(z) &= -\cos(z) \Rightarrow f^{(3)}(0) = -1 \\ f^{(4)}(z) &= \sin(z) \Rightarrow f^{(4)}(0) = 0 \\ &\vdots \end{aligned}$$

Notice, we have the following pattern

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n, \quad n = 0, 1, 2, \dots$$

Therefore, the Maclaurin series for $\sin(z)$ is

$$\sin(z) = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

which matches the known Maclaurin series for $\sin(z)$. \square

Exercise 5.59.11. Show that when $z \neq 0$,

$$(i) \quad \frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots;$$

$$(ii) \quad \frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \dots$$

Solution to (i). We can express e^z as its Maclaurin series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Dividing this series by z^2 , we get

$$\frac{e^z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots$$

This series converges for all $z \neq 0$. □

Solution to (ii). The Maclaurin series for $\sin(z)$ is given by

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Substituting z^2 for z , we have

$$\sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}.$$

Dividing this series by z^4 , we get

$$\frac{\sin(z^2)}{z^4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n-2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2(2n-1)} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \dots$$

This series converges for all $z \neq 0$. □

Exercise 5.59.13. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Solution. We can factor the denominator as follows:

$$4z - z^2 = z(4 - z).$$

Thus, we can rewrite the expression as

$$\frac{1}{4z - z^2} = \frac{1}{z(4 - z)} = \frac{1}{4z} \cdot \frac{1}{1 - z/4}.$$

The series expansion for $\frac{1}{1-x}$ is given by

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

which converges for $|x| < 1$. In our case, we have $x = z/4$, so the series converges for $|z/4| < 1$, or equivalently, $|z| < 4$.

Therefore, we can write

$$\frac{1}{4z - z^2} = \frac{1}{4z} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$
□

Exercise 6.74.3. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz,$$

taken counterclockwise around the circle (i) $|z - 2| = 2$; (ii) $|z| = 4$.

Solution to (i). The poles of the integrand are at $z = 1$ and $z = \pm 3i$. The circle $|z - 2| = 2$ contains the pole at $z = 1$ but not the poles at $z = 3i$ and $z = -3i$. We can use the residue theorem to evaluate the integral. The function $f(z)$ can be written as

$$f(z) = \frac{\Phi(z)}{z - 1} \quad \text{where} \quad \Phi(z) = \frac{3z^3 + 2}{z^2 + 9}.$$

Since $\Phi(z)$ is analytic on the contour $|z - 2| = 2$, we can find the residue at the pole $z = 1$. The residue at $z = 1$ is given by

$$\operatorname{Res}_{z=1} f(z) = \lim_{z \rightarrow 1} (z - 1)f(z) = \lim_{z \rightarrow 1} (z - 1) \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} = \lim_{z \rightarrow 1} \frac{3z^3 + 2}{z^2 + 9} = \frac{1}{2}.$$

By the residue theorem, the value of the integral is given by

$$I = 2\pi i \cdot \operatorname{Res}_{z=1} \left(\frac{3z^3 + 2}{(z - 1)(z^2 + 9)} \right) = 2\pi i \cdot \frac{1}{2} = \pi i. \quad \square$$

Solution to (ii). The circle $|z| = 4$ contains all three poles: $z = 1$, $z = 3i$, and $z = -3i$. We can find the residues at each of these poles.

For the pole at $z = 1$, we already calculated the residue, specifically,

$$\operatorname{Res}_{z=1} f(z) = \frac{1}{2}.$$

Now, we calculate the residues at the poles $z = 3i$ and $z = -3i$.

The residue at $z = \pm 3i$ are given by

$$\begin{aligned} \operatorname{Res}_{z=3i} f(z) &= \lim_{z \rightarrow 3i} (z - 3i)f(z) = \lim_{z \rightarrow 3i} \frac{3z^3 + 2}{(z - 1)(z + 3i)} = \frac{3(3i)^3 + 2}{(3i - 1)(3i + 3i)} = \frac{15 + 49i}{12} \\ \operatorname{Res}_{z=-3i} f(z) &= \lim_{z \rightarrow -3i} (z + 3i)f(z) = \lim_{z \rightarrow -3i} \frac{3z^3 + 2}{(z - 1)(z - 3i)} = \frac{3(-3i)^3 + 2}{(-3i - 1)(-3i - 3i)} = \frac{15 - 49i}{12}. \end{aligned}$$

Now, we can sum the residues

$$\operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=3i} f(z) + \operatorname{Res}_{z=-3i} f(z) = \frac{1}{2} + \frac{15 + 49i}{12} + \frac{15 - 49i}{12} = 3.$$

By the residue theorem, the value of the integral is given by

$$I = 2\pi i \cdot \left(\operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=3i} f(z) + \operatorname{Res}_{z=-3i} f(z) \right) = 2\pi i \cdot 3 = 6\pi i. \quad \square$$