Funds of Anal I: Homework 3

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Exercise 2.2.1

What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\varepsilon > 0$ such that $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \varepsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution 2.2.1

A series (x_n) *verconges* to x if $|x_n - x|$ is bounded. Meaning, the sequence is vercongent *if and only if* it is bounded.

Exercise 2.2.2

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(i) $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(ii) $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$.

(iii) $\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution 2.2.2

The definition of convergence of a sequence is as follows

$$\boxed{\lim_{n\to\infty} a_n = a \Leftrightarrow \forall \varepsilon > 0, \exists N \text{ such that } \forall n > N \Rightarrow |a_n - a| < \varepsilon}$$

(i) *Proof.* Given $\varepsilon > 0$, choose

$$N = \frac{3}{25\varepsilon} - \frac{4}{5}.$$

Suppose $n > N > 0$. Then,

$$|a_n - a| = \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \frac{3}{5(5n+4)} < \frac{3}{5(5N+4)} < \frac{3}{5\left(5\left(\frac{3}{25\varepsilon} - \frac{4}{5}\right) + 4\right)} = \frac{3}{5\left(\frac{3}{5\varepsilon}\right)} = \varepsilon. \qquad \square$$

(ii) *Proof.* Given $\varepsilon > 0$, choose

$$N = \frac{1}{\varepsilon}.$$

Suppose $n > N > 0$. Then,

$$|a_n - a| = \left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{1}{n} < \frac{1}{N} < \varepsilon. \qquad \square$$

(iii) *Proof.* Given $\varepsilon > 0$, choose

$$N = \frac{1}{\varepsilon^3}.$$

Suppose $n > N > 0$. Then,

$$|a_n - a| = \left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \leq \frac{1}{\sqrt[3]{n}} < \frac{1}{\sqrt[3]{N}} < \varepsilon. \qquad \square$$

Exercise 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(i) At every college in the United States, there is a student who is at least seven feet tall.

(ii) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(iii) There exists a college in the United States where every student is at least six feet tall.

Solution 2.2.3

(i) Find a college in the United States where all students are less than seven feet tall.

(ii) Find a college in the United States where every professor does not only give A's or B's.

(iii) Show that all colleges in the United States have at least one student that's shorter than six feet tall.

Exercise 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(i) A sequence with an infinite number of ones that does not converge to one.

(ii) A sequence with an infinite number of ones that converges to a limit not equal to one.

(iii) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution 2.2.4

(i) $a_n = (-1)^n$.

(ii) Impossible, if $\lim_{n\to\infty} a_n = a \neq 1$ then for any $n \geq N$ we can find a n with $a_n = 1$, meaning $|1 - a| < \varepsilon$ is impossible.

(iii) $a_n = (1, 2, 1, 1, 3, 1, 1, 1, \dots)$.

Exercise 2.2.6

Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.27. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Solution 2.2.6

Proof. Let $\varepsilon > 0$. Since $(a_n) \to a$, then, by the definition of the limit, we get $\exists N_1$ such that $\forall n > N \Rightarrow |a_n - a| < \varepsilon/2$. We get a similar thing for $(a_n) \to b$, $\exists N_2$ such that $\forall n > N \Rightarrow |a_n - b| < \varepsilon/2$

Let $N = \max(\{N_1, N_2\})$. Then, for all $n > N$, we get

$$|a - b| \leq |(a_n - a) + (a_n - b)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $a = b$. □

Exercise 2.3.1

(i) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(ii) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Solution 2.3.1

(i) *Proof.* Let $\varepsilon > 0$. Since $(x_n) \to 0$, by the definition of convergence, there exists N such that for all $n > N$, we get $|x_n| < \varepsilon^2$. This means $x_n < \varepsilon^2$ for a large enough n. Taking the square root of both sides we get $\sqrt{x_n} < \varepsilon$ (this is allowed, as $x_n \geq 0$).

Thus, for all $n > N$, we get $|\sqrt{x_n} - 0| < \varepsilon$, showing that if $(x_n) \to 0$, then $(\sqrt{x_n}) \to 0$. □

(ii) *Proof.* Let $\varepsilon > 0$. Since $(x_n) \to x$, by the definition of convergence, there exists N such that for all $n > N$, we get $|x_n - x| < \varepsilon^2$. Multiplying by $(\sqrt{x_n} + \sqrt{x})$ gives $|x_n - x| < (\sqrt{x_n} + \sqrt{x})\varepsilon$. Since x_n is convergent, it is bounded $|x_n| \leq M$ implying $\sqrt{|x_n|} \leq \sqrt{M}$. This implies that

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{M} + \sqrt{x}} < \varepsilon.$$

Thus, for all $n > N$, we get $|\sqrt{x_n} - x| < \varepsilon$, showing that if $(x_n) \to x$, then $(\sqrt{x_n}) \to x$. □

Exercise 2.3.3

Theorem (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = z_n = l$, then $\lim_{n\to\infty} y_n = l$ as well.

Solution 2.3.3

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = z_n = l$, then there exists N_1, N_2 such that for all $n > N_1$, $|x_n - l| < \varepsilon/2$ and for all $n > N_2$ $|z_n - l| < \varepsilon/2$. Define $N = \max(\{N_1, N_2\})$. Then, for all $n > N$, we get $|x_n - l| < \varepsilon/2$ and $|z_n - l| < \varepsilon/2$.

Using the triangle inequality, we get

$$|x_n - z_n| \leq |x_n - l| + |z_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $x_n \leq y_n \leq z_n$, it follows that

$$|y_n - l| \leq |y_n - x_n| + |x_n - l| < |z_n - x_n| + |x_n - l| < \varepsilon.$$

Thus, for all $n > N$, we get $|y_n - l| < \varepsilon$, showing that if $\lim_{n\to\infty} x_n = z_n = l$ and $x_n \leq y_n \leq z_n$, then $\lim_{n\to\infty} y_n = l$. $\square$