Mathematical Image Modeling: Homework 1

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Jason Murphey 10:00

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Problem 1. Let $\langle \cdot, \cdot \rangle$ be a real inner product on a vector space V, with induced norm $\|\cdot\|$. Prove the following (called the Cauchy–Schwarz inequality):

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|v\| \|w\| \quad \text{for all } v, w \in V.$$

Hint. Consider the quadratic polynomial $p(t) = \langle v + tw, v + tw \rangle$, where $t \in \mathbb{R}$.

Solution. Let $\mathbf{v}, \mathbf{w} \in V$. Consider the quadratic polynomial

$$p(t) = \langle \mathbf{v} + t\mathbf{w}, \mathbf{v} + t\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle + 2t \langle \mathbf{v}, \mathbf{w} \rangle + t^2 \langle \mathbf{w}, \mathbf{w} \rangle = \|\mathbf{v}\|^2 + 2t \langle \mathbf{v}, \mathbf{w} \rangle + t^2 \|\mathbf{w}\|^2.$$

Since V is an inner product space, we have $p(t) \geq 0$, since $\langle v, w \rangle \geq 0$ and $\|v\|, \|w\| \geq 0$. Therefore, we have

$$\|\mathbf{w}\|^2 t^2 + 2 \langle \mathbf{v}, \mathbf{w} \rangle t + \|\mathbf{v}\|^2 \geq 0.$$

The discriminant of this quadratic polynomial must be less than or equal to zero, so we have

$$b^2 - 4ac = (2 \langle \mathbf{v}, \mathbf{w} \rangle)^2 - 4\|\mathbf{w}\|^2 \|\mathbf{v}\|^2 \leq 0.$$

Solving this inequality gives

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \|\mathbf{w}\|^2 \|\mathbf{v}\|^2.$$

Taking the square root of both sides, the Cauchy–Schwarz inequality holds. □

Problem 2. Show that a line segment in $\mathbb{R}^2$ has two-dimensional Lebesgue measure equal to zero. *Optional*: Prove that the same result is true even if the line has infinite length.

Solution. Let A be the set of a line segment in $\mathbb{R}^2$. Without loss of generality, we can assume that the line segment is horizontal. Let the length of the line segment be L. For A to have measure zero, for any $\varepsilon > 0$, there exists a countable collection of open intervals such that

$$A \subseteq \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} \text{area}(I_i) < \varepsilon.$$

Let A be the collection of rectangles of width L and height δ, where $\delta = \frac{\varepsilon}{L}$. Then, we have

$$\text{area}(A) = L \cdot \delta = L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

Therefore, the line segment has measure zero. □

Problem 3. Find a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that

$$\lim_{n \to \infty} f_n(x) = 0 \quad \text{for all } x \in [0, 1],$$

but

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, \mathrm{d}x = 1.$$

Why does this not contradict the Lebesgue Dominated Convergence Theorem?

Solution. Take the sequence of functions defined by

$$f_n(x) = \begin{cases} n, & \text{if } 0 \leq x \leq 1/n \\ 0, & \text{if } 1/n < x \leq 1 \end{cases}.$$

For all $x \in [0, 1]$, we have

$$\lim_{n \to \infty} f_n(x) = 0.$$

Integrating $f_n(x)$ over $[0,1]$, we have

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_0^{1/n} n\,\mathrm{d}x + \int_{1/n}^1 0\,\mathrm{d}x = n \cdot \frac{1}{n} + 0 = 1.$$

The Lebesgue Dominated Convergence Theorem states that given a converging sequence of functions (f_n) on a measurable space (S, Σ, μ) that converges pointwise to a function f and is dominated by an integrable function $g(x)$ (i.e., $|f(x)| \leq g(x)$), then the following must hold

$$\lim_{n\to\infty} \int_S f_n(x)\,\mathrm{d}\mu = \int_S f(x)\,\mathrm{d}\mu.$$

We don't get any contradictions because there is no integrable function $g(x)$ that dominates the sequence of functions $f_n(x)$ on $[0,1]$. □

Problem 4. Let $V = L^1(\mathbb{R}^n)$ and $W = L^\infty(\mathbb{R}^n)$. Show that if $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ then the mapping T defined by

$$[Tf](x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y,$$

is a bounded linear transformation from V to W, with

$$\|T\|_{L^1 \to L^\infty} \leq \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.$$

Solution. The norm of K in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is defined as

$$\|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} := \inf\{\alpha > 0 : |\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x)| > \alpha\}| = 0\} < \infty.$$

This means that $\|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n}$ is the smallest number such that $|K(x,y)| \leq \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $f \in L^1(\mathbb{R}^n)$. Then, for every $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
|[Tf](x)| &= \left| \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}^n} |K(x,y)||f(y)|\mathrm{d}y \\
&\leq \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \int_{\mathbb{R}^n} |f(y)|\mathrm{d}y \\
&= \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

We have the norm $\|T\|_{L^1 \to L^\infty}$ defined as

$$\|T\|_{L^1 \to L^\infty} := \sup\{\|Tf\|_{L^\infty} : \|f\|_{L^1} = 1\}.$$

Therefore, we have

$$\|Tf\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} |[Tf](x)| \leq \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)} = \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)},$$

since $\|f\|_{L^1} = 1$. Thus, T is a bounded linear transformation from V to W. □

Problem 5. Let W be a closed subset of a normed space V. Show that if $w_k \in W$ and $\lim_{k\to\infty} w_k = w$, then $w \in W$. Recall that by definition W is closed if its complement is open.

Solution. Assume $w_k \in W$ for all $k \in \mathbb{Z}$. Since W is closed, then W^c is open. Since $w_k \to w$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have $\|w_k - w\| < \varepsilon$. Suppose, for the sake of contradiction, that $w \notin W$. Then, $w \in W^c$. Since W^c is open, there exists $\delta > 0$ such that the open ball $B(w, \delta) \subseteq W^c$. However, since $w_k \to w$, there exists $N' \in \mathbb{N}$ such that for all $k \geq N'$, we have $\|w_k - w\| < \delta$. This implies that $w_k \in B(w, \delta) \subseteq W^c$ for all $k \geq N'$, contradicting the assumption that $w_k \in W$ for all k. Therefore, $w \notin W^c$, and thus $w \in W$. □