Abstract Linear Algebra: Homework 8

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Problem 1. *This problem will provide another of Cauchy-Schwarz inequality.*

Let V be an inner produce space over $\mathbb{C}$. For any $\mathbf{x}, \mathbf{y} \in V$, define $G = \begin{pmatrix} \langle \mathbf{x}, \mathbf{x} \rangle & \langle \mathbf{x}, \mathbf{y} \rangle \\ \langle \mathbf{y}, \mathbf{x} \rangle & \langle \mathbf{y}, \mathbf{y} \rangle \end{pmatrix} \in \mathbb{C}^{2 \times 2}$.

(i) Prove that g is a (Hermitian) positive semi-definite matrix.

(ii) Prove that eigenvalues of a Hermitian positive semi-definite matrix are all non-negative.

(iii) Prove the Cauchy-Schwarz inequality, i.e $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$. (Hint: What is the determinant of G? How do we relate determinant of a matrix with its eigenvalues?)

Solution to (i). The conjugate transpose of our matrix is

$$G^* = \begin{pmatrix} \overline{\langle \mathbf{x}, \mathbf{x} \rangle} & \overline{\langle \mathbf{y}, \mathbf{x} \rangle} \\ \overline{\langle \mathbf{x}, \mathbf{y} \rangle} & \overline{\langle \mathbf{y}, \mathbf{y} \rangle} \end{pmatrix}.$$

We know that $\overline{\langle \mathbf{x}, \mathbf{x} \rangle}$ is real, so it equals its own conjugate. We also know that $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$. So, we get

$$G^* = \begin{pmatrix} \overline{\langle \mathbf{x}, \mathbf{x} \rangle} & \overline{\langle \mathbf{y}, \mathbf{x} \rangle} \\ \overline{\langle \mathbf{x}, \mathbf{y} \rangle} & \overline{\langle \mathbf{y}, \mathbf{y} \rangle} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}, \mathbf{x} \rangle & \langle \mathbf{x}, \mathbf{y} \rangle \\ \langle \mathbf{y}, \mathbf{x} \rangle & \langle \mathbf{y}, \mathbf{y} \rangle \end{pmatrix} = G.$$

So, G is Hermitian.

Now, we need to show that G is positive semi-definite. We do this by showing that for any $\mathbf{v} \in \mathbb{C}^2$, we have $\mathbf{v}^* G \mathbf{v} \geq 0$. We have

$$\begin{aligned} \mathbf{v}^* G \mathbf{v} &= \begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}, \mathbf{x} \rangle & \langle \mathbf{x}, \mathbf{y} \rangle \\ \langle \mathbf{y}, \mathbf{x} \rangle & \langle \mathbf{y}, \mathbf{y} \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \overline{a} a \langle \mathbf{x}, \mathbf{x} \rangle + \overline{a} b \langle \mathbf{x}, \mathbf{y} \rangle + \overline{b} a \langle \mathbf{y}, \mathbf{x} \rangle + \overline{b} b \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle a\mathbf{x} + b\mathbf{y}, a\mathbf{x} + b\mathbf{y} \rangle \geq 0. \end{aligned}$$

Since this is true for any $\mathbf{v} \in \mathbb{C}^2$, we conclude that G is positive semi-definite. □

Solution to (ii). Let $\mathbf{v}$ be an eigenvector of G with eigenvalue λ, i.e., $G\mathbf{v} = \lambda \mathbf{v}$. Since G is positive semi-definite, we have $\mathbf{v}^* G \mathbf{v} \geq 0$. Expanding using $G\mathbf{v} = \lambda \mathbf{v}$, we get $\mathbf{v}^*(\lambda \mathbf{v}) = \lambda(\mathbf{v}^* \mathbf{v})$. Since $\mathbf{v}^* \mathbf{v}$ is the inner product of $\mathbf{v}$ with itself, we get $\mathbf{v}^* \mathbf{v} = \|\mathbf{v}\|^2$. So, we have $\lambda \|\mathbf{v}\|^2 \geq 0$. Since $\|\mathbf{v}\|^2$ is non-negative, we can conclude that $\lambda \geq 0$.

Therefore, all eigenvalues of G are non-negative. □

Solution to (iii). The determinant of G is

$$\det(G) = \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle - |\langle \mathbf{x}, \mathbf{y} \rangle|^2.$$

Since G is positive semi-definite, its determinant must be non-negative

$$\begin{aligned} & 0 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle - |\langle \mathbf{x}, \mathbf{y} \rangle|^2 \\ \Rightarrow\ & |\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \\ \Rightarrow\ & |\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \cdot \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} \\ \Rightarrow\ & |\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|. \end{aligned}$$

This is the Cauchy-Schwarz inequality. □

Problem 2. *Note this problem gives the formula for the orthogonal projection when a general (not necessarily orthogonal) basis of the subspace is given.*

Consider the standard inner product on $\mathbb{C}^n$. Let $W \subseteq \mathbb{C}^n$ be a subspace. Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ is a basis for W. Denote $B = [\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_m] \in \mathbb{C}^{n\times m}$.

(i) Prove that B^*B is (Hermitian) positive definite. (Note B^*B is often referred as the Gramian matrix related to $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$.)

(ii) Prove that eigenvalues of a Hermitian positive definite matrix are all positive.

(iii) Prove that B^*B is invertible.

(iv) Let $\mathbf{x} \in \mathbb{C}^n$ and let $\mathbf{x}_W$ be the orthogonal projection of $\mathbf{x}$ onto W. Prove that $\mathbf{x}_W = B(B^*B)^{-1}B^*\mathbf{x}$.

(v) Let $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. By using the formula in Part (iv), find the orthogonal projection of $\mathbf{x}_3$ onto the subspace spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$.

Solution to (i). Since transposition and conjugation distribute over matrix multiplication, we have $(B^*B)^* = B^*B$. So, B^*B is Hermitian.

Expanding $\mathbf{v}^*(B^*B)\mathbf{v} \geq 0$, we get

$$\mathbf{v}^*(B^*B)\mathbf{v} = (B\mathbf{v})^*(B\mathbf{v}) = \|B\mathbf{v}\|^2 \geq 0.$$

Since B is an $n \times n$ matrix whose columns form a basis of W, the map $B : \mathbb{C}^m \to W$ is injective. Thus, if $\mathbf{v} \neq \mathbf{0}$, then $B\mathbf{v} \neq \mathbf{0}$, which implies that $\|B\mathbf{v}\|^2 > 0$.

Therefore, B^*B is a (Hermitian) positive definite matrix. □

Solution to (ii). Let A be a Hermitian positive definite matrix. Consider an eigenpair $(\lambda, \mathbf{v})$, meaning $A\mathbf{v} = \lambda\mathbf{v}$. Taking the inner product with $\mathbf{v}$, $\langle A\mathbf{v}, \mathbf{v}\rangle = \lambda\langle \mathbf{v}, \mathbf{v}\rangle$. Since A is a positive definite, we have $\langle A\mathbf{v}, \mathbf{v}\rangle > 0$. The right hand side becomes $\lambda\langle \mathbf{v}, \mathbf{v}\rangle = \lambda\|\mathbf{v}\|^2 > 0$. Therefore, we get $\lambda > 0$, since $\|\mathbf{v}\|^2 > 0$.

Therefore, all eigenvalues of a Hermitian positive definite matrix are positive. □

Solution to (iii). Since B^*B is positive definite, all of its eigenvalues are strictly positive, which implies that B^*B is invertible. Specifically, since B^*B is full rank, its determinant is nonzero

$$\det(B^*B) = \prod_{i=1}^{m} \lambda_i > 0.$$

Therefore, B^*B is invertible. □

Solution to (vi). The orthogonal projection $\mathbf{x}_W$ of $\mathbf{x}$ onto W is defined as the unique vector in W minimizing the distance $\|\mathbf{x} - \mathbf{x}_W\|$. Since $\mathbf{x}_W \in W$, we can write it as $\mathbf{x}_W = B\mathbf{c}$, for some coefficient vector $\mathbf{c} \in \mathbb{C}^m$.

The vector $\mathbf{x} - B\mathbf{c}$ is orthogonal to W, which means that it is orthogonal to each column of B, $B^*(\mathbf{x} - B\mathbf{c}) = \mathbf{0}$. Expanding this, we get $B^*\mathbf{x} - B^*B\mathbf{c} = \mathbf{0}$. Since B^*B is invertible, we can solve for $\mathbf{c}$ to get $\mathbf{c} = (B^*B)^{-1}B^*\mathbf{x}$. Thus, we have

$$\mathbf{x}_W = B(B^*B)^{-1}B^*\mathbf{x}.$$ □

Solution to (v). The columns of B are $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The conjugate transpose of B is

$$B^* = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We now compute B^*B

$$B^*B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 & 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Using the formula for the inverse of a 2×2 matrix

$$(B^*B)^{-1} = \frac{1}{(2 \cdot 2 - 1 \cdot 1)} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Next, we compute $B^*\mathbf{x}_3$

$$B^*\mathbf{x}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 \\ 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

We then compute $(B^*B)^{-1}B^*\mathbf{x}_3$

$$(B^*B)^{-1}B^*\mathbf{x}_3 = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 8 - 3 \\ -4 + 6 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Now we compute the orthogonal projection $\mathbf{x}_W$:

$$\mathbf{x}_W = B \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5/3 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 7/3 \\ 2/3 \\ \frac{5}{3} \end{pmatrix}.$$

Thus, the orthogonal projection of $\mathbf{x}_3$ onto the subspace spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$ is:

$$\mathbf{x}_W = \begin{pmatrix} 7/3 \\ 2/3 \\ 5/3 \end{pmatrix}.$$

□

Problem 3. Find the QR-decomposition for the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$.

Solution. Using the Gram-Schmidt process, we can find the orthonormal basis for the column space of A. We start with the first column, given $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Let $\mathbf{u}_1 = \mathbf{v}_1$. We then have to normalize $\mathbf{u}_1$ to get

$$\hat{\mathbf{u}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

Next, we compute $\mathbf{u}_2$ as

$$\mathbf{u}_2 = \mathbf{v}_2 - \langle \hat{\mathbf{u}}_1, \mathbf{v}_2 \rangle \hat{\mathbf{u}}_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \sqrt{2} \cdot \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We then normalize $\mathbf{u}_2$ to get

$$\hat{\mathbf{u}}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

Finally, we compute $\mathbf{u}_3$ as

$$\begin{aligned} \mathbf{u}_3 = \mathbf{v}_3 - \langle \hat{\mathbf{u}}_1, \mathbf{v}_3 \rangle \hat{\mathbf{u}}_1 - \langle \hat{\mathbf{u}}_2, \mathbf{v}_3 \rangle \hat{\mathbf{u}}_2 &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \sqrt{2} \cdot \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} - \sqrt{2} \cdot \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

We then normalize $\mathbf{u}_3$ to get

$$\hat{\mathbf{u}}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We now have the orthonormal basis $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$. Therefore, the matrix Q and Q^T is

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{pmatrix} \quad \text{and} \quad Q^T = \begin{pmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix R is given by

$$\begin{aligned} R = Q^T A &= \begin{pmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \left(-\sqrt{2}/2\right)\cdot(-1) + \left(\sqrt{2}/2\right)\cdot(1) & \left(\sqrt{2}/2\right)\cdot(2) & \left(\sqrt{2}/2\right)\cdot(2) \\ \left(\sqrt{2}/2\right)\cdot(-1) + \left(\sqrt{2}/2\right)\cdot(1) & \left(\sqrt{2}/2\right)\cdot(2) & \left(\sqrt{2}/2\right)\cdot(2) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, the QR-decomposition of A is

$$A = QR = \begin{pmatrix} 0 & 0 & 1 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

□

Problem 4. Let $V = \mathbb{C}^{n\times n}$ with the inner product $\langle A, B\rangle = \operatorname{Tr}(B^* A)$. Find the orthogonal complement of the subspace of diagonal matrices.

Solution. Let D be the subspace of V consisting of diagonal matrices, i.e., $D = \{A \in \mathbb{C}^{n\times n} \mid A_{ij} = \delta_{ij}\}$. Its dimension is n since a diagonal matrix is determined by its n diagonal entries.

The orthogonal complement $D^\perp$ consists of all matrices $X \in \mathbb{C}^{n\times n}$ that satisfy $\langle A, X\rangle = \mathbf{0}$. Expanding the inner product, we get $\langle A, X\rangle = \operatorname{Tr}(X^* A)$.

For this to be zero for all diagonal matrices A, we consider a diagonal matrix $A = \operatorname{diag}(a_1, \cdots, a_n)$, so that $A_{ij} = a_{ij}\delta_{ij}$. Then,

$$\langle A, X\rangle = \operatorname{Tr}(X^* A) = \sum_{i=1}^{n} a_i \left(X^*\right)_{ii}.$$

Since this must be zero for all choices of a_i, we conclude that $\left(X^*\right)_{ii} = 0$, i.e., $X_{ii} = 0$ for all i. Therefore, $D^\perp$ consists of all matrices with zero diagonal entries. □

Problem 5. Let $A \in \mathbb{C}^{m\times n}$. Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be equipped with the standard inner product. Prove the following statements.

(i) $\operatorname{Null}(A) = (\operatorname{Range}(A^*))^\perp$.

(ii) $\operatorname{Null}(A^* A) = \operatorname{Null}(A)$.

(iii) $\operatorname{Rank}(A^* A) = \operatorname{Rank}(A) = \operatorname{Rank}(A^*)$.

(iv) $\operatorname{Range}(A^* A) = \operatorname{Range}(A^*)$.

Solution to (i). The orthogonal complement of $\text{Range}(A^*)$ consists of all vectors $\mathbf{x} \in \mathbb{C}^n$ such that $\langle A^*\mathbf{y}, \mathbf{x}\rangle = \mathbf{0}$, for all $\mathbf{y} \in \mathbb{C}^m$. Expanding the inner product, we get $\langle A^*\mathbf{y}, \mathbf{x}\rangle = (A^*\mathbf{y})^*\mathbf{x} = \mathbf{y}^*(A\mathbf{x})$. Since this must hold for all $\mathbf{y}$, it follows that $A\mathbf{x} = \mathbf{0}$, meaning $\mathbf{x} \in \text{Null}(A)$. Thus, $\text{Null}(A) \subseteq (\text{Range}(A^*))^\perp$

Conversely, if $\mathbf{x} \in (\text{Range}(A^*))^\perp$, then $\mathbf{x}$ satisfies $\langle A^*\mathbf{x}, \mathbf{y}\rangle = \mathbf{0}$, for all $\mathbf{y} \in \mathbb{C}^m$, which, again implies that $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \text{Null}(A)$. Thus, $(\text{Range}(A^*))^\perp \subseteq \text{Null}(A)$.

Therefore, we conclude that $\text{Null}(A) = (\text{Range}(A^*))^\perp$. □

Solution to (ii). Suppose $\mathbf{x} \in \text{Null}(A)$. Then, $A\mathbf{x} = \mathbf{0}$. We have $A^*A\mathbf{x} = A^*(A\mathbf{x}) = A^*\mathbf{0} = \mathbf{0}$. So, $\mathbf{x} \in \text{Null}(A^*A)$, which implies that $\text{Null}(A) \subseteq \text{Null}(A^*A)$.

Conversely, suppose $\mathbf{x} \in \text{Null}(A^*A)$. Then, $A^*A\mathbf{x} = \mathbf{0}$. Taking the inner product with $\mathbf{x}$, we get $\langle A^*A\mathbf{x}, \mathbf{x}\rangle = \mathbf{0}$. Using the definition of the inner product, $(A\mathbf{x})^*(A\mathbf{x}) = \|A\mathbf{x}\|^2 = \mathbf{0}$. Thus, $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \text{Null}(A)$. Therefore, $\text{Null}(A^*A) \subseteq \text{Null}(A)$.

Therefore, we conclude that $\text{Null}(A^*A) = \text{Null}(A)$. □

Solution to (iii). By the rank-nullity theorem, we have $\dim(\text{Null}(A)) + \dim(\text{Range}(A)) = n$. Since we just proved $\text{Null}(A^*A) = \text{Null}(A)$, we get $\dim(\text{Null}(A^*A)) = \dim(\text{Null}(A))$. Applying rank-nullity to A^*A, we get $\dim(\text{Null}(A^*A)) + \dim(\text{Range}(A^*A)) = n$. Since $\dim(\text{Null}(A^*A)) = \dim(\text{Null}(A))$, it follows that $\dim(\text{Range}(A^*A)) = \dim(\text{Range}(A))$. Therefore, we have $\text{Rank}(A^*A) = \text{Rank}(A)$.

Similarly, since A^*A and AA^* have the same rank (because their null spaces have the same dimension), we also obtain $\text{Rank}(A^*) = \text{Rank}(A)$.

Therefore, we conclude that $\text{Rank}(A^*A) = \text{Rank}(A) = \text{Rank}(A^*)$. □

Solution to (iv). Since A^*A maps $\mathbb{C}^n$ to $\mathbb{C}^n$, its range is contained in $\text{Range}(A^*)$. That is, $\text{Range}(A^*A) \subseteq \text{Range}(A^*)$.

For the reverse inclusion, let $\mathbf{y} \in \text{Range}(A^*)$, so there exists $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{y} = A^*\mathbf{x}$. Then, we have $\mathbf{y} = A^*A(A^t\mathbf{x})$, where A^t is the least-squares solution of $A\mathbf{x} = \mathbf{y}$. This shows that every element in $\text{Range}(A^*)$ can be written as $A^*A\mathbf{x}$, so $\text{Range}(A^*) \subseteq \text{Range}(A^*A)$.

Therefore, we conclude that $\text{Range}(A^*A) = \text{Range}(A^*)$. □

Problem 6. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Let $(\lambda, \mathbf{v})$ be an eigenvalue/eigenvector pair of A. Prove that $(\overline{\lambda}, \mathbf{v})$ is an eigenvalue/eigenvector pair of A^*.

Solution. Since $(\lambda, \mathbf{v})$ is an eigenpair of A, we have $A\mathbf{v} = \lambda\mathbf{v}$. Taking the inner product of both sides with $\mathbf{v}$, we get $\langle A\mathbf{v}, \mathbf{v}\rangle = \langle \lambda\mathbf{v}, \mathbf{v}\rangle = \lambda\langle \mathbf{v}, \mathbf{v}\rangle$. Since A is normal, we also consider $\langle \mathbf{v}, A^*\mathbf{v}\rangle = \langle A\mathbf{v}, \mathbf{v}\rangle$. Substituting $A\mathbf{v} = \lambda\mathbf{v}$, we get $\langle \mathbf{v}, A^*\mathbf{v}\rangle = \lambda\langle \mathbf{v}, \mathbf{v}\rangle$. But the left-hand side can be rewritten as $\langle \mathbf{v}, A^*\mathbf{v}\rangle = \langle A^*\mathbf{v}, \mathbf{v}\rangle$. Since the inner product satisfies $\langle A^*\mathbf{v}, \mathbf{v}\rangle = \overline{\langle \mathbf{v}, A^*\mathbf{v}\rangle}$, we take the conjugate to get $\langle A^*\mathbf{v},\ \rangle = \overline{\lambda}\langle \mathbf{v}, \mathbf{v}\rangle$.

Since $\langle \mathbf{v}, \mathbf{v}\rangle = \|\mathbf{v}\|^2 \neq \mathbf{0}$, we conclude that $A^*\mathbf{v} = \overline{\lambda}\mathbf{v}$. Therefore, $(\overline{\lambda}, \mathbf{v})$ is an eigenpair of A^*. □

Problem 7. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Prove that eigenvectors of A associated with distinct eigenvalues are orthogonal.

Solution. Let $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ be two distinct eigenpairs of A. First, we compute $\langle A\mathbf{v}_1, \mathbf{v}_2\rangle = \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle = \lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle$. Now, we compute $\langle \mathbf{v}_1, A^*\mathbf{v}_2\rangle$. Since A is normal, its eigenvectors satisfy $A^*\mathbf{v}_2 = \overline{\lambda_2}\mathbf{v}_2$. Thus, $\langle \mathbf{v}_1, A^*\mathbf{v}_2\rangle = \langle \mathbf{v}_1, \overline{\lambda_2}\mathbf{v}_2\rangle = \lambda_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle$.

Since A is normal, we know $\langle A\mathbf{v}_1, \mathbf{v}_2\rangle = \langle \mathbf{v}_1, A^*\mathbf{v}_2\rangle$. Therefore, we have $\lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \overline{\lambda_2}\langle \mathbf{v}_1, \mathbf{v}_2\rangle$. Rearranging, we get $(\lambda_1 - \overline{\lambda_2})\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \mathbf{0}$. Since λ_1 and λ_2 are distinct, we have $\lambda_1 - \overline{\lambda_2} \neq \mathbf{0}$.

Therefore, eigenvectors corresponding to distinct eigenvalues are orthogonal. □

Problem 8. True or False. (No explanation is needed)

(i) Suppose $A \in \mathbb{C}^{n\times n}$. Then $\text{Range}(A^*A) = \text{Range}(A^*) = \text{Range}(A)$.

(ii) A set of orthonormal vectors must be linearly independent.

(iii) A set of orthogonal vectors must be linearly independent.

(iv) Every linear transformation on V has a unique adjoint.

(v) For every linear transformation $T : V \to V$ and any given ordered basis B for V, we have $[T^*]_B = ([T]_B)^*$.

(vi) For any linear transformation T and U on V and scalars a and b, we have

$$(aT + bU)^* = aT^* + bU^*.$$

(vii) Every self-adjoint linear transformation on V is normal.

(viii) Linear transformations and their adjoints on V have the same eigenvalues.

(ix) Linear transformations and their adjoints on V have the same eigenvectors.

Solution to (i). False. □

Solution to (ii). True. □

Solution to (iii). False. □

Solution to (iv). True. □

Solution to (v). True. □

Solution to (vi). True. □

Solution to (vii). True. □

Solution to (viii). False. □

Solution to (ix). False. □