Introduction to Abstract Algebra I: Homework 1

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Victor Ostrik

Hashem A. Damrah
UO ID: 952102243

Exercise 0.1. Describe the set by listing its elements: $\{x \in \mathbb{R} \mid x^2 = 3\}$.

Solution. The set can be described by listing its elements as $\{\sqrt{3}, -\sqrt{3}\}$. ☐

Exercise 0.2. Describe the set by listing its elements: $\{m \in \mathbb{Z} \mid m^2 + m = 6\}$.

Solution. The equation $m^2 + m - 6 = 0$ can be factored as $(m-2)(m+3) = 0$. Therefore, the solutions are $m = 2$ and $m = -3$. Thus, the set can be described by listing its elements as $\{2, -3\}$. ☐

Exercise 0.4. Describe the set by listing its elements: $\{x \in \mathbb{Z} \mid x^2 - 10x + 16 \leq 0\}$.

Solution. The inequality $x^2 - 10x + 16 \leq 0$ can be factored as $(x-2)(x-8) \leq 0$. The solutions to the equation $(x-2)(x-8) = 0$ are $x = 2$ and $x = 8$. The inequality holds for values of x between 2 and 8, inclusive. Therefore, the integer solutions are $x = 2, 3, 4, 5, 6, 7, 8$. Thus, the set can be described by listing its elements as $\{2, 3, 4, 5, 6, 7, 8\}$. ☐

Exercise 0.15. Show that $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has the same cardinality as $\mathbb{R}$. [Hint: Find an elementary function of calculus that maps an interval one-to-one onto $\mathbb{R}$, and then translate and scale appropriately to make the domain the set S.]

Solution. Take the function $f : S \to \mathbb{R}$, defined by $f(x) = \tan(\pi x - \pi/2)$. As x approaches 0 from the right, $f(x)$ approaches $-\infty$, and as x approaches 1 from the left, $f(x)$ approaches ∞. The function is continuous and strictly increasing on the interval $(0, 1)$. Therefore, f is a bijection from S to $\mathbb{R}$, showing that S has the same cardinality as $\mathbb{R}$. ☐

Exercise 0.24. Find the number of different partitions of a set having 2 elements.

Solution. A set with 2 elements, say $\{a, b\}$, can be partitioned in the following ways:

(i) $\{\{a\}, \{b\}\}$

(ii) $\{\{a, b\}\}$

Therefore, there are a total of 2 different partitions of a set having 2 elements. ☐

Exercise 0.26. Find the number of different partitions of a set having 4 elements.

Solution. A set with 4 elements, say $\{a, b, c, d\}$, can be partitioned in the following ways:

(i) $\{\{a\}, \{b\}, \{c\}, \{d\}\}$

(ii) $\{\{a, b\}, \{c\}, \{d\}\}$ (and all permutations of this form)

(iii) $\{\{a, b, c\}, \{d\}\}$ (and all permutations of this form)

(iv) $\{\{a, b\}, \{c, d\}\}$ (and all permutations of this form)

(v) $\{\{a, b, c, d\}\}$

Counting all unique partitions, we find there are a total of 15 different partitions of a set having 4 elements. ☐

Exercise 0.30. Determine whether $x \mathscr{R} y$ in $\mathbb{R}$ if $x \geq y$ is an equivalence relation on the set. Describe the partition arising from the relation.

Solution. The relation $x \mathscr{R} y$ defined by $x \geq y$ is not an equivalence relation because it fails the symmetry property. For example, if $x = 3$ and $y = 2$, then $3 \geq 2$ (so $3 \mathscr{R} 2$), but $2 \not\geq 3$.

Since it is not an equivalence relation, there is no partition arising from this relation. ☐

Exercise 0.32. Determine whether $(x_1, y_1)\mathscr{R}(x_2, y_2)$ in $\mathbb{R} \times \mathbb{R}$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$ is an equivalence relation on the set. Describe the partition arising from the relation.

Solution. The relation $(x_1, y_1)\mathscr{R}(x_2, y_2)$ defined by $x_1^2 + y_1^2 = x_2^2 + y_2^2$ is an equivalence relation because it satisfies the following properties:

(i) Reflexivity: For any point (x, y), we have $x^2 + y^2 = x^2 + y^2$, so $(x, y)\mathscr{R}(x, y)$.

(ii) Symmetry: If $(x_1, y_1)\mathscr{R}(x_2, y_2)$, then $x_1^2 + y_1^2 = x_2^2 + y_2^2$. This implies that $x_2^2 + y_2^2 = x_1^2 + y_1^2$, so $(x_2, y_2)\mathscr{R}(x_1, y_1)$.

(iii) Transitivity: If $(x_1, y_1)\mathscr{R}(x_2, y_2)$ and $(x_2, y_2)\mathscr{R}(x_3, y_3)$, then $x_1^2 + y_1^2 = x_2^2 + y_2^2$ and $x_2^2 + y_2^2 = x_3^2 + y_3^2$. Therefore, $x_1^2 + y_1^2 = x_3^2 + y_3^2$, so $(x_1, y_1)\mathscr{R}(x_3, y_3)$.

The partition arising from this equivalence relation consists of sets of points in $\mathbb{R} \times \mathbb{R}$ that lie on circles centered at the origin with radius $\sqrt{r}$ for each non-negative real number r. Each equivalence class corresponds to a circle defined by the equation $x^2 + y^2 = \sqrt{r}$ for some fixed $r \geq 0$. □

Exercise 0.34. Determine whether $n\mathscr{R}m$ in $\mathbb{Z}^+$ if n and m have the same final digit in the usual base ten notation is an equivalence relation on the set. Describe the partition arising from the relation.

Solution. The relation $n\mathscr{R}m$ defined by n and m having the same final digit in base ten is an equivalence relation because it satisfies the following properties:

(i) Reflexivity: For any positive integer n, the final digit of n is the same as itself, so $n\mathscr{R}n$.

(ii) Symmetry: If $n\mathscr{R}m$, then n and m have the same final digit. This implies that m and n also have the same final digit, so $m\mathscr{R}n$.

(iii) Transitivity: If $n\mathscr{R}m$ and $m\mathscr{R}p$, then n and m have the same final digit, and m and p have the same final digit. Therefore, n and p must also have the same final digit, so $n\mathscr{R}p$.

The partition arising from this equivalence relation consists of sets of positive integers that share the same final digit. There are ten equivalence classes corresponding to the final digits 0 through 9:

(i) Class for final digit 0: $\{0, 10, 20, 30, \cdots\}$.

(ii) Class for final digit 1: $\{1, 11, 21, 31, \cdots\}$.

(iii) Class for final digit 2: $\{2, 12, 22, 32, \cdots\}$.

(iv) Class for final digit 3: $\{3, 13, 23, 33, \cdots\}$.

(v) Class for final digit 4: $\{4, 14, 24, 34, \cdots\}$.

(vi) Class for final digit 5: $\{5, 15, 25, 35, \cdots\}$.

(vii) Class for final digit 6: $\{6, 16, 26, 36, \cdots\}$.

(viii) Class for final digit 7: $\{7, 17, 27, 37, \cdots\}$.

(ix) Class for final digit 8: $\{8, 18, 28, 38, \cdots\}$.

(x) Class for final digit 9: $\{9, 19, 29, 39, \cdots\}$. □

Exercise 1.7. Determine whether the operation *, defined on $\mathbb{Z}$ by letting $a * b = a - b$, is associative, whether the operation is commutative, and whether the set has an identity element.

Solution. The operation is not associative since $(3 * 2) * 1 = 1 * 1 = 0$, while $3 * (2 * 1) = 3 * 1 = 2$.

The operation is also not commutative since $3 * 2 = 3 - 2 = 1$, while $2 * 3 = 2 - 3 = -1$.

Finally, the set does not have an identity element for this operation. An identity element e would need to satisfy $a * e = a$ for all $a \in \mathbb{Z}$. This implies $a - e = a \Rightarrow e = 0$. However, checking with $e = 0$, we have $a * 0 = a - 0 = a$, which holds true, but we also need to check if $0 * a = -a$, which does not equal a unless $a = 0$. Therefore, there is no identity element that works for all integers. ☐

Exercise 1.8. Determine whether the operation *, defined on $\mathbb{Q}$ by letting $a * b = 2ab + 3$, is associative, whether the operation is commutative, and whether the set has an identity element.

Solution. The operation is not associative since $(a * b) * c = (2ab + 3) * c = 2(2ab + 3)c + 3 = 4abc + 6c + 3$, while $a * (b * c) = a * (2bc + 3) = 2a(2bc + 3) + 3 = 4abc + 6a + 3$.

The operation is commutative since $a * b = 2ab + 3 = b * a$.

To find the identity element e, we need $a*e = a$ for all $a \in \mathbb{Q}$. This gives $a*e = 2ae+3 = a$. Rearranging, we get $2ae - a + 3 = 0 \Rightarrow a(2e - 1) + 3 = 0$. For this to hold for all a, we must have $2e - 1 = 0$ and $3 = 0$, which is impossible. Therefore, there is no identity element in $\mathbb{Q}$ for this operation. ☐

Exercise 1.10. Determine whether the operation *, defined on $\mathbb{Z}^+$ by letting $a * b = 2^{ab}$, is associative, whether the operation is commutative, and whether the set has an identity element.

Solution. The operation is not associative since $(a*b)*c = 2^{(2^{ab})c} = 2^{c \cdot 2^{ab}}$, while $a*(b*c) = 2^{a(2^{bc})} = 2^{a \cdot 2^{bc}}$.

The operation is commutative since $a * b = 2^{ab} = 2^{ba} = b * a$.

If an identity e existed, we would need $a * e = a$ for all a, i.e. $2^{ae} = a$. Taking $\log_2$ gives $ae = \log_2(a)$, which has no single solution e valid for all a. Therefore, no identity element exists. ☐