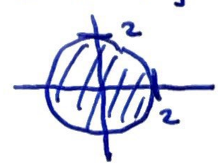
Defn: A set D in $\mathbb{R}^2$ is <u>closed</u> if it contains its boundary.

Ex: a) $D = \{(x,y) : x^2+y^2 \leq 4\}$

The boundary is $x^2+y^2=4$ which is in D.

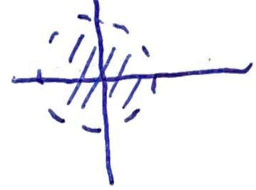
D is closed



b) $D = \{(x,y) : x^2+y^2 < 4\}$

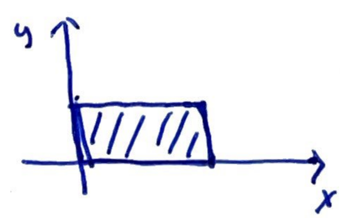
The boundary is $x^2+y^2=4$.

D is not closed



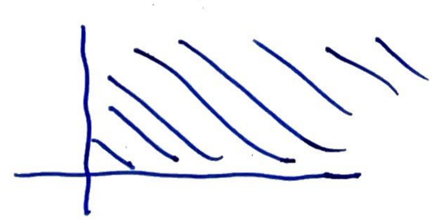
Defn: A set D in $\mathbb{R}^2$ is <u>bounded</u> if there exists a circular disk entirely containing D.

Ex: a) $D = \{(x,y) : 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$



The set is bounded. The circle $C = \{(x,y) : (x-1)^2+(y-\frac{1}{2})^2 \leq \frac{\sqrt{5}}{2}\}$ contains D. The set is also closed.

b) $D = \{(x,y) : x \geq 0,\ y \geq 0\}$



D is not bounded

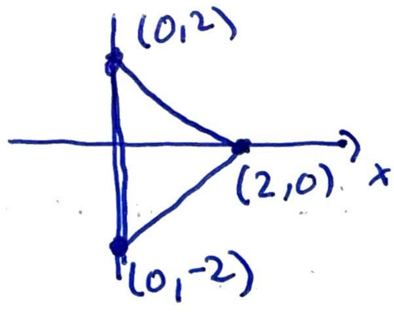
D is closed. It contains boundary $x=0$ and $y=0$.

Extreme Value Theorem: If $f(x,y)$ is continuous on a closed and bounded set D, then it has an absolute maximum value and absolute minimum value in D.

Steps

1. Evaluate f at critical points in D.
2. Maximize and Minimize f on the boundary of D.
3. Largest value is absolute max. Smallest value is absolute min.

Ex: Find the absolute max and absolute min values of $f(x,y) = x^2+y^2-2x$ on D where D is the closed triangular region with vertices $(2,0)$, $(0,-2)$, and $(0,2)$.



Solve
$\nabla f = \vec{0}$

$\langle 2x-2,\ 2y \rangle = \vec{0}$

Critical Point is $(1,0)$

$f(1,0) = 1+0-2 = -1$

Boundary

① $x = 0$ for $-2 \leq y \leq 2$

$f(0,y) = y^2$ on $[-2,2]$

By EVT, $f(0,y)$ has absolute extrema on closed interval. Check critical points and end points.

$f_y = 2y = 0$

critical Point $(0,0)$

$f(0,-2) = 4$

$f(0,0) = 0$

$f(0,2) = 4$

② Line between $(0,-2)$ and $(2,0)$

$y = x - 2$ $[0,2]$

$f(x, x-2) = x^2 + (x-2)^2 - 2x$ on $[0,2]$

$f_x(x, x-2) = 2x + 2(x-2) - 2$

$= 4x - 6 = 0$ $x = \frac{3}{2}$

$f(\frac{3}{2}, -\frac{1}{2}) = \frac{9}{4} + \frac{1}{4} - 3 = \frac{5}{2} - \frac{6}{2} = \frac{-1}{2}$

$f(2,0) = 4 - 4 = 0$

③ Line between $(0,2)$ and $(2,0)$

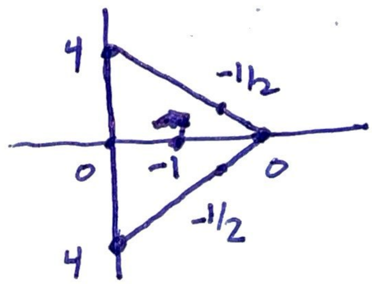
$y = 2 - x$ for $0 \le x \le 2$

$f(x, 2-x) = x^2 + (2-x)^2 - 2x$

$= f(x, x-2)$ 　 f is even in y coordinate.

Therefore another critical point at $x = \frac{3}{2}$.

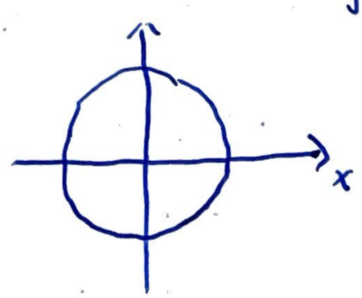
$f\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{2}$



Absolute min of -1 at $(1,0)$.

Absolute max of 4 at $(0,-2)$ and $(0,2)$.

Ex: Find absolute extrema of $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on $D = \{(x,y) \mid x^2 + y^2 \le 16\}$



Solve

$\nabla f = \vec{0}$

$\langle 4x - 4, 6y \rangle = \vec{0}$

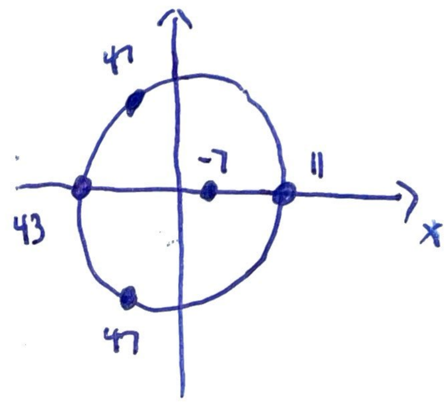
Critical Point is $(1,0)$

$f(1,0) = 2 - 4 - 5 = -7$

Boundary: $x^2+y^2=16$.

Use Lagrange multiplier method to maximize/minimize f subject to $g(x,y)=x^2+y^2=16$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 16 \end{cases}$$



① $4x-4=2\lambda x$

② $6y=2\lambda y$

③ $g=16$

From ②, $2y(3-\lambda)=0$. Either $y=0$ or $\lambda=3$.

If $y=0$, then from ③, $x^2=16$ and $x=\pm 4$

$(-4,0)$ and $(4,0)$ are solutions.

If $\lambda=3$, then from ①, $4x-4=6x$ and $x=-2$

Then from ③, $4+y^2=16$ and $y=\pm\sqrt{12}=\pm 2\sqrt{3}$

$(-2,2\sqrt{3})$ and $(-2,-2\sqrt{3})$ are solutions.

$f(-4,0)=32+16-5=43$

$f(4,0)=32-16-5=11$

$f(-2,2\sqrt{3})=8+36+8-5=47$

$f(-2,-2\sqrt{3})=47$

Absolute min of -7 at $(1,0)$.

Absolute max of 47 at $(-2,\pm 2\sqrt{3})$