Fundamentals of Analysis I: Homework 1

Due on October 9, 2024 at 13:00

Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

SECTION 1.2

Exercise 1.2.1

(i) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

(ii) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution to (i). Suppose $\sqrt{3}$ is rational. This means that there exists $p, q \in \mathbb{Z}$ such that $\frac{p}{q} = \sqrt{3}$. Suppose p and q have no common factors. Then we have $p^2 = 3q^2$, which means p^2 is divisible by 3. This implies that p is divisible by 3, so $p = 3k$ for some $k \in \mathbb{Z}$. Substituting this back into the equation, we have $9k^2 = 3q^2$, which simplifies to $3k^2 = q^2$. This means q is also divisible by 3, which contradicts our assumption that p and q have no common factors. Therefore, $\sqrt{3}$ is irrational.

The same argument does show that $\sqrt{6}$ is irrational. □

Solution to (ii). The fact that breaks the proof for Theorem 1.1.1 is that $\sqrt{4}$ is a perfect square, meaning that $p^2 = 4q^2$ doesn't imply that p is a multiple of 4. In fact, $p = 2q$ implies that p is a multiple of 2, which isn't a contradiction. □

Exercise 1.2.5 Let A and B be subsets of $\mathbb{R}$.

(i) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(ii) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(iii) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution to (i). Suppose $x \in (A \cap B)^c$. Then $x \notin (A \cap B)$, meaning $x \notin A$ or $x \notin B$, which is equivalent to $x \in A^c \cup B^c$. Therefore, $(A \cap B)^c \subseteq A^c \cup B^c$. □

Solution to (ii). Suppose $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, which is equivalent to $x \notin A$ or $x \notin B$. This means $x \notin A \cap B$, so $x \in (A \cap B)^c$. Then, $A^c \cup B^c \subseteq (A \cap B)^c$.

Therefore, since $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$, we have $(A \cap B)^c = A^c \cup B^c$. □

Solution to (iii). Suppose $x \in (A \cup B)^c$. Then $x \notin (A \cup B)$, meaning $x \notin A$ and $x \notin B$. This is equivalent to $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$. Then, $(A \cup B)^c \subseteq A^c \cap B^c$.

Now suppose $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, which is equivalent to $x \notin A$ and $x \notin B$. This means $x \notin A \cup B$, so $x \in (A \cup B)^c$. Then, $A^c \cap B^c \subseteq (A \cup B)^c$.

Therefore, since $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$, we have $(A \cup B)^c = A^c \cap B^c$. □

Exercise 1.2.6

(i) Verify the triangle inequality in the special case where a and b have the same sign.

(ii) Find an efficient proof for all the cases at once by first demonstrating $(a + b)^2 \leq (|a| + |b|)^2$.

(iii) Prove $|a - b| \leq |a - c| + |c - d| + |d - b|$, for all a, b, c, and d.

(iv) Prove $||a| - |b|| \leq |a - b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution to (i). Let $a, b \in \mathbb{R}$ such that $a, b > 0$.

(i) Using the triangle inequality, we have $|a + b| \leq |a| + |b|$, but since $a, b > 0$, we have $|a| = a$ and $|b| = b$. Therefore, we have $|a + b| = a + b = |a| + |b|$.

(ii) Both a and b will be positive but I'll add a negative to make things easier. Using the triangle inequality, we have $|(-a)+(-b)| \leq |-a|+|-b|$. Factoring out the negative, we have $|-(a+b)| \leq |-a|+|-b|$. Applying the definition of absolute value, we have $|-(a+b)| = a+b$, $|-a| = a$, and $|-b| = b$. Therefore, we have $|(-a)+(-b)| = a+b = |-a|+|-b|$. □

Solution to (ii). Simplifying the expression $(a+b)^2 \leq (|a|+|b|)^2$ gives us $2ab \leq 2|a|\cdot|b|$, which is always true, as the left side can be negative, but the right side will always be positive. As we've just squared both sides which keeps the inequality, the original inequality $a+b \leq |a|+|b|$ is true. □

Solution to (iii). Notice that $(a-c)+(c-d)+(d-b) = a-b$. We can use the triangle inequality for multiple terms $|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$. □

Solution to (iv). Assume $|a| > |b|$ since $||a|-|b||$. Then $||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$. □

Exercise 1.2.7 Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) \mid x \in A\}$.

(i) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(ii) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(iii) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(iv) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Solution to (i). The value of $f(A) = f([0,2]) = [0,4]$ and the value of $f(B) = f([1,4]) = [1,16]$. We have $f(A \cap B) = f([0,2] \cap [1,4]) = f([1,2]) = [1,4]$ and $f(A) \cap f(B) = f([0,2]) \cap f([1,4]) = [0,4] \cap [1,16] = [1,4]$. Therefore, $f(A \cap B) = f(A) \cap f(B)$.

We have $f(A \cup B) = f([0,2] \cup [1,4]) = f([0,4]) = [0,16]$ and $f(A) \cup f(B) = [0,4] \cup [1,16] = [0,16]$. Therefore, $f(A \cup B) = f(A) \cup f(B)$. □

Solution to (ii). Let $A = [-1]$ and $B = [1]$. The value of $f(A) = f([-1]) = [1]$ and $f(B) = [1] = [1]$. We have $f(A \cap B) = f([-1] \cap [1]) = f(\emptyset)$ and $f(A) \cap f(B) = f([-1]) \cap f([1]) = [1] \cap [1] = [1]$. □

Solution to (iii). Suppose $g(x) \in g(A \cap B)$. Then, $x \in A \cap B$, meaning $x \in A$ and $x \in B$.This implies that $g(x) \in g(A)$ and $g(x) \in g(B)$. Then, $g(x) \in g(A) \cap g(B)$. Therefore, $g(A \cap B) \subseteq g(A) \cap g(B)$. □

Solution to (iv). Given an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cup B) = g(A) \cup g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Suppose $g(x) \in g(A \cup B)$. Then, $x \in A \cup B$, meaning $x \in A$ or $x \in B$. This implies that $g(x) \in g(A)$ or $g(x) \in g(B)$. Then, $g(x) \in g(A) \cup g(B)$. Therefore, $g(A \cup B) \subseteq g(A) \cup g(B)$.

Suppose $g(x) \in g(A) \cup g(B)$. Then, $g(x) \in g(A)$ or $g(x) \in g(B)$, meaning $x \in A$ or $x \in B$. This implies that $x \in A \cup B$. Then, $g(x) \in g(A \cup B)$. Therefore, $g(A) \cup g(B) \subseteq g(A \cup B)$.

Since $g(A \cup B) \subseteq g(A) \cup g(B)$ and $g(A) \cup g(B) \subseteq g(A \cup B)$, we have $g(A \cup B) = g(A) \cup g(B)$. □

Exercise 1.2.11 Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that ..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(i) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $\frac{a+1}{n} < b$.

(ii) There exists a real number $x > 0$ such that $x < \frac{1}{n}$ for all $n \in \mathbb{N}$.

(iii) There exists two real numbers $a < b$ such that if $c < b$ then $c < a$ for all $c \in \mathbb{Q}$.

Solution to (i). For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $\frac{a+1}{n} \geq b$. *Intuition:* False. □

Solution to (ii). There exists a real number $x > 0$ such that $x \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. *Intuition:* True. □

Solution to (iii). There exists two real numbers $a < b$ such that if $c < b$ then $c < a$ for all $c \in \mathbb{Q}$. *Intuition:* False. □

Exercise 1.2.12 Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{2y_n - 6}{3}$.

(i) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

(ii) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

Solution to (i). I'll use mathematical induction.
Base case: Setting $n = 1$, we get $y_1 = 6 > -6$ as required.
Induction step: Suppose that $y_n > -6$ up to some number k. Then, by induction, we get

$$y_{k+1} > -6 \Rightarrow \frac{2y_k - 6}{3} > -6 \Rightarrow 2y_k - 6 > -18 \Rightarrow 2y_k > -12 \Rightarrow y_k > -6.$$

Therefore, by induction, $y_n > -6$ for all n. □

Solution to (ii). I'll use mathematical induction.
Base case: Setting $n = 1$, we get $y_1 = 6$. Setting $n = 2$, we get $y_2 = 2$, giving us $6 \geq 2$ as required.
Induction step: Suppose that $y_n \geq y_{n+1}$ up to some number k. Then, by induction, we get

$$\begin{aligned} y_k \geq y_{k+1} &\Rightarrow 2y_k \geq 2y_{k+1} \\ &\Rightarrow 2y_k - 6 \geq 2y_{k+1} - 6 \\ &\Rightarrow \frac{2y_k - 6}{3} \geq \frac{2y_{k+1} - 6}{3} \\ &\Rightarrow y_{k+1} \geq y_{k+2}. \end{aligned}$$

Therefore, by induction, $y_n \geq y_{n+1}$, for all n. □

Exercise 1.2.13

(i) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c,$$

for any finite $n \in \mathbb{N}$.

(ii) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statements holds for every real value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \dots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(iii) Nevertheless, the infinite version of De Morgan's Law stated in (ii) is a valid statement. Provide a proof that does not use induction.

Solution to (i). I'll use mathematical induction.

Base case: Exercise 1.2.5 will be our base case.

Induction step: Assume that the statement $(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots A_n^c$ is true. Then

$$\begin{aligned} ((A_1 \cup A_2 \cup \dots \cup A_n) \cup A_{n+1})^c &= (A_1 \cap A_2 \cap \cdots A_n)^c \cap A_{n+1}^c \\ &= A_1^c \cap A_2^c \cap \cdots A_n^c \cap A_{n+1}^c. \end{aligned}$$

□

Solution to (ii). The collection of sets are $B_1 = \{1, 2, \dots\}, B_2 = \{2, 3, \dots\}, \dots$. If you take their intersection until n, you will always get a number, i.e., $n = 100$, you get the singleton set $\{100\}$. But the intersection of all the sets as $n \to \infty$, you get $\emptyset$. □

Solution to (iii). Suppose $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$. Then, $x \notin \bigcup_{i=1}^{\infty} A_i$, meaning, $x \notin A_i$ for all $i \in \mathbb{N}$. This implies that $x \in A_i^c$ for all $i \in \mathbb{N}$, so $x \in \bigcap_{i=1}^{\infty} A_i^c$. Therefore, $\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$.

Suppose $x \in \bigcap_{i=1}^{\infty} A_i^c$, meaning $i \in \mathbb{N}$, $x \notin A_i$, for some $i \in \mathbb{N}$. This implies that $x \in A_i^c$, for some $i \in \mathbb{N}$. Therefore, $x \notin \bigcup_{i=1}^{\infty} A_i$, meaning $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$. Therefore, $\bigcap_{i=1}^{\infty} A_i^c \subseteq \left(\bigcup_{i=1}^{\infty} A_i\right)^c$.

Since $\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$ and $\bigcap_{i=1}^{\infty} A_i^c \subseteq \left(\bigcup_{i=1}^{\infty} A_i\right)^c$, we have $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$. □

SECTION 1.3

Exercise 1.3.3

(i) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} \mid b \text{ is a lower bound for } A\}$. Show that $\sup(B) = \inf(A)$.

(ii) Use (i) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution to (i). By definition, $\sup(B)$ is the greatest lower bound for A, meaning that it equals $\inf(A)$. □

Solution to (i). Part (i) proves that the greatest lower bound exists using the least upper bound. □

Exercise 1.3.8 Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(i) $\left\{\frac{m}{n} \mid m, n \in \mathbb{N} \text{ with } m < n\right\}$.

(ii) $\left\{\frac{(-1)^m}{n} \mid m, n \in \mathbb{N}\right\}$.

(iii) $\left\{\frac{n}{3n+1} \mid n \in \mathbb{N}\right\}$.

(iv) $\left\{\frac{m}{m+n} \mid m, n \in \mathbb{N}\right\}$.

Solution to (i). $\sup(S) = 1$ and $\inf(S) = 0$. □

Solution to (ii). $\sup(S) = 1$ and $\inf(S) = -1$. □

Solution to (iii). $\sup(S) = \frac{1}{3}$ and $\inf(S) = \frac{1}{4}$. □

Solution to (iv). $\sup(S) = 1$ and $\inf(S) = 0$. □