

	#	Points
Correct Exer.	9	42
Complete Exer.	9	50

9P

## Contour Integration; Homework 1

Due on April 9, 2025 at 23:59

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**Exercise 1.2.11.** Solve the equation  $z^2 + z + 1 = 0$  for  $z = (x, y)$  by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0),$$

and then solving a pair of simultaneous equations in  $x$  and  $y$ .

*Suggestion:* Use the fact that no real number  $x$  satisfies the given equation to show that  $y \neq 0$ .

Ans.  $z = \left( -\frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right)$ .

**Solution.** Expanding the left-hand side and simplifying, we get

$$\begin{aligned} z^2 + z + 1 = 0 &\Rightarrow (x^2 - y^2, 2xy) + (x, y) + (1, 0) = (0, 0) \\ &\Rightarrow (x^2 - y^2 + x + 1, 2xy + y) = (0, 0). \end{aligned}$$

Therefore, we have the system of equations

$$\begin{aligned} x^2 - y^2 + x + 1 &= 0 \\ 2xy + y &= 0. \end{aligned}$$

From the second equation, we can factor out  $y$  to get  $y(2x + 1) = 0$ . This gives us two cases to consider: either  $y = 0$  or  $2x + 1 = 0$ .

If  $y = 0$ , then substituting it back into the first equation gives us

$$x^2 + x + 1 = 0,$$

which doesn't have any real solutions, as the discriminant is negative. Therefore, we must have  $y \neq 0$ .

In the case where  $2x + 1 = 0$ , we can solve for  $x$  to get  $x = -1/2$ . Substituting this value into the first equation gives us

$$\begin{aligned} 0 &= \left( -\frac{1}{2} \right)^2 - y^2 - \frac{1}{2} + 1 \\ 0 &= \frac{1}{4} - y^2 - \frac{1}{2} + 1 \\ 0 &= -y^2 + \frac{3}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm \frac{\sqrt{3}}{2}. \end{aligned}$$

Thus, the solutions to the equation  $z^2 + z + 1 = 0$  are

$$z = \left( -\frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

□

**Exercise 1.3.1.** Reduce each of these quantities to a real number

$$(i) \frac{1+2i}{3-4i} + \frac{2-i}{5i}; \quad (ii) \frac{5i}{(1-i)(2-i)(3-i)}; \quad (iii) (1-i)^4.$$

Ans. (i)  $-2/5$ , (ii)  $-1/2$ , (iii)  $-4$ .

**Solution to (i).** Multiplying the numerator and denominator of the first term by the conjugate of the denominator, we have

$$\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{3+4i+6i-8}{9+16} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i.$$

Now, we can simplify the second term

$$\frac{2-i}{5i} = \frac{(2-i)(-i)}{5i(-i)} = \frac{-2i-1}{5} = -\frac{1}{5} - \frac{2}{5}i.$$

Adding these two results together, we have

$$\frac{1+2i}{3-4i} + \frac{2-i}{5i} = \left(-\frac{1}{5} + \frac{2}{5}i\right) + \left(-\frac{1}{5} - \frac{2}{5}i\right) = -\frac{2}{5}. \quad \square$$

**Solution to (ii).** Expanding the denominator, we have

$$\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(1-3i)(3-i)} = -\frac{5i}{10i} = -\frac{1}{2}. \quad \square$$

**Solution to (iii).** Expanding the polynomial  $(1-i)^2$ , we have

$$(1-i)^2 = (1-i)(1-i) = 1-2i+i^2 = 1-2i-1 = -2i.$$

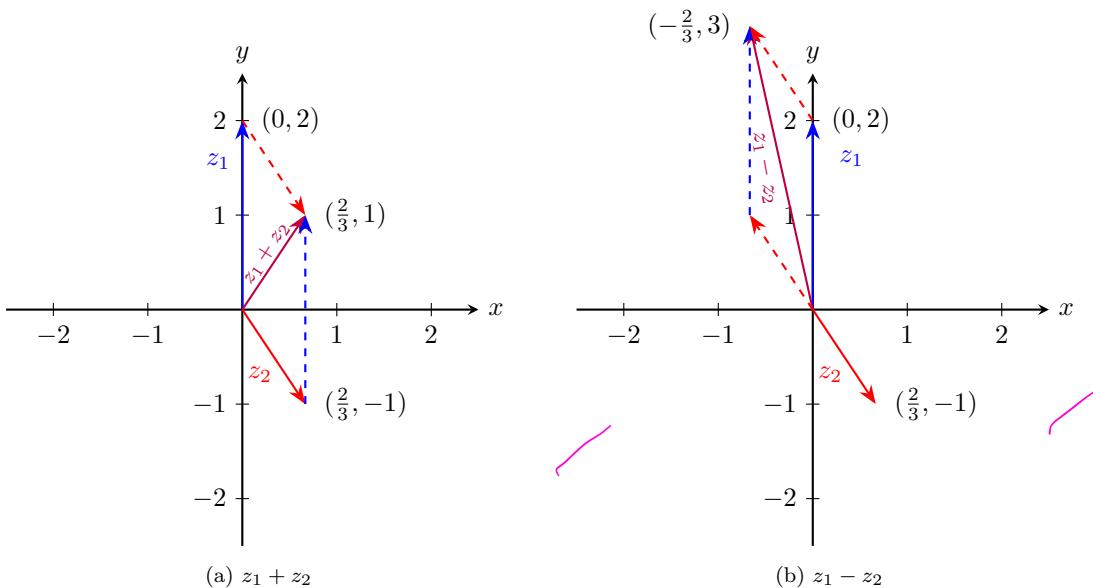
Now, we can expand  $(1-i)^4$  as follows

$$(1-i)^4 = (1-i)^2(1-i)^2 = (-2i)(-2i) = 4i^2 = -4. \quad \square$$

**Exercise 1.4.1.** Locate the numbers  $z_1 + z_2$  and  $z_1 - z_2$  vertically when

- (i)  $z_1 = 2i$ ,  $z_2 = \frac{2}{3} - i$ ; (ii)  $z_1 = (-\sqrt{3}, 1)$ ,  $z_2 = (\sqrt{3}, 0)$ ;
- (iii)  $z_1 = (-3, 1)$ ,  $z_2 = (1, 4)$ ; (iv)  $z_1 = x_1 + y_1i$ ,  $z_2 = x_1 - y_1i$ .

**Solution to (i).** Graphing the complex numbers  $z_1 = 2i$  and  $z_2 = \frac{2}{3} - i$  on the complex plane, we have



In this case, we can compute

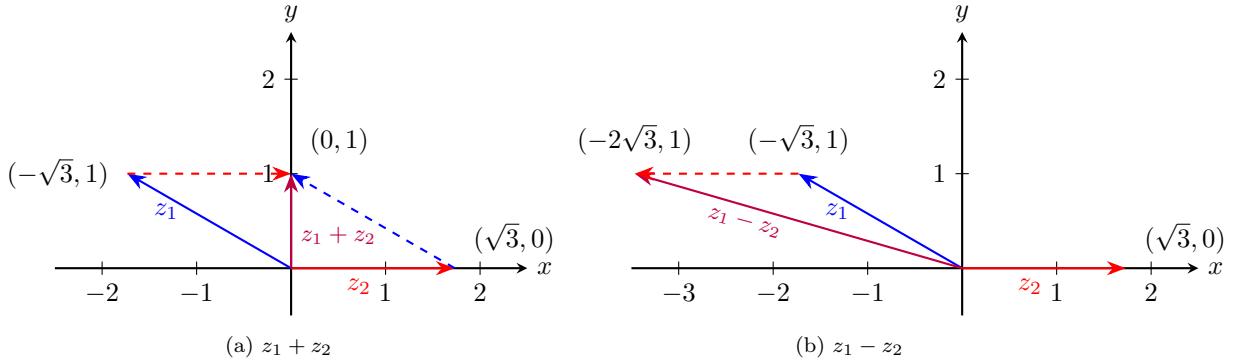
$$z_1 + z_2 = (0 + 2i) + \left(\frac{2}{3} - i\right) = \frac{2}{3} + i.$$

This gives us the point  $(2/3, 1)$ . Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (0 + 2i) + \left(-\frac{2}{3} + i\right) = -\frac{2}{3} + 3i.$$

This gives us the point  $(-2/3, 3)$ .  $\square$

**Solution to (ii).** Graphing the complex numbers  $z_1 = (-\sqrt{3}, 1)$  and  $z_2 = (\sqrt{3}, 0)$  on the complex plane, we have



In this case, we can compute

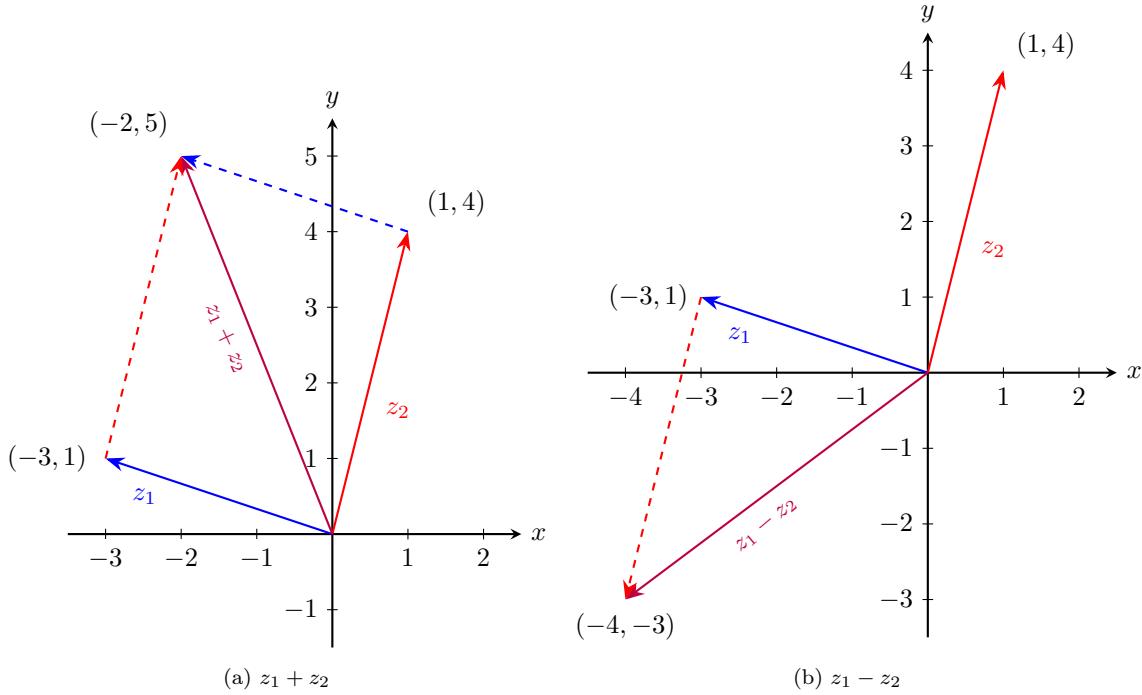
$$z_1 + z_2 = (-\sqrt{3}, 1) + (\sqrt{3}, 0) = (0, 1).$$

This gives us the point  $(0, 1)$ . Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (-\sqrt{3}, 1) + (-\sqrt{3}, 0) = (-2\sqrt{3}, 1).$$

This gives us the point  $(-2\sqrt{3}, 1)$ .  $\square$

**Solution to (iii).** Graphing the complex numbers  $z_1 = (-3, 1)$  and  $z_2 = (1, 4)$  on the complex plane, we have



In this case, we can compute

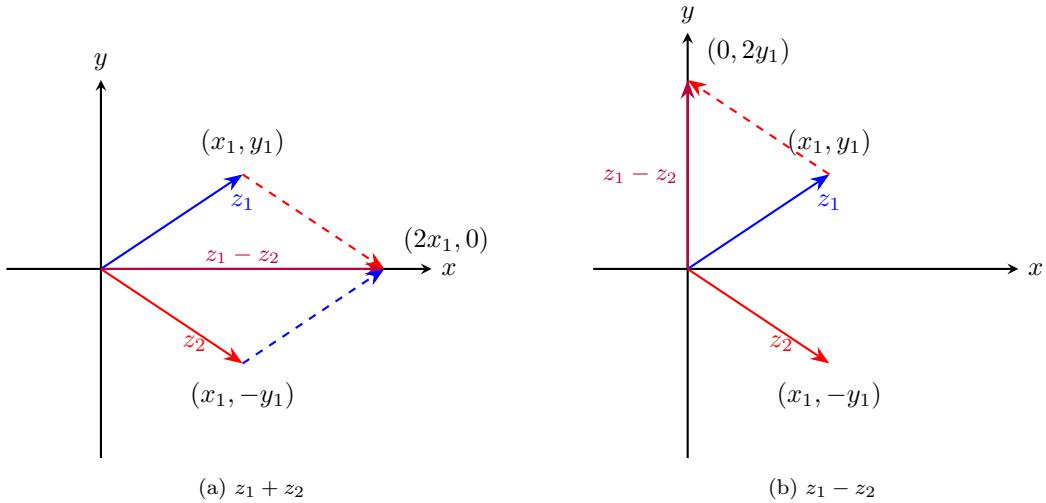
$$z_1 + z_2 = (-3, 1) + (1, 4) = (-2, 5).$$

This gives us the point  $(-2, 5)$ . Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (-3, 1) + (-1, -4) = (-4, -3).$$

This gives us the point  $(-4, -3)$ . □

**Solution to (iv).** Graphing the complex numbers  $z_1 = x_1 + y_1i$  and  $z_2 = x_1 - y_1i$  on the complex plane, we have



In this case, we can compute

$$z_1 + z_2 = (x_1 + y_1i) + (x_1 - y_1i) = (2x_1 + 0i).$$

This gives us the point  $(2x_1, 0)$ . Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (x_1 + y_1 i) + (x_1 - y_1 i) = (0, 2y_1).$$

This gives us the point  $(0, 2y_1)$ . □

**Exercise 1.4.4.** Verify that  $\sqrt{2}|z| \geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ .

*Suggestion:* Reduce this inequality to  $(|x| - |y|)^2 \geq 0$ .

**Solution.** We know that  $\operatorname{Re}(z) = |x|$  and  $\operatorname{Im}(z) = |y|$ . We also know that  $|z| = \sqrt{x^2 + y^2}$ . Therefore, we can rewrite the inequality as

$$\begin{aligned} 2|z| &\geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \\ \Rightarrow 2\sqrt{x^2 + y^2} &\geq |x| + |y| \\ \Rightarrow 2(x^2 + y^2) &\geq (|x| + |y|)^2 \\ \Rightarrow 2(x^2 + y^2) &\geq x^2 + 2|x||y| + y^2 \\ \Rightarrow 2x^2 + 2y^2 - x^2 - 2|x||y| - y^2 &\geq 0 \\ \Rightarrow (x^2 + y^2) - 2|x||y| &\geq 0 \\ \Rightarrow (x - |y|)(x + |y|) &\geq 0. \end{aligned}$$

Since  $x^2 + y^2 = |x|^2 + |y|^2$ , we can re-write the inequality as

$$\begin{aligned} (x - |y|)(x + |y|) &\geq 0 \\ \Rightarrow (|x| - |y|)^2 &\geq 0. \end{aligned}$$

Therefore, we have  $(|x| - |y|)^2 \geq 0$ , which is always true. This means that  $\sqrt{2}|z| \geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$  is true for all complex numbers  $z$ . □

**Exercise 1.4.6.** Using the fact that  $|z_1 - z_2|$  is the distance between two points  $z_1$  and  $z_2$ , give a geometric argument that

(i)  $|z - 4i| + |z + 4i| = 10$  represents an ellipse whose foci are  $(0, \pm 4)$ .

(ii)  $|z - 1| = |z + i|$  represents the line through the origin whose slope is  $-1$ .

**Solution to (i).** The modulus  $|z - 4i|$  represents the Euclidean distance between the complex number  $z$  and the point  $4i$ . Similarly,  $|z + 4i|$  represents the distance from  $z$  to  $-4i$ .

Since the sum of the distances from any point  $z$  on the curve to the fixed points  $(0, 4)$  and  $(0, -4)$  is a constant, this satisfies the definition of an ellipse, where the sum of distances to the foci is constant.

The foci are at  $(0, \pm 4)$ . The given sum of distances is 10, which corresponds to  $2a$  in the standard form of an ellipse equation. The foci are at a distance  $c = 4$  from the center  $(0, 0)$ , and using the standard ellipse relation  $a^2 = b^2 + c^2$ , we get  $a = 5$ ,  $b = 3$ , and  $c = 4$ . Thus, the ellipse has semi-major axis  $a = 5$  and semi-minor axis  $b = 3$ . □

**Solution to (ii).** The expression  $|z - 1|$  represents the distance from  $z$  to 1. The expression  $|z + i|$  represents the distance from  $z$  to  $-i$ .

The equation states that any point  $z = x + yi$  is equidistant from these two fixed points. The midpoint of  $(1, 0)$  and  $(0, -1)$  is

$$\left( \frac{1+0}{2}, \frac{0+(-1)}{2} \right) = \left( \frac{1}{2}, -\frac{1}{2} \right).$$

The slope of the segment joining  $(1, 0)$  and  $(0, -1)$  is  $m = 1$ . The perpendicular bisector has a slope of  $-1$ , so its equation is

$$y + \frac{1}{2} = -1(x - \frac{1}{2}) \Rightarrow y = -x + 1.$$

Thus, the given equation represents the line through the origin with slope  $-1$ . □

**Exercise 1.5.1(iv).** Use properties of conjugates and moduli established in Sec. 5 to show that

$$(iv) \quad |(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|.$$

**Solution to (iv).** Expanding the left-hand side, we have

$$|(2\bar{z} + 5)(\sqrt{2} - i)| = |2\bar{z} + 5| \cdot |\sqrt{2} - i|.$$

Taking the norm of the complex number  $\sqrt{2} - i$ , we have

$$|\sqrt{2} - i| = \sqrt{(\sqrt{2})^2 + (-i)^2} = \sqrt{2 + 1} = \sqrt{3}.$$

Therefore, we have

$$|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3} \cdot |2\bar{z} + 5|.$$

Since  $|z| = |\bar{z}|$ , we can replace  $|2\bar{z} + 5|$  with  $|2z + 5|$  to get

$$|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|. \quad \checkmark$$

□

**Exercise 1.5.10.** Prove that

- (i)  $z$  is real if and only if  $\bar{z} = z$ .
- (ii)  $z$  is either real or pure imaginary if and only if  $z^2 = z^2$ .

**Solution to (i).** Assume  $z$  is real. This means that  $z = x + 0i$  for some real number  $x$ . The complex conjugate of  $z$  is  $\bar{z} = x - 0i = x$ . Therefore, we have  $\bar{z} = z$ .

Conversely, assume  $\bar{z} = z$ . This means that  $z = x + yi$  and  $\bar{z} = x - yi$ . By assumption, we have  $\bar{z} = z$  giving us  $x + yi = x - yi$ . This implies that  $yi = -yi$ , giving us  $y = -y$ . This only holds true if  $y = 0$ . Therefore, we have  $z = x + 0i$  for some real number  $x$ . This means that  $z$  is real.

Thus,  $z$  is real if and only if  $\bar{z} = z$ . □

**Solution to (ii).** Assume  $z$  is either real or pure imaginary. This gives us two cases, when  $z$  is real,  $z = x + 0i$  for some real number  $x$ , and when  $z$  is pure imaginary,  $z = 0 + yi$  for some real number  $y$ . Notice that the first case is already proven in part (i), i.e.,  $z = \bar{z}$ , giving us  $z^2 = \bar{z}^2$ . In the second case, we have  $z = 0 + yi$  and  $\bar{z} = 0 - yi = -yi$ . Therefore, we have  $\bar{z}^2 = (-yi)^2 = -y^2$ . On the other hand,  $z^2 = (0 + yi)^2 = -y^2$ . Therefore, we have  $\bar{z}^2 = z^2$ .

Assume  $\bar{z}^2 = z^2$ . Let  $z = x + yi$  and  $\bar{z} = x - yi$ . By assumption, we have

$$z^2 = \bar{z}^2 \Rightarrow (x + yi)^2 = (x - yi)^2 \Rightarrow x^2 - y^2 + 2xyi = x^2 - y^2 - 2xyi.$$

Comparing the real parts, we have  $x^2 - y^2 = x^2 - y^2$ , which is always true. Comparing the imaginary parts, we have  $2xy = -2xy$ . This implies that  $4xy = 0$ . This means that either  $x = 0$  or  $y = 0$ . If  $x = 0$ , then  $z$  is pure imaginary. If  $y = 0$ , then  $z$  is real. Therefore,  $z$  is either real or pure imaginary.

Thus,  $z$  is either real or pure imaginary if and only if  $\bar{z}^2 = z^2$ . □

**Exercise 1.5.14.** Using expressions (6), Sec. 5, for  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$ , show that the hyperbola  $x^2 - y^2 = 1$  can be written as

$$z^2 + \bar{z}^2 = 2.$$

**Solution.** Substituting  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$  for  $x$  and  $y$  into the equation for the hyperbola, we have

$$x^2 - y^2 = 1 \Rightarrow \operatorname{Re}(z)^2 - \operatorname{Im}(z)^2 = 1 \Rightarrow \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 = 1. \quad \checkmark$$

Expanding the left-hand side, we have

$$\left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 = \frac{(z+\bar{z})^2}{4} - \frac{(z-\bar{z})^2}{-4} = \frac{(z+\bar{z})^2 + (z-\bar{z})^2}{4}.$$

Therefore, we have  $(z+\bar{z})^2 + (z-\bar{z})^2 = 4$ . Expanding the squares, we have

$$\begin{aligned} z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2 &= 4 \\ \Rightarrow 2z^2 + 2\bar{z}^2 &= 4 \\ \Rightarrow z^2 + \bar{z}^2 &= 2. \end{aligned}$$

□

**Exercise (Extra).** Given  $a+bi$ ,  $a, b$  are real numbers, find  $c$  and  $d$  such that  $(c+di)^2 = a+bi$ .

**Solution.**

$$\begin{aligned} (c+di)^2 &= a+bi \\ (c^2 - d^2) + i(2cd) &= a+bi \end{aligned}$$

So we end up with the system of equations

$$\begin{aligned} c^2 - d^2 &= a \\ 2cd &= b. \end{aligned}$$

Solving the second one for  $c$ , and substituting this into the first one,

$$\begin{aligned} c &= \frac{b}{2d} \\ \left(\frac{b}{2d}\right)^2 - d^2 &= a \\ \frac{b^2}{4d^2} - d^2 &= a \\ \frac{b^2}{4} &= ad^2 + d^4 \\ d^4 + ad^2 - \frac{b^2}{4} &= 0. \end{aligned}$$

This is a function with quadratic form, so to solve we complete the square and factor

$$\begin{aligned} 0 &= \left(d^4 + ad^2 + \frac{a^2}{4}\right) - \frac{b^2}{4} - \frac{a^2}{4} \\ 0 &= \left(d^2 + \frac{a}{2}\right)^2 - \frac{(a^2 + b^2)}{4} \\ \frac{\sqrt{a^2 + b^2}}{2} &= d^2 + \frac{a}{2} \\ \frac{\sqrt{a^2 + b^2} - a}{2} &= d^2 \\ \pm \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}} &= d \\ &= \pm \frac{\sqrt{\sqrt{a^2 + b^2} - a}}{\sqrt{2}} \\ &= \pm \frac{\sqrt{|z|-a}}{\sqrt{2}} \cdot \text{circled } \frac{b}{|b|} \text{ careful} \end{aligned}$$

Plugging this back into the second equation to solve for  $c$

$$c = \frac{b}{2 \left( \pm \frac{\sqrt{|z|-a}}{\sqrt{2}} \right)}$$

$$= \pm \frac{b\sqrt{2}}{2\sqrt{|z|-a}} = \pm \frac{b\sqrt{2}}{2\sqrt{\sqrt{a^2+b^2}-a}}.$$

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The sign of these picked will depend on the sign of  $b$ , as the sign of  $a$  only depends on the magnitudes of  $c$  and  $d$ .  $\square$