Several-Variab Calc II: Homework 4

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Problem 1. Use cylindrical and spherical coordinates to find the volume of a sphere with radius a.

Solution. Graphing the sphere $x^2 + y^2 + z^2 = a^2$ gives us the following

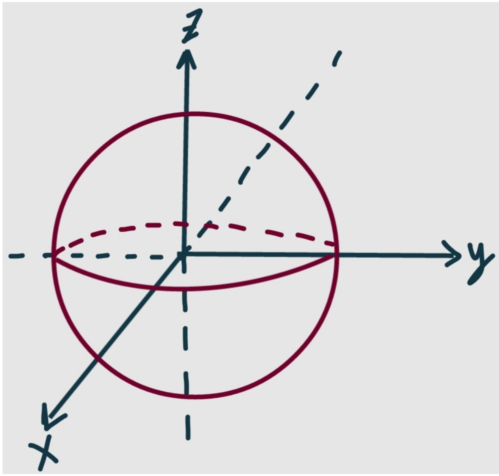


Figure 1: Sphere $x^2 + y^2 + z^2 = a^2$

Cylindrical coordinates: Converting the sphere $x^2 + y^2 + z^2 = a^2$ to cylindrical coordinates gives $r^2 + z^2 = a^2$ and solving for z gives us $z = \sqrt{a^2 - r^2}$. The volume element in cylindrical coordinates is $\mathrm{d}V = r\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$. The bounds for θ are $0 \leq \theta \leq 2\pi$, the bounds for r are $0 \leq r \leq a$, and the bounds for z are $-\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}$.

Spherical coordinates: Converting the sphere $x^2 + y^2 + z^2 = a^2$ to spherical coordinates gives $\rho^2 = a^2$ and solving for ρ gives us $\rho = a$. The volume element in spherical coordinates is $\mathrm{d}V = \rho^2 \sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$. The bounds for ρ are $0 \leq \rho \leq a$, the bounds for φ are $0 \leq \varphi \leq \pi$, and the bounds for θ are $0 \leq \theta \leq 2\pi$.

Expanding and evaluating both triple integrals gives us

$$\begin{aligned}
V = \iiint_E \mathrm{d}V &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^a rz\Big|_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^a r(\sqrt{a^2-r^2} + \sqrt{a^2-r^2})\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^a 2r(\sqrt{a^2-r^2})\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} -\frac{2}{3}(a^2-r^2)^{3/2}\Big|_0^a \mathrm{d}\theta \\
&= \int_0^{2\pi} -\frac{2}{3}(a^2-a^2)^{3/2} + \frac{2}{3}(a^2-0)^{3/2}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{2}{3}a^3\,\mathrm{d}\theta \\
&= \frac{2}{3}a^3\theta\Big|_0^{2\pi} \\
&= \frac{4}{3}\pi a^3,
\end{aligned}$$

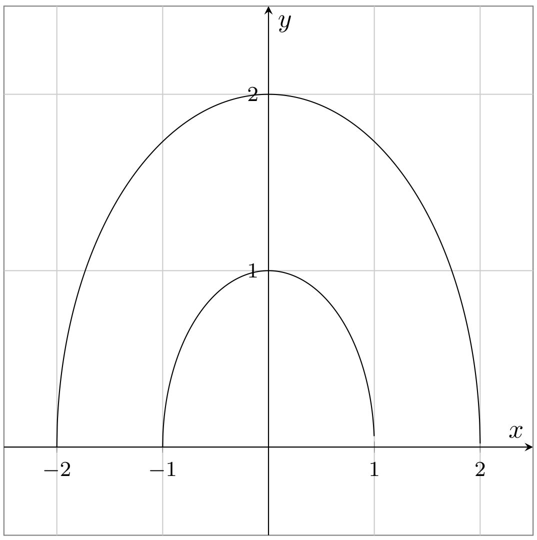
and

$$\begin{aligned}
V = \iiint_E \mathrm{d}V &= \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^{\pi} \sin(\varphi)\,\mathrm{d}\varphi \cdot \int_0^a \rho^2\,\mathrm{d}\rho \\
&= 2\pi \cdot (-\cos(\varphi))\Big|_0^{\pi} \cdot \frac{1}{3}\rho^3\Big|_0^a
\end{aligned}$$

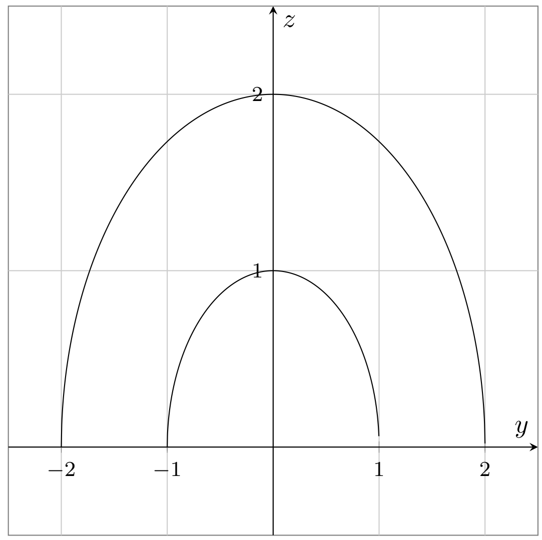
$$
\begin{aligned}
&= 2\pi \cdot (-\cos(\pi) - (-\cos(0))) \cdot \frac{1}{3}a^3 \\
&= 2\pi \cdot (1 + 1) \cdot \frac{1}{3}a^3 \\
&= \frac{4}{3}\pi a^3. \qquad \square
\end{aligned}
$$

Problem 2. Evaluate $\iiint_E ze^{(x^2+y^2+z^2)^2}\, \mathrm{d}V$ where E is the solid that lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ for $y \geq 0$ and $z \geq 0$.

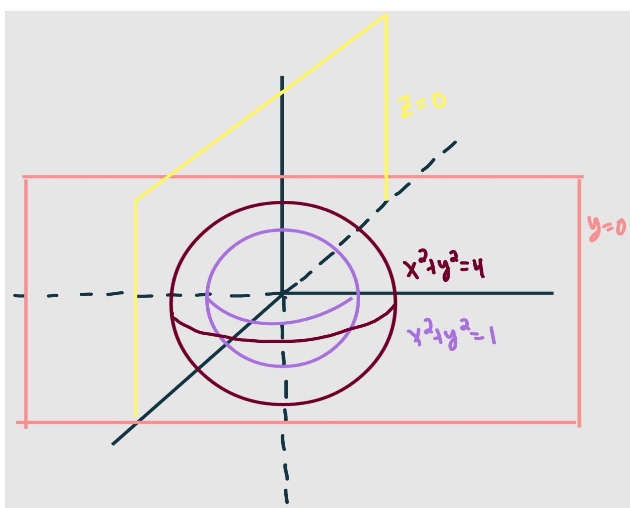
Solution. Graphing the solid E and the bounds of integration gives us



(a) Bounds of integration on the xy-plane



(b) Bounds of integration on the yz-plane



(c) Solid E

Converting the spheres to spherical coordinates gives us $\rho^2 = 1$ and $\rho^2 = 4$ and solving for ρ gives us $\rho = 1$ and $\rho = 2$. The bounds for ρ are $1 \leq \rho \leq 2$, the bounds for θ are $0 \leq \theta \leq \pi$, and the bounds for φ are $0 \leq \varphi \leq \frac{\pi}{2}$. Converting the integrand to spherical coordinates gives us $f(x, y, z) = ze^{(x^2+y^2+z^2)^2} =$

$\rho\cos(\theta)e^{\rho^4}$. Expanding and evaluating the triple integral gives us

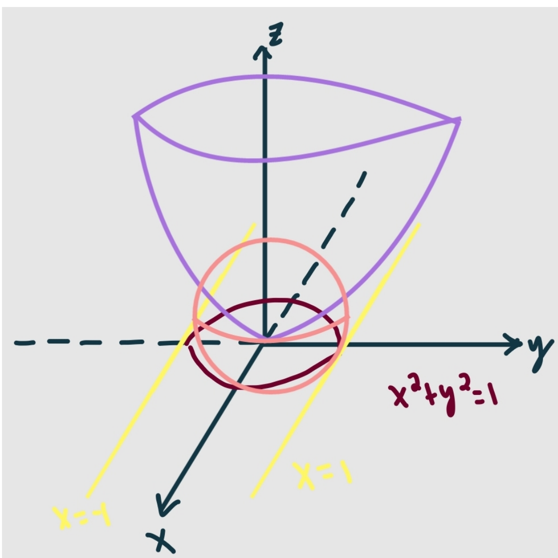
$$\begin{aligned}
\iiint_E ze^{(x^2+y^2+z^2)^2}\,\mathrm{d}V &= \int_0^{\pi}\int_0^{\pi/2}\int_1^2 \rho\cdot\cos(\varphi)\cdot e^{\pi^4}\cdot\rho^2\cdot\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \int_0^{\pi}\mathrm{d}\theta\cdot\int_0^{\pi/2}\cos(\varphi)\sin(\varphi)\,\mathrm{d}\varphi\cdot\int_1^2 e^{\rho^4}\,\mathrm{d}\rho \\
&= \pi\cdot\frac{1}{2}\sin^2(\varphi)\bigg|_0^{\pi/2}\cdot\frac{1}{4}e^{\rho^4}\bigg|_1^2 \\
&= \frac{\pi}{2}\cdot\frac{1}{4}(e^{16}-e) = \frac{\pi}{8}(e^{16}-e).
\end{aligned}$$

□

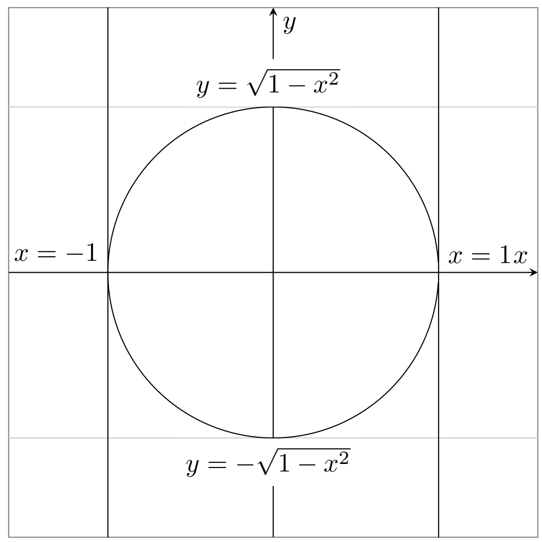
Problem 3. Evaluate

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} x^2+y^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Solution. Graphing the given bounds gives us



(a) Solid E



(b) Bounds of integration on the xy-plane

The bounds for ρ are $0 \le \rho \le 2\cos(\varphi)$, the bounds for φ are $0 \le \varphi \le \frac{\pi}{4}$, and the bounds for θ are $0 \le \theta \le 2\pi$. Converting the integrand to spherical coordinates gives us $f(x,y,z) = \rho^2\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta) = \rho^2\sin^2(\varphi)$.Expanding and evaluating the triple integral gives us

$$\begin{aligned}
V = \iiint_E f(x,y)\,\mathrm{d}V &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos(\varphi)} \rho^4\sin^3(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\mathrm{d}\theta\cdot\int_0^{\pi/4}\sin^3(\varphi)\cdot\frac{1}{5}\rho^5\bigg|_0^{2\cos(\varphi)}\,\mathrm{d}\varphi \\
&= 2\pi\cdot\int_0^{\pi/4}\frac{32}{5}\sin^3(\varphi)\cdot\cos^5(\varphi)\,\mathrm{d}\varphi \\
&= \frac{64\pi}{5}\cdot\int_0^{\pi/4}\cos^5(\varphi)(1-\cos^2(\varphi))\sin(\varphi)\,\mathrm{d}\varphi \\
&= \frac{64\pi}{5}\cdot\left(-\int_{\cos(0)}^{\cos(\pi/4)} u^5(1-u^2)\,\mathrm{d}\varphi\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{64\pi}{5} \cdot \left(- \int_1^{1/\sqrt{2}} u^5 - u^7 \, \mathrm{d}\varphi \right) \\
&= \frac{64\pi}{5} \cdot \left(- \frac{u^6}{6} + \frac{u^8}{8} \bigg|_1^{1/\sqrt{2}} \right) \\
&= \frac{64\pi}{5} \cdot \left(\left[- \frac{1/\sqrt[6]{2}}{6} + \frac{1/\sqrt[8]{2}}{8} \right] - \left[-\frac{1}{6} + \frac{1}{8} \right] \right) \\
&= \frac{64\pi}{5} \cdot \frac{11}{384} = \frac{11\pi}{30}.
\end{aligned}
$$

□