Differential Geometry: Homework 2

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34/40 = 85%

Exercise 2.2.1. Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a regular surface, and find parametrizations whose coordinate neighborhoods cover it.

Solution. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. We first show that this set is a regular surface.

Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2 + y^2 - 1$. Then $S = f^{-1}(0)$, and f is differentiable. The gradient is $\nabla f(x, y, z) = \langle 2x, 2y, 0\rangle$. For any point on S, we have $x^2 + y^2 = 1$, which implies that x and y cannot both be zero. Thus, $\nabla f \neq 0$ on all of S. Therefore, by proposition 1, S is a regular surface. +3

Now, we construct parametrizations for S whose coordinate neighborhoods cover it. We split S into two coordinate neighborhoods

$$\begin{aligned}
U_+ &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z > 0\} \\
U_- &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z < 0\} \\
U_0 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z = 0\}.
\end{aligned}$$

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Define the parametrizations

$$\begin{aligned}
\Phi_1(\theta, z) &: (-\pi, \pi) \times (0, \infty) \to U_+ \subset \mathbb{R}^3 \\
\Phi_2(\theta, z) &: (-\pi, \pi) \times (-\infty, 0) \to U_- \subset \mathbb{R}^3 \\
\Phi_3(\theta, z) &: (-\pi, \pi) \times \{0\} \to U_0 \subset \mathbb{R}^3.
\end{aligned}$$

Each Φ_i is smooth, has injective differential (since the vectors $\partial_\theta \Phi_i = \langle -\sin\theta, \cos\theta, 0\rangle$ and $\partial_z \Phi_i = \langle 0, 0, 1\rangle$ are linearly independent), and maps into S. Also check condition 2.

The union of the images of Φ_1, Φ_2, and Φ_3 covers the entire cylinder S. Therefore, we have constructed parametrizations for S whose coordinate neighborhoods cover it. □

Exercise 2.2.3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Solution. The two-sheeted cone is not a regular surface because it does not satisfy the regularity condition. The set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0\}$ is not a regular surface because the gradient of the function
+4 $f(x, y, z) = x^2 + y^2 - z^2$ is zero at the origin $(0, 0, 0)$, which is a point in S. Thus, the cone does not have a well-defined tangent plane at the vertex. □

Exercise 2.2.8. Let $\mathbf{x}(u, v)$ be as in Def. 1. Verify that $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one if and only if

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

Solution. Assume that $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one, where $d\mathbf{x}_q$ is given by

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$$d\mathbf{x}_q = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{pmatrix}.$$

This map is one-to-one if and only if the two column vectors $\partial \mathbf{x}/\partial u$ and $\partial \mathbf{x}/\partial v$ are linearly independent in $\mathbb{R}^3$. This is equivalent to their cross product (or wedge product) being non-zero

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

Therefore, $d\mathbf{x}_q$ is one-to-one if and only if $\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v \neq 0$. □

Exercise 2.2.11. Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ is a regular surface and check that parts (i) and (ii) are parametrizations for S:

(i) $\mathbf{x}(u, v) = \langle u + v, u - v, 4uv\rangle$, $(u, v) \in \mathbb{R}^2$.

(ii) $\mathbf{x}(u, v) = \langle u\cosh(v), u\sinh(v), u^2\rangle$, $(u, v) \in \mathbb{R}^2$, $u \neq 0$.

Solution to (i). We first show that the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ is a regular surface. The function $f(x, y, z) = z - x^2 + y^2$ is a differentiable function and 0 is a regular value of f. The gradient of f is given by

$$\nabla f(x, y, z) = \langle -2x, 2y, 1\rangle,$$

+2 which is non-zero for all points in S except for the points $(0, 0, z)$, where $z \in \mathbb{R}$. Thus, the set S is a regular surface.

Now, we check that the parametrization $\mathbf{x}(u, v) = \langle u + v, u - v, 4uv\rangle$ is a parametrization for S. Clearly, $\mathbf{x}(u, v)$ is a smooth function from $\mathbb{R}^2 \to \mathbb{R}^3$. Now, we check that the image of $\mathbf{x}$ lies in S. Let $x = u + v$, $y = u - v$, and $z = 4uv$. Then, we have

$$x^2 - y^2 = (u + v)^2 - (u - v)^2 = 4uv = z \in S.$$

Therefore, the image of $\mathbf{x}$ lies in S. Next, we compute the partial derivatives

$$\frac{\partial \mathbf{x}}{\partial u} = \langle 1, 1, 4v\rangle \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v} = \langle 1, -1, 4u\rangle.$$

To verify regularity, we check that the partial derivatives $\partial\mathbf{x}/\partial u$ and $\partial\mathbf{x}/\partial v$ are linearly independent. Since
+6 their cross product is given by

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} = \langle -8v, 8u, -2\rangle,$$

which is non-zero for all $(u, v) \in \mathbb{R}^2$ (except for $u = 0$ and $v = 0$, and since $u = 0$ and $v = 0$ are not in the domain of $\mathbf{x}$), we conclude that $\partial\mathbf{x}/\partial u$ and $\partial\mathbf{x}/\partial v$ are linearly independent. Thus, $\mathbf{x}$ is a regular parametrization whose image lies in S, so it parametrizes (part of) the surface S. □

Solution to (ii). Again, clearly, $\mathbf{x}(u, v) = \langle u\cosh(v), u\sinh(v), u^2\rangle$ is a smooth function from $\mathbb{R}^2 \to \mathbb{R}^3$. Now, we check that the image of $\mathbf{x}$ lies in S. Let $x = u\cosh(v)$, $y = u\sinh(v)$, and $z = u^2$. Then, we have

$$x^2 - y^2 = u^2\cosh^2(v) - u^2\sinh^2(v) = u^2(\cosh^2(v) - \sinh^2(v)) = u^2 = z \in S.$$

Therefore, the image of $\mathbf{x}$ lies in S. Next, we compute the partial derivatives

$$\frac{\partial \mathbf{x}}{\partial u} = \langle \cosh(v), \sinh(v), 2u\rangle \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v} = \langle u\sinh(v), u\cosh(v), 0\rangle.$$

Again, we take the cross product

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} = \langle 2u\cosh(v), 2u\sinh(v), u^2\rangle.$$

Since $u \neq 0$, the cross product is non-zero for all $(u, v) \in \mathbb{R}^2$, meaning that $\partial\mathbf{x}/\partial u$ and $\partial\mathbf{x}/\partial v$ are linearly independent. Thus, $\mathbf{x}$ is a regular parametrization whose image lies in S, so it parametrizes (part of) the surface S. □

Exercise 2.2.12. Show that $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{x}(u, v) = \langle a\sin(u)\cos(v), b\sin(u)\sin(v), c\cos(u)\rangle, \quad a, b, c \neq 0,$$

where $0 < u < \pi$, $0 < v < 2\pi$, is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u = \text{const}$ on the ellipsoid.

Solution. Clearly, $\mathbf{x}(u,v) = \langle a\sin(u)\cos(v), b\sin(u)\sin(v), c\cos(u)\rangle$ is a smooth function from $U \subset \mathbb{R}^2$ to $\mathbb{R}^3$. Now, we check that the image of $\mathbf{x}$ lies in the ellipsoid. Let $x = a\sin(u)\cos(v)$, $y = b\sin(u)\sin(v)$, and $z = c\cos(u)$. Then, we have

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \frac{a^2\sin^2(u)\cos^2(v)}{a^2} + \frac{b^2\sin^2(u)\sin^2(v)}{b^2} + \frac{c^2\cos^2(u)}{c^2} \\
&= \sin^2(u)\cos^2(v) + \sin^2(u)\sin^2(v) + \cos^2(u) \\
&= \sin^2(u)(\cos^2(v) + \sin^2(v)) + \cos^2(u) \\
&= \sin^2(u) + \cos^2(u) \\
&= 1.
\end{aligned}$$

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Therefore, the image of $\mathbf{x}$ lies on the ellipsoid.

Lastly, we check regularity. The partial derivatives are given by

$$\frac{\partial \mathbf{x}}{\partial u} = \langle a\cos(u)\cos(v), b\cos(u)\sin(v), -c\sin(u)\rangle \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v} = \langle -a\sin(u)\sin(v), b\sin(u)\cos(v), 0\rangle.$$

The cross product is given by

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} = \langle -bc\sin(u)\cos(v), ac\sin(u)\sin(v), ab\sin^2(u)\rangle,$$

which is non-zero for all $(u,v) \in U$ (except for $u = 0$ and $u = \pi$, and since $0 < u < \pi$, we have $u \neq 0$ and $u \neq \pi$). Thus, $\partial\mathbf{x}/\partial u$ and $\partial\mathbf{x}/\partial v$ are linearly independent, so $\mathbf{x}$ is a regular parametrization whose image lies on the ellipsoid, so it parametrizes (part of) the surface of the ellipsoid.

Geometrically, the curves $u = \text{const}$ on the ellipsoid are circles of latitude, which are the circles obtained by fixing the angle u and varying the angle v. These circles lie in planes parallel to the xy-plane and are centered at the z-axis. The radius of these circles depends on the value of u, with the largest circle corresponding to $u = \pi/2$ (the equator) and the smallest circle corresponding to $u = 0$ or $u = \pi$ (the poles). □ +2

Exercise 2.2.16. One way to define a system of coordinates for the sphere S^2, given by $x^2+y^2+(z-1)^2 = 1$, is to consider the so-called *stenographic projection* $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ which carries a point $p = (x,y,z)$ of the sphere S^2 minus the north pole $N = (0,0,2)$ onto the intersection of the xy-plane with the straight line which connects N to p. Let $(u,v) = \pi(x,y,z)$, where $(x,y,z) \in S^2 \setminus \{N\}$ and $(u,v) \in xy$-plane.

(i) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$\pi^{-1} = \begin{cases} x = \dfrac{4u}{u^2+v^2+4} \\[2ex] y = \dfrac{4v}{u^2+v^2+4} \\[2ex] z = \dfrac{2(u^2+v^2)}{u^2+v^2+4}. \end{cases}$$

(ii) Show that it is possible, using stereographic projection, to cover the sphere with two coordinate neighborhoods.

Solution to (i). To derive the formula for $\pi^{-1} : \mathbb{R}^2 \to S^2$, consider the line from the north pole $N = (0,0,2)$ to a point $(u,v,0)$ on the xy-plane. A point (x,y,z) on this line can be written as

$$(x,y,z) = (tu, tv, 2-2t),$$

for some parameter $t \in (0,1)$. We now find t such that this point lies on the sphere S^2, defined by

$$x^2 + y^2 + (z-1)^2 = 1.$$

Substituting the parameterized coordinates

$$\begin{aligned}(tu)^2 + (tv)^2 + (2-2t-1)^2 &= 1\\ t^2(u^2+v^2) + (1-2t)^2 &= 1\\ t^2(u^2+v^2+4) - 4t + 1 &= 1\\ t^2(u^2+v^2+4) - 4t &= 0\\ t(t(u^2+v^2+4)-4) &= 0.\end{aligned}$$

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Discarding the solution $t = 0$ (which gives the north pole), we get

$$t = \frac{4}{u^2+v^2+4}.$$

Now substitute back into the parameterized line

$$x = tu = \frac{4u}{u^2+v^2+4}, \quad y = tv = \frac{4v}{u^2+v^2+4}, \quad \text{and} \quad z = 2 - 2t = \frac{2(u^2+v^2)}{u^2+v^2+4}.$$

Therefore, the inverse stereographic projection is

$$\pi^{-1}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right).$$ □

Solution to (ii). The stereographic projection from the *north pole* $N = (0,0,2)$ defines a smooth bijection

$$\pi_N : S^2 \setminus \{N\} \to \mathbb{R}^2,$$

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by projecting each point on the sphere (except N) onto the xy-plane along the line connecting that point to N. This provides a coordinate chart covering all of S^2 except the north pole.

Similarly, we can define a second stereographic projection from the *south pole* $S = (0,0,0)$

$$\pi_S : S^2 \setminus \{S\} \to \mathbb{R}^2,$$

which maps all of the sphere except the south pole onto the xy-plane. The union of the domains of these two maps is

$$(S^2 \setminus \{N\}) \cup (S^2 \setminus \{S\}) = S^2,$$

so together, π_N and π_S form two coordinate neighborhoods whose union covers the entire sphere.

Therefore, it is possible to cover S^2 with two coordinate neighborhoods using stereographic projection. □