Abstract Linear Algebra: Homework 5

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Problem 1. Verify that the determinant of $\begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix}$ is $\prod_{1 \leq i < j \leq 3} (t_j - t_i)$.

Solution. Using the Cofactor Expansion on the first row, we have

$$\begin{aligned}
\det(A) = \sum_{i=1}^{3} (-1)^{1+i} \cdot \det(A_{1i}) &= \det(A_{11}) - \det(A_{12}) + \det(A_{13}) \\
&= \begin{vmatrix} t_2 & t_3 \\ t_2^2 & t_3^2 \end{vmatrix} - \begin{vmatrix} t_1 & t_3 \\ t_1^2 & t_3^2 \end{vmatrix} + \begin{vmatrix} t_1 & t_2 \\ t_1^2 & t_2^2 \end{vmatrix} \\
&= \left(t_2 t_3^2 - t_3 t_2^2\right) - \left(t_1 t_3^2 - t_3 t_1^2\right) + \left(t_1 t_2^2 - t_2 t_1^2\right) \\
&= t_2 t_3^2 - t_3 t_2^2 - t_1 t_3^2 + t_3 t_1^2 + t_1 t_2^2 - t_2 t_1^2,
\end{aligned}$$

where $C_i = 1$ for $i = 1, 2, 3$.

For the product, we have the following possible pairs: $(1, 2)$, $(1, 3)$, and $(2, 3)$. We can expand the product as follows

$$\prod_{1 \leq i < j \leq 3} (t_j - t_i) = (t_2 - t_1)(t_3 - t_1)(t_3 - t_2) = \det(A) = t_2 t_3^2 - t_3 t_2^2 - t_1 t_3^2 + t_3 t_1^2 + t_1 t_2^2 - t_2 t_1^2.$$

Therefore,

$$\det(A) = \prod_{1 \leq i < j \leq 3} (t_j - t_i).$$

□

Problem 2. Use the method introduced in the class to find a polynomial $p(x)$ in $\mathbb{P}^3(\mathbb{R})$ such that $p(1) = 1$, $p(2) = 3$, $p(3) = -1$, and $p(4) = 2$.

Solution. Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be the polynomial we are looking for. We know that

$$p(x) = \sum_{i=1}^{n} c_i p_i(x) \quad \text{where} \quad p_i(x) = \prod_{\substack{j \neq i \\ j=1}}^{n+1} \frac{x - t_j}{t_i - t_j},$$

where $c_i = p(t_i)$ and t_i are the points we are given. In our case, we have $n = 3$, $c_1 = 1$, $c_2 = 3$, $c_3 = -1$, $c_4 = 2$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, and $t_4 = 4$.

$p_1(x)$: Possible pairs are $(1, 2)$, $(1, 3)$, and $(1, 4)$.

$$p_1(x) = \prod_{j=2}^{4} \frac{x - t_j}{t_1 - t_j} = \frac{(x - t_2)(x - t_3)(x - t_4)}{(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)} = \frac{(x - 2)(x - 3)(x - 4)}{(1 - 2)(1 - 3)(1 - 4)} = -\frac{(x - 2)(x - 3)(x - 4)}{6}.$$

$p_2(x)$: Possible pairs are $(2, 1)$, $(2, 3)$, and $(2, 4)$.

$$p_2(x) = \prod_{\substack{j \neq 2 \\ j=1}}^{4} \frac{x - t_j}{t_2 - t_j} = \frac{(x - t_1)(x - t_3)(x - t_4)}{(t_2 - t_1)(t_2 - t_3)(t_2 - t_4)} = \frac{(x - 1)(x - 3)(x - 4)}{(2 - 1)(2 - 3)(2 - 4)} = \frac{(x - 1)(x - 3)(x - 4)}{2}.$$

$p_3(x)$: Possible pairs are $(3, 1)$, $(3, 2)$, and $(3, 4)$.

$$p_3(x) = \prod_{\substack{j \neq 3 \\ j=1}}^{4} \frac{x - t_j}{t_3 - t_j} = \frac{(x - t_1)(x - t_2)(x - t_4)}{(t_3 - t_1)(t_3 - t_2)(t_3 - t_4)} = \frac{(x - 1)(x - 2)(x - 4)}{(3 - 1)(3 - 2)(3 - 4)} = -\frac{(x - 1)(x - 2)(x - 4)}{2}.$$

$p_4(x)$: Possible pairs are $(4,1)$, $(4,2)$, and $(4,3)$.

$$p_4(x) = \prod_{j=1}^{3} \frac{x - t_j}{t_4 - t_j} = \frac{(x-t_1)(x-t_2)(x-t_3)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} = \frac{(x-1)(x-2)(x-3)}{6}.$$

Therefore,

$$\begin{aligned} p(x) = \sum_{i=1}^{4} c_i p_i(x) &= 1 \cdot p_1(x) + 3 \cdot p_2(x) - 1 \cdot p_3(x) + 2 \cdot p_4(x) \\ &= -\frac{(x-2)(x-3)(x-4)}{6} + \frac{3(x-1)(x-3)(x-4)}{2} + \frac{(x-1)(x-2)(x-4)}{2} \\ &\qquad + \frac{(x-1)(x-2)(x-3)}{3}. \end{aligned}$$

Verifying that the values are correct,

$$\begin{aligned} p(1) &= -\frac{(1-2)(1-3)(1-4)}{6} + 0 + 0 + 0 = -\frac{(-1)(-2)(-3)}{6} = \frac{6}{6} = 1 \\ p(2) &= 0 + \frac{3(2-1)(2-3)(2-4)}{2} + 0 + 0 = \frac{3(1)(-1)(-2)}{2} = \frac{6}{2} = 3 \\ p(3) &= 0 + 0 + \frac{(3-1)(3-2)(3-4)}{2} + 0 = \frac{(2)(1)(-1)}{2} = \frac{-2}{2} = -1 \\ p(4) &= 0 + 0 + 0 + \frac{(4-1)(4-2)(4-3)}{3} = \frac{(3)(2)(1)}{3} = 2. \end{aligned}$$

Therefore, the polynomial we are looking for is

$$p(x) = -\frac{(x-2)(x-3)(x-4)}{6} + \frac{3(x-1)(x-3)(x-4)}{2} + \frac{(x-1)(x-2)(x-4)}{2} + \frac{(x-1)(x-2)(x-3)}{3}. \quad \square$$

Problem 3. Let f be the linear functional on $\mathbb{R}^2$ defined by $f(x_1, x_2) = 2x_1 - 3x_2$. Let T be a linear transformation defined by $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$. Let T^t be the transpose linear transformation of T on the dual space of $\mathbb{R}^2$. Find the formula for the linear functional $T^t(f) : \mathbb{R}^2 \to \mathbb{R}$.

Solution. Let $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of $\mathbb{R}^2$. Then, $[T]_B$ and $[T]_B^T$ are given by

$$[T]_B = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad [T^t]_B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The linear functional f is given by

$$[f]_B = \begin{pmatrix} 2 & -3 \end{pmatrix}.$$

In order to get $T^t(f)$, we need to multiply $[T^t]_B$ by $[f]_B$ to get

$$[T^t(f)]_B = [f]_B [T^t]_B = [f]_B [T]_B^t = \begin{pmatrix} 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \end{pmatrix}.$$

Therefore, the linear functional $T^t(f)$ is given by

$$T^t f(x_1, x_2) = 5x_1 - x_2. \quad \square$$

Problem 4. Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by $f(p) = \int_a^b p(x)\,\mathrm{d}x$. Let D be the differentiation operator on V, and $D^t : V^* \to V^*$ be the transpose linear transformation of D on the dual space V^*. Find the formula for the linear functional $D^t(f) : V \to \mathbb{R}$.

Solution. The transpose linear transformation satisfies the property $D^t f(p) = f(Dp)$. Therefore, we get

$$D^t(f) = f(D) = \int_a^b Dp(x)\,\mathrm{d}x = \int_a^b p'(x)\,\mathrm{d}x = p(b) - p(a). \qquad \square$$

Problem 5. Let $V = \mathbb{R}^{n\times n}$ and let $B \in \mathbb{R}^{n\times n}$ be a fixed matrix. Let $T : V \to V$ be the linear transformation defined by $T(A) = AB - BA$, and $f : V \to \mathbb{R}$ be the trace linear functional defined by $f(C) = \mathrm{Tr}(C)$. Let $T^t : V^* \to V^*$ be t the transpose linear transformation of T the dual space V^*. Find the formula for the linear functional $T^t(f) : V \to \mathbb{R}$.

Solution. As in problem 4, the transpose linear transformation satisfies the property $T^t f(A) = f(TA)$. Therefore, we get

$$T^t(f) = f(T) = \mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA).$$

Using the property of the trace, we know that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, so we have

$$T^t(f) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0. \qquad \square$$

Problem 6. Let $\mathbb{R}^\infty$ be a vector space of infinite sequences $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ of real numbers.

(i) Define a linear transformation $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$T(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, \alpha_1, \alpha_2, \alpha_3, \cdots).$$

Find the eigenvalue(s) and eigenvectors of T or prove that there are no eigenvalues or eigenvectors for T.

(ii) Define a linear transformation $U : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$U(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\alpha_2, \alpha_3, \alpha_4, \cdots).$$

Find the eigenvalue(s) and eigenvectors of U or prove that there are no eigenvalues or eigenvectors for U.

Solution to (i). Suppose λ and $\mathbf{v}$ are an eigenvalue and eigenvector of T, respectively, such that $T(\mathbf{v}) = \lambda\mathbf{v}$. Then, we have

$$T(\mathbf{v}) = \lambda\mathbf{v} \Leftrightarrow T(\alpha_1, \alpha_2, \alpha_3, \cdots) = \lambda(\alpha_1, \alpha_2, \alpha_3, \cdots) \Leftrightarrow (0, \alpha_1, \alpha_2, \alpha_3, \cdots) = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots).$$

This implies that $0 = \lambda\alpha_1$, $\alpha_1 = \lambda\alpha_2$, $\alpha_2 = \lambda\alpha_3$, and so on. If $\alpha_1 \neq 0$, then $\lambda = 0$ making the entire eigenvector zero. If $\alpha_1 = 0$, then $0 = \lambda\alpha_2$. Applying this recursively, we get $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = 0$. Therefore, there are no eigenvalues or eigenvectors for T. $\square$

Solution to (ii). Suppose λ and $\mathbf{v}$ are an eigenvalue and eigenvector of U, respectively, such that $U(\mathbf{v}) = \lambda\mathbf{v}$. Then, we have

$$U(\mathbf{v}) = \lambda\mathbf{v} \Leftrightarrow U(\alpha_1, \alpha_2, \alpha_3, \cdots) = \lambda(\alpha_1, \alpha_2, \alpha_3, \cdots) \Leftrightarrow (\alpha_2, \alpha_3, \alpha_4, \cdots) = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots).$$

This implies that $\alpha_2 = \lambda\alpha_1$, $\alpha_3 = \lambda\alpha_2$, $\alpha_4 = \lambda\alpha_3$, and so on. If $\alpha_1 = 0$, then $\alpha_2 = \alpha_3 = \alpha_4 = \cdots = 0$. If $\alpha_1 \neq 0$, then $\lambda = {}^{\alpha_2}/_{\alpha_1}$. Applying this recursively, we get

$$\lambda = \frac{\alpha_2}{\alpha_1} = \frac{\alpha_3}{\alpha_2} = \frac{\alpha_4}{\alpha_3} = \cdots.$$

Therefore, $0 \neq \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \cdots$. Thus, the eigenvalues of U are all non-zero real numbers and the eigenvectors are

$$\mathbf{v} = (\alpha, \alpha, \alpha, \cdots) = \alpha(1, 1, 1, \cdots), \quad \forall \alpha \in \mathbb{R} - \{0\}. \qquad \square$$

Problem 7. Let $A \in \mathbb{C}^{n \times n}$. Let $\lambda_1, \cdots, \lambda_n$ be all eigenvalues of A.

(i) Prove that the determinant of A equals to the product of all eigenvalues of A, i.e.,

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

(ii) Use Part (i) to prove that A is invertible if and only if $\lambda_i \neq 0$ for all $i = 1, \cdots, n$.

Solution to (i). The characteristic polynomial of A is defined as

$$p_A(\lambda) = \det(A - \lambda I).$$

Since the eigenvalues $\lambda_1, \cdots, \lambda_n$ are the roots of this polynomial, we know that $p_A(\lambda)$ can be written as

$$p_A(\lambda) = c_n \prod_{i=1}^{n} (\lambda - \lambda_i),$$

where c_n is the leading coefficient of the polynomial (problem 1 was just a special case of this). From the determinant properties, we know that the term with λ^n in $\det(A - \lambda I)$ comes from the product of the diagonal entries when expanding along rows/columns. The coefficient of $(-\lambda)^n$ is always $(-1)^n$, meaning that

$$p_A(\lambda) = \det(A - \lambda I) = (-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i).$$

Evaluating at $\lambda = 0$ gives

$$p_A(0) = \det(A) = (-1)^n \prod_{i=1}^{n} (-\lambda_i) = \prod_{i=1}^{n} \lambda_i.$$ □

Solution to (ii). A matrix A is invertible if and only if $\det(A) \neq 0$. From part (i), we know that

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

For this product to be nonzero, none of the eigenvalues can be zero.

Conversely, if all $\lambda_i \neq 0$, then the product is nonzero, and thus

$$\det(A) = \prod_{i=1}^{n} \lambda_i \neq 0,$$

making A invertible. □

Problem 8. Let λ_1 and λ_2 be distinct eigenvalues of a linear transformation $T : V \to V$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors associated with λ_1 and λ_2 respectively. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Solution. Our goal is to show that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ implies $c_1 = c_2 = 0$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors associated with the eigenvalues λ_1 and λ_2, we have

$$T(\mathbf{v}_1) = \lambda_1 \quad \text{and} \quad T(\mathbf{v}_2) = \lambda_2.$$

Applying the linear transformation to the linear combination, we get

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = T(0) = 0.$$

By the linearity of T, we have

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) = c_1\lambda_1 + c_2\lambda_2 = 0.$$

This gives us the following system of equations

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= 0 \\ c_1\lambda_1 + c_2\lambda_2 &= 0. \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, we can subtract λ_1 times the first equation from the second equation, expanding, and simplifying to get

$$c_2(\lambda_2 - \lambda_1)\mathbf{v}_2 = 0.$$

Again, we know that $\lambda_2 - \lambda_1 \neq 0$ and since $\mathbf{v}_2$ is an eigenvector, it also cannot be zero. Therefore, we have $c_2 = 0$. Since Substituting this back into the first equation, we get that $c_1 = 0$. Hence, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. □

Problem 9. Let T be the linear transformation on $\mathbb{R}^4$ which is represented in standard ordered basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}.$$

Under what condition on a, b, and c is T diagonalizable? Explain your answer.

Solution. Finding the determinant of the matrix gives us $\det(A) = \lambda^4$. From the characteristic polynomial, $\lambda^4 = 0$, the only eigenvalue is $\lambda = 0$ with algebraic multiplicity 4.

To be diagonalizable, T must have a basis of eigenvectors, meaning that the geometric multiplicity of $\lambda = 0$ must be 4. The geometric multiplicity is the dimension of the null space of A, which is given by

$$\dim(\text{Null}(A)) = \dim(\text{Null}(A^t)) = 4 - \text{Rank}(A).$$

Solving for $A\mathbf{v} = \mathbf{0}$, we set

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{0} \Leftrightarrow \begin{pmatrix} ax_1 = 0 \\ bx_2 = 0 \\ cx_3 = 0 \end{pmatrix}.$$

Thus, the solution exists if and only if $a = b = c = 0$. Therefore, T is diagonalizable if and only if $a = b = c = 0$. □

Problem 10. Let T be a linear transformation on an n-dimensional vector space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.

Solution. □