Math 432/532: Introduction to Topology II

HW #3

For each of these problems, we will suppose that X is a topological space and $f : X \to X$ is a map such that $f(p) \neq p$ for all $p \in X$, but $f \circ f = \mathrm{id}_X$. (In particular, $f(U) = f^{-1}(U)$ for any set U.) Define an equivalence relation on X by putting $p \sim f(p)$ for all $p \in X$ (so every equivalence class has exactly two elements). Let Y be the identification space $X/\sim$. Let $\pi : X \to Y$ be the identification map.

1. Assume that X is Hausdorff. The purpose of this problem is to prove that Y is Hausdorff, as well.

   (i) Show that, given any natural number n and any n-tuple of pairwise distinct points $p_1, \ldots, p_n \in X$, we can find pairwise disjoint open sets $U_1, \ldots, U_n$ with $p_i \in U_i$ for all i. (Note that the $n = 2$ case is the definition of Hausdorffness. For larger n, use induction. Note also that this part of the problem does not involve the map f or the space Y.)

   (ii) Let q_1 and q_2 be two distinct points in Y. Choose points $p_1, p_2 \in X$ with $\pi(p_i) = q_i$. In particular, this implies that the four points $p_1, f(p_1), p_2, f(p_2)$ are pairwise distinct. By part (i), we can find pairwise disjoint open sets U_1, U_2, V_1, and V_2 with $p_1 \in U_1$, $p_2 \in U_2$, $f(p_1) \in V_1$, and $f(p_2) \in V_2$. Show that we can make these choices in such a way that $V_1 = f(U_1)$ and $V_2 = f(U_2)$.

   (iii) Show that Y is Hausdorff.

2. Assume that X is a surface. The purpose of this problem is to prove that Y is a surface, as well. We have already established that, if X is Hausdorff, so is Y. So we just need to show that, if every point in X has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$, then the same is true of Y.

   (i) Show that the identification map π is open.

   (ii) Show that, if $U \subset X$ is an open set and $U \cap f(U) = \emptyset$, then $\pi : U \to \pi(U)$ is a homeomorphism.

   (iii) Show that, if $p \in X$, there exists an open neighborhood U of p such that U is homeomorphic to an open subset of $\mathbb{R}^2$ and $U \cap f(U) = \emptyset$.

   (iv) Show that Y is a surface.

For the remaining problems, suppose that $X = |K|$ is a combinatorial surface and $f = |\varphi|$ for some isomorphism $\varphi : K \to K$.

3. It's tempting to try to define a triangulation of Y whose simplices are in bijection with equivalence classes of simplices of K. Find an example that shows that such a simplicial complex might not exist! (If one subdivides K first, then it works, but you don't have to show that.)

4. Suppose that K is equipped with an orientation. We say that φ is **orientation preserving** if it takes positively oriented triangles to positively oriented triangles. That is, if v_1, v_2, v_3 span a triangle in K and appear in positive cyclic order, then $\varphi(v_1), \varphi(v_2), \varphi(v_3)$ also appear in positive cyclic order. Give an example that is orientation preserving, and an example that is not.

5. Assume that φ is orientation preserving. Also assume that the procedure described in Problem 3 actually works, so that we have a simplicial complex L and a homeomorphism $|L| \cong X/\sim$. Show that the orientation of K induces an orientation of L.