Math 307, Homework #5
Due Wednesday, November 6
SOLUTIONS TO SELECTED PROBLEMS

1. Show that $Q \Rightarrow S$, $R \Rightarrow T \vdash (Q \vee R) \Rightarrow (S \vee T)$.

   Proof:

1.	$Q \Rightarrow S$	hyp.
2.	$R \Rightarrow T$	hyp.
3.	$Q \vee R$	dis. hyp.
4.	$\sim(S \vee T)$	dis hyp.
5.	$\sim S \wedge \sim T$	GSP, For 4, deMorgan taut.
6.	$\sim S$	LCS, For 5.
7.	$\sim T$	RCS, For 5.
8.	$\sim Q$	MT, For 1, For 6
9.	R	DI, For 3, For 8
10.	T	MP, For 2, For 9
11.	$T \wedge \sim T$	CI, For 10, For 7
12.	$S \vee T$	II, discharge For (4)
13.	$(Q \vee R) \Rightarrow (S \vee T)$	DT, discharge For 3.

2. Show that $(Q \wedge \sim T) \Rightarrow (Y \vee \sim P)$, $Y \Rightarrow (V \vee \sim X) \vdash (P \wedge X) \Rightarrow [Q \Rightarrow (T \vee V)]$.

   Proof:

1.	$(Q \wedge \sim T) \Rightarrow (Y \vee \sim P)$	hyp.
2.	$Y \Rightarrow (V \vee \sim X)$	hyp.
3.	$P \wedge X$	dis. hyp.
4.	Q	dis hyp.
5.	$\sim(T \vee V)$	dis. hyp.
6.	$\sim T \wedge \sim V$	GSP, For 5, deMorgan taut.
7.	$\sim T$	LCS, For 6.
8.	$\sim V$	RCS, For 6.
9.	$Q \wedge \sim T$	CI, For 4, For 7
10.	$Y \vee \sim P$	MP, For 1, For 9
11.	P	LCS, For 3
12.	Y	DI, For 10, For 11
13.	$V \vee \sim X$	MP, For 2, For 12
14.	$\sim X$	DI, For 13, For 8
15.	X	RCS, For 3
16.	$X \wedge \sim X$	CI, For 15, For 14
17.	$T \vee V$	II, discharge For 5
18.	$Q \Rightarrow (T \vee V)$	DT, discharge For 4
19.	$(P \wedge X) \Rightarrow (Q \Rightarrow (T \vee V))$	DT, discharge For 3.

3. Give a line proof that $(A \subseteq X \wedge B \subseteq X) \Rightarrow (A \cup B \subseteq X)$.

   Proof:

   1. Assume $A \subseteq X$ and $B \subseteq X$.
   2. Assume $x \in A \cup B$.
   3. Then $x \in A$ or $x \in B$.
   4. If $x \in A$, then $x \in X$ since $A \subseteq X$.
   5. If $x \in B$, then $x \in X$ since $B \subseteq X$.
   6. Both cases lead to $x \in X$.
   7. We have now shown $A \cup B \subseteq X$.

4. Give a line proof that $(X \subseteq A \wedge X \subseteq B) \Rightarrow (X \subseteq A \cap B)$.

   Proof:

   1. Assume $X \subseteq A$ and $X \subseteq B$.
   2. Assume $x \in X$.
   3. Then $x \in A$, since $X \subseteq A$.
   4. Also $x \in B$, since $X \subseteq B$.
   5. So $x \in A \cap B$.
   6. We have now shown $X \subseteq A \cap B$.

For any $n \in \mathbb{Z}$, recall that $M_n = \{x \in \mathbb{Z} \,|\, x \equiv_n 0\}$; that is, M_n is the set of all multiples of n.

In each of questions 5–7, if you choose to prove the claim, give a line proof. To disprove an equality, explain which subset direction you are disproving and give a specific element which disproves it.

5. Prove or disprove: If $a, b \in \mathbb{N}$ and $a \leq b$ then $M_a \subseteq M_b$.

   The statement is false. Take $a = 2$ and $b = 3$. Then $2 \leq 3$, but $2 \in M_2$ and $2 \notin M_3$; so $M_2 \not\subseteq M_3$.

6. Prove or disprove: $M_4 \cap M_6 = M_{24}$.

   The statement is false. $12 \in M_4$ and $12 \in M_6$, so $12 \in M_4 \cap M_6$. But $12 \notin M_{24}$.

7. Prove or disprove: $M_4 \cap M_9 = M_{36}$.

   Proof:

   1. Assume $x \in M_4 \cap M_9$.
   2. Then $4|x$ and $9|x$.
   3. So $x = 4a$ and $x = 9b$ for some $a, b \in \mathbb{Z}$.
   4. Then $4a = x = 3(3b)$, so $3|4a$.
   5. Since 3 is prime, Property (P) implies $3|4$ or $3|a$.
   6. But $3 \nmid 4$, so $3|a$.
   7. So $a = 3d$ for some $d \in \mathbb{Z}$.
   8. Plugging back in, we get $9b = x = 4a = 4(3d) = 12d$.
   9. Divide both sides by 3 to get $3b = 4d$.
   10. So $3|4d$, hence $3|4$ or $3|d$.
   11. But $3 \nmid 4$, so $3|d$.
   12. So $d = 3u$ for some $u \in \mathbb{Z}$.
   13. Plugging back in to the equation from (8), $x = 12(3u) = 36u$.
   14. So $36|u$, hence $u \in M_{36}$.
   15. We have therefore shown $M_4 \cap M_9 \subseteq M_{36}$.
   16. For the other direction, assume $x \in M_{36}$.
   17. So $x = 36m$ for some $m \in \mathbb{Z}$.
   18. Then $x = 9(4m)$ and $x = 4(9m)$, so $9|x$ and $4|x$.
   19. Hence $x \in M_9 \cap M_4$.
   20. So $M_{36} \subseteq M_9 \cap M_4$.
   21. We have shown the subset in both directions, so the two sets are equal.

8. Give a line proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof:

1. Let $x \in A \cup (B \cap C)$.
2. Then $x \in A$ or $x \in B \cap C$.
3. Case 1: $x \in A$.
4. Then $x \in A \cup B$, since $A \subseteq A \cup B$.
5. But we also have $x \in A \cup C$, since $A \subseteq A \cup C$.
6. Therefore $x \in (A \cup B) \cap (A \cup C)$.
7. Case 2: $x \in B \cap C$.
8. Then $x \in B$ and $x \in C$.
9. Since $x \in B$, $x \in A \cup B$ (since $B \subseteq A \cup B$).
10. Since $x \in C$, $x \in A \cup C$ (since $C \subseteq A \cup C$).
11. So $x \in (A \cup B) \cap (A \cup C)$.
12. Both cases lead to $x \in (A \cup B) \cap (A \cup C)$.
13. So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
14. Now assume $y \in (A \cup B) \cap (A \cup C)$.
15. Then $y \in A \cup B$ and $y \in A \cup C$.
16. So $y \in A$ or $y \in B$; likewise, $y \in A$ or $y \in C$.
17. Assume $y \notin A$.
18. Then $y \in B$ and $y \in C$.
19. So $y \in B \cap C$.
20. We therefore have $y \notin A \Rightarrow y \in B \cap C$.
21. This is equivalent to $y \in A$ or $y \in B \cap C$.
22. So $y \in A \cup (B \cap C)$.
23. Hence, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$, and we have now shown equality.

For sets A and X, recall that $X - A = \{x \mid x \in X \wedge x \notin A\}$.

9. Give a line proof showing that $X - (A \cup B) = (X - A) \cap (X - B)$, for all sets A, B, and X.

Proof:

1. Let $x \in X - (A \cup B)$.
2. Then $x \in X$ and $x \notin A \cup B$.
3. So $x \notin A$ and $x \notin B$.
4. Since $x \in X$ and $x \notin A$, $x \in X - A$.
5. Since $x \in X$ and $x \notin B$, $x \in X - B$.
6. So $x \in (X - A) \cap (X - B)$.
7. Thus, $X - (A \cup B) \subseteq (X - A) \cup (X - B)$.
8. Now let $y \in (X - A) \cap (X - B)$.
9. Then $y \in X - A$ and $y \in X - B$.
10. So $y \in X$ and $y \notin A$, and $y \in X$ and $y \notin B$.
11. Then $y \notin (A \cup B)$.
12. Thus, $y \in X - (A \cup B)$.
13. We have now shown $(X - A) \cup (X - B) \subseteq X - (A \cup B)$.
14. Since we proved the subset in both directions, the two sets are equal.

10. Give a line proof showing that $X - (A \cap B) = (X - A) \cup (X - B)$, for all sets A, B, and X.

Proof:

1. Let $x \in X - (A \cap B)$.
2. Then $x \in X$ and $x \notin A \cap B$.
3. So $x \notin A$ or $x \notin B$.
4. Case 1: $x \notin A$.
5. Then $x \in X$ and $x \notin A$, so $x \in X - A$.
6. Therefore $x \in (X - A) \cup (X - B)$.
7. Case 2: $x \notin B$.
8. Then $x \in X$ and $x \notin B$, so $x \in X - B$.
9. Therefore $x \in (X - A) \cup (X - B)$.
10. Both cases lead to $x \in (X - A) \cup (X - B)$.
11. We have now shown $X - (A \cap B) \subseteq (X - A) \cup (X - B)$.
12. Now let $y \in (X - A) \cup (X - B)$.
13. Then $y \in X - A$ or $y \in X - B$.
14. Case 1: $y \in X - A$.
15. Then $y \in X$ and $y \notin A$.
16. Since $y \notin A$, we also have $y \notin A \cap B$ (since $A \cap B \subseteq A$).
17. So $y \in X - (A \cap B)$.
18. Case 2: $y \in X - B$.
19. Then $y \in X$ and $y \notin B$.
20. Since $y \notin B$, we also have $y \notin A \cap B$ (since $A \cap B \subseteq B$).
21. So $y \in X - (A \cap B)$.
22. Both cases lead to $y \in X - (A \cap B)$.
23. So $(X - A) \cup (X - B) \subseteq X - (A \cap B)$, and we have now shown the two sets are equal.

11. If f is a function from S to T and $A \subseteq S$, define $f(A) = \{x \mid (\exists y \in A)[x = f(y)]\}$. This is called the **image of A under f**.

(a) Give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ where $f(M_3 \cap \mathbb{N}) = \mathbb{N}$.

An easy example is given by

$$f(n) = \begin{cases} \frac{n}{3} & \text{if } 3|n, \\ 1 & \text{otherwise.} \end{cases}$$

(b) Suppose $f\colon S \to T$ and $A \subseteq S$, $B \subseteq S$.

(i) Prove that $f(A \cup B) = f(A) \cup f(B)$

Proof:

1. Let $x \in f(A \cup B)$.
2. Then $x = f(u)$ for some $u \in A \cup B$.
3. $u \in A$ or $u \in B$.
4. Case 1: $u \in A$.
5. Then $x \in f(A)$, so $x \in f(A) \cup f(B)$.
5. Case 2: $u \in B$.
6. Then $x \in f(B)$, so again $x \in f(A) \cup f(B)$.
7. Both cases lead to $x \in f(A) \cup f(B)$.
8. So $f(A \cup B) \subseteq f(A) \cup f(B)$.
9. Now let $y \in f(A) \cup f(B)$.
10. Then $y \in f(A)$ or $y \in f(B)$.
11. Case 1: $y \in f(A)$.
12. Then $y = f(a)$ for some $a \in A$.
13. Since $a \in A \cup B$, we have $y \in f(A \cup B)$.
14. Case 2: $y \in f(B)$.
15. Then $y = f(b)$ for some $b \in B$.
16. Since $b \in A \cup B$, we have $y \in f(A \cup B)$.
17. Both cases lead to $y \in f(A \cup B)$.
18. So $f(A) \cup f(B) \subseteq f(A \cup B)$, and we have now shown equality.

(ii) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$

Proof:

1. Let $x \in f(A \cap B)$.
2. Then $x = f(u)$ for some $u \in A \cap B$.
3. Since $u \in A$, we have $x \in f(A)$.
4. Since $u \in B$, we have $x \in f(B)$.
5. So $x \in f(A) \cap f(B)$
6. Thus, $f(A \cap B) \subseteq f(A) \cap f(B)$.

(iii) Give an example of sets S, T, A, B, and a function f for which $f(A) \cap f(B) \not\subseteq f(A \cap B)$. [Hint: Start by trying to prove that $f(A) \cap f(B) \subseteq f(A \cap B)$, and see where you get stuck.]

An easy example is to let $S = \{1, 2\}$ and $T = \{0\}$. Define $f\colon S \to T$ by $f(1) = f(2) = 0$. Let $A = \{1\}$ and $B = \{2\}$. Then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$. But $f(A) = \{0\} = f(B)$, so $f(A) \cap f(B) = \{0\}$. But $\{0\} \not\subseteq \emptyset$.

12. Yoda has a bunch of eggs, and you have to figure out how many.

(a) He tells you two facts: When you separate the eggs into groups of 11, there are 3 left over. When you separate the eggs into groups of 8, there are 4 left over.

Given these facts, what is the least number of eggs that Yoda could have?

(b) Suppose Yoda also tells you that he has between 100 and 200 eggs. Given this additional piece of information, you can determine exactly how many eggs Yoda has. How many?

(c) The next day Yoda comes back with a lot more eggs. This time he says: When I separate the eggs into groups of 11, there are 3 left over. When I separate them into groups of 300, there are 51 left over. What is the least number of eggs that Yoda could have?

Solution to (c):

Let n be the number of eggs. Then $n = 11a + 3$ and $n = 300b + 51$, for some $a, b \in \mathbb{Z}$. So we have

$$11a + 3 = n = 300b + 51.$$

Going into $\mathbb{Z}_{11}$ this becomes $3 = 3b + 7$ (note that $300 \equiv_{11} 3$ and $51 \equiv_{11} 7$). So $3b = 3 - 7 = 7$. But $3^{-1} = 4$ in $\mathbb{Z}_{11}$, so multiplying both sides by 4 gives $b = 6$.

Going back to $\mathbb{Z}$, we now know that $b = 11u + 6$, for some $u \in \mathbb{Z}$. Plug this back in the formula for n to get

$$n = 300(11u + 6) + 51 = 3300u + 1851.$$

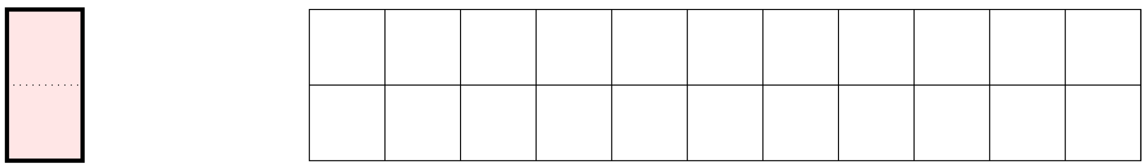
Notice that Yoda's first congruence $n \equiv_{11} 3$ actually FOLLOWS from this formula, since

$$n = 3300u + 1851 = 3300u + 1848 + 3 = 11(30u + 167) + 3.$$

This is because we already "solved" the first congruence when we went into $\mathbb{Z}_{11}$ and found that $b = 6$.

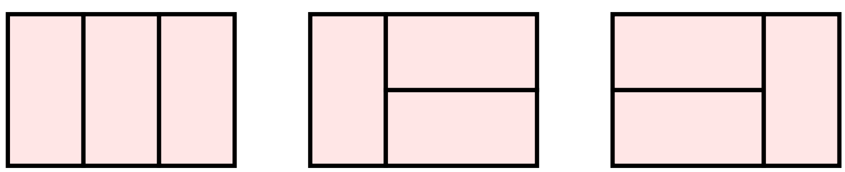
In any case, we now know that $n = 3300u + 1851$ for some $u \in \mathbb{Z}$. These are all the numbers that satisfy Yoda's two congruences. The smallest possibility is when $u = 0$, which gives 1851.

13. You have a huge collection of 1×2 dominoes (or tiles), and a 2×11 checkerboard:



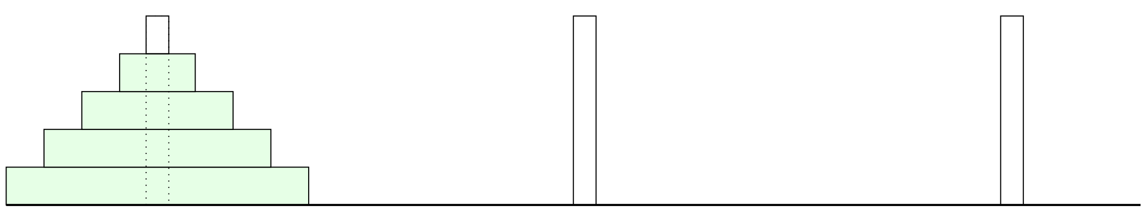
Your goal in this problem is to determine how many different ways there are to tile the checkerboard using your dominoes.

To solve this problem, it is best to solve some smaller versions first. Let S_n denote the number of different ways to tile a $2 \times n$ checkerboard. For example, $S_3 = 3$ as we see below:



Make a table showing the values of S_n for $1 \leq n \leq 11$. As you are doing this, try to find a systematic way of finding all the tilings. Note any patterns that you find, and see if you can explain them. As a check, you should get $S_6 = 13$.

14. A certain puzzle has three pegs, the leftmost peg starting out with a tower of n disks. The object of the puzzle is to move this tower to the rightmost peg. The rules are that you can only move one disk at a time, the disk has to be moved from one peg to another, and you can never place a larger disk on top of a smaller disk. The following picture shows the starting position of the puzzle when $n = 4$:



Your goal in this problem is to determine the minimum number of moves needed to solve the puzzle when there are 11 disks. Approach this in a similar way to what we did in problem #13, by making a table showing the minimum number of moves to solve the n-disk game for $1 \leq n \leq 11$.