Math 307, Homework #8
Due Wednesday, November 27
SOLUTIONS TO SELECTED PROBLEMS

1. In this problem we have four functions, as indicated in the diagram below:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D. \end{tikzcd}$$

You are given that $q \circ f = g \circ p$. Let $X \subseteq C$.

(a) Prove that $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$.

Proof:

Assume $u \in f(p^{-1}(X))$. Then $u = f(s)$ for some $s \in p^{-1}(X)$. Then $p(s) \in X$, so $g(p(s)) \in g(X)$. But $g(p(s)) = q(f(s)) = q(u)$, so we have $q(u) \in g(X)$. That is, $u \in q^{-1}(g(X))$. This shows $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$.

(b) If p is onto and q is one-to-one, prove $f(p^{-1}(X)) = q^{-1}(g(X))$.

Proof:

We have already proven $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$ in (a). Let $u \in q^{-1}(g(X))$. Then $q(u) \in g(X)$, so $q(u) = g(x)$ for some $x \in X$. Since p is onto, $x = p(a)$ for some $a \in A$. Then

$$q(f(a)) = g(p(a)) = g(x) = q(u).$$

Since q is one-to-one, we conclude $f(a) = u$.

But $p(a) = x \in X$, so $a \in p^{-1}(X)$. Therefore $u \in f(p^{-1}(X))$, and we have now shown the two sets are equal.

The solutions to induction problems are written in a less formal style than before. Make sure that you understand what is the inductive statement $P(n)$ in each case!

2. For all $n \geq 2$, $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{(n-1)(3n+2)}{4n(n+1)}$.

Proof:

I. When $n = 2$ the left-hand-side is $\frac{1}{2^2-1} = \frac{1}{3}$ and the right-hand-side is $\frac{1 \cdot 8}{4 \cdot 2 \cdot 3}$. Certainly these are equal.

II. Assume $n \in \mathbb{N}$, $n \geq 2$, and $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{(n-1)(3n+2)}{4n(n+1)}$.

Add $\frac{1}{(n+1)^2-1}$ to both sides of the induction hypothesis, to get

$$\begin{aligned} \sum_{k=2}^{n+1} \frac{1}{k^2-1} &= \frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{(n+1)^2-1} \\ &= \frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{n^2+2n} \\ &= \frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{n(n+2)}. \end{aligned}$$

Now, we have

$$\begin{aligned}n^2(3(n+1)+2) = n^2(3n+5) = 3n^3+5n^2 &= 3n^3 - n^2 - 2n + 6n^2 - 2n - 4 + 4n + 4 \\ &= (3n^2 - n - 2)(n+2) + 4n + 4 \\ &= (n-1)(3n+2)(n+2) + 4(n+1).\end{aligned}$$

Divide by $4n(n+1)(n+2)$ to get

$$\tfrac{n(3(n+1)+2)}{4(n+1)(n+2)} = \tfrac{(n-1)(3n+2)}{4n(n+1)} + \tfrac{1}{n(n+2)} = \sum_{k=2}^{n+1} \tfrac{1}{k^2-1}.$$

This completes the induction step.

III. By PMI, we have shown $\sum\limits_{k=2}^{n} \frac{1}{k^2-1} = \frac{(n-1)(3n+2)}{4n(n+1)}$ for all $n \geq 2$.

3. Let $a_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1}n$. Prove by induction that $a_{2n} = -n$ for all $n \geq 1$.

   Proof:

   I. $a_{2\cdot 1} = a_2 = 1 - 2 = -1$.

   II. Assume $n \in \mathbb{N}$, $n \geq 1$, and $a_{2n} = -2n$. Then

$$\begin{aligned}a_{2(n+1)} = a_{2n+2} &= 1 - 2 + \cdots + (-1)^{2n+1}(2n) + (-1)^{2n+2}(2n+1) + (-1)^{2n+3}(2n+2) \\ &= a_{2n} + (-1)^{2n+2}(2n+1) + (-1)^{2n+3}(2n+2) \\ &= a_{2n} + (2n+1) - (2n+2) \\ &= a_{2n} - 1 \\ &= -n - 1 \qquad \text{by the induction hypothesis} \\ &= -(n+1).\end{aligned}$$

   III. By PMI, we conclude $a_{2n} = -n$ for all $n \geq 1$.

4. Suppose $f\colon \mathbb{Z} \to \mathbb{Z}$ is a function with the property that $(\forall x, y \in \mathbb{Z})[f(x+y) = f(x) + f(y)]$.

   (a) Prove by induction that $(\forall n \in \mathbb{N})[n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]]$.

   Proof:

   I. For all $x \in \mathbb{Z}$, $f(1 \cdot x) = f(x) = 1 \cdot f(x)$.

   II. Assume $n \in \mathbb{N}$, $n \geq 1$, and $(\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]$. Let $x \in \mathbb{Z}$. Then

$$f((n+1)x) = f(nx + x) = f(nx) + f(x) = n \cdot f(x) + f(x) = (n+1) \cdot f(x)$$

   where the third equality is by the induction hypothesis.

   III. By PMI, for every natural number $n \geq 1$ we have $(\forall x \in \mathbb{Z})(f(nx) = n \cdot f(x))$.

   (b) Prove that $(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k]$.

   Proof:

   I. Let $k \in \mathbb{N}$ and let $x \in f(M_k)$. Then $x = f(y)$ for some $y \in M_k$. Since $k|y$, we have $y = kz$ for some $z \in \mathbb{Z}$. So $x = f(y) = f(kz) = k \cdot f(z)$ by (a). Therefore $x \in M_k$, hence $f(M_k) \subseteq M_k$.

5. For all $n \geq 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.

   Proof:

   I. $\sqrt{2} - \frac{1}{\sqrt{2}} = \sqrt{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} < 1$. So $\sqrt{2} < 1 + \frac{1}{\sqrt{2}}$.

II. Assume $n \geq 2$ and $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$. Add $\frac{1}{\sqrt{n+1}}$ to both sides of the induction hypothesis to get

$$\sqrt{n} + \tfrac{1}{\sqrt{n}} < \tfrac{1}{\sqrt{1}} + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{3}} + \cdots + \tfrac{1}{\sqrt{n+1}}.$$

We know $0 < n$, so $n^2 < n^2 + n = n(n+1)$. Take square roots to get $n < \sqrt{n}\sqrt{n+1}$, then add 1 to both sides to get

$$n + 1 < \sqrt{n}\sqrt{n+1} + 1.$$

Dividing by $\sqrt{n+1}$ yields

$$\sqrt{n+1} < \sqrt{n} + \tfrac{1}{\sqrt{n+1}}.$$

We now have a chain

$$\sqrt{n+1} < \sqrt{n} + \tfrac{1}{\sqrt{n+1}} < \tfrac{1}{\sqrt{1}} + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{3}} + \cdots + \tfrac{1}{\sqrt{n+1}}$$

and this completes the induction step.

III. By PMI, we have $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ for all $n \geq 2$.

6. For all $n \geq 2$, $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n^2}\right) = \frac{n+1}{2n}$.

   Proof:

   I. $\frac{2^2-1}{2^2} = \frac{3}{4} = \frac{2+1}{2 \cdot 2}$.

   II. Assume $n \in \mathbb{N}$, $n \geq 2$, and $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n^2}\right) = \frac{n+1}{2n}$.

   Multiply both sides of the induction hypothesis by $\frac{(n+1)^2-1}{(n+1)^2}$. This gives

$$\left(\tfrac{2^2-1}{2^2}\right) \cdot \left(\tfrac{3^2-1}{3^2}\right) \cdots \left(\tfrac{(n+1)^2-1}{(n+1)^2}\right) = \tfrac{n+1}{2n} \cdot \tfrac{(n+1)^2-1}{(n+1)^2} = \tfrac{n+1}{2n} \cdot \tfrac{n^2+2n}{(n+1)^2} = \tfrac{1}{2} \cdot \tfrac{n+2}{n+1} = \tfrac{n+2}{2(n+1)}.$$

   III. By PMI, we conclude $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n^2}\right) = \frac{n+1}{2n}$ for all $n \geq 2$.

7. For all $n \geq 2$, $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$.

   Proof:

   I. $\frac{1}{2} + \frac{2}{3} = \frac{7}{6} < \frac{4}{3}$ since $7 \cdot 3 < 6 \cdot 4$.

   II. Assume $n \geq 2$ and $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$. Add $\frac{n+1}{n+2}$ to both sides of the induction hypothesis to get

$$\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \cdots + \tfrac{n+1}{n+2} < \tfrac{n^2}{n+1} + \tfrac{n+1}{n+2}.$$

   Now, $2n < 3n$ so

$$n^2(n+2) + (n+1)^2 = n^3 + 2n^2 + n^2 + 2n + 1 = n^3 + 3n^3 + 2n + 1 < n^3 + 3n^2 + 3n + 1 = (n+1)^3.$$

   Divide both sides of this inequality by $(n+1)(n+2)$ to get

$$\tfrac{n^2}{n+1} + \tfrac{n+1}{n+2} < \tfrac{(n+1)^2}{n+2}.$$

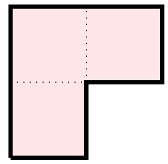
   We now have the chain

$$\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \cdots + \tfrac{n+1}{n+2} < \tfrac{n^2}{n+1} + \tfrac{n+1}{n+2} < \tfrac{(n+1)^2}{n+2}.$$

   This completes the induction step.

   III. By PMI, we conclude that $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$ for all $n \geq 2$.

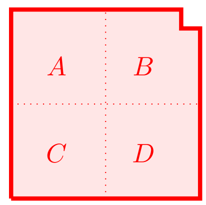
8. You have a huge collection of “trionimo” tiles that look like this:



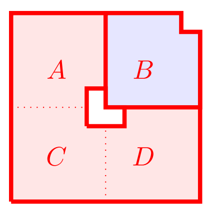
Prove by induction that for all $k \in \mathbb{N}$ such that $k \geq 1$, a $2^k \times 2^k$ checkerboard with the upper-right corner square removed can be tiled using trionimos. [Hint to get started: As scratchwork, do the cases $k = 1$, $k = 2$, and $k = 3$ by hand. Look for a link between the $2^{k+1} \times 2^{k+1}$ case and the $2^k \times 2^k$ case.]

I. The case $k = 1$ is trivial.

II. Assume that $k \geq 1$ and a $2^k \times 2^k$ checkerboard with upper right corner removed can be tiled using trionimos. Take a $2^{k+1} \times 2^{k+1}$ checkerboard and break it up into four quadrants as shown below:



Start by tiling region B by trionimos, which we can do by induction. Then place a single trionimo in the center, as shown below:



Next, tile the remaining area of region A by induction. Repeat for the remaining areas of regions C and D. This completes the tiling of the $2^{k+1} \times 2^{k+1}$ checkerboard with upper right corner removed.

9. There is a famous proof that all horses are the same color. Let $P(n)$ be the statement "for all sets of n horses, all the horses in the set have the same color". We will prove this by induction. The base case $n = 1$ is clear, since in a set consisting of exactly 1 horse all the horses have the same color. Now assume that $P(n)$ is true, and let S be a set of $n + 1$ horses. Label the horses $1, 2, \ldots, n + 1$. Then the first n horses constitute a set of n horses, so by the induction hypothesis they all have the same color. Likewise, the last n horses are a set of n horses; so by induction *they* all have the same color. But if the first n horses all have the same color, and the last n horses all have the same color, then since these two sets overlap the two colors must be identical. So all the horses in S have the same color, and we are done by induction.

   Find the mistake in the above proof.

   The mistake is in the clause "since these two sets overlap". When $n = 1$ and there are only two horses in the set, the two sets do NOT overlap—and so one cannot conclude that all the horses have the same color.

   One can think of an induction proof as establishing

$$P(1), \quad P(1) \Rightarrow P(2), \quad P(2) \Rightarrow P(3), \quad P(3) \Rightarrow P(4), \quad \ldots$$

   This is an interesting example where $P(1)$ is true, and $P(n) \Rightarrow P(n+1)$ is true for all $n \geq 2$, but where $P(1) \Rightarrow P(2)$ is FALSE.