Introduction to Topology II: Homework 3

Due on January 28, 2026 at 23:59

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For each of these problems, we will suppose that X is a topological space and $f : X \to X$ is a map such that $f(p) \neq p$ for all $p \in X$, but $(f \circ f)(x) = \mathrm{Id}_X$. (In particular, $f(U) = f^{-1}(U)$ for any set U.) Define an equivalence relation on X by putting $p \sim f(p)$ for all $p \in X$ (so every equivalence class has exactly two elements). Let Y be the identification space $X/\sim$. Let $\pi : X \to Y$ be the identification map.

Problem 1. Assume that X is Hausdorff. The purpose of this problem is to prove that Y is Hausdorff, as well.

(i) Show that, given any natural number n and any n-tuple of pairwise distinct points $p_1, \cdots, p_n \in X$, we can find pairwise disjoint open sets $U_1, \cdots, U_n$ with $p_i \in U_i$ for all i. (Note that the $n = 2$ case is the definition of Hausdorffness. For larger n, use induction. Note also that this part of the problem does not involve the map f or the space Y.)

(ii) Let q_1 and q_2 be two distinct points in Y. Choose points $p_1, p_2 \in X$ with $\pi(p_i) = p_i$. In particular, this implies that the four points $p_1, f(p_1), p_2, f(p_2)$ are pairwise disjoint. By part (i), we can find pairwise disjoint open sets U_1, U_2, V_1, and V_2 with $p_1 \in U_1$, $p_2 \in U_2$, $f(p_1) \in V_1$, and $f(p_2) \in V_2$. Show that we can make these choices in such a way that $V_1 = f(U_1)$ and $V_2 = f(U_2)$.

(iii) Show that Y is Hausdorff.

Solution to (i). We will use induction on n. The base case $n = 2$ is true by the definition of Hausdorffness. Now, assume that the statement is true for some $n \geq 2$; we will show that it is true for $n + 1$. Let $p_1, \cdots, p_{n+1} \in X$ be pairwise distinct points. By the inductive hypothesis, we can find pairwise disjoint open sets $U_1, \cdots, U_n$ with $p_i \in U_i$ for all $1 \leq i \leq n$. Since X is Hausdorff, for each $1 \leq i \leq n$, we can find disjoint open sets V_i and W_i with $p_{n+1} \in V_i$ and $p_i \in W_i$. Now, let $U_{n+1} = \bigcap_{i=1}^{n} V_i$ and redefine $U_i = U_i \cap W_i$ for all $1 \leq i \leq n$. Then, the sets $U_1, \cdots, U_{n+1}$ are pairwise disjoint open sets with $p_i \in U_i$ for all $1 \leq i \leq n+1$. This completes the inductive step, and thus the proof. $\square$

Solution to (ii). Since q_1 and q_2 are distinct points in Y, their preimages under the identification map π are distinct sets in X. Specifically, we have $\pi^{-1}(q_1) = \{p_1, f(p_1)\}$ and $\pi^{-1}(q_2) = \{p_2, f(p_2)\}$. By part (i), we can find pairwise disjoint open sets U_1, U_2, V_1, and V_2 in X such that $p_1 \in U_1$, $p_2 \in U_2$, $f(p_1) \in V_1$, and $f(p_2) \in V_2$. Now, we can define $V_1 = f(U_1)$ and $V_2 = f(U_2)$. Since f is a homeomorphism, the sets V_1 and V_2 are open in X. Furthermore, since the original sets were pairwise disjoint, the new sets remain disjoint as well. Thus, we have constructed the desired open sets. $\square$

Solution to (iii). To show that Y is Hausdorff, we need to demonstrate that for any two distinct points $q_1, q_2 \in Y$, there exist disjoint open neighborhoods around each point. Let $p_1, p_2 \in X$ be such that $\pi(p_i) = q_i$ for $i = 1, 2$. By part (ii), we can find pairwise disjoint open sets U_1, U_2, V_1, and V_2 in X such that $p_1 \in U_1$, $p_2 \in U_2$, $f(p_1) \in V_1$, and $f(p_2) \in V_2$, with $V_1 = f(U_1)$ and $V_2 = f(U_2)$. Now, consider the images of these sets under the identification map π. The sets $\pi(U_1)$ and $\pi(U_2)$ are open in Y because π is an open map (as shown in part (i)). Furthermore, since the original sets were disjoint in X, their images under π will also be disjoint in Y. Therefore, we have found disjoint open neighborhoods around q_1 and q_2, which shows that Y is Hausdorff. $\square$

Problem 2. Assume that X is a surface. The purpose of this problem is to prove that Y is a surface, as well. We have already established that, if X is Hausdorff, so is Y. So we just need to show that, if every point in X has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$, then the same is true of Y.

(i) Show that the identification map π is open.

(ii) Show that, if $U \subset X$ is an open set and $U \cap f(U) = \emptyset$, then $\pi : U \to \pi(U)$ is a homeomorphism.

(iii) Show that, if $p \in X$, there exists an open neighborhood U of p such that U is homeomorphic to an open subset of $\mathbb{R}^2$ and $U \cap f(U) = \emptyset$.

(iv) Show that Y is a surface.

For the remaining problems, suppose that $X = |K|$ is a combinatorial surface and $f = |\varphi|$ for some isomorphism $\varphi : K \to K$.

Solution to (i). The identification map π is open if and only if for every open set $U \subset X$, the image $\pi(U)$ is open in Y. Let U be an open set in X. By the definition of the quotient topology on Y, a set is open in Y if and only if its preimage under π is open in X. Since $\pi^{-1}(\pi(U)) = U \cup f(U)$, and both U and $f(U)$ are open in X (as f is a homeomorphism), their union is also open. Therefore, $\pi(U)$ is open in Y, which shows that π is an open map. $\square$

Solution to (ii). If $U \subset X$ is open and $U \cap f(U) = \emptyset$, then the restriction of π to U, denoted $\pi|_U \colon U \to \pi(U)$, is a bijection. To see this, note that for any point $x \in U$, $\pi(x) = \pi(f(x))$ only if $f(x) \in U$, which contradicts the assumption that $U \cap f(U) = \emptyset$. Thus, $\pi|_U$ is injective. It is also surjective onto $\pi(U)$ by definition. Since π is an open map (from part (i)), the restriction $\pi|_U$ is also open. Therefore, $\pi|_U$ is a homeomorphism between U and $\pi(U)$. $\square$

Solution to (iii). Since X is a surface, then it is locally Euclidean. Therefore, for any point $p \in X$, there exists an open neighborhood V of p that is homeomorphic to an open subset of $\mathbb{R}^2$. Let $U = V \setminus f(V)$. Since f is a homeomorphism, $f(V)$ is also open in X, and thus U is open as the difference of two open sets. Additionally, we have $U \cap f(U) = \emptyset$ by construction. Finally, since $U \subset V$ and V is homeomorphic to an open subset of $\mathbb{R}^2$, it follows that U is also homeomorphic to an open subset of $\mathbb{R}^2$. Thus, we have found the desired neighborhood U of p. $\square$

Solution to (iv). To show that Y is a surface, we need to demonstrate that every point in Y has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$. Let $q \in Y$ be an arbitrary point. Choose a point $p \in X$ such that $\pi(p) = q$. By part (iii), there exists an open neighborhood U of p in X such that U is homeomorphic to an open subset of $\mathbb{R}^2$ and $U \cap f(U) = \emptyset$. By part (ii), the restriction $\pi|_U \colon U \to \pi(U)$ is a homeomorphism. Therefore, $\pi(U)$ is an open neighborhood of q in Y, and it is homeomorphic to an open subset of $\mathbb{R}^2$. Since q was arbitrary, this shows that every point in Y has the required property, and thus Y is a surface. $\square$

Problem 3. It's tempting to try to define a triangulation of Y whose simplices are in bijection with equivalence classes of simplices of K. Find an example that shows that such a simplicial complex might not exist! (If one subdivides K first, then it works, but you don't have to show that.)

Solution. For the orientation preserving map, take K to be the boundary of a triangle with vertices v_1, v_2, and v_3. Define the map $\varphi : K \to K$ by $\varphi(v_1) = v_2$, $\varphi(v_2) = v_3$, and $\varphi(v_3) = v_1$. This map is an isomorphism of the simplicial complex K that preserves the orientation of the triangle. The identification space $Y = |K|/\sim$ will be homeomorphic to a circle, which can be triangulated with a single edge and two vertices. For the non-orientation preserving map, take K to be the boundary of a square with vertices v_1, v_2, v_3, and v_4. Define the map $\varphi : K \to K$ by $\varphi(v_1) = v_2$, $\varphi(v_2) = v_1$, $\varphi(v_3) = v_4$, and $\varphi(v_4) = v_3$. This map is an isomorphism of the simplicial complex K that reverses the orientation of the square. The identification space $Y = |K|/\sim$ will be homeomorphic to a figure-eight shape, which cannot be triangulated in a way that corresponds directly to equivalence classes of simplices in K without further subdivision. $\square$

Problem 4. Suppose that K is equipped with an orientation. We say that φ is *orientation preserving* if it takes positively oriented triangles to positively oriented triangles. That is, if v_1, v_2, v_3 span a triangle in K and appear in positive cyclic order, then $\varphi(v_1)$, $\varphi(v_2)$, $\varphi(v_3)$ also appear in positive cyclic order. Give an example that is orientation preserving, and an example that is not.

Solution. An example of an orientation preserving map is just the identity map, $f : Y \to Y$, defined by $f(p_i) = p_i$. An example of a non-orientation preserving map is the reflection map across the x-axis in $\mathbb{R}^2$, defined by $f(x, y) = (x, -y)$. This map reverses the orientation of any triangle in the plane. $\square$

Problem 5. Assume that φ is orientation preserving. Also assume that the procedure described in Problem 3 actually works, so that we have a simplicial complex L and a homeomorphism $|L|= X/{\sim}$. Show that the orientation of K induces an orientation of L.

Solution. Let φ be an orientation preserving isomorphism of the simplicial complex K. We want to show that the orientation of K induces an orientation of the simplicial complex L formed by the quotient $X/{\sim}$. To do this, we will define the orientation of L based on the orientation of K. Consider a triangle in L represented by the equivalence class of a triangle in K. Since φ is orientation preserving, the orientation of the triangle in K will be preserved under the identification. Therefore, we can assign the same orientation to the corresponding triangle in L. $\square$