DRP Tasks

For Hashem

1. Let $z = x + iy$ be the standard coordinate on $\mathbb{C}$. Show that the standard volume form (symplectic form) $\omega = \mathrm{d}x \wedge \mathrm{d}y$ can be written as $\frac{1}{2i}\mathrm{d}\bar{z} \wedge \mathrm{d}z$, whereas the standard metric $\mathrm{d}x^2 + \mathrm{d}y^2$ is equal to $\mathrm{d}\bar{z} \cdot \mathrm{d}z$ (here $\cdot$ is understood to be a symmetric product). The latter is what the M.R.S. paper denotes $|\mathrm{d}z|^2$.

2. Let $\beta \in (0, 1]$. Under the map $f_\beta : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^\beta$, check that $f_\beta^*(\omega) = \beta^2 r^{2\beta-2}\omega$, which is a rescaling of $r^{2\beta-2}\omega$. See if you can understand what this has to do with a cone.

3. Consider the maps : $\mathbb{R}_{\geq 0} \times S^1 \xrightarrow{\phi} \mathbb{C}$ given by $\phi(r, \theta) = re^{i\theta}$ and $\mathbb{R} \times S^1 \xrightarrow{\tau} \mathbb{R}_{\geq 0} \times S^1$ given by $\tau(y, \theta) = (e^y, \theta)$. Calculate the pullback forms, $\phi^*(\omega)$ and then $\tau^*(\phi^*(\omega))$.

4. Now let $g_\beta = r^{2\beta-2}|\mathrm{d}z|^2$ be the conical metric from the M.R.S. paper, which has corresponding symplectic form
$$\omega_\beta = \frac{r^{2\beta-2}}{2i}\mathrm{d}\bar{z} \wedge \mathrm{d}z = r^{2\beta-1}\mathrm{d}r \wedge \mathrm{d}\theta$$
(verify the last equality). Calculate these pullbacks $\phi^*\omega_\beta$ and $\tau^*\phi^*\omega_\beta$.

5. Section 1.3 of Huybrechts states $\partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\,\partial_y)$. Use this to show $\partial_{\bar{z}}\,\partial_z = \frac{1}{4}(\partial_{xx} + \partial_{yy})$. Hence the Laplacian is $\Delta = 4\,\partial_{\bar{z}}\,\partial_z$.

6. (Harder) Calculate the pushforward of the vector fields ∂_z and $\partial_{\bar{z}}$ under the inverse map ϕ^{-1}. Use this to derive the formula for the Laplacian in polar coordinates
$$\Delta = \partial_{rr} + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\partial_{\theta\theta}\,.$$

7. When evaluating Δ on a radially symmetric function, we can ignore the last term $\frac{1}{r^2}\,\partial_{\theta\theta}$. Calculate the pushforward of $\partial_{rr} + \frac{1}{r}\,\partial_r$ under the inverse map $\tau^{-1}(r, \theta) = (\log r, \theta)$.

8. Now incorporate the $\partial_{\theta\theta}$ term in the above to recover the standard Laplacian on the infinite cylinder $C = \mathbb{R} \times S^1$.

For Jack

1. In the 1D case, you fixed the temperature at the endpoints. This is called a Dirichlet boundary condition, and it represents having heat sinks on either end. The other option is to use a Neumann boundary condition, which means instead of fixing the temperature at the endpoints, you fix the first spatial derivatives. If you set these equal to zero, this ensures the graph of the function looks flat at the edges. Physically, it represents having insulation at either end of the rod. Try playing around with this and see what the simulation looks like.

2. We want to simulate heat flow for the function $u(x, y, t)$ where x, y are spatial coordinates on the square $D^2 = [0, 1]^2$ and t is time. The boundary ∂D^2 has four components

$$\partial D^2 = (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1]) \cup ([0, 1] \times \{0\}) \cup ([0, 1] \times \{1\}).$$

   Fix the boundary values

$$f(0, y) = 0, \quad f(1, y) = 1 - 2y, \quad f(x, 0) = \sin\left(\frac{\pi}{2}x\right), \quad f(x, 1) = -\sin\left(\frac{\pi}{2}x\right)$$

   so that $u(x, y, t) = f(x, y)$ on ∂D for all time t. Away from the boundaries, use the initial condition $u(x, y, 0) = 0$. Using as many samples as you can reasonably get away with, use heat flow to approximate the long-term equilibrium state. (For simplicity, let's assume the diffusivity constant is 1, so the heat equation is $\partial_t u = \Delta u$, or in other notation $u_t = u_{xx} + u_{yy}$).

3. (Optional) Try recreating the 1D simulation for the wave equation $u_{tt} = \Delta u$. To do this, you will need to keep track of not only the values of u at the various points, but also those of u_t. You'll need an initial condition which specifies both. This is harder, but the result is pretty spectacular.

4. Let's try something similar to (2) but in polar coordinates $(r, \theta) \in [0, 1] \times [-\pi, \pi]$. We'll begin with the radially symmetric boundary condition $u(1, \theta, t) = 2$ for all θ and t, and initial condition $u(r, \theta, 0) = 0$ for all $r < 1$. The heat equation turns into $u_{tt} = u_{rr} + \frac{1}{r}u_r$ (I'm dropping the $\partial_{\theta\theta}$ part because the function is radially symmetric). Note that you will likely run into trouble near $r = 0$; I want to see exactly what this looks like and whether you can find a way around it.

5. Once we figure out how to approach (4), do it again but with boundary condition $u(1, \theta, t) = \sin(5\theta)$. This isn't radially symmetric, so we'll need to impose the gluing condition $u(r, \pi, t) = u(r, -\pi, t)$ for all r, t, and the heat equation will become $\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$.