Funds of Anal I: Homework 8

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Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

Exercise 3.3.5

(i) The arbitrary intersection of compact sets is compact.

(ii) The arbitrary union of compact sets is compact.

(iii) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(iv) If $f_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution 3.3.5

(i) True, since the intersection will be closed and bounded.

(ii) False, since $\bigcup_{n=1}^{\infty} [0, n]$ is unbounded.

(iii) False, since $(0, 1] \cap [0, 1] = (0, 1]$ is not closed, as 0 is a limit point of $(0, 1]$, since $\left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$ is a sequence that converges to 0.

(iv) False, since $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$.

Exercise 3.3.9

Follow these steps to prove that being compact implies every open cover has a finite subcover.

Assume K is compact, and let $\{O_\lambda \mid \lambda \in \Lambda\}$. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(i) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim_{n\to\infty} |I_n| = 0$.

(ii) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

(iii) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Solution 3.3.9

(i) Bisect I_0 into two intervals. Let I_1 be the interval where $I_1 \cap K$ cannot be finitely covered. Repeating this process gets us $\lim\limits_{n\to\infty} |I_n| = \lim\limits_{n\to\infty} |I_0| \left(\dfrac{1}{2}\right)^n = 0$.

(ii) The nested compact set property $K_n = I_n \cap K$ gives $x \in \bigcap\limits_{n=1}^{\infty} K_n$, meaning $x \in K$ and $x \in I_n$ for all n.

(iii) Since $x \in O_{\lambda_0}$ and $|I_n| \to 0$ with $x \in I_n$ for all n, there exists an N where $n > N$ implies $I_n \subseteq O_{\lambda_0}$ contradicting the assumption that $I_n \cap K$ cannot be finitely covered since $\{O_{\lambda_0}\}$ is a finite subcover for $I_n \cap K$.

Exercise 4.2.2

For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ε challenge.

(i) $\lim_{x\to 3}(5x-6)=9$, where $\varepsilon=1$.

(ii) $\lim_{x\to 4}\sqrt{x}=2$, where $\varepsilon=1$.

(iii) $\lim_{x\to \pi}[[x]]=3$, where $\varepsilon=1$.

(iv) $\lim_{x\to \pi}[[x]]=3$, where $\varepsilon=0.01$.

Solution 4.2.2

(i) The largest possible δ-neighborhood is

$$|(5x-6)-9|=|5x-15|=5\cdot|x-3|<5\delta\Rightarrow\delta=\frac{1}{5}.$$

(ii) Expanding $|\sqrt{x}-2|<1$ gives us

$$1<\sqrt{x}<3\Rightarrow\delta=3.$$

(iii) To satisfy $|[[x]]-1|<1$, we require

$$[[x]]=3,$$

which happens when $3\leq x<4$. Thus,

$$|x-\pi|<\min(\{\pi-3,4-\pi\})\Rightarrow\delta=\min(\{\pi-3,4-\pi\})=\pi-3\approx 0.1416.$$

(iv) For $\varepsilon=0.01$, we still need $|[[x]]-3|<0.01$. Since $[[x]]$ is piecewise constant, this requires $[[x]=3$. This occurs only when $3\leq x<4$. The analysis of $|x-\pi|<\delta$ is the same as before

$$\delta=\min(\{\pi-3,4-\pi\})=\pi-3\approx 0.1416.$$

Exercise 4.2.5

Use Definition 4.2.1 to supply a proper proof for the following limit statements

(i) $\lim_{x \to 2}(3x + 4) = 10$.

(ii) $\lim_{x \to 0} x^3 = 0$.

(iii) $\lim_{x \to 2}(x^2 + x - 1) = 5$.

(iv) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

Solution 4.2.5

(i) *Proof.* Let $\varepsilon > 0$. Define $\delta = 3\varepsilon$. Then, for all $x \in A$, we get Let ε be an arbitrary positive number. Let $\delta = 3\varepsilon$. Let x be arbitrary. Suppose $0 < |x - 2| < \delta$ Multiplying both sides by 3 gives us

$$3 \cdot |x - 2| < 3\delta \Rightarrow |3x - 6| < 3 \cdot \frac{\varepsilon}{3} \Rightarrow |(3x + 4) - 10| < \varepsilon.$$

Therefore, $0 < |x - 2| < \delta \Rightarrow |(3x + 4) - 10| < \varepsilon$.

It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[0 < |x - 2| < \delta \Rightarrow |(3x + 4) - 10| < \varepsilon]$. □

(ii) *Proof.* Let ε be an arbitrary positive number. Let $\delta = \varepsilon^{1/3}$. Let x be arbitrary. Suppose $0 < |x| < \delta$. Cubing both sides gives us

$$0 < (|x|)^3 < \delta^3 \Rightarrow |x^3 - 0| < \varepsilon.$$

Therefore, $0 < |x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon$.

It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[0 < |x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon]$. □

(iii) *Proof.* Let ε be an arbitrary positive number. Let $\delta = \min(\{1, \frac{\varepsilon}{6}\})$. Let x be arbitrary. Suppose $0 < |x - 2| < \delta$. Multiplying both sides by 6 gives us $6 \cdot |x - 2| < 6\delta$. Playing with the inequality on the left hand side gives us

$$6\delta > 6 \cdot |x - 2| = |6x - 12| = |(3 + 3)(x - 2)| \geq |(x + 3)(x - 2)|.$$

Therefore, we get

$$|(x + 3)(x - 2)| < 6\delta \Rightarrow |x^2 + x - 6| < 6 \cdot \frac{\varepsilon}{6} \Rightarrow |(x^2 + x - 1) - 5| < \varepsilon.$$

Therefore $0 < |x - 2| < 6 \Rightarrow |(x^2 + x - 1) - 5| < \varepsilon$.

It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[0 < |x - 2| < \delta \Rightarrow |(x^2 + x - 1) - 5| < \varepsilon]$. □

(iv) *Proof.* Let ε be an arbitrary positive number. Let $\delta = \min(\{1, 6\varepsilon\})$. Let x be arbitrary. Suppose $0 < |x - 3| < \delta$. Dividing the middle inequality by 6 gives us

$$\frac{|x - 3|}{6} = \frac{|x - 3|}{3 \cdot 2} \leq \frac{|x - 3|}{3 \cdot |x|} = \left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon.$$

Therefore $0 < |x - 3| < 6 \Rightarrow \left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$.

It follows that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)[0 < |x - 3| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon]$. □

Exercise 4.2.6

(i) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.

(ii) $\lim_{x \to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.

(iii) If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} 3[f(x) - 2]^2 = 3(L - 2)^2$.

(iv) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$ for any function g (with the domain equal to the domain of f).

Solution 4.2.6

(i) True, since $|x - a| < \delta_2 < \delta$.

(ii) False. Counterexample: If $x = 0$, then $f(x) = 1$. Otherwise, $f(x) = 0$. The definition of the limit of a function states that $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$ for all x that's not equal to a.

(iii) True, as you can use the Algebraic Limit Theorem.

(iv) False. Counterexample: Given two functions $f(x) = x$ and $g(x) = {}^1\!/\!{}_x$, then the limit $\lim_{x \to 0} f(x) = 0$, but $\lim_{x \to 0} g(x)$ does not exist, as ${}^1\!/\!{}_0$ isn't defined as it's not continuous.