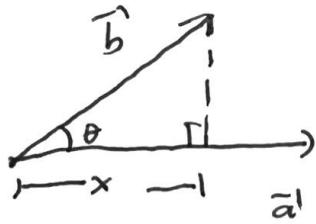


The scalar projection of \vec{b} onto \vec{a} or component of \vec{b} along \vec{a} is the signed magnitude of the vector projection. It is denoted $\text{comp}_{\vec{a}} \vec{b}$.



$$\cos \theta = \frac{x}{|\vec{b}|}$$

$$\begin{aligned} x &= \text{comp}_{\vec{a}} \vec{b} = |\vec{b}| \cos \theta \\ &= |\vec{b}| \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) \\ &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \end{aligned}$$

Note: If $0 \leq \theta < \frac{\pi}{2}$, $\vec{a} \cdot \vec{b} > 0$ and $\text{comp}_{\vec{a}} \vec{b} > 0$

If $\frac{\pi}{2} < \theta \leq \pi$, $\vec{a} \cdot \vec{b} \leq 0$ and $\text{comp}_{\vec{a}} \vec{b} < 0$

Since the scalar projection of \vec{b} onto \vec{a} is the directed length of the vector projection, the vector projection can be found by multiplying the unit vector in the direction of \vec{a} by the scalar projection.

$$\begin{aligned} \text{proj}_{\vec{a}} \vec{b} &= (\text{comp}_{\vec{a}} \vec{b}) \hat{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} \end{aligned}$$

Ex: Find the scalar projection and vector projection of $\vec{b} = \langle 2, 1, 3 \rangle$ onto $\vec{a} = \langle 4, 0, 3 \rangle$.

$$\vec{a} \cdot \vec{b} = 8 + 9 = 17$$

$$|\vec{a}| = \sqrt{16+9} = 5$$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{17}{5}$$

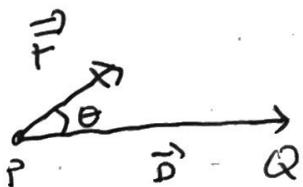
$$\text{proj}_{\vec{a}} \vec{b} = (\text{comp}_{\vec{a}} \vec{b}) \hat{a} = \frac{17}{25} \langle 4, 0, 3 \rangle$$

Work

The amount of work done by a constant force \vec{F} moving an object a distance d is $w = \vec{F} \cdot \vec{d}$ where the force is acting in the direction of motion.

Suppose the force and displacement have different directions.

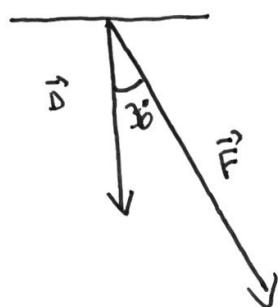
The force, \vec{F} , moves an object from P to Q resulting in the displacement $\vec{D} = \vec{PQ}$



The work done by the force is the product of the projection of the magnitude of \vec{F} acting in the direction of \vec{D} and the distance traveled, $|\vec{D}|$.

$$w = (\text{comp}_{\vec{D}} \vec{F}) |\vec{D}| = \frac{\vec{F} \cdot \vec{D}}{|\vec{D}|} |\vec{D}| = \vec{F} \cdot \vec{D}$$

Ex: A boat sails south with the help of a wind blowing in the direction $S36^\circ E$ with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.



$$\begin{aligned}
 W &= \vec{F} \cdot \vec{D} \\
 &= |\vec{F}| |\vec{D}| \cos(36^\circ) \\
 &= 400 (120) \cos(36^\circ)
 \end{aligned}$$

§12.4: Cross Product

Consider nonzero vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

Find \vec{c} so that $\vec{c} = \langle c_1, c_2, c_3 \rangle$ is orthogonal to \vec{a} and \vec{b} .

$$\begin{aligned}
 \vec{a} \cdot \vec{c} &= 0 & \left\{ \begin{aligned} a_1 c_1 + a_2 c_2 + a_3 c_3 &= 0 \\ b_1 c_1 + b_2 c_2 + b_3 c_3 &= 0 \end{aligned} \right. \\
 \vec{b} \cdot \vec{c} &= 0
 \end{aligned}$$

1) Scale and add equations to eliminate c_1 ,

$$a_1 b_1 c_1 + a_2 b_1 c_2 + a_3 b_1 c_3 = 0$$

$$-a_1 b_1 c_1 - a_1 b_2 c_2 - a_1 b_3 c_3 = 0$$

$$(a_2 b_1 - a_1 b_2) c_2 + (a_3 b_1 - a_1 b_3) c_3 = 0$$

2) Let $c_2 = a_3 b_1 - a_1 b_3$ and $c_3 = a_1 b_2 - a_2 b_1$

3) Use $\vec{a} \cdot \vec{c} = \vec{0}$ to solve for c_1 ,

$$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

$$a_1 c_1 + \underline{a_2 c_3 b_1} - a_1 a_2 b_3 + a_1 a_3 b_2 - \underline{a_2 a_3 b_1} = 0$$

$$\Rightarrow c_1 = a_2 b_3 - a_3 b_2$$

Defn': Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in \mathbb{R}^3 .

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is the cross product or vector product of \vec{a} and \vec{b} .

Determinants from matrix theory can be used to remember definition

$$2 \times 2 \text{ determinant: } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$$

$$= \langle a_2 b_3 - a_3 b_2, - (a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$$