SOLUTIONS TO HOMEWORK 2

Warning: Very little proofreading has been done.

1. Section 4.4

Exercise 4.4.2.

(a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?
(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?
(c) Is $h(x) = x \sin(1/x)$ uniformly continuous on $(0, 1)$?

Solution. (a) No. Let $\varepsilon = 1/2$. Let $x_n = 1/n$ and $y_n = 1/(n + 1)$. Then $|x_n - y_n| \to 0$ as $n \to \infty$, but $|f(x_n) - f(y_n)| = 1 \geq \varepsilon$. By Theorem 4.4.5, f is not uniformly continuous. □

Solution. (b) Yes. In fact, g is uniformly continuous on $[0, 1]$ by Theorem 4.4.7. In particular, it is uniformly continuous on any subset of $[0, 1]$. □

Solution. (c) Yes. Define $h(0) = 0$ and extend the definition of h on $[0, 1]$. Then, by squeeze theorem, $\lim_{x\to 0} h(x) = 0 = h(0)$, so that h is continuous on $[0, 1]$. Consequently, h is uniformly continuous on $[0, 1]$ and, as a result, on $(0, 1)$. □

Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution. On $[1, \infty)$: let $\varepsilon > 0$. Choose $\delta = \varepsilon/2$. Then, for all $x, y \in [1, \infty)$ and $|x - y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \frac{x + y}{x^2 y^2} |x - y| = \left(\frac{1}{x^2 y} + \frac{1}{x y^2} \right) |x - y| \\ &\leq (1 + 1)|x - y| = 2|x - y| < 2\delta = \varepsilon. \end{aligned}$$

This shows that f is uniformly continuous on $[1, \infty)$.

On $(0, 1]$: let $\varepsilon = 1/2$ and $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n + 1}$. Then $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = 1 \geq \varepsilon$, which shows that f is not uniformly continuous on $[1, \infty)$. □

Exercise 4.4.6. Given an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.
(b) A uniform continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.
(c) A continuous function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution. (a) Let $f(x) = \sin(1/x)$ and let $x_n = 1/(n\pi + \pi/2)$. Then f is continuous on $(0, 1)$, (x_n) is Cauchy, but $f(x_n) = \sin(n\pi + \pi/2) = (-1)^n$ is not Cauchy. □

Solution. (b) This is impossible. Indeed, f uniform continuous means that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in (0, 1)$. If $\{x_n\}$ is Cauchy, then there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < \delta$, which then implies that $|f(x_n) - f(x_m)| < \varepsilon$, so that $\{f(x_n)\}$ is Cauchy. □

Solution. (c) This is impossible. Since (x_n) converges, it must be bounded. Let $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Then f is uniformly continuous on $[-M, M]$. Consequently, $f(x_n)$ is Cauchy, following the same argument as in part (b). □

Exercise 4.4.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution. Since $\sqrt{x}$ is continuous on $[0, 2]$, it is uniformly continuous on $[0, 2]$. Let $\varepsilon > 0$. Choose $\delta_1 > 0$ such that $|\sqrt{x} - \sqrt{y}| < \varepsilon$ whenever $|x - y| < \delta_1$ with $x, y \in [0, 2]$.

We now show that f is uniformly continuous on $[1, \infty)$. Let $\varepsilon > 0$ and choose $\delta_2 = \varepsilon$. If $x, y \in [1, \infty)$ and $|x - y| < \delta_2$, then

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x}\sqrt{y}} \leq |x - y| < \delta = \varepsilon.$$

Now, chose $\delta = \min\{1, \delta_1, \delta_2\}$. If $x, y \in [0, \infty)$ and $|x - y| < \delta$, then x, y must be both in $[0, 2]$ or both be in $[1, \infty)$, so that $|f(x) - f(y)| < \varepsilon$. This completes the proof. $\square$

Exercise 4.4.9. (Lipschitz Functions). A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{|f(x) - f(y)|}{x - y} \right| \leq M$$

for all $x \neq y$. ....

(a) Show that if $f : A \mapsto \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.
(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution. (a) Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$. Then, for all $x, y \in A$ satisfying $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \varepsilon.$$

This proves that f is uniformly continuous. $\square$

Solution. (b) The converse is not true. For example, $\sqrt{x}$ is uniformly continuous on $[0, 1]$, but

$$\left| \frac{|f(x) - f(y)|}{x - y} \right| = \frac{|\sqrt{x} - \sqrt{y}|}{|x - y|} = \frac{1}{\sqrt{x} + \sqrt{y}}$$

which is unbounded as can be seen by setting $y = 0$ and $x = 1/n$, $n \to \infty$. $\square$

2. Section 5.2

Exercise 5.2.2. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.
(b) A functions f not differentiable at zero and a functions g differentiable at zero where fg is differentiable at zero.
(c) A functions f not differentiable at zero and a functions g differentiable at zero where $f+g$ is differentiable at zero.

Solution. (a) Let $f(x) = g(x) = |x|$. Then f and g are not differentiable at zero, but $f(x)g(x) = x^2$ is differentiable at zero. $\square$

Solution. (b) Let $f(x) = |x|$ and $g(x) = x$. Then f is not differentiable at zero whereas g is. For fg we have

$$\lim_{x \to 0} \frac{fg(x) - fg(0)}{x} = \lim_{x \to 0} \frac{x|x|}{x} = \lim_{x \to 0} |x| = 0,$$

which shows that fg is differentiable at $x = 0$. $\square$

Solution. (c) This one is impossible, since if g is differentiable at zero and $f + g$ is differentiable at zero, then $f(x) = (f(x) + g(x)) - g(x)$ must be differentiable at zero by Theorem 5.2.4. $\square$

Exercise 5.2.5 For this problem, assume the usual differentiation formulas from elementary calculus are known.

For $a, x \in \mathbb{R}$ define

$$f_a(x) = \begin{cases} x^a & x \geq 0 \\ 0 & x < 0. \end{cases}$$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?
(c) For which values of a is f twice–differentiable?

Solution to (a). If $a \leq 0$, then $f_a(0)$ is not defined. Therefore f_a can't be continuous at 0. (Remember that it is also not correct to say that f_a is discontinuous at 0!)

Suppose therefore that $a > 0$. Then $f_a(0) = 0$. We prove that f_a is continuous at 0. Let $\varepsilon > 0$. Define $\delta = \varepsilon^{1/a}$. Let $x \in \mathbb{R}$ satisfy $|x| < \delta$. If $x \leq 0$, it is obvious that $|f_a(x) - 0| = 0 < \varepsilon$. If $x > 0$, then

$$|f_a(x) - 0| = x^a < \delta^a = \left(\varepsilon^{1/a}\right)^a = \varepsilon.$$

So f_a is continuous at 0. □

Solution to (b). We first claim that if $a \leq 0$, then $\lim_{x\to 0} f_a(x)$ does not exist. To see this, observe that the sequence $x_n = -\frac{1}{n}$ is never zero, converges to 0, and $\lim_{n\to\infty} f_a(x_n) = 0$. However, the sequence $y_n = \frac{1}{n}$ is never zero, converges to 0, and $f_a(y_n) = n^{-a} \geq 1$ for all $n \in \mathbb{N}$ (because $-a \geq 0$), so that $f_a(y_n) \not\to 0$.

Now we consider $f_a'(0)$. For $x \neq 0$ and $a > 0$, one easily checks that

$$\frac{f_a(x) - f_a(0)}{x} = f_{a-1}(x).$$

If $a > 1$, then $a - 1 > 0$, and we saw in Part (a) that $\lim_{x\to 0} f_{a-1}(x) = 0$. So $f_a'(0) = 0$. If $a \leq 1$, then $a - 1 \leq 0$, and we saw above that $\lim_{x\to 0} f_{a-1}(x)$ does not exist. So $f_a'(0)$ does not exist.

The usual differentiation formula from calculus gives $f_a'(x) = ax^{a-1}$ for $x > 0$, while clearly $f_a'(x) = 0$ for $x < 0$. (This is true for all $a \in \mathbb{R}$.) Combining this with the computation above, if $a > 1$ we get $f_a'(x) = af_{a-1}(x)$ for all $x \in \mathbb{R}$, so f_a' is continuous. □

Solution to (c). Using $f_a' = af_{a-1}$ for $a > 1$, apply Part (b) to conclude that f_a' is differentiable on $\mathbb{R}$ if and only if $a - 1 > 1$, that is, $a > 2$. □