Fundamentals of Analysis II: Homework 9

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Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

Exercise 6.4.8. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Solution. We use the Weierstrass M-test to show that f is defined on all of $\mathbb{R}$. Notice that

$$\left| \frac{\sin(x/k)}{k} \right| \leq \frac{|x|}{k^2}.$$

Therefore, we get

$$\sum_{k=1}^{\infty} \frac{|x|}{k^2} = |x| \sum_{k=1}^{\infty} \frac{1}{k^2},$$

converges by the p-series test. Therefore,

$$\sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

converges uniformly on all of $\mathbb{R}$.

To determine continuity, we check uniform convergence. The Weierstrass M-test applies here. Since

$$\left| \frac{\sin(x/k)}{k} \right| \leq \frac{|x|}{k^2},$$

and $\sum_k 1/k^2$ converges, the original series converges uniformly by the Weierstrass M-test. Uniform convergence of a series of continuous functions ensures continuity of $f(x)$. Thus, $f(x)$ is continuous on all of $\mathbb{R}$.

To determine differentiability, we differentiate each term of the series to get

$$f'(x) = \sum_{k=1}^{\infty} \frac{\cos(x/k)}{k^2}.$$

To check whether this series converges uniformly, we use the bound

$$\left| \frac{\cos(x/k)}{k^2} \right| \leq \frac{1}{k^2}.$$

Again, the Weierstrass M-test applies, and since $\sum_k 1/k^2$ converges, the series converges uniformly. Thus, $f'(x)$ is continuous on all of $\mathbb{R}$.

To determine twice-differentiability, we differentiate again to get

$$f''(x) = \sum_{k=1}^{\infty} -\frac{\sin(x/k)}{k^3}.$$

To check uniform convergence, we use the bound

$$\left| -\frac{\sin(x/k)}{k^3} \right| \leq \frac{|x|}{k^3}.$$

The Weierstrass M-test applies again, and since $\sum_k 1/k^3$ converges, the series converges uniformly. Thus, $f''(x)$ is continuous on all of $\mathbb{R}$. □

Exercise 6.4.9. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(i) Show that h is a continuous function defined on all of $\mathbb{R}$.

(ii) Is h differentiable? If so, is the derivative function f' continuous?

Solution to (i). For any fixed $x \in \mathbb{R}$, we note that $x^2 + n^2 \geq n^2$ for all $n \geq 1$, so

$$\frac{1}{x^2+n^2} \leq \frac{1}{n^2}.$$

Since $\sum_n 1/n^2$ is a convergent p-series, the given series converges absolutely for all x. By the Weierstrass M-test, the series converges uniformly on all of $\mathbb{R}$. Since the series converges uniformly, it is continuous on all of $\mathbb{R}$.

To prove continuity, we check uniform convergence. The Weierstrass M-test applies

$$\left|\frac{1}{x^2+n^2}\right| \leq \frac{1}{n^2}.$$

Since $\sum_n 1/n^2$ converges, the series converges uniformly on any bounded subset of $\mathbb{R}$, implying that $h(x)$ is continuous everywhere. Thus, $h(x)$ is continuous on all of $\mathbb{R}$. $\square$

Solution to (ii). To check differentiability, we differentiate each term of the series to get

$$h'(x) = \sum_{n=1}^{\infty} -\frac{2x}{(x^2+n^2)^2}.$$

To check whether this series converges uniformly, we use the bound

$$\left|-\frac{2x}{(x^2+n^2)^2}\right| \leq \frac{2|x|}{n^4}.$$

The Weierstrass M-test applies again, and since $\sum_n 1/n^4$ converges, the series converges uniformly. Thus, $h(x)$ is differentiable on all of $\mathbb{R}$ and $h'(x)$ is continuous on all of $\mathbb{R}$. $\square$

Exercise 6.5.1. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

(i) Is g defined on $(-1,1)$? Is it continuous on this set? Is g defined on $(-1,1]$? Is it continuous on this set? What happens on $[-1,1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(ii) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution to (i). We're given the following Taylor Series

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = \ln(1+x), \text{ for } |x| < 1.$$

Convergence on $(-1,1)$: A power series $\sum a_n x^n$ converges absolutely inside its radius of convergence R, which is given by

$$R = \limsup_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

For this series, the ratio test gives

$$\left|\frac{(-1)^{n+1}\frac{x^n}{n}}{(-1)^{n+2}\frac{x^{n+1}}{n+1}}\right| = |\frac{n+1}{xn}|.$$

As $n \to \infty$, this approaches $\frac{1}{|x|}$. Therefore, the series converges absolutely for $|x| < 1$. Thus, $g(x)$ is defined on $(-1, 1)$.

At $x = 1$: The series becomes the alternating harmonic series

$$g(1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2),$$

which is conditionally convergent. Thus, $g(x)$ is defined at $x = 1$.

At $x = -1$: The series becomes

$$g(-1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}.$$

This series diverges, so $g(x)$ is not defined at $x = -1$.

Convergence for $|x| > 1$: For $|x| > 1$, the terms x^n/n grow too large, and the series diverges by the nth-term test. Thus, $g(x)$ is not defined for $|x| > 1$.

Therefore, $g(x)$ is defined on $(-1, 1]$ and continuous on $(-1, 1)$. It is not defined at $x = -1$ and diverges at $x = -1$. The power series for $g(x)$ converges for $|x| < 1$ and diverges for $|x| > 1$. □

Solution to (ii). To find $g'(x)$, we differentiate the power series term by term:

$$g'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nx^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

This is a geometric series. For $|x| < 1$, we recognize this as the Taylor series for

$$g'(x) = \frac{1}{1+x}.$$

Thus, $g'(x)$ is defined on $(-1, 1)$.

At $x = 1$: The series becomes the alternating series

$$g'(1) = \sum_{n=0}^{\infty} (-1)^n,$$

which does not converge. Thus, $g'(1)$ is not defined.

At $x = -1$: As $x \to -1^+$, the series $\sum (-1)^n x^n$ diverges. Thus, $g'(-1)$ is not defined.

Therefore, $g'(x)$ is defined on $(-1, 1)$, not defined at $x = \pm 1$, and the formula for $g'(x)$ is given by

$$g'(x) = \frac{1}{1+x}, \text{ for } |x| < 1.$$ □

Exercise 6.5.2. Find suitable coefficients (a_n) so that the resulting power series $\sum_n a_n x^n$ has the given properties, or explain why such a request is impossible.

(i) Converges for every value of $x \in \mathbb{R}$.

(ii) Diverges for every value of $x \in \mathbb{R}$.

(iii) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.

Solution to (i). Let

$$a_n = \frac{1}{n!}.$$

This results in the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

which converges for all $x \in \mathbb{R}$. □

Solution to (ii). Impossible, as $x = 0$ will always converge. □

Solution to (iii). Let

$$a_n = \frac{1}{n^2},$$

with the convention that $a_0 = 1$. This gives the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n^2}.$$

We first analyze the radius of convergence R. By the root test, we compute

$$\limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{1/n} = \limsup_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1,$$

since $\left(\frac{1}{n^2}\right)^{1/n} \to 1$ as $n \to \infty$. Thus, the radius of convergence is $R = 1$.

Therefore, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Next, we analyze what happens when $|x| = 1$. In this case, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^2},$$

which is a convergent p-series with $p = 2 > 1$. Hence, the series converges absolutely when $|x| = 1$.

Therefore, the series converges absolutely for all $x \in [-1, 1]$ and diverges off of this set. □

Exercise 6.5.8(i).

(i) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

Solution to (i). Define the sequence $c_n = a_n - b_n$, and consider the power series formed by their difference

$$\sum_{n=0}^{\infty} (a_n - b_n) x^n = \sum_{n=0}^{\infty} c_n x^n.$$

Since the original series are equal for all $x \in (-R, R)$, we have

$$\sum_{n=0}^{\infty} c_n x^n = 0,$$

for all $x \in (-R, R)$.

Thus, the function defined by this power series is identically zero on $(-R, R)$.

Recall that a power series defines an analytic function on its interval of convergence. Analytic functions are uniquely determined by their power series expansion. Moreover, if a power series converges to zero on an interval $(-R, R)$, all its coefficients must vanish.

Thus, since

$$\sum_{n=0}^{\infty} c_n x^n = 0,$$

for all $x \in (-R, R)$, it follows that

$$c_n = 0 \quad \text{for all } n \geq 0.$$

Since $c_n = a_n - b_n = 0$, we conclude that $a_n = b_n$ for all $n \geq 0$. □

Exercise 6.5.10. Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-R, R)$, and assume $(x_n) \to 0$ with $x_n \neq 0$. If $g(x_n) = 0$ for all $n \in \mathbb{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

Solution. Recall that a power series defines an analytic function. Since $g(x)$ is given by a power series converging on $(-R, R)$, it defines an analytic function on that interval. Analytic functions are infinitely differentiable and determined completely by their Taylor series at any point in their domain.

A key fact about analytic functions (also called the identity theorem) says: If an analytic function is zero on a set that has a limit point inside its domain, then the function is identically zero on its domain. In this case, we know the following

(i) $g(x)$ is analytic on $(-R, R)$,

(ii) $\{x_n \mid n \in \mathbf{n}\}$ is a set of points in $(-R, R)$,

(iii) $x_n \neq 0$, $x_n \to 0$, so 0 is a limit point of this set,

(iv) $g(x_n) = 0$ for all n.

Thus, by the identity theorem, $g(x) \equiv 0$ on $(-R, R)$.

Hence, $g(x) = 0$ for all $x \in (-R, R)$. Since $g(x) = \sum_{n=0}^{\infty} b_n x^n$, it follows that all coefficients must vanish, giving us $b_n = 0$, for all $n \geq 0$. $\square$