1. State the rank-nullity theorem for an $m \times n$ matrix with real entries.

Let $A \in \mathbb{R}^{m\times n}$. Then rank$(A)$ + nullity$(A) = n$.

2. Given that $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, find a basis for Null$(A)$ and a basis of Range(A).

1) A basis for Range(A) is given by $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ -5 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 5 \\ -2 \end{pmatrix} \right\}$

2). $\begin{pmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{9}R_3 \to R_3} \begin{pmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[-5R_3+R_1 \to R_1]{-8R_3+R_2 \to R_2} \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 5 & -7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{\frac{1}{5}R_2 \to R_2} \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow A\vec{x} = \vec{0} \iff \begin{aligned} x_1 + 2x_2 + 4x_4 &= 0 \\ x_3 - \tfrac{7}{5}x_4 &= 0 \\ x_5 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -2x_2 - 4x_4 \\ x_3 &= \tfrac{7}{5}x_4 \\ x_5 &= 0 \end{aligned}$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 - 4x_4 \\ x_2 \\ \tfrac{7}{5}x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ A basis for Null(A) is given by $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{pmatrix} \right\}$

3. Let A be an $m \times n$ matrix with real entires. Prove that $A = 0$ if and only if $\mathrm{Tr}(A^T A) = 0$.

Denote $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ $A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$

Denote $A^T A = (c_{ij})_{n \times n}$

Then $c_{ij} = \underbrace{(a_{1i} \ a_{2i} \ \cdots \ a_{mi})}_{\text{the i-th row of } A^T} \overbrace{\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}}^{\text{the j-th column of } A}$

$= a_{1i}a_{1j} + a_{2i}a_{2j} + \cdots + a_{mi}a_{mj}$

In particular $c_{kk} = a_{1k}a_{1k} + a_{2k}a_{2k} + \cdots + a_{mk}a_{mk}$

$= a_{1k}^2 + a_{2k}^2 + \cdots + a_{mk}^2 = \sum_{l=1}^{m} a_{lk}^2$

$\mathrm{Tr}(A^T A) = \sum_{k=1}^{n} c_{kk} = \sum_{k=1}^{n} \left(\sum_{l=1}^{m} a_{lk}^2 \right) = 0 \iff a_{lk} = 0$ for all $l = 1, \cdots, m$ and $k = 1, \cdots, k$.

$\iff A = 0$. ■

4. True or False. No explanation needed.

F 1). If A is a 3×3 matrix, then $\det(3A) = 9 \det A$.

F 2). If A, B are invertible $n \times n$ matrices, then $[(AB)^T]^{-1} = (B^T)^{-1}(A^T)^{-1}$.

$= (B^T A^T)^{-1}$

$= (A^T)^{-1} (B^T)^{-1}$