Introduction to Abstract Algebra I: Homework 9

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Exercise 12.32. Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$.

Solution. Let $a \in G$. Since H is normal in G, the left cosets of H in G are the same as the right cosets. The index $m = (G : H)$ represents the number of distinct cosets of H in G. Therefore, the cosets can be represented as $H, aH, a^2H, \ldots, a^{m-1}H$. Since there are m distinct cosets, we have $a^mH = H$. This implies that $a^m \in H$. Thus, for every $a \in G$, we have $a^m \in H$. $\square$

Exercise 12.37. Show that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

Solution. Let H and N be subgroups of a group G, and let N be normal in G. We need to show that $H \cap N$ is normal in H. Take any $h \in H$ and any $x \in H \cap N$. Since $x \in N$ and N is normal in G, we have

$$hxh^{-1} \in N.$$

Additionally, since $h, x \in H$ and H is a subgroup, we have

$$hxh^{-1} \in H.$$

Therefore, we have $hxh^{-1} \in H \cap N$. Thus, $H \cap N$ is normal in H.

For an example where $H \cap N$ is not normal in G, consider the group $G = S_3$, the symmetric group on 3 elements. Let $H = \langle (1), (12) \rangle$ and let $N = A_3 = \langle (1), (123), (132) \rangle$. Here, N is normal in G, but the intersection $H \cap N = \langle (1) \rangle$ is not normal in G, since conjugating (12) by (123) gives (13), which is not in H. Thus, this example shows that $H \cap N$ need not be normal in G. $\square$

Exercise 12.39.

(i) Show that all automorphisms of a group G form a group under function composition.

(ii) Show that the inner automorphisms of a group G form a normal subgroup of the group of all automorphisms of G under function composition. [Warning: Be sure to show that the inner automorphisms do form a subgroup.]

Solution to (i). The identity automorphism ι_G defined by $\iota_G(a) = a$ for all $a \in G$ is in $\mathrm{Aut}(G)$, serving as the identity element. For any $\varphi \in \mathrm{Aut}(G)$, its inverse φ^{-1} is also an automorphism since it is a bijection and satisfies the homomorphism property. Therefore, every element in $\mathrm{Aut}(G)$ has an inverse in $\mathrm{Aut}(G)$.

Next, we show closure. Let $\mathrm{Aut}(G)$ be the collection of all automorphisms of G. Take $\varphi, \psi \in \mathrm{Aut}(G)$. We need to show that the composition $\varphi \circ \psi \in \mathrm{Aut}(G)$. By function composition, we have

$$(\varphi \circ \psi)(ab) = \varphi(\psi(ab)) = \varphi(\psi(a)\psi(b)) = \varphi(\psi(a))\varphi(\psi(b)) = (\varphi \circ \psi)(a)(\varphi \circ \psi)(b).$$

Thus, $\varphi \circ \psi$ is a homomorphism. Since both φ and ψ are bijections, their composition is also a bijection. Therefore, $\varphi \circ \psi$ is an automorphism of G. Thus, $\mathrm{Aut}(G)$ is closed under function composition.

Next, we show that it contains inverses. Since $\varphi \in \mathrm{Aut}(G)$ is an automorphism (an isomorphism from G to G), there exists an inverse, φ^{-1} that's also an automorphism. From this, we get

$$(\varphi \circ \varphi^{-1})(x) = \varphi(\varphi^{-1}(x)) = \iota_G(x).$$

Therefore, $\mathrm{Aut}(G)$ contains all its inverses.

Lastly, we show associativity. Given $\varphi, \psi, \rho \in \mathrm{Aut}(G)$, we have

$$(\varphi \circ \psi) \circ \rho = \varphi \circ (\psi \circ \rho).$$

Therefore, $\mathrm{Aut}(G)$ is associative.

Thus, $\mathrm{Aut}(G)$ is a group under function composition. $\square$

Solution to (ii). Clearly, $\mathrm{Inn}(G)$ is non-empty, since $\iota_e(x) = exe^{-1} = e$ for all $x \in G$, where e is the identity element of G. Thus, $\iota_e \in \mathrm{Inn}(G)$.

Next, we show that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ is closed. Take $\iota_a, \iota_b \in \mathrm{Inn}(G)$. Then, we have

$$(\iota_a \circ \iota_b)(x) = \iota_a(\iota_b(x)) = \iota_a(bxb^{-1}) = \iota_a(b)\iota_a(x)\iota_a(b^{-1}) = aba^{-1}xab^{-1}a^{-1} = \iota_{ab}(x).$$

Therefore, $\iota_a \circ \iota_b \in \mathrm{Inn}(G)$, showing closure.

Lastly, we show that $\mathrm{Inn}(G)$ contains inverses. Take $\iota_a \in \mathrm{Inn}(G)$. Then, we have

$$\iota_a^{-1}(x) = a^{-1}xa = \iota_{a^{-1}}(x).$$

Therefore, $\iota_a^{-1} \in \mathrm{Inn}(G)$, showing that $\mathrm{Inn}(G)$ contains inverses.

Thus, $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$.

Lastly, we show that $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$. Take $\varphi \in \mathrm{Aut}(G)$ and $\iota_a \in \mathrm{Inn}(G)$. Then, we have

$$(\varphi \circ \iota_a \circ \varphi^{-1})(x) = \varphi(\iota_a(\varphi^{-1}(x))) = \varphi(a\varphi^{-1}(x)a^{-1}) = \varphi(a)x\varphi(a)^{-1} = \iota_{\varphi(a)}(x).$$

Therefore, $\varphi \circ \iota_a \circ \varphi^{-1} \in \mathrm{Inn}(G)$. Thus, $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$. □

Exercise 13.12. Classify the group $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(3,3,3)\rangle$ according to the fundamental theorem of finitely generated abelian groups.

Solution. Here, we have the group $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and the subgroup $N = \langle(3,3,3)\rangle$. The subgroup N is generated by the element $(3,3,3)$, which can be expressed as $N = \{(3k,3k,3k) \mid k \in \mathbb{Z}\}$. Take the homomorphism $\varphi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3$ defined by

$$\varphi(a,b,c) = (a-c, b-c, c \mod 3).$$

Clearly, φ is a surjective homomorphism. The kernel of φ is given by

$$\ker(\varphi) = \{(a,b,c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mid a-c=0, b-c=0, c \equiv 0 \mod 3\} = \{(3k,3k,3k) \mid k \in \mathbb{Z}\} = N.$$

By the First Isomorphism Theorem, we have

$$(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/\langle(3,3,3)\rangle \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_3.$$ □

Exercise 13.14. Classify the group $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2)/\langle(1,1,1)\rangle$ according to the fundamental theorem of finitely generated abelian groups.

Solution. Here, we have the group $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ and the subgroup $N = \langle(1,1,1)\rangle$. The subgroup N is generated by the element $(1,1,1)$, which can be expressed as $N = \{(k,k,k \mod 2) \mid k \in \mathbb{Z}\}$. Take the homomorphism $\varphi : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z} \times \mathbb{Z}_2$ defined by

$$\varphi(a,b,c) = (a-b, (b-c) \mod 2).$$

Clearly, φ is a surjective homomorphism. The kernel of φ is given by

$$\ker(\varphi) = \{(a,b,c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \mid a-b=0, b-c \equiv 0 \mod 2\} = \{(k,k,k \mod 2) \mid k \in \mathbb{Z}\} = N.$$

By the First Isomorphism Theorem, we have

$$(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2)/\langle(1,1,1)\rangle \cong \mathbb{Z} \times \mathbb{Z}_2.$$ □

Exercise 13.16. Find both the center and the commutator subgroup of $\mathbb{Z}_3 \times S_3$.

Solution. The center of the group $\mathbb{Z}_3 \times S_3$ is given by

$$Z(\mathbb{Z}_3 \times S_3) = Z(\mathbb{Z}_3) \times Z(S_3) = \mathbb{Z}_3 \times \{\iota\} \cong \mathbb{Z}_3.$$

The commutator of the group $\mathbb{Z}_3 \times S_3$ is given by

$$(\mathbb{Z}_3 \times S_3)' = \mathbb{Z}_3' \times S_3' = \{e\} \times A_3 \cong A_3.$$ □

Exercise 13.17. Find both the center and the commutator subgroup of $S_3 \times D_4$.

Solution. The center of the group $S_3 \times D_4$ is given by

$$Z(S_3 \times D_4) = Z(S_3) \times Z(D_4) = \{\iota\} \times \{e, r^2\} \cong \mathbb{Z}_2.$$

The commutator of the group $S_3 \times D_4$ is given by

$$(S_3 \times D_4)' = S_3' \times D_4' = A_3 \times \{e, r^2\} \cong \mathbb{Z}_3 \times \mathbb{Z}_2.$$ □

Exercise 13.37. Let $\varphi : G \to G'$ be a group homomorphism, and let N be a normal subgroup of G. Show that $\varphi[N]$ is a normal subgroup of $\varphi[G]$.

Solution. Let $\varphi : G \to G'$ be a group homomorphism, and let N be a normal subgroup of G. First, we show that $\varphi[N]$ is a subgroup of $\varphi[G]$. Since N is a subgroup of G, for any $n_1, n_2 \in N$, we have $n_1 n_2^{-1} \in N$. Applying the homomorphism φ, we get

$$\varphi(n_1 n_2^{-1}) = \varphi(n_1)\varphi(n_2)^{-1} \in \varphi[N].$$

Thus, $\varphi[N]$ is closed under the group operation and contains inverses, making it a subgroup of $\varphi[G]$.

Next, we show that $\varphi[N]$ is normal in $\varphi[G]$. Take any $g' \in \varphi[G]$ and $n' \in \varphi[N]$. There exist $g \in G$ and $n \in N$ such that $\varphi(g) = g'$ and $\varphi(n) = n'$. Since N is normal in G, we have $gng^{-1} \in N$. Applying the homomorphism φ, we get

$$\varphi(gng^{-1}) = \varphi(g)\varphi(n)\varphi(g)^{-1} = g'n'(g')^{-1} \in \varphi[N].$$

Thus, $\varphi[N]$ is normal in $\varphi[G]$. □

Exercise 13.38. Let $\varphi : G \to G'$ be a group homomorphism, and let N' be a normal subgroup of G'. Show that $\varphi^{-1}[N']$ is a normal subgroup of G.

Solution. Let $\varphi : G \to G'$ be a group homomorphism, and let N' be a normal subgroup of G'. First, we show that $\varphi^{-1}[N']$ is a subgroup of G. For any $a, b \in \varphi^{-1}[N']$, we have $\varphi(a), \varphi(b) \in N'$. Since N' is a subgroup of G', we have $\varphi(a)\varphi(b)^{-1} \in N'$. Applying the inverse homomorphism, we get

$$\varphi^{-1}(\varphi(a)\varphi(b)^{-1}) = ab^{-1} \in \varphi^{-1}[N'].$$

Thus, $\varphi^{-1}[N']$ is closed under the group operation and contains inverses, making it a subgroup of G.

Next, we show that $\varphi^{-1}[N']$ is normal in G. Take any $g \in G$ and $n \in \varphi^{-1}[N']$. There exists $n' \in N'$ such that $\varphi(n) = n'$. Since N' is normal in G', we have $\varphi(g)n'\varphi(g)^{-1} \in N'$. Applying the inverse homomorphism, we get

$$\varphi^{-1}(\varphi(g)n'\varphi(g)^{-1}) = gng^{-1} \in \varphi^{-1}[N'].$$

Thus, $\varphi^{-1}[N']$ is normal in G. □

Exercise 13.39. Show that if G is nonabelian, then the factor group $G/Z(G)$ is not cyclic. [Hint: Show the equivalent contrapositive, namely, that if $G/Z(G)$ is cyclic then G is abelian (and hence $Z(G) = G$).]

Solution. Assume $G/Z(G)$ is cyclic. Then, there exists an element $gZ(G) \in G/Z(G)$ such that every element of $G/Z(G)$ can be written as $(gZ(G))^n$ for some integer n. This means that for any $a \in G$, there exists an integer n such that

$$aZ(G) = (gZ(G))^n = g^n Z(G).$$

Therefore, we can express a as $a = g^n z$, for some $z \in Z(G)$. Now, take any two elements $a, b \in G$. We can write them as $a = g^n z_1$ and $b = g^m z_2$ for some integers n, m and $z_1, z_2 \in Z(G)$. Then, we have

$$ab = (g^n z_1)(g^m z_2) = g^{n+m} z_1 z_2.$$

Similarly, we have

$$ba = (g^m z_2)(g^n z_1) = g^{m+n} z_2 z_1.$$

Since $z_1, z_2 \in Z(G)$, they commute with all elements of G, including each other. Thus, we have $z_1 z_2 = z_2 z_1$. Therefore, we get

$$ab = g^{n+m} z_1 z_2 = g^{m+n} z_2 z_1 = ba.$$

Hence, G is abelian. Thus, if $G/Z(G)$ is cyclic, then G is abelian. The contrapositive statement is that if G is nonabelian, then $G/Z(G)$ is not cyclic. $\square$