

Introduction to Statistics I: Homework 6

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Problem 1. A system consisting of one original unit plus a spare can function for a random amount of time X . If the probability density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

what is the probability that the system functions for at least 5 months?

Solution. To determine the normalizing constant, we first set

$$\int_0^\infty Cxe^{-x/2} dx = 1.$$

To evaluate the integral, it is convenient to use Feynman's technique (differentiation under the integral sign). Define

$$I(\alpha) = \int_0^\infty xe^{-\alpha x} dx.$$

Differentiating with respect to α under the integral sign gives

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial \alpha} (xe^{-\alpha x}) dx = - \int_0^\infty x^2 e^{-\alpha x} dx.$$

The last integral can be computed by integrating by parts twice, or by recalling the standard Gamma-type formula, resulting in

$$\int_0^\infty x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3}.$$

Hence

$$\frac{dI}{d\alpha} = -\frac{2}{\alpha^3}.$$

Integrating with respect to α gives

$$I(\alpha) = \int -\frac{2}{\alpha^3} d\alpha = \frac{1}{\alpha^2} + C_1$$

To determine C_1 , observe that as $\alpha \rightarrow \infty$ the integrand $xe^{-\alpha x}$ tends pointwise to 0 and is dominated by an integrable function, so $I(\alpha) \rightarrow 0$. Therefore $C_1 = 0$, and we have

$$I(\alpha) = \frac{1}{\alpha^2}.$$

Now the normalizing constant can be found from

$$\int_0^\infty Cxe^{-x/2} dx = CI(1/2) = C \cdot \frac{1}{(1/2)^2} = 4C,$$

so $4C = 1$ and therefore $C = 1/4$. The density is thus

$$f(x) = \frac{1}{4}xe^{-x/2}, \quad x > 0.$$

To compute the probability that the system functions for at least five months, we evaluate

$$P(X \geq 5) = \int_5^\infty \frac{1}{4}xe^{-x/2} dx.$$

An antiderivative of $xe^{-x/2}$ is $-2e^{-x/2}(x+2)$, so

$$\int_5^\infty xe^{-x/2} dx = \left[-2e^{-x/2}(x+2) \right]_5^\infty = 14e^{-5/2}.$$

Thus

$$P(X \geq 5) = \frac{1}{4} \cdot 14e^{-5/2} = \frac{7}{2}e^{-5/2}.$$

Thus, we have around a 28.7% chance the system lasts at least five months. \square

Problem 2. The probability density function of X , the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10, \\ 0 & x \leq 10. \end{cases}$$

- (i) Find $P\{X > 15\}$.
- (ii) What is the cumulative distribution function of X ?
- (iii) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours?
What assumptions are you making?

Solution to (i). The density is normalized since

$$\int_{10}^{\infty} \frac{10}{x^2} dx = 10 \left[-\frac{1}{x} \right]_{10}^{\infty} = 10 \left(0 - \left[-\frac{1}{10} \right] \right) = 1,$$

so it is a valid pdf. The tail probability for $x > 10$ is

$$P(X > x) = \int_x^{\infty} \frac{10}{t^2} dt = 10 \left[-\frac{1}{t} \right]_x^{\infty} = \frac{10}{x}.$$

Hence

$$P\{X > 15\} = \frac{10}{15} = \frac{2}{3}.$$

Thus there is a 66.67% chance that a device lasts more than 15 hours. \square

Solution to (ii). The cumulative distribution function is obtained by integrating the pdf from the left endpoint of the support. For $x \leq 10$ we have $F(x) = 0$. For $x > 10$,

$$F(x) = \int_{10}^x \frac{10}{t^2} dt = 10 \left[-\frac{1}{t} \right]_{10}^x = 1 - \frac{10}{x}.$$

Thus the CDF is

$$F(x) = \begin{cases} 0, & x \leq 10, \\ 1 - \frac{10}{x}, & x > 10, \end{cases}$$

which satisfies $F(10^+) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$. \square

Solution to (iii). Let a “success” be the event that a device lasts at least 15 hours. From part (i) the single-device success probability is $p = P(X \geq 15) = 2/3$. Assuming the six devices are independent and identically distributed, the number of successes among the six is Binomial($n = 6, p = 2/3$). Therefore the probability that at least three of the six last at least 15 hours is

$$\sum_{k=3}^6 \binom{6}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{6-k} = \frac{656}{729}.$$

The assumptions are that each device’s lifetime follows the given pdf and that the six lifetimes are independent. \square

Problem 3. Compute $E[X]$ if X has the probability density function given by

$$(i) \quad \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$(ii) \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Recall from Problem 1 that for $\alpha > 0$

$$I(\alpha) = \int_0^\infty xe^{-\alpha x} dx = \frac{1}{\alpha^2}.$$

Differentiate $I(\alpha)$ with respect to α under the integral sign:

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial \alpha} (xe^{-\alpha x}) dx = - \int_0^\infty x^2 e^{-\alpha x} dx.$$

Hence

$$\int_0^\infty x^2 e^{-\alpha x} dx = - \frac{d}{d\alpha} \left(\frac{1}{\alpha^2} \right) = \frac{2}{\alpha^3}.$$

For the expectation in Problem 3(i) we need the case $\alpha = 1/2$:

$$\int_0^\infty x^2 e^{-x/2} dx = \frac{2}{(1/2)^3} = 16,$$

and therefore

$$E[X] = \frac{1}{4} \int_0^\infty x^2 e^{-x/2} dx = \frac{1}{4} \cdot 16 = 4.$$

Thus, the expected value is $E[X] = 4$. □

Solution to (ii). We require

$$1 = \int_{-1}^1 c(1-x^2) dx = c \left[x - \frac{x^3}{3} \right]_{-1}^1 = \frac{4c}{3},$$

so $c = 3/4$. The expectation is

$$E[X] = \int_{-1}^1 x \cdot \frac{3}{4}(1-x^2) dx = \frac{3}{4} \int_{-1}^1 x(1-x^2) dx.$$

The integrand is odd and the interval is symmetric, hence the integral vanishes and $E[X] = 0$. □

Problem 4. The lifetime in hours of an electronic tube is a random variable having a probability density function given by $f(x) = xe^{-x}$, $x \geq 0$. Compute the expected lifetime of such a tube.

Solution. Recall from Problem 1 that

$$I(\alpha) = \int_0^\infty xe^{-\alpha x} dx = \frac{1}{\alpha^2}, \quad \alpha > 0.$$

Differentiate $I(\alpha)$ under the integral sign:

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial \alpha} (xe^{-\alpha x}) dx = - \int_0^\infty x^2 e^{-\alpha x} dx.$$

Hence

$$\int_0^\infty x^2 e^{-\alpha x} dx = - \frac{d}{d\alpha} \left(\frac{1}{\alpha^2} \right) = \frac{2}{\alpha^3}.$$

For the given density we have $\alpha = 1$, so

$$E[X] = \int_0^\infty x^2 e^{-x} dx = \frac{2}{1^3} = 2.$$

Also, you could have noticed that the integral is just the definition of the Gamma function at 3, i.e.,

$$E[X] = \int_0^\infty x^2 e^{-x} dx = \Gamma(3) = 2!,$$

but this way is way less fun. \square

Problem 5. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute the following probabilities.

$$(i) P\{X < 20\}$$

$$(ii) P\{4 < X < 12\}$$

Solution to (i). Since $X \sim N(10, 36)$, we standardize using

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 10}{6}, \quad Z \sim N(0, 1).$$

Then

$$P\{X < 20\} = P\left\{\frac{X - 10}{6} < \frac{20 - 10}{6}\right\} = P\{Z < 10/6\} = P\{Z < 5/3\}.$$

Thus the answer is

$$P\{X < 20\} = \Phi\left(\frac{5}{3}\right),$$

where Φ denotes the standard normal CDF, which is given by

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

Now, let's try and find a numerical approximation for $\Phi(5/3)$. Using Feynman's technique (once again, this technique is super useful), define

$$J(\alpha) = \int_0^a e^{-\alpha t^2} dt, \quad \alpha > 0.$$

Differentiate under the integral sign:

$$\frac{dJ}{d\alpha} = - \int_0^a t^2 e^{-\alpha t^2} dt.$$

Instead of solving this differential equation directly, note the standard Gaussian integral on $[0, \infty)$:

$$\int_0^\infty e^{-\alpha t^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}.$$

Using the substitution $u = t\sqrt{\alpha}$ in the finite integral gives the closed form

$$J(\alpha) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \operatorname{erf}(a\sqrt{\alpha}),$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

Put $\alpha = 1/2$ and recall the relationship between Φ and the integral from 0 to a :

$$\Phi(a) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^a e^{-t^2/2} dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} J(1/2).$$

Hence

$$\Phi(a) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{a}{\sqrt{2}}\right) = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{a}{\sqrt{2}}\right)\right).$$

Now, we can expand $e^{-t^2/2}$ as

$$e^{-t^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} t^{2n},$$

and integrate termwise to get

$$\Phi(a) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{a^{2n+1}}{2n+1},$$

which is a perfectly valid Feynman-style series representation you can use for hand computations (it converges rapidly for moderate a).

Using the erf representation,

$$\Phi\left(\frac{5}{3}\right) = \frac{1}{2} \left(1 + \text{erf}\left(\frac{5}{3\sqrt{2}}\right)\right) \approx 0.9522096477.$$

Thus, there is around a 95.22% chance that X is less than 20. \square

Solution to (ii). Again using the standardization $Z = (X - 10)/6$, we have

$$P\{4 < X < 12\} = P\left\{\frac{4 - 10}{6} < Z < \frac{12 - 10}{6}\right\} = P\left\{-1 < Z < \frac{1}{3}\right\}.$$

Therefore

$$P\{4 < X < 12\} = \Phi\left(\frac{1}{3}\right) - \Phi(-1),$$

where Φ is the standard normal CDF (I'm not computing this again). \square

Problem 6. Suppose that X is a normal random variable with mean 8. If $P\{X > 14\} = 0.2$ approximately what is $\text{Var}(X)$?

Solution. Since X is normal with mean $\mu = 8$ and unknown variance σ^2 , we standardize:

$$P\{X > 14\} = P\left\{\frac{X - 8}{\sigma} > \frac{14 - 8}{\sigma}\right\} = P\left\{Z > \frac{6}{\sigma}\right\},$$

where $Z \sim N(0, 1)$. We are given that this probability equals 0.2, so

$$P\{Z > 6/\sigma\} = 0.2.$$

This is equivalent to

$$\Phi\left(\frac{6}{\sigma}\right) = 0.8.$$

From standard normal tables, we have

$$\Phi^{-1}(0.8) \approx 0.841621.$$

Therefore

$$\frac{6}{\sigma} = 0.841621 \Rightarrow \sigma = \frac{6}{0.841621} \approx 7.128.$$

Hence

$$\text{Var}(X) = \sigma^2 \approx (7.128)^2 \approx 50.8.$$

Thus, the variance is approximately 50.8. \square

Problem 7. Every day Jo practices her tennis serve by continually serving until she has had a total of 50 successful serves. If each of her serves is, independently of previous serves, successful with probability 0.4, approximately what is the probability that she will need more than 100 serves to accomplish her goal? Use all possible improvements.

Solution. Let N be the total number of serves Jo needs to obtain 50 successes. Equivalently, after 100 serves she will still need more than 0 additional serves precisely when the number of successes in the first 100 serves is at most 49. If S_{100} denotes the number of successes in 100 independent Bernoulli($p = 0.4$) trials, then

$$P\{N > 100\} = P\{S_{100} \leq 49\},$$

and $S_{100} \sim \text{Binomial}(100, 0.4)$.

The binomial has mean and variance

$$\mu := E[S_{100}] = 100 \cdot 0.4 = 40 \quad \text{and} \quad \sigma^2 := \text{Var}(S_{100}) = 100 \cdot 0.4 \cdot 0.6 = 24,$$

so $\sigma = \sqrt{24} \approx 4.8989795$. Using the normal approximation with the continuity correction (the usual improvement), we approximate

$$P\{S_{100} \leq 49\} \approx \Phi\left(\frac{49.5 - \mu}{\sigma}\right) = \Phi\left(\frac{49.5 - 40}{\sqrt{24}}\right) = \Phi(1.93899\dots),$$

where Φ is the standard normal CDF. Numerically,

$$\Phi(1.93899\dots) \approx 0.97376.$$

For comparison, the exact binomial tail evaluates to

$$P\{S_{100} \leq 49\} = \sum_{k=0}^{49} \binom{100}{k} 0.4^k 0.6^{100-k} \approx 0.9729008,$$

so the continuity-corrected normal approximation is very accurate. Therefore the probability that Jo needs more than 100 serves is approximately 97.38%. \square

Problem 8. A fire station is to be located on a road of infinite length stretching from point 0 outward to ∞ . If the distance of a fire from point 0 is exponentially distributed with rate λ , where should the fire station be located? That is, we want to minimize $E[|X - a|]$, where X is exponential with rate λ .

Solution. To minimize $g(a) := E[|X - a|]$ it is convenient to use the standard fact (which we derive briefly) that any minimizer of $E|X - a|$ is a median of the distribution of X . For a continuous X one may differentiate under the expectation: since

$$\frac{\partial}{\partial a} |x - a| = -\text{sgn}(x - a),$$

dominated convergence gives

$$g'(a) = \frac{d}{da} E[|X - a|] = -E[\text{sgn}(X - a)] = -(P\{X > a\} - P\{X < a\}) = 2F_X(a) - 1,$$

where F_X is the CDF of X . Thus $g'(a) = 0$ iff $F_X(a) = 1/2$. Hence, any minimizer of $g(a)$ is a median of X .

For $X \sim \text{Exponential}(\lambda)$ we have

$$F_X(a) = 1 - e^{-\lambda a}, \quad a \geq 0.$$

Setting $F_X(a) = 1/2$ gives $1 - e^{-\lambda a} = 1/2$, so

$$e^{-\lambda a} = 1/2 \Rightarrow a = \frac{\ln(2)}{\lambda}.$$

Therefore the fire station should be located at $a = \ln(2)/\lambda$, the median of the exponential distribution.

Writing $m = \ln(2)/\lambda$ and evaluating

$$E[|X - m|] = \int_0^m (m - x) \lambda e^{-\lambda x} dx + \int_m^\infty (x - m) \lambda e^{-\lambda x} dx,$$

and using $e^{-\lambda m} = 1/2$, we get

$$E[|X - m|] = m - \frac{1}{\lambda} + \frac{2e^{-\lambda m}}{\lambda} = m.$$

Thus at the optimal location the minimal expected distance equals the median, i.e.,

$$\min_{a \geq 0} E[|X - a|] = \frac{\ln(2)}{\lambda}. \quad \square$$

Problem 9. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is the conditional probability that a repair takes at least 11 hours, given that its duration exceeds 9 hours?

Solution. If X is exponentially distributed with rate $\lambda = 1/2$, then

$$P\{X > x\} = e^{-\lambda x} = e^{-x/2}.$$

We want

$$P\{X \geq 11 \mid X > 9\} = \frac{P\{X \geq 11\}}{P\{X > 9\}}.$$

Using the exponential tail formula,

$$P\{X \geq 11\} = e^{-11/2} \quad \text{and} \quad P\{X > 9\} = e^{-9/2}..$$

Thus

$$P\{X \geq 11 \mid X > 9\} = \frac{e^{-11/2}}{e^{-9/2}} = e^{-1} = \frac{1}{e}.$$

This is exactly the memoryless property of the exponential distribution: after waiting 9 hours, the additional time until completion is still exponential with the same rate. \square

Problem 10. If X is uniformly distributed over the interval $(0, 3)$, find the probability density function for $Y = e^X$

Solution. Since X is uniformly distributed on $(0, 3)$, its density is

$$f_X(x) = \frac{1}{3}, \quad 0 < x < 3.$$

We define $Y = e^X$. Because the function $y = e^x$ is strictly increasing, we may use the standard change-of-variables formula

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|,$$

where $g(x) = e^x$ and hence $g^{-1}(y) = \ln(y)$.

The range of Y is determined by the range of X :

$$X \in (0, 3) \Rightarrow Y \in (e^0, e^3) = (1, e^3).$$

Thus,

$$f_Y(y) = f_X(\ln y) \left| \frac{d}{dy} (\ln y) \right| = \frac{1}{3} \cdot \frac{1}{y}, \quad 1 < y < e^3..$$

Therefore, the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{3y}, & 1 < y < e^3, \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

Problem 11. Show that if a random variable X has pdf $f_X(x)$ then $Y = aX + b$ has pdf $f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$, where a and b are real-valued constants.

Solution. Let X be a continuous random variable with pdf $f_X(x)$ and let $Y = aX + b$ for real constants a and b with $a \neq 0$. We derive the pdf of Y from its distribution function. For any $y \in \mathbb{R}$,

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y).$$

If $a > 0$ then $aX + b \leq y$ is equivalent to $X \leq (y - b)/a$, so

$$F_Y(y) = P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right).$$

Differentiating with respect to y and using the chain rule gives

$$f_Y(y) = \frac{d}{dy}F_Y(y) = F'_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \frac{1}{a}f_X\left(\frac{y-b}{a}\right).$$

If $a < 0$ then $aX + b \leq y$ is equivalent to $X \geq (y - b)/a$ (the inequality reverses), so

$$F_Y(y) = P\left(X \geq \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right).$$

Differentiating yields

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -F'_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right),$$

since $-1/a = 1/|a|$ when $a < 0$.

Combining both cases, we have shown that for any $a \neq 0$,

$$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right). \quad \square$$