Midterm 2 Solution **Fall 2024**

1. (4 pts) State the definition of the limit point of a set in $\mathbb{R}$.

   Solution. Let A be a set in $\mathbb{R}$. x is a limit point of A if, for every $\varepsilon > 0$, the neighborhood $V_\varepsilon(x)$ contains a element $y \in A$ and $y \neq x$. □

2. (4 pts each) Give an example for each statement and provide a short justification:

   (i) A sequence that does not contain a convergence subsequence.

   Solution. The sequence $\mathbb{N} = \{1, 2, 3, \ldots\}$ is unbounded and does not contain a convergence subsequence. □

   (ii) A conditional convergence infinite series (convergence but not absolutely).

   Solution. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test, but the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

   (iii) A convergence sequence that is not monotone.

   Solution. Consider $a_n = \frac{(-1)^n}{n}$. The sequence converges to zero but alternates from positive to negative so it is not monotone. □

   (iv) A set S in $\mathbb{R}$ contains exactly three limit points, none belong to the set.

   Solution. Here is one example: $\{1 + \frac{1}{n+1}, 2 + \frac{1}{n+1}, 3 + \frac{1}{n+1} : n \in \mathbb{N}\}$. □

3. (10 pts) Let s be a limit point of a set S. Prove that every ε-neighborhood $V_\varepsilon(s)$ contains infinitely many points in S and none equal to s.

   Solution. By definition, every ε-nbd of s contains an element $s_1 \neq s$ and $s_1 \in S$. Let $0 < \varepsilon_1 < \min\{\varepsilon, |s - s_1|\}$. There exists $s_2 \in s$ in $V_{\varepsilon_1}(s) \subset V_\varepsilon(s)$, so that $s_2 \neq s$. Assume we have found n elements $s_1, s_2, \ldots, s_m$ in S that in $V_\varepsilon(s)$. Choose $\varepsilon_m > 0$ satisfying $\varepsilon_m < \min\{\varepsilon, |s - s_1|, \ldots, |s - s_m|\}$. There is an element s_{m+1} in $V_{\varepsilon_m}(s)$ and $s_{m+1} \neq s$. By induction, this shows that there are infinitely many elements $\{s_n\}$ in $V_\varepsilon(s)$.

   Note. Several students used "Theorem. s is a limit point of S if and only if $s = \lim s_n$ for some sequence $\{s_n\}$ in S satisfying $s_n \neq s$ for all n." Then state simply that $V_\varepsilon(s)$ must contain infinite elements because s_n is an infinite sequence. This needs explanation: it is infinite because the condition $s_n \neq s$ for all n. Indeed, assume there are only finite many elements of S in $V_\varepsilon(s)$, call them $s_{n_1}, s_{n_2}, \ldots, s_{n_m}$, none equal to s, so that $|s_{n_i} - s| > 0$. Let $\varepsilon_0 > 0$ satisfies $\varepsilon_0 < \min\{|s_{n_1} - s|, |s_{n_2} - s|, \ldots, |s_{n_m} - s|\}$. Then the neighborhood $V_{\varepsilon_0}(s)$ contains no point in S other than possibly s itself, which contradicts to s being a limit point.

   The condition $s_n \neq s$ for all n is need since, otherwise, consider the sequece $\{(-1)^n\}$, or $s_{2n} = 1$ and $s_{2n-1} = -1$. Every nbd $V_\varepsilon(1)$ contains infinitely many elements $s_{2n} = 1$, but 1 is not a limit point. □

4. (10 pts) (a) Using the definition of open sets, prove that $(1,7)=\{x\in\mathbb{R}:1<x<7\}$ is open.

   Solution. Let $x\in(1,7)$. We need to find a neighborhood $V_\varepsilon(x)$ in $(1,7)$. Let $0<\varepsilon<\min\{x-1,7-x\}$. Then $V_\varepsilon(x)$ is a subset of $(1,7)$. Hence, $(1,7)$ is open. $\square$

   (b) Prove that every sequence in $[-1,3]$ contains a convergence subsequence whose limit is in $[-1,3]$.

   Solution. Let $\{x_n\}$ be a sequence in $[-1,3]$. Then it is a bounded sequence. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergence subsequence, $x_{n_k}\to x$, and x is a limit point of $[-1,3]$. Since $[-1,3]$ is closed, $x\in[-1,3]$. $\square$

5. (10 pts) Do **ONLY ONE** of the following problems.

   (a) Assume $b_n\geq 0$ and $\sum\limits_{n=0}^{\infty} b_n$ converges. If $a_n\geq 0$ and $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=r>0$, prove $\sum\limits_{n=0}^{\infty} a_n$ converges. [Hint: one example is $b_n=\frac{1}{n^2}$, for which $\frac{a_n}{b_n}=n^2a_n$]

   Solution. Let $\varepsilon=r/2>0$. since $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=r$, there is $N\in\mathbb{N}_0$ such that, for $n>N$,

$$\left|\frac{a_n}{b_n}-r\right|\leq\frac{r}{2}\qquad\Leftrightarrow\qquad -\frac{r}{2}<\frac{a_n}{b_n}-r\leq\frac{r}{2},$$

   which shows, in particular, for $n>N$,

$$\frac{a_n}{b_n}<r+\frac{r}{2}=\frac{3r}{2}\qquad\Leftrightarrow\qquad a_n<\frac{3r}{2}b_n.$$

   For $n\geq m\geq N$, this implies

$$\sum_{k=m+1}^{n} a_n\leq\frac{3r}{2}\sum_{k=m+1}^{n} b_n.$$

   This inequality allows us to deduce the convergence of $\sum a_n$ from that of $\sum b_n$ by the Cauchy criterion. $\square$

   (b) Prove the MCT: A bounded increasing sequence must converge.

   Solution. Let $\{a_n\}$ be a bounded increasing sequence. By the Axiom of Completeness, the least upper bound $a=\sup\{a_n\}$ exists. Let $\varepsilon>0$. Then $a-\varepsilon$ is no longer an upper bound. Hence, there is an $N\in\mathbb{N}_0$ such that $a_N>a-\varepsilon$. Since $\{a_n\}$ is increasing, for all $n>N$, we have

$$a-\varepsilon<a_N<a_n\leq\sup\{a_m\}=a<a+\varepsilon.$$

   That is, for $n\geq N$, $|a_n-a|<\varepsilon$. Thus, a_n converges to a. $\square$