Multi-Variable Calculus I: Homework 1

Due on October 8, 2024 at 8:00 AM

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Problem 1

Consider the following regions of $\mathbb{R}^3$ described by an inequality. Describe each region in words and sketch it to the best of your ability. Note: Inequalities define a solid region in space rather than a surface.

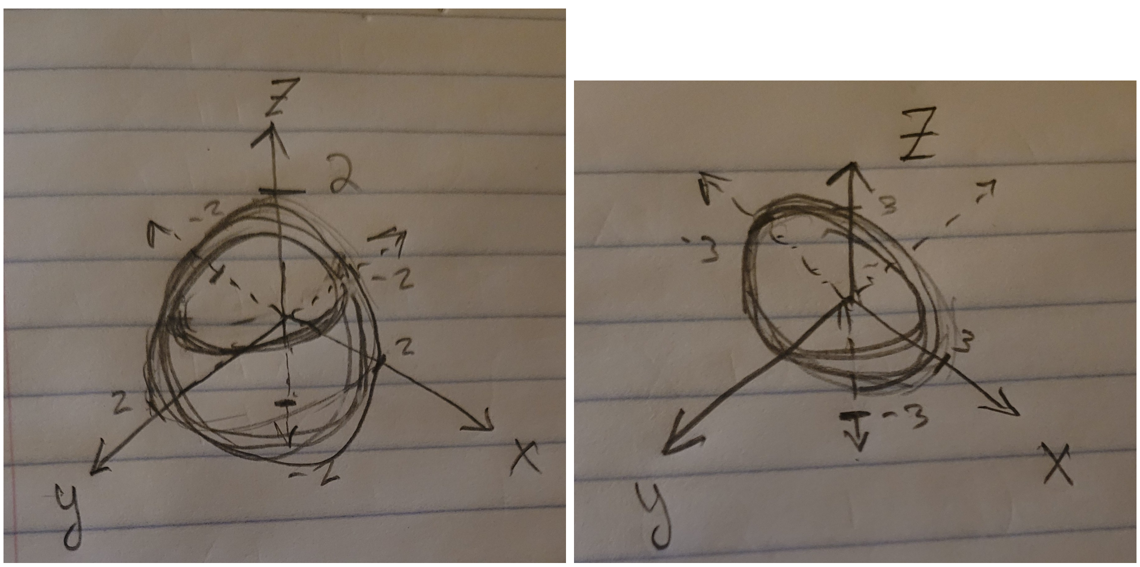
① $4 < x^2 + y^2 + z^2$.

② $x^2 + z^2 \leq 9$.

Solution 1

① The inequality represents the points (x, y, z) whose distance from the origin is more than 2. This creates a sphere of radius 2 centered at the origin (see figure 1a). All the points are outside the sphere.

② Since $y = 0$ in this inequality, that means the inequality is being drawn on the xz-plane. This is just a circle of radius 3 centered at the origin (see figure 1b). NOTE: The disk should be drawn on the xz-plane, but my artistic skills are very subpar.



(a) (b)

Figure 1

Problem 2

Consider the vectors $\mathbf{a} = \langle -2, 5, 4 \rangle$ and $\mathbf{b} = \langle 4, 8, 1 \rangle$.

1. Find $2\mathbf{a} - 3\mathbf{b}$.
2. Find the length of the vectors $\mathbf{a}$ and $\mathbf{b}$.
3. Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.
4. Find a unit vector in the direction of $\mathbf{a}$.
5. Find $\text{comp}_{\mathbf{a}}(\mathbf{b})$ and $\text{proj}_{\mathbf{a}}(\mathbf{b})$.

Solution 2

1. $2\langle -2, 5, 4 \rangle - 3\langle 4, 8, 1 \rangle = \langle -4, 10, 8 \rangle - \langle 12, 24, 3 \rangle = \langle -16, -14, 5 \rangle$.
2. $|a| = \sqrt{(-2)^2 + 5^2 + 4^2} = \sqrt{45}$.
   $|b| = \sqrt{4^2 + 8^2 + 1^2} = \sqrt{81} = 9$.
3. $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \dfrac{36}{\sqrt{45} \cdot 9} = \dfrac{4\sqrt{5}}{15}$. This gives us $\arccos\left(\dfrac{4\sqrt{5}}{15}\right) \approx 54°$.
4. The unit vector for $\mathbf{a}$ is $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\langle -2, 5, 4 \rangle}{\sqrt{45}} = \left\langle -\dfrac{2}{\sqrt{45}}, \dfrac{5}{\sqrt{45}}, \dfrac{4}{\sqrt{45}} \right\rangle$.
5. $\text{comp}_{\mathbf{a}}(\mathbf{b}) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{36}{\sqrt{45}}$.
   $\text{proj}_{\mathbf{a}}(\mathbf{b}) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{36}{45}\langle -2, 5, 4 \rangle = \left\langle -\dfrac{8}{5}, 4, \dfrac{16}{5} \right\rangle$.

Problem 3

Find the values of x such that the vectors $\langle 6, 3x, x \rangle$ and $\langle -3, 1, x \rangle$ are orthogonal.

Solution 3

Both vectors are orthogonal if and only if their dot product is zero. Thus, solving for x gives us

$$\begin{aligned} \langle 6, 3x, x \rangle \cdot \langle -3, 1, x \rangle = 0 &\implies = 6(-3) + 3x(1) + x(x) &&= 0 \\ &\implies -18 + 3x + x^2 &&= 0 \\ &\implies (x-3)(x+6) &&= 0. \end{aligned}$$

This gives us $x = 3$ and $x = -6$. Therefore, the following vector pairs are orthogonal to one another

$$\begin{aligned} \langle 6, 9, 3 \rangle \quad &\text{and} \quad \langle -3, 1, 3 \rangle \\ \langle 6, -18, -6 \rangle \quad &\text{and} \quad \langle -3, 1, -6 \rangle. \end{aligned}$$

Problem 4

If $\mathbf{r} = \langle x, y, z \rangle$ is an arbitrary position vector and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are constant position vectors, show that $(\mathbf{r} - \mathbf{u}) \cdot (\mathbf{r} - \mathbf{v}) = 0$ defines a sphere. Find the center and radius of the sphere.

Solution 4

Expanding the expression gives us

$$\begin{aligned}
&(\mathbf{r} - \mathbf{u}) \cdot (\mathbf{r} - \mathbf{v}) &&= 0 \\
\implies &\langle x - u_1, y - u_2, z - u_3 \rangle \cdot \langle x - v_1, y - v_2, z - v_3 \rangle &&= 0 \\
\implies &(x - u_1)(x - v_1) + (y - u_2)(y - v_2) + (z - u_3)(z - v_3) &&= 0 \\
\implies &x^2 - (u_1 + v_1)x + u_1v_1 + y^2 - (u_2 + v_2)y + u_2v_2 + z^2 - (u_3 + v_3)z + u_3v_3 &&= 0 \\
\implies &x^2 - (u_1 + v_1)x + y^2 - (u_2 + v_2)y + z^2 - (u_3 + v_3)z &&= D_1,
\end{aligned}$$

where D_n is the sum of all the constant terms. Lastly, we need to factor everything by completing the square, giving us

$$\begin{aligned}
&x^2 - (u_1 + v_1)x + y^2 - (u_2 + v_2)y + z^2 - (u_3 + v_3)z &&= D_1 \\
\implies &x^2 - (u_1 + v_1)x + \left(\frac{u_1 + v_1}{2}\right)^2 + y^2 - (u_2 + v_2)y + \left(\frac{u_2 + v_2}{2}\right)^2 + z^2 - (u_3 + v_3)z + \left(\frac{u_3 + v_3}{2}\right)^2 &&= D_2 \\
\implies &\left(x - \frac{u_1 + v_1}{2}\right)^2 + \left(y - \frac{u_2 + v_2}{2}\right)^2 + \left(z - \frac{u_3 + v_3}{2}\right)^2 &&= D_3,
\end{aligned}$$

where $\frac{u_1 + v_1}{2} = h$, $\frac{u_2 + v_2}{2} = k$, $\frac{u_3 + v_3}{2} = l$, and $\sqrt{D_3} = r$. This gives us the following center and radius for the sphere:

$$\begin{aligned}
\text{Center} &= (h, k, l) = \left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}, \frac{u_3 + v_3}{2}\right) \\
\text{Radius} &= r = \sqrt{D_3}.
\end{aligned}$$

Problem 5

Show that if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then $\mathbf{u}$ and $\mathbf{v}$ have the same length.

Solution 5

Suppose $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal. Then, by definition, we get

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= 0\\ \mathbf{u}^2-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}^2 &= 0\\ \mathbf{u}^2-\mathbf{v}^2 &= 0\\ \mathbf{u}^2 &= \mathbf{v}^2\\ \sqrt{\mathbf{u}\cdot\mathbf{u}} &= \sqrt{\mathbf{v}\cdot\mathbf{v}}.\end{aligned}$$

Expanding $\mathbf{u}\cdot\mathbf{u}$ and $\mathbf{v}\cdot\mathbf{v}$, we get

$$\underbrace{\sqrt{u_1^2+u_2^2+\cdots+u_n^2}}_{|\mathbf{u}|} = \underbrace{\sqrt{v_1^2+v_2^2+\cdots+v_n^2}}_{|\mathbf{v}|}.$$

Therefore, $\mathbf{u}$ and $\mathbf{v}$ have the same length if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.