Introduction to Proof: Homework 6

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Problem 1

In each case, use mathematical notation to write the negation of the given statement, in such a way that no quantifier is immediately preceded by a negation sign. For parts (i) - (iv) decide which is true: the given statement or its negation.

(i) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y = 0]$.

(ii) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = 0]$.

(iii) $(\exists x, y \in \mathbb{R})[x^2 + y^2 = -1]$.

(iv) $(\forall x \in \mathbb{R})[x > 0 \Rightarrow (\forall y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \Rightarrow y = z]]$.

(v) $(\forall \varepsilon \in \mathbb{R})[\varepsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})[0 < \delta \wedge (\forall x \in \mathbb{R})[1 - \delta < x < 1 + \delta \Rightarrow |f(x) - 5| < \varepsilon]]]$.

(vi) $(\forall a, b \in \mathbb{R})(a < b) \Rightarrow (\exists c \in \mathbb{R}) \left[a < c < b \wedge f'(c) = \frac{f(b)-f(a)}{b-a}\right]$.

Part (vi) is a statement which is true for differentiable functions $f : \mathbb{R} \to \mathbb{R}$, and it is a well-known theorem taught in every calculus class. What is the common name of this theorem?

Solution 1

(i) The negated statement is $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y \neq 0]$. The original statement is true, because for every real number x, there is a real number $y = -x$ such that $x + y = 0$.

(ii) The negated statement is $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y \neq 0]$. The negation is true because there is no real number x such that for every real number y, $x + y = 0$.

(iii) The negated statement is $(\forall x, y \in \mathbb{R})[x^2 + y^2 \neq -1]$. The negation is true because the sum of two squares is always non-negative.

(iv) The negated statement is $(\exists x \in \mathbb{R})[x > 0 \wedge (\exists y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \wedge y \neq z]]$. The original statement is true. For $x > 0$, if $y^2 = x$ and $z^2 = x$, then $y = z$ must hold when both y and z are positive.

(v) The negated statement is $(\exists \varepsilon \in \mathbb{R})[\varepsilon > 0 \wedge (\forall \delta \in \mathbb{R})[0 < \delta \Rightarrow (\exists x \in \mathbb{R})[1-\delta < x < 1+\delta \wedge |f(x)-5| \geq \varepsilon]]]$. Whether the original statement or its negation is true depends on the behavior of the function $f(x)$ near $x = 1$. If $f(x)$ is continuous at $x = 1$ and $f(1) = 5$, then the original statement is true.

(vi) The negated statement is $(\exists a, b \in \mathbb{R}) \left[a < b \wedge (\forall c \in \mathbb{R}) \left[a < c < b \Rightarrow f'(c) \neq \frac{f(b)-f(a)}{b-a}\right]\right]$. The original statement is true for differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and is known as the “Mean Value Theorem”.

Problem 2

In each part below I give the definition for a mathematical concept we have encountered, but using the shorthand notation in quantifiers. Fill in each box with the appropriate mathematical term or phrase that best completes the definition. In parts (ii)-(iv), $f : S \to T$ and $A \subseteq S$.

(i) $\boxed{} \Leftrightarrow (\forall x \in A)[x \in B]$.

(ii) $\boxed{} \Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$.

(iii) $\boxed{} \Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$.

(iv) $\boxed{} = \{z \mid (\exists v \in A)[z = f(v)]\}$.

Solution 2

(i) $A \subseteq B \Leftrightarrow (\forall x \in A)[x \in B]$.

(ii) The function f is one-to-one $\Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$.

(iii) The function f is onto $\Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$.

(iv) Image of A under f, $f(A) = \{z \mid (\exists v \in A)[z = f(v)]\}$.

Problem 3

Suppose $f : S \to T$ is one-to-one, $A \subseteq S$, and $B \subseteq S$. Give a line proof showing that $f(A) \cap f(B) \subseteq f(A \cap B)$.

Solution 3

1. Assume $z \in f(A) \cap f(B)$.
2. Then $z \in f(A)$ and $z \in f(B)$.
3. Then $f^{-1}(z) \in A$ and $f^{-1}(z) \in B$.
4. So $f^{-1}(z) \in A \cap B$.
5. Then $z = f(f^{-1}(z)) \in f(A \cap B)$.
6. Therefore, $f(A) \cap f(B) \subseteq f(A \cap B)$.

Problem 4

Give a line proof showing that if $A \cap B = \emptyset$ and $B \cup C = A \cup D$ then $B \subseteq D$.

Solution 4

1. Assume $x \in B$.
2. Then, $x \notin A$ since A and B are disjoint.
3. Then, since $B \cup C = A \cup D$, $x \in D$.
4. Therefore, $B \subseteq D$.

Problem 5

Let $f : S \to T$ and suppose f is onto. Let $A \subseteq S$. Give a line proof that $T - f(A) \subseteq f(S - A)$.

Solution 5

1. Suppose $y \in T - f(A)$.
2. Since f is onto, there exists some $x \in S$ such that $f(x) = y$.
3. Since $y \notin f(A)$, $x \notin A$.
4. Then, $x \in S - A$.
5. Therefore, $f(x) \in f(S - A)$, where $f(x) = y$.
6. Therefore, $T - f(A) \subseteq f(S - A)$.

Problem 6

Give a line proof showing $A \cap (X - B) = (A \cap X) - (A \cap B)$.

Solution 6

1. Suppose $x \in A \cap (X - B)$.
2. Then, $x \in A$ and $x \in X - B$.
3. Since $x \in X - B$, then $x \in X$ and $x \notin B$.
4. Then, $x \in A \cap X$ and $x \notin A \cap B$.
5. Therefore, $x \in (A \cap X) - (A \cap B)$.
6. Therefore, $A \cap (X - B) \subseteq (A \cap X) - (A \cap B)$.
7. Suppose $x \in (A \cap X) - (A \cap B)$.
8. Then, $x \in A \cap X$ and $x \notin A \cap B$.
9. Since $x \in A \cap X$, then $x \in A$ and $x \in X$.
10. Since $x \notin A \cap B$, then $x \notin B$, since $x \in A$ on the previous line.
11. Then, $x \in A$ and $x \in X - B$.
12. Therefore, $x \in A \cap (X - B)$.
13. Therefore, $(A \cap X) - (A \cap B) \subseteq A \cap (X - B)$.
14. Therefore, $A \cap (X - B) = (A \cap X) - (A \cap B)$.

Problem 7

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2e^{-x} + 5$.

(i) Prove that f is one-to-one.

(ii) Is f onto? Justify your answer.

(iii) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 5x^3 + 41$. Prove that g is onto.

Solution 7

(i) Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Then, $2e^{-a} + 5 = 2e^{-b} + 5$. This implies $e^{-a} = e^{-b}$, which implies $a = b$. Therefore, f is one-to-one.

(ii) Let $y \in \mathbb{R}$. Suppose $y = f(x) = 2e^{-x} + 5$. Then $e^{-x} = \frac{y-5}{2}$. The function e^{-x} only takes positive values, specifically $e^{-x} > 0$ for all $x \in \mathbb{R}$. This implies that

$$\frac{y-5}{2} > 0 \Rightarrow y > 5.$$

Therefore, f is not onto.

(iii) Let $y \in \mathbb{R}$. Suppose $y = g(x) = 5x^3 + 41$. Then,

$$x = \sqrt[3]{\frac{y-41}{5}}.$$

Since the cube root function is defined for all real numbers, g is onto.

Problem 8

Suppose $f : A \to B$ and $g : B \to C$.

(i) If f and g are one-to-one, prove that $g \circ f$ is onto.

(ii) If f and g are both onto, prove that $g \circ f$ is onto.

Solution 8

(i) *Proof.* Assume f and g are both one-to-one. Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then, using the definition of composition, we have $g(f(x_1)) = g(f(x_2))$. Since g is one-to-one, this implies $f(x_1) = f(x_2)$. Since f is one-to-one, this implies $x_1 = x_2$. Therefore, $g \circ f$ is one-to-one. □

(ii) *Proof.* Assume f and g are both onto. Let $z \in C$. Since g is onto, then there exists some y such that $g(y) = z$. Since f is onto, then there exists some x such that $f(x) = y$. Therefore, $(g \circ f)(x) = g(f(x)) = g(y) = z$. Therefore, $g \circ f$ is onto. □

Problem 9

Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that $(\forall x, y \in \ R)[x < y \Rightarrow f(x) < f(y)]$.

(i) Prove that f is one-to-one.

(ii) Give an example of a function f satisfying the given property but which is not onto.

Solution 9

(i) *Proof.* Assume $f(x_1) = f(x_2)$, for some $x_1, x_2 \in \mathbb{R}$. By the trichotomy property of real numbers, we know that $x_1 < x_2$, $x_1 = x_2$, or $x_1 > x_2$. If $x_1 < x_2$, then by the property of f, $f(x_1) < f(x_2)$. This contradicts our assumption that $f(x_1) = f(x_2)$. If $x_1 > x_2$, then by the property of f, $f(x_1) > f(x_2)$. This also contradicts our assumption that $f(x_1) = f(x_2)$. Therefore, $x_1 = x_2$. Therefore, f is one-to-one. ☐

(ii) The function $f(x) = \dfrac{x}{x-1}$, which holds the property that if $x_1 < x_2$, for some $x_1, x_2 \in \mathbb{R}$, then $f(x_1) < f(x_2)$. But $f(x)$ is not onto, since there doesn't exist any $x \in \mathbb{R}$ such that $f(x) = 1$.

Problem 10

(i) If $f : \mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = x^2 - 1$ and $g : \mathbb{Z} \to \mathbb{Z}$ is given by $g(x) = 3x + 2$, determine $(g \circ f)(0)$ and $(g \circ f)(2)$. Determine an algebraic formula for $(g \circ f)(x)$ for any integer x.

(ii) Suppose that $f : S \to T$ and $g : T \to U$. If $A \subseteq S$, give a line proof that $(g \circ f)(A) = g(f(A))$.

Solution 10

(i) The composition $(g \circ f)(x)$ is given by

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 3(x^2 - 1) + 2 = 3x^2 - 1.$$

Therefore, $(g \circ f)(0) = -1$ and $(g \circ f)(2) = 11$.

(ii) Here's the line proof for $(g \circ f)(A) = g(f(A))$.

1. By definition, $(g \circ f)(A) = \{(g \circ f) \mid a \in A\}$.
2. Then, $(\forall a \in A)[(g \circ f)(a) = g(f(a)]$.
3. Thus, $(g \circ f)(A) = \{g(f(a)) \mid a \in A\}$.
4. By definition, $g(f(A)) = \{g(f(a)) \mid a \in A\}$.
5. Therefore, $g(f(A)) = \{g(t) \mid t \in f(A)\} = \{g(f(a)) \mid a \in A\}$.
6. Therefore, $(g \circ f)(A) = g(f(A))$.

Problem 11

Consider the function $f : \mathbb{Z}_7 \to \mathbb{Z}_7$ given by $f(x) = x^3 + 1$. Answer the following questions

(i) Is f is one-to-one? Explain why or why not.

(ii) Is f onto? Explain why or why not.

(iii) Determine $f(S)$, where $S = \{0, 2, 4, 6\}$.

(iv) What is $f^{-1}(\{0\})$.

(v) If $A = \{1, 2, 3, 4\}$ and $B = \{0, 4, 5, 6\}$, determine $f^{-1}(A)$ and $f^{-1}(B)$. Also, determine $f^{-1}(A \cap B)$.

Solution 11

(i) To check if f is one-to-one, we must create a table of values to see if any two distinct elements in the domain map to the same element in the codomain. The table is shown below.

x	0	1	2	3	4	5	6
$f(x)$	1	2	2	0	2	0	1

Since $f(3) = f(5) = 0$, $f(0) = f(6) = 1$, and $f(1) = f(2) = f(4) = 2$, f is not one-to-one.

(ii) To check if f is onto, we must check if every element in the codomain is mapped to by some element in the domain. The table of values shows that f is not onto, since f does not map 3, 4, 5, or 6 to any element in the codomain.

(iii) The set $f(S)$ is given by $\{f(0), f(2), f(4), f(6)\} = \{1, 2, 2, 1\} = \{0, 1, 2\}$.

(iv) The set $f^{-1}(\{0\})$ is given by $\{x \in \mathbb{Z}_7 \mid f(x) = 0\} = \{3, 5\}$.

(v) The set $f^{-1}(A) = \{f(1), f(2), f(3), f(4)\} = \{2, 2, 0, 4\} = \{0, 2, 4\}$ and $f^{-1}(B) = \{f(0), f(4), f(5), f(6)\} = \{1, 2, 0, 1\}$. The set $f^{-1}(A \cap B) = \{f(4)\} = \{2\}$.

Problem 12

Suppose $f : S \to T$, $A \subseteq S$, and $B \subseteq T$. Give a line proof for each of the following

(i) $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

(ii) $f(A) \cap B = \emptyset \Rightarrow A \subseteq -f^{-1}(B)$.

Solution 12

(i) Here's the line proof for $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

1. Suppose $x \in A$.
2. Then, $f(x) \in f(A)$.
3. Since $f(A) \subseteq B$, then $f(x) \in B$.
4. Then, $x \in f^{-1}(B)$.
5. Then, $A \subseteq f^{-1}(B)$.
6. Therefore, $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

(ii) Here's the line proof for $f(A) \cap B = \emptyset \Rightarrow A \subseteq -f^{-1}(B)$.

1. Suppose $x \in A$.
2. Then, $f(x) \in f(A)$.
3. Then, $f(x) \notin B$.
4. Then, $x \notin f^{-1}(B)$.
5. Then, $A \subseteq -f^{-1}(B)$.
6. Therefore, $f(A) \cap B = \emptyset \Rightarrow A \subseteq -f^{-1}(B)$.

Problem 13

Suppose $f : S \to T$, $A \subseteq T$, $B \subseteq T$, and $C \subseteq S$. Give a line proof of each of the following

(i) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(iii) $f(f^{-1}(A)) \subseteq A$.

(iv) $C \subseteq f^{-1}(f(c))$.

(v) If f is onto, then $f(f^{-1}(A)) = A$.

(vi) If f is one-to-one, then $C = f^{-1}(f(C))$.

Solution 13

(i) Here's the line proof for $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

1. Suppose $x \in f^{-1}(A \cap B)$.
2. Then, $f(x) \in A \cap B$.
3. Then, $f(x) \in A$ and $f(x) \in B$.
4. Then, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.
5. Therefore, $x \in f^{-1}(A) \cap f^{-1}(B)$.
6. Therefore, $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.
7. Suppose $x \in f^{-1}(A) \cap f^{-1}(B)$.
8. Then, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.
9. Then, $f(x) \in A$ and $f(x) \in B$.
10. Then, $f(x) \in A \cap B$.
11. Therefore, $x \in f^{-1}(A \cap B)$.
12. Therefore, $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.
13. Therefore, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) Here's the line proof for $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Problem 13 continued on next page . . .

1. Suppose $x \in f^{-1}(A \cup B)$.
2. Then, $f(x) \in A \cup B$.
3. Then, $f(x) \in A$ or $f(x) \in B$.
4. Then, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.
5. Therefore, $x \in f^{-1}(A) \cup f^{-1}(B)$.
6. Therefore, $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.
7. Suppose $x \in f^{-1}(A) \cup f^{-1}(B)$.
8. Then, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.
9. Then, $f(x) \in A$ or $f(x) \in B$.
10. Then, $f(x) \in A \cup B$.
11. Therefore, $x \in f^{-1}(A \cup B)$.
12. Therefore, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.
13. Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(iii) Here's the line proof for $f(f^{-1}(A)) \subseteq A$.

1. Suppose $y \in f(f^{-1}(A))$.
2. Then, $y = f(x)$ for some $x \in f^{-1}(A)$.
3. Then, $x \in f^{-1}(A)$.
4. Then, $f(x) \in A$.
5. Therefore, $y \in A$.
6. Therefore, $f(f^{-1}(A)) \subseteq A$.

(iv) Here's the line proof for $C \subseteq f^{-1}(f(C))$.

1. Suppose $x \in C$.
2. Then, $f(x) \in f(C)$.
3. Then, $x \in f^{-1}(f(C))$.
4. Therefore, $C \subseteq f^{-1}(f(C))$.

(v) Here's the line proof for if f is onto, then $f(f^{-1}(A)) = A$.

1. From (iii), we know that $f(f^{-1}(A)) \subseteq A$.
2. If f is onto, then for $y \in A$, there exists $x \in f^{-1}(A)$ such that $f(x) = y$.
3. Thus, $y \in f(f^{-1}(A))$.
4. So $A \subseteq f(f^{-1}(A))$.
5. Therefore, $f(f^{-1}(A)) = A$.

(vi) Here's the line proof for if f is one-to-one, then $C = f^{-1}(f(C))$.

Problem 13 continued on next page . . .

1. From (iv), we know that $C \subseteq f^{-1}(f(C))$.
2. If f is one-to-one and $x \in f^{-1}(f(C))$, then $f(x) = f(c)$, for some $c \in C$.
3. This implies that $x = c$.
4. So, $f^{-1}(f(C)) \subseteq C$.
5. Therefore, $C = f^{-1}(f(C))$.

Problem 14

Construct an example of a function $f : \{0, 1, 2\} \to \{0, 1\}$ and a subset $A \subseteq \{0, 1\}$ where $f(f^{-1}(A)) \neq A$. Also construct an example of a function $g : \{0, 1, 2\} \to \{0, 1\}$ and a subset $C \subseteq \{0, 1, 2\}$ where $C \neq g^{-1}(g(C))$.

Solution 14

We cannot construct such examples, because for any function $f : S \to T$ and any subset $A \subseteq T$, we have $f(f^{-1}(A)) = A$.

We cannot construct such examples, because for any function $g : S \to T$ and any subset $C \subseteq S$, we have $C = g^{-1}(g(C))$.

Problem 15

Suppose $f, g : \mathbb{Z} \to \mathbb{Z}$ are two functions such that $f(M_3) \subseteq M_6$, $g(M_2) \subseteq M_7$, and $g^{-1}(M_5) = M_3$. Prove that for all $x \in \mathbb{Z}$, if $3 \mid x$ then $35 \mid g(f(x))$.

Solution 15

1. Assume $x \in \mathbb{Z}$ such that $3 \mid x$.
2. By the property of f, $f(M_3) \subseteq M_6$.
3. Thus, $f(x) \in M_6$.
4. Since $6 \mid f(x)$, we have $2 \mid f(x)$ and $3 \mid f(x)$.
5. Because $2 \mid f(x)$, $f(x) \in M_2$.
6. By condition 2, $g(M_2) \subseteq M_7$.
7. Thus, $g(f(x)) \in M_7$.
8. This means $7 \mid g(f(x))$.
9. By condition 3, $g^{-1}(M_5) = M_3$.
10. So, $f(x) \in M_3$, which means $g(f(x) \in M_5$.
11. This means $5 \mid g(f(x)$.
12. Since $g(f(x))$ is divisible by both 5 and 7, then $35 \mid g(f(x))$.
13. Therefore, for all $x \in \mathbb{Z}$, if $3 \mid x$ then $35 \mid g(f(x))$.

Problem 16

Given $[Q \wedge S] \Rightarrow R$ and $\neg S \Rightarrow T$, prove $[P \Rightarrow Q] \Rightarrow [\neg T \Rightarrow [\neg P \vee R]]$

Solution 16

1.	$[Q \wedge S] \Rightarrow R$	Hypothesis
2.	$\neg S \Rightarrow T$	Hypothesis
3.	Assume $P \Rightarrow Q$.	Dischargeable Hypothesis
4.	Assume $\neg T$.	Dischargeable Hypothesis
5.	Assume $P \wedge \neg R$.	Dischargeable Hypothesis
6.	$\neg R$.	RCS, for 5
7.	P	LCS, for 5
8.	$\neg[Q \wedge S]$.	MT, for 6, for 1
9.	$\neg[Q \wedge S] \Leftrightarrow \neg Q \vee \neg S$.	Tautology
10.	$\neg Q \vee \neg S$.	MPB, for 8, for 9
11.	Q.	MP, for 7, for 3
12.	$\neg S$.	DI, for 10, for 11
13.	T.	MP, for 12, for 2
14.	$\neg T \wedge T$	CI, for 4, for 13
15.	$\neg[P \wedge \neg R]$.	II, discharge for 5 [5 - 14 unusable]
16.	$\neg[P \wedge \neg R] \Leftrightarrow \neg P \vee R$.	Tautology
17.	$\neg P \vee R$.	MPB, for 15, for 16
15.	$\neg T \Rightarrow [\neg P \vee R]$.	DT, discharge for 4 [4 - 17 unusable]
16.	$[P \Rightarrow Q] \Rightarrow [\neg T \Rightarrow [\neg P \vee R]]$.	DT, discharge for 3 [3 - 15 unusable]