Abstract Linear Algebra: Homework 6

Due on February 12, 2025 at 12:00

Xiaojing Chen–Murphy

Hashem A. Damrah
UO ID: 952102243

Problem 1. Let T be the linear transformation on $\mathbb{R}^4$ which is represented in standard ordered basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}.$$

Under what condition on a, b, and c is T diagonalizable? Explain your answer.

Solution. Computing the determinant using cofactor expansion along the first column gives us

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ b & -\lambda & 0 \\ 0 & c & -\lambda \end{vmatrix} = -\lambda \left[(-\lambda)(\lambda^2) - 0 - 0\right] = \lambda^4.$$

The roots of the characteristic polynomial are all $\lambda = 0$ with multiplicity of 4.

To determine diagonalizability, we check the geometric multiplicity of $\lambda = 0$, which is the dimension of the null space. This gives the system of equations

$$ax_1 = 0, \quad bx_2 = 0, \quad \text{and} \quad cx_3 = 0.$$

For T to be diagonalizable, we must have $\dim(\text{Null}(A)) = 4$, which happens if and only if $a = b = c = 0$. Otherwise, the geometric multiplicity is strictly less than 4, and the matrix is not diagonalizable. □

Problem 2. Let T be a linear transformation on the n-dimensional vector space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.

Solution. Since T has n distinct eigenvalues, say $\lambda_1, \lambda_2, \cdots, \lambda_n$, there exist corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ such that for each i, we get $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$.

For $n = 1$, we get the set $\{\mathbf{v}_1\}$, which is a single nonzero vector. Therefore, it is linearly independent.

Assume that for any set of n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$, the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is linearly independent. Now consider a set of $n+1$ distinct eigenvalues $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$ with corresponding eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n, \mathbf{v}_{n+1}$. Suppose there exists a linear dependence among these eigenvectors

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v}_{n+1} = \mathbf{0} \tag{1}$$

We need to show that all i, $c_i = 0$.

Applying T to both sides and subtracting λ_{k+1} times equation 1 from the left side gives us

$$(c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n + c_{n+1}\lambda_{n+1}\mathbf{v}_{n+1}) - \lambda_{n+1}(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v}_{n+1}) = \mathbf{0}.$$

Simplifying gives us

$$c_1(\lambda_1 - \lambda_{n+1})\mathbf{v}_1 + \cdots + c_n(\lambda_n - \lambda_{n+1})\mathbf{v}_n = \mathbf{0}.$$

By the inductive hypothesis, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is linearly independent, so the coefficients must be zero

$$c_1(\lambda_1 - \lambda_{n+1}) = 0, \quad \cdots, \quad c_n(\lambda_n - \lambda_{n+1}) = \mathbf{0}.$$

Since the eigenvalues are distinct, $\lambda_i - \lambda_{n+1} \neq 0$ for all i, that implies that $c_1 = c_2 = \cdots = c_n = 0$. Returning back to equation 1, we see that

$$c_n\mathbf{v}_n + c_{n+1}\mathbf{v}_{n+1} = \mathbf{0}.$$

Since $\mathbf{v}_{n+1} \neq \mathbf{0}$, we must have $c_n = c_{n+1} = 0$. Therefore, set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent.

Since V is n-dimensional and we have found n linearly independent eigenvectors, they form a basis for V. With respect to this basis of eigenvectors, the matrix representation of T is diagonal, with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ as the diagonal entries. That is, in this basis, T is represented by the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Therefore, T is diagonalizable. ☐

Problem 3. Let T be an invertible linear transformation on a vector space V. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Solution. By definition, λ is an eigenvalue of T if there exists a nonzero vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$. Since T is invertible, we can apply T^{-1} to both sides to get $\mathbf{v} = \lambda T^{-1}(\mathbf{v})$. Rearranging gives us $T^{-1}(\mathbf{v}) = \lambda^{-1}\mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, λ^{-1} is an eigenvalue of T^{-1} with eigenvector $\mathbf{v}$.

Assume that λ^{-1} is an eigenvalue of T^{-1}. Then, by definition, there exists a nonzero vector $\mathbf{v} \in V$ such that $T^{-1}(\mathbf{v}) = \lambda^{-1}\mathbf{v}$. Applying T to both sides, we get $T(T^{-1}(\mathbf{v})) = T(\lambda^{-1}\mathbf{v}) \Rightarrow \mathbf{v} = \lambda^{-1}T(\mathbf{v})$. Multiplying both sides by λ, we obtain $\lambda\mathbf{v} = T(\mathbf{v})$. Since $\mathbf{v} \neq \mathbf{0}$, this shows that λ is an eigenvalue of T with eigenvector $\mathbf{v}$.

Therefore, λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. ☐

Problem 4. Let $A, B \in \mathbb{R}^{n\times n}$. This problem is to conclude that AB and BA have exactly the same set of eigenvalues.

(i) Assume that $\lambda I - AB$ is invertible. Prove that

$$(\lambda I - BA)\left[I + B(\lambda I - AB)^{-1}A\right] = \lambda I.$$

(ii) Use part (i) to prove that AB and BA have the same eigenvalues. (Note: The algebraic multiplicity of the same eigenvalues may not be the same.)

Solution to (i). Expanding the left-hand side gives us

$$(\lambda I - BA)I + (\lambda I - BA)B(\lambda I - AB)^{-1}A = \lambda I - BA + (\lambda I - BA)B(\lambda I - AB)^{-1}A.$$

Expanding the second term

$$(\lambda I - BA)B(\lambda I - AB)^{-1}A = \lambda B(\lambda I - AB)^{-1}A - BAB(\lambda I - AB)^{-1}A.$$

Notice that we can rewrite $BAB(\lambda I - AB)^{-1}A$ as $B(AB)(\lambda I - AB)^{-1}A$ and since we know that $AB(\lambda I - AB)^{-1} = I - \lambda(\lambda I - AB)^{-1}$, we can substitute this back into our original equation to get

$$\lambda I - BA + \lambda B(\lambda I - AB)^{-1}A - B(I - \lambda(\lambda I - AB)^{-1})A.$$

Distributing B in the last term gives us

$$B(I - \lambda(\lambda I - AB)^{-1})A = BA - B\lambda(\lambda I - AB)^{-1}A.$$

Substituting this back, we get

$$\lambda I - BA + \lambda B(\lambda I - AB)^{-1}A - BA + B\lambda(\lambda I - AB)^{-1}A.$$

Now, observe that the terms $\lambda B(\lambda I - AB)^{-1}A$ and $B\lambda(\lambda I - AB)^{-1}A$ are the same, so they cancel out, leaving us with

$$\lambda I - BA + BA - BA = \lambda I.$$

Therefore, we have shown that

$$(\lambda I - BA)\left[I + B(\lambda I - AB)^{-1}A\right] = \lambda I. \qquad \square$$

Solution to (ii). Taking determinants on both sides from the equation proved in part (i), we get

$$\det\left((\lambda I - BA)\left[I + B(\lambda I - AB)^{-1}A\right]\right) = \det(\lambda I).$$

Using the determinant property $\det(AB) = \det(A)\det(B)$, we rewrite this as

$$\det(\lambda I - BA) \cdot \det\left(I + B(\lambda I - AB)^{-1}A\right) = \det(\lambda I).$$

If $\lambda I - AB$ is invertible, we analyze $I + B(\lambda I - AB)^{-1}A$. Suppose $I + B(\lambda I - AB)^{-1}A$ is invertible, then

$$\det(\lambda I - BA) = \frac{\det(\lambda I)}{\det(I + B(\lambda I - AB)^{-1}A)}.$$

Since the right-hand side is nonzero when $I + B(\lambda I - AB)^{-1}A$ is invertible, this means that $\lambda I - BA$ is invertible whenever $\lambda I - AB$ is invertible.

Eigenvalues correspond to values of λ for which the matrix $\lambda I - AB$ is not invertible, meaning

$$\det(\lambda I - AB) = 0.$$

From our determinant equation, if $\lambda I - AB$ is singular (i.e., $\det(\lambda I - AB) = 0$), then $\lambda I - BA$ must also be singular (i.e., $\det(\lambda I - BA) = 0$). This shows that if λ is an eigenvalue of AB, then it is also an eigenvalue of BA.

Conversely, if $\lambda I - BA$ is singular, then $I + B(\lambda I - AB)^{-1}A$ must be non-invertible, implying that $\lambda I - AB$ is also singular. This proves that if λ is an eigenvalue of BA, then it is also an eigenvalue of AB.

Thus, AB and BA have exactly the same eigenvalues. $\square$

Problem 5. Let $A \in \mathbb{C}^{n\times n}$. Let g be a polynomial over $\mathbb{C}$. Prove that c is an eigenvalue of $g(A)$ if and only if $c = g(\lambda)$ for some eigenvalue λ of A.

Solution. Let $A \in \mathbb{C}^{n\times n}$and let g be a polynomial over $\mathbb{C}$. Suppose $g(A)$ is formed by evaluating g at A, meaning that

$$g(A) = \sum_{k=0}^{m} a_k A^k.$$

Suppose c is an eigenvalue of $g(A)$. By definition, there exists a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ such that $g(A)\mathbf{v} = c\mathbf{v}$. We need to show that $c = g(\lambda)$ for some eigenvalue λ of A. To do so, we note that $\mathbf{v}$ is an eigenvector of $g(A)$, corresponding to the eigenvalue λ. To justify this, recall that any polynomial of a matrix acts naturally on its eigenvectors. Since matrix polynomials respect the structure of eigenspaces, applying $g(A)$ to an eigenvector $\mathbf{v}$ results in $g(A)\mathbf{v} = g(\lambda)\mathbf{v}$. Since we already assumed $g(A)\mathbf{v} = c\mathbf{v}$, comparing the two equations gives $c\mathbf{v} = g(\lambda)\mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, we must have $c = g(\lambda)$.

Suppose λ is an eigenvalue of A and $c = g(\lambda)$. Since λ is an eigenvalue of A, there exists a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Applying $g(A)$ to $\mathbf{v}$, we compute $g(A)\mathbf{v} = g(\lambda)\mathbf{v} = c\mathbf{v}$. This shows that $\mathbf{v}$ is an eigenvector of $g(A)$ with eigenvalue c, meaning that c is an eigenvalue of $g(A)$.

Therefore, c is an eigenvalue of $g(A)$ if and only if $c = g(\lambda)$ for some eigenvalue λ of A. $\square$

Problem 6. Suppose $V = W_1 \oplus W_2$. Prove that for any $\mathbf{v} \in V$, there exists a unique pair of vectors $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.

Solution. To prove the statement, we must show that for every $\mathbf{v} \in V$, there exists a unique pair $(\mathbf{w}_1, \mathbf{w}_2)$, with $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.

Since $V = W_1 \oplus W_2$, by definition of the direct sum, every vector $\mathbf{v} \in V$ can be written as a sum of two vectors $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in W_1$ where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$. Thus, such a decomposition always exists.

Suppose there exist two different representations $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}_1' + \mathbf{w}_2'$, where $\mathbf{w}_1, \mathbf{w}_1' \in W_1$ and $\mathbf{w}_2, \mathbf{w}_2' \in W_2$. Then we have $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2'$. Rearranging gives us $\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{w}_2 - \mathbf{w}_2'$. Since $\mathbf{w}_1 - \mathbf{w}_1' \in W_1$ and $\mathbf{w}_2 - \mathbf{w}_2' \in W_2$ and because $W_1 \cap W_2 = \{\mathbf{0}\}$, we must have $\mathbf{w}_1 - \mathbf{w}_1' = \mathbf{0}$ and $\mathbf{w}_2 - \mathbf{w}_2' = \mathbf{0}$. That is, $\mathbf{w}_1 = \mathbf{w}_1'$ and $\mathbf{w}_2 = \mathbf{w}_2'$. Thus, the pair $(\mathbf{w}_1, \mathbf{w}_2)$ is unique. □

Problem 7. True or False. (No explanation needed.)

(i) Let $A \in \mathbb{C}^{n \times n}$, then A has exactly n eigenvalues (counting the multiplicities).

(ii) Let $T : V \to V$ be a linear transformation, where $\dim(V) = n$. Then T is diagonalizable if and only if T has n distinct eigenvalues.

(iii) Similar matrices always have the same eigenvalues.

(iv) Similar matrices always have the same eigenvectors.

(v) The sum of two eigenvectors of a linear transformation T is always an eigenvector of T.

(vi) If λ_1 and λ_2 are distinct eigenvalues of a linear transformation, then $E_{\lambda_1} \cap E_{\lambda_2} = \{\mathbf{0}\}$.

(vii) A linear transformation T on a finite-dimensional vector space is diagonalizable if and only if the algebraic multiplicity of each eigenvalue λ equals to its geometric multiplicity.

(viii) Suppose $W_1, W_2, \cdots, W_m \subset V$ are subspaces. Then $W_1 + W_2 + \cdots + W_m$ is a direct sum if $W_i \cap W_j = \{\mathbf{0}\}$ for any $i \neq j$.

Solution to (i). True. □

Solution to (ii). False. □

Solution to (iii). True. □

Solution to (iv). False. □

Solution to (v). False. □

Solution to (vi). True. □

Solution to (vii). True. □

Solution to (viii). False. □