$$\Rightarrow (L_A)^* = L_{A^T}.$$

2). Let $A \in \mathbb{C}^{n\times n}$. Consider standard inner product on $\mathbb{C}^n$

and $L_A : \mathbb{C}^n \to \mathbb{C}^n$ w/ $L_A(\vec{x}) = A\vec{x}$.

Then $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$: $(L_A(\vec{x}), \vec{y}) = (A\vec{x}, \vec{y}) = y^* A\vec{x}$

$$= (A^*\vec{y})^* \vec{x}$$

$$= (\vec{x}, \underbrace{A^*\vec{y}}_{(L_A)^*(\vec{y})})$$

$$\Rightarrow (L_A)^* = L_{A^*}.$$

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Proof of the theorem.

For each $\vec{y} \in V$, consider the function $g_{\vec{y}} : \vec{V} \to F$ defined

by $g_y(\vec{x}) = (T(\vec{x}), \vec{y})$. for any $\vec{x} \in V$.

Then for any $\vec{x}_1, \vec{x}_2 \in V$ and $c \in F$:

$$g_{\vec{y}}(c\vec{x}_1 + \vec{x}_2) = (T(c\vec{x}_1 + \vec{x}_2), \vec{y})$$

$$= (cT(\vec{x}_1) + T(\vec{x}_2), \vec{y})$$

$$= c(T(\vec{x}_1), \vec{y}) + (T(\vec{x}_2), \vec{y})$$

$$= c\, g_{\vec{y}}(\vec{x}_1) + g_{\vec{y}}(\vec{x}_2)$$

$\Rightarrow g_{\vec{y}}$ is a linear functional on V.

Then by the first result in the section, there exists a unique $y' \in V$ such that

$$g_y(\vec{x}) = (T(\vec{x}), \vec{y}) = (\vec{x}, \vec{y}').$$

Then define $T^*: V \to V$ such that $T^*(\vec{y}) = \vec{y}'$.

① Note T^* is a well defined function b/c of the uniqueness of $\vec{y}'$.

② T^* is linear because for any $\vec{y}_1, \vec{y}_2 \in V$, $c \in F$ and any $\vec{x} \in V$:

To show properties about T^*, we must put it in an inner product. Recall $\forall \vec{x} \in V$, if $(\vec{x}, \vec{v}) = (\vec{x}, \vec{w})$, then $\vec{v} = \vec{w}$.

$$(\vec{x}, T^*(c\vec{y}_1 + \vec{y}_2)) = (T(\vec{x}), c\vec{y}_1 + \vec{y}_2)$$

$$= (T(\vec{x}), c\vec{y}_1) + (T(\vec{x}), \vec{y}_2)$$

$$= \bar{c}\,(T(\vec{x}), \vec{y}_1) + (T(\vec{x}), \vec{y}_2)$$

$$= \bar{c}\,(\vec{x}, T^*(\vec{y}_1)) + (\vec{x}, T^*(\vec{y}_2))$$

$$= (\vec{x}, cT^*(\vec{y}_1)) + (\vec{x}, T^*(\vec{y}_2))$$

$$= (\vec{x}, cT^*(\vec{y}_1) + T^*(\vec{y}_2))$$

$$\Rightarrow \forall \vec{x} \in V: \quad (\vec{x}, T^*(c\vec{y}_1 + \vec{y}_2)) = (\vec{x}, cT^*(\vec{y}_1) + T^*(\vec{y}_2))$$

$$\Rightarrow T^*(c\vec{y}_1 + \vec{y}_2) = cT(\vec{y}_1^*) + T^*(\vec{y}_2).$$

③ Finally we need to prove T^* is unique. In fact, suppose $U: V \to V$ also satisfies: $(T(\vec{x}), \vec{y}) = (\vec{x}, U(\vec{y}))$ $\forall \vec{x}, \vec{y} \in V$

Then $(\vec{x}, U(\vec{y})) = (\vec{x}, T^*(\vec{y}))$ for all $\vec{x} \in V$,

$\Rightarrow U(\vec{y}) = T^*(\vec{y})$ for all $\vec{y}$

$\Rightarrow U = T^*$.

Theorem: Let V be a finite-dimensional inner product space. Let B be an orthonormal basis of V. If T is a linear transformation on V. Then $[T^*]_B = ([T]_B)^*$.

<u>Proof</u>: Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be the orthonormal basis of V.

Denote $[T]_B = A = (a_{ij})$.

Then $T(\vec{v}_j) = \sum_{k=1}^{n} a_{kj} \vec{v}_k$ for each $j=1,\cdots,n$.

For each j: $\left(T(\vec{v}_j), \vec{v}_i\right) = \left(\sum_{i=1}^{n} a_{kj} \vec{v}_k, \vec{v}_i\right) = a_{ij}$, $\forall i=1,\cdots,n$

Denote $[T^*]_B = C = (c_{ij})$

Similarly: $c_{ij} = (T^*(\vec{v}_j), \vec{v}_i)$

$$c_{ij} = (T^*(\vec{v}_j), \vec{v}_i) = \overline{(\vec{v}_i, T^*(\vec{v}_j))} = \overline{(T(\vec{v}_i), \vec{v}_j)}$$
$$= \overline{a_{ji}}$$

$\Rightarrow$ $C = A^*$. i.e. $[T^*]_B = [T]_B^*$. $\boxtimes$

Proposition. Let V be a finite dimensional inner product space.

Let T and U be linear transformations on V. Then

1). $(U+T)^* = U^* + T^*$

2). $(cT)^* = \bar{c}\, T^*$ $\quad \forall c \in F$

3). $(TU)^* = U^* T^*$

4) $(T^*)^* = T$

5) $(I^*)^* = I$.

Proof For 3): $(TU(\vec{x}), \vec{y}) = (T(U(\vec{x})), \vec{y})$

$$= (U(\vec{x}), T^*(\vec{y}))$$

$$= (\vec{x}, U^* T^*(\vec{y}))$$

$$= (\vec{x}, (TU)^*(\vec{y})).$$

Others: exercise !

Definition. Let V be a finite dimensional inner product space. Let T be a linear transformation on V.

T is called normal if $T^*T = TT^*$.

Definition. Let $A \in \mathbb{C}^{n\times n}$. A is called normal if $A^*A = AA^*$

Definition: Let $A \in \mathbb{R}^{n\times n}$. A is called normal if $A^TA = AA^T$.

Definition. Let V be a finite dimensional inner product space. Let T be a linear transformation on V. T is called self-adjoint if $T^* = T$.

Examples of normal linear transformation or normal matrices.

1). Self-adjoint linear transformations are normal

2). Hermitian matrices: $A^* = A$.

3). Symmetric matrices: $A^T = A$

4). Unitary matrix: $A^*A = AA^* = I$

5) Orthogonal matrix: $A^TA = AA^T = I$

6) Skew-Hermitian: $A^* = -A$

7) Skew-Symmetric: $A^T = -A$.

Lemma: Let T be a normal linear transformation on V. Let $(\lambda, \vec{v})$ be an eigenvalue/eigenvector pair of T. Then $(\bar{\lambda}, \vec{v})$ is an eigenvalue/eigenvector pair of T^*.

Proof: $T\vec{v} = \lambda\vec{v} \iff (T-\lambda I)\vec{v} = \vec{0}$

$\iff ((T-\lambda I)\vec{v}, (T-\lambda I)\vec{v}) = 0$

$\iff (\vec{v}, (T-\lambda I)^*(T-\lambda I)\vec{v}) = 0$

Note: $(T-\lambda I)^*(T-\lambda I) = (T^* - \bar{\lambda} I)(T-\lambda I)$

$= T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda I$

$\overset{T \text{ is normal}}{=} TT^* - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda I$

$$= (T-\lambda I)T^* - (T-\lambda I)\bar{\lambda} I$$

$$= (T-\lambda I)(T^* - \bar{\lambda} I)$$

$$= (T-\lambda I)(T-\lambda I)^*$$

$$= [(T-\lambda I)^*]^*(T-\lambda I)^*$$

$$\Leftrightarrow (\vec{v}, [(T-\lambda I)^*]^*(T-\lambda I)^*) = 0$$

$$\Leftrightarrow ((T-\lambda I)^*\vec{v}, (T-\lambda I)^*\vec{v}) = 0$$

$$\Leftrightarrow (T-\lambda I)^*\vec{v} = \vec{0}$$

$$\Leftrightarrow (T^* - \bar{\lambda} I)\vec{v} = \vec{0}$$

$$\Leftrightarrow T^*\vec{v} = \bar{\lambda}\vec{v} \qquad \blacksquare$$

Lemma: Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Let $(\lambda, \vec{v})$ be an eigenvalue/eigenvector pair of A. Then $(\bar{\lambda}, \vec{v})$ is an eigenvalue/eigenvector pair of A^*.

Proof: Homework.

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