Functional Complex Variables I: Homework 7

Due on May 28, 2025 at 23:59

Weiyong He

Hashem A. Damrah
UO ID: 952102243

Exercise 4.49.7. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \overline{z}\,\mathrm{d}z.$$

Solution. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then, we have

$$\overline{z} = x - iy \quad \text{and} \quad \mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y.$$

Now compute the integrand:

$$\overline{z}\,\mathrm{d}z = (x - iy)(dx + idy) = x\,\mathrm{d}x + xi\,\mathrm{d}y - iy\,\mathrm{d}x - i^2 y\,\mathrm{d}y.$$

Since $i^2 = -1$, we get

$$\overline{z}\,\mathrm{d}z = x\,\mathrm{d}x + ix\,\mathrm{d}y - iy\,\mathrm{d}x + y\,\mathrm{d}y = (x\,\mathrm{d}x + y\,\mathrm{d}y) + i(x\,\mathrm{d}y - y\,\mathrm{d}x).$$

So,

$$\frac{1}{2i}\int_C \overline{z}\,\mathrm{d}z = \frac{1}{2i}\int_C [(x\,\mathrm{d}x + y\,\mathrm{d}y) + i(x\,\mathrm{d}y - y\,\mathrm{d}x)] = \frac{1}{2i}\left[\int_C (x\,\mathrm{d}x + y\,\mathrm{d}y) + i\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)\right].$$

The first integral $\int_C (x\,\mathrm{d}x + y\,\mathrm{d}y)$ is zero because it represents the line integral of the gradient of the scalar function $1/2(x^2 + y^2)$, and over a closed path the integral of a gradient is zero

$$\int_C (x\,\mathrm{d}x + y\,\mathrm{d}y) = 0.$$

So we're left with just

$$\frac{1}{2i}\int_C \overline{z}\,\mathrm{d}z = \frac{1}{2i}\cdot i\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x) = \frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

Notice that this is an expression for the area of the planar region bounded by a positively oriented curve,

$$A = \frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

Therefore,

$$A = \frac{1}{2i}\int_C \overline{z}\,\mathrm{d}z. \qquad \square$$

Exercise 4.52.1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(i) $\displaystyle\int_C \frac{e^{-z}}{z - (\pi i/2)}\,\mathrm{d}z$; (ii) $\displaystyle\int_C \frac{\cos(z)}{z(z^2 + 8)}\,\mathrm{d}z$; (iii) $\displaystyle\int_C \frac{z}{2z + 1}\,\mathrm{d}z$;

(iv) $\displaystyle\int_C \frac{\cosh(z)}{z^4}\,\mathrm{d}z$; (v) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,\mathrm{d}z$.

Solution to (i). Let $f(z) = e^{-z}$, which is entire (analytic everywhere), and note that $\pi i/2$ lies inside the square since its imaginary part is $\frac{\pi}{2} < 2$. By Cauchy's Integral Formula, we have

$$\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\,\mathrm{d}z = 2\pi i \cdot e^{-\frac{\pi i}{2}} = 2\pi i \cdot \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = 2\pi i \cdot (-i) = 2\pi. \qquad \square$$

Solution to (ii). The singularities are at $z = 0$ and $z = \pm 2\sqrt{2}i$. All of these are within the square (since $\sqrt{8} \approx 2.828 < 4$). Let

$$f(z) = \frac{\cos(z)}{z^2 + 8},$$

which is analytic at $z = 0$. Then by Cauchy's Integral Formula, we have

$$\int_C \frac{\cos(z)}{z(z^2 + 8)} \,\mathrm{d}z = 2\pi i \cdot \frac{\cos(0)}{0^2 + 8} = 2\pi i \cdot \frac{1}{8} = \frac{\pi i}{4}. \qquad \square$$

Solution to (iii). The integrand can be rewritten as

$$\frac{z}{2z + 1} = \frac{1}{2} \cdot \frac{2z}{2z + 1} = \frac{1}{2} \cdot \left(1 - \frac{1}{2z + 1}\right).$$

Now integrate term-by-term, we have

$$\int_C \frac{z}{2z + 1} \,\mathrm{d}z = \frac{1}{2} \int_C \left(1 - \frac{1}{2z + 1}\right) \mathrm{d}z.$$

The integral of 1 over a closed contour is 0, and $1/(2z + 1)$ has a simple pole at $z = -\frac{1}{2}$, which lies inside the square. So:

$$\int_C \frac{z}{2z + 1} \,\mathrm{d}z = -\frac{1}{2} \int_C \frac{1}{2z + 1} \,\mathrm{d}z = -\frac{1}{2} \cdot 2\pi i \cdot \frac{1}{2} = -\frac{\pi i}{2}. \qquad \square$$

Solution to (iv). We use the fact that

$$\int_C \frac{f(z)}{(z - z_0)^n} \,\mathrm{d}z = \frac{2\pi i}{(n - 1)!} f^{(n-1)}(z_0),$$

for $n = 4$ and $z_0 = 0$. Here, $f(z) = \cosh(z)$ is entire, so, we have

$$\int_C \frac{\cosh(z)}{z^4} \,\mathrm{d}z = \frac{2\pi i}{3!} \cosh^{(3)}(0).$$

Recall the Taylor series expansion for $\cosh(z)$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!},$$

which imply that $\cosh^{(3)}(0) = 0$. Therefore,

$$\int_C \frac{\cosh(z)}{z^4} \,\mathrm{d}z = 0. \qquad \square$$

Solution to (v). Let $f(z) = \tan(z/2)$, which is analytic inside the square (its singularities occur at $z = (2n + 1)\pi$, none of which are inside the square since $\pi > 3$).

The integrand has a pole of order 2 at $z = x_0 \in (-2, 2)$, and we can use the derivative form of Cauchy's Integral Formula,

$$\int_C \frac{f(z)}{(z - x_0)^2} \,\mathrm{d}z = 2\pi i \cdot f'(x_0).$$

Computing the derivative of $f(z)$, we have

$$f'(z) = \frac{1}{2} \sec^2\left(\frac{z}{2}\right) \Rightarrow f'(x_0) = \frac{1}{2} \sec^2\left(\frac{x_0}{2}\right).$$

Therefore, we have

$$\int_C \frac{\tan(z/2)}{(z - x_0)^2} \,\mathrm{d}z = \pi i \cdot \sec^2\left(\frac{x_0}{2}\right). \qquad \square$$

Exercise 4.52.2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(i) $g(z) = \dfrac{1}{z^2+4}$; (ii) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Solution to (i). Factoring the denominator, we have $z^2+4 = (z+2i)(z-2i)$. The singularities are at $z = 2i$ and $z = -2i$. The contour $|z-i| = 2$ is centered at $z = i$ and has radius 2. Therefore, $|2i - i| = 1 < 2$ and $|-2i-i| = 3 > 2$, so the singularity at $z = 2i$ lies *inside* the contour, while the singularity at $z = -2i$ lies *outside*.

Since only $z = 2i$ is inside, we write

$$g(z) = \frac{1}{(z-2i)(z+2i)} = \frac{1}{z+2i} \cdot \frac{1}{z-2i}.$$

The function $f(z) = 1/(z+2i)$ is analytic on and inside the contour (since $z = -2i$ is outside). So we can use Cauchy's Integral Formula for the simple pole at $z = 2i$

$$\int_{|z-i|=2} \frac{f(z)}{z-2i}\,\mathrm{d}z = 2\pi i \cdot f(2i) = 2\pi i \cdot \frac{1}{2i+2i} = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}. \qquad \square$$

Solution to (ii). Again, factoring the denominator, we have $(z^2+4)^2 = [(z-2i)(z+2i)]^2$. So the integrand has a pole of order 2 at $z = 2i$, which is *inside* the circle, and another at $z = -2i$, which is *outside* the circle. We can rewrite the integrand as

$$g(z) = \frac{1}{[(z-2i)^2(z+2i)^2]} = \frac{1}{(z+2i)^2} \cdot \frac{1}{(z-2i)^2}.$$

Let

$$f(z) = \frac{1}{(z+2i)^2},$$

which is analytic inside and on the circle (since $z = -2i$ is outside). We apply the Cauchy Integral Formula for derivatives, to get

$$\int_{|z-i|=2} \frac{f(z)}{(z-2i)^2}\,\mathrm{d}z = 2\pi i \cdot f'(2i).$$

Computing the derivative of f, we have $f'(z) = -2(z+2i)^{-3}$. So,

$$f'(2i) = -2(4i)^{-3} = -2 \cdot \frac{1}{64i^3} = -2 \cdot \frac{1}{64(-i)} = \frac{2}{64i} = \frac{1}{32i}.$$

Therefore, we have

$$\int_{|z-i|=2} \frac{1}{(z^2+4)^2}\,\mathrm{d}z = 2\pi i \cdot \frac{1}{32i} = \frac{\pi}{16}. \qquad \square$$

Exercise 4.52.3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,\mathrm{d}s \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

Solution. Assume that $|2| < 3$. Since $z = 2$ is strictly *inside* the contour C (because $|2| < 3$), we apply Cauchy's Integral Formula, to get

$$\int_C \frac{f(s)}{s-z}\,\mathrm{d}s = 2\pi i \cdot f(z).$$

Therefore, we have

$$g(2) = 2\pi i \cdot f(2) = 2\pi i \cdot (2(2)^2 - 2 - 2) = 2\pi i \cdot (8 - 2 - 2) = 2\pi i \cdot 4 = 8\pi i.$$

Assume that $|z| > 3$, i.e., z is *outside* the contour C. In this case, the function $f(s)/(s-z)$ is analytic in s on and inside the contour C, because z is outside the region enclosed by C and f is entire.

Since the integrand is analytic inside and on C, and C is a closed curve, the Cauchy-Goursat Theorem implies that

$$g(z) = \int_C \frac{f(s)}{s-z}\,\mathrm{d}s = 0. \qquad \square$$

Exercise 4.52.4. Let C be any simple closed contour, described in the positive sense in the z-plane, and write

$$g(z) = \int_C \frac{s^3 + 2z}{(s-z)^3}\,\mathrm{d}s.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

Solution. If z is *outside* the contour C, then the integrand is analytic on and inside C since the denominator never vanishes. Therefore, by Cauchy's Theorem, we have

$$g(z) = \int_C \frac{s^3 + 2z}{(s-z)^3}\,\mathrm{d}s = 0.$$

Now, for when z is *inside* the contour C. We can write the integrand as

$$\frac{s^3 + 2z}{(s-z)^3} = \frac{s^3}{(s-z)^3} + \frac{2z}{(s-z)^3}.$$

Let us evaluate each term using Cauchy's Integral Formula for derivatives to get

$$\int_C \frac{f(s)}{(s-z)^n}\,ds = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z), \quad n \geq 1.$$

For the first term, we have $f(s) = s^3$ and we compute the second derivative of f at $s = z$ to get

$$f'(s) = 3s^2 \quad \text{and} \quad f''(s) = 6s \Rightarrow f''(z) = 6z.$$

Thus, we have

$$\int_C \frac{s^3}{(s-z)^3}\,\mathrm{d}s = \frac{2\pi i}{2!} \cdot 6z = \pi i \cdot 6z = 6\pi i z.$$

For the second term, we have $f(s) = 2z$, which is constant with respect to s, so, we have

$$\int_C \frac{2z}{(s-z)^3}\,ds = 2z \cdot \int_C \frac{1}{(s-z)^3}\,\mathrm{d}s.$$

But $1/(s-z)^3$ is the third derivative kernel for the constant function, which is 0, giving us

$$\int_C \frac{1}{(s-z)^3}\,\mathrm{d}s = 0.$$

So the second term vanishes. Therefore, we have

$$g(z) = \int_C \frac{s^3 + 2z}{(s-z)^3}\,\mathrm{d}s = 6\pi i z + 0 = 6\pi i z. \qquad \square$$

Exercise 4.52.7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,\mathrm{d}z = 2\pi i.$$

Then, write this integral in terms of θ to dive the integration formula

$$\int_0^\pi e^{a\cos(\theta)} \cos(a\sin(\theta))\,\mathrm{d}\theta = \pi.$$

Solution. Note that the function $\frac{e^{az}}{z}$ is analytic everywhere inside and on C, except for a simple pole at $z = 0$, which lies inside C. We apply the Cauchy Integral Formula, to get

$$\int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z = 2\pi i f(z_0),$$

for a function f analytic on and inside C, and z_0 inside C. In our case, $f(z) = e^{az}$, and $z_0 = 0$, so

$$\int_C \frac{e^{az}}{z} \, \mathrm{d}z = 2\pi i e^{a \cdot 0} = 2\pi i.$$

Let $z = e^{i\theta}$, with $-\pi \leq \theta \leq \pi$, then $dz = ie^{i\theta}\mathbf{d}\theta$. So, we have

$$\int_C \frac{e^{az}}{z} \, \mathrm{d}z = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} \cdot ie^{i\theta}\mathbf{d}\theta = i \int_{-\pi}^{\pi} e^{ae^{i\theta}} \, \mathrm{d}\theta.$$

Now write $e^{ae^{i\theta}}$ in terms of real and imaginary parts, to get

$$e^{ae^{i\theta}} = e^{a(\cos\theta + i\sin\theta)} = e^{a\cos\theta} \cdot e^{ia\sin\theta} = e^{a\cos\theta}\left[\cos(a\sin\theta) + i\sin(a\sin\theta)\right].$$

So,

$$\begin{aligned} i\int_{-\pi}^{\pi} e^{ae^{i\theta}} \, \mathrm{d}\theta &= i\int_{-\pi}^{\pi} e^{a\cos\theta}\left[\cos(a\sin\theta) + i\sin(a\sin\theta)\right] \mathrm{d}\theta \\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta) \, \mathrm{d}\theta. \end{aligned}$$

We know this entire expression equals $2\pi i$, so

$$i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta - \int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta) \, \mathrm{d}\theta = 2\pi i.$$

Equating real and imaginary parts, the imaginary part gives

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta = 2\pi.$$

Since the integrand is even (both cos and $e^{a\cos\theta}$ are even functions),

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta = 2\int_{0}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta.$$

So,

$$2\int_{0}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta = 2\pi \Rightarrow \int_{0}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) \, \mathrm{d}\theta = \pi.$$ □