SOLUTIONS TO HOMEWORK 1

Warning: Very little proofreading has been done.

1. Section 4.3

Exercise 4.3.3a Supply a proof for Theorem 4.3.9 using the ε-δ characterization of continuity.

Solution. We have to prove the following. Let $A, B \subseteq \mathbb{R}$, let $f\colon A \to \mathbb{R}$ and $g\colon B \to R$ be functions, and assume that the range $f(A) = \{f(x)\colon x \in A\}$ of f is contained in B (so that the composition $g \circ f(x) = g(f(x))$ is well defined on A). If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Let $\varepsilon > 0$. Since g is continuous at $f(c)$, choose $\rho > 0$ such that for all $y \in B$ with $|y - f(c)| < \rho$, we have $|g(y) - g(f(c))| < \varepsilon$. Since f is continuous at c, choose $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \rho$. Let $x \in A$ satisfy $|x - c| < \delta$. Then $|f(x) - f(c)| < \rho$, so $|g(f(x)) - g(f(c))| < \varepsilon$. $\square$

Example 4.3.6abc Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 but such that $f(x) + g(x)$ and $f(x)g(x)$ are both continuous at 0.
(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.
(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

Solution to (a). Let

$$f(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases} \quad \text{and} \quad g(x) = -f(x) = \begin{cases} -1 & x \geq 0, \\ 1 & x < 0. \end{cases}.$$

Then $f(x) + g(x) = 0$ and $f(x)g(x) = -1$ are both continuous at 0. $\square$

Solution to (b). This is not possible. If $f(x) + g(x)$ were continuous, then, by the Algebraic Continuous Theorem, $g(x) = (f(x) + g(x)) - f(x)$ would have to be continuous. $\square$

Solution to (c). This is not possible if $f(0) \neq 0$, since if $h(x) = f(x)g(x)$ were continuous at 0, then $g(x) = h(x)/f(x)$ would be continuous by the Algebraic Continuous Theorem.

It is possible if $f(0) = 0$ so that $\lim_{x \to 0} f(x) = 0$. If g is bounded in a neighborhood of 0 but not continuous at 0, then $f(x)g(x)$ is continuous at 0. Indeed, if $|g(x)| \leq M$ in the neighborhood $V_\rho(0)$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta < \rho$ and, for every $|x| < \delta$, $|f(x) - f(0)| = |f(x)| < \varepsilon/M$. Then, for $|x| < \delta$,

$$|f(x)g(x) - f(0)g(0)| = |f(x)| \cdot |g(x)| \leq M|f(x)| < \varepsilon.$$

This proves that $f(x)g(x)$ is continuous at 0. $\square$

Example 4.3.8 Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.
(b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.
(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution to (a). This is true. Since g is continuous, $g(1) = \lim_{n\to\infty} g(1 - 1/n) \geq 0$ by the Order Limit Theorem (Theorem 2.3.4). $\square$

Solution to (b). This is true. By Theorem 3.2.10, for every $y \in \mathbb{R}$, there is a sequence y_n or rational numbers that converges to y. Since $g(y_n) = 0$ and g is continuous, $g(y) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(0) = 0$. $\square$

Alternative solution to (b). Let $x \in \mathbb{R}$. We prove that for every $\varepsilon > 0$, we have $|f(x)| < \varepsilon$. So let $\varepsilon > 0$. Choose $\delta > 0$ such that whenever $y \in \mathbb{R}$ satisfies $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ (see Theorem 3.2.10), there is $y \in \mathbb{Q}$ such that $|y - x| < \delta$. Then

$$|f(x)| = |0 - f(x)| = |f(y) - f(x)| < \varepsilon.$$

$\square$

Solution to (c). This is true. Let $\varepsilon = g(x_0)/2 > 0$. Choose $\delta > 0$ such that, for all $x \in V_\delta(x_0)$, $|g(x) - g(x_0)| < \varepsilon$, which implies that

$$g(x) > g(x_0) - \varepsilon = g(x_0) - g(x_0)/2 = g(x_0)/2 > 0.$$

This shows that $g(x)$ is positive for all $x \in V_\delta(x_0)$, or $x \in (x_0 - \delta, x_0 + \delta)$. The interval contains uncountably many points. $\square$

Example 4.3.9 Assume $h : \mathbb{R} \mapsto \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Shown that K is a closed set.

Solution. Let y be a limit point of K. We need to show that $y \in K$. By Theorem 3.2.5, there is a subsequence of (y_n) in K such that $y_n \to y$ as $n \in \infty$. Then $g(y_n) = 0$ and, since g is continuous, $g(y) = \lim_{n\to\infty} g(y) = \lim_{n\to\infty} 0 = 0$. This shows that y is a zero of g, so that $y \in K$. $\square$

2. Section 4.5

Exercise 4.5.3. A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Solution. We prove f is continous at $a < x_0 < b$. The proof applies to $x = a$ or $x = b$ (one-sided limit). Since f is increasing, we assume $f(x_0) < f(b)$ as otherwise f is a constant on $[x_0, b]$. Then, for $0 < \varepsilon < f(b) - f(x_0)$, $f(x_0) < f(x_0) + \varepsilon \leq f(b)$. By the intermediate value theorem, there is y_1 such that $f(x_0) + \varepsilon = f(y_1)$ and $x_0 < y_1 < b$. Thus,

$$f(x_0) \leq f(x) \leq f(y_1) = f(x_0) + \varepsilon, \quad \text{for } x \text{ satisfying } x_0 < x \leq y_1.$$

Similarly, consider $f(a) < f(x_0) - \varepsilon < f(x_0)$ for $0 < \varepsilon < f(x_0) - f(a)$, there is a y_2, such that $f(x_0) - \varepsilon = f(y_2)$ for $a < y_2 < x_0$. Thus,

$$f(x_0) \geq f(x) \geq f(y_2) = f(x_0) - \varepsilon, \quad \text{for } x \text{ satisfying } y_2 \leq x < x_0.$$

Hence, choosing $\delta = \min\{y_1 - x_0, x_0 - y_2\}$, we obtain

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for } |x - x_0| < \delta.$$

□

Exercise 4.5.6a. Let $f : [0, 1] \mapsto \mathbb{R}$ be continuous with $f(0) = f(1)$. (a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

Solution. Let $F(x) = f(x + \frac{1}{2}) - f(x)$ on $[0, \frac{1}{2}$. Then, the problem is equivalent to showing that there exists $x \in [0, \frac{1}{2}]$ such that $F(x) = 0$. Now, $F(0) = f(\frac{1}{2}) - f(0)$ and $F(1) = f(1) - f(\frac{1}{2}) = f(0) - f(\frac{1}{2}) = -F(0)$. If $F(0) = 0$, we are done. If $F(0) \neq 0$, then $F(0)$ and $F(1)$ have opposite sign, so that there must be a $c \in (0, 1)$ such that $F(c) = 0$ by the intermediate value theorem, □

Exercise 4.5.7. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at lease one value of $x \in [0, 1]$.

Solution. Let $F(x) = f(x) - x$. Then F is a continuous function. If either $F(0) = 0$ or $F(1) = 0$, we are done. Assume otherwise, then $F(0) = f(0) - 0 > 0$ and $F(1) = f(1) - 1 < 0$. By the intermediate value theorem, F must have a zero in $(0, 1)$, which is a fixed point of f. □