Chapter 1

Introduction to Soergel Bimodules

Exercise 7.8. Show that the axioms of a 2-category imply the following equalities.

Solution to 7.8. ☐

Exercise 7.16. We can view the algebra $A = \mathbb{R}[x]/(x^2)$ as an object in the monoidal category of $\mathbb{R}$-vector spaces. Let $\cap : A \otimes A \to \mathbb{R}$ denote the linear map which sends $f \otimes g$ to the coefficient of x in fg. Let $\cup : \mathbb{R} \to A \otimes A$ denote the map which sends 1 to $x \otimes 1 + 1 \otimes x$.

1. We wish to encode these maps diagrammatically, drawing $\cap$ as a cap and $\cup$ as a cup. Justify this diagrammatic notation, by checking the isotopy relations.
2. Draw a sequence of nested circles, as in an archery target. Evaluate this morphism.

Solution to 7.16(i). ☐

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Exercise 7.17. This question is about the Temperley–Lieb category.

1. Finish the proof that the isotopy relation holds in vector spaces.
2. There is a map $V \otimes V \to V \otimes V$ which sends $x \otimes y \mapsto y \otimes x$. Draw this as an element of the Temperley–Lieb category (a linear combination of diagrams).
3. Find an endomorphism of 2 strands which is killed by placing a cap on top. Can you find one which is an idempotent? Also find an endomorphism killed by putting a cup on bottom.
4. (Harder) Find an idempotent endomorphism of 3 strands which is killed by a cap on top (for either of the two placements of the cap).

Solution to 7.17(i). ☐

Solution to 7.17(ii). ☐

Solution to 7.17(iii). ☐

Exercise 7.19. One can think about the right mate and the left mate as "twisting" or "rotating" α by 180° to the right or to the left. Visualize what it would mean to twist α by 360° to the right, yielding another 2-morphism $\alpha^{\vee\vee} : E \to F$. Verify that ${}^{\vee}\alpha = \alpha^{\vee}$, if and only if $\alpha = \alpha^{\vee\vee}$. Thus cyclicity is the same as "360 degree rotation invariance," which one might expect from any planar picture.

Solution to 7.19. ☐

Exercise 7.20. Suppose that B is an object in a monoidal category with biadjoints, and $\Phi : B \otimes B \otimes B \to \Bbbk$ is a cyclic morphism. What should it mean to "rotate" Φ by $120°$? Suppose that $\mathrm{Hom}(B \otimes B \otimes B, \Bbbk)$ is one-dimensional over $\mathbb{C}$. What can you say about the $120°$ rotation of Φ, vis a vis Φ? What if $\mathrm{Hom}(B \otimes B \otimes B, \Bbbk)$ is one-dimensional over $\mathbb{R}$?

Solution to 7.20. $\square$

Exercise 9.25.

Solution to 9.25. $\square$

Exercise 9.26.

Solution to 9.26. $\square$

Exercise 9.27.

Solution to 9.27. $\square$

Exercise 9.28.

Solution to 9.28. $\square$

Chapter 2

Research Papers

Exercise 1. Compute the value of a bigon at $q = 1$ or at general q.

Solution to 1. □

Exercise 2. Look at (2.9). Can you find associativity and coassociativity inside? Use only these relations and (2.4) to prove (2.9).

Solution to 2. □

Exercise 3. Write down what (2.10) means explicitly for some small values of k, l, r, s, until you get a feeling for how it works. You'll definitely want an exampel where k-l+r-s is at least 2 eventually. Then try to verify it using vectors for small values.

Solution to 3. □

Exercise 4. Try to prove Lemma 2.9 from Light Ladders and Clasp Conjectures

Solution to 4. □

Exercise 5. Remember how for the Temperley-Lieb algebra you described the "Crossing" $v \otimes w \mapsto w \otimes v$ as a linear combination of other maps. Let's do this again, but with webs this time. You're going to have to use $q = 1$ do this exercise, so forget about the q-deformation.

Consider the map $\Lambda^1 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^1 V$ which just swaps the tensor factors. This is a linear combination of:

1. The web which merges $1, 2$ into 3 and then splits 3 into $2, 1$.
2. The web which splits $1, 2$ into $1, 1, 1$ and then merges $1, 1, 1$ into $2, 1$.

Find the linear combo.

Solution to 5(i). □

Solution to 5(ii). □

Exercise 6. Consider the map $\Lambda^2 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^2 V$ which just swaps the tensor factors. This is a linear combination of:

1. the web which merges $2, 2$ into 4 and then splits 4 into $2, 2$.
2. the web which splits $2, 2$ into $2, 1, 1$ and then merges $2, 1, 1$ into $3, 1$ and then splits back to $2, 1, 1$ and marges back to $2, 2$.

3. the identity of $2, 2$.

Find the linear combo.

Solution to 6(i). $\square$

Solution to 6(ii). $\square$

Solution to 6(iii). $\square$

Chapter 3

Misc

Exercise 1. Find a formula for the product $[n][3]$ when $n \geq 3$ and $[n][4]$ when $n \geq 4$. Generalize this.

Solution to 1. $\square$

Exercise 2. What is $[n][n] - [n+1][n-1]$?

Solution to 2. $\square$

Exercise 3. What is $[n][k] - [n+1][k-1]$ for $k < n$?

Solution to 3. $\square$