SOLUTIONS TO HOMEWORK 6

Warning: Little proofreading has been done.

1. Section 2.7

Exercise 2.7.4 Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both divergent but where $\sum x_n y_n$ converges.
(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.
(c) Two sequence (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but where $\sum y_n$ diverges.
(d) A sequence $(x)n)$ satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.

Solution. (a) Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n}$. Then both $\sum x_n$ and $\sum y_n$ diverge but $\sum x_n y_n = \sum \frac{1}{n^2}$ converges.

(b) Let $x_n = (-1)^n/n$ and $(y_n) = (-1)^n$. Then $\sum x_n$ converges and $(y)_n$ is bounded, but $\sum x_n y_n = \sum \frac{1}{n}$ diverges.

(c) This is impossible. In fact, since $\sum y_n = \sum(x_n + y_n) - \sum x_n$ and both series in the right hand side converge, $\sum y_n$ converges by Theorem 2.7.1.

(d) Let $x_1 = 1$, $x_{2n} = \frac{1}{2n}$ and $x_{2n+1} = \frac{1}{2n} - \frac{1}{2n+1} = \frac{1}{2n(2n+1)}$. Then $0 \leq x_n \leq \frac{1}{n}$. However,

$$x_{2n} - x_{2n+1} = \frac{1}{2n} - (\frac{1}{2n} - \frac{1}{2n+1}) = \frac{1}{2n+1},$$

so that the partial sum

$$\sum_{n=1}^{2m+1} (-1)^n x_n = -1 + (x_2 + x_3) + (x_4 + x_5) + \ldots + (x_{2m} + x_{2m+1}) = -1 + \sum_{n=1}^{m} \frac{1}{2n+1},$$

which is unbounded, hence $\sum(-1)^n x_n$ diverges. □

Exercise 2.7.7 (a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l > 0$, then the series $\sum a_n$ diverges.
(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution. (a) Since $a_n > 0$ and $\lim na_n = l$. Let $\varepsilon = l/2 > 0$. Then there is $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$|a_n - l| < \varepsilon = l/2,$$

which implies that $|a_n - l| \geq l/2$ or

$$|a_n| = |l - (l - a_n)| \geq l - |a_n - l| \geq l - l/2 = l/2.$$

Hence, for $n \geq N$,

$$\sum_{k=N}^{n} a_k \geq \frac{l}{2} \sum_{k=N}^{n} 1 = \frac{l}{2}(n - N)$$

is unbounded. By Cauchy criterion, the series $\sum a_n$ diverges. □

(b) Since $\sum na_n^2$ converges, $n^2 a_n \to 0$ as $n \to \infty$ by Theorem 2.7.3. In particular, the sequence is bounded. That is, there is $M > 0$ such that

$$0 < n^2 a_n \leq M, \qquad \text{or} \qquad a_n \leq M/n^2 \qquad \text{for all } n \geq 1.$$

Since $\sum 1/n^2$ converges, $\sum a_n$ converges by the Comparison Test. □

Exercise 2.7.8. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.
(b) If $\sum a_n$ and $\sum b_n$ converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution. (a) Since $\sum |a_n|$ converges, $|a_n| \to 0$. In particular, $|a_n|$ is bounded. There is an $M > 0$ such that $|a_n| \leq M$. Hence,

$$\sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} |a_k| \cdot |a_k| \leq M \sum_{k=1}^{\infty} |a_k|.$$

By comparison test, $\sum a_k^2$ converges.

(b) This is not true. For example, let $a_n = b_n = (-1)^n/\sqrt{n}$. Then $\sum a_n$ and $\sum b_n$ converge by the Theorem of Alternating Series, but $\sum a_n b_n = \sum \frac{1}{n}$ diverges.

(c) This is true. If $\sum n^2 a_n$ were to converge, then as in Exercise 2.7.7b, we see that there would be $M > 0$ such that $|a_n| \leq M/n^2$ for all $n \in \mathbb{N}$, so that $\sum |a_n|$ converges by Comparison Test. The is a contradiction to the assumption that $\sum a_n$ converges conditionally.

□

2. Section 3.2

Exercise 3.2.1

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?
(b) Give an example of an infinite collection of nested open sets

$$O_1 \supset O_2 \supset O_3 \supset \cdots$$

whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbb{R}$.

Solution. (a) We need N to be finite so that we can say that $\inf(\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}) > 0$. This is true because

$$\inf(\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}) = \min(\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}).$$

(b) We have seen that $[-1, 1]$ is closed, and that for $n \in \mathbb{N}$, the interval $\left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ is open. Clearly

$$\bigcap_{n=1}^{\infty} \left(-1 - \tfrac{1}{n}, 1 + \tfrac{1}{n}\right) = [-1, 1].$$

□

Exercise 3.2.3 Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ε-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$.
(b) $\mathbb{N}$.
(c) $\{x \in \mathbb{R} \colon x \neq 0\}$.

Solution. (a) The set $\mathbb{Q}$ is not open. Consider $0 \in \mathbb{Q}$. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Set

$$a = \frac{\sqrt{2}}{2n}.$$

We know $\sqrt{2} \notin \mathbb{Q}$, so also $a \notin \mathbb{Q}$. But $0 < a < \frac{1}{n} < \varepsilon$, so $a \in V_\varepsilon(0)$.

The set $\mathbb{Q}$ is also not closed. We know $\sqrt{2} \notin \mathbb{Q}$. However, we claim that $\sqrt{2}$ is a limit point of $\mathbb{Q}$. To see this, let $\varepsilon > 0$. The order density of $\mathbb{Q}$ implies that there is $r \in \mathbb{Q}$ such that $\sqrt{2} < r < \sqrt{2} + \varepsilon$. Then $r \in V_\varepsilon(\sqrt{2})$ but $r \neq \sqrt{2}$, as desired.

We can also see that $\mathbb{Q}$ is not closed by combining Theorem 3.2.10 and Theorem 3.2.5 to show that $\sqrt{2}$ is a limit point of $\mathbb{Q}$.

(b) The set $\mathbb{N}$ is not open. Consider $1 \in \mathbb{N}$. Let $\varepsilon > 0$. Then $a = 1 + \min\left(\frac{1}{2}, \frac{1}{2}\varepsilon\right)$ satisfies $a \notin \mathbb{N}$ but $a \in V_\varepsilon(0)$.

The set $\mathbb{N}$ is closed since it has no limit point. It can also be seen by showing that $\mathbb{R} \setminus \mathbb{N}$ is open. First observe that $(-\infty, 1)$ is open. One can prove this directly, or write it as the union of bounded open intervals as follows:

$$(-\infty, 1) = \bigcup_{n=1}^{\infty} (-n, 1).$$

We can now write $\mathbb{R} \setminus \mathbb{N}$ as a union of open sets:

$$\mathbb{R} \setminus \mathbb{N} = (-\infty, 1) \cup \bigcup_{n=1}^{\infty} (n,\, n+1).$$

(c) The set $\{x \in \mathbb{R} \colon x \neq 0\}$ is open, since we can write it as a union of sets that are known to be open:

$$\{x \in \mathbb{R} \colon x \neq 0\} = \bigcup_{n=1}^{\infty} (-n, 0) \cup (0, n).$$

However, this set is not closed. Clearly $0 \notin \{x \in \mathbb{R} \colon x \neq 0\}$, but we claim that 0 is a limit point of $\{x \in \mathbb{R} \colon x \neq 0\}$. So let $\varepsilon > 0$. Then $\frac{1}{2}\varepsilon \in V_\varepsilon(0)$, proving the claim.

$\square$

Exercise 3.2.6 Decide whether the following statements are true of false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.
(b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."
(c) Every nonempty open set contains a rational number.
(d) Every bounded infinite closed set contains a rational number.

Solution. (a) This is true. Let the set be denoted by S. For each $r \in \mathbb{Q}$, there is an $\varepsilon(r) > 0$ such that $V_{\varepsilon(r)}(r) \subset S$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we see that

$$S = \bigcup_{r \in \mathbb{Q}} V_{\varepsilon(r)}(r).$$

By the density of $\mathbb{Q}$, S contains all irrational numbers and, hence, all real numbers. Then $S = \mathbb{R}$.

(b) This is false. For example, let $F_n = \{m \in \mathbb{N} : m \geq n\}$; that is, $F_n = \{n, n+1, n+2, \ldots\}$. Then F_n contains only isolated points, so that F_n is a closed set. Evidently, $F_1 \supset F_2 \supset F_3 \supset \ldots$. However, $\bigcap_n F_n = \emptyset$.

(c) This is true. Let O be such a set. Since it is nonempty, it contains at least one element a. By definition, there is an ε-neighborhood $V_\varepsilon(a) \subset O$. By the density of $\mathbb{Q}$, there are rational numbers in $V_\varepsilon(a)$ and, hence, in O.

(d) This is false. For example, let $S = \{\sqrt{2}\} \cup \{\sqrt{2}(1 + 1/n) : n \in \mathbb{N}\}$. Then S is infinite, bounded (by $\sqrt{2}$ from below and $2\sqrt{2}$ from above), and is closed (with $\sqrt{2}$ as the only limiting point). However, S does not contain any rational point. $\square$