Example: This example is to find a polynomial $P_0(x)$ of degree at most n such that given $n+1$ distinct constants $t_1, t_2, \cdots, t_{n+1}$ and $n+1$ constants $c_1, c_2, \cdots, c_{n+1}$, $P_0(x)$ satisfies:

$$P_0(t_1)=c_1, \quad P_0(t_2)=c_2, \quad \cdots, \quad P_0(t_{n+1}) = c_{n+1} \qquad (*)$$

Method 1: (Math 341 Method):

Suppose $P_0(x)= a_0 + a_1 x + \cdots + a_n x^n$ such that $P_0(x)$ satisfies $(*)$.

Then

$$\begin{aligned} a_0 + a_1 t_1 + \cdots + a_n t_1^n &= c_1 \\ a_0 + a_1 t_2 + \cdots + a_n t_2^n &= c_2 \\ &\vdots \\ a_0 + a_1 t_{n+1} + \cdots + a_n t_{n+1}^n &= c_{n+1} \end{aligned} \iff \underbrace{\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ 1 & t_2 & t_2^2 & \cdots & t_2^n \\ \vdots & & & & \\ 1 & t_{n+1} & t_{n+1}^2 & \cdots & t_{n+1}^n \end{pmatrix}}_{A} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{pmatrix}}_{\vec{c}}$$

Note that A is a Vandermonde matrix, and $\det A = \prod_{1 \le i < j \le n+1} (t_j - t_i)$

As $t_1, \cdots, t_{n+1}$ are distinct, $\det A \neq 0$

$\Rightarrow$ The above linear system about $a_0, \cdots, a_n$ has a unique solution: $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = A^{-1}\vec{c}$.

end of Jan 29

Method 2. (Math 441 method of using linear functionals). For simpler notations, we will work with $n=2$ case.

Refrashing the problem for $n=2$:

Given t_1, t_2, t_3 distinct constants, and c_1, c_2, c_3 constants, find $P_0(x) \in \mathbb{P}^2(\mathbb{R})$ such that $P_0(t_1)= c_1$, $P_0(t_2)= c_2$, $P_0(t_3)= c_3$.

Idea!

Idea of Proof: If $\{L_1, L_2, L_3\} \subseteq (\mathbb{P}^2(\mathbb{R}))^*$ and $\{P_1(x), P_2(x), P_3(x)\} \in \mathbb{P}^2(\mathbb{R})$ are dual bases. Then $P_0(x) = \sum_{i=1}^{3} L_i(P_0(x))\, P_i(x)$.

To choose $L_i \in (\mathbb{P}^2(\mathbb{R}))^*$, note that $P_0(t_i)$ is the evaluation of a polynomial at t_i, so a natural choice for L_i would be: $L_{t_i}: \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}$

$$p(x) \mapsto p(t_i)$$

Then $L_i(P_0(x)) = C_i$, for $i=1,2,3$.

The dual basis $\{P_1(x), P_2(x), P_3(x)\} \subseteq \mathbb{P}^2(\mathbb{R})$ of $\{L_{t_1}, L_{t_2}, L_{t_3}\}$,

Satisfies: $L_{t_i}(P_j(x)) = P_j(t_i) = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j. \end{cases}$

For example: $P_1(x)$ satisfies $P_1(t_1)=1$ $P_1(t_2)=0$ $P_1(t_3)=0$

$\Rightarrow$ t_2, t_3 are roots of $P_1(x)$ $\Rightarrow$ $P_1(x) = a\cdot(x-t_2)\cdot(x-t_3)$

and as $P(t_1) = a_1(t_1-t_2)(t_1-t_3) = 1$ $\Rightarrow$ $a_1 = \dfrac{1}{(t_1-t_2)(t_2-t_3)}$.

Solutions: Denote: $L_{t_i}: \mathbb{P}^2(\mathbb{R}) \longrightarrow \mathbb{R}$ for $i=1,2,3$.

$$p(x) \longmapsto p(t_i)$$

Recall that L_{t_i} is a linear functional on $\mathbb{P}^2(\mathbb{R})$.

Claim 1: $\{L_{t_1}, L_{t_2}, L_{t_3}\}$ are linearly independent.

Suppose $a_1 L_{t_1} + a_2 L_{t_2} + a_3 L_{t_3} = 0$.

Then: for $f_1 = 1$, $f_2 = x$, $f_3 = x^2 \in \mathbb{P}^2(\mathbb{R})$

$$(a_1 L_{t_1} + a_2 L_{t_2} + a_3 L_{t_3})(f_1) = a_1 + a_2 + a_3 = 0$$

$$(a_1 L_{t_1} + a_2 L_{t_2} + a_3 L_{t_3})(f_2) = a_1 t_1 + a_2 t_2 + a_3 t_3 = 0$$

$$(a_1 L_{t_1} + a_2 L_{t_2} + a_3 L_{t_3})(x^2) = a_1 t_1^2 + a_2 t_2^2 + a_3 t_3^2 = 0$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A^{-1}\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $\det A = (t_2-t_1)(t_3-t_1)(t_3-t_1) \neq 0$ as t_1, t_2, t_3 are distinct.

Vandermonde matrix

$\Rightarrow \{L_{t_1}, L_{t_2}, L_{t_3}\} \subseteq V^*$ is linearly independent

As $\dim V^* = 3$, $\{L_{t_1}, L_{t_2}, L_{t_3}\}$ is a basis for V^*.

Define: $p_1(x) = \dfrac{(x-t_2)(x-t_3)}{(t_1-t_2)(t_1-t_3)}$, $p_2(x) = \dfrac{(x-t_1)(x-t_3)}{(t_2-t_1)(t_2-t_3)}$, $p_3(x) = \dfrac{(x-t_1)(x-t_2)}{(t_3-t_1)(t_3-t_2)}$

Claim: $\{p_1(x), p_2(x), p_3(x)\}$ is a basis for $\mathbb{P}^2(\mathbb{R})$.

Proof of the claim: Suppose $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0$

in particular:

$$b_1p_1(t_1) + b_2p_2(t_1) + b_3p_3(t_1) = 0$$

$$b_1p_1(t_2) + b_2p_2(t_2) + b_3p_3(t_2) = 0$$

$$b_1p_1(t_3) + b_2p_3(t_3) + b_3p_3(t_3) = 0.$$

Note that:

$p_1(t_1) = 1$	$p_2(t_1) = 0$	$p_3(t_1) = 0$
$p_1(t_2) = 0$	$p_2(t_2) = 1$	$p_3(t_2) = 0$
$p_1(t_3) = 0$	$p_2(t_3) = 1$	$p_3(t_3) = 0$

plug those values into the above system: $b_1 = 0$, $b_2 = 0$, $b_3 = 0$

$\Rightarrow$ $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent and thus a basis for $\mathbb{P}^2(\mathbb{R})$.

As: $L_{t_i}(p_j(x)) = p_j(t_i) = \delta_{ij}$ for all $i, j = 1, 2, 3$

$\Rightarrow \{L_{t_1}, L_{t_2}, L_{t_3}\} \subseteq (\mathbb{P}^2(\mathbb{R}))^*$ and $\{p_1(x), p_2(x), p_3(x)\}$ are dual bases.

If $p_0(x) \in \mathbb{P}^2(\mathbb{R})$ satisfies: $p_0(t_1) = c_1$, $p_0(t_2) = c_2$, $p_0(t_3) = c_3$.

then
$$p_0(x) = \sum_{i=1}^{3} L_{t_i}(p_0(x))\, p_i(x)$$

$$= \sum_{i=1}^{3} p_0(t_i)\, p_i(x)$$

$$= c_1p_1(x) + c_2p_2(x) + c_3p_3(x).$$

$\boxtimes$

Proposition: Given that $t_1, t_2, \cdots, t_{n+1}$ are distinct constants and $c_1, c_2, \cdots, c_{n+1}$ are constants.

then $P_0(x) \in \mathbb{P}^n(\mathbb{R})$ satisfies $P_0(t_1) = c_1$, $P_0(t_2) = c_2$, $\cdots$, $P_0(t_{n+1}) = c_{n+1}$ have

the following form:
$$P_0(x) = \sum_{k=1}^{n} c_k p_k(x)$$
$$\text{where } \quad p_k(x) = \frac{\prod_{i=1, i\neq k}^{n+1} (x - t_i)}{\prod_{i=1, i\neq k}^{n+1} (t_k - t_i)}.$$

Definition: (transpose of a linear map or the dual map of a linear map).

Let V, W be finite dimensional vector spaces. Let B_1 and B_2 be ordered bases for V and W respectively. Let B_1^* and B_2^* be dual bases corresponding to B_1, and B_2 respectively. For any linear transformation $T: V \to W$, we may define $T^t: W^* \to V^*$

by: $T^t(\varphi) = \varphi \circ T$. for any $\varphi \in W^*$.

Then T^t is a linear transformation and $[T^t]_{B_2^*}^{B_1^*} = \left([T]_{B_1}^{B_2}\right)^T$.

where $\left([T]_{B_1}^{B_2}\right)^T$ stands for the transpose of $[T]_{B_1}^{B_2}$.

__Proof__: 1) Check T^t is well-define. i.e. check $T^t(\varphi) \in V^*$ for all $\varphi \in W^*$.

As $T: V \to W$ is a linear transformation, and $\varphi: W \to F$ is also a linear transformation $\varphi \circ T: V \to F$ is also a linear transfo

$$T^t(\varphi) = \varphi \circ T \in V^*$$

2). Check: $T^t: W^* \longrightarrow V^*$ is a linear transformation.

$\forall \varphi, \psi \in W^*$ and $c \in F$, need to show $T^t(c\varphi + \psi) = cT^t(\varphi) + T^t(\psi)$

In fact: $\forall \vec{x} \in V$: $\quad T^t(c\varphi + \psi)(\vec{x})$

// end of Jan 31