

Ex: Determine if the limit exists

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+4y^2}}$$

$$f(x,y) = \frac{xy}{\sqrt{x^2+4y^2}}$$

Domain: $D = \{(x,y) \in \mathbb{R}^2; (x,y) \neq (0,0)\}$

Path 1: $x=0$

$$f(0,y) = \frac{0(y)}{\sqrt{0+4y^2}} = \frac{0}{2|y|} = 0 \text{ if } y \neq 0$$

$$\lim_{y \rightarrow 0} f(0,y) = \lim_{y \rightarrow 0} 0 = 0$$

Path 2: $y = mx \quad m \in \mathbb{R}$

$$f(x, mx) = \frac{x(mx)}{\sqrt{x^2+4m^2x^2}} = \frac{mx^2}{|x|\sqrt{1+4m^2}} = \frac{m|x|}{\sqrt{1+4m^2}}$$

$$\lim_{x \rightarrow 0} f(x, mx) = \lim_{x \rightarrow 0} \frac{m|x|}{\sqrt{1+4m^2}} = 0 \quad \text{for all } m \in \mathbb{R}$$

Along every line through $(0,0)$, the function approached 0 as $(x,y) \rightarrow (0,0)$.

It appears the limit exists. How to prove it?

Squeeze Theorem: If $g(x,y) \leq f(x,y) \leq h(x,y)$ for all (x,y) arbitrarily close to (a,b) except possibly at (a,b) and $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = L$ and $\lim_{(x,y) \rightarrow (a,b)} h(x,y) = L$, then $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$.

$$f(x,y) = \frac{xy}{\sqrt{x^2+4y^2}}$$

Find bounds for f near $(0,0)$.

$$x^2 + 4y^2 \geq x^2 \quad \text{for all } (x,y)$$

$$\sqrt{x^2+4y^2} \geq \sqrt{x^2} = |x| \quad \text{since } g = \sqrt{x} \text{ is an increasing function.}$$

$$\text{Then } \frac{|x|}{\sqrt{x^2+4y^2}} \leq 1 \quad \text{for all } (x,y) \neq (0,0)$$

$$\frac{|x||y|}{\sqrt{x^2+4y^2}} \leq |y|$$

$$-|y| \leq \frac{xy}{\sqrt{x^2+4y^2}} \leq |y| \quad \text{for all } (x,y) \neq (0,0)$$

$$\lim_{(x,y) \rightarrow (0,0)} |y| = 0$$

Since $-|y| \leq \frac{xy}{\sqrt{x^2+y^2}} \leq |y|$ and $\lim_{(x,y) \rightarrow (0,0)} |y| = \lim_{(x,y) \rightarrow (0,0)} -|y| = 0$,

then $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$ by Squeeze Theorem

Prove $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$ by definition of limit.

Show for every $\epsilon > 0$, there exists $\delta > 0$

such that if $0 < \sqrt{x^2+y^2} < \delta$ then $| \frac{xy}{\sqrt{x^2+y^2}} - 0 | < \epsilon$

Since for each ϵ , there exists a δ , δ will typically be a function of ϵ .

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|x||y|}{\sqrt{x^2+y^2}} < \frac{|x||y|}{\sqrt{x^2}} = \frac{|x||y|}{|x|} = |y| = \sqrt{y^2} < \sqrt{x^2+y^2} < \epsilon \text{ if } \delta = \epsilon$$

For every $\epsilon > 0$, there exists $\delta = \epsilon > 0$

such that if $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{|x||y|}{\sqrt{x^2}} = |y| < \sqrt{x^2+y^2} < \delta = \epsilon$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Continuity

Defn: The function, $f(x, y)$, is continuous at (a, b) if $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$.

Note: Polynomials, Rational functions, root functions, trig functions, exponential functions are continuous on their domain.

Ex: a) $\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + 2xy - y^2}{x^4 + y^4 + 2}$

$$f = \frac{x^2 + 2xy - y^2}{x^4 + y^4 + 2}$$

f is rational and continuous on its domain $D = \mathbb{R}^2$

By continuity,

$$\lim_{(x,y) \rightarrow (1,2)} f(x, y) = f(1, 2)$$

$$= \frac{1+4-4}{1+16+2} = \frac{1}{19}$$

b) $\lim_{(x,y) \rightarrow (3,5)} e^{y/x}$

$$f(x, y) = e^{y/x} = g(h(x, y))$$

$g(t) = e^t$ continuous on \mathbb{R}

$$h(x, y) = \frac{y}{x}$$
 discontinuous on line $x=0$

Since g is continuous on range of h , f is continuous on its domain

$$\lim_{(x,y) \rightarrow (3,5)} e^{y/x} = e^{5/3}$$

$$D = \{(x, y) \in \mathbb{R}^2; x \neq 0\}$$