Several-Variab Calc II: Homework 7

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Problem 1. Evaluate the line integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

(i) $\mathbf{F} = \langle xy^2 + 4xy, 2y + x^2 \rangle$ and C is the path $y = x^2$ from $(-2, 4)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-2, 4)$.

(ii) $\mathbf{F} = \langle y \sin(x) - y^4, y^2 - \cos(x) \rangle$ and C is the union of the half circle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$ and the line segment from $(-2, 0)$ to $(2, 0)$.

Solution to (i). The region D is the region enclosed by the path $y = x^2$ from $(-2, 4)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-2, 4)$. The region D is shown in Figure 1.

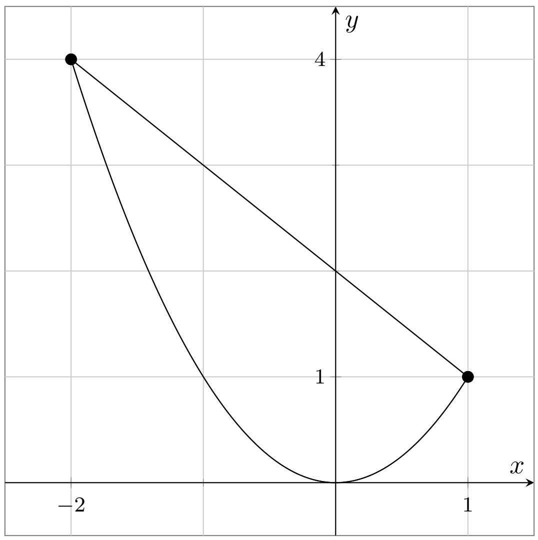

Figure 1: Region D for Problem 1(i)

Since D is positively oriented, piecewise-smooth, and simply connected, we can use Green's Theorem to evaluate the line integral, which states that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A,$$

where $P = xy^2 + 4xy$ and $Q = 2y + x^2$. We have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (2x) - (2xy + 4x) = 2xy - 2x.$$

Finding the equation of the line segment from $(1, 1)$ to $(-2, 4)$, we have $y = -x + 2$. Therefore, we get the bounds for x as $-2 \leq x \leq 1$ and $x^2 \leq y \leq -x + 2$. Therefore, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A \\
&= \int_{-2}^{1} \int_{x^2}^{-x+2} 2xy - 2x \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-2}^{1} xy^2 - 2xy \Big|_{x^2}^{-x+2} \mathrm{d}x \\
&= \int_{-2}^{1} \left[x(-x+2)^2 - 2x(-x+2) \right] - \left[x(x^2)^2 - 2x(x^2) \right] \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&= \int_{-2}^{1} \left[x^3 - 2x^2\right] - \left[x^5 - 2x^3\right] \,\mathrm{d}x \\
&= \int_{-2}^{1} -x^5 + 3x^3 - 2x^2 \,\mathrm{d}x \\
&= -\frac{x^6}{6} + \frac{3x^4}{4} - \frac{2x^3}{3}\bigg|_{-2}^{1} \\
&= \left[-\frac{(1)^6}{6} + \frac{3(1)^4}{4} - \frac{2(1)^3}{3}\right] - \left[-\frac{(-2)^6}{6} + \frac{3(-2)^4}{4} - \frac{2(-2)^3}{3}\right] \\
&= -\frac{27}{4}.
\end{aligned}$$

□

Solution to (ii). The region D is the region enclosed by the half circle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$ and the line segment from $(-2, 0)$ to $(2, 0)$. The region D is shown in Figure 2.

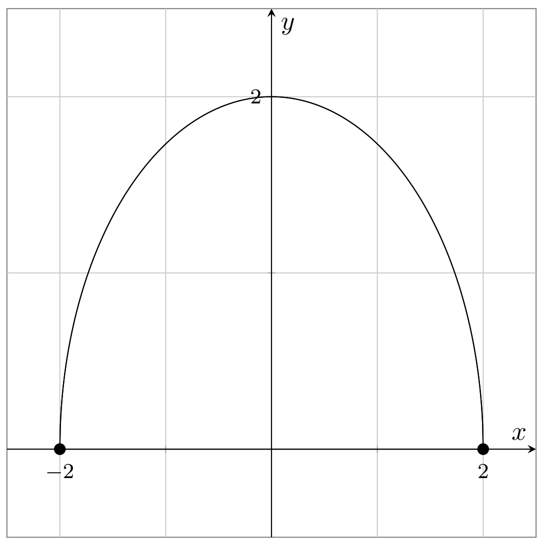

Figure 2: Region D for Problem 1(ii)

Since D is positively oriented, piecewise-smooth, and simply connected, we can use Green's Theorem to evaluate the line integral, which states that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A,$$

where $P = y\sin(x) - y^4$ and $Q = y^2 - \cos(x)$. We have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (\sin(x)) - \left(\sin(x) - 4y^3\right) = -4y^3 = -4r^3\sin^3(\theta).$$

Converting to polar coordinates, we get the bounds $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A \\
&= \int_0^{\pi} \int_0^2 -4r^3\sin^3(\theta) \cdot r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= -4 \int_0^{\pi} \sin^3(\theta) \,\mathrm{d}\theta \cdot \int_0^2 r^4 \,\mathrm{d}r
\end{aligned}$$

$$= \left(-4 \cdot \frac{32}{5}\right) \cdot \int_0^\pi \sin(\theta)(1-\cos^2(\theta))\,\mathrm{d}\theta.$$

Using the substitution $u = \cos(\theta)$, we get $\mathrm{d}u = -\sin(\theta)\,\mathrm{d}t$. The bounds for u are $u(0) = 1$ and $u(\pi) = -1$. Therefore, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \left(-4 \cdot \frac{32}{5}\right) \cdot \int_0^\pi \sin(\theta)(1-\cos^2(\theta))\,\mathrm{d}\theta \\
&= -\frac{128}{5} \cdot \int_1^{-1} -1 + u^2\,\mathrm{d}u \\
&= -\frac{128}{5} \cdot \int_{-1}^{1} 1 - u^2\,\mathrm{d}u \\
&= -\frac{128}{5} \cdot \left(u - \frac{u^3}{3}\Big|_{-1}^{1}\right) \\
&= -\frac{128}{5} \cdot \left(\left[1 - \frac{1}{3}\right] - \left[-1 + \frac{1}{3}\right]\right) \\
&= -\frac{128}{5} \cdot \left(2 - \frac{2}{3}\right) = -\frac{512}{15}.
\end{aligned}$$

□

Problem 2. If a closed and bounded region, D, has a constant density, ρ, then the center of mass is called the centroid.

(i) Use Green's Theorem to show that the centroid of D has coordinates

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,\mathrm{d}y \quad \text{and} \quad \bar{y} = -\frac{1}{2A}\oint_C y^2\,\mathrm{d}x,$$

where A is the area of D and C is the closed boundary of D with positive orientation.

(ii) Use these line integrals to find the centroid of the quarter circular region $D = \{(x,y) \mid x^2 + y^2 \leq a^2, x \geq 0, y \geq 0\}$.

Solution to (i). The centroid $(\bar{x}, \bar{y})$ of a region D with uniform density is given by by

$$\bar{x} = \frac{1}{A}\iint_D x\,\mathrm{d}A \quad \text{and} \quad \bar{y} = \frac{1}{A}\iint_D y\,\mathrm{d}A.$$

Converting the double integral for $\bar{x}$ to a line integral, we set $P = 0$ and $Q = x^2/2$. Therefore, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x.$$

Therefore,

$$\bar{x} = \frac{1}{A}\iint_D x\,\mathrm{d}A = \frac{1}{2A}\oint_C x^2\,\mathrm{d}y.$$

Similarly, converting the double integral for $\bar{y}$ to a line integral, we set $P = y^2/2$ and $Q = 0$. Therefore, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y.$$

Therefore, we have

$$\bar{y} = \frac{1}{A}\iint_D y\,\mathrm{d}A = -\frac{1}{2A}\oint_C y^2\,\mathrm{d}x.$$

Therefore, the centroid of D has coordinates

$$\bar{x} = \frac{1}{A}\iint_D x\,\mathrm{d}A = \frac{1}{2A}\oint_C x^2\,\mathrm{d}y \quad \text{and} \quad \bar{y} = \frac{1}{A}\iint_D y\,\mathrm{d}A = -\frac{1}{2A}\oint_C y^2\,\mathrm{d}x.$$

□

Solution to (ii). The region D is the quarter circular region $D = \{(x,y) \mid x^2+y^2 \leq a^2, x \geq 0, y \geq 0\}$. The region D is shown in Figure 3.

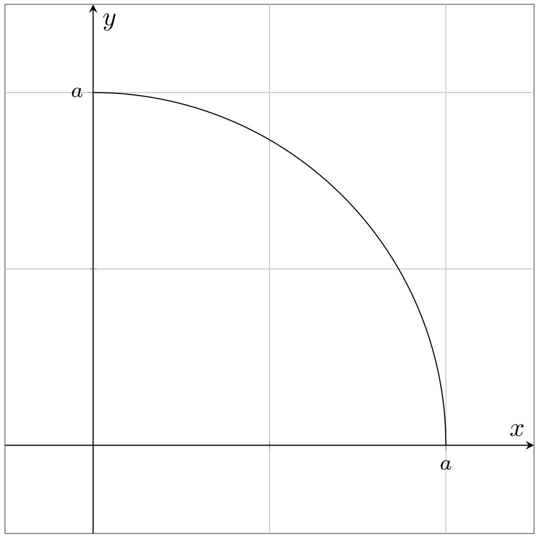

Figure 3: Region D for Problem 2(ii)

The area of D is $A = \pi a^2/4$.

We break the boundary of D into three parts: C_1 is the circular arc $r = a$, C_2 is the line segment $y = 0$, and C_3 is the line segment $x = 0$. Note that the paths C_2 and C_3 don't contribute to the line integrals for the centroid. We know that $x = a\cos(\theta)$ and $y = a\sin(\theta)$, giving us the differential $\mathrm{d}y = a\cos(\theta)\,\mathrm{d}\theta$. The bounds are clearly $0 \leq \theta \leq \pi/2$. Substituting these into the line integrals for the centroid, we get

$$\begin{aligned}\oint_{C_1} x^2\,\mathrm{d}y &= \int_0^{\pi/2} (a\cos(\theta))^2 \cdot a\cos(\theta)\,\mathrm{d}\theta \\ &= a^3 \int_0^{\pi/2} \cos^3(\theta)\,\mathrm{d}\theta \\ &= a^3 \int_0^{\pi/2} \cos(\theta)(1-\sin^2(\theta))\,\mathrm{d}\theta.\end{aligned}$$

Using the substitution $u = \sin(\theta)$, we get $\mathrm{d}u = \cos(\theta)\,\mathrm{d}t$. The bounds for u are $u(0) = 0$ and $u(\pi/2) = 1$. Therefore, we have

$$\begin{aligned}\oint_{C_1} x^2\,\mathrm{d}y &= a^3 \int_0^{\pi/2} \cos(\theta)(1-\sin^2(\theta))\,\mathrm{d}\theta \\ &= a^3 \int_0^1 (1-u^2)\,\mathrm{d}u \\ &= a^3 \left(u - \frac{u^3}{3}\bigg|_0^1 \right) \\ &= a^3\left(1 - \frac{1}{3}\right) = \frac{2a^3}{3}.\end{aligned}$$

Therefore, the center for the x-coordinate is

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,\mathrm{d}y = \frac{1}{2\cdot \pi a^2/4} \cdot \frac{2a^3}{3} = \frac{4a}{3\pi}.$$

Similarly, we have

$$\begin{aligned}\oint_{C_1} y^2\,\mathrm{d}x &= \int_0^{\pi/2} (a\sin(\theta))^2 \cdot -a\sin(\theta)\,\mathrm{d}\theta \\ &= -a^3 \int_0^{\pi/2} \sin^3(\theta)\,\mathrm{d}\theta \\ &= -a^3 \int_0^{\pi/2} \sin(\theta)(1-\cos^2(\theta))\,\mathrm{d}\theta.\end{aligned}$$

Using the substitution $u = \cos(\theta)$, we get $\mathrm{d}u = -\sin(\theta)\,\mathrm{d}t$. The bounds for u are $u(0) = 1$ and $u\left(\pi/2\right) = 0$. Therefore, we have

$$\begin{aligned}\oint_{C_1} y^2\,\mathrm{d}x &= -a^3 \int_0^{\pi/2} \sin(\theta)(1-\cos^2(\theta))\,\mathrm{d}\theta \\ &= -a^3 \int_1^0 1 - u^2\,\mathrm{d}u \\ &= -a^3 \left(u - \frac{u^3}{3}\bigg|_1^0 \right) \\ &= -a^3 \left(0 - \frac{0^3}{3} - 1 + \frac{1^3}{3} \right) \\ &= -a^3 \left(-\frac{1}{3} \right) = \frac{a^3}{3}.\end{aligned}$$

Therefore, the center for the y-coordinate is

$$\bar{y} = -\frac{1}{2A}\oint_C y^2\,\mathrm{d}x = -\frac{1}{2\cdot \pi a^2/4} \cdot \left(-\frac{a^3}{3}\right) = \frac{4a}{3\pi}.$$

Therefore, the centroid of the quarter circular region D is

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right). \qquad \square$$

Problem 3. Find parametric equations and the parameter domain that define the following surfaces. Then use the parametric equations to find the surface area of the surfaces.

Note: All steps used for computing the normal vector must be included.

(i) The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$.

(ii) The portion of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 16$.

(iii) The portion of the cylinder $x^2 + z^2 = 4$ that is above $z = 0$ and inside the cylinder $x^2 + y^2 = 4$.

Solution to (i). The equation has a radius of $r = 2$ centered at the origin and the given constraints are $z = -1$ and $z = \sqrt{3}$, which define a spherical cap. Using spherical coordinates, we have

$$x = 2\sin(\theta)\cos(\varphi), \quad y = 2\sin(\theta)\sin(\varphi), \quad \text{and} \quad z = 2\cos(\theta).$$

From $z = 2\cos(\theta)$, we set bounds for θ using

$$\begin{aligned}-1 = 2\cos(\theta) &\Rightarrow \cos(\theta) = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \\ \sqrt{3} = 2\cos(\theta) &\Rightarrow \cos(\theta) = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}.\end{aligned}$$

Thus, we get the bounds $\pi/6 \leq \theta \leq 2\pi/3$ and $0 \leq \varphi \leq 2\pi$.

We know that

$$\begin{aligned}
\mathbf{r}_\theta &= \left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right\rangle = \langle 2\cos(\theta)\cos(\varphi), 2\cos(\theta)\sin(\varphi), -2\sin(\theta) \rangle \\
\mathbf{r}_\varphi &= \left\langle \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi} \right\rangle = \langle -2\sin(\theta)\sin(\varphi), 2\sin(\theta)\cos(\varphi), 0 \rangle.
\end{aligned}$$

The normal vector is

$$\begin{aligned}
\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_\varphi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos(\theta)\cos(\varphi) & 2\cos(\theta)\sin(\varphi) & -2\sin(\theta) \\ -2\sin(\theta)\sin(\varphi) & 2\sin(\theta)\cos(\varphi) & 0 \end{vmatrix} \\
&= \langle (2\cos(\theta)\sin(\varphi))(0) - (-2\sin(\theta))(2\sin(\theta)\cos(\varphi)), \\
&\quad - \left[(2\cos(\theta)\cos(\varphi))(0) - (-2\sin(\theta))(-2\sin(\theta)\sin(\varphi)) \right], \\
&\quad (2\cos(\theta)\cos(\varphi))(2\sin(\theta)\cos(\varphi)) - (2\cos(\theta)\sin(\varphi))(-2\sin(\theta)\sin(\varphi)) \rangle \\
&= \langle 4\sin^2(\theta)\cos(\varphi), 4\sin^2(\theta)\sin(\varphi), 4\cos(\theta)\sin(\theta)\cos^2(\varphi) + 4\cos(\theta)\sin(\theta)\sin^2(\varphi) \rangle.
\end{aligned}$$

The magnitude of the normal vector is

$$\begin{aligned}
|\mathbf{r}_\theta \times \mathbf{r}_\varphi| &= \sqrt{\left(4\sin^2(\theta)\cos(\varphi)\right)^2 + \left(4\sin^2(\theta)\sin(\varphi)\right)^2 + \left(4\cos(\theta)\sin(\theta)\cos^2(\varphi) + 4\cos(\theta)\sin(\theta)\sin^2(\varphi)\right)^2} \\
&= \sqrt{16\sin^4(\theta)\cos^2(\varphi) + 16\sin^4(\theta)\sin^2(\varphi) + 16\cos^2(\theta)\sin^2(\theta)} \\
&= 4\sqrt{\sin^2(\theta)\left(\sin^2(\theta)\cos^2(\varphi) + \sin^2(\theta)\sin^2(\varphi) + \cos^2(\theta)\right)} \\
&= 4\sin(\theta)\sqrt{\sin^2(\theta)\cos^2(\varphi) + \sin^2(\theta)\sin^2(\varphi) + \cos^2(\theta)} \\
&= 4\sin(\theta)\sqrt{\sin^2(\theta)\left(\cos^2(\varphi) + \sin^2(\varphi)\right) + \cos^2(\theta)} \\
&= 4\sin(\theta)\sqrt{\sin^2(\theta) + \cos^2(\theta)} \\
&= 4\sin(\theta).
\end{aligned}$$

Therefore, the surface area element is

$$\mathrm{d}S = |\mathbf{r}_\theta \times \mathbf{r}_\varphi| \, \mathrm{d}A = 4\sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi.$$

The surface area integral is

$$\begin{aligned}
A &= \int_{\pi/6}^{2\pi/3} \int_0^{2\pi} 4\sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi \\
&= 4 \int_{\pi/6}^{2\pi/3} \sin(\theta)\, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \\
&= -8\pi\cos(\theta)\Big|_{\pi/6}^{2\pi/3} \\
&= -8\pi\left(\cos\left(\frac{2\pi}{3}\right) - \cos\left(\frac{\pi}{6}\right)\right) \\
&= -8\pi\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \\
&= 4\pi + 4\sqrt{3}\pi.
\end{aligned}$$

□

Solution to (ii). Using cylindrical coordinates, we have

$$x = r^2, \quad y = r\cos(\theta), \quad \text{and} \quad z = r\sin(\theta),$$

we get the parametric equation for the surface area as

$$\mathbf{r} = \langle r^2, r\cos(\theta), r\sin(\theta)\rangle.$$

The bounds for r are $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$.

We know that

$$\begin{aligned}
\mathbf{r}_r &= \left\langle \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r} \right\rangle = \langle 2r, \cos(\theta), \sin(\theta)\rangle \\
\mathbf{r}_\theta &= \left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right\rangle = \langle 0, -r\sin(\theta), r\cos(\theta)\rangle.
\end{aligned}$$

The normal vector is

$$\begin{aligned}
\mathbf{N} = \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2r & \cos(\theta) & \sin(\theta) \\ 0 & -r\sin(\theta) & r\cos(\theta) \end{vmatrix} \\
&= \langle (\cos(\theta))(r\cos(\theta)) - (\sin(\theta))(-r\sin(\theta)), -\left[(2r)(r\cos(\theta)) - (\sin(\theta))(0)\right], 2r(-r\sin(\theta)) - (\sin(\theta))(0)\rangle \\
&= \langle r\cos^2(\theta) + r\sin^2(\theta), -2r^2\cos(\theta), 2r^2\sin(\theta)\rangle \\
&= \langle r, -2r\cos(\theta), 2r^2\sin(\theta)\rangle
\end{aligned}$$

The magnitude of the normal vector is

$$\begin{aligned}
|\mathbf{r}_r \times \mathbf{r}_\theta| &= \sqrt{r^2 + 4r^2\cos^2(\theta) + 4r^2\sin^2(\theta)} \\
&= \sqrt{r^2 + 4r^2} = \sqrt{5r^2} = r\sqrt{5}.
\end{aligned}$$

Therefore, the surface area element is

$$\mathrm{d}S = |\mathbf{r}_r \times \mathbf{r}_\theta|\,\mathrm{d}A = r\sqrt{5}\,\mathrm{d}r\,\mathrm{d}\theta.$$

The surface area integral is

$$\begin{aligned}
A &= \int_0^{2\pi} \int_0^4 r\sqrt{5}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \sqrt{5}\int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^4 r\,\mathrm{d}r \\
&= 2\pi\sqrt{5} \cdot \frac{4^2}{2} = 16\pi\sqrt{5}.
\end{aligned}$$

□

Solution to (iii). Using cylindrical coordinates, we have

$$x = 2\cos(\theta), \quad y = y, \quad \text{and} \quad z = 2\sin(\theta),$$

where $r = 2$. Therefore, we get the parametric equation for the surface area as

$$\mathbf{r} = \langle 2\cos(\theta), y, 2\sin(\theta)\rangle.$$

The bounds for y are $0 \leq y \leq 2$ and $0 \leq \theta \leq 2\pi$.

We know that

$$\begin{aligned}
\mathbf{r}_y &= \left\langle \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y} \right\rangle = \langle 0, 1, 0 \rangle \\
\mathbf{r}_\theta &= \left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right\rangle = \langle -2\sin(\theta), 0, 2\cos(\theta) \rangle.
\end{aligned}$$

The normal vector is

$$\begin{aligned}
\mathbf{N} = \mathbf{r}_y \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -2\sin(\theta) & 0 & 2\cos(\theta) \end{vmatrix} \\
&= \langle (1)(2\cos(\theta)), -\left[(0)(2\cos(\theta)) - (0)(-2\sin(\theta))\right], (0)(0) - (1)(-2\sin(\theta)) \rangle \\
&= \langle 2\cos(\theta), 0, 2\sin(\theta) \rangle.
\end{aligned}$$

The magnitude of the normal vector is

$$\begin{aligned}
|\mathbf{r}_y \times \mathbf{r}_\theta| &= \sqrt{(2\cos(\theta))^2 + (2\sin(\theta))^2} \\
&= \sqrt{4\cos^2(\theta) + 4\sin^2(\theta)} = \sqrt{4} = 2.
\end{aligned}$$

Therefore, the surface area element is

$$\mathrm{d}S = |\mathbf{r}_y \times \mathbf{r}_\theta| \, \mathrm{d}A = 2 \, \mathrm{d}y \, \mathrm{d}\theta.$$

The surface area integral is

$$\begin{aligned}
A &= \int_0^{2\pi} \int_0^2 2 \, \mathrm{d}y \, \mathrm{d}\theta \\
&= 2 \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^2 \mathrm{d}y \\
&= 4\pi \cdot 2 = 8\pi.
\end{aligned}$$

□