Fundamentals of Analysis I: Homework 6

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SECTION 2.7

Exercise 2.7.4 Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(i) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(ii) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(iii) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge, but $\sum y_n$ diverges.

(iv) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution to (i). Let $x_n = 1/n$ and $y_n = 1/n$. Then, $\sum x_n = \sum 1/n$ and $\sum y_n = \sum 1/n$ diverges, but $\sum x_n y_n = \sum 1/n^2$ converges to $\frac{\pi^2}{6}$. □

Solution to (ii). Let $x_n = (-1)^n/n$ and $y_n = (-1)^n$. Then, $\sum x_n = \sum \frac{(-1)^n}{n}$ converges, and (y_n) is bounded, but $\sum x_n y_n = \sum 1/n$ diverges. □

Solution to (iii). Impossible, since the algebraic limit theorem for series tells us that $\sum (x_n + y_n) - \sum x_n = \sum y_n$ converges. □

Solution to (iv). The sequence

$$x_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

satisfies $0 \leq x_n \leq 1/n$, but $\sum (-1)^n x_n$, since it's always bouncing between 0 and $1/n$. □

Exercise 2.7.7

(i) Show that if $a_n > 0$ and $\lim_{n \to \infty} (n a_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(ii) Assume $a_n > 0$ and $\lim_{n \to \infty} (n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution to (i). Setting $\varepsilon = \frac{|l|}{2}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|n a_n - l| < \frac{|l|}{2} \Rightarrow \frac{|l|}{2} < n a_n < \frac{3|l|}{2}.$$

Therefore, for all $n \geq N$,

$$a_n > \frac{|l|}{2n}.$$

Now, since $\sum \frac{1}{n}$ (the harmonic series) diverges, and a_n is greater than a constant multiple of $\frac{1}{n}$ for large n, we conclude by the Comparison Test that $\sum a_n$ must also diverge. □

Solution to (ii). Setting $\varepsilon = \frac{|l|}{2}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|n^2 a_n - l| < \varepsilon \Rightarrow \frac{|l|}{2n^2} < a_n < \frac{3|l|}{2n^2}.$$

If that's the case, then for all $n \geq N$, we have

$$a_n > \frac{|l|}{2n^2}.$$

Since $\sum \frac{1}{n^2}$ converges, and $a_n \geq 0$, for all n, then, we can use the limit comparison test to conclude that $\sum a_n$ converges

$$\lim_{n\to\infty} \frac{\frac{|l|}{2n^2}}{\frac{1}{n^2}}.$$

Using the properties from the algebraic limit theorem, we have

$$\lim_{n\to\infty} \frac{\frac{|l|}{2n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{|l|}{2n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{|l|}{2} = \frac{|l|}{2}.$$

Since $\frac{|l|}{2}$ is a positive constant, then $\sum a_n$ converges. $\square$

Exercise 2.7.8 Consider each of the following proportions. Provide short proofs for those that are true and counterexamples for any that are not.

(i) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(ii) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(iii) If $\sum a_n$ converges conditionally, then $\sum a_n^2$ diverges.

Solution to (i). The statement is true.

If $\sum a_n$ converges absolutely, then $\sum |a_n|$ converges. Since $|a_n| \leq |a_n|^2$ for all n, then by the comparison test, $\sum a_n^2$ converges absolutely. $\square$

Solution to (ii). The statement is false.

Counterexample: Let $a_n = \frac{(-1)^n}{\sqrt{n}}$, which converges conditionally by the alternating series test, and let $b_n = a_n$, which converges to 0. Then

$$a_n b_n = \frac{(-1)^n}{\sqrt{n}} \cdot \frac{(-1)^n}{\sqrt{n}} = \frac{1}{n}.$$

Now, $\sum \frac{1}{n}$ diverges by the p-series test, as it is a p-series with $p = 1 \not> 1$. Therefore, $\sum a_n b_n$ diverges. $\square$

Solution to (iii). The statement is true.

If $\sum a_n$ converges conditionally, then $\sum |a_n|$ diverges. Since $a_n^2 \leq |a_n|$ but a_n^2 does not tend to zero fast enough to allow $\sum a_n^2$ to converge when $\sum |a_n|$ diverges, $\sum a_n^2$ must diverge. $\square$

SECTION 3.2

Exercise 3.2.1

(i) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be finite get used?

(ii) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \dots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty, and not all of $\mathbb{R}$.

Solution to (i). Taking the $\min(\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N\})$ is only possible for finite sets. $\square$

Solution to (ii). Let $O_n = (-1/n, 1 + 1/n)$. Then, $\bigcap_{n=1}^{\infty} O_n = [0, 1]$. $\square$

Exercise 3.2.3 Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ε-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(i) $\mathbb{Q}$.

(ii) $\mathbb{N}$.

(iii) $\{x \in \mathbb{R} \mid x \neq 0\}$.

Solution to (i). $\mathbb{Q}$ is neither open nor closed, as $(a, b) \subseteq \mathbb{Q}$ is impossible since $\mathbb{Q}$ contains no irrationals but (a, b) does. It's not closed since every irrationals are limit points of $\mathbb{Q}$, as they can be reached as a limit of rational numbers. Take $\sqrt{2}$ for instance. □

Solution to (ii). $\mathbb{N}$ is not open since it doesn't have any boundaries. It is closed, since it doesn't have any limit points. □

Solution to (iii). $\{x \in \mathbb{R} \mid x \neq 0\}$ is open, since it's the complement of $\{0\}$, which is closed. It's not closed, since 0 is a limit point of the set. □

Exercise 3.2.6 Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(i) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

(ii) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set".

(iii) Every nonempty open set contains a rational number.

(iv) Every bounded infinite closed set contains a rational number.

Solution to (i). The statement is false.

Counterexample: Let $A = (-\infty, \sqrt{5}) \cup (\sqrt{5}, \infty)$. Then, A is open and contains every rational number except for $\sqrt{5}$ and it's not all of $\mathbb{R}$. □

Solution to (ii). The statement is false.

Counterexample: Let $C_n = [n, \infty)$ is closed. It has $C_{n+1} \supseteq C_n$ and $C_n \neq \emptyset$ but $\bigcap_{n=1}^{\infty} C_n = \emptyset$. □

Solution to (iii). The statement is true.

Let $x \in A$. Since A is open, we have $(a, b) \subseteq A$ with $x \in (a, b)$, and by the density of the rationals, there exists $q \in \mathbb{Q}$ such that $q \in (a, b) \subseteq A$. □

Solution to (iv). The statement is false.

Counterexample: Let $A = \{1/n + \sqrt{2} \mid n \in \mathbb{N}\} \cup \{\sqrt{2}\}$. Then, A is closed, bounded, and infinite, but contains no rational numbers. □