

# Abstract Linear Algebra: Homework 4

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**Problem 1.** Let  $T : \mathbb{R}^{1 \times 3} \rightarrow \mathbb{R}^{2 \times 3}$  be a linear transformation defined by

$$T(x, y, z) = (x + z, 2x - z).$$

If  $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ ,  $\mathcal{B}' = \{\gamma_1, \gamma_2\}$ , where

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0), \quad \gamma_1 = (0, 1), \quad \text{and} \quad \gamma_2 = (1, 0).$$

Find the matrix  ${}_{\mathcal{B}'}^T \leftarrow {}_{\mathcal{B}}$ .

*Solution.* Evaluating  $T$  at  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  we get

$$\begin{aligned} T(\alpha_1) &= T(1, 0, -1) = (0, 3) = 0\gamma_1 + 3\gamma_2 \\ T(\alpha_2) &= T(1, 1, 1) = (2, 1) = 1\gamma_1 + 2\gamma_2 \\ T(\alpha_3) &= T(1, 0, 0) = (1, 2) = 2\gamma_1 + 1\gamma_2. \end{aligned}$$

Therefore, the matrix  ${}_{\mathcal{B}'}^T \leftarrow {}_{\mathcal{B}}$  is

$${}_{\mathcal{B}'}^T \leftarrow {}_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}. \quad \square$$

**Problem 2.** Let  $D$  be the differentiation operator on  $\mathbb{P}^3(\mathbb{R})$ , i.e.

$$D(g(x)) = g'(x) \text{ for } g(x) \in \mathbb{P}^3(\mathbb{R}).$$

(Note:  $D$  is a linear transformation on  $\mathbb{P}^3(\mathbb{R})$ )

- (i) Let  $\mathcal{B} = \{1, x, x^2, x^3\}$  be the standard basis for  $\mathbb{P}^3(\mathbb{R})$ . Find the matrix  $[D]_{\mathcal{B}}$ .
- (ii) Let  $\mathcal{B}' = \{x^3, x^2, x, 1\}$  be the basis for  $\mathbb{P}^3(\mathbb{R})$ . Find the matrix  $[D]_{\mathcal{B}'}$ .

*Solution.* Just as in problem 1, we evaluate  $D$  at the elements of  $\mathcal{B}$  to get

$$\begin{aligned} D(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ D(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ D(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ D(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3. \end{aligned}$$

From that, we get

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad [D]_{\mathcal{B}'} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}. \quad \square$$

**Problem 3.** Let  $T$  be a linear transformation on the vector space  $V = \mathbb{R}^{2 \times 2}$  defined by

$$T(A) = 2A + A^T.$$

Let  $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ . Find the matrix representation  $[T]_{\mathcal{B}}$ .

*Solution.* We evaluate  $T$  at the elements of  $\mathcal{B}$  to get

$$\begin{aligned} T(E_{11}) &= 2E_{11} + E_{11}^T = 3E_{11} + 0E_{12} + 0E_{21} + 0E_{22} \\ T(E_{12}) &= 2E_{12} + E_{12}^T = 0E_{11} + 2E_{12} + E_{21} + 0E_{22} \end{aligned}$$

.

Therefore, the matrix representation  $[T]_{\mathcal{B}}$  is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \quad \square$$

**Problem 4.** Let  $V$  be a two-dimensional vector space over  $\mathbb{F}$ , and let  $\mathcal{B}$  be an ordered basis for  $V$ . If  $T$  is a linear transformation on  $V$  and  $[T]_{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , prove that  $T^2 - (a+d)T + (ad-bc)I = 0$ .

*Solution.* The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation. Given an  $A \in \mathbb{R}^{2 \times 2}$  and its characteristic polynomial is  $P_A(\lambda) = \det(\lambda I - A)$ , then substituting  $A$  for  $\lambda$  in  $P_A(\lambda)$  results in the zero matrix.

Since  $[T]_{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , the characteristic polynomial is

$$P_T(\lambda) = \det(\lambda I - [T]_{\mathcal{B}}) = \det \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a+d)\lambda + (ad - bc).$$

By the Cayley-Hamilton theorem, the matrix  $[T]_{\mathcal{B}}$  satisfies its characteristic polynomial, giving us

$$[T]_{\mathcal{B}}^2 - (a+d)[T]_{\mathcal{B}} + (ad - bc)I = 0.$$

Since  $[T]_{\mathcal{B}}$  is a matrix representation of the linear transformation  $T$ , the equation above is equivalent to  $T^2 - (a+d)T + (ad - bc)I = 0$ .  $\square$

**Problem 5.** Suppose that  $T$  is a linear transformation on a two-dimensional vector space such that  $T$  is neither the zero nor the identity linear transformation. Prove that if  $T^2 = T$ , there is an ordered basis  $\mathcal{B}$  for

$V$  such that  $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ .

(Hint: Construct a basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$  such that  $T(\mathbf{v}_1) = \mathbf{v}_1$  and  $T(\mathbf{v}_2) = \mathbf{0}$ .)

*Solution.* Since  $T^2 = T$ , this means  $T$  is idempotent. The eigenvalues of an idempotent operator must satisfy the equation

$$\lambda^2 = \lambda.$$

Therefore, the eigenvalues of  $T$  are  $\lambda = 0$  and  $\lambda = 1$ . Since 1 is an eigenvalue for  $T$ , there exists a non-zero vector  $\mathbf{v}_1 \neq \mathbf{0}$  such that  $T(\mathbf{v}_1) = \mathbf{v}_1$ . Since  $T$  has two eigenvalues, there must exist another nonzero vector  $\mathbf{v}_2$  such that  $T(\mathbf{v}_2) = \mathbf{0}$ . This,  $\mathbf{v}_1 \in \text{Range}(T)$  and  $\mathbf{v}_2 \in \text{Ker}(T)$ .

Now, we must show that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent. Suppose that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly dependent, then there exist scalars  $\alpha$  and  $\beta$  such that  $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{0}$ . Applying  $T$  to both sides of the equation gives

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = T(\mathbf{0}) = \mathbf{0}.$$

By the linearity of  $T$ , we have

$$\alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) = \alpha\mathbf{v}_1 + \beta\mathbf{0} = \alpha\mathbf{v}_1 = \mathbf{0}.$$

Since  $\mathbf{v}_1 \neq \mathbf{0}$ , this implies that  $\alpha = 0$ . But this means that  $\mathbf{v}_2 = \mathbf{0}$ , which is a contradiction. Therefore,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent.

Since  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent, they form a basis for  $V$ . Let  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$  be the basis for  $V$ . In the basis, we write

$$T(\mathbf{v}_1) = 1\mathbf{v}_1 + 0\mathbf{v}_2 \quad \text{and} \quad T(\mathbf{v}_2) = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Therefore, the matrix representation of  $T$  in the basis  $\mathcal{B}$  is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad \square$$

**Problem 6.** Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{R}$ . Let  $T$  be a linear transformation on  $V$ . If

$T^n = 0$ , and  $T^{n-1} \neq 0$ , prove that there exists a basis  $\mathcal{B}$  such that  $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ .

(Hint: Construct a set of the form  $\{x, T(x), T^2(x), \dots, T^{n-1}(x)\}$  and show that this set is a basis of  $V$ .)

*Solution.* Since  $T^{n-1} \neq 0$ , there exists some nonzero vector  $\mathbf{x} \in V$  such that  $T^{n-1}(\mathbf{x}) \neq 0$ . Define the set of vectors

$$S = \{\mathbf{x}, T(\mathbf{x}), T^2(\mathbf{x}), \dots, T^{n-1}(\mathbf{x})\}.$$

We will show that  $S$  is a basis for  $V$ .

Suppose there exist scalars  $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$  such that

$$c_0\mathbf{x} + c_1T(\mathbf{x}) + c_2T^2(\mathbf{x}) + \cdots + c_{n-1}T^{n-1}(\mathbf{x}) = \mathbf{0}.$$

Applying  $T^{n-1}$  to both sides of the equation gives

$$c_0T^{n-1}(\mathbf{x}) + c_1T^n(\mathbf{x}) + c_2T^{n+1}(\mathbf{x}) + \cdots + c_{n-1}T^{2n-2}(\mathbf{x}) = \mathbf{0}.$$

Since  $T^{n-1}(\mathbf{x}) \neq 0$ , it follows that  $c_0 = 0$ . Applying  $T^{n-2}$  to the original equation gives

$$c_1T^{n-2}(\mathbf{x}) = 0.$$

Since  $T^{n-2}(\mathbf{x}) \neq 0$ , it follows that  $c_1 = 0$ . Continuing this process until we reach  $c_{n-1}$ , we find that  $c_0 = c_1 = \cdots = c_{n-1} = 0$ . Therefore,  $S$  is linearly independent.

Since  $S$  contains  $n$  linearly independent vectors and  $V$  is an  $n$ -dimensional vector space,  $S$  is a basis for  $V$ .

Let  $\mathcal{B} = \{\mathbf{x}, T(\mathbf{x}), T^2(\mathbf{x}), \dots, T^{n-1}(\mathbf{x})\}$  be the basis for  $V$ . Since  $T(\mathbf{x}) = T(\mathbf{x}) \in \mathcal{B}$ , it can be represented as  $\mathbf{e}_2 = (0, 1, 0, \dots, 0)^T$ . Similarly,  $T^2(\mathbf{x}) = T^2(\mathbf{x}) \in \mathcal{B}$  can be represented as  $\mathbf{e}_3 = (0, 0, 1, 0, \dots, 0)^T$ .

Continuing this process, we find that  $T^{n-1}(\mathbf{x}) = T^{n-1}(\mathbf{x}) \in \mathcal{B}$  can be represented as  $\mathbf{e}_n = (0, 0, 0, \dots, 1)^T$ . Therefore, the matrix representation of  $T$  in the basis  $\mathcal{B}$  is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \quad \square$$

**Problem 7.** Suppose  $A \in \mathbb{R}^{n \times n}$  is an invertible matrix. Prove that  $AB$  and  $BA$  are similar matrices for any  $B \in \mathbb{R}^{n \times n}$ .

*Solution.* Since  $A$  is invertible, let  $P = A$ . We can then compute

$$A^{-1}(AB)A = (A^{-1}A)BA = BA.$$

Thus, we can express  $BA$  as

$$BA = A^{-1}(AB)A.$$

Therefore,  $AB$  and  $BA$  are similar matrices.  $\square$

**Problem 8.** Let  $A, B \in \mathbb{R}^{n \times n}$ . Prove the following statements.

- (i) If  $A$  and  $B$  are similar, then  $\text{Tr}(A) = \text{Tr}(B)$ .
- (ii)  $AB - BA = I$  is impossible.

*Hint:* You may use the results from Homework 2 Problem 1.

*Solution to (i).* By definition, two matrices are similar if there exists an invertible matrix  $P$  such that  $B = P^{-1}AP$ . Using the cyclic property of the trace, we have

$$\text{Tr}(B) = \text{Tr}(P^{-1}AP) = \text{Tr}(A).$$

Therefore, if  $A$  and  $B$  are similar, then  $\text{Tr}(A) = \text{Tr}(B)$ .  $\square$

*Solution to (ii).* Assume, for contradiction, that there exists  $A, B \in \mathbb{R}^{n \times n}$  such that

$$AB - BA = I.$$

Using the linearity of the trace

$$\text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA) = I.$$

From Homework 2 Problem 1, we know that  $\text{Tr}(AB) = \text{Tr}(BA)$ , so

$$\text{Tr}(AB) - \text{Tr}(AB) = 0 \neq n = \text{Tr}(I).$$

This is a contradiction, so the equation  $AB - BA = I$  is impossible.  $\square$

**Problem 9.** True or False. (No explanation needed.)

In the following statements (i) - (iii): Let  $V$  and  $W$  be finite-dimensional vector spaces. Let  $T, U : V \rightarrow W$  be linear transformations. Let  $\beta$  and  $\gamma$  be ordered basis for  $V$  and  $W$ , respectively.

- (i) Let  $[T]_{\beta}^{\gamma} = [U]_{\beta}^{\gamma}$  implies  $T = U$ .

- (ii) If  $\dim(V) = n$  and  $\dim(W) = m$ , then  $[T]_{\beta}^{\gamma} \in R^{n \times m}$ .
- (iii)  $[T(\mathbf{v})]_{\gamma} = [T]_{\beta}^{\gamma}[\mathbf{v}]_{\beta}$  for all  $\mathbf{v} \in V$ .
- (iv) Let  $A \in R^{n \times n}$ . If  $A^2 = I$ , then  $A = I$  or  $A = -I$ .
- (v) Let  $A \in R^{m \times n}$ . Suppose  $L_A : R^n \rightarrow R^m$  is defined by  $L_A(\mathbf{v}) = A\mathbf{v}$  for any  $\mathbf{v} \in R^n$ . Then  $[L_A]_{\beta} = A$ , where  $\beta$  is the standard basis for  $R^n$ .

*Solution to (i).* It's true. If  $[T]_{\beta}^{\gamma} = [U]_{\beta}^{\gamma}$ , then  $T$  and  $U$  have the same matrix representation with respect to the bases  $\beta$  and  $\gamma$ . Since a linear transformation is uniquely determined by its action on a basis, having the same matrix implies that  $T$  and  $U$  act identically on all basis vectors, and hence they must be the same transformation, i.e.,  $T = U$ .  $\square$

*Solution to (ii).* It's false. The matrix representation  $[T]_{\beta}^{\gamma}$  has dimensions  $m \times n$ , not  $n \times m$ , because  $T$  maps from an  $n$ -dimensional space to an  $m$ -dimensional space. The correct statement would be  $[T]_{\beta}^{\gamma} \in R^{m \times n}$ .  $\square$

*Solution to (iii).* It's true. This is a fundamental property of matrix representations of linear transformations. Given a vector  $\mathbf{v} \in V$ , its image under  $T$  has coordinates  $[T(\mathbf{v})]_{\gamma}$ , which are obtained by multiplying the matrix representation  $[T]_{\beta}^{\gamma}$  by the coordinate vector  $[\mathbf{v}]_{\beta}$ .  $\square$

*Solution to (iv).* It's false. The equation  $A^2 = I$  means  $A$  is an involutory matrix, but this does not imply that  $A$  must be either  $I$  or  $-I$ . For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

satisfies  $A^2 = I$  but is neither  $I$  nor  $-I$ .  $\square$

*Solution to (v).* It's true. By definition, the standard matrix representation of the linear map  $L_A$  induced by  $A$  is exactly  $A$ , since the standard basis vectors get mapped directly according to the columns of  $A$ .  $\square$