

Multi-Variable Calculus I: Homework 3

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Jennifer Thorenson 08:00

Hashem A. Damrah
UO ID: 952102243

Problem 1

Use traces to identify the surfaces. Sketch the region bounded by the surfaces and determine the curve of intersection. Find a vector function that parametrizes the curves of intersection.

(i) $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 6$ for $z \geq 0$.

(ii) $z = x^2 + 3y^2$ and $z = 12 - 3x^2 - y^2$.

Solution 1

- (i) The first equation is the equation of a cone and the second equation is the equation of a sphere.

To find the curve where the cone and the sphere intersect, substitute $z = \sqrt{x^2 + y^2}$ into the equation of the sphere to get

$$x^2 + y^2 + x^2 + y^2 = 6 \Rightarrow 2x^2 + 2y^2 = 6 \Rightarrow x^2 + y^2 = 3.$$

So, their intersection creates a circle in the xy -plane with a radius of $\sqrt{3}$. Therefore, the curve of intersection is parameterized by the vector function

$$\mathbf{r}(t) = \langle \sqrt{3} \cos(t), \sqrt{3} \sin(t), \sqrt{3} \rangle, \quad 0 \leq t \leq 2\pi.$$

- (ii) The first equation is the equation of an elliptic paraboloid and the second equation is the equation of an elliptic paraboloid.

To find the curve where the two paraboloids intersect, set the two equations equal to each other

$$x^2 + 3y^2 = 12 - 3x^2 - y^2 \Rightarrow x^2 + y^2 = 3.$$

So, their intersection creates a circle in the xy -plane with a radius of $\sqrt{3}$. The circle of radius $\sqrt{3}$ in the xy -plane can be parameterized as

$$x = \sqrt{3} \cos(t) \quad \text{and} \quad y = \sqrt{3} \sin(t).$$

Substitute these into either equation to find z

$$z = 3 \cos^2(t) + 9 \sin^2(t).$$

Therefore, the curve of intersection is parameterized by the vector function

$$\mathbf{r}(t) = \langle \sqrt{3} \cos(t), \sqrt{3} \sin(t), 3 \cos^2(t) + 9 \sin^2(t) \rangle, \quad 0 \leq t \leq 2\pi.$$

Problem 2

Consider the vector function $\mathbf{r}(t) = \left\langle \sqrt{16 - t^2}, t^2 - 2t + 1, \frac{t+3}{t^2 - 2t - 3} \right\rangle$.

- (i) Find the domain of $\mathbf{r}(t)$.
- (ii) Find $\mathbf{r}'(t)$.
- (iii) Find the vector equation for the tangent line to the curve at the point $(4, 1, -1)$.

Solution 2

(i) The domain of $\mathbf{r}(t)$ is the intersection of the domain of all the function components of $\mathbf{r}(t)$.

The domain of $f(t)\mathbf{i}$ is $t \geq 4$, or $[4, \infty)$.

The domain of $g(t)\mathbf{j}$ is $t \in \mathbb{R}$, or $(-\infty, \infty)$.

The domain of $h(t)\mathbf{k}$ is $t \neq -1, 3$, or $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

Therefore, the domain of $\mathbf{r}(t)$ is [NOTE: I left out the $(-\infty, \infty)$ interval, as it doesn't change anything]

$$\begin{aligned} [-4, 4] \cap [(-\infty, -1) \cup (-1, 3) \cup (3, \infty)] &= [4, \infty) \cap [(-\infty, -1) \cup (-1, 3) \cup (3, \infty)] \\ &= [-4, -1) \cap (-1, 3) \cap (3, 4] \end{aligned}$$

Therefore, the domain of $\mathbf{r}(t)$ is $[-4, 4] \setminus \{-1, 3\}$.

(ii) The derivative of $\mathbf{r}(t)$ is just the derivatives of the components, giving us

$$\begin{aligned} f'(t)\mathbf{i} &= \frac{d}{dt} \left[\sqrt{16 - t^2} \right] = \frac{-t}{\sqrt{16 - t^2}} \\ g'(t)\mathbf{j} &= \frac{d}{dt} [t^2 - 2t + 1] = 2t - 2 \\ h'(t)\mathbf{k} &= \frac{d}{dt} \left[\frac{t+3}{t^2 - 2t - 3} \right] = \frac{(t^2 - 2t - 3) - (t+3)(2t-2)}{(t^2 - 2t - 3)^2} \\ &= -\frac{t^2 + 6t - 3}{(t^2 - 2t - 3)^2}. \end{aligned}$$

Therefore, we get

$$\mathbf{r}'(t) = \left\langle -\frac{t}{\sqrt{16 - t^2}}, 2t - 2, -\frac{t^2 + 6t - 3}{(t^2 - 2t - 3)^2} \right\rangle.$$

(iii) Since $\mathbf{r}(0) = \langle 4, 1, -1 \rangle$ and $\mathbf{r}'(0) = \langle 0, -2, -1/2 \rangle$, the unit tangent vector at the point $(4, 1, -1)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\langle 0, -2, -1/2 \rangle}{\sqrt{0^2 + (-2)^2 + (-\frac{1}{2})^2}} = \left\langle 0, -2 \cdot \frac{2}{\sqrt{17}}, -\frac{1}{2} \cdot \frac{2}{\sqrt{17}} \right\rangle = \left\langle 0, -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle.$$

Problem 3

In general, the magnitude of a vector function is a scalar function. We may want to know the rate at which the magnitude of the position vector, $\mathbf{r}(t)$ changes along the space curve defined by the function. Suppose $\mathbf{r}(t) \neq 0$ is a differentiable vector function. Show that

$$\frac{d}{dt} [\mathbf{r}(t)] = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Hint: rewrite $|\mathbf{r}(t)|^2$ using the dot product and differentiate the result.

Solution 3

Proof. The magnitude of a vector $\mathbf{r}(t)$ is given by $|\mathbf{r}(t)|$. The square of this magnitude can be written using the dot product

$$|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t).$$

Now, differentiate both sides of the equation $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$ with respect to t

$$\frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Now, observe that

$$\frac{d}{dt} [|\mathbf{r}(t)|^2] = 2|\mathbf{r}(t)| \frac{d}{dt} [|\mathbf{r}(t)|].$$

Equating this with the result from Step 2 gives

$$2|\mathbf{r}(t)| \frac{d}{dt} [|\mathbf{r}(t)|] = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Finally, divide both sides by $2|\mathbf{r}(t)|$ (assuming $\mathbf{r}(t) \neq 0$) to isolate $\frac{d}{dt} |\mathbf{r}(t)|$

$$\frac{d}{dt} [|\mathbf{r}(t)|] = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{|\mathbf{r}(t)|}.$$

Thus, we have shown that

$$\frac{d}{dt} [|\mathbf{r}(t)|] = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t). \quad \square$$