Math 432/532: Introduction to Topology II

Solutions to HW #1

1. Show that, if X has finitely many connected components, then each component is both open and closed. On the other hand, find an example of a space X none of whose connected components are open sets.

Solution: Let $\{C_i \mid i \in I\}$ be the connected components. We proved in class that these components are closed and disjoint and that they cover X. In particular, for all $i \in I$, we have

$$C_i = \bigcap_{j \in I \setminus \{i\}} (X \setminus C_j).$$

If I is finite, this is a finite intersection of open sets, so C_i is open. On the other hand, if $X = \mathbb{Q}$, the connected components are single points, so they are not open.

2. Fix real numbers $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Use connectedness of the interval $[a, b]$ to prove the intermediate value theorem, which says that there exists an element $c \in (a, b)$ with $f(c) = 0$.

Solution: Suppose not. Then we have a continuous surjection $g : [a, b] \to \{-1, 1\}$ given by the formula $g(x) = f(x)/|x|$. Since $[a, b]$ is connected and $\{-1, 1\}$ is not, this is impossible. (Note: There are many equally good ways to prove this statement, using the various conditions that are equivalent to connectedness.)

3. Prove that, if $f : X \to Y$ is surjective and X is path-connected, then so is Y.

Solution: Let $y_1, y_2 \in Y$ be given. Since f is surjective, we can choose x_i so that $f(x_i) = y_i$ for $i \in \{1, 2\}$. Since X is path-connected, there exists a path $\gamma : [0, 1] \to X$ from x_1 to x_2. Then $f \circ \gamma : [0, 1] \to Y$ is a path from y_1 to y_2.

4. Prove that, if X and Y are path-connected, then so is $X \times Y$.

Solution: Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be given. If $\gamma : [0, 1] \to X$ is a path from x_1 to x_2 and $\eta : [0, 1] \to Y$ is a path from y_1 to y_2, then $\gamma \times \eta : [0, 1] \to X \times Y$ is a path from (x_1, y_1) to (x_2, y_2).

5. Suppose that $X = A \cup B$ with A, B, and $A \cap B$ all path-connected. Show that, if $A \cap B$ is nonempty, then X is also path-connected.

Solution: Let $x_1, x_2 \in X$ be given. If they both lie in A or both lie in B, then we know that they can be connected by a path. So, let's assume that $x_1 \in A$ and $x_2 \in B$. Since $A \cap B$ is nonempty, we can choose $x_3 \in A \cap B$. By path-connectedness of A, there is a path from x_1 to x_3. By path-connectedness of B, there is a path from x_3 to x_2. By a lemma that we proved in class, these paths can be concatenated to obtain a path from x_1 to x_2.

6. For each description below, name a familiar space that is homeomorphic to the corresponding identification space (no proofs required).

(i) The cylinder $S^1 \times [0, 1]$ with each of its boundary circles collapsed to a point. (That is, $(x, s) \sim (y, t)$ if and only if $s = t \in \{0, 1\}$.)

(ii) The torus $S^1 \times S^1$ with both a longitude $(1, 0) \times S^1$ and a meridian $S^1 \times (1, 0)$ collapsed to a point.

(iii) The Möbius strip M with its boundary circle collapsed to a point.

Solution: (i) and (ii) are both homeomorphic to S^2, while (iii) is homeomorphic to $\mathbb{R}P^2$.

7. Give an example of an identification map $f : X \to Y$ and a subspace $A \subset X$ such that the surjection $f : A \to f(A)$ is not an identification map.

Solution: Take $X = [0, 1]$, $Y = S^1 \subset \mathbb{C}$, and $f(t) = e^{2\pi it}$. Now take $A = [0, 1)$. Then $f(A) = S^1$ and $f : A \to f(A)$ is a bijection. A bijective map is an identification map if and only if it is a homeomorphism, and this is not one.

8. Define $f : S^2 \to \mathbb{R}^4$ by the formula $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. You can convince yourself (and you may assume) that $f(x, y, z) = f(a, b, c)$ only if $(a, b, c) = (x, y, z)$ or $(a, b, c) = (-x, -y, -z)$. Show that f descends to a map $g : \mathbb{R}P^2 \to \mathbb{R}^4$, and that g is a homeomorphism from $\mathbb{R}P^2$ onto its image.

Solution: The second sentence implies that f descends to an injection $g : \mathbb{R}P^2 \to \mathbb{R}^4$, and therefore a bijection onto its image. Since S^2 is compact, so is $\mathbb{R}P^2$. Since $\mathbb{R}^4$ is Hausdorff, so is $g(\mathbb{R}P^2)$. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.