Math 307, Homework #9
Due Sunday, December 8
SOLUTIONS TO SELECTED PROBLEMS

1. Prove that $\frac{d^n}{dx^n}(x^2e^x) = (x^2 + 2nx + n(n-1))e^x$, for all $n \in \mathbb{N}$.

   Proof:

   I. The $n = 1$ case is trivial, since $(x^2e^x)' = 2ze^x + x^2e^x = (x^2 + 2 \cdot 1 \cdot x + 1 \cdot (1-1))e^x$.

   II. Assume $n \in \mathbb{N}$ and $\frac{d^n}{dx^n}(x^2e^x) = (x^2 + 2nx + n(n-1))e^x$. Then

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(x^2e^x) = \frac{d}{dx}\left(\frac{d^n}{dx^n}(x^2e^x)\right) &= \frac{d}{dx}\left((x^2 + 2nx + n(n-1))e^x\right) \\
&= (2x + 2n)e^x + (x^2 + 2nx + n(n-1))e^x \\
&= (x^2 + 2(n+1)x + (2n + n^2 - n))e^x \\
&= (x^2 + 2(n+1)x + (n^2 + n))e^x \\
&= (x^2 + 2(n+1)x + (n+1)n)e^x.
\end{aligned}$$

   This completes the induction step, which completes the proof.

2. Here are two facts that hold for all real numbers x and y:

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{(addition formula for sine)} \\
|x+y| &\le |x| + |y| && \text{(triangle inequality).}
\end{aligned}$$

   Using these together with induction, prove that $(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})\big[\,|\sin(nx)| \le n|\sin(x)|\,\big]$.

   Proof: Let $x \in \mathbb{R}$.

   I. Certainly $|\sin(1 \cdot x)| = |\sin(x)| \le 1 \cdot |\sin(x)|$.

   II. Assume $n \in \mathbb{N}$ and $|\sin(nx)| \le n|\sin(x)|$. Then

$$\begin{aligned}
|\sin((n+1)x)| = |\sin(nx + x)| &= |\sin(nx)\cos(x) + \cos(nx)\sin(x)| \\
&\le |\sin(nx)\cos(x)| + |\cos(nx)\sin(x)| \\
&= |\sin(nx)| \cdot |\cos(x)| + |\cos(nx)| \cdot |\sin(x)| \\
&\le |\sin(nx)| \cdot 1 + 1 \cdot |\sin(x)| && \text{since } |\cos(y)| \le 1 \text{ for any real } y \\
&= |\sin(nx)| + |\sin(x)| \\
&\le n|\sin(x)| + |\sin(x)| && \text{by the induction hypothesis} \\
&= (n+1)|\sin(x)|.
\end{aligned}$$

   This completes the induction step, and the proof.

3. Let f_n denote the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \ge 2$. Prove that $f_n < (\frac{7}{4})^n$ for all $n \in \mathbb{N}$.

   Proof:

   I. $f_1 = 1 < (\frac{7}{4})^1$ and $f_2 = f_0 + f_1 = 1 < (\frac{7}{4})^2$.

II. Assume $n \in \mathbb{N}$ and $n \geq 2$ and $f_k < (\frac{7}{4})^k$ for all natural numbers k such that $1 \leq k \leq n$. Note that $\frac{11}{4} < \frac{49}{16}$, since $\frac{11}{4} = \frac{4 \cdot 11}{4 \cdot 4} = \frac{44}{16} < \frac{49}{16}$. Then

$$f_{n+1} = f_n + f_{n-1} < (\tfrac{7}{4})^n + (\tfrac{7}{4})^{n-1} = (\tfrac{7}{4})^{n-1} \cdot (\tfrac{7}{4} + 1) = (\tfrac{7}{4})^{n-1} \cdot \tfrac{11}{4} < (\tfrac{7}{4})^{n-1} \cdot \tfrac{49}{16} = (\tfrac{7}{4})^{n-1} \cdot (\tfrac{7}{4})^2 = (\tfrac{7}{4})^{n+1}$$

This completes the proof.

4. Consider the sequence given by $a_n = 2a_{n-1} + 4a_{n-2}$ and initial conditions $a_0 = 0$, $a_1 = 3$. Prove that $3|a_n$ for all $n \geq 0$.

   Proof:

   I. Surely $3|a_0$ and $3|a_1$.

   II. Assume $n \in \mathbb{N}$, $n \geq 1$, and $3|a_k$ for all k such that $0 \leq k \leq n$. Then $a_n = 3u$ for some $u \in \mathbb{Z}$ and $a_{n-1} = 3t$ for some $t \in \mathbb{Z}$. So

   $$a_{n+1} = 2a_n + 4a_{n-1} = 2(3u) + 4(3t) = 3(2u + 4t)$$

   and so $3|a_{n+1}$. The result follows by induction.

5. Imagine that you have an infinite supply of 6-cent and 11-cent stamps. Prove that for all $n \geq 50$, you can make a combination of your stamps that exactly totals n cents. Use 2nd principle of mathematical induction.

   Proof:

   I. $50 = 4(11) + 1(6)$, $51 = 3(11) + 3(6)$, $52 = 2(11) + 5(6)$, $53 = 1(11) + 7(6)$, $54 = 9(6)$, $55 = 5(11)$.

   II. Assume $n \in \mathbb{N}$, $n \geq 55$, and that we can make k cents for every k such that $50 \leq k \leq n$. Then $n - 5 \geq 50$, so by our induction hypothesis we can make $n - 5$ cents. By adding another 6 cent stamp, we can make $n + 1$ cents. The result follows by 2nd principle of mathematical induction.

6. Define the "Tribonacci" sequence as follows: $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 4$. Prove that for all natural numbers $n \geq 1$ one has $T_n < 2^n$.

   Proof:

   I. $T_1 = 1 < 2^1$, $T_2 = 1 < 2^2$, and $T_3 = 1 < 2^3$.

   II. Assume $n \in \mathbb{N}$, $n \geq 3$, and $T_k < 2^k$ for all k such that $1 \leq k \leq n$. Then

   $$T_{n+1} = T_n + T_{n-1} + T_{n-2} < 2^n + 2^{n-1} + 2^{n-2} = 2^{n-2}(4 + 2 + 1) = 2^{n-2} \cdot 7 < 2^{n-2} \cdot 2^3 = 2^{n+1}.$$

   The result follows by 2nd principle of mathematical induction.

7. Recall the Fibonacci sequence $f_0, f_1, f_2, \ldots$. The following is a faulty proof by strong induction that all Fibonacci numbers are even:

   I. $f_0 = 0$, which is even.

   II. Suppose $n \in \mathbb{N}$ and f_k is even for all $0 \leq k \leq n$.

   Then $f_{n+1} = f_n + f_{n-1}$. By the induction hypothesis, both f_n and f_{n-1} are even. So f_{n+1} is the sum of two even numbers, hence even.

   III. By PSMI, f_n is even for all $n \in \mathbb{N}$.

   Goal: Find the mistake in the above proof.

   The problem here is that the base case needed to be verified for both f_0 and f_1. In fact, the result is false for f_1 (because $f_1 = 1$ and this is not even).

To look a little deeper, the error in the above proof is the line $f_{n+1} = f_n + f_{n-1}$. This does not hold true when $n = 0$; in fact, it doesn't even make sense when $n = 0$ since f_{-1} is not defined. To use this recurrence relation the induction hypothesis should have said "Assume $n \in \mathbb{N}$, $n \geq 1$, and f_k is even for all $0 \leq k \leq n$." This makes step II correct, but it then requires that $n = 1$ be checked as part of the base case.

8. In this question we will prove that you can lift any cow. Let $P(n)$ be the statement "You can lift the cow on day n of its life". We will use induction to prove $(\forall n \in \mathbb{N})(n \geq 1 \Rightarrow P(n))$.

   I. When the cow is born it is very small, and of course you can lift it. This proves $P(1)$.

   II. Suppose that you can lift the cow on day n. It grows very little by the next day, so you will still be able to lift it on day $n + 1$.

   III. By induction, we have proven that you can lift the cow on every day of its life. Boy, you are strong.

   Explain what you think is wrong with this proof.

   In reality, there *will* be one day where you can lift the cow, but only barely, and where the next day you will not be able to lift it at all. The induction step is therefore incorrect.

9. Prove that $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

   Proof:

   We first work on $(\subseteq)$. Let $u \in (A \cap B) \times C$. Then $u = (x, y)$ where $x \in A \cap B$ and $y \in C$. Since $x \in A$ and $y \in C$, we have $u \in A \times C$. Since $x \in B$ and $y \in C$, we have $u \in B \times C$. So $u \in (A \times C) \cap (B \times C)$, hence we have shown $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

   Now let $u \in (A \times C) \cap (B \times C)$. Then $u \in A \times C$, so $u = (x, y)$ for some $x \in A$ and $y \in C$. Also $(x, y) \in B \times C$, so $x \in B$. Hence $x \in A \cap B$, so $u \in (A \cap B) \times C$. Thus, we have shown $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

10. Suppose $A \subseteq X$ and $B \subseteq Y$. Prove that $(X \times Y) - [(A \times Y) \cup (X \times B)] = (X - A) \times (Y - B)$.

    Proof:

    Let $u \in (X \times Y) - [(A \times Y) \cup (X \times B)]$. Then $u \in X \times Y$, so $u = (x, y)$ for some $x \in X$ and $y \in Y$. We know $u \notin A \times Y$ and $u \notin X \times B$. The former implies either $x \notin A$ or $y \notin Y$, but we already know $y \in Y$; so $x \notin A$. Similarly, since $u \notin X \times B$ we have either $x \notin X$ or $y \notin B$; but we already had $x \in X$, so $y \notin B$. Thus $x \in X$ and $x \notin A$, so $x \in X - A$. And $y \in Y$ and $y \notin B$, so $y \in Y - B$. Hence, $u = (x, y) \in (X - A) \times (Y - B)$. We have now shown $(X \times Y) - [(A \times Y) \cup (X \times B)] \subseteq (X - A) \times (Y - B)$.

    Now let $u \in (X - A) \times (Y - B)$. Then $u = (x, y)$ where $x \in X - A$ and $y \in Y - B$. Since $x \in X$ and $y \in Y$, we have $u \in X \times Y$. But $x \notin A$, so $u \notin A \times Y$. And $y \notin B$, so $u \notin X \times B$. Therefore $u \notin (A \times Y) \cup (X \times B)$, so $u \in (X \times Y) - [(A \times Y) \cup (X \times B)]$. We have shown that $(X - A) \times (Y - B) \subseteq (X \times Y) - [(A \times Y) \cup (X \times B)]$.

    We have now proved that $(X \times Y) - [(A \times Y) \cup (X \times B)] \subseteq (X - A) \times (Y - B)$ and $(X - A) \times (Y - B) \subseteq (X \times Y) - [(A \times Y) \cup (X \times B)]$, so the two sets are equal.