Math 307, Midterm Exam 2

Fall 2024

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Instructions: You can use a notecard. No calculators are allowed. Common tautologies which we use often (e.g., DeMorgan's Laws) can be used without giving truth tables; more complicated tautologies need to be verified. This is a 50 minute exam.

1. (10 points) Let A, B, and C be sets. Prove that $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

   Proof:

   1. Assume $x \in (A \cup B) - C$.
   2. Thus $x \in (A \cup B)$ and $x \notin C$.
   3. Hence $x \in A$ or $x \in B$.
   4. Case 1: If $x \in A$ then $x \in A - C$, so $x \in (A - C) \cup (B - C)$.
   5. Case 2: If $x \in B$ then $x \in B - C$, so $x \in (A - C) \cup (B - C)$.
   6. Both cases lead to $x \in (A - C) \cup (B - C)$.
   7. Therefore $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

   Comment: In fact $(A \cup B) - C = (A - C) \cup (B - C)$. Provet it!

2. (10 points) Give a line proof showing that $(\forall n \in \mathbb{N})[5n \equiv_3 20 \Rightarrow n^2 + n \equiv_9 2]$.

   (Hint: factor polynomial $n^2 + n - 2$).

   Proof:

   1. Assume $n \in \mathbb{N}$ and $5n \equiv_3 20$.
   2. Then $3 | 5n - 20 = 5(n - 4)$.
   3. 3 is prime, so by property (P) we know $3|5$ or $3|n - 4$.
   4. But $3 \nmid 5$, so $3|n - 4$.
   5. Thus $n - 4 = 3k$ for some $k \in \mathbb{Z}$ whence $n = 3k + 4$.
   6. Hence $n^2 + n - 2 = (n + 2)(n - 1) = (3k + 6)(3k + 3) = 9(k + 2)(k + 1)$.
   7. Therefore $9|n^2 + n - 2$.
   8. We showed that $n^2 + n \equiv_9 2$.

   Comment: Line 6 can be replaced by a less mysterious computation:

   $$n^2+n-2 = (3k+4)^2+(3k+4)-2 = 9k^2+2\cdot3k\cdot4+16+3k+4-2 = 9k^2+27k+18 = 9(k^2+3k+2).$$

3. (16 points) Let $f\colon S \to T$. Recall that for any set A, we denote by $I_A : A \to A$ the identity function, i.e. $I_A(x) = x$ for all $x \in A$.

   (a) Assume there is a function $g\colon T \to S$ such that $f \circ g = I_T$. Prove that f is onto.

   Proof:

   1. Let $y \in T$.
   2. Set $x = g(y)$.
   3. Then $f(x) = f(g(y)) = (f \circ g)(y) = I_T(y) = y$.
   4. Thus $(\forall y \in T)(\exists x \in S)(y = f(x))$.
   5. So f is onto.

   (b) Assume there is a function $g\colon T \to S$ such that $g \circ f = I_S$. Prove that f is one-to-one.

   Proof:

   1. Let $x, y \in S$ and $f(x) = f(y)$.
   2. Set $z = f(x) = f(y)$.
   3. Then $x = I_S(x) = (g \circ f)(x) = g(f(x)) = g(z)$.
   4. Also $y = I_S(y) = (g \circ f)(y) = g(f(y)) = g(z)$.
   5. Thus $x = g(z) = y$.
   6. Hence $(\forall x, y \in S)((f(x) = f(y)) \Rightarrow (x = y))$.
   7. So f is one-to-one.

4. (12 points) Prove by induction that

$$(\forall n \in \mathbb{N})\ 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

No points will be given for a non-induction proof.

Proof:

For any $n \in \mathbb{N}$ consider the following statement

$$P(n) : 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

I. The statement $P(1)$ says $1 \cdot 2^1 = (1-1)2^{1+1} + 2$ and it is true since both sides equal 2.

II. Assume that $P(n)$ is true, i.e.

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

Let us add $(n+1) \cdot 2^{n+1}$ to both sides. We get

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n + (n+1) \cdot 2^{n+1} = (n-1)2^{n+1} + 2 + (n+1) \cdot 2^{n+1}$$

$$= (n-1+n+1)2^{n+1} + 2 = 2n \cdot 2^{n+1} + 2 = n \cdot 2^{n+2} + 2 = ((n+1)-1) \cdot 2^{(n+1)+1} + 2.$$

Thus we proved that $P(n+1)$ is true. Hence $P(n) \Rightarrow P(n+1)$.

III. By the Principle of Mathematical Induction we proved $P(n)$ for all $n \in \mathbb{N}$.

5. (12 points) In each part, fill in the missing right-hand-side with a mathematical proposition which completes the definition:

   (a) $3|n \Leftrightarrow (\exists k)(k \in \mathbb{Z} \wedge n = 3k)$

   (b) $[A \subseteq B] \Leftrightarrow (\forall x)[(x \in A) \Rightarrow (x \in B)]$

   (c) $A - B = \{x | x \in A \wedge x \notin B\}$

   In parts (d)–(f), $f\colon S \to T$.

   (d) $f^{-1}(A) = \{x \in S | f(x) \in A\}$

   (e) f is one-to-one $\Leftrightarrow (\forall x, y \in S)[(x \neq y) \Rightarrow (f(x) \neq f(y))]$

   (f) f is onto $\Leftrightarrow (\forall y \in T)[(\exists x \in S)(f(x) = y)]$