

# Mathematical Image Modeling: Homework 3

Due on January 28, 2026 at 23:59

*Jason Murphey 10:00*

**Hashem A. Damrah**

UO ID: 952102243



**Problem 1.** Show that  $|e^{i\theta} - 1| \leq \theta$  for  $\theta \in \mathbb{R}$ . (*Hint:* You can use the fundamental theorem of calculus.)

*Solution.* We have

$$e^{i\theta} - 1 = \int_0^\theta \frac{d}{dt} e^{it} dt = \int_0^\theta i e^{it} dt.$$

Thus,

$$|e^{i\theta} - 1| = \left| \int_0^\theta i e^{it} dt \right| \leq \int_0^\theta |i e^{it}| dt \leq \int_0^\theta 1 dt = \theta. \quad \square$$

**Problem 2.** Prove that

$$(f * g)'(x) = (f' * g)(x),$$

where  $'$  denotes derivative and  $*$  denotes convolution. Just treat this as a formal identity, i.e. assume everything converges nicely and you can pass derivatives through the integral sign.

This identity has an important consequence, namely: “the convolution of  $f$  and  $g$  is as smooth as the smoother of  $f$  and  $g$ ”.

*Solution.* We have

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy.$$

Computing the derivative, we have

$$(f * g)'(x) = \frac{d}{dx} \int_{\mathbb{R}} f(x - y)g(y) dy = \int_{\mathbb{R}} \frac{d}{dx} f(x - y)g(y) dy = \int_{\mathbb{R}} f'(x - y)g(y) dy = (f' * g)(x). \quad \square$$

**Problem 3.** Prove that if  $f(x) = 0$  for  $|x| > R$  and  $g(x) = 0$  for  $|x| > T$ , then  $(f * g)(x) = 0$  for  $|x| > R + T$ .

*Solution.* We have

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy.$$

If  $|x| > R + T$ , then for any  $y$  such that  $g(y) \neq 0$ , we have  $|y| \leq T$ . Thus,

$$|x - y| \geq |x| - |y| > (R + T) - T = R.$$

Since  $f(x - y) = 0$  for  $|x - y| > R$ , it follows that  $f(x - y) = 0$  whenever  $g(y) \neq 0$ . Therefore, the integrand  $f(x - y)g(y)$  is zero for all  $y$ , and hence

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy = 0. \quad \square$$

**Problem 4.** Show that if  $g(x) = f(x + y)$  for some  $y \in \mathbb{R}$ , then  $\hat{g}(\xi) = e^{iy\xi} \hat{f}(\xi)$ .

*Solution.* We have

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-ix\xi} dx = \int_{\mathbb{R}} f(x + y) e^{-ix\xi} dx.$$

Making the substitution  $u = x + y$ , we have  $dx = du$  and thus

$$\hat{g}(\xi) = \int_{\mathbb{R}} f(u) e^{-i(u-y)\xi} du = e^{iy\xi} \int_{\mathbb{R}} f(u) e^{-iu\xi} du = e^{iy\xi} \hat{f}(\xi). \quad \square$$

**Problem 5.** Show that limits are unique in a metric space. That is, if  $(X, d)$  is a metric space and  $\{x_n\}$  is a sequence satisfying  $x_n \rightarrow x \in X$  and  $x_n \rightarrow y \in X$ , then  $x = y$ .

*Solution.* Since  $\{x_n\} \rightarrow x$ , we have for every  $\varepsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $d(x_n, x) < \varepsilon/2$ . Similarly, since  $\{x_n\} \rightarrow y$ , we have for every  $\varepsilon > 0$ , there exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $d(x_n, y) < \varepsilon/2$ . Let  $N = \max(N_1, N_2)$ . Then for all  $n \geq N$ , we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we conclude that  $d(x, y) = 0$ . By the properties of a metric, this implies that  $x = y$ .  $\square$

**Problem 6.** (optional for 410, required for 510). Taking the following fact for granted, complete the proof of the Riemann–Lebesgue lemma (i.e.  $f \in L^1 \Rightarrow \hat{f} \in C_0$ ):

**Fact:** For any  $f \in L^1$  and any  $\varepsilon > 0$ , there exists a function  $g \in L^1$  satisfying (i)  $fg \in L^1$  and  $g' \in L^1$  and (ii)  $\|f - g\|_{L^1} < \varepsilon$ .

Recall that in class we proved that for  $g \in L^1$  satisfying (i), we have  $\hat{g} \in C_0$ .

*Solution.* Let  $f \in L^1$  and let  $\varepsilon > 0$ . By the given fact, there exists  $g \in L^1$  such that  $fg \in L^1$ ,  $g' \in L^1$ , and  $\|f - g\|_{L^1} < \varepsilon$ . Since we have already established that for such a function  $g$ ,  $\hat{g} \in C_0$ , it follows that  $\hat{g}$  is continuous and vanishes at infinity.

Now, we need to show that  $\hat{f} \in C_0$ . We can write

$$\hat{f}(\xi) = \hat{g}(\xi) + (\hat{f}(\xi) - \hat{g}(\xi)).$$

Using the properties of the Fourier transform, we have

$$|\hat{f}(\xi) - \hat{g}(\xi)| = \left| \int_{\mathbb{R}} (f(x) - g(x)) e^{-ix\xi} dx \right| \leq \int_{\mathbb{R}} |f(x) - g(x)| dx = \|f - g\|_{L^1} < \varepsilon.$$

Since  $\hat{g}(\xi)$  is continuous and vanishes at infinity, for any  $\delta > 0$ , there exists  $M > 0$  such that for all  $|\xi| > M$ ,  $|\hat{g}(\xi)| < \delta$ . Therefore, for all  $|\xi| > M$ , we have

$$|\hat{f}(\xi)| \leq |\hat{g}(\xi)| + |\hat{f}(\xi) - \hat{g}(\xi)| < \delta + \varepsilon.$$

Since  $\varepsilon$  and  $\delta$  were arbitrary, this shows that  $\hat{f}(\xi)$  also vanishes at infinity. Thus, we conclude that  $\hat{f} \in C_0$ .  $\square$