Fundamentals of Analysis II: Homework 4

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Exercise 5.3.3. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(i) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

(ii) Argue that at some point c we have $h'(c) = 1/3$.

(iii) Argue that $h'(x) = 1/4$ at some point in the domain.

Solution to (i). Consider $g(x) = h(x) = x$. It's continuous. It's values at the endpoints are $g(0) = 1$ and $g(0) = -1$. By the IVT, there exists a point $d \in [0, 3]$ where $g(d) = 0$, i.e., $h(d) = d$. ☐

Solution to (ii). Just apply the IVT on the interval $[0, 3]$ to get a $c \in (0, 3)$ where

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3 - 0} = \frac{1}{3}.$$ ☐

Solution to (iii). By part (ii), we know there exists a $c \in (0, 3)$ where $h'(c) = 0$. We can also find a $d \in (1, 3)$ with $h'(d) = 0$. Then, using Darboux's Theorem, there must exist a point $x \in (c, d)$ where $h'(x) = 1/4$. ☐

Exercise 5.3.4. Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

(i) If $f(x_n) = 0$ for all $n \in \mathbb{N}$, show $f(0) = 0$ and $f'(0) = 0$.

(ii) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Solution to (i). Since $f'(0)$ exists and $f(x_n) = 0$ for all $n \in \mathbb{N}$, we have

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} \frac{0 - 0}{x_n - 0} = 0.$$ ☐

Solution to (ii). Using the MVT over $[0, x_n]$, there must exist a $c_n \in (0, x_n)$ such that

$$f'(c_n) = \frac{f(x_n)}{x_n}.$$

Then, like we did in part (i), we have

$$f''(0) = \lim_{n\to\infty} \frac{f'(c_n) - f'(0)}{c_n - 0} = \lim_{n\to\infty} \frac{\frac{f(x_n)}{x_n} - 0}{c_n - 0} = 0.$$ ☐

Exercise 5.3.6.

(i) Let $g : [0, a] \to \mathbb{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show that $|g(x)| \leq Mx$ for all $x \in [0, a]$.

(ii) Let $h : [0, a] \to \mathbb{R}$ be twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.

(iii) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Solution to (i). Since $g(x)$ is differentiable on $(0, a)$ and continuous on $[0, a]$, for each $x \in (0, a]$, there exists some $c \in (0, x)$ such that

$$g(x) = g(0) + g'(c)(x - 0) \Rightarrow g(x) = g'(c)x \Rightarrow |g(x)| = |g'(c)x| \Rightarrow |g(x)| \leq Mx.$$ ☐

Solution to (ii). Applying the MVT to h' on $[0, x]$, there exists some $c \in (0, x)$ such that

$$h'(x) = h'(0) + h''(c)(x - 0) \Rightarrow h'(x) = h''(c)x \Rightarrow |h(x)| \leq Mx.$$

Now, applying the MVT to h on $[0, x]$, there exists some $d \in (0, x)$ such that

$$h(x) = h(0) + h'(d)(x - 0) \Rightarrow h(x) = h'(d)x \Rightarrow |h(x)| \leq Mx^2.$$

However, this overestimates the bound. To refine it, observe that the MVT picks d so that

$$|h'(d)| \leq Md \leq Mx.$$

Thus, since d is chosen in the middle of the interval, we get the tighter bound

$$|h(x)| \leq \frac{Mx^2}{2}. \qquad \square$$

Solution to (iii). I conjecture: If $f : [0, a] \to \mathbb{R}$ is three times differentiable with $f(0) = f'(0) = f''(0) = 0$ and $|f'''(x)| \leq M$ for all $x \in [0, a]$, then

$$|f(x)| \leq \frac{Mx^3}{6}.$$

Proof. Applying the MVT to f'' on $[0, x]$, then there exists some $c \in (0, x)$ such that

$$f''(x) = f''(0) + f'''(c)x \Rightarrow |f''(x)| = |f'''(c)x| \leq Mx.$$

Applying the MVT to f' on $[0, x]$, there exists some $d \in (0, x)$ such that

$$f'(x) = f'(0) + f''(d)x \Rightarrow |f'(x)| = |f''(d)x| \leq Mdx \leq \frac{Mx^2}{2}.$$

Finally, applying the MVT to f on $[0, x]$, there exists some $\varepsilon \in (0, x)$ such that

$$f(x) = f(0) + f'(\varepsilon)x \Rightarrow |f(x)| = |f(\varepsilon)x|.$$

Using the bound $|f'(\varepsilon)| \leq {}^{M\varepsilon^2}\!/_2$, we get

$$|f(x)| \leq \frac{M\varepsilon^2}{2}x.$$

Since $\varepsilon \in (0, x)$ we refine the bound as before to get

$$|f(x)| \leq \frac{Mx^3}{6}. \qquad \square$$

Exercise 5.3.8. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x \to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Solution. Define $g(x) = f(x) - f(0)$. Then, $g(x)$ is continuous on an interval containing 0 and differentiable for $x \neq 0$, with $g(0) = 0$. Then, using l'Hospital's Rule, we have

$$\lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} \frac{g'(x)}{1} = L. \qquad \square$$

Exercise 7.2.1. Let f be a bounded function on $[a, b]$, and let P be an arbitrary partition of $[a, b]$. First, explain why $U(f) \geq L(f, P)$. Now, prove Lemma 7.2.6.

Solution. Since for each subinterval $[x_{i-1}, x_i]$, we have

$$m_i \leq f(x) \leq M_i \text{ for all } x \in [x_{i-1}, x_i].$$

It follows that $m_i \leq M_i$ for each i. Multiplying by δx_i, which is always positive, and summing over all subintervals, we get

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = U(f, P).$$

Thus, $U(f) \geq L(f, P)$.

The upper integral and lower integral are defined as

$$U(f) = \inf_P U(f, P) \quad \text{and} \quad L(f) = \sup_P L(f, P).$$

Since for any partition P, we established that $U(f, P) \geq L(f, P)$, it follows that

$$\inf_P U(f, P) \geq \sup_P L(f, P) \Rightarrow U(f) \geq L(f) \Leftrightarrow U(f) \geq L(f, P).$$

This proves that the upper integral is always at least the lower integral for any bounded function f on $[a, b]$. □

Exercise 7.2.2. Consider $f(x) = 1/x$ over the interval $[1, 4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(i) Compute $L(f, P)$, $U(f, P)$, and $U(f, P) - L(f, P)$.

(ii) What happens to the value of $U(f, P) - L(f, P)$ when we add the point 3 to the partition?

(iii) Find a partition P' of $[1, 4]$ for which $U(f, P') - L(f, P') < 2/5$.

Solution to (i). This creates the subintervals

$$\left[1, \frac{3}{2}\right], \quad \left[\frac{3}{2}, 2\right], \quad \text{and} \quad [2, 4].$$

$$\begin{aligned} M_i &= \sup f(x) \text{ on } [x_{i-1}, x_i] \\ m_i &= \inf f(x) \text{ on } [x_{i-1}, x_i] \\ \Delta x_i &= x_i - x_{i-1}. \end{aligned}$$

Therefore, computing each of these values, we get

$$\left.\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i \Delta x_i = \frac{1}{3} + \frac{1}{4} + \frac{1}{2} = \frac{13}{12} \\ U(f, P) &= \sum_{i=1}^{n} \frac{1}{2} + \frac{1}{3} + 1 = \frac{11}{6} \end{aligned}\right\} \Rightarrow U(f, P) - L(f, P) = \frac{11}{6} - \frac{13}{12} = \frac{3}{4}.$$ □

Solution to (ii). If we add $x = 3$, we create a new subinterval $[2, 3]$ and $[3, 4]$. This reduces the maximum and minimum function values over each subinterval, leading to a smaller difference $U(f, P) - L(f, P)$. Since we are refining the partition, $U(f, P)$ decreases and $L(f, P)$ increases, making $U(f, P) - L(f, P)$ smaller. □

Solution to (iii). Let $P' = \{1, 5/4, 3/2, 2, 3, 4\}$. Then, we have

$$\left.\begin{aligned} L(f, P') &= \sum_{i=1}^{n} m_i \Delta x_i = \frac{12}{60} + \frac{10}{60} + \frac{15}{60} + \frac{20}{60} + \frac{15}{60} = \frac{6}{5} \\ U(f, P') &= \sum_{i=1}^{n} M_i \Delta x_i = \frac{15}{60} + \frac{12}{60} + \frac{20}{60} + \frac{30}{60} + \frac{20}{60} = \frac{97}{60} \end{aligned}\right\} \Rightarrow U(f, P') - L(f, P') = \frac{97}{60} - \frac{6}{5} = \frac{5}{12} < \frac{1}{2}.$$ □