

Introduction to Toplogy I: Homework 2

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Exercise 1.5. In Example 1.8(a) we saw a sequence in $C([0, 1])$ that converges in the L^1 metric but not in the sup metric. Prove that the reverse cannot happen: every sequence that converges in the sup metric converges in the L^1 metric.

Solution. Let (f_n) be a sequence in $C([0, 1])$ and suppose

$$d_\infty(f_n, f) := \sup_{x \in [0, 1]} |f_n(x) - f(x)| \rightarrow 0,$$

as $n \rightarrow \infty$, for some function f in $C([0, 1])$. For each n and every $x \in [0, 1]$ we have the pointwise inequality

$$|f_n(x) - f(x)| \leq d_\infty(f_n, f).$$

Integrating both sides over $[0, 1]$ yields

$$\int_0^1 |f_n(x) - f(x)| dx \leq \int_0^1 d_\infty(f_n, f) dx = d_\infty(f_n, f) \cdot 1.$$

Hence, with $d_1(g, h) := \int_0^1 |g(x) - h(x)| dx$, we get

$$d_1(f_n, f) \leq d_\infty(f_n, f) \rightarrow 0.$$

Therefore $d_1(f_n, f) \rightarrow 0$, i.e. $f_n \rightarrow f$ in the L^1 metric on $C([0, 1])$. □

Exercise 1.6. Let (X, d) be a metric space. Prove the reverse *triangle inequality*:

$$|d(p, q) - d(p, r)| \leq d(q, r),$$

for all $p, q, r \in X$. Include an appropriate picture.

Solution. Let $p, q, r \in X$. By the triangle inequality we have

$$d(p, q) \leq d(p, r) + d(r, q),$$

which can be rewritten as

$$d(p, q) - d(p, r) \leq d(r, q) = d(q, r).$$

Similarly, swapping q and r gives

$$d(p, r) \leq d(p, q) + d(q, r) \Rightarrow d(p, r) - d(p, q) \leq d(q, r),$$

or equivalently

$$-d(q, r) \leq d(p, q) - d(p, r).$$

Combining these two inequalities yields the reverse triangle inequality:

$$|d(p, q) - d(p, r)| \leq d(q, r).$$

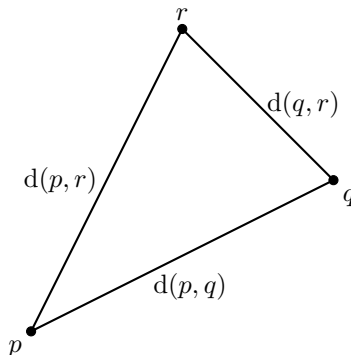


Figure 1: Visualization of the reverse triangle inequality in a metric space. □

Exercise 1.7. Let (X, d_X) and (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f : X \rightarrow Y$ be continuous at a point $p \in X$, and let $g : Y \rightarrow Z$ be continuous at $f(p)$. Prove that $g \circ f$ is continuous at p .

Solution. Let (X, d_X) , (Y, d_Y) and (Z, d_Z) be metric spaces, let $f : X \rightarrow Y$ be continuous at $p \in X$, and let $g : Y \rightarrow Z$ be continuous at $f(p)$. We show $g \circ f$ is continuous at p .

Fix $\varepsilon > 0$. Since g is continuous at $f(p)$, there exists $\delta_2 > 0$ such that

$$d_Y(y, f(p)) < \delta_2 \Rightarrow d_Z(g(y), g(f(p))) < \varepsilon.$$

Since f is continuous at p , there exists $\delta_1 > 0$ such that

$$d_X(x, p) < \delta_1 \Rightarrow d_Y(f(x), f(p)) < \delta_2.$$

Now let $x \in X$ satisfy $d_X(x, p) < \delta_1$. Then by the previous implication $d_Y(f(x), f(p)) < \delta_2$, and therefore by the choice of δ_2 we have $d_Z(g(f(x)), g(f(p))) < \varepsilon$. This shows that for every $\varepsilon > 0$, take $\delta = \delta_1$ to get

$$d_X(x, p) < \delta \Rightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon.$$

Therefore, $g \circ f$ is continuous at p . □

Exercise 2.1. Sketch each subset of \mathbb{R}^2 and find its closure, interior, and boundary in the Euclidean metric:

- (i) $A_1 = \{(x, y) \mid 0 < x \leq 1, 0 \leq y < 1\}$.
- (ii) $A_2 = \{(x, y) \mid 0 < x \leq 1, y = 0\}$.
- (iii) $A_3 = \{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$.
- (iv) $A_4 = \{(x, y) \mid x \neq 0 \text{ or } y = 0\}$.

Solution to (i). The boundary of A_1 is the rectangle formed by the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The interior is the open rectangle bounded by those same lines, and the closure is the union of the interior and the boundary, that is, the closed rectangle $[0, 1] \times [0, 1]$.

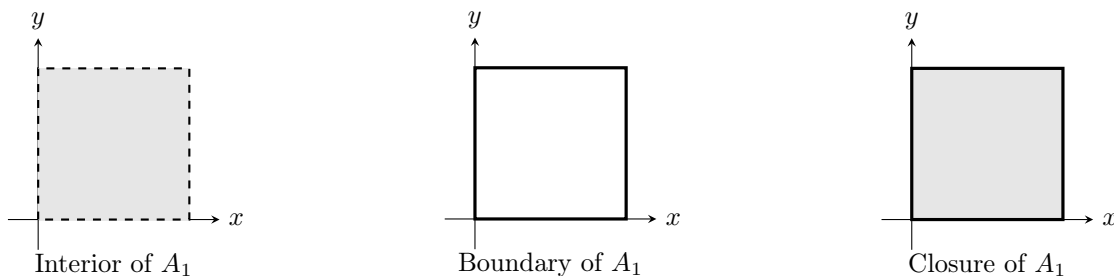


Figure 2: Visualization of the interior, boundary, and closure of $A_1 = [0, 1] \times [0, 1]$. □

Solution to (ii). The set $A_2 = \{(x, y) \mid 0 < x \leq 1, y = 0\}$ is a half-open line segment on the x -axis. Its interior is empty, since it has no open neighborhood contained in A_2 . Its boundary is the closed segment $[0, 1] \times \{0\}$, and its closure is that same closed segment.

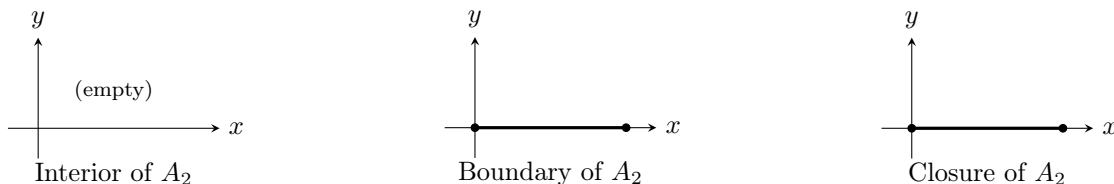


Figure 3: Visualization of the interior, boundary, and closure of $A_2 = \{(x, y) \mid 0 < x \leq 1, y = 0\}$. □

Solution to (iii). The set $A_3 = \{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ is dense in \mathbb{R}^2 , since every open ball contains points with rational coordinates. However, its interior is empty because any open set in \mathbb{R}^2 also contains points whose x and y coordinates are both irrational. Therefore, its boundary and closure are both equal to \mathbb{R}^2 .

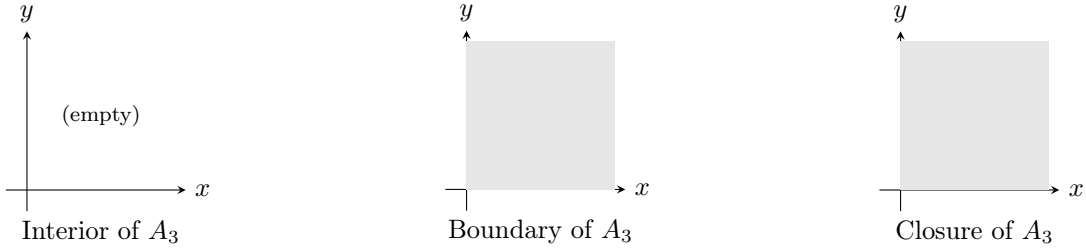


Figure 4: Visualization of the interior, boundary, and closure of $A_3 = \{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$. \square

Solution to (iv). The set $A_4 = \{(x, y) \mid x \neq 0 \text{ or } y = 0\}$ is the entire plane except for the y -axis, with the x -axis added back in. Its interior is A_4 itself, since removing a one-dimensional subset from \mathbb{R}^2 does not affect openness. The boundary consists of the y -axis, and the closure is the whole plane \mathbb{R}^2 .

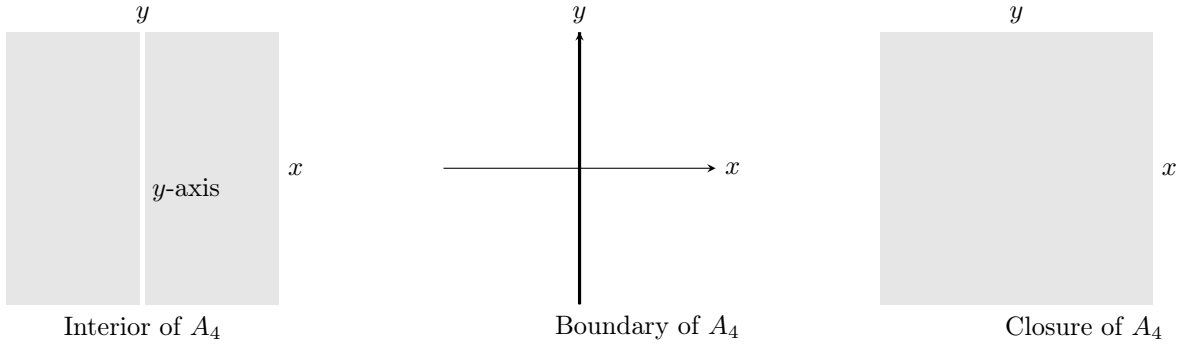


Figure 5: Visualization of the interior, boundary, and closure of $A_4 = \{(x, y) \mid x \neq 0 \text{ or } y = 0\}$. \square

Exercise 2.3. Let (X, d) be a metric space and $A \subset X$. Use Proposition 2.6 to prove that

$$\partial A = \partial(X \setminus A) = \overline{A} \cap \overline{X \setminus A}.$$

Solution. Assume $x \in \partial A$. By definition, that means $x \in \overline{A} \setminus \text{int } A$. Then, $x \in \overline{A}$ and $x \notin \text{int } A$. By Proposition 2.6, we have $x \notin \text{int } A$ if and only if $x \in \overline{X \setminus A}$. Therefore, $x \in \overline{A}$ and $x \in \overline{X \setminus A}$, which implies

$$x \in \overline{A} \cap \overline{X \setminus A}.$$

Hence, $\partial A \subset \overline{A} \cap \overline{X \setminus A}$.

Now, for the converse. Assume $x \in \overline{A} \cap \overline{X \setminus A}$. Then, $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. By Proposition 2.6 again, $x \in \overline{X \setminus A}$ if and only if $x \notin \text{int } A$. Therefore, $x \in \overline{A}$ and $x \notin \text{int } A$, which means $x \in \partial A$. Hence,

$$\overline{A} \cap \overline{X \setminus A} \subset \partial A.$$

Combining both inclusions, we conclude that

$$\partial A = \overline{A} \cap \overline{X \setminus A}.$$

Finally, since the expression is symmetric in A and $X \setminus A$, it follows that

$$\partial A = \partial(X \setminus A). \quad \square$$

Exercise 2.6. We saw in Example 2.8 that the inclusion $\text{int } A \cup \text{int } B \subset \text{int}(A \cup B)$ can be strict. Prove however that if $\overline{A} \cap \overline{B} = \emptyset$ then $\text{int } A \cup \text{int } B = \text{int}(A \cup B)$.

Solution. We always have

$$\text{int } A \cup \text{int } B \subset \text{int}(A \cup B),$$

so it remains to prove the reverse inclusion under the assumption

$$\overline{A} \cap \overline{B} = \emptyset.$$

Let $x \in \text{int}(A \cup B)$. Then there exists an open neighbourhood U of x with $U \subset A \cup B$. We will show that $x \in \text{int } A$ or $x \in \text{int } B$.

Suppose, for contradiction, that $x \notin \text{int } A$ and $x \notin \text{int } B$. Since $x \notin \text{int } A$, every neighbourhood of x meets the complement $X \setminus A$. In particular U meets $X \setminus A$. But $U \subset A \cup B$, so any point of $U \cap (X \setminus A)$ must lie in B . Hence $U \cap B \neq \emptyset$, and because this holds for every neighbourhood of x we conclude $x \in \overline{B}$. By the same argument (swapping A and B) the assumption $x \notin \text{int } B$ implies $x \in \overline{A}$. Thus

$$x \in \overline{A} \cap \overline{B},$$

contradicting the hypothesis that $\overline{A} \cap \overline{B} = \emptyset$.

Therefore at least one of $x \in \text{int } A$ or $x \in \text{int } B$ must hold, so

$$x \in \text{int } A \cup \text{int } B.$$

This proves $\text{int}(A \cup B) \subset \text{int } A \cup \text{int } B$, and combined with the trivial inclusion we obtain

$$\text{int}(A \cup B) = \text{int } A \cup \text{int } B.$$

□