Abstract Linear Algebra: Homework 7

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Problem 1. Use the following conclusion to solve the given problems.

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ with $n \geq m$. Then $\det(\lambda I_n - AB) = \lambda^{n-m}\det(\lambda I_m - BA)$.

(i) Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}^T\mathbf{x} = 1$. Find the eigenvalues for $I_n - 2\mathbf{x}\mathbf{x}^T$.

(ii) Let $\mathbf{x} = \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ and $\mathbf{y} = \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix} \in \mathbb{R}^n$. Find the eigenvalues for $I_n - \mathbf{x}\mathbf{y}^T$.

Solution to (i). We set

$$A = \mathbf{x} \in \mathbb{R}^{n\times 1} \quad \text{and} \quad B = 2\mathbf{x}^T \in \mathbb{R}^{1\times n}.$$

Then,

$$AB = \mathbf{x}(2\mathbf{x}^T) = 2\mathbf{x}\mathbf{x}^T \in \mathbb{R}^{n\times n} \quad \text{and} \quad BA = (2\mathbf{x}^T)\mathbf{x} = 2(\mathbf{x}^T\mathbf{x}) \in \mathbb{R}^{1\times 1}.$$

Since we are given $\mathbf{x}^T\mathbf{x} = 1$, it follows that $BA = 2(1) = 2$. Applying the determinant formula with $m = 1$, we get

$$\det(\lambda I_n - 2\mathbf{x}\mathbf{x}^T) = \lambda^{n-1}\det(\lambda I_1 - BA).$$

Since $BA = 2$, we get

$$\det(\lambda I_1 - BA) = \det(\lambda - 2) = (\lambda - 2).$$

Thus,

$$\det(\lambda I_n - 2\mathbf{x}\mathbf{x}^T) = \lambda^{n-1}(\lambda - 2).$$

The characteristic equation is:

$$\lambda^{n-1}(\lambda - 2) = 0.$$

Therefore, the eigenvalues of $I_n - 2\mathbf{x}\mathbf{x}^T$ are 1 with multiplicity $n - 1$ and -1 with multiplicity 1. □

Solution to (ii). We set

$$A = \mathbf{x} \in \mathbb{R}^{n\times 1} \quad \text{and} \quad B = \mathbf{y}^T \in \mathbb{R}^{1\times n}.$$

Thus, $AB = \mathbf{x}\mathbf{y}^T$ is an $n \times n$ matrix, while $BA = \mathbf{y}^T\mathbf{x}$ is a 1×1 scalar (a rank-1 matrix). Using the determinant identity with $m = 1$, we get

$$\det(\lambda I_n - AB) = \lambda^{n-1}\det(\lambda I_1 - BA).$$

Since $BA = \mathbf{y}^T\mathbf{x}$ is a 1×1 matrix, we can write

$$\det(\lambda I_1 - BA) = \det(\lambda - \mathbf{y}^T\mathbf{x}) = (\lambda - \mathbf{y}^T\mathbf{x}).$$

Thus,

$$\det(\lambda I_n - \mathbf{x}\mathbf{y}^T) = \lambda^{n-1}(\lambda - \mathbf{y}^T\mathbf{x}).$$

The characteristic equation is

$$\lambda^{n-1}(\lambda - \mathbf{y}^T\mathbf{x}) = 0.$$

Therefore, the eigenvalues of $I_n - \mathbf{x}\mathbf{y}^T$ are 0 with multiplicity $n - 1$ and $\mathbf{y}^T\mathbf{x}$ with multiplicity 1. □

Problem 2. Prove that an upper triangular matrix with zeros in all the diagonal entries is nilpotent. (Note: A matrix A is nilpotent if and only if there exists a positive integer k such that $A^k = 0$.)

Solution. Since A is upper triangular, it has the form:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Each entry on the main diagonal is zero: $a_{ii} = 0$ for all $1 \leq i \leq n$. We prove by induction that for each k, the matrix A^k is still upper triangular, and its first nonzero entries shift further above the main diagonal as k increases.

Base Case ($k = 1$): A is upper triangular with all zeros on the diagonal.

Induction Step: Suppose A^k is upper triangular and has zeros on the first k diagonals, meaning that its nonzero entries are restricted to positions where $j - i \geq k$. Now consider $A^{k+1} = A^k A$. The (i, j) entry of A^{k+1} is given by

$$(A^{k+1})_{ij} = \sum_{m=1}^{n} (A^k)_{im} A_{mj}.$$

By the induction hypothesis, $(A^k)_{im} = 0$ unless $m - i \geq k$, and since $A_{mj} = 0$ unless $j - m \geq 1$, it follows that A^{k+1} has zeros on the first $k + 1$ diagonals.

By induction, A^k has zeros for all entries where $j - i < k$. In particular, for $k = n$, this means $A^n = 0$, since there are no positions satisfying $j - i \geq n$ in an $n \times n$ matrix. Therefore, A is nilpotent. □

Problem 3. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Prove that $A^{-1} = g(A)$, for some polynomial $g(x)$ with $\deg(g(x)) = n - 1$.

Solution. Since A is a square matrix of size n, its characteristic polynomial is given by $p_A(x) = \det(xI - A)$. By the Cayley-Hamilton theorem, the matrix A satisfies its own characteristic equation $p_A(A) = 0$. Explicitly, if we write out the characteristic polynomial and substitute A into it, we get

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0.$$

Since A is invertible, we must have $c_0 \neq 0$, as otherwise, the equation above would imply that A is singular, contradicting the fact that A is invertible. Rearranging the equation, we get

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A = -c_0 I.$$

Multiplying both sides by $-\frac{1}{c_0}$ gives

$$\begin{aligned} -\frac{1}{c_0}\left(A^n + c_{n-1}A^{n-1} + \cdots + c_1 A\right) &= I \\ \Rightarrow A\left(-\frac{1}{c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I)\right) &= I \\ \Rightarrow -\frac{1}{c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I) &= A^{-1}. \end{aligned}$$

The left-hand side is a polynomial in A of degree at most $n - 1$, so defining

$$g(x) = -\frac{1}{c_0}(x^{n-1} + c_{n-1}x^{n-2} + \cdots + c_1),$$

we obtain $A^{-1} = g(A)$. □

Problem 4. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Define

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2 \tag{1}$$

Prove that 1 is an inner product on $\mathbb{R}^2$.

Solution. We define the inner product $\langle \cdot, \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbf{F}$ as in 1. We

For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we have

$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= (x_1 + z_1)y_1 - (x_2 + z_2)y_1 - (x_1 + z_1)y_2 + 4(x_2 + z_2)y_2 \\ &= x_1 y_1 + z_1 y_1 - x_2 y_1 - z_2 y_1 - x_1 y_2 - z_1 y_2 + 4x_2 y_2 + 4z_2 y_2 \\ &= (x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2) + (z_1 y_1 - z_2 y_1 - z_1 y_2 + 4z_2 y_2) \\ &= \langle \mathbf{x} + \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle. \end{aligned}$$

For all $c \in \mathbf{F}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we

$$\begin{aligned} \langle c\mathbf{x}, \mathbf{y} \rangle &= (cx_1)y_1 - (cx_2)y_1 - (cx_1)y_2 + 4(cx_2)y_2 \\ &= c(x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2) \\ &= c\langle \mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we have

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2 \\ &= y_1 x_1 - y_2 x_1 - y_1 x_2 + 4y_2 x_2 \\ &= \overline{\langle \mathbf{y}, \mathbf{x} \rangle} \\ &= \langle \mathbf{y}, \mathbf{x} \rangle, \end{aligned}$$

since we're in $\mathbb{R}$ and the conjugate is the identity.

For all $\mathbf{x} \in \mathbb{R}^2$, we have

$$\begin{aligned} \langle \mathbf{x}, \mathbf{x} \rangle &= x_1 x_1 - x_2 x_1 - x_1 x_2 + 4x_2 x_2 \\ &= x_1^2 - 2x_1 x_2 + 4x_2^2 \\ &= (x_1 - 2x_2)^2 \geq 0, \end{aligned}$$

with equality if and only if $\mathbf{x} = \mathbf{0}$.

Therefore, $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^2$. □

Problem 5. Let $A \in \mathbb{C}^{n \times n}$ and assume A is Hermitian positive-definite. Prove that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* A\mathbf{x}$ defines an inner product on $\mathbb{C}^n$.

Solution. We define the inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* A\mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. We need to verify that this inner product satisfies the properties of positivity, linearity, and conjugate symmetry.

For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$ and $c \in \mathbb{C}$, we have

$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= \mathbf{z}^* A(\mathbf{x} + \mathbf{y}) \\ &= \mathbf{z}^* A\mathbf{x} + \mathbf{z}^* A\mathbf{y} \\ &= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle. \end{aligned}$$

For all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and $c \in \mathbb{C}$, we have

$$\langle c\mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* A(c\mathbf{x})$$

$$\begin{aligned} &= \mathbf{y}^*(cA\mathbf{x}) \\ &= c\mathbf{y}^*A\mathbf{x} \\ &= c\langle \mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

For all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, we have

$$\begin{aligned} \overline{\langle \mathbf{x}, \mathbf{y} \rangle} &= \overline{\mathbf{y}^*A\mathbf{x}} \\ &= (\mathbf{y}^*A\mathbf{x})^* \\ &= \mathbf{x}^*A^*\mathbf{y} \\ &= \mathbf{x}^*A\mathbf{y} \\ &= \langle \mathbf{y}, \mathbf{x} \rangle. \end{aligned}$$

For all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, we have

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^*A\mathbf{x} \geq 0,$$

since A is positive-definite, this is true for all $\mathbf{x} \neq \mathbf{0}$, with equality if and only if $\mathbf{x} = \mathbf{0}$.

Therefore, $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$. □

Problem 6. If V is a vector space over $\mathbb{R}$, verify the following polarization identity for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2.$$

Solution. We begin with the definition of the norm induced by the inner product

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle.$$

Expanding the squared norms of the sum and difference of $\mathbf{x}$ and $\mathbf{y}$, we compute

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle. \end{aligned}$$

Since we are working in $\mathbb{R}^n$, the inner product satisfies symmetry, meaning $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$, so we rewrite the equation as

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2.$$

Similarly, we expand $\|\mathbf{x} - \mathbf{y}\|^2$

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 &= \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle. \end{aligned}$$

Using symmetry again, this simplifies to

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2.$$

Now, subtracting the two equations and dividing by 4, we get

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 &= (\|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2) - (\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2) \\ &= 4\langle \mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

Dividing both sides by 4, we obtain the desired polarization identity

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2.$$ □

Problem 7. Let V be an inner product space. Prove the following triangular inequality for any $\mathbf{x}, \mathbf{y} \in V$:

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Solution. We start with the definition of the norm induced by the inner product

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

For any $\mathbf{x}, \mathbf{y} \in V$, we expand the squared norm of their sum

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle. \end{aligned}$$

Since the inner product satisfies conjugate symmetry, we have $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$. Using the Cauchy–Schwarz inequality, we obtain the following

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2\mathrm{Re}(\langle \mathbf{x}, \mathbf{y} \rangle) + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2. \end{aligned}$$

The right-hand side is a perfect square

$$\|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Thus, we obtain

$$\|\mathbf{x}+\mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Taking the square root of both sides (using the fact that norms are always nonnegative),

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

□

Problem 8. Let $\mathbf{x}_1, \cdots, \mathbf{x}_n$ be orthonormal vectors in $\mathbb{R}^n$. Show that $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ are also orthonormal if and only if $A \in \mathbb{R}^{n \times n}$ is orthogonal.

Solution. Consider the transformed vectors $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ for some matrix $A \in \mathbb{R}^{n \times n}$. These vectors are orthonormal if and only if

$$(\forall i, j \in \{1, \cdots, n\})[\langle A\mathbf{x}_i, A\mathbf{x}_j \rangle = \delta_{ij}].$$

Using the definition of the inner product in $\mathbb{R}^n$, we get

$$\langle A\mathbf{x}_i, A\mathbf{x}_j \rangle = (A\mathbf{x}_i)^T (A\mathbf{x}_j).$$

Rewriting in matrix form, we obtain

$$(A\mathbf{x}_i)^T (A\mathbf{x}_j) = \mathbf{x}_i^T A^T A \mathbf{x}_j.$$

For this to hold for all i, j, we require

$$\mathbf{x}_i^T A^T A \mathbf{x}_j = \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}.$$

This is equivalent to the matrix equation

$$A^T A = I_n.$$

By definition, a matrix A satisfying $A^T A = I_n$ is an orthogonal matrix.

Therefore, the set $\{A\mathbf{x}_1, \cdots, A\mathbf{x}_n\}$ is orthonormal if and only if A is an orthogonal matrix. □

Problem 9. True or False (No explanation needed.)

(i) An inner product is a scalar-valued function on the set of ordered pairs of vectors.

(ii) An inner product is linear in both components.

(iii) If $(\mathbf{v}, \mathbf{w}) = \mathbf{0}$ for all $\mathbf{v}$ in an inner product space, then $\mathbf{w} = \mathbf{0}$.

(iv) A set of orthonormal vectors must be linearly independent.

(v) A set of orthogonal vectors must be linearly independent.

(vi) A matrix in $\mathbb{R}^{n\times n}$ is orthogonal if and only if its column vectors are orthogonal.

Solution to (i). True. □

Solution to (ii). False. □

Solution to (iii). True. □

Solution to (iv). True. □

Solution to (v). False. □

Solution to (vi). False. □