§12.3: The Dot Product

Defn: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ is the dot product of $\vec{a}$ and $\vec{b}$.

Ex: $\vec{a} = \langle 2, 4, -3 \rangle$ $\quad$ $\vec{b} = \langle 5, 3, 7 \rangle$

$$\vec{a} \cdot \vec{b} = 2(5) + 4(3) + (-3)(7)$$

$$= 10 + 12 - 21 = 1$$

Properties of Dot Product

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and K is a scalar, then.

1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

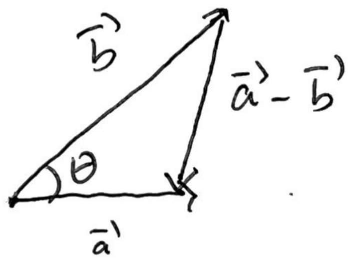
3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4) $(K\vec{a}) \cdot \vec{b} = K(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (K\vec{b})$

5) $\vec{a} \cdot \vec{0} = 0$

Geometry of Dot Product

Position $\vec{a}$ and $\vec{b}$ to share initial points. Let θ be angle between vectors such that $0 \le \theta \le \pi$.



By Law of Cosines,

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

Note:
$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot (\vec{a} - \vec{b}) - \vec{b} \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\ &= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \end{aligned}$$

Therefore

$$|\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Two nonzero vectors are <u>perpendicular</u> or <u>orthogonal</u> if the angle between them is $\theta = \frac{\pi}{2}$.

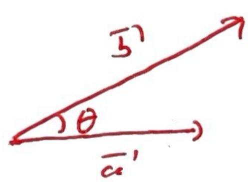
If $\theta = \frac{\pi}{2}$, then $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\left(\frac{\pi}{2}\right)$
$$= 0$$

~~Thm: Two nonzero ve~~

Thm: Two vectors are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$

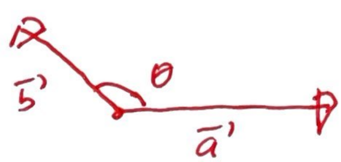
Suppose $\vec{a}, \vec{b} \neq \vec{0}$

If $0 \le \theta < \frac{\pi}{2}$, then $\cos(\theta) > 0$ and $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$
$$> 0$$



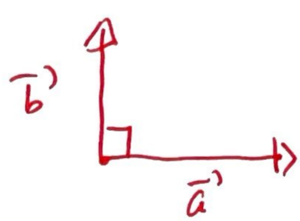
$\vec{a} \cdot \vec{b} > 0$

If $\frac{\pi}{2} < \theta \le \pi$, then $\cos(\theta) < 0$ and $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$
$$< 0$$



$\vec{a} \cdot \vec{b} < 0$

If $\theta = \frac{\pi}{2}$, then $\cos(\theta) = 0$ and $\vec{a} \cdot \vec{b} = 0$



$\vec{a} \cdot \vec{b} = 0$

Ex: Find the angle between $\vec{a} = \langle 1, 0, 1 \rangle$ and $\vec{b} = \langle -2, 4, -4 \rangle$

$\vec{a} \cdot \vec{b} = -2 + 0 - 4 = -6$

$|\vec{a}| = \sqrt{1+1} = \sqrt{2}$

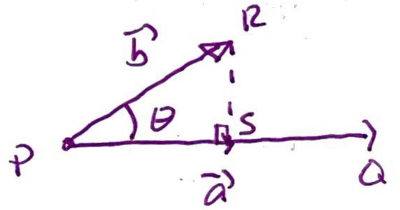
$|\vec{b}| = \sqrt{4+16+16} = \sqrt{36} = 6$

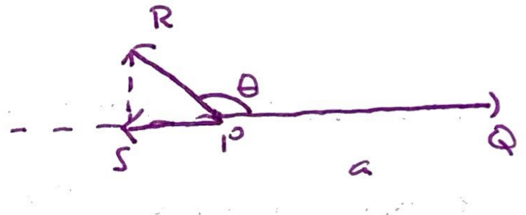
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-6}{6\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\theta = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

Projections

Consider vectors $\vec{a} = \vec{PQ}$ and $\vec{b} = \vec{PR}$ with angle θ.





The <u>vector projection</u> of $\vec{b}$ onto $\vec{a}$ denoted $\text{proj}_{\vec{a}}\vec{b}$ is $\vec{PS}$. It is parallel to $\vec{a}$. If $0 \le \theta < \frac{\pi}{2}$, then $\vec{a}$ and $\text{proj}_{\vec{a}}\vec{b}$ have the same direction.

If $\frac{\pi}{2} < \theta \le \pi$, then $\vec{a}$ and $\text{proj}_{\vec{a}}\vec{b}$ have opposite directions. If $\theta = \frac{\pi}{2}$, $\text{proj}_{\vec{a}}\vec{b} = \vec{0}$