Functional Complex Variables I: Homework 1

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Exercise 1.2.11. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0),$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}\right)$.

Solution. Expanding the left-hand side and simplifying, we get

$$\begin{aligned} z^2 + z + 1 = 0 &\Rightarrow (x^2 - y^2, 2xy) + (x, y) + (1, 0) = (0, 0) \\ &\Rightarrow (x^2 - y^2 + x + 1, 2xy + y) \qquad = (0, 0). \end{aligned}$$

Therefore, we have the system of equations

$$\begin{aligned} x^2 - y^2 + x + 1 &= 0 \\ 2xy + y &= 0. \end{aligned}$$

From the second equation, we can factor out y to get $y(2x + 1) = 0$. This gives us two cases to consider: either $y = 0$ or $2x + 1 = 0$.

If $y = 0$, then substituting it back into the first equation gives us

$$x^2 + x + 1 = 0,$$

which doesn't have any real solutions, as the discriminant is negative. Therefore, we must have $y \neq 0$.

In the case where $2x + 1 = 0$, we can solve for x to get $x = -1/2$. Substituting this value into the first equation gives us

$$\begin{aligned} 0 &= \left(-\frac{1}{2}\right)^2 - y^2 - \frac{1}{2} + 1 \\ 0 &= \frac{1}{4} - y^2 - \frac{1}{2} + 1 \\ 0 &= -y^2 + \frac{3}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm\frac{\sqrt{3}}{2}. \end{aligned}$$

Thus, the solutions to the equation $z^2 + z + 1 = 0$ are

$$z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \qquad \square$$

Exercise 1.3.1. Reduce each of these quantities to a real number

$$\text{(i) } \frac{1 + 2i}{3 - 4i} + \frac{2 - i}{5i}; \qquad \text{(ii) } \frac{5i}{(1 - i)(2 - i)(3 - i)}; \qquad \text{(iii) } (1 - i)^4.$$

Ans. (i) $-2/5$, (ii) $-1/2$, (iii) -4.

Solution to (i). Multiplying the numerator and denominator of the first term by the conjugate of the denominator, we have

$$\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{3+4i+6i-8}{9+16} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i.$$

Now, we can simplify the second term

$$\frac{2-i}{5i} = \frac{(2-i)(-i)}{5i(-i)} = \frac{-2i-1}{5} = -\frac{1}{5} - \frac{2}{5}i.$$

Adding these two results together, we have

$$\frac{1+2i}{3-4i} + \frac{2-i}{5i} = \left(-\frac{1}{5} + \frac{2}{5}i\right) + \left(-\frac{1}{5} - \frac{2}{5}i\right) = -\frac{2}{5}. \qquad \square$$

Solution to (ii). Expanding the denominator, we have

$$\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(1-3i)(3-i)} = -\frac{5i}{10i} = -\frac{1}{2}. \qquad \square$$

Solution to (iii). Expanding the polynomial $(1-i)^2$, we have

$$(1-i)^2 = (1-i)(1-i) = 1 - 2i + i^2 = 1 - 2i - 1 = -2i.$$

Now, we can expand $(1-i)^4$ as follows

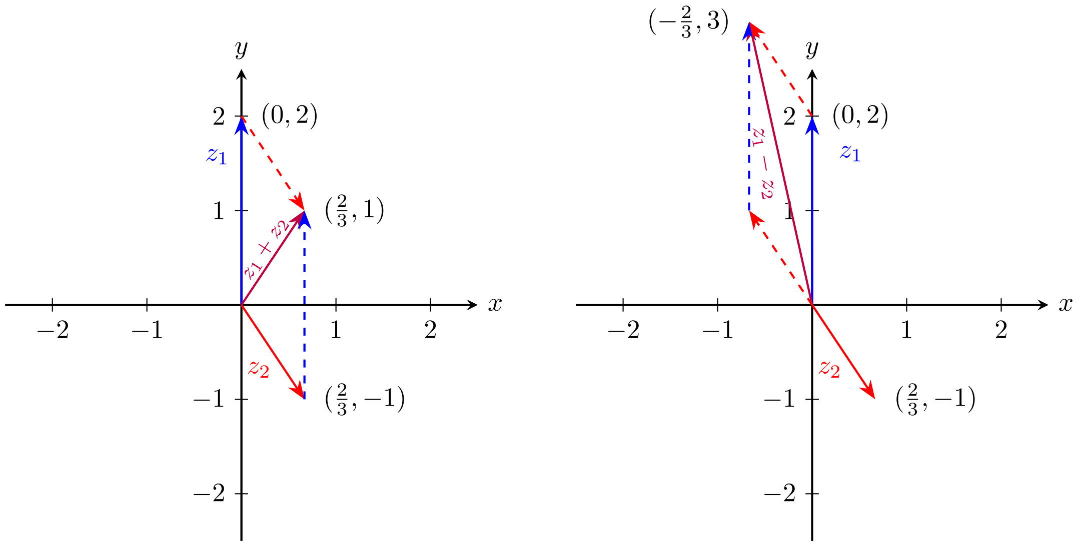
$$(1-i)^4 = (1-i)^2(1-i)^2 = (-2i)(-2i) = 4i^2 = -4. \qquad \square$$

Exercise 1.4.1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vertically when

(i) $z_1 = 2i, \quad z_2 = \dfrac{2}{3} - i$; (ii) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

(iii) $z_1 = (-3, 1), \quad z_2 = (1, 4)$; (iv) $z_1 = x_1 + y_1 i, \quad z_2 = x_1 - y_1 i$.

Solution to (i). Graphing the complex numbers $z_1 = 2i$ and $z_2 = \frac{2}{3} - i$ on the complex plane, we have



(a) $z_1 + z_2$ (b) $z_1 - z_2$

In this case, we can compute

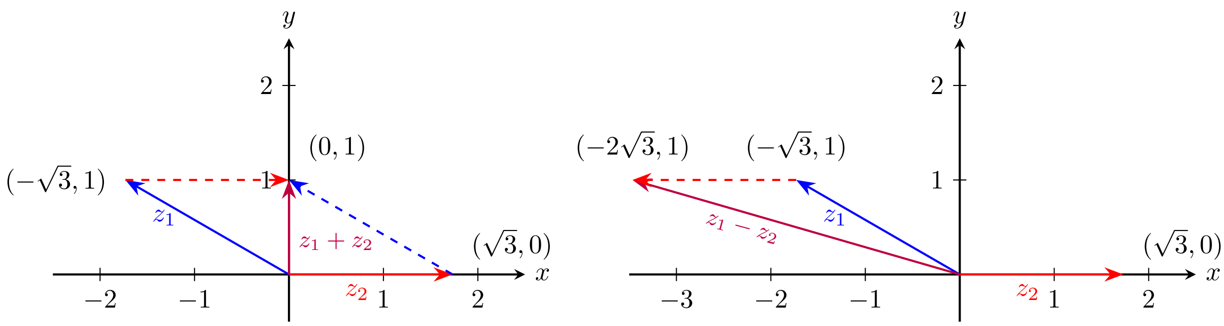
$$z_1 + z_2 = (0 + 2i) + \left(\frac{2}{3} - i\right) = \frac{2}{3} + i.$$

This gives us the point $(2/3, 1)$. Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (0 + 2i) + \left(-\frac{2}{3} + i\right) = -\frac{2}{3} + 3i.$$

This gives us the point $(-2/3, 3)$. $\square$

Solution to (ii). Graphing the complex numbers $z_1 = (-\sqrt{3}, 1)$ and $z_2 = (\sqrt{3}, 0)$ on the complex plane, we have



(a) $z_1 + z_2$ (b) $z_1 - z_2$

In this case, we can compute

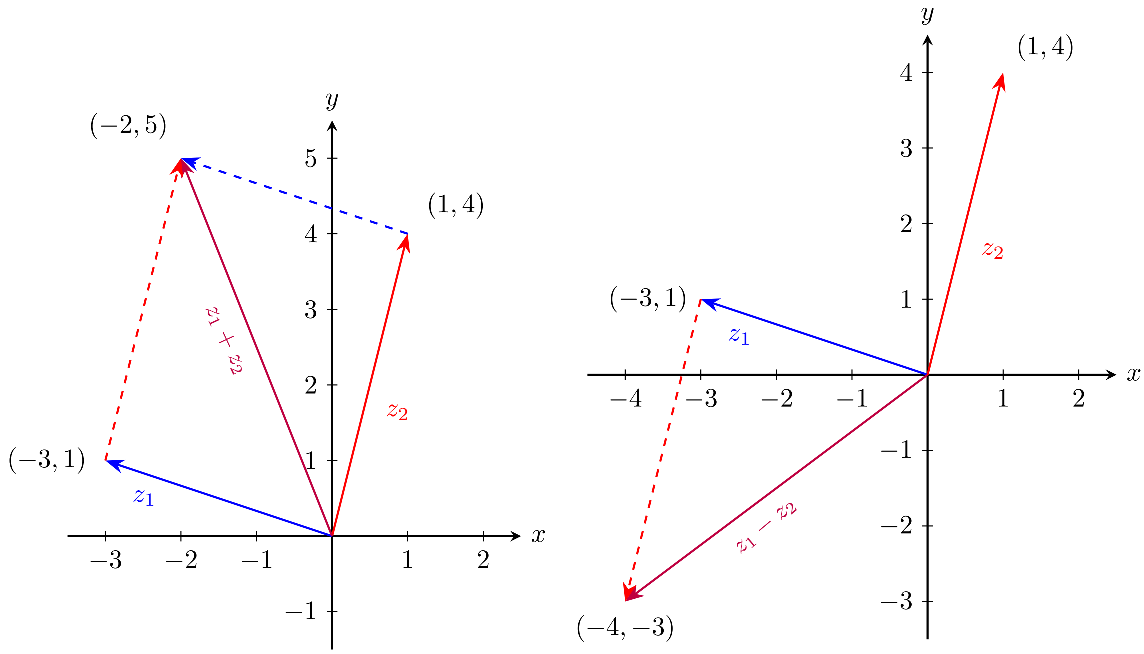
$$z_1 + z_2 = \left(-\sqrt{3}, 1\right) + \left(\sqrt{3}, 0\right) = (0, 1).$$

This gives us the point $(0, 1)$. Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = \left(-\sqrt{3}, 1\right) + \left(-\sqrt{3}, 0\right) = (-2\sqrt{3}, 1).$$

This gives us the point $(-2\sqrt{3}, 1)$. $\square$

Solution to (iii). Graphing the complex numbers $z_1 = (-3, 1)$ and $z_2 = (1, 4)$ on the complex plane, we have



(a) $z_1 + z_2$ (b) $z_1 - z_2$

In this case, we can compute

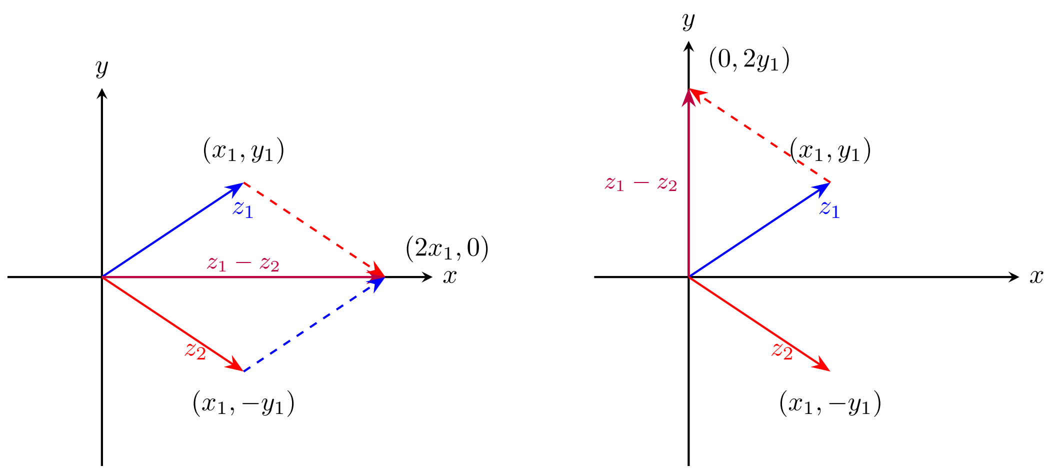
$$z_1 + z_2 = (-3, 1) + (1, 4) = (-2, 5).$$

This gives us the point $(-2, 5)$. Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (-3, 1) + (-1, -4) = (-4, -3).$$

This gives us the point $(-4, -3)$. $\square$

Solution to (iv). Graphing the complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_1 - y_1 i$ on the complex plane, we have



(a) $z_1 + z_2$ (b) $z_1 - z_2$

In this case, we can compute

$$z_1 + z_2 = (x_1 + y_1 i) + (x_1 - y_1 i) = (2x_1 + 0i).$$

This gives us the point $(2x_1, 0)$. Similarly, we can compute

$$z_1 - z_2 = z_1 + (-z_2) = (x_1 + y_1 i) + (x_1 - y_1 i) = (0, 2y_1).$$

This gives us the point $(0, 2y_1)$. □

Exercise 1.4.4. Verify that $\sqrt{2}|z| \geq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Solution. We know that $\mathrm{Re}(z) = |x|$ and $\mathrm{Im}(z) = |y|$. We also know that $|z| = \sqrt{x^2 + y^2}$. Therefore, we can rewrite the inequality as

$$\begin{aligned}
& 2|z| \geq |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \\
\Rightarrow\quad & 2\sqrt{x^2 + y^2} \geq |x| + |y| \\
\Rightarrow\quad & 2(x^2 + y^2) \geq (|x| + |y|)^2 \\
\Rightarrow\quad & 2(x^2 + y^2) \geq x^2 + 2|x||y| + y^2 \\
\Rightarrow\quad & 2x^2 + 2y^2 - x^2 - 2|x||y| - y^2 \geq 0 \\
\Rightarrow\quad & (x^2 + y^2) - 2|x||y| \geq 0 \\
\Rightarrow\quad & (x - |y|)(x + |y|) \geq 0.
\end{aligned}$$

Since $x^2 + y^2 = |x|^2 + |y|^2$, we can re-write the inequality as

$$\begin{aligned}
& (x - |y|)(x + |y|) \geq 0 \\
\Rightarrow\quad & (|x| - |y|)^2 \geq 0.
\end{aligned}$$

Therefore, we have $(|x| - |y|)^2 \geq 0$, which is always true. This means that $\sqrt{2}|z| \geq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$ is true for all complex numbers z. □

Exercise 1.4.6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(i) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$.

(ii) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

Solution to (i). The modulus $|z - 4i|$ represents the Euclidean distance between the complex number z and the point $4i$. Similarly, $|z + 4i|$ represents the distance from z to $-4i$.

Since the sum of the distances from any point z on the curve to the fixed points $(0, 4)$ and $(0, -4)$ is a constant, this satisfies the definition of an ellipse, where the sum of distances to the foci is constant.

The foci are at $(0, \pm 4)$. The given sum of distances is 10, which corresponds to $2a$ in the standard form of an ellipse equation. The foci are at a distance $c = 4$ from the center $(0, 0)$, and using the standard ellipse relation $a^2 = b^2 + c^2$, we get $a = 5$, $b = 3$, and $c = 4$. Thus, the ellipse has semi-major axis $a = 5$ and semi-minor axis $b = 3$. □

Solution to (ii). The expression $|z - 1|$ represents the distance from z to 1. The expression $|z + i|$ represents the distance from z to $-i$.

The equation states that any point $z = x + yi$ is equidistant from these two fixed points. The midpoint of $(1, 0)$ and $(0, -1)$ is

$$\left(\frac{1 + 0}{2}, \frac{0 + (-1)}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right).$$

The slope of the segment joining $(1, 0)$ and $(0, -1)$ is $m = 1$. The perpendicular bisector has a slope of -1, so its equation is

$$y + \frac{1}{2} = -1(x - \frac{1}{2}) \Rightarrow y = -x + 1.$$

Thus, the given equation represents the line through the origin with slope -1. □

Exercise 1.5.1(iv). Use properties of conjugates and moduli established in Sec. 5 to show that

(iv) $|(2\overline{z}+5)(\sqrt{2}-i)| = \sqrt{3}|2z+5|$.

Solution to (iv). Expanding the left-hand side, we have

$$|(2\overline{z}+5)(\sqrt{2}-i)| = |2\overline{z}+5| \cdot |\sqrt{2}-i|.$$

Taking the norm of the complex number $\sqrt{2}-i$, we have

$$|\sqrt{2}-i| = \sqrt{(\sqrt{2})^2+(-i)^2} = \sqrt{2+1} = \sqrt{3}.$$

Therefore, we have

$$|(2\overline{z}+5)(\sqrt{2}-i)| = \sqrt{3} \cdot |2\overline{z}+5|.$$

Since $|z| = |\overline{z}|$, we can replace $|2\overline{z}+5|$ with $|2z+5|$ to get

$$|(2\overline{z}+5)(\sqrt{2}-i)| = \sqrt{3}|2z+5|. \qquad \square$$

Exercise 1.5.10. Prove that

(i) z is real if and only if $\overline{z} = z$.

(ii) z is either real or pure imaginary if and only if $\overline{z}^2 = z^2$.

Solution to (i). Assume z is real. This means that $z = x + 0i$ for some real number x. The complex conjugate of z is $\overline{z} = x - 0i = x$. Therefore, we have $\overline{z} = z$.

Conversely, assume $\overline{z} = z$. This means that $z = x + yi$ and $\overline{z} = x - yi$. By assumption, we have $\overline{z} = z$ giving us $x + yi = x - yi$. This implies that $yi = -yi$, giving us $y = -y$. This only holds true if $y = 0$. Therefore, we have $z = x + 0i$ for some real number x. This means that z is real.

Thus, z is real if and only if $\overline{z} = z$. $\square$

Solution to (ii). Assume z is either real or pure imaginary. This gives us two cases, when z is real, $z = x+0i$ for some real number x, and when z is pure imaginary, $z = 0 + yi$ for some real number y. Notice that the first case is already proven in part (i), i.e., $z = \overline{z}$, giving us $z^2 = \overline{z}^2$. In the second case, we have $z = 0 + yi$ and $\overline{z} = 0 - yi = -yi$. Therefore, we have $\overline{z}^2 = (-yi)^2 = -y^2$. On the other hand, $z^2 = (0 + yi)^2 = -y^2$. Therefore, we have $\overline{z}^2 = z^2$.

Assume $\overline{z}^2 = z^2$. Let $z = x + yi$ and $\overline{z} = x - yi$. By assumption, we have

$$z^2 = \overline{z}^2 \Rightarrow (x+yi)^2 = (x-yi)^2 \Rightarrow x^2 - y^2 + 2xyi = x^2 - y^2 - 2xyi.$$

Comparing the real parts, we have $x^2 - y^2 = x^2 - y^2$, which is always true. Comparing the imaginary parts, we have $2xy = -2xy$. This implies that $4xy = 0$. This means that either $x = 0$ or $y = 0$. If $x = 0$, then z is pure imaginary. If $y = 0$, then z is real. Therefore, z is either real or pure imaginary.

Thus, z is either real or pure imaginary if and only if $\overline{z}^2 = z^2$. $\square$

Exercise 1.5.14. Using expressions (6), Sec. 5, for $\text{Re}(z)$ and $\text{Im}(z)$, show that the hyperbola $x^2 - y^2 = 1$ can be written as

$$z^2 + \overline{z}^2 = 2.$$

Solution. Substituting $\text{Re}(z)$ and $\text{Im}(z)$ for x and y into the equation for the hyperbola, we have

$$x^2 - y^2 = 1 \Rightarrow \text{Re}(z)^2 - \text{Im}(z)^2 = 1 \Rightarrow \left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2 = 1.$$

Expanding the left-hand side, we have

$$\left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2 = \frac{(z+\overline{z})^2}{4} - \frac{(z-\overline{z})^2}{-4} = \frac{(z+\overline{z})^2 + (z-\overline{z})^2}{4}.$$

Therefore, we have $(z+\overline{z})^2 + (z-\overline{z})^2 = 4$. Expanding the squares, we have

$$\begin{aligned} & z^2 + 2z\overline{z} + \overline{z}^2 + z^2 - 2z\overline{z} + \overline{z}^2 &= 4 \\ \Rightarrow \quad & 2z^2 + 2\overline{z}^2 &= 4 \\ \Rightarrow \quad & z^2 + \overline{z}^2 &= 2. \end{aligned}$$

□

Exercise (Extra). Given $a + bi$, a, b are real numbers, find c and d such that $(c + di)^2 = a + bi$.

Solution.

$$\begin{aligned} (c+di)^2 &= a + bi \\ (c^2 - d^2) + i(2cd) &= a + bi \end{aligned}$$

So we end up with the system of equations

$$\begin{aligned} c^2 - d^2 &= a \\ 2cd &= b. \end{aligned}$$

Solving the second one for c, and substituting this into the first one,

$$\begin{aligned} c &= \frac{b}{2d} \\ \left(\frac{b}{2d}\right)^2 - d^2 &= a \\ \frac{b^2}{4d^2} - d^2 &= a \\ \frac{b^2}{4} &= ad^2 + d^4 \\ d^4 + ad^2 - \frac{b^2}{4} &= 0. \end{aligned}$$

This is a function with quadratic form, so to solve we complete the square and factor

$$\begin{aligned} 0 &= \left(d^4 + ad^2 + \frac{a^2}{4}\right) - \frac{b^2}{4} - \frac{a^2}{4} \\ 0 &= \left(d^2 + \frac{a}{2}\right)^2 - \frac{(a^2+b^2)}{4} \\ \frac{\sqrt{a^2+b^2}}{2} &= d^2 + \frac{a}{2} \\ \frac{\sqrt{a^2+b^2} - a}{2} &= d^2 \\ \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}} &= d \\ &= \pm\frac{\sqrt{\sqrt{a^2+b^2}-a}}{\sqrt{2}} \\ &= \pm\frac{\sqrt{|z|-a}}{\sqrt{2}}. \end{aligned}$$

Plugging this back into the second equation to solve for c

$$\begin{aligned} c &= \frac{b}{2\left(\pm\frac{\sqrt{|z|-a}}{\sqrt{2}}\right)} \\ &= \pm\frac{b\sqrt{2}}{2\sqrt{|z|-a}} = \pm\frac{b\sqrt{2}}{2\sqrt{\sqrt{a^2+b^2}-a}}. \end{aligned}$$

The sign of these picked will depend on the sign of b, as the sign of a only depends on the magnitudes of c and d. $\square$