

Introduction to Proof: Homework 6

Due on November 14, 2024 at 11:59 PM

Victor Ostrik 12:00

Hashem A. Damrah

UO ID: 952102243

Problem 1 In each case, use mathematical notation to write the negation of the given statement, in such a way that no quantifier is immediately preceded by a negation sign. For parts (i) - (iv) decide which is true: the given statement or its negation.

(i) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y = 0]$.

(ii) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = 0]$.

(iii) $(\exists x, y \in \mathbb{R})[x^2 + y^2 = -1]$.

(iv) $(\forall x \in \mathbb{R})[x > 0 \Rightarrow (\forall y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \Rightarrow y = z]]$.

(v) $(\forall \varepsilon \in \mathbb{R})[\varepsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})[0 < \delta \wedge (\forall x \in \mathbb{R})[1 - \delta < x < 1 + \delta \Rightarrow |f(x) - 5| < \varepsilon]]]$.

(vi) $(\forall a, b \in \mathbb{R})(a < b) \Rightarrow (\exists c \in \mathbb{R}) \left[a < c < b \wedge f'(c) = \frac{f(b)-f(a)}{b-a} \right]$.

Part (vi) is a statement which is true for differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$, and it is a well-known theorem taught in every calculus class. What is the common name of this theorem?

Solution to (i). The negated statement is $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y \neq 0]$. The original statement is true, because for every real number x , there is a real number $y = -x$ such that $x + y = 0$. \square

Solution to (ii). The negated statement is $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y \neq 0]$. The negation is true because there is no real number x such that for every real number y , $x + y = 0$. \square

Solution to (iii). The negated statement is $(\forall x, y \in \mathbb{R})[x^2 + y^2 \neq -1]$. The negation is true because the sum of two squares is always non-negative. \square

Solution to (iv). The negated statement is $(\exists x \in \mathbb{R})[x > 0 \wedge (\exists y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \wedge y \neq z]]$. The original statement is true. For $x > 0$, if $y^2 = x$ and $z^2 = x$, then $y = z$ must hold when both y and z are positive. \square

Solution to (v). The negated statement is $(\exists \varepsilon \in \mathbb{R})[\varepsilon > 0 \wedge (\forall \delta \in \mathbb{R})[0 < \delta \Rightarrow (\exists x \in \mathbb{R})[1 - \delta < x < 1 + \delta \wedge |f(x) - 5| \geq \varepsilon]]]$. Whether the original statement or its negation is true depends on the behavior of the function $f(x)$ near $x = 1$. If $f(x)$ is continuous at $x = 1$ and $f(1) = 5$, then the original statement is true. \square

Solution to (vi). The negated statement is $(\exists a, b \in \mathbb{R}) \left[a < b \wedge (\forall c \in \mathbb{R}) \left[a < c < b \Rightarrow f'(c) \neq \frac{f(b)-f(a)}{b-a} \right] \right]$. The original statement is true for differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and is known as the “Mean Value Theorem”. \square

Problem 2 In each part below I give the definition for a mathematical concept we have encountered, but using the shorthand notation in quantifiers. Fill in each box with the appropriate mathematical term or phrase that best completes the definition. In parts (ii)-(iv), $f : S \rightarrow T$ and $A \subseteq S$.

(i) $\Leftrightarrow (\forall x \in A)[x \in B]$.

(ii) $\Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$.

(iii) $\Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$.

(iv) $= \{z \mid (\exists v \in A)[z = f(v)]\}$.

Solution to (i). $A \subseteq B \Leftrightarrow (\forall x \in A)[x \in B]$. \square

Solution to (ii). The function f is one-to-one $\Leftrightarrow (\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$. \square

Solution to (iii). The function f is onto $\Leftrightarrow (\forall t \in T)(\exists s \in S)[f(s) = t]$. \square

Solution to (iv). Image of A under f , $f(A) = \{z \mid (\exists v \in A)[z = f(v)]\}$. \square

Problem 3 Suppose $f : S \rightarrow T$ is one-to-one, $A \subseteq S$, and $B \subseteq S$. Give a line proof showing that $f(A) \cap f(B) \subseteq f(A \cap B)$.

Solution.

1. Assume $z \in f(A) \cap f(B)$.
2. Then $z \in f(A)$ and $z \in f(B)$.
3. Then $f^{-1}(z) \in A$ and $f^{-1}(z) \in B$.
4. So $f^{-1}(z) \in A \cap B$.
5. Then $z = f(f^{-1}(z)) \in f(A \cap B)$.
6. Therefore, $f(A) \cap f(B) \subseteq f(A \cap B)$.

Problem 4 Give a line proof showing that if $A \cap B = \emptyset$ and $B \cup C = A \cup D$ then $B \subseteq D$.

Solution.

1. Assume $x \in B$.
2. Then, $x \notin A$ since A and B are disjoint.
3. Then, since $B \cup C = A \cup D$, $x \in D$.
4. Therefore, $B \subseteq D$.

Problem 5 Let $f : S \rightarrow T$ and suppose f is onto. Let $A \subseteq S$. Give a line proof that $T - f(A) \subseteq f(S - A)$.

Solution.

1. Suppose $y \in T - f(A)$.
2. Since f is onto, there exists some $x \in S$ such that $f(x) = y$.
3. Since $y \notin f(A)$, $x \notin A$.
4. Then, $x \in S - A$.
5. Therefore, $f(x) \in f(S - A)$, where $f(x) = y$.
6. Therefore, $T - f(A) \subseteq f(S - A)$.

Problem 6 Give a line proof showing $A \cap (X - B) = (A \cap X) - (A \cap B)$.

- Solution.*
1. Suppose $x \in A \cap (X - B)$.
 2. Then, $x \in A$ and $x \in X - B$.
 3. Since $x \in X - B$, then $x \in X$ and $x \notin B$.
 4. Then, $x \in A \cap X$ and $x \notin A \cap B$.
 5. Therefore, $x \in (A \cap X) - (A \cap B)$.
 6. Therefore, $A \cap (X - B) \subseteq (A \cap X) - (A \cap B)$.
 7. Suppose $x \in (A \cap X) - (A \cap B)$.
 8. Then, $x \in A \cap X$ and $x \notin A \cap B$.
 9. Since $x \in A \cap X$, then $x \in A$ and $x \in X$.
 10. Since $x \notin A \cap B$, then $x \notin B$, since $x \in A$ on the previous line.
 11. Then, $x \in A$ and $x \in X - B$.
 12. Therefore, $x \in A \cap (X - B)$.
 13. Therefore, $(A \cap X) - (A \cap B) \subseteq A \cap (X - B)$.
 14. Therefore, $A \cap (X - B) = (A \cap X) - (A \cap B)$.

Problem 7 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 2e^{-x} + 5$.

- (i) Prove that f is one-to-one.
- (ii) Is f onto? Justify your answer.
- (iii) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = 5x^3 + 41$. Prove that g is onto.

Solution to (i). Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Then, $2e^{-a} + 5 = 2e^{-b} + 5$. This implies $e^{-a} = e^{-b}$, which implies $a = b$. Therefore, f is one-to-one. \square

Solution to (ii). Let $y \in \mathbb{R}$. Suppose $y = f(x) = 2e^{-x} + 5$. Then $e^{-x} = \frac{y-5}{2}$. The function e^{-x} only takes positive values, specifically $e^{-x} > 0$ for all $x \in \mathbb{R}$. This implies that

$$\frac{y-5}{2} > 0 \Rightarrow y > 5.$$

Therefore, f is not onto. \square

Solution to (iii). Let $y \in \mathbb{R}$. Suppose $y = g(x) = 5x^3 + 41$. Then,

$$x = \sqrt[3]{\frac{y-41}{5}}.$$

Since the cube root function is defined for all real numbers, g is onto. \square

Problem 8 Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$.

- (i) If f and g are one-to-one, prove that $g \circ f$ is onto.
- (ii) If f and g are both onto, prove that $g \circ f$ is onto.

Solution to (i). Assume f and g are both one-to-one. Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then, using the definition of composition, we have $g(f(x_1)) = g(f(x_2))$. Since g is one-to-one, this implies $f(x_1) = f(x_2)$. Since f is one-to-one, this implies $x_1 = x_2$. Therefore, $g \circ f$ is one-to-one. \square

Solution to (ii). Assume f and g are both onto. Let $z \in C$. Since g is onto, then there exists some y such that $g(y) = z$. Since f is onto, then there exists some x such that $f(x) = y$. Therefore, $(g \circ f)(x) = g(f(x)) = g(y) = z$. Therefore, $g \circ f$ is onto. \square

Problem 9 Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ has the property that $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) < f(y)]$.

(i) Prove that f is one-to-one.

(ii) Give an example of a function f satisfying the given property but which is not onto.

Solution to (i). Assume $f(x_1) = f(x_2)$, for some $x_1, x_2 \in \mathbb{R}$. By the trichotomy property of real numbers, we know that $x_1 < x_2$, $x_1 = x_2$, or $x_1 > x_2$. If $x_1 < x_2$, then by the property of f , $f(x_1) < f(x_2)$. This contradicts our assumption that $f(x_1) = f(x_2)$. If $x_1 > x_2$, then by the property of f , $f(x_1) > f(x_2)$. This also contradicts our assumption that $f(x_1) = f(x_2)$. Therefore, $x_1 = x_2$. Therefore, f is one-to-one. \square

Solution to (ii). The function $f(x) = \frac{x}{x-1}$, which holds the property that if $x_1 < x_2$, for some $x_1, x_2 \in \mathbb{R}$, then $f(x_1) < f(x_2)$. But $f(x)$ is not onto, since there doesn't exist any $x \in \mathbb{R}$ such that $f(x) = 1$. \square

Problem 10

(i) If $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is given by $f(x) = x^2 - 1$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ is given by $g(x) = 3x + 2$, determine $(g \circ f)(0)$ and $(g \circ f)(2)$. Determine an algebraic formula for $(g \circ f)(x)$ for any integer x .

(ii) Suppose that $f : S \rightarrow T$ and $g : T \rightarrow U$. If $A \subseteq S$, give a line proof that $(g \circ f)(A) = g(f(A))$.

Solution to (i). The composition $(g \circ f)(x)$ is given by

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 3(x^2 - 1) + 2 = 3x^2 - 1.$$

Therefore, $(g \circ f)(0) = -1$ and $(g \circ f)(2) = 11$. \square

Solution to (ii). 1. By definition, $(g \circ f)(A) = \{(g \circ f)(a) \mid a \in A\}$.

2. Then, $(\forall a \in A)[(g \circ f)(a) = g(f(a))]$.

3. Thus, $(g \circ f)(A) = \{g(f(a)) \mid a \in A\}$.

4. By definition, $g(f(A)) = \{g(f(a)) \mid a \in A\}$.

5. Therefore, $g(f(A)) = \{g(t) \mid t \in f(A)\} = \{g(f(a)) \mid a \in A\}$.

6. Therefore, $(g \circ f)(A) = g(f(A))$.

Problem 11 Consider the function $f : \mathbb{Z}_7 \rightarrow \mathbb{Z}_7$ given by $f(x) = x^3 + 1$. Answer the following questions

(i) Is f is one-to-one? Explain why or why not.

(ii) Is f onto? Explain why or why not.

(iii) Determine $f(S)$, where $S = \{0, 2, 4, 6\}$.

(iv) What is $f^{-1}(\{0\})$.

(v) If $A = \{1, 2, 3, 4\}$ and $B = \{0, 4, 5, 6\}$, determine $f^{-1}(A)$ and $f^{-1}(B)$. Also, determine $f^{-1}(A \cap B)$.

Solution to (i). To check if f is one-to-one, we must create a table of values to see if any two distinct elements in the domain map to the same element in the codomain. The table is shown below.

x	0	1	2	3	4	5	6
$f(x)$	1	2	2	0	2	0	1

Since $f(3) = f(5) = 0$, $f(0) = f(6) = 1$, and $f(1) = f(2) = f(4) = 2$, f is not one-to-one. \square

Solution to (ii). To check if f is onto, we must check if every element in the codomain is mapped to by some element in the domain. The table of values shows that f is not onto, since f does not map 3, 4, 5, or 6 to any element in the codomain. \square

Solution to (iii). The set $f(S)$ is given by $\{f(0), f(2), f(4), f(6)\} = \{1, 2, 2, 1\} = \{0, 1, 2\}$. \square

Solution to (iv). The set $f^{-1}(\{0\})$ is given by $\{x \in \mathbb{Z}_7 \mid f(x) = 0\} = \{3, 5\}$. \square

Solution to (v). The set $f^{-1}(A) = \{f(1), f(2), f(3), f(4)\} = \{2, 2, 0, 4\} = \{0, 2, 4\}$ and $f^{-1}(B) = \{f(0), f(4), f(5), f(6)\} = \{1, 2, 0, 1\}$. The set $f^{-1}(A \cap B) = \{f(4)\} = \{2\}$. \square

Problem 12 Suppose $f : S \rightarrow T$, $A \subseteq S$, and $B \subseteq T$. Give a line proof for each of the following

(i) $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

(ii) $f(A) \cap B = \emptyset \Rightarrow A \subseteq -f^{-1}(B)$.

Solution to (i).

1. Suppose $x \in A$.
2. Then, $f(x) \in f(A)$.
3. Since $f(A) \subseteq B$, then $f(x) \in B$.
4. Then, $x \in f^{-1}(B)$.
5. Then, $A \subseteq f^{-1}(B)$.
6. Therefore, $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$.

Solution to (ii).

1. Suppose $x \in A$.
2. Then, $f(x) \in f(A)$.
3. Then, $f(x) \notin B$.
4. Then, $x \notin f^{-1}(B)$.
5. Then, $A \subseteq -f^{-1}(B)$.
6. Therefore, $f(A) \cap B = \emptyset \Rightarrow A \subseteq -f^{-1}(B)$.

Problem 13 Suppose $f : S \rightarrow T$, $A \subseteq T$, $B \subseteq T$, and $C \subseteq S$. Give a line proof of each of the following

(i) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(iii) $f(f^{-1}(A)) \subseteq A$.

(iv) $C \subseteq f^{-1}(f(C))$.

(v) If f is onto, then $f(f^{-1}(A)) = A$.

(vi) If f is one-to-one, then $C = f^{-1}(f(C))$.

Solution to (i). 1. Suppose $x \in f^{-1}(A \cap B)$.

2. Then, $f(x) \in A \cap B$.

3. Then, $f(x) \in A$ and $f(x) \in B$.

4. Then, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.

5. Therefore, $x \in f^{-1}(A) \cap f^{-1}(B)$.

6. Therefore, $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

7. Suppose $x \in f^{-1}(A) \cap f^{-1}(B)$.

8. Then, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.

9. Then, $f(x) \in A$ and $f(x) \in B$.

10. Then, $f(x) \in A \cap B$.

11. Therefore, $x \in f^{-1}(A \cap B)$.

12. Therefore, $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

13. Therefore, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Solution to (ii). 1. Suppose $x \in f^{-1}(A \cup B)$.

2. Then, $f(x) \in A \cup B$.

3. Then, $f(x) \in A$ or $f(x) \in B$.

4. Then, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

5. Therefore, $x \in f^{-1}(A) \cup f^{-1}(B)$.

6. Therefore, $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

7. Suppose $x \in f^{-1}(A) \cup f^{-1}(B)$.

8. Then, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

9. Then, $f(x) \in A$ or $f(x) \in B$.

10. Then, $f(x) \in A \cup B$.

11. Therefore, $x \in f^{-1}(A \cup B)$.

12. Therefore, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

13. Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

- Solution to (iii).*
1. Suppose $y \in f(f^{-1}(A))$.
 2. Then, $y = f(x)$ for some $x \in f^{-1}(A)$.
 3. Then, $x \in f^{-1}(A)$.
 4. Then, $f(x) \in A$.
 5. Therefore, $y \in A$.
 6. Therefore, $f(f^{-1}(A)) \subseteq A$.

- Solution to (iv).*
1. Suppose $x \in C$.
 2. Then, $f(x) \in f(C)$.
 3. Then, $x \in f^{-1}(f(C))$.
 4. Therefore, $C \subseteq f^{-1}(f(C))$.

- Solution to (v).*
1. From (iii), we know that $f(f^{-1}(A)) \subseteq A$.
 2. If f is onto, then for $y \in A$, there exists $x \in f^{-1}(A)$ such that $f(x) = y$.
 3. Thus, $y \in f(f^{-1}(A))$.
 4. So $A \subseteq f(f^{-1}(A))$.
 5. Therefore, $f(f^{-1}(A)) = A$.

- Solution to (vi).*
1. From (iv), we know that $C \subseteq f^{-1}(f(C))$.
 2. If f is one-to-one and $x \in f^{-1}(f(C))$, then $f(x) = f(c)$, for some $c \in C$.
 3. This implies that $x = c$.
 4. So, $f^{-1}(f(C)) \subseteq C$.
 5. Therefore, $C = f^{-1}(f(C))$.

Problem 14 Construct an example of a function $f : \{0, 1, 2\} \rightarrow \{0, 1\}$ and a subset $A \subseteq \{0, 1\}$ where $f(f^{-1}(A)) \neq A$. Also construct an example of a function $g : \{0, 1, 2\} \rightarrow \{0, 1\}$ and a subset $C \subseteq \{0, 1, 2\}$ where $C \neq g^{-1}(g(C))$.

Solution. We cannot construct such examples, because for any function $f : S \rightarrow T$ and any subset $A \subseteq T$, we have $f(f^{-1}(A)) = A$.

We cannot construct such examples, because for any function $g : S \rightarrow T$ and any subset $C \subseteq S$, we have $C = g^{-1}(g(C))$. \square

Problem 15 Suppose $f, g : \mathbb{Z} \rightarrow \mathbb{Z}$ are two functions such that $f(M_3) \subseteq M_6$, $g(M_2) \subseteq M_7$, and $g^{-1}(M_5) = M_3$. Prove that for all $x \in \mathbb{Z}$, if $3 \mid x$ then $35 \mid g(f(x))$.

- Solution.*
1. Assume $x \in \mathbb{Z}$ such that $3 \mid x$.
 2. By the property of f , $f(M_3) \subseteq M_6$.
 3. Thus, $f(x) \in M_6$.
 4. Since $6 \mid f(x)$, we have $2 \mid f(x)$ and $3 \mid f(x)$.
 5. Because $2 \mid f(x)$, $f(x) \in M_2$.
 6. By condition 2, $g(M_2) \subseteq M_7$.
 7. Thus, $g(f(x)) \in M_7$.
 8. This means $7 \mid g(f(x))$.
 9. By condition 3, $g^{-1}(M_5) = M_3$.
 10. So, $f(x) \in M_3$, which means $g(f(x)) \in M_5$.
 11. This means $5 \mid g(f(x))$.
 12. Since $g(f(x))$ is divisible by both 5 and 7, then $35 \mid g(f(x))$.
 13. Therefore, for all $x \in \mathbb{Z}$, if $3 \mid x$ then $35 \mid g(f(x))$.

Problem 16 Given $[Q \wedge S] \Rightarrow R$ and $\neg S \Rightarrow T$, prove $[P \Rightarrow Q] \Rightarrow [\neg T \Rightarrow [\neg P \vee R]]$

<i>Solution.</i>	1.	$[Q \wedge S] \Rightarrow R$	Hypothesis
	2.	$\neg S \Rightarrow T$	Hypothesis
	3.	Assume $P \Rightarrow Q$.	Dischargeable Hypothesis
	4.	Assume $\neg T$.	Dischargeable Hypothesis
	5.	Assume $P \wedge \neg R$.	Dischargeable Hypothesis
	6.	$\neg R$.	RCS, for 5
	7.	P	LCS, for 5
	8.	$\neg[Q \wedge S]$.	MT, for 6, for 1
	9.	$\neg[Q \wedge S] \Leftrightarrow \neg Q \vee \neg S$.	Tautology
	10.	$\neg Q \vee \neg S$.	MPB, for 8, for 9
	11.	Q .	MP, for 7, for 3
	12.	$\neg S$.	DI, for 10, for 11
	13.	T .	MP, for 12, for 2
	14.	$\neg T \wedge T$	CI, for 4, for 13
	15.	$\neg[P \wedge \neg R]$.	II, discharge for 5 [5 - 14 unusable]
	16.	$\neg[P \wedge \neg R] \Leftrightarrow \neg P \vee R$.	Tautology
	17.	$\neg P \vee R$.	MPB, for 15, for 16
	15.	$\neg T \Rightarrow [\neg P \vee R]$.	DT, discharge for 4 [4 - 17 unusable]
	16.	$[P \Rightarrow Q] \Rightarrow [\neg T \Rightarrow [\neg P \vee R]]$.	DT, discharge for 3 [3 - 15 unusable]