SOLUTIONS TO HOMEWORK 7

Warning: Little proofreading has been done.

1. Section 3.2

Exercise 3.2.4 Let A be bounded above, so that $s = \sup(A)$ exists.

(a) Show that $s \in \overline{A}$.
(b) Can an open set contain its supremum?

Solution. (a) If $s \in A$ then obviously $s \in \overline{A}$. So assume $s \notin A$; we prove that s is a limit point of A.

Let $\varepsilon > 0$. Then $s - \varepsilon$ is not an upper bound for A. Therefore there is $x \in A$ such that $x > s - \varepsilon$. We have $x \leq s$ because s is an upper bound for A, and $x \neq s$ because $s \notin A$. So

$$x \in (s - \varepsilon,\, s) \cap A \subseteq (s - \varepsilon,\, s + \varepsilon) \cap (A \setminus \{s\}).$$

So this last set is not empty. This proves that s is a limit point of A.

(b) No. If $s = \sup(A)$ belongs to A, then s mush have a ϵ-neighborhood that is in A. However, since s is an upper bound, this means that $s + \epsilon/2$ will be a point in A, which contradicts the fact that s is the least upper bound. □

Remark. It is *not* true in general that $\sup(A)$ is a limit point of A. Take $A = \{0\}$. Then $\sup(A) = 0$ but A has *no* limit points. □

Exercise 3.2.5 Prove Theorem 3.2.8 in the book: A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Solution. Suppose F is closed, and let (x_n) be a Cauchy sequence in F. Then (x_n) converges (by completeness). Set $x = \lim x_n$. We show that $x \in F$. If for all $n \in \mathbb{N}$ we have $x_n \neq x$, then Theorem 3.2.5 of the book shows that x is a limit point of F. Since F is closed, this implies $x \in F$. Otherwise, there is n such that $x_n = x$. Then $x \in F$ since $x_n \in F$.

Conversely, suppose every Cauchy sequence in F has a limit in F. Since every convergent sequence is Cauchy, it follows that every convergent sequence in F has a limit in F. Let x be a limit point of F; we must show $x \in F$. Theorem 3.2.5 of the book provides a sequence (x_n) in $F \setminus \{x\}$ such that $\lim x_n = x$. The hypothesis now implies $x \in F$.

□

Exercise 3.2.7 Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

Solution. (a) Let x be a limit point of L. Let $\varepsilon > 0$. The ε-neighborhood $V_\varepsilon(x)$ contains a number $y \neq x$ and $y \in L$. Then y is a limit point of A. Choose $\delta > 0$ small enough such that $V_\delta(y)$ is a subset of $V_\varepsilon(x)$ and it does not contain x (more precisely, choose δ such that $0 < \delta < \min\{\varepsilon - |y - x|, |y - x|\}$). Then $V_\delta(y)$ contains a number $z \neq y$ and $z \in A$ since y is a limit point of A. In particular, this shows that $V_\varepsilon(x)$ contains $z \in A$ and $z \neq x$, which proves that x is a limit point of A.

□

Exercise 3.2.11

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(b) Does the result about closures in part (a) extend to infinite unions of sets?

Solution to (a). We first show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. It follows from Theorem 3.2.12 that $\overline{A}$ is the smallest closed set which contains A, and also that $\overline{A \cup B}$ is a closed set which contains $A \cup B$. So $\overline{A \cup B}$ is a closed set which contains A, and therefore $\overline{A \cup B}$ contains $\overline{A}$.

The proof that $\overline{B} \subseteq \overline{A \cup B}$ is essentially the same.

We conclude that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

We now show the reverse inclusion. Let L be the set of limit points of $A \cup B$. Then $\overline{A \cup B} = A \cup B \cup L$. If $x \in A$ or $x \in B$, then obviously $x \in A \cup B \subseteq \overline{A} \cup \overline{B}$. If $x \in L$, then part (a) shows that x is a limit point of A or of B. Thus $x \in \overline{A}$ or $x \in \overline{B}$. In either case, $x \in \overline{A} \cup \overline{B}$.

(b) No, the result does not extend. Let $X \subseteq \mathbb{R}$ be any set which is not closed. (For example, take $X = \mathbb{Q}$.) The sets $\{x\}$, for $x \in \mathbb{R}$, are all closed. (Proof: For trivial reasons, a one point set has no limit points.) We clearly have

$$X = \bigcup_{x \in X} \{x\}.$$

However,

$$\bigcup_{x \in X} \overline{\{x\}} = \bigcup_{x \in X} \{x\} = X,$$

which, because X is not closed, is a proper subset of $\overline{X}$. ☐

2. Section 3.3

Exercise 3.3.1 Show that if $K \subseteq \mathbb{R}$ is compact, then $\sup(K)$ and $\inf(K)$ both exist and are elements of K.

Solution. The set K is bounded by Theorem 3.3.4. Therefore $\sup(K)$ and $\inf(K)$ both exist. Exercise 3.2.4 shows that $\sup(K) \in \overline{K}$. An almost identical argument shows that $\inf(K) \in \overline{K}$. Since K is closed by Theorem 3.3.4, this gives $\sup(K) \in K$ and $\inf(K) \in K$. ☐

Exercise 3.3.2abe Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbb{Z}$.
(b) $\mathbb{Q} \cap [0, 1]$.
(e) $\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$.

Solution to (a). This set is not compact since it is unbounded. Note that the set is closed since it has no limit points. ☐

Solution to (b). This set is not compact.

Set $x = \frac{1}{2}\sqrt{2}$. Then $x \in [0, 1]$ but $x \notin \mathbb{Q}$. Using the density of $\mathbb{Q}$ in $\mathbb{R}$, for each $n \in \mathbb{N}$ choose $x_n \in \mathbb{Q}$ such that

$$\tfrac{1}{2}\sqrt{2} - \tfrac{1}{n+1} < x_n < \tfrac{1}{2}\sqrt{2}.$$

Then $x_n \in \mathbb{Q} \cap [0, 1]$ for all $n \in \mathbb{N}$. (Note that $\frac{1}{2}\sqrt{2} - \frac{1}{2} > 0$.) One easily checks that $\lim_{n\to\infty} x_n = \frac{1}{2}\sqrt{2}$, so also every subsequence converges to $\frac{1}{2}\sqrt{2}$. In particular, there is no subsequence converging to any point of $\mathbb{Q} \cap [0, 1]$. ☐

Solution to (e). To simplify notation, define

$$A = \left\{1, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \dots\right\} = \left\{\tfrac{n}{n+1} : n \in \mathbb{N}\right\} \cup \{1\}.$$

We show that A is compact. Clearly A is bounded, so it is enough to show that A is closed. The set A has one limit point 1, since $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Since every subsequence of a convergence sequence converges to the same limit, 1 is the only limit point of A. Since A contains 1 by definition, A is closed. ☐

Remark. It is also easy to show that A is compact using open covers. Here is an outline. Let $\mathcal{U}$ be an open cover of A. Then there is $U \in \mathcal{U}$ such that $1 \in U$. Since there is $\varepsilon > 0$ such that $(1 - \varepsilon,\ 1 + \varepsilon) \subseteq U$, it follows that U contains all but finitely many of the points in A. One needs only finitely many more elements of $\mathcal{U}$ to cover the rest of A. ☐

Exercise 3.3.3 Prove the converse of the part of Theorem 3.3.4 proved in the text, by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Solution. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K. Then $(x_n)_{n\in\mathbb{N}}$ is bounded, so by the Bolzano-Weierstrass Theorem there is a convergent subsequence $\left(x_{n_k}\right)_{k\in\mathbb{N}}$. Since K is closed, $\lim_{k\to\infty} x_{n_k} \in K$. $\square$