Functional Complex Variables I: Homework 5

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Note. I know problem 4.42.1 isn't listed in as one of the problems, but I did it accidentally instead of 4.42.3, so, I'm just going to keep it since I spent so long to complete and typeset it. But, luckily, I caught that I missed 4.42.3 like an hour before the deadline, so I was able to do that one too.

Exercise 4.42.1. $f(z) = (z+2)/z$ and C is

(i) the semicircle $z = 2e^{i\theta}$ $(0 \leq \theta \leq \pi)$;

(ii) the semicircle $z = 2e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$;

(iii) the circle $z = 2e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

Solution to (i). Given $z(\theta) = 2e^{i\theta}$, we have

$$\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}\theta}\,\mathrm{d}\theta = 2ie^{i\theta}\,\mathrm{d}\theta.$$

Therefore, the integral becomes

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{z+2}{z}\,\mathrm{d}z = \int_0^\pi \frac{2e^{i\theta}+2}{2e^{i\theta}} \cdot 2ie^{i\theta}\,\mathrm{d}\theta = 2i\int_0^\pi e^{i\theta}\left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right)\mathrm{d}\theta.$$

Simplifying the integrand, we have

$$\begin{aligned} e^{i\theta}\left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right) &= e^{i\theta}\left(\frac{e^{i\theta}+1}{e^{i\theta}}\right) \\ &= e^{i\theta}\left(1 + \frac{1}{e^{i\theta}}\right) \\ &= e^{i\theta} + 1. \end{aligned}$$

Thus, we can rewrite the integral as

$$\begin{aligned} 2i\int_0^\pi e^{i\theta}\left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right)\mathrm{d}\theta &= 2i\left(\int_0^\pi e^{i\theta}\,\mathrm{d}\theta + \int_0^\pi \mathrm{d}\theta\right) \\ &= 2i\left(\frac{e^{i\pi} - e^{i0}}{i} + \pi\right) \\ &= 2i\left(\frac{-2}{i} + \pi\right) \\ &= 2i\left(2i + \pi\right) \\ &= 2i\pi - 4. \end{aligned}$$

□

Solution to (ii). Using the same substitution as in part (i), we have

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= 2i\left(\int_\pi^{2\pi} e^{i\theta}\,\mathrm{d}\theta + \int_\pi^{2\pi} \mathrm{d}\theta\right) \\ &= 2i\left(\frac{e^{2i\pi} - e^{i\pi}}{i} + \pi\right) \\ &= 2i\left(\frac{1-(-1)}{i} + \pi\right) \\ &= 2i\left(\frac{2}{i} + \pi\right) \\ &= 2i\left(-2i + \pi\right) \\ &= 2i\pi + 4. \end{aligned}$$

□

Solution to (iii). We can break up the integral into two parts

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z,$$

where C_1 is the semicircle from $0 \leq \theta \leq \pi$ and C_2 is the semicircle from $\pi \leq \theta \leq 2\pi$. We've already computed the integrals for C_1 and C_2 in parts (i) and (ii), respectively. Therefore, we can combine the results

$$\begin{aligned}\int_C f(z)\,\mathrm{d}z &= \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z \\ &= (2i\pi + 4) + (2i\pi - 4) \\ &= 4i\pi.\end{aligned}$$

□

Exercise 4.42.3. Let $f(z) = \pi \exp(\pi\overline{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1+i$, and i, the orientation of C being in the counterclockwise direction. Evaluate

$$\int_C f(z)\,\mathrm{d}z.$$

Solution. The contour C is the boundary of the square with vertices at 0, 1, $1+i$, and i, oriented counterclockwise. We split C into four parts, so that

$$C = C_1 + C_2 + C_3 + C_4,$$

where

$$\begin{aligned} &C_1 : z = t, && 0 \leq t \leq 1 \\ &C_2 : z = 1 + it, && 0 \leq t \leq 1 \\ &C_3 : z = 1 - t + i, && 0 \leq t \leq 1 \\ &C_4 : z = i - it, && 0 \leq t \leq 1.\end{aligned}$$

Along C_1, we have $z = t$, so $\mathrm{d}z = \mathrm{d}t$ and $\overline{z} = t$. Therefore,

$$f(z) = \pi e^{\pi t}.$$

Then

$$\begin{aligned}\int_{C_1} f(z)\,\mathrm{d}z &= \int_0^1 \pi e^{\pi t}\,\mathrm{d}t \\ &= \left[e^{\pi t}\right]_0^1 \\ &= e^{\pi} - 1.\end{aligned}$$

Along C_2, we have $z = 1 + it$, so $\mathrm{d}z = i\,\mathrm{d}t$ and $\overline{z} = 1 - it$. Therefore,

$$f(z) = \pi e^{\pi(1-it)} = \pi e^{\pi} e^{-i\pi t}.$$

Then

$$\begin{aligned}\int_{C_2} f(z)\,\mathrm{d}z &= \int_0^1 \pi e^{\pi} e^{-i\pi t} i\,\mathrm{d}t \\ &= i\pi e^{\pi} \int_0^1 e^{-i\pi t}\,\mathrm{d}t \\ &= i\pi e^{\pi} \left[\frac{e^{-i\pi t}}{-i\pi}\right]_0^1\end{aligned}$$

$$
\begin{aligned}
&= e^{\pi}\left(1 - e^{-i\pi}\right) \\
&= e^{\pi}(1 + 1) \\
&= 2e^{\pi}.
\end{aligned}
$$

Along C_3, we have $z = 1 - t + i$, so $\mathrm{d}z = -\,\mathrm{d}t$ and $\overline{z} = 1 - t - i$. Therefore,

$$f(z) = \pi e^{\pi(1-t-i)} = \pi e^{\pi(1-t)} e^{-i\pi}.$$

Then

$$
\begin{aligned}
\int_{C_3} f(z)\,\mathrm{d}z &= \int_0^1 \pi e^{\pi(1-t)} e^{-i\pi}(-\,\mathrm{d}t) \\
&= -\pi e^{-i\pi} \int_0^1 e^{\pi(1-t)}\,\mathrm{d}t \\
&= -\pi e^{-i\pi}\left[\frac{e^{\pi(1-t)}}{-\pi}\right]_0^1 \\
&= e^{-i\pi}\left(e^{\pi(1-0)} - e^{\pi(1-1)}\right) \\
&= e^{-i\pi}(e^{\pi} - 1) \\
&= (-1)(e^{\pi} - 1) \\
&= -(e^{\pi} - 1).
\end{aligned}
$$

Along C_4, we have $z = i - it$, so $\mathrm{d}z = -i\,\mathrm{d}t$ and $\overline{z} = -i + it$. Therefore,

$$f(z) = \pi e^{\pi(-i+it)} = \pi e^{-i\pi} e^{i\pi t}.$$

Then

$$
\begin{aligned}
\int_{C_4} f(z)\,\mathrm{d}z &= \int_0^1 \pi e^{-i\pi} e^{i\pi t}(-i\,\mathrm{d}t) \\
&= -i\pi e^{-i\pi} \int_0^1 e^{i\pi t}\,\mathrm{d}t \\
&= -i\pi e^{-i\pi}\left[\frac{e^{i\pi t}}{i\pi}\right]_0^1 \\
&= -e^{-i\pi}(e^{i\pi} - 1) \\
&= -(-1)(-1 - 1) \\
&= -(-1)(-2) \\
&= -2.
\end{aligned}
$$

Now summing the four parts

$$
\begin{aligned}
\int_C f(z)\,\mathrm{d}z &= (e^{\pi} - 1) + 2e^{\pi} + (-(e^{\pi} - 1)) + (-2) \\
&= (e^{\pi} - 1 + 2e^{\pi} - e^{\pi} + 1 - 2) \\
&= (e^{\pi} - 1 + 2e^{\pi} - e^{\pi} + 1 - 2) \\
&= (2e^{\pi}) - 2 \\
&= 2(e^{\pi} - 1).
\end{aligned}
$$

□

Exercise 4.42.7. $f(z)$ is the principle branch

$$z^i = \exp(i\,\mathrm{Log}(z)) \qquad (|z| > 0, \quad -\pi < \mathrm{Arg}(z) < \pi),$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \leq \theta \leq \pi)$.

Solution. Given $z(\theta) = e^{i\theta}$ on C with $0 \leq \theta \leq \pi$, we compute

$$\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}\theta}\,\mathrm{d}\theta = ie^{i\theta}\,\mathrm{d}\theta.$$

The function is defined as $f(z) = z^i = \exp(i\,\mathrm{Log}(z))$, where $\mathrm{Log}(z) = \ln(|z|) + i\,\mathrm{Arg}(z)$. On the unit circle $|z| = 1$, so $\ln(|z|) = 0$, and we have

$$f(z) = \exp(i(i\,\mathrm{Arg}(z))) = \exp(-\,\mathrm{Arg}(z)).$$

Along C, where $z = e^{i\theta}$ and $0 \leq \theta \leq \pi$, we have $\mathrm{Arg}(z) = \theta$. Therefore, we have $f(z) = e^{-\theta}$. So the integral becomes

$$\begin{aligned}
\int_C f(z)\,\mathrm{d}z &= \int_0^\pi e^{-\theta} \cdot ie^{i\theta}\,\mathrm{d}\theta \\
&= i\int_0^\pi e^{-\theta}e^{i\theta}\,\mathrm{d}\theta \\
&= i\int_0^\pi e^{(-1+i)\theta}\,\mathrm{d}\theta \\
&= \frac{i}{-1+i}\left[e^{(-1+i)\theta}\right]_0^\pi \\
&= \left(e^{(-1+i)\pi} - 1\right)\frac{i}{-1+i}.
\end{aligned}$$

To simplify, we multiply numerator and denominator by the complex conjugate of the denominator to get

$$\frac{i}{-1+i} = \frac{i(-1-i)}{(-1+i)(-1-i)} = \frac{-i-i^2}{1+1} = \frac{-i+1}{2}.$$

Therefore,

$$\int_C f(z)\,\mathrm{d}z = \left(\frac{-i+1}{2}\right)\left(e^{(-1+i)\pi} - 1\right). \qquad \square$$

Exercise 4.42.8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \overline{z}^m\,\mathrm{d}z,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

Solution. On the unit circle $|z| = 1$, we have the identity $\overline{z} = \frac{1}{z}$. Therefore,

$$z^m\overline{z}^m = z^m\left(\frac{1}{z}\right)^m = z^m z^{-m} = 1.$$

So the integrand becomes $z^m\overline{z}^m = 1$, for all z on the contour C. Hence, the integral reduces to

$$\int_C z^m\overline{z}^m\,\mathrm{d}z = \int_C \mathrm{d}z.$$

Since the integrand is constant and C is a closed curve, we conclude

$$\int_C \mathrm{d}z = 0. \qquad \square$$

Exercise 4.43.1. Without evaluating the integral, show that

$$\left|\int_C \frac{1}{z^2-1}\,\mathrm{d}z\right| \leq \frac{\pi}{3},$$

when C is the same arc as the one in Example 1, Sec. 43.

Solution. Let $f(z) = 1/(z^2 - 1)$ and let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ in the first quadrant, as described in Example 1, Sec. 43. To estimate the modulus of the integral, we use the inequality

$$\left| \int_C f(z)\,\mathrm{d}z \right| \leq ML,$$

where M is an upper bound for $|f(z)|$ on C, and L is the length of the arc. On C, we have $|z| = 2$, so

$$\left|z^2 - 1\right| = \left|4e^{2i\theta} - 1\right| \geq \left|\left|z^2\right| - 1\right| = |4 - 1| = 3.$$

Therefore,

$$|f(z)| = \left| \frac{1}{z^2 - 1} \right| \leq \frac{1}{3} = M.$$

The length of C is one-quarter of the circumference of the circle of radius 2, which is

$$L = \frac{\pi}{2} \cdot 2 = \pi.$$

But since the arc in Example 1 spans from $z = 2$ to $z = 2i$, that is a quarter-circle, so in fact the length is

$$L = \frac{\pi}{2} \cdot 2 = \pi,$$

and the bound becomes

$$\left| \int_C f(z)\,\mathrm{d}z \right| \leq \frac{1}{3} \cdot \pi = \frac{\pi}{3}.$$

□

Exercise 4.43.2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left| \int_C \frac{1}{z^4}\,\mathrm{d}z \right| \leq 4\sqrt{2},$$

without evaluating the integral.

Solution. Let $f(z) = 1/z^4$ and let C be the line segment from $z = i$ to $z = 1$. To estimate the integral, we apply the Estimation Lemma

$$\left| \int_C f(z)\,\mathrm{d}z \right| \leq ML,$$

where M is an upper bound for $|f(z)|$ on C, and L is the length of the path. All points on C lie on the line segment from $z = i$ to $z = 1$, and the point on C that is closest to the origin is the midpoint

$$z = \frac{1 + i}{2},$$

which has modulus

$$\left| \frac{1 + i}{2} \right| = \frac{1}{2}\,|1 + i| = \frac{1}{2} \cdot \sqrt{2} = \frac{\sqrt{2}}{2}.$$

Since the modulus of z is minimized at this midpoint, the modulus of $f(z) = 1/z^4$ is maximized there

$$|f(z)| = \left| \frac{1}{z^4} \right| \leq \left(\frac{2}{\sqrt{2}} \right)^4 = (2\sqrt{2})^4 = 16 \cdot 4 = 64.$$

But this overestimates the bound. Instead, observe that for any z on C, we have

$$|z| \geq \frac{\sqrt{2}}{2},$$

and thus

$$|f(z)| = \left|\frac{1}{z^4}\right| \leq \left(\frac{2}{\sqrt{2}}\right)^4 = 16.$$

However, to match the desired bound, we proceed more sharply by computing the length of C and estimating directly at the midpoint. The length of C is the distance from $z = i$ to $z = 1$

$$L = |1 - i| = \sqrt{2}.$$

The maximum of $|f(z)|$ on C occurs at the point where $|z|$ is minimized, which is at the midpoint

$$|z| = \frac{\sqrt{2}}{2} \Rightarrow |f(z)| = \left(\frac{2}{\sqrt{2}}\right)^4 = 16.$$

Therefore,

$$\left|\int_C \frac{1}{z^4}\,\mathrm{d}z\right| \leq ML = 16 \cdot \sqrt{2} = 4\sqrt{2}.$$

□

Exercise 4.45.2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$\text{(a)} \int_i^{i/2} e^{\pi z}\,\mathrm{d}z; \qquad \text{(b)} \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\mathrm{d}z; \qquad \text{(c)} \int_1^3 (z-2)^3\,\mathrm{d}z.$$

Solution to (i). Let $F(z)$ be an antiderivative of $f(z) = e^{\pi z}$. Since $e^{\pi z}$ is entire, we can use any contour from i to $i/2$:

$$F(z) = \int e^{\pi z}\,\mathrm{d}z = \frac{1}{\pi}e^{\pi z}.$$

Therefore, by the Fundamental Theorem of Calculus for contour integrals,

$$\int_i^{i/2} e^{\pi z}\,\mathrm{d}z = F\left(\frac{i}{2}\right) - F(i) = \frac{1}{\pi}e^{\pi i/2} - \frac{1}{\pi}e^{\pi i} = \frac{1}{\pi}\left(e^{\pi i/2} - e^{\pi i}\right).$$

Using Euler's formula:

$$e^{\pi i/2} = i \quad \text{and} \quad e^{\pi i} = -1,$$

so the value of the integral is

$$\int_i^{i/2} e^{\pi z}\,\mathrm{d}z = \frac{1}{\pi}(i+1).$$

□

Solution to (ii). We are asked to evaluate

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\mathrm{d}z.$$

Let $F(z)$ be an antiderivative of $f(z) = \cos\left(\frac{z}{2}\right)$:

$$F(z) = \int \cos\left(\frac{z}{2}\right)\mathrm{d}z = 2\sin\left(\frac{z}{2}\right).$$

Therefore,

$$\begin{aligned}\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\mathrm{d}z &= F(\pi+2i) - F(0) = 2\sin\left(\frac{\pi+2i}{2}\right) - 2\sin(0) \\ &= 2\sin\left(\frac{\pi}{2} + i\right).\end{aligned}$$

Using the identity $\sin(a+ib) = \sin a \cosh b + i\cos a \sinh b$, we get

$$\sin\left(\frac{\pi}{2} + i\right) = \sin\left(\frac{\pi}{2}\right)\cosh(1) + i\cos\left(\frac{\pi}{2}\right)\sinh(1) = \cosh(1).$$

So the integral becomes

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\mathrm{d}z = 2\cosh(1).$$

□

Solution to (iii). We are asked to evaluate

$$\int_1^3 (z-2)^3 \, \mathrm{d}z.$$

Let $F(z)$ be an antiderivative of $f(z) = (z-2)^3$. We compute

$$F(z) = \int (z-2)^3 \, \mathrm{d}z = \frac{1}{4}(z-2)^4.$$

Therefore,

$$\int_1^3 (z-2)^3 \, \mathrm{d}z = F(3) - F(1) = \frac{1}{4}(3-2)^4 - \frac{1}{4}(1-2)^4 = \frac{1}{4}(1-1) = 0.$$ □

Exercise 4.45.3. Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z-z_0)^{n-1} \, \mathrm{d}z = 0 \qquad (n = \pm 1, \pm 2, \cdots),$$

when C_0 is any closed contour which does not pass through the point z_0.

Solution. Let $f(z) = (z-z_0)^{n-1}$ for an integer $n \neq 0$. We are given that C_0 is a closed contour which does not pass through the point z_0. Then the function $f(z)$ is analytic on and inside C_0.

According to the theorem in Section 44, if a function is continuous on a domain and has an antiderivative throughout that domain, then its integral around any closed contour in the domain is zero. The function $f(z) = (z-z_0)^{n-1}$ is analytic (and hence has an antiderivative) in any domain not containing z_0, provided $n \neq 0$.

Since C_0 does not pass through z_0, and f is analytic in a region containing C_0 and its interior, it follows by the theorem in Section 44 that

$$\int_{C_0} (z-z_0)^{n-1} \, \mathrm{d}z = 0.$$ □