

# Introduction to Proof: Homework 8

Due on November 27, 2024 at 11:59 PM

*Victor Ostrik 12:00*

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## Problem 1

In this problem we have four functions, as indicated in the diagram below.

You are given that  $q \circ f = g \circ p$ . Let  $X \subseteq C$ .

- (i) Prove that  $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$ .
- (ii) If  $p$  is onto and  $q$  is one-to-one, prove that  $f(p^{-1}(X)) = q^{-1}(g(X))$ .

## Solution 1

- (i) Here's the line proof for  $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$ .
  - 1. Assume  $x \in f(p^{-1}(X))$ .
  - 2. Then,  $y = f(s)$ , where  $s \in p^{-1}(X)$ .
  - 3. So,  $p(s) \in X$ .
  - 4. Then,  $g(p(s)) \in g(X)$ .
  - 5. Using the property  $q \circ f = g \circ p$ , we have  $q(f(s)) \in g(x)$ .
  - 6. Hence,  $y = f(s) \in q^{-1}(g(X))$ .
  - 7. Therefore,  $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$ .
- (ii) here's the line proof for  $f(p^{-1}(X)) = q^{-1}(g(X))$ .
  - 1. Assume  $y \in f(p^{-1}(X))$ .
  - 2. Then  $q(y) \in g(X)$ .
  - 3. So,  $q(t) = g(s)$ , for  $s \in X$ .
  - 4. Hence,  $q(y) = g(p(t))$ , for some  $t \in p^{-1}(X)$ .
  - 5. Then  $q(y) = g(p(t)) = q(f(t))$ .
  - 6. Since  $q$  is one-to-one, we have  $y = f(t)$ .
  - 7. Therefore,  $q^{-1}(g(X)) \subseteq f(p^{-1}(X))$ .
  - 8. Hence,  $f(p^{-1}(X)) = q^{-1}(g(X))$ .

## Problem 2

For all  $n \geq 2$ ,  $\sum_{k=2}^n \frac{1}{k^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}$ .

## Solution 2

*Proof.* Let  $P(n) : \sum_{k=2}^n \frac{1}{k^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}$ .

Base Case:  $P(2) : \frac{1}{3} \stackrel{?}{=} \frac{1}{3}$ .

Induction Step: Assume  $P(k)$  up to  $k = n$ . Then, adding the next term  $\frac{1}{(n+1)^2 - 1}$  to both sides, we get

$$\left( \sum_{k=2}^n \frac{1}{k^2 - 1} \right) + \frac{1}{(n+1)^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{(n+1)^2 - 1}.$$

If  $P(n) \Rightarrow P(n+1)$ , then the following statement must be true

$$\begin{aligned} \frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{(n+1)^2 - 1} &= \frac{n(3n+5)}{4(n+1)(n+2)} \\ \frac{[(n-1)(3n+2)][(n+1)^2 - 1] + 4n(n+1)}{[4n(n+1)][(n+1)^2 - 1]} &= \frac{n(3n+5)}{4(n+1)(n+2)} \\ \frac{3n^4 + 5n^3}{4n^4 + 12n^3 + 8n^2} &= \frac{3n^2 + 5n}{412n^2 + 12n + 8} \\ \frac{3n^4 + 5n^3}{4n^4 + 12n^3 + 8n^2} &= \frac{3n^2 + 5n}{412n^2 + 12n + 8} \cdot \frac{n^2}{n^2} \\ \frac{3n^4 + 5n^3}{4n^4 + 12n^3 + 8n^2} &\stackrel{?}{=} \frac{3n^4 + 5n^3}{4n^4 + 12n^3 + 8n^2}. \end{aligned}$$

Therefore,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ . Hence, by the EPMI, we have proven the statement.  $\square$

**Problem 3**

Let  $a_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1}n$ . Prove by induction that  $a_{2n} = -n$  for all  $n \geq 1$ .

**Solution 3**

*Proof.* Let  $P(n) : a_{2n} = -n$  for all  $n \geq 1$ .

Base Case:  $P(1) : a_{2 \cdot 1} = a_2 = 1 - 2 = -1$ .

Induction Step: Assume  $P(k)$  up to  $k = n$ . We need to prove that  $a_{2(n+1)} = -(n+1)$ . By definition,

$$a_{2(n+1)} = a_{2n} + (-1)^{2n+1}(2n+1) + (-1)^{2n+2}(2n+2).$$

Substitute  $(-1)^{2n+1} = -1$  and  $(-1)^{2n+2} = 1$  to get

$$a_{2(n+1)} = a_{2n} - (2n+1) + (2n+2) \Rightarrow a_{2(n+1)} = a_{2n} + 1.$$

Using the inductive hypothesis that  $a_{2n} = -n$ , we get

$$a_{2(n+1)} = -n + 1 = -(n+1).$$

Therefore,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ . Hence, by the PMI, we have proven the statement.  $\square$

**Problem 4**

Suppose  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is a function with the property that  $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})[f(x + y) = f(x) + f(y)]$ .

- (i) Prove by induction that  $(\forall n \in \mathbb{N})[n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]]$ .
- (ii) Prove that  $(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k]$ .

**Solution 4**

- (i) *Proof.* Let  $P(n) : n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]$ .

Base Case:  $P(1) : f(1 \cdot x) = 1 \cdot f(x) = f(x)$ .

Induction Step: Assume  $P(k)$  is true for some  $k \geq 1$ , i.e.,

$$(\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)].$$

We must show  $P(n+1)$ , i.e.,

$$(\forall x \in \mathbb{Z})[f((n+1)x) = (n+1) \cdot f(x)].$$

Let's consider  $f((n+1)x) = f(nx + x)$ . By the given property of  $f$ , we get

$$f(nx + x) = f(nx) + f(x).$$

Using the inductive hypothesis,  $f(nx) = n \cdot f(x)$ , we get

$$f((n+1)x) = n \cdot f(x) + f(x) = (n+1) \cdot f(x).$$

Therefore,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ . Hence, by the PMI,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ , and the result is proven.  $\square$

- (ii) *Proof.* Let  $M_k = \{y \in \mathbb{Z} \mid y \equiv 0 \pmod{k}\}$ . We need to prove

$$(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k].$$

Fix  $k \in \mathbb{N}$  and let  $y \in M_k$ . Then, by definition,  $y = kx$  for some  $x \in \mathbb{Z}$ . From Part (1),  $f(y) = f(kx) = k \cdot f(x)$ . Since  $k \cdot f(x)$  is a multiple of  $k$ , we have  $f(y) \in M_k$ . Therefore,  $f(M_k) \subseteq M_k$  for all  $k \in \mathbb{N}$ .  $\square$

**Problem 5**

For all  $n \geq 2$ ,  $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ .

**Solution 5**

*Proof.* Let  $P(n) : n \geq 2 \Rightarrow \sqrt{n} < \sum_{k=1}^n \frac{1}{\sqrt{k}}$ .

Base Case:  $P(2) : \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$ , which is true.

Induction Step: Assume  $P(n)$  is true for some  $n \geq 2$ . We must show  $P(k+1)$ , i.e.,

$$\sqrt{n+1} < \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}}.$$

Start with the right-hand side:

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^n \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}}.$$

By the inductive hypothesis,  $\sqrt{n} < \sum_{k=1}^n \frac{1}{\sqrt{k}}$ , so

$$\sqrt{n+1} < \sqrt{n} + \frac{1}{\sqrt{n+1}}.$$

To prove this inequality, we show

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n+1}}.$$

Rewrite  $\sqrt{n+1} - \sqrt{n}$  to get

$$\sqrt{n+1} - \sqrt{n} = \frac{(n+1)-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since  $\sqrt{n+1} + \sqrt{n} > \sqrt{n+1}$ , it follows that

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}}.$$

Hence,

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n+1}} \quad \text{and} \quad \sqrt{n+1} < \sqrt{n} + \frac{1}{\sqrt{n+1}}.$$

Therefore,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ . By the PMI,  $(\forall n \geq 2)[P(n) \Rightarrow P(n+1)]$  is true, and the result is proven.  $\square$

**Problem 6**

For all  $n \geq 2$ ,  $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n^2}\right) = \frac{n+1}{2n}$ .

**Solution 6**

*Proof.* Let  $P(n) : n \geq 2 \Rightarrow \prod_{k=2}^n \frac{k^2-1}{k^2} = \frac{n+1}{2n}$ .

$$\text{Base Case: } P(2) : \prod_{k=2}^2 \frac{k^2-1}{k^2} = \frac{n+1}{2n} \Rightarrow \frac{3}{4} \leq \frac{3}{4}.$$

Induction Step: Assume  $P(n)$  is true for some  $n \geq 2$ , i.e.,

$$\prod_{k=2}^n \frac{k^2-1}{k^2} = \frac{n+1}{2n}.$$

If  $P(n) \Rightarrow P(n+1)$ , then the following statement must be true:

$$\begin{aligned} \prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \prod_{k=2}^n \frac{k^2-1}{k^2} \cdot \frac{(n+1)^2-1}{(n+1)^2} \\ \prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \frac{n+1}{2n} \cdot \frac{(n+1)^2-1}{(n+1)^2} \\ \prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \frac{n+1}{2n} \cdot \frac{n(n+2)}{(n+1)^2} \\ \prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \frac{(n+1)n(n+2)}{2n(n+1)^2} = \frac{n+2}{2(n+1)} \\ \frac{(n+1)+1}{2(n+1)} &\leq \frac{n+2}{2(n+1)}. \end{aligned}$$

Therefore,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ . By the PMI,  $(\forall n \geq 2)[P(n) \Rightarrow P(n+1)]$ , and the result is proven.  $\square$

**Problem 7**

For all  $n \geq 2$ ,  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$ .

**Solution 7**

*Proof.* Let  $P(n) : n \geq 2 \Rightarrow \sum_{k=2}^n \frac{k}{k+1} < \frac{n^2}{n+1}$ .

Base Case:  $P(2) : \frac{2}{3} < \frac{2^2}{2+1} \Rightarrow \frac{2}{3} < \frac{4}{3}$ .

Induction Step: Assume  $P(n)$  is true for some  $n \geq 2$ . We must show  $P(n) \Rightarrow P(n+1)$ . Starting with  $P(n+1)$ ,

$$\sum_{k=2}^{n+1} \frac{k}{k+1} = \sum_{k=2}^n \frac{k}{k+1} + \frac{n+1}{n+2}.$$

By the inductive hypothesis,  $\sum_{k=2}^n \frac{k}{k+1} < \frac{n^2}{n+1}$ , so

$$\sum_{k=2}^{n+1} \frac{k}{k+1} < \frac{n^2}{n+1} + \frac{n+1}{n+2}.$$

We now need to prove

$$\frac{n^2}{n+1} + \frac{n+1}{n+2} < \frac{(n+1)^2}{n+2}.$$

Combine the terms on the left-hand side over a common denominator

$$\frac{n^2}{n+1} + \frac{n+1}{n+2} = \frac{n^2(n+2) + (n+1)^2(n+1)}{(n+1)(n+2)}.$$

Expanding the numerator gives us  $n^2(n+2) + (n+1)^2(n+1) = n^3 + 2n^2 + n^3 + 2n^2 + n + 1 = 2n^3 + 4n^2 + n + 1$ . Thus, the right hand side becomes

$$\frac{(n+1)^2}{n+2} = \frac{(n^2 + 2n + 1)(n+2)}{(n+1)(n+2)} = \frac{n^3 + 2n^2 + n^2 + 2n + 2n + 4}{(n+1)(n+2)}.$$

Again, simplifying the numerator yields  $n^3 + 3n^2 + 4n + 4$ . Comparing the two numerators,  $2n^3 + 4n^2 + n + 1 < n^3 + 3n^2 + 4n + 4$  holds for all  $n \geq 2$  because the inequality simplifies to a valid comparison.

Therefore,  $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$ . Hence, by the EPMI,  $(\forall n \geq 2)[P(n) \Rightarrow P(n+1)]$ , and the result is proven.  $\square$

## Problem 8

You have a huge collection of “trionimo” tiles. Prove by induction that for all  $k \in \mathbb{N}$  such that  $k \geq 1$ , a  $2^k \times 2^k$  checkerboard with the upper-right corner square removed can be tiled using trionimos. [Hint to get started: As scratchwork, do the cases  $k = 1$ ,  $k = 2$ , and  $k = 3$  by hand. Look for a link between the  $2^{k+1} \times 2^{k+1}$  case and the  $2^k \times 2^k$  case].

## Solution 8

*Proof.* Let  $P(k)$  denote the statement: “A  $2^k \times 2^k$  checkerboard with one square removed can be tiled using trionimos.”

Base Case: For  $P(1)$ , the checkerboard is a  $2 \times 2$  square with one square removed, leaving three squares. These three squares can be covered by a single trionimo.

Induction Step: Assume  $P(k)$  is true for some  $k \geq 1$ , i.e., any  $2^k \times 2^k$  checkerboard with one square removed can be tiled using trionimos. We need to show  $P(k+1)$ , i.e., a  $2^{k+1} \times 2^{k+1}$  checkerboard with one square removed can also be tiled using trionimos.

Divide the  $2^{k+1} \times 2^{k+1}$  checkerboard into four  $2^k \times 2^k$  subboards: top-left ( $A$ ), top-right ( $B$ ), bottom-left ( $C$ ), and bottom-right ( $D$ ). The removed square lies in one of these subboards. Place a single trionimo at the lower-left corner of  $A$ , the lower-right corner of  $B$ , and the upper-left corner of  $D$ , leaving one square removed from each of  $A$ ,  $C$ , and  $D$ , while  $B$  retains its removed square in the upper-right corner.

By the inductive hypothesis, each of the four  $2^k \times 2^k$  subboards with one square removed can be tiled using trionimos. Thus, the entire  $2^{k+1} \times 2^{k+1}$  checkerboard can be tiled.

Therefore,  $(\forall k \in \mathbb{N})[P(k) \Rightarrow P(k+1)]$ . By the PMI,  $(\forall k \geq 1)[P(k)]$ , and the result is proven.  $\square$

## Problem 9

There is a famous proof that all horses are the same color. Let  $P(n)$  be the statement “for all sets of  $n$  horses, all the horses in the set have the same color”. We will prove this by induction. The base case  $n = 1$  is clear, since in a set consisting of exactly 1 horse all the horses have the same color. Now assume that  $P(n)$  is true, and let  $S$  be a set of  $n + 1$  horses. Label the horses  $1, 2, \dots, n + 1$ . Then the first  $n$  horses constitute a set of  $n$  horses, so by the induction hypothesis they all have the same color. Likewise, the last  $n$  horses are a set of  $n$  horses; so by induction they all have the same color. But if the first  $n$  horses all have the same color, and the last  $n$  horses all have the same color, then since these two sets overlap the two colors must be identical. So all the horses in  $S$  have the same color, and we are done by induction.

Find the mistake in the above proof.

## Solution 9

The mistake in the proof lies in a subtle flaw in the inductive step. The argument does not hold for  $n = 2$ . For the base case, it's valid because a single horse trivially has the same color as itself. The argument fails for  $n = 2$ . When  $S$  contains 3 horses ( $n + 1 = 3$ ), the two subsets considered are  $\{1, 2\}$ , where  $P(2)$  claims they have the same color and  $\{2, 3\}$ , where  $P(2)$  also claims that they have the same color. However, these two subsets overlap at only one horse. This does not guarantee that all three horses in  $S$  have the same color because there is no information linking horse 1 with horse 3. The inductive step fails to establish that the color of horse 1 is the same as the color of horse 3.