1. The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is defined as the sum of the diagonal elements of A, i.e. $\mathrm{Tr}(A) = \sum_{j=1}^{n} a_{jj}$. Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any $n \times n$ matrices A and B.

Proof: Denote $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $AB = (c_{ij})_{n\times n}$, and $BA = (d_{ij})_{n\times n}$.

Then $c_{ij} = (a_{i1} \ a_{i2} \ \cdots \ a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj} \Rightarrow c_{ll} = \sum_{k=1}^{n} a_{lk} b_{kl}$

$d_{ij} = (b_{i1} \ b_{i2} \ \cdots \ b_{in}) \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = \sum_{k=1}^{n} b_{ik} a_{kj} \Rightarrow d_{ll} = \sum_{k=1}^{n} b_{lk} a_{kl}$

make an index change $k \to i$, $l \to j$

$\Rightarrow \mathrm{Tr}(AB) = \sum_{l=1}^{n} c_{ll} = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{lk} b_{kl} \right) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ij}$

$\mathrm{Tr}(BA) = \sum_{l=1}^{n} d_{ll} = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} b_{lk} a_{kl} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ji} b_{ij}$

make an index change $k \to j$, $l \to i$

If may cannot see the two summations are the same, you may add this step.

$\Rightarrow \mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$\boxtimes$

2. State the replacement theorem.

Check the textbook or lecture notes.

3. Let V be a vector space. Prove that the zero vector in V is unique.

Proof: Suppose there are zeros 0 and $0'$ in V.

b/c $0'$ is a zero in V.

Then by Axiom III $\quad 0' = 0' + 0 = 0$ $\blacksquare$

b/c 0 is a zero in V

4. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{C}\}$. Define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \text{ and } c(a_1, a_2) = (a_1, 0).$$

Determine whether or not V is a vector space over $\mathbb{C}$ with these operations. Justify your answer.

Answer: V is not a vector space, b/c Axiom V and Axiom VII (one of the distributive laws) are not satisfied.

One only need use one of these two to explain V is not a vector space.

For Axiom V: $\forall (a_1, a_2) \in V$ with $a_2 \neq 0$: $\quad 1 \cdot (a_1, a_2) = (a_1, 0) \neq (a_1, a_2)$.

For Axiom VII: $\forall c, d \in \mathbb{C}$, and $\forall (a_1, a_2) \in V$ with $a_1 \neq 0$

$$(c+d)(a_1, a_2) = (a_1, 0)$$

$$c(a_1, a_2) + d(a_1, a_2) = (a_1, 0) + (a_1, 0) = (2a_1, 0)$$

$\Rightarrow$ If $a_1 \neq 0$: $\quad (a_1, 0) \neq (2a_1, 0)$

$\Rightarrow \quad (c+d)(a_1, a_2) \neq c(a_1, a_2) + d(a_1, a_2), \ \forall a_1 \neq 0$

5. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{C}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{C}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Determine whether or not V is a vector space over $\mathbb{C}$ with these operations. Justify your answer.

Answer: V is not a vector space because Axiom I (the commutative law for $+$) is not satisfied.

For $(a_1, a_2), (b_1, b_2) \in V$:

$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$, $(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2)$

Then take $(a_1, a_2) = (1, 1)$, and $(b_1, b_2) = (0, 0)$:

$(1, 1) + (0, 0) = (1, 1)$, $(0, 0) + (1, 1) = (2, 3)$

$\Rightarrow (1, 1) + (0, 0) \neq (0, 0) + (1, 1)$.

6. If W_1 and W_2 are subspaces of a vector space V, prove that $W_1 \cap W_2$ is a subspace of V.

Proof: ① As $W_1, W_2 \subseteq V$ are subspaces: $\vec{0} \in W_1$ and $\vec{0} \in W_2$

$\Rightarrow \vec{0} \in W_1 \cap W_2 \Rightarrow W_1 \cap W_2 \neq \phi$.

② $\forall \vec{x}, \vec{y} \in W_1 \cap W_2$ and $\forall c \in F$.

$c\vec{x} + \vec{y} \in W_1$ b/c W_1 is a subspace of V

$c\vec{x} + \vec{y} \in W_2$ b/c W_2 is a subspace of V

$\Rightarrow c\vec{x} + \vec{y} \in W_1 \cap W_2$

By ① and ②, $W_1 \cap W_2 \subseteq V$ is a subspace.

7. Consider the following subsets in $\mathbb{C}^n$:

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n : a_1 + \cdots + a_n = 0 \right\}, \; W_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n : a_1 + \cdots + a_n = c, \text{ where } c \neq 0 \right\}$$

Prove that W_1 is a subspace of $\mathbb{C}^n$, but W_2 is not a subspace of $\mathbb{C}^n$.

Proof: i) ① $\vec{0} \in W_1 \Rightarrow W_1 \neq \emptyset$

② $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in W_1$ and $c \in \mathbb{C}$

$\Rightarrow a_1 + a_2 + \cdots + a_n = 0$ as $\vec{a}, \vec{b} \in W_1$

$b_1 + b_2 + \cdots + b_n = 0$

$$c\vec{a} + \vec{b} = c\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} ca_1 + b_1 \\ \vdots \\ ca_n + b_n \end{pmatrix}$$

$$(ca_1 + b_1) + (ca_2 + b_2) + \cdots + (ca_n + b_n) = c(a_1 + a_2 + \cdots + a_n) + b_1 + \cdots + b_n = 0$$

$\Rightarrow c\vec{a} + \vec{b} \in W_1$

By ① and ②, W_1 is a subspace.

ii) For W_2, since: $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \notin W_2$ b/c $0 + 0 + \cdots + 0 = 0 \neq c$ as $c \neq 0$

$\Rightarrow \vec{0} \notin W_2 \Rightarrow W_2$ does not contain the zero vector

$\Rightarrow W_2$ is not a vector space, thus not a subspace.

(One can also argue that W_2 is not closed under "+" or scalar multiplication).

8. Let S be the subset of all symmetric matrices in $\mathbb{R}^{n\times n}$, i.e. $S = \{A \in \mathbb{R}^{n\times n} : A^T = A\}$. Prove that S is a subspace of $\mathbb{R}^{n\times n}$.

Proof: ① Denote the zero matrix by O. Then $O^T = O$ Thus $O \in S$

$\Rightarrow$ S is non-empty

② For any $A, B \in S$ then $A^T = A$ and $B^T = B$.

For any $A, B \in S$ and $c \in \mathbb{R}$:

$$(cA+B)^T = cA^T + B^T = cA + B$$

$\Rightarrow$ $cA + B \in S$.

By ① and ②, S is a subspace.

9. Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ be an ordered basis for V. Prove that for any $\mathbf{x} \in V$, there exists a unique set of scalars $\{a_1, a_2, \cdots, a_n\}$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n.$$

Proof: As $B = \{\vec{x}_1, \cdots, \vec{x}_n\}$ is a basis of V, B is linearly independent and $\text{Span}\, B = V$.

"Existence": As $V = \text{Span}\{\vec{x}_1, \cdots, \vec{x}_n\}$, there exists $a_1, \cdots, a_n$ such that

$$\vec{x} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n$$

"Uniqueness": Suppose $\vec{x} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n = b_1\vec{x}_1 + \cdots + b_n\vec{x}_n$

$\Rightarrow$ $a_1\vec{x}_1 + \cdots + a_n\vec{x}_n = b_1\vec{x}_1 + \cdots + b_n\vec{x}_n$

$\Leftrightarrow$ $(a_1 - b_1)\vec{x}_1 + \cdots + (a_n - b_n)\vec{x}_n = \vec{0}$

As $\{\vec{x}_1, \cdots, \vec{x}_n\}$ is linearly independent

$a_1 - b_1 = \cdots = a_n - b_n = 0$ $\Rightarrow$ $a_1 = b_1, \cdots, a_n = b_n$.

10. True or False. (No explanation needed).

F 1). A vector space may have more than one zero vector.

F 2). If f and g polynomials of degree n, then $f + g$ is a polynomial of degree n.

T 3). If V is a vector space and W is a subset of V that is a vector space, then W is a subspace.

F 4). If W and U are subspaces of V, then $W \cup U$ is a subspace of V.