MATH 410 - WINTER 2026 - HOMEWORK 3

1. Show that $|e^{i\theta} - 1| \leq \theta$ for $\theta \in \mathbb{R}$. (*Hint*: You can use the fundamental theorem of calculus.)

2. Prove that

$$(f * g)'(x) = (f' * g)(x),$$

where $'$ denotes derivative and $*$ denotes convolution. Just treat this as a formal identity, i.e. assume everything converges nicely and you can pass derivatives through the integral sign.

This identity has an important consequence, namely: "the convolution of f and g is as smooth as the smoother of f and g".

3. Prove that if $f(x) = 0$ for $|x| > R$ and $g(x) = 0$ for $|x| > T$, then $f * g(x) = 0$ for $|x| > R + T$.

4. Show that if $g(x) = f(x + y)$ for some $y \in \mathbb{R}$, then $\hat{g}(\xi) = e^{iy\xi}\hat{f}(\xi)$.

5. Show that limits are unique in a metric space. That is, if (X, d) is a metric space and $\{x_n\}$ is a sequence satisfying $x_n \to x \in X$ and $x_n \to y \in X$, then $x = y$.

6 (optional for 410, required for 510). Taking the following fact for granted, complete the proof of the Riemann–Lebesgue lemma (i.e. $f \in L^1 \implies \hat{f} \in C_0$):

Fact. For any $f \in L^1$ and any $\varepsilon > 0$, there exists a function $g \in L^1$ satisfying (i) $xg \in L^1$ and $g' \in L^1$ and (ii) $\|f - g\|_{L^1} < \varepsilon$.

Recall that in class we proved that for $g \in L^1$ satisfying (i), we have $\hat{g} \in C_0$.