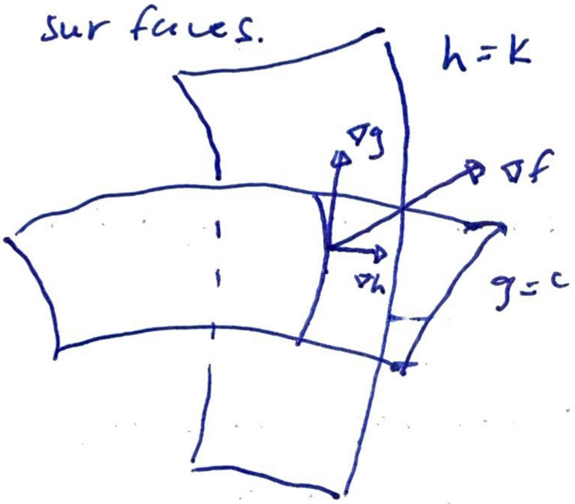
Two Constraints

Minimize and Maximize $f(x,y,z)$ subject to $g(x,y,z) = c$ and $h(x,y,z) = K$.

The constraints $g = c$ and $h = K$ define level surfaces of the functions g and h. Minimize and Maximize f subject to the curve of intersection of the level surfaces.



① If f has an extreme value at P along the curve, then ∇f is orthogonal to the curve at P.

② ∇g is orthogonal to $g = c$ at all points. Therefore ∇g is orthogonal to the curve at P.

③ ∇h is orthogonal to $h = K$ at all points. Therefore ∇h is orthogonal to the curve at P.

Then ∇f lies in the plane containing ∇g and ∇h.

Find (x,y,z) and λ and μ such that

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = c \\ h = K \end{cases}$$

λ and μ are the Lagrange multipliers

Ex: maximize and minimize $f = x + 2y$ subject to

$x + y + z = 1$ and $y^2 + z^2 = 4$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$\langle 1, 2, 0 \rangle = \lambda \langle 1, 1, 1 \rangle + \mu \langle 0, 2y, 2z \rangle$

(1) $1 = \lambda$

(2) $2 = \lambda + 2\mu y$ $\Rightarrow$ $2\mu y = 1$ $\quad y = \frac{1}{2\mu}$

(3) $0 = \lambda + 2\mu z$ $\Rightarrow$ $2\mu z = -1$ $\quad z = \frac{-1}{2\mu}$ $\quad z = -y$

(4) $g = 1$

(5) $h = 4$

Sub $y = \frac{1}{2\mu}$ and $z = \frac{-1}{2\mu}$ into (5)

$$\frac{1}{4\mu^2} + \frac{1}{4\mu^2} = 4$$

$$\frac{1}{2\mu^2} = 4$$

$$\mu^2 = \frac{1}{8} \qquad \mu = \pm\frac{1}{\sqrt{8}}$$

If $\mu = \frac{1}{\sqrt{8}}$, $y = \frac{\sqrt{8}}{2} = \sqrt{2}$ and $z = \frac{-\sqrt{8}}{2} = -\sqrt{2}$

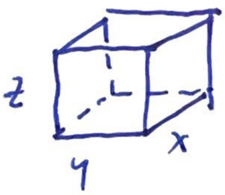
$x = 1 - y - z = 1$ $\qquad (1, \sqrt{2}, -\sqrt{2})$

If $\mu = \frac{-1}{\sqrt{8}}$, $y = -\sqrt{2}$ and $z = \sqrt{2}$ $\qquad (1, -\sqrt{2}, \sqrt{2})$

$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}$ max

$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$ min

Ex: Find the maximum and minimum volume of a rectangular box with surface area 1500 cm^2 and edge length 200 cm.



Volume : $V = xyz$

Area : $A = 2xy + 2xz + 2yz = 1500$

Edge Length: $L = 4x + 4y + 4z = 200$

Maximize/Minimize $V = xyz$ subject to

$g = xy + xz + yz = 750$ and $h = x + y + z = 50$

Solve

$$\begin{cases} \nabla V = \lambda \nabla g + \mu \nabla h \\ g = 750 \\ h = 50 \end{cases}$$

$$\langle yz, xz, xy \rangle = \lambda \langle y+z, x+z, x+y \rangle + \mu \langle 1, 1, 1 \rangle$$

① $yz = \lambda(y+z) + \mu$

② $xz = \lambda(x+z) + \mu$

③ $xy = \lambda(x+y) + \mu$

④ $g = 750$

⑤ $h = 50$

① − ② : $yz - xz = \lambda(y+z) - \lambda(x+z)$

$$z(y-x) = \lambda(y+z-x-z)$$
$$= \lambda(y-x)$$

$$z(y-x) - \lambda(y-x) = 0$$

$(z-\lambda)(y-x) = 0$ Either $x=y$ or $z=\lambda$

① − ③ : $yz - xy = \lambda(y+z) - \lambda(x+y)$

$$y(z-x) = \lambda(z-x)$$

$(y-\lambda)(z-x) = 0$ Either $x=z$ or $y=\lambda$

② − ③ : $xz - xy = \lambda(x+z) - \lambda(x+y)$

$$x(z-y) = \lambda(z-y)$$

$(x-\lambda)(z-y) = 0$ Either $y=z$ or $x=\lambda$

If $x=y=z$ (Box is a cube), then

$$g = xy + xz + yz = x^2 + x^2 + x^2 = 3x^2 = 750$$
$$x^2 = 250$$
$$x = 5\sqrt{10}$$

$$h = x+y+z = 3x = 50$$
$$x = \frac{50}{3}$$

A cube does not satisfy the constraints. The box must have at least one distinct side length. Suppose x is distinct so that $x \neq y$ and $x \neq z$. Then $z = \lambda$ and $y = \lambda$ $(y=z)$.

Constraints:

$$h = x + y + z = 50$$

$$x + 2\lambda = 50$$

$$x = 50 - 2\lambda$$

$$g = xy + xz + yz = 750$$

$$x\lambda + x\lambda + \lambda^2 = 750$$

$$2\lambda x + \lambda^2 = 750$$

$$2\lambda(50 - 2\lambda) + \lambda^2 = 750$$

$$-3\lambda^2 + 100\lambda - 750 = 0$$

By Quad Formula, $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$

If $\lambda = y = z = \dfrac{50 + 5\sqrt{10}}{3}$, then $x = 50 - 2\lambda = \dfrac{50 - 10\sqrt{10}}{3}$

$$V = xyz = \left(\frac{50 - 10\sqrt{10}}{3}\right)\left(\frac{50 + 5\sqrt{10}}{3}\right)^2 \approx 2947.94$$ Min Volume

If $\lambda = y = z = \dfrac{50 - 5\sqrt{10}}{3}$, then $x = 50 - 2\lambda = \dfrac{50 + 10\sqrt{10}}{3}$

$$V = xyz = \left(\frac{50 + 10\sqrt{10}}{3}\right)\left(\frac{50 - 5\sqrt{10}}{3}\right)^2 \approx 3533.54$$ Max Volume