Functional Complex Variables I: Final Exam

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Problem 1. Find all the solutions of $\sin(z) = i$.

Solution. Using the complex exponential form of the sine function, we have

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = i.$$

Multiplying both sides by $2i$ gives $e^{iz} - e^{-iz} = -2$. Let $w = e^{iz}$, then we get the quadratic $w^2 + 2w - 1 = 0$. Using the quadratic formula, we find $w = -1 \pm \sqrt{2}$. Since $w = e^{iz}$, taking logarithms gives and multiplying by $-i$ gives $z = -i\ln(w)$, giving us two branches of solutions, $-1 + \sqrt{2}$ and $-1 - \sqrt{2}$.

For the first branch, $w = -1 + \sqrt{2} > 0$, we get $\arg(w) = 0$. So

$$\ln(w) = \ln\left(-1 + \sqrt{2}\right) + 2\pi i k,$$

and

$$z = -i\ln\left(-1 + \sqrt{2}\right) + 2\pi k, \quad k \in \mathbb{Z}.$$

For the second branch, $w = -1 - \sqrt{2} < 0$, we get $\arg(w) = \pi$. So

$$\ln(w) = \ln\left(-1 - \sqrt{2}\right) + i\pi + 2\pi i k,$$

and

$$z = -i\ln\left(-1 - \sqrt{2}\right) + \pi + 2\pi k, \quad k \in \mathbb{Z}.$$

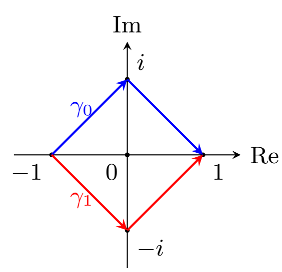
Therefore, the complete set of solutions is

$$z = -i\ln\left(-1 + \sqrt{2}\right) + 2\pi k \quad \text{or} \quad z = -i\ln\left(-1 - \sqrt{2}\right) + \pi + 2\pi k, \quad k \in \mathbb{Z}.$$ □

Problem 2. Let γ_0 be the path which consists of line segments from -1 to i and then from i to 1. Let γ_1 be the path which consists of line segments from -1 to $-i$ and then from $-i$ to 1. Evaluate the following two integrals

$$\int_{\gamma_0} z^{-1}\,\mathrm{d}z; \quad \int_{\gamma_1} z^{-1}\,\mathrm{d}z.$$

Solution. Graphing the contour paths, we have



For the first integral, parametrizing the contour γ_0, we have two segments

$$z_1(t) = -1(1-t) + it \text{ where } t \in [0,1] \quad \text{and} \quad z_2(t) = i(1-t) + t \text{ where } t \in [0,1].$$

This gives us the following differentials

$$\mathrm{d}z_1 = (1+i)\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}z_2 = (-i+1)\,\mathrm{d}t.$$

Therefore, we can compute the integral over γ_0 as follows

$$\int_{\gamma_0} z^{-1}\,\mathrm{d}z = \int_0^1 \frac{1}{z_1(t)}\,\mathrm{d}z_1 + \int_0^1 \frac{1}{z_2(t)}\,\mathrm{d}z_2$$

$$= \int_0^1 \frac{1+i}{-(1-t)+it}\,\mathrm{d}t + \int_0^1 \frac{-i+1}{i(1-t)+t}\,\mathrm{d}t.$$

Using the substitution $u_1 = -(1-t)+it$ in the first integral and $u_2 = i(1-t)+t$ in the second integral, we can simplify these integrals to get

$$\begin{aligned}
\int_0^1 \frac{1+i}{-(1-t)+it}\,\mathrm{d}t + \int_0^1 \frac{-i+1}{i(1-t)+t}\,\mathrm{d}t &= \int_{-1}^{i} \frac{1}{u}\,\mathrm{d}u + \int_i^1 \frac{1}{u}\,\mathrm{d}u \\
&= \ln(u)\Big|_{-1}^{i} + \ln(u)\Big|_{i}^{1} \\
&= -\frac{i\pi}{2} - \frac{i\pi}{2} \\
&= -i\pi.
\end{aligned}$$

For the second integral, we can use a similar parametrization for γ_1

$$z_1(t) = -1(1-t) - it \text{ where } t \in [0,1] \quad \text{and} \quad z_2(t) = -i(1-t)+t \text{ where } t \in [0,1].$$

This gives us the differentials

$$\mathrm{d}z_1 = (1-i)\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}z_2 = (i+1)\,\mathrm{d}t.$$

Therefore, we can compute the integral over γ_1 as follows

$$\begin{aligned}
\int_{\gamma_1} z^{-1}\,\mathrm{d}z &= \int_0^1 \frac{1}{z_1(t)}\,\mathrm{d}z_1 + \int_0^1 \frac{1}{z_2(t)}\,\mathrm{d}z_2 \\
&= \int_0^1 \frac{1-i}{-(1-t)-it}\,\mathrm{d}t + \int_0^1 \frac{i+1}{-i(1-t)+t}\,\mathrm{d}t.
\end{aligned}$$

Using the substitutions $u_1 = -(1-t)-it$ in the first integral and $u_2 = -i(1-t)+t$ in the second integral, we can simplify these integrals to get

$$\begin{aligned}
\int_0^1 \frac{-1-i}{-(1-t)-it}\,\mathrm{d}t + \int_0^1 \frac{i+1}{-i(1-t)+t}\,\mathrm{d}t &= \int_{-1}^{-i} \frac{1}{u}\,\mathrm{d}u + \int_{-i}^1 \frac{1}{u}\,\mathrm{d}u \\
&= \ln(u)\Big|_{-1}^{-i} + \ln(u)\Big|_{-i}^{1} \\
&= \frac{i\pi}{2} + \frac{i\pi}{2} \\
&= i\pi.
\end{aligned}$$

Therefore, we conclude that

$$\int_{\gamma_0} z^{-1}\,\mathrm{d}z = -i\pi \quad \text{and} \quad \int_{\gamma_1} z^{-1}\,\mathrm{d}z = i\pi. \qquad \square$$

Problem 3. Let $C_r = \{z \mid |z| = r\}$, for $0 < r \neq 2$ your answer might depend on r). Find the integral

$$\int_{C_r} \frac{1}{z^3 - 2z^2}\,\mathrm{d}z.$$

Solution. We begin by factoring the denominator to get

$$\frac{1}{z^3-2z^2} = \frac{1}{z^2(z-2)}.$$

The integrand has singularities at $z = 0$ (a pole of order 2) and $z = 2$ (a simple pole). The integral depends on the location of these singularities relative to the contour C_r. In the case where $0 < r < 2$, C_r encloses the singularity at $z = 0$, but not $z = 2$. Since the integrand is holomorphic on and inside C_r except at $z = 0$, we compute the residue at $z = 0$. Expanding using partial fraction decomposition, we have

$$f(z) = \frac{1}{z^2(z-2)} = \frac{A}{z-2} + \frac{B}{z} + \frac{C}{z^2},$$

for constants A, B, C. Multiply both sides by $z^2(z-2)$ and solve to get

$$1 = Az^2 + Bz(z-2) + C(z-2).$$

Then, expand and collect terms to get

$$1 = Az^2 + Bz^2 - 2Bz + Cz - 2C = (A+B)z^2 + (C-2B)z - 2C.$$

By comparing and solving for the coefficients, we have

$$A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad \text{and} \quad C = -\frac{1}{2}.$$

So,

$$\frac{1}{z^2(z-2)} = \frac{1}{4(z-2)} - \frac{1}{4z} - \frac{1}{2z^2}.$$

On the circle C_r with $r < 2$, only the terms with singularities at $z = 0$ contribute to the integral. The term $1/4(z-2)$ is analytic inside C_r, so its integral is 0. We compute

$$\int_{C_r} \frac{1}{z^2(z-2)}\,\mathrm{d}z = \int_{C_r} \left(-\frac{1}{4z} - \frac{1}{2z^2}\right)\mathrm{d}z = -\frac{1}{4}\int_{C_r} \frac{1}{z}\,\mathrm{d}z - \frac{1}{2}\int_{C_r} \frac{1}{z^2}\,\mathrm{d}z.$$

Now, by Cauchy's integral formula, we have

$$\int_{C_r} \frac{1}{z^2(z-2)}\,\mathrm{d}z = -\frac{1}{4}(2\pi i) = -\frac{\pi i}{2}.$$

In the case where $r > 2$, both $z = 0$ and $z = 2$ lie inside C_r, so all three terms contribute, giving us

$$\int_{C_r} \frac{1}{z^2(z-2)}\,\mathrm{d}z = \int_{C_r} \left(\frac{1}{4(z-2)} - \frac{1}{4z} - \frac{1}{2z^2}\right)\mathrm{d}z = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) - 0 = 0.$$

Therefore, we conclude that

$$\int_{C_r} \frac{1}{z^3 - 2z^2}\,\mathrm{d}z = \begin{cases} -\dfrac{\pi i}{2} & \text{if } 0 < r < 2 \\ 0 & \text{if } r > 2 \end{cases}. \qquad \square$$

Problem 4. Let f be an entire function such that $\lim_{z\to\infty} f(z)z^{-2} = 0$. Use Cauchy's integral formula to show that $f''(z) = 0$ for any z, hence $f(z) = az + b$ for some constants $a, b \in \mathbb{C}$.

Solution. Since f is entire, it is holomorphic on all of $\mathbb{C}$. Fix any $z \in \mathbb{C}$ and let $R > 0$ be large enough so that z lies inside the disk $D(0, R)$. Then, by Cauchy's integral formula for the second derivative, we have

$$f''(z) = \frac{2!}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - z)^3}\,\mathrm{d}\zeta.$$

Since $|\zeta - z| \geq |\zeta| - |z| = R - |z|$ for $|\zeta| = R$, and since f is entire with the property that $\lim_{\zeta\to\infty} \dfrac{f(\zeta)}{\zeta^2} = 0$, we can write

$$|f(\zeta)| \leq \varepsilon\,|\zeta|^2 = \varepsilon R^2, \quad \text{for all } |\zeta| = R \text{ with } R \text{ sufficiently large,}$$

for any $\varepsilon > 0$. Thus, we estimate

$$|f''(z)| \leq \frac{2}{2\pi} \int_{|\zeta|=R} \frac{|f(\zeta)|}{|\zeta - z|^3} \, |\mathrm{d}\zeta| \leq \frac{2}{2\pi} \cdot \frac{\varepsilon R^2}{(R - |z|)^3} \cdot 2\pi R = \frac{2\varepsilon R^3}{(R - |z|)^3}.$$

Now, taking the limit as $R \to \infty$, we note that the right-hand side tends to 0:

$$\lim_{R \to \infty} \frac{2\varepsilon R^3}{(R - |z|)^3} = 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $|f''(z)| = 0$, so $f''(z) = 0$ for all $z \in \mathbb{C}$.

Therefore, f is a polynomial of degree at most 1, so there exist constants $a, b \in \mathbb{C}$ such that

$$f(z) = az + b.$$

□

Problem 5. Let f be an entire function such that $|f(z)| \geq 1$ and $f(0) = i$. Prove that $f(z) = i$ for all z.

Solution. Since f is entire, it is holomorphic on all of $\mathbb{C}$. Moreover, the condition $|f(z)| \geq 1$ for all $z \in \mathbb{C}$ implies that the function f never vanishes, so we may define the function

$$g(z) = \frac{1}{f(z)},$$

for all $z \in \mathbb{C}$. Then g is entire, since the reciprocal of a non-vanishing holomorphic function is holomorphic.

Furthermore, for all $z \in \mathbb{C}$, we have

$$|g(z)| = \left| \frac{1}{f(z)} \right| \leq \frac{1}{1} = 1,$$

so g is a bounded entire function. By Liouville's theorem in Sec. 53, g is constant. Hence, $f(z)$ is constant as well.

Since $f(0) = i$, we conclude that

$$f(z) = i \quad \text{for all } z \in \mathbb{C}.$$

□

Problem 6. Compute (use contour integral)

$$\int_0^\infty \frac{1}{1 + x^6} \, \mathrm{d}x.$$

Solution. We begin by observing that the integrand is an even function, so we can extend the domain of integration to the entire real line and halve the result, i.e.,

$$\int_0^\infty \frac{1}{1 + x^6} \, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{1 + x^6} \, \mathrm{d}x.$$

We evaluate the full integral using contour integration. Let

$$f(z) = \frac{1}{1 + z^6},$$

and integrate this function over the semicircular contour in the upper half-plane of radius R, consisting of the real interval $[-R, R]$ and the semicircular arc γ_R in the upper half-plane. As $R \to \infty$, the integral over γ_R vanishes, so the integral over the real axis is given by the sum of the residues inside the contour.

The function $f(z)$ has poles at the sixth roots of -1, that is, at

$$z_k = e^{i(2k+1)\pi/6}, \quad k = 0, 1, 2, 3, 4, 5.$$

The three poles in the upper half-plane are

$$z_0 = e^{i\pi/6}, \quad z_1 = e^{i\pi/2}, \quad \text{and} \quad z_2 = e^{i5\pi/6}.$$

These are all simple poles, and we compute the residue at each using the formula

$$\operatorname*{Res}_{z=z_k}\left(\frac{1}{1+z^6}\right) = \frac{1}{6z_k^5}.$$

Thus, the integral over the real line is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,\mathrm{d}x = 2\pi i \sum_{k=0}^{2} \operatorname*{Res}_{z=z_k} f(z) = 2\pi i \cdot \frac{1}{6}\left(\frac{1}{z_0^5} + \frac{1}{z_1^5} + \frac{1}{z_2^5}\right).$$

We compute each of the terms to get

$$\begin{aligned}
z_0^5 &= e^{i5\pi/6} && \Rightarrow \frac{1}{z_0^5} = e^{-i5\pi/6} \\
z_1^5 &= e^{i5\pi/2} = e^{i\pi/2} && \Rightarrow \frac{1}{z_1^5} = e^{-i\pi/2} \\
z_2^5 &= e^{i25\pi/6} = e^{i\pi/6} && \Rightarrow \frac{1}{z_2^5} = e^{-i\pi/6}.
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,\mathrm{d}x = 2\pi i \cdot \frac{1}{6}\left(e^{-i5\pi/6} + e^{-i\pi/2} + e^{-i\pi/6}\right).$$

Compute the sum of exponentials to get

$$\begin{aligned}
e^{-i5\pi/6} &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i \\
e^{-i\pi/2} &= -i \\
e^{-i\pi/6} &= \frac{\sqrt{3}}{2} - \frac{1}{2}i,
\end{aligned}$$

and summing,

$$e^{-i5\pi/6} + e^{-i\pi/2} + e^{-i\pi/6} = \left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - 1 - \frac{1}{2}\right)i = -2i.$$

Substituting into the integral,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,\mathrm{d}x = 2\pi i \cdot \frac{-2i}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}.$$

Finally, since our original integral was half of this,

$$\int_{0}^{\infty} \frac{1}{1+x^6}\,\mathrm{d}x = \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}.$$

□

Problem 7. Compute (use contour integral)

$$\int_0^{2\pi} \frac{1}{2+\cos(t)}\,\mathrm{d}t.$$

Solution. We begin by transforming the integral using the complex exponential substitution $z = e^{it}$, which maps the interval $t \in [0, 2\pi]$ onto the unit circle $|z|= 1$ traversed once counterclockwise. Recall the identities

$$\cos(t) = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \Rightarrow \mathrm{d}t = \frac{\mathrm{d}z}{iz}.$$

Substituting into the integrand, we have

$$\frac{1}{2 + \cos t} = \frac{1}{2 + \frac{1}{2}(z + z^{-1})} = \frac{2}{4 + z + z^{-1}}.$$

Therefore, the integral becomes

$$\int_0^{2\pi} \frac{1}{2 + \cos t}\,\mathrm{d}t = \oint_{|z|=1} \frac{2}{4 + z + \frac{1}{z}} \cdot \frac{1}{iz}\,\mathrm{d}z.$$

Combining terms gives us

$$\frac{2}{iz\left(4 + z + \frac{1}{z}\right)} = \frac{2}{i\left(z^2 + 4z + 1\right)}.$$

Thus, the integral reduces to:

$$\oint_{|z|=1} \frac{2}{i(z^2 + 4z + 1)}\,\mathrm{d}z.$$

Now, we can evaluate this integral using the Residue theorem. The integrand has simple poles at the roots of the denominator, which are

$$z^2 + 4z + 1 = 0 \Rightarrow z = -2 \pm \sqrt{3}.$$

Only one of these poles lies within the unit circle, since $z_0 = -2 + \sqrt{3} \approx -0.2679$, since $\left|-2 + \sqrt{3}\right| < 1$. We compute the residue of the integrand at z_0. For a simple pole, the residue is given by

$$\operatorname*{Res}_{z=z_0}\left(\frac{2}{i(z^2 + 4z + 1)}\right) = \frac{2}{i} \cdot \frac{1}{(z_0 - z_1)},$$

where $z_1 = -2 - \sqrt{3}$ is the other root. Hence,

$$z_0 - z_1 = 2\sqrt{3} \Rightarrow \operatorname*{Res}_{z=z_0} = \frac{2}{i \cdot 2\sqrt{3}} = \frac{1}{i\sqrt{3}} = -\frac{i}{\sqrt{3}}.$$

Therefore, by the Residue theorem, we have

$$\oint_{|z|=1} \frac{2}{i(z^2 + 4z + 1)}\,\mathrm{d}z = 2\pi i \cdot \left(-\frac{i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

Finally, we conclude that

$$\int_0^{2\pi} \frac{1}{2 + \cos(t)}\,\mathrm{d}t = \frac{2\pi}{\sqrt{3}}.$$

□

Problem 8. Find the Taylor series of $f(z) = 3/(3 + z)^2$ at $z = 0$. What is its convergence radius?

Solution. We begin by rewriting the function in a form suitable for expansion as a binomial series. Observe that

$$f(z) = \frac{3}{(3 + z)^2} = \frac{3}{9\left(1 + \frac{z}{3}\right)^2} = \frac{1}{3} \cdot \left(1 + \frac{z}{3}\right)^{-2}.$$

Now we apply the generalized binomial series

$$(1 + w)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} w^n, \quad \text{for } |w| < 1,$$

with $\alpha = -2$ and $w = z/3$. We compute

$$\left(1 + \frac{z}{3}\right)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{z}{3}\right)^n.$$

Using the identity

$$\binom{-2}{n} = (-1)^n \binom{n+1}{1} = (-1)^n (n+1),$$

we obtain

$$f(z) = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3^{n+1}} z^n.$$

To determine the radius of convergence, we locate the nearest singularity of the function $f(z) = 3/(3+z)^2$. The only singularity is at $z = -3$, so the distance from the origin is $R = |-3| = 3$.

Hence, we get

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3^{n+1}} z^n \quad \text{and} \quad R = 3. \qquad \square$$