Chapter 2

Matrix Lie Groups Solutions

Exercise 11. *Connectedness of* $\mathrm{SO}(n)$. Show that $\mathrm{SO}(n)$ is connected, following the outline below.

For the $n = 1$ case, there is not much to show, since a 1×1 matrix with determinant one must be 1. Assume, then, that $n \geq 2$ Let $\mathbf{e}_1$ denote the vector

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

Given any unit vector $\mathbf{v} \in \mathbb{R}^n$, show that there exists a continuous path $R(t)$ in $\mathrm{SO}(n)$ such that $R(0) = I$ and $R(1)\mathbf{e}_1 = \mathbf{v}$. (Thus any unit vector can be "continuously rotated" to $\mathbf{e}_1$.)

Now show that any element R of $\mathrm{SO}(n)$ can be connected to an element of $\mathrm{SO}(n-1)$, and proceed by induction.

Solution. Let $\mathcal{B} = \{\mathbf{v}, \mathbf{e}_1\}$ be a basis for a two-dimensional plane. By the Gram-Schmitt process, we can construct an orthonormal basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2\}$ for the same plane. Let $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2$ be defined as

$$\mathbf{u}_2 = \frac{\mathbf{e}_1 - \langle \mathbf{e}_1, \mathbf{v} \rangle \mathbf{v}}{\|\mathbf{e}_1 - \langle \mathbf{e}_1, \mathbf{v} \rangle \mathbf{v}\|}.$$

Let $\theta(t) : [0, 1] \to \mathbb{R}$, defined by

$$\theta(0) = 0 \quad \text{and} \quad \theta(1) = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{e}_1}{\|\mathbf{v}\| \|\mathbf{e}_1\|} \right)$$

Then, we can construct a rotation $R(t) \in \mathrm{SO}(n)$ as a block matrix that acts as a rotation by $\theta(t)$ in the plane spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$, and as the identity on the orthogonal complement. This defines a continuous path with $R(0) = I$ and $R(1)\mathbf{v} = \mathbf{e}_1$.

We'll use induction to show that $\mathrm{SO}(n)$ is connected for all $n \geq 1$. The base case is trivial, which is trivially connected.

Assume $\mathrm{SO}(n)$ is connected for some $n \geq 1$. We need to show that $\mathrm{SO}(n+1)$ is connected. Let $R \in \mathrm{SO}(n+1)$. Consider the first column of R, which is a unit vector $\mathbf{v} \in \mathbb{R}^{n+1}$.

I'm not sure how to proceed from here. □

Exercise 12. *The polar decomposition of* $\mathrm{SL}(n, \mathbb{R})$. Show that every element A of $\mathrm{SL}(n, \mathbb{R})$ can be written uniquely in the form $A = RH$, where $R \in \mathrm{SO}(n)$, and H is a symmetric, positive-definite matrix with determinant one (That is, $H^T = H$, and $\langle \mathbf{x}, H\mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$).

Hint: If A could be written in this form, then we would have

$$A^T A = H^T R^T R H = H R^{-1} R H = H^2.$$

Thus H would have to be the unique positive-definite symmetric square root of $A^T A$.

Note: A similar argument gives polar decompositions for $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{GL}(n, \mathbb{C})$. For example, every element A of $\mathrm{SL}(n, \mathbb{C})$ can be written uniquely as $A = UH$, with $U \in \mathrm{SU}(n)$, and H is a self-adjoint positive definite matrix with determinant one.

Solution. Consider the matrix $A^T A$, which is symmetric and positive definite because for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, we have

$$(A^T A)^T = (A)^T (A^T)^T = A^T A \quad \text{and} \quad \mathbf{x}^T A^T A \mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 > 0.$$

Since $A \in \mathrm{SL}(n, \mathbb{R})$, we have $\det(A) = 1$, implying $\det(A^T A) = \det(A^T) \cdot \det(A) = 1 \cdot 1 = 1$ since determinant is multiplicative. By the spectral theorem, $A^T A$ has an orthonormal eigenbasis with positive eigenvalues, so it admits a unique positive-definite square root, denoted H, such that

$$H = \sqrt{A^T A} \implies H^2 = A^T A.$$

Define $R = AH^{-1}$. We check that R is orthogonal

$$R^T R = (H^{-1} A^T)(A H^{-1}) = H^{-1} A^T A H^{-1} = H^{-1} H^2 H^{-1} = I.$$

Thus, $R \in \mathrm{SO}(n)$ since $\det(R) = {}^{\det(A)}/_{\det(H)} = {}^1/_1 = 1$, proving existence.

Suppose $A = R_1 H_1 = R_2 H_2$ are two such decompositions. Then,

$$H_1^{-1} R_1^{-1} R_2 H_2 = I.$$

Multiplying on the right by H_2^{-1} and on the left by H_1, we obtain

$$H_1 H_1^{-1} R_1^{-1} R_2 H_2 H_2^{-1} = H_1 H_2^{-1} = I,$$

so $H_1 = H_2$. This implies $R_1 = R_2$, proving uniqueness.

Thus, every element of $\mathrm{SL}(n, \mathbb{R})$ has a unique polar decomposition. □

Exercise 13. *The connectedness of* $\mathrm{SL}(n, \mathbb{R})$. Using the polar decomposition of $\mathrm{SL}(n, \mathbb{R})$ and the connectedness of $\mathrm{SO}(n)$, show that $\mathrm{SL}(n, \mathbb{R})$ is connected.

Hint: Recall that if H is a real, symmetric matrix, then there exists a *real* orthogonal matrix R_1 such that $H = R_1 D R_1^{-1}$, where D is diagonal.

Solution. Since we are dealing with $\mathrm{SL}(n, \mathbb{R})$, we add the restriction that H is of determinant one. By the polar decomposition, we can write $A = RH$, where $R \in \mathrm{SO}(n)$ and H is a symmetric, positive-definite matrix with determinant one.

Also, by the hint, we can write $H = R_1 D R_1^{-1}$, where $R_1 \in \mathrm{O}(n)$ and D is a diagonal matrix. The space of symmetric matrices with determinant 1 that are also positive definite forms a connected space. This follows because the space of positive-definite diagonal matrices with determinant 1 is connected, and conjugation by an orthogonal matrix does not change connectivity.

By exercise 11, we know that $\mathrm{SO}(n)$ is connected. Since each element in $\mathrm{SL}(n, \mathbb{R})$ can be written as RH, where $R \in \mathrm{SO}(n)$ and H belongs to a connected space, and the product of connected spaces is connected, we conclude that $\mathrm{SL}(n, \mathbb{R})$ is connected. □

Exercise 14. *The connectedness of* $\mathrm{GL}(n, \mathbb{R})^+$. Show that $\mathrm{GL}(n, \mathbb{R})^+$ is connected.

Solution. For any $A \in \mathrm{GL}(n, \mathbb{R})$, the polar decomposition expresses A uniquely as $A = U_A P_A$, $U_A \in \mathrm{O}(n)$ (i.e., $U_A U_A^T = I$), and P_A is a symmetric positive-definite matrix (i.e., $P_A = \sqrt{A^\top A}$, and P_A has only positive eigenvalues).

Since $A, B \in \mathrm{GL}(n, \mathbb{R})^+$, we know that $\det(A) > 0$ and $\det(B) > 0$, which implies U_A, U_B have determinant $+1$, so $U_A, U_B \in \mathrm{SO}(n)$.

Given $A, B \in \mathrm{GL}(n, \mathbb{R})^+$ with their polar decompositions $A = U_A P_A$ and $B = U_B P_B$, we can construct a continuous path from A to B as follows. Since $\mathrm{SO}(n)$ is path-connected, there exists a smooth path U_t in $\mathrm{SO}(n)$ such that $U_0 = U_A$ and $U_1 = U_B$. One explicit choice is the geodesic interpolation, $U_t = U_A \exp\left(t \log\left(U_A^\top U_B\right)\right)$, which remains in $\mathrm{SO}(n)$ for all $t \in [0, 1]$. Since the space of symmetric positive-definite matrices is also path-connected, we use the interpolation $P_t = (1 - t)P_A + tP_B$. This remains positive definite for all $t \in [0, 1]$ because the sum of two positive-definite matrices with positive weights remains positive definite. Now, we can define the path $A_t = U_t P_t$, for $t \in [0, 1]$ Since U_t remains in $\mathrm{SO}(n)$ and P_t remains positive definite, each A_t is invertible with $\det(A_t) > 0$, ensuring $A_t \in \mathrm{GL}(n, \mathbb{R})^+$ for all t.

Verifying continuity, we get:

1. The function $t \mapsto U_t$ is continuous because it is constructed from matrix exponentiation, which is smooth.

2. The function $t \mapsto P_t$ is trivially continuous as it is a convex combination of continuous matrices.

3. Since matrix multiplication is continuous, the final path $t \mapsto A_t = U_t P_t$ is continuous.

Thus, $\mathrm{GL}(n, \mathbb{R})^+$ is connected. □

Exercise 15. Show that the set of translations is a normal subgroup of the Euclidean group, and also of the Poincaré group. Show that $(\mathrm{E}(n)/\text{translations}) \cong \mathrm{O}(n)$.

Solution. □

Exercise 16. *Harder.* Show that every Lie group homomorphism $\phi : \mathbb{R} \to S^1$ is of the form $\phi(x) = e^{iax}$ for some $a \in \mathbb{R}$. In particular, every such homomorphism is smooth.

Solution. □