

1. **Classify** by Jordan Canonical form (i.e. up to similarity and up to the order of the Jordan blocks) for all 6×6 matrices which have characteristic polynomial $(x - 2)^2(x + 3)^4$.

Possible Jordan blocks for $\lambda=2$ with multiplicity 2 are: $J_1=2$, $J_2=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Possible Jordan blocks for $\lambda=-3$ with multiplicity 4 are:

$$J_3=-3, \quad J_4=\begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}, \quad J_5=\begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}, \quad J_6=\begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

\Rightarrow Possible Jordan Canonical form of the given 6×6 matrix:

$$\begin{pmatrix} J_1 & & & & & \\ & J_1 & & & & \\ & & J_3 & & & \\ & & & J_3 & & \\ & & & & J_3 & \\ & & & & & J_3 \end{pmatrix}, \quad \begin{pmatrix} J_1 & & & & & \\ & J_1 & & & & \\ & & J_3 \rightarrow 1 \times 1 & & & \\ & & & J_3 \rightarrow 1 \times 1 & & \\ & & & & J_4 \rightarrow 1 \times 3 & \end{pmatrix}, \quad \begin{pmatrix} J_1 & & & & & \\ & J_1 & & & & \\ & & J_4 \rightarrow 2 \times 2 & & & \\ & & & J_4 \rightarrow 2 \times 2 & & \end{pmatrix}, \quad \begin{pmatrix} J_1 & & & & & \\ & J_1 & & & & \\ & & J_3 \rightarrow 1 \times 1 & & & \\ & & & J_5 \rightarrow 3 \times 3 & & \end{pmatrix}, \quad \begin{pmatrix} J_1 & & & & & \\ & J_1 & & & & \\ & & & & & 4 \times 4 = J_6 \end{pmatrix}$$

$$\begin{pmatrix} J_2 & & & & & \\ & J_3 & & & & \\ & & J_3 & & & \\ & & & J_3 & & \\ & & & & J_3 & \\ & & & & & J_3 \end{pmatrix}, \quad \begin{pmatrix} J_2 & & & & & \\ & J_3 & & & & \\ & & J_3 & & & \\ & & & J_3 & & \\ & & & & J_4 & \end{pmatrix}, \quad \begin{pmatrix} J_2 & & & & & \\ & & & & & \\ & & J_4 & & & \\ & & & J_4 & & \end{pmatrix}, \quad \begin{pmatrix} J_2 & & & & & \\ & J_3 & & & & \\ & & & & & \\ & & & J_5 & & \end{pmatrix}, \quad \begin{pmatrix} J_2 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & J_6 \end{pmatrix}$$

2. Find the Jordan canonical form of the matrix

$$A = \begin{pmatrix} a & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Step 1: Compute eigenvalue $\lambda = a$ with multiplicity n

Step 2: Compute eigenvectors: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ \rightarrow expect two cycles \leftrightarrow two Jordan blocks.
(linearly independent ones).

Step 3: Compute the two cycles.

1). For the initial vector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, Solve $(A - aI)\vec{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{v}_1$, we get $\vec{x} = \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow$ 3rd entry.

Solve $(A - aI)\vec{x} = \vec{v}_2$, we get $\vec{x} = \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow (2k-1)\text{-th entry}$

Follow the pattern: if $n = 2m$: $\vec{v}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \end{pmatrix} \rightarrow (2k-1)\text{-th entry}, k=1, 2, \dots, m$

if $n = 2m+1$: $\vec{v}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \end{pmatrix} \rightarrow (2k-1)\text{-th entry}, k=1, 2, \dots, m, m+1$

2). For the cycle with initial vector $\vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$: Solve $(A - aI)\vec{x} = \vec{w}_1 \Rightarrow$ we get $\vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

\vdots

Solve $(A - aI)\vec{x} = \vec{w}_{k-1} \Rightarrow$ we get $\vec{w}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$ \rightarrow $(2k)$ -th entry

If $n=2m$: $\vec{w}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow$ $(2k)$ -th entry, $k=1, \dots, m$

If $n=2m+1$: $\vec{w}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow$ $(2k)$ -th entry, $k=1, \dots, m$.

\Rightarrow If $n=2m$: $\rho = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_m)$, where \vec{v}_k and \vec{w}_k were specified above

$$J = \begin{pmatrix} J_1 & \\ & J_1 \end{pmatrix} \text{ where } J_1 = \begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & 1 \\ & & & a \end{pmatrix} \in \mathbb{R}^{m \times m}$$

$$\text{then } A = PJP^{-1}$$

If $n=2m+1$: $\rho = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{v}_{m+1}, \vec{w}_1, \dots, \vec{w}_m)$ where \vec{v}_k and \vec{w}_k were specified above

\downarrow \downarrow
 $k=1, \dots, m+1$ $k=1, \dots, m$

$$J = \begin{pmatrix} J_2 & \\ & J_1 \end{pmatrix} \text{ where } J_2 = \begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & 1 \\ & & & a \end{pmatrix} \in \mathbb{R}^{(m+1) \times (m+1)}$$

$$J_1 = \begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & 1 \\ & & & a \end{pmatrix} \in \mathbb{R}^{m \times m}$$

3. Let $A = \begin{bmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{bmatrix}$.

- 1). Find a basis for each generalized eigenspace of A consisting of a union of disjoint cycles of generalized eigenvectors.
- 2). Find a Jordan canonical form J of A using your basis in Part 1).
- 3). Find the minimal polynomial of A .

1) Step 1: Find eigenvalues and eigenvectors of A :

$$\lambda = 2 \text{ (multiplicity 2)}, \quad \lambda = -1 \text{ (multiplicity 1)}$$

$$\text{Solve } (A - 2I)\vec{x} = \vec{0} : \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad E_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Solve } (A + I)\vec{x} = \vec{0} : \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad E_{-1} = \text{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

Step 2: Two cycles: ①. For the cycle w/ initial vector \vec{v}_1 , the length should be 2.

$$\text{Solve } (A - 2I)\vec{x} = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ we get } \vec{x} = \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{A cycle corresponding to } \lambda = 2 \text{ is: } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

② For the cycle w/ initial vector \vec{w}_1 , the length should be 1

$$\Rightarrow \text{A cycle corresponding to } \lambda = -1 \text{ is: } \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

A basis of generalized eigenvectors consisting of disjoint cycles is given by: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$

$$2). \quad J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix} \text{ s.t. } A = PJP^{-1}$$

$$3). \quad \text{minimal polynomial: } m(x) = (x-2)^2(x+1).$$

4. Let D be the differential linear operator on the vector space $V = \text{span}\{1, t, t^2, e^t, te^t\}$, i.e.

$$D(f(x)) = \frac{d}{dx}(f(x)), \quad \text{for any } f(x) \in V.$$

- 1). Let $\mathcal{B} = \{1, t, t^2, e^t, te^t\}$. Find the matrix representation $A = [D]_{\mathcal{B}}$.
- 2). Find a basis for each generalized eigenspace of D consisting of a union of disjoint cycles of generalized eigenvectors.
- 3). Find a Jordan canonical form J of D using your basis in Part 2).
- 4). Find the minimal polynomial of D .

(Sketch of solutions, detailed computation work was not included)

$$1) \quad [D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \text{(Need to show more work!)}$$

$$2) \quad \text{Eigenvalues: } \lambda_1 = 0 \quad (\text{multiplicity} = 3) \quad \text{Solve } A\vec{x} = \vec{0} \Rightarrow \vec{v}_i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow E_0 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

eigenvectors

$$\text{Eigenvalues } \lambda_2 = 1 \quad (\text{multiplicity} = 2) \quad \text{Solve } (A - I)\vec{x} = \vec{0} \Rightarrow \vec{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow E_1 = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

For $\lambda = 0$: one cycle corresponding to $\lambda = 0$ with length 3, and the Jordan block should be a 3×3 block.

$$\text{Solve } (A - 0I)\vec{x} = \vec{v}_1 \Rightarrow \vec{x} = \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Solve } (A - 0I)\vec{x} = \vec{v}_2 \Rightarrow \vec{x} = \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\text{the Cycles corresponding to } \lambda = 0 : \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right\}$$

For $\lambda = 1$: one cycle corresponding to $\lambda = 1$ with length 2 and the Jordan block should be a 2×2 block.

$$\text{Solve } (A - I)\vec{x} = \vec{w}_1 : \vec{x} \stackrel{4 \text{ of } 8}{=} \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The cycle corresponding to $\lambda=1$: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

3). $[D]_B = PJP^{-1}$ where $J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4). minimal polynomial for D : $m(x) = x^3(x-1)^2$

5. Let $A \in \mathbb{C}^{n \times n}$. Assume $(A + I_n)^m = 0$. Prove that A is invertible and find $\det(A)$.

Proof: Let $g(x) = (x + 1)^m$. Since $g(A) = (A + I)^m = 0$, $g(x)$ annihilate A . Let $m(x)$ be the minimal polynomial and let $f(x)$ be the characteristic polynomial of A . Then $m(x)|g(x)$. Thus $m(x) = (x + 1)^m$ for some $m \leq n$. Thus the only root $m(x)$ has is -1 . Since $m(x)$ and $f(x)$ have the same roots (may not have the same multiplicity for each root), thus $f(x)$ also has only -1 as its roots. Thus $f(x) = (x + 1)^n$. Therefore all the eigenvalue of A are -1 . Thus $\det(A) = (-1)^n \neq 0$. Thus A is invertible.

6. Let V be the vector space of all polynomials over \mathbb{R} . Let D be the differentiation operator on the vector space V . Find the minimal polynomial of D on V or prove that D has no minimal polynomial on V .

Answer: D does not have a minimal polynomial on V .

Proof by contradiction. Suppose towards a contradiction that $m(x)$ is the minimal polynomial of D . Then

$$m(x) = x^m + a_{m-1}x^{m-1} + \dots + a_1x + a_0 \quad \text{where } m \in \mathbb{Z}^+.$$

$$\text{Then } m(D) = D^m + a_{m-1}D^{m-1} + \dots + a_1D + a_0I$$

$$\text{i.e. } \forall p(x) \in V: m(D)(p(x)) = 0.$$

On the other hand: let $p_0(t) = t^m$. Then $D(p_0(x)) = mt^{m-1}$

\vdots

$$D^k(p_0(x)) = m(m-1)\dots(m-k+1)t^{m-k}$$

\vdots

$$D^m(p_0(x)) = m!$$

$$\Rightarrow 0 = m(D)(p_0(t)) = D^m(p_0(t)) + a_{m-1}D^{m-1}(p_0(t)) + \dots + a_1D(p_0(t)) + a_0p_0(t)$$

$$= m! + a_{m-1}(m \dots 2)t + a_{m-2}(m \dots 3)t^2 + \dots + a_1mt^{m-1} + a_0t^m$$

$$\neq 0 \quad \text{b/c the constant term is } m! \neq 0$$

Contradiction !

7. Suppose $A \in \mathbb{C}^{n \times n}$ satisfies $A^2 = A$. Prove that A is diagonalizable.

Let $g(x) = x^2 - x = x(x-1)$. Then $g(A) = A^2 - A = 0$. $\Rightarrow g(x)$ annihilates A .

Let $m(x)$ be the minimal polynomial of A . then $m(x) \mid g(x)$

$\Rightarrow m(x) = x$ or $m(x) = x-1$ or $m(x) = x(x-1)$

In any of the above three cases, $m(x)$ has no repeated roots.

$\Rightarrow A$ is diagonalizable. (Note: This is using the result of: A is diagonalizable if and only if its minimal polynomial has no repeated roots).

8. Give an example of two matrices who have the same characteristic polynomial but distinct minimal polynomials.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ,$$

Characteristic polynomials: $f_A(x) = f_B(x) = (x-1)^2$

minimal polynomials: A: $m_A(x) = (x-1)^2$

B: $m_B(x) = (x-1)$.

(You may use many of other examples)