Several-Variab Calc II: Homework 8

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Problem 1. Evaluate $\iint_S x \, \mathrm{d}S$ where S is the portion of the plane $4x + 2y + z = 8$ in octant one.

Solution. Solving for z, we get $z = 8 - 4x - 2y$. The region lies in the first octant, meaning $x \geq 0$, $y \geq 0$, and $z \geq 0$. Finding the intersection points with each axis, we get

$$\begin{aligned} x = 0 = y &\Rightarrow z = 8 \\ x = 0 = z &\Rightarrow y = 4 \\ y = 0 = z &\Rightarrow x = 2. \end{aligned}$$

Thus, the region is a triangle with vertices at $(0, 0, 8)$, $(0, 4, 0)$, and $(2, 0, 0)$. The normal vector to the plane is $\mathbf{N} = \nabla f = \langle 4, 2, 1 \rangle$. Therefore, the surface element is given by

$$\left| \frac{\partial z}{\partial(x, y)} \right| \mathrm{d}x \, \mathrm{d}y = \sqrt{1^2 + 2^2 + 4^2} \, \mathrm{d}x \, \mathrm{d}y = \sqrt{21} \, \mathrm{d}x \, \mathrm{d}y.$$

Therefore, we get

$$\mathrm{d}S = \sqrt{21} \, \mathrm{d}x \, \mathrm{d}y.$$

The equation of the line connecting $(2, 0)$ and $(0, 4)$ is $y = -2x + 4$. Therefore, the bounds of integration are $0 \leq x \leq 2$ and $0 \leq y \leq -2x + 4$. Thus, we get

$$\begin{aligned} \iint_S x \, \mathrm{d}S &= \int_0^2 \int_0^{-2x+4} x\sqrt{21} \, \mathrm{d}y \, \mathrm{d}x \\ &= \sqrt{21} \cdot \int_0^2 xy \Big|_0^{-2x+4} \mathrm{d}x \\ &= \sqrt{21} \cdot \int_0^2 -2x^2 + 4x \, \mathrm{d}x \\ &= \sqrt{21} \cdot \left(-\frac{2}{3}x^3 + 2x^2 \right) \Big|_0^2 \\ &= \sqrt{21} \left(-\frac{16}{3} + 8 \right) = \frac{8\sqrt{21}}{3}. \end{aligned}$$

□

Problem 2. Evaluate $\iint_S x^2 + y^2 + z^2 \, \mathrm{d}S$ where S is the boundary surface of the solid above the plane $z = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

Solution. In spherical coordinates, we have $x = 2\sin(\theta)\cos(\varphi)$, $y = 2\sin(\theta)\sin(\varphi)$, and $z = 2\cos(\theta)$. The cap is bounded by $z \geq 1$, or $2\cos(\theta) \geq 1 \Rightarrow \cos(\varphi) \geq 1/2$. This gives us the bounds $0 \leq \theta \leq \pi/3$. The surface element is given by

$$\mathrm{d}S = 4\sin(\theta) \, \mathrm{d}\theta \, \mathrm{d}\varphi.$$

On the spherical cap, we have

$$\begin{aligned} I_1 = \iint x^2 + y^2 + z^2 \, \mathrm{d}S &= \int_0^{2\pi} \int_0^{\pi/3} 4 \cdot 4\sin(\theta) \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ &= 16 \cdot \int_0^{2\pi} \mathrm{d}\varphi \cdot \int_0^{\pi/3} \sin(\theta) \, \mathrm{d}\theta \\ &= 16 \cdot 2\pi \cdot \cos(\theta) \Big|_{\pi/3}^{0} \\ &= 32\pi \cdot \left(\cos(0) - \cos\left(\frac{\pi}{3} \right) \right) \end{aligned}$$

$$= 32\pi \cdot \left(-\frac{1}{2} + 1\right) = 16\pi.$$

On the disk, $x^2 + y^2 + z^2 = 4$ and $z = 1$, we have $x^2 + y^2 = 3$. Using polar coordinates, $x = \sqrt{3}\cos(\varphi)$ and $y = \sqrt{3}\sin(\varphi)$, we get the bounds $0 \leq \varphi \leq 2\pi$ and $0 \leq r \leq \sqrt{3}$. The surface

$$\mathrm{d}S = r \,\mathrm{d}r \,\mathrm{d}\varphi.$$

Therefore, we get

$$\begin{aligned}
I_2 = \iint x^2 + y^2 + z^2 \,\mathrm{d}S &= \int_0^{2\pi} \int_0^{\sqrt{3}} (r^2 + 1) \cdot r \,\mathrm{d}r \,\mathrm{d}\varphi \\
&= \int_0^{2\pi} \mathrm{d}\varphi \cdot \int_0^{\sqrt{3}} r^3 + r \,\mathrm{d}r \\
&= 2\pi \cdot \left(\frac{r^4}{4} + \frac{r^2}{2}\right)_0^{\sqrt{3}} \\
&= 2\pi \cdot \left(\frac{9}{4} + \frac{3}{2}\right) = \frac{15\pi}{2}.
\end{aligned}$$

Therefore, the total integral is

$$I = I_1 + I_2 = 16\pi + \frac{15\pi}{2} = \frac{47\pi}{2}.$$ □

Problem 3. A funnel has the shape of the cone $z = \sqrt{x^2 + y^2}$ for $1 \leq z \leq 3$. The mass density of the funnel is $\rho(x, y, z) = 5 - z$. Find the mass of the funnel.

Solution. Rewriting in cylindrical coordinates, we have $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = r$. For $1 \leq z \leq 3$, we have $1 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. The normal vector is given by the gradient of the function $F(x, y, z) = z - \sqrt{x^2 + y^2}$,

$$\nabla F = \left\langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}, 1 \right\rangle.$$

It's magnitude is given by

$$\begin{aligned}
|\nabla F| &= \sqrt{\left(-\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(-\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1^2} \\
&= \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + 1} = \sqrt{2}.
\end{aligned}$$

Therefore, the surface element is given by

$$\mathrm{d}S = |\nabla F| \,\mathrm{d}r \,\mathrm{d}\theta = \sqrt{2} \,\mathrm{d}r \,\mathrm{d}\theta.$$

The mass is given by

$$\begin{aligned}
M = \iint_S \rho \,\mathrm{d}S &= \int_0^{2\pi} \int_1^3 (5 - r) \cdot r\sqrt{2} \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \sqrt{2} \cdot 2\pi \cdot \int_1^3 5r - r^2 \,\mathrm{d}r \\
&= 2\sqrt{2}\pi \cdot \left(\frac{5r^2}{2} - \frac{r^3}{3}\right)_1^3
\end{aligned}$$

$$= 2\sqrt{2}\pi \cdot \left(\frac{27}{2} - \frac{13}{6}\right)$$
$$= 2\sqrt{2}\pi \cdot \frac{34\sqrt{2}}{3} = \frac{68\sqrt{2}\pi}{3}. \qquad \square$$

Problem 4. Evaluate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where $\mathbf{F} = \langle x - 2y, 2z + 8x, y + z \rangle$ downward across the portion of the plane $4x + y + z = 12$ that lies in the first octant.

Solution. The normal vector to the plane is $\mathbf{N} = \nabla f = \langle 4, 1, 1 \rangle$. The region lies in the first octant, meaning $x \geq 0$, $y \geq 0$, and $z \geq 0$. Finding the intersection points with each axis, we get

$$x = 0 = y \Rightarrow z = 12$$
$$x = 0 = z \Rightarrow y = 12$$
$$y = 0 = z \Rightarrow x = 3.$$

Rewriting the plane equation, we get $z = 12 - 4x - y$. The normal vector to the plane is $\mathbf{N} = \nabla f = \langle 4, 1, 1 \rangle$. Since we want a downward orientation, we check the z-component of ∇F, Since it is positive, the normal vector is already pointing upward. To orient it downward, we take the negative of the normal vector, $\mathbf{N} = \langle -4, -1, -1 \rangle$. Therefore, the surface element is given by

$$\mathrm{d}\mathbf{S} = \langle -4, -1, -1 \rangle \, \mathrm{d}A.$$

Computing the dot product, we get

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (x - 2y)(-4) + (2z + 8x)(-1) + (y + z)(-1) \, \mathrm{d}A = 10y - 36 \, \mathrm{d}A.$$

The equation of the line connecting $(3, 0)$ and $(0, 12)$ is $y = -4x + 12$. Therefore, the bounds of integration are $0 \leq x \leq 3$ and $0 \leq y \leq -4x + 12$. Thus, we get

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_0^3 \int_0^{-4x+12} 10y - 36 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^3 5y^2 - 36y \Big|_0^{-4x+12} \mathrm{d}x \\
&= \int_0^3 5(-4x + 12)^2 - 36(-4x + 12) \, \mathrm{d}x \\
&= \int_0^3 5(16x^2 - 96x + 144) - 432 + 144x \, \mathrm{d}x \\
&= \int_0^3 80x^2 - 336x + 288 \, \mathrm{d}x \\
&= \frac{80}{3}x^3 - 168x^2 + 288x \Big|_0^3 = 72.
\end{aligned} \qquad \square$$

Problem 5. Evaluate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where $\mathbf{F} = \langle xz, x, y \rangle$ and S is the hemisphere $x^2 + y^2 + z^2 = 9$ with $y \geq 0$ oriented in the direction of the positive y-axis.

Solution. The hemisphere $x^2 + y^2 + z^2 = 9$ can be described using spherical coordinates

$$x = 3\sin(\varphi)\cos(\theta), \quad y = 3\sin(\varphi)\sin(\theta), \quad \text{and} \quad z = 3\cos(\varphi).$$

where $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq 2\pi$. This gives us the parameterization

$$\mathbf{r}(\theta, \varphi) = \langle 3\sin(\varphi)\cos(\theta), 3\sin(\varphi)\sin(\theta), 3\cos(\varphi) \rangle.$$

We know that

$$\begin{aligned}
\mathbf{r}_\varphi &= \langle 3\cos(\varphi)\cos(\theta), 3\cos(\varphi)\sin(\theta), -3\sin(\varphi)\rangle \\
\mathbf{r}_\theta &= \langle -3\sin(\varphi)\sin(\theta), 3\sin(\varphi)\cos(\theta), 0\rangle.
\end{aligned}$$

The normal vector is given by

$$\begin{aligned}
\mathbf{n} = \mathbf{r}_\varphi \times \mathbf{r}_\theta &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 3\cos(\varphi)\cos(\theta) & 3\cos(\varphi)\sin(\theta) & -3\sin(\varphi) \\ -3\sin(\varphi)\sin(\theta) & 3\sin(\varphi)\cos(\theta) & 0 \end{vmatrix} \\
&= \langle 9\sin^2(\varphi)\cos(\theta), 9\sin^2(\varphi)\sin(\theta), 9\cos(\varphi)\sin(\varphi)\cos^2(\theta) + 9\cos(\varphi)\sin(\varphi)\sin^2(\theta)\rangle \\
&= \langle 9\sin^2(\varphi)\cos(\theta), 9\sin^2(\varphi)\sin(\theta), 9\cos(\varphi)\sin(\varphi)\rangle.
\end{aligned}$$

Since the y-component is positive, we have the normal vector pointing in the positive y-direction. Taking the dot product with $\mathbf{F}$, we get

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{n} &= xz\cdot 9\sin^2(\varphi)\cos(\theta) + x\cdot 9\sin^2(\varphi)\sin(\theta) + y\cdot 9\cos(\varphi)\sin(\varphi) \\
&= (3\sin(\varphi)\cos(\theta))(3\cos(\varphi))(9\sin^2(\varphi)\cos(\theta)) \\
&\quad + (3\sin(\varphi)\cos(\theta))(9\sin^2(\varphi)\sin(\theta)) \\
&\quad + (3\sin(\varphi)\sin(\theta))(9\cos(\varphi)\sin(\varphi)) \\
&= 81\cos^2(\theta)\cos(\varphi)\sin^3(\varphi) + 27\cos(\theta)\sin(\theta)\sin^3(\varphi) + 27\sin(\theta)\cos(\varphi)\sin^2(\varphi).
\end{aligned}$$

Therefore, the integral is given by

$$\begin{aligned}
\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \int_0^\pi\int_0^{2\pi} 81\cos^2(\theta)\cos(\varphi)\sin^3(\varphi) + 27\cos(\theta)\sin(\theta)\sin^3(\varphi) + 27\sin(\theta)\cos(\varphi)\sin^2(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \int_0^\pi\int_0^{2\pi} 81\cos^2(\theta)\cos(\varphi)\sin^3(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad + \int_0^\pi\int_0^{2\pi} 27\cos(\theta)\sin(\theta)\sin^3(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad + \int_0^\pi\int_0^{2\pi} 27\sin(\theta)\cos(\varphi)\sin^2(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi.
\end{aligned}$$

The first integral is zero since

$$\begin{aligned}
\int_0^\pi \cos(\varphi)\sin^3(\varphi)\,\mathrm{d}\varphi &= \int_0^\pi \cos(\varphi)(1-\cos^2(\varphi))\sin(\varphi)\,\mathrm{d}\varphi \\
&= \int_1^{-1} -u(1-u^2)\,\mathrm{d}u \\
&= \frac{u^2}{2} - \frac{u^4}{4}\bigg|_1^{-1} = 0.
\end{aligned}$$

The second is also zero since it's integrating over a full period of $\sin(\theta)$ and $\cos(\theta)$. Lastly, the third integral is also zero since it's integrating over a full period of $\sin(\theta)$. Therefore, we get

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = 0. \qquad \square$$