Differential Geometry I: Homework 1

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Exercise 1.1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the ***line with two origins***.)

Solution. Define

$$X = \{(x, 1) \mid x \in \mathbb{R}\} \cup \{(x, -1) \mid x \in \mathbb{R}\} \cong \mathbb{R} \times \{\pm 1\},$$

with the subspace topology from $\mathbb{R}^2$. Let $q : X \to M$ be the quotient map under the equivalence relation $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Denote $0_+ = q(0, 1)$ and $0_- = q(0, -1)$.

Let $p \in M$. If $p = q(x_0, y_0)$ with $x_0 \neq 0$, then there exists $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon)$ does not contain 0. Define

$$U = q((x_0 - \varepsilon, x_0 + \varepsilon) \times \{1, -1\}).$$

The map $\Phi : U \to (x_0 - \varepsilon, x_0 + \varepsilon)$ defined by $\Phi(q(x, y)) = x$ is well-defined, bijective, continuous, and its inverse $\Phi^{-1}(x) = q(x, 1)$ is continuous. Hence U is homeomorphic to an open interval in $\mathbb{R}$.

But if $p = 0_+$, choose $\varepsilon > 0$ and define

$$U_+ = q((-\varepsilon, \varepsilon) \times \{1\}) = \{0_+\} \cup \{q(x, 1) \mid 0 < |x| < \varepsilon\}.$$

Define $\Phi_+ : U_+ \to (-\varepsilon, \varepsilon)$ by

$$\Phi_+(0_+) = 0, \quad \Phi_+(q(x, 1)) = x \text{ for } x \neq 0.$$

Then Φ_+ is bijective, continuous, and has a continuous inverse $\Phi_+^{-1}(x) = q(x, 1)$. Thus, U_+ is homeomorphic to an open interval. The same argument applies for $p = 0_-$ using

$$U_- = q((-\varepsilon, \varepsilon) \times \{-1\}), \quad \Phi_-(0_-) = 0, \ \Phi_-(q(x, -1)) = x.$$

Therefore, every point of M has a neighborhood homeomorphic to an open subset of $\mathbb{R}$, and hence M is locally Euclidean of dimension 1.

Now, we show that M is second-countable. For each $a \in \mathbb{Q}$ and $r \in \mathbb{Q}^+$, define

$$U_{a,r}^+ = q((a - r, a + r) \times \{1\}), \quad U_{a,r}^- = q((a - r, a + r) \times \{-1\}),$$

and for the two origins, define for each $\varepsilon \in \mathbb{Q}^+$,

$$U_+(\varepsilon) = q((-\varepsilon, \varepsilon) \times \{1\}), \quad U_-(\varepsilon) = q((-\varepsilon, \varepsilon) \times \{-1\}).$$

Let

$$\mathcal{B} = \{U_{a,r}^+, U_{a,r}^- \mid a \in \mathbb{Q}, r \in \mathbb{Q}^+\} \cup \{U_+(\varepsilon), U_-(\varepsilon) \mid \varepsilon \in \mathbb{Q}^+\}.$$

Since $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable, $\mathcal{B}$ is countable. Each $B \in \mathcal{B}$ is open in M because $q^{-1}(B)$ is open in X. To see that $\mathcal{B}$ is a basis, take any open $O \subset M$ and any $p \in O$. Then $q^{-1}(O)$ is open in X, so there exists an open interval $(a - r, a + r)$ (with $a, r \in \mathbb{Q}$) such that $q((a - r, a + r) \times \{y_0\}) \subset O$. Hence O is a union of elements of $\mathcal{B}$, and $\mathcal{B}$ is a countable basis. Thus, M is second-countable.

Lastly, we show that M is not Hausdorff. Suppose U and V are disjoint open neighborhoods of 0_+ and 0_-, respectively. Then $q^{-1}(U)$ is an open subset of X containing $(0, 1)$, so there exists $\delta > 0$ such that $(-\delta, \delta) \times \{1\} \subset q^{-1}(U)$. Similarly, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times \{-1\} \subset q^{-1}(V)$. Let $0 < |x| < \min(\delta, \varepsilon)$. Then

$$q(x, 1) = q(x, -1) \in U \cap V,$$

contradicting that U and V are disjoint. Hence, no disjoint open neighborhoods can separate 0_+ and 0_-, and M fails to be Hausdorff.

Therefore, the space M is locally Euclidean and second-countable, but not Hausdorff. $\square$

Exercise 1.2. Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Solution. Define the space

$$X = \bigsqcup_{\alpha \in A} \mathbb{R}_\alpha,$$

where A is an uncountable index set and each $\mathbb{R}_\alpha$ is a copy of the real line. The topology on X is the disjoint union topology, where a set $U \subseteq X$ is open if and only if $U \cap \mathbb{R}_\alpha$ is open in $\mathbb{R}_\alpha$ for each $\alpha \in A$. To show that X is locally Euclidean, take any point $p \in X$. Then, there exists a unique $\alpha_0 \in A$ such that $p \in \mathbb{R}_{\alpha_0}$. Notice that each $\mathbb{R}_\alpha$ is open in X, since for any open set $U \subseteq \mathbb{R}_\alpha$, we have $U = U \cap \mathbb{R}_\alpha$ and $U \cap \mathbb{R}_\beta = \emptyset$ for all $\beta \neq \alpha$. Thus, U is open in X by the definition of the disjoint union topology. Therefore, $\mathbb{R}_{\alpha_0}$ is an open neighborhood of p in X. Since there exists a homeomorphism

$$\Phi : \mathbb{R}_{\alpha_0} \to \mathbb{R}, \quad \Phi(x) = x,$$

the set $\mathbb{R}_{\alpha_0}$ is homeomorphic to $\mathbb{R}$. Hence, X is locally homeomorphic to $\mathbb{R}$. Therefore, X is locally Euclidean of dimension 1.

To show that X is Hausdorff, take any two distinct points $p, q \in X$. Then, there exist unique $\alpha_1, \alpha_2 \in A$ such that $p \in \mathbb{R}_{\alpha_1}$ and $q \in \mathbb{R}_{\alpha_2}$. If $\alpha_1 \neq \alpha_2$, then $\mathbb{R}_{\alpha_1}$ and $\mathbb{R}_{\alpha_2}$ are disjoint open neighborhoods of p and q, respectively. If $\alpha_1 = \alpha_2$, then $p, q \in \mathbb{R}_{\alpha_1}$. Since $\mathbb{R}_{\alpha_1}$ is homeomorphic to $\mathbb{R}$, which is Hausdorff, there exist disjoint open neighborhoods U_p and U_q of p and q in $\mathbb{R}_{\alpha_1}$. Since $\mathbb{R}_{\alpha_1}$ is open in X, both U_p and U_q are open in X. Thus, in either case, we can find disjoint open neighborhoods of any two distinct points in X. Therefore, X is Hausdorff.

Finally, we show that X is not second-countable. Suppose for contradiction that X has a countable basis $\mathcal{B} = \{B_i\}_{i \in I}$, where I is a countable index. That means, for every $\alpha \in A$, if $p_\alpha \in \mathbb{R}_\alpha$, then there exists $i \in I$ such that $p_\alpha \in B_i$ and $B_i \subseteq \mathbb{R}_\alpha$. But since A is uncountable and I is countable, by the pigeonhole principle, there exists some $i_0 \in I$ such that $B_{i_0} \subseteq \mathbb{R}_\alpha$ for uncountably many $\alpha \in A$. This is a contradiction since B_{i_0} can only be a subset of one $\mathbb{R}_\alpha$. Therefore, X cannot have a countable basis, and hence is not second-countable.

In conclusion, the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable. $\square$

Exercise 1.3. A topological manifold is said to be ***σ-compact*** if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

Solution. Assume M is a locally Euclidean Hausdorff topological manifold. Then, there exists a countable amount of charts $\{(U_\alpha, \Phi_\alpha)\}_{\alpha=1}$ such that

$$M = \bigcup_{\alpha=1}^{\infty} U_\alpha,$$

where each $\Phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$ is a homeomorphism, and U_α is an open subset of M. Since M is Hausdorff, for any point $p_{\alpha_1} \in U_\alpha$, there exists an open ball $B_r(p)$ that's fully contained in U_α such that $\overline{B_r(p_{\alpha_1})}$ is compact. Continuing this process, we can find a countable collection of open balls $\{B_{r_i}(p_{\alpha_i})\}_{i=1}^{\infty}$ such that

$$U_\alpha = \bigcup_{i=1}^{\infty} B_{r_i}(p_{\alpha_i}).$$

We do this for every α, and we get a countable collection of open balls $\{B_{r_i}(p_{\alpha_i})\}_{i=1}^{\infty}$ such that

$$M = \bigcup_{\alpha=1}^{\infty} \bigcup_{i=1}^{\infty} B_{r_i}(p_{\alpha_i}).$$

Since each $\overline{B_{r_i}(p_{\alpha_i})}$ is compact, M is the union of countably many compact subsets. Thus, M is σ-compact.

Now, for the converse, suppose M is a locally Euclidean Hausdorff space that is σ-compact. Then, there exists a countable collection of compact subsets $\{K_n\}_{n=1}^{\infty}$ such that

$$M = \bigcup_{n=1}^{\infty} K_n.$$

To show that M is second-countable, we will construct a countable basis for its topology. Since each K_n is compact, there exists a finite collection of charts $\{(U_{\alpha_1}, \Phi_{\alpha_1}), \cdots, (U_{\alpha_{m_n}}, \Phi_{\alpha_{m_n}})\}$ such that for every $i \leq n$, $\Phi_{\alpha_i} : U_{\alpha_i} \subset M \to \mathbb{R}^n$ is a homeomorphism. Define B_{α_i} to be the collection of the pre-image of all countably many open balls with rational radii and centers with rational coordinates in $\mathbb{R}^n$. Then, the collection

$$\mathcal{B}_n = \bigcup_{i=1}^{m_n} \{\Phi_{\alpha_i}^{-1}(B) \mid B \in B_{\alpha_i}\},$$

is a countable basis for the subspace topology on K_n. Since M is the union of the K_n, the collection

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n.$$

is a countable basis for the topology on M. Thus, M is second-countable.

Therefore, a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact. $\square$

Exercise 1.7. Let N denote the ***north pole*** $(0, \cdots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the ***south pole*** $(0, \cdots, 0, -1)$. Define the ***stereographic projection*** $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \cdots, x^{n+1}) = \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma} = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(i) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called ***stereographic projection from the south pole***.)

(ii) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u^2| + 1}.$$

(iii) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of two charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \sigma)$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called ***stereographic coordinates***.)

(iv) Show that this smooth structure is the same as the one defined in Example 1.31.

Solution to (i). Let $x = (x^1, \cdots, x^n, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$. The line from N to x can be written as $\ell(t) = N + t(x - N)$. We seek the point $\ell(t)$ whose last coordinate is zero, i.e., the intersection with the hyperplane $x^{n+1} = 0$. Setting the last coordinate of $\ell(t)$ to zero gives:

$$1 + t(x^{n+1} - 1) = 0 \Rightarrow t = \frac{1}{1 - x^{n+1}}.$$

Substituting this back into the first n coordinates of $\ell(t)$, we have:

$$\ell(t) = \left(\frac{x^1}{1 - x^{n+1}}, \cdots, \frac{x^n}{1 - x^{n+1}}, 0\right) = \frac{(x^1, \cdots, x^n, 0)}{1 - x^{n+1}} = (\sigma(x), 0).$$

Thus, $\sigma(x) = u$ where $(u, 0)$ is the intersection point.

To find $\widetilde{\sigma}(x)$, note that it is defined by $\widetilde{\sigma}(x) = -\sigma(-x)$. Applying σ to the antipodal point $-x$ gives

$$\sigma(-x) = \frac{(-x^1, \cdots, -x^n)}{1 + x^{n+1}} = -\frac{(x^1, \cdots, x^n)}{1 + x^{n+1}}.$$

But $\widetilde{\sigma}(x) = -\sigma(-x)$, giving us

$$\widetilde{\sigma}(x) = -\sigma(-x) = -\left(-\frac{(x^1, \cdots, x^n)}{1 + x^{n+1}}\right) = \frac{(x^1, \cdots, x^n)}{1 + x^{n+1}}.$$

Therefore, $\widetilde{\sigma}(x) = u$ where $(u, 0)$ is the intersection of the line through S and x with the hyperplane $x^{n+1} = 0$. ☐

Solution to (ii). If σ is a bijection, there its inverse σ^{-1} is well-defined. Solving for each coordinate of u, we get

$$u^i = \frac{x^i}{1 - x^{n+1}}.$$

Since the radius of $\mathbb{S}^n$ is 1, we have

$$\begin{aligned}
& \sum_{i=1}^{n} (x^i)^2 + \left(x^{n+1}\right)^2 && = 1 \\
\Rightarrow \quad & \sum_{i=1}^{n} (u^i)^2 \left(1 - x^{n+1}\right)^2 + \left(x^{n+1}\right)^2 && = 1 \\
\Rightarrow \quad & \left(1 - x^{n+1}\right)^2 \sum_{i=1}^{n} (u^i)^2 + \left(x^{n+1}\right)^2 && = 1 \\
\Rightarrow \quad & \left(1 - x^{n+1}\right)^2 |u|^2 + \left(x^{n+1}\right)^2 && = 1 \\
\Rightarrow \quad & |u|^2\left(1 - 2x^{n+1} + (x^{n+1})^2\right) + \left(x^{n+1}\right)^2 && = 1 \\
\Rightarrow \quad & |u|^2 - 2|u|^2 x^{n+1} + |u|^2\left(x^{n+1}\right)^2 + \left(x^{n+1}\right)^2 && = 1 \\
\Rightarrow \quad & \left(|u|^2 + 1\right)\left(x^{n+1}\right)^2 - 2|u|^2 x^{n+1} + (|u|^2 - 1) && = 0.
\end{aligned}$$

This is a quadratic equation in x^{n+1}. From the equation, we know $A = |u|^2 + 1$, $B = -2|u|^2$, and $C = |u|^2 - 1$. There are two solutions to this equation, but we want the one less than 1 since $x^{n+1} \neq 1$ (as N is not in the domain of σ). Let $r^2 = |u|^2$ and $t = x^{n+1}$. From this, we get the following quadratic

$$(r^2 + 1)t^2 - 2r^2 t + (r^2 - 1) = 0.$$

Using the quadratic formula, we find

$$\begin{aligned}
t &= \frac{2r^2 \pm \sqrt{(-2r^2)^2 - 4(r^2 + 1)(r^2 - 1)}}{2(r^2 + 1)} \\
&= \frac{2r^2 \pm \sqrt{4r^4 - 4(r^4 - 1)}}{2(r^2 + 1)} \\
&= \frac{2r^2 \pm 2}{2(r^2 + 1)} \\
&= \frac{r^2 \pm 1}{r^2 + 1}.
\end{aligned}$$

Therefore, the two roots are

$$x^{n+1} = 1 \quad \text{and} \quad x^{n+1} = \frac{|u|^2 - 1}{|u|^2 + 1}.$$

Taking the second root, we can substitute back to find x^i for $i = 1, \cdots, n$:

$$1 - x^{n+1} = 1 - \frac{|u|^2-1}{|u|^2+1} = \frac{2}{|u|^2+1}.$$

Plugging that into the equation for x^i, we get

$$x^i = u^i\left(1 - x^{n+1}\right) = u^i \frac{2}{|u|^2+1} = \frac{2u^i}{|u|^2+1}.$$

Thus, we have

$$\sigma^{-1}(u^1, \cdots, u^n) = \left(\frac{2u^1}{|u|^2+1}, \cdots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right) = \frac{(2u^1, \cdots, 2u^n, |u|^2-1)}{|u|^2+1}.$$

Since σ^{-1} exists and is well-defined, σ is bijective. □

Solution to (iii). We compute the transition map

$$\Phi = \sigma \circ \widetilde{\sigma}^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}.$$

Let $u = (u^1, \cdots, u^n) \in \mathbb{R}^n \setminus \{0\}$ and set $r^2 = |u|^2$. Recall that

$$\widetilde{\sigma}^{-1}(u) = \left(\frac{2u}{1 + r^2}, \frac{1 - r^2}{1 + r^2}\right).$$

Applying σ to $\widetilde{\sigma}^{-1}(u)$ gives

$$\Phi(u) = \sigma(\widetilde{\sigma}^{-1}(u)) = \frac{\frac{2u}{1 + r^2}}{1 - \frac{1 - r^2}{1 + r^2}} = \frac{\frac{2u}{1 + r^2}}{\frac{2r^2}{1 + r^2}} = \frac{u}{r^2} = \frac{u}{|u|^2}.$$

Therefore, $\Phi(u)$ is the *inversion in the unit sphere*. Note that this map is its own inverse and smooth on $\mathbb{R}^n \setminus \{0\}$, so the stereographic charts σ and $\widetilde{\sigma}$ are smoothly compatible. □

Solution to (iv). Recall that in Example 1.31, the smooth structure on $\mathbb{S}^n$ was defined using the charts

$$\varphi_i^{\pm} : U_i^{\pm} \cap \mathbb{S}^n \to \mathbb{B}^n, \quad \varphi_i^{\pm}(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}),$$

where $U_i^{\pm} = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \pm x^i > 0\}$. Each $\varphi_i^{\pm}$ is a homeomorphism with smooth inverse

$$(\varphi_i^{\pm})^{-1}(u^1, \ldots, u^n) = (u^1, \ldots, u^{i-1}, \pm\sqrt{1 - |u|^2}, u^i, \ldots, u^n),$$

so that $\{(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})\}$ defines the standard smooth structure on $\mathbb{S}^n$.

Now, the smooth structure defined by the stereographic projections

$$\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad \widetilde{\sigma} : \mathbb{S}^n \setminus \{S\} \to \mathbb{R}^n,$$

is generated by the charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma})$. To show that this smooth structure coincides with the standard one, it suffices to check that the transition maps between these charts and the $\varphi_i^{\pm}$ are smooth.

For instance, consider $\sigma \circ (\varphi_{n+1}^{+})^{-1}$. For $u \in \mathbb{B}^n$,

$$(\varphi_{n+1}^{+})^{-1}(u) = (u^1, \ldots, u^n, \sqrt{1 - |u|^2}),$$

and thus

$$\sigma((\varphi_{n+1}^{+})^{-1}(u)) = \frac{(u^1, \ldots, u^n)}{1 - \sqrt{1 - |u|^2}},$$

which is a smooth map on $\mathbb{B}^n \setminus \{0\}$. A similar computation shows that $\varphi_{n+1}^{+} \circ \sigma^{-1}$ and the corresponding maps for $\widetilde{\sigma}$ are smooth as well.

Since all transition maps between the stereographic charts and the charts $\varphi_i^{\pm}$ are smooth, the two atlases are smoothly compatible. Hence, they determine the same maximal smooth structure on $\mathbb{S}^n$. □

Exercise 2.3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(i) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the ***nth power map*** for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.

(ii) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the ***antipodal map*** $\alpha(x) = -x$.

(iii) $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$ where we think of $\mathbb{S}^3$ as the subset $\{(w, z) \mid |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.

Solution to (i). Since $\mathbb{S}^1 \subseteq \mathbb{C}$, z can be represented in polar coordinates as $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$, where $\theta \in (0, 2\pi]$. Re-writing p_n in terms of θ, we have

$$p_n(z) = p_n\left(e^{i\theta}\right) = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

Computing $\sigma(e^{i\theta})$, $\widetilde{\sigma}(e^{i\theta})$, $\sigma^{-1}(e^{i\theta})$, and $\widetilde{\sigma}^{-1}(e^{i\theta})$ using the equations from Problem 1.7, we get

$$\sigma(e^{i\theta}) = \frac{\cos(\theta)}{1 - \sin(\theta)}, \quad \widetilde{\sigma}(e^{i\theta}) = \frac{\cos(\theta)}{1 + \sin(\theta)},$$
$$\sigma^{-1}(u) = \frac{2u + i(u^2 - 1)}{u^2 + 1}, \quad \widetilde{\sigma}^{-1}(u) = \frac{2u - i(u^2 - 1)}{u^2 + 1},$$

for $u \in \mathbb{R}$. Now, we compute the following coordinate representations of p_n:

$$\sigma \circ p_n \circ \sigma^{-1}, \quad \sigma \circ p_n \circ \widetilde{\sigma}^{-1}, \quad \widetilde{\sigma} \circ p_n \circ \sigma^{-1}, \quad \widetilde{\sigma} \circ p_n \circ \widetilde{\sigma}^{-1}.$$

Define $\widehat{p_n} : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{p_n}(u) = \left(\sigma \circ p_n \circ \sigma^{-1}\right)(u).$$

Computing $\widehat{p_n}(u)$, we get

$$\begin{aligned}
\widehat{p_n}(u) &= \sigma\left(p_n\left(\sigma^{-1}(u)\right)\right) \\
&= \frac{\operatorname{Re}(p_n(\sigma^{-1}(u))}{1 - \operatorname{Im}(p_n(\sigma^{-1}(u)))} \\
&= \frac{\operatorname{Re}\left(\left(\frac{2u+i(u^2-1)}{u^2+1}\right)^n\right)}{1 - \operatorname{Im}\left(\left(\frac{2u+i(u^2-1)}{u^2+1}\right)^n\right)}.
\end{aligned}$$

Notice that this is a quotient of polynomials in u, sines, and cosines, which are all smooth in $\mathbb{R}$. Also, raising to the n-th power in $\mathbb{C}$ is smooth, taking real and imaginary parts are linear maps, and dividing by a non-zero denominator is also smooth. Thus, $\widehat{p_n}$ is smooth on $\mathbb{R}$.

The remaining coordinate representations differ from this one by smooth transition maps between stereographic charts (proved in Problem 1.7), so they are also smooth. Hence p_n is a smooth map $\mathbb{S}^1 \to \mathbb{S}^1$. $\square$

Solution to (ii). Using the same stereographic charts σ and $\widetilde{\sigma}$ from Problem 1.7, we compute the following coordinate representations of α:

$$\widehat{\alpha} = (\widetilde{\sigma} \circ \alpha \circ \sigma^{-1})(u) = \widetilde{\sigma}(\alpha(\sigma^{-1}(u))) = \widetilde{\sigma}(-\sigma^{-1}(u)) = -u,$$

for $u \in \mathbb{R}^n$. By symmetry, the other three coordinate representations

$$\sigma \circ \alpha \circ \sigma^{-1}, \quad \sigma \circ \alpha \circ \widetilde{\sigma}^{-1}, \quad \widetilde{\sigma} \circ \alpha \circ \widetilde{\sigma}^{-1},$$

are also $-u$. Since $-u$ is smooth on $\mathbb{R}^n$, all four coordinate representations of α are smooth. Hence, α is a smooth map $\mathbb{S}^n \to \mathbb{S}^n$. $\square$

Solution to (iii). □

Exercise 2.4. Show that the inclusion map $\overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.

Proof. Let $\iota : \overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ denote the inclusion map $\iota(x) = x$. To check smoothness it suffices to check smoothness of every coordinate representation

$$\psi \circ \iota \circ \varphi^{-1},$$

where (U, φ) is a chart of the manifold-with-boundary $\overline{\mathbb{B}}^n$ with $U \subset \overline{\mathbb{B}}^n$ and ψ is a chart of $\mathbb{R}^n$. We may take ψ to be the identity chart on $\mathbb{R}^n$ (so $\psi = \mathrm{id}_{\mathbb{R}^n}$), and hence the coordinate representation reduces to

$$\mathrm{id}_{\mathbb{R}^n} \circ \iota \circ \varphi^{-1} = \iota \circ \varphi^{-1} = \varphi^{-1}.$$

By definition of a chart, $\varphi : U \to V \subset \mathbb{H}^n$ is a diffeomorphism onto an open set V of the half-space $\mathbb{H}^n$, so $\varphi^{-1} : V \to U$ is smooth. Therefore every coordinate representation of ι is smooth, and thus the inclusion $\iota : \overline{\mathbb{B}}^n \to \mathbb{R}^n$ is a smooth map. □