Several-Variab Calc II: Homework 6

Due on February 18, 2025 at 9:00

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Problem 1. Evaluate $\displaystyle\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ for the vector field $\mathbf{F} = (x+z)\boldsymbol{\imath} + 2y\boldsymbol{\jmath} + (y+2x)\mathbf{k}$ where C is

(i) the line segment from $(-1,0,1)$ and $(0,-1,2)$.

(ii) the curve $\mathbf{r}(t) = \left\langle t-1, t^2-2t, t^3+1\right\rangle$ from $0 \le t \le 1$.

Solution to (i). Let C be the line segment from $P(-1,0,1)$ to $Q(0,-1,2)$. Then, we have

$$\begin{aligned} &\mathbf{r}(t) = (1-t)P + tQ = (1-t)(-1,0,1) + t(0,-1,2) = \langle -1+t, -t, 1+t\rangle \\ \Rightarrow\ &\mathbf{r}'(t) = \langle 1,-1,1\rangle. \end{aligned}$$

Converting $\mathbf{F}$ from a vector function of x and y to a vector function of t, we have

$$\mathbf{F}(x,y) = \langle x+z, 2y, y+2x\rangle = \langle (-1+t)+(-1+t), -2t, -t+(-1+t)\rangle = \langle 2t, -2t, -2+t\rangle = \mathbf{F}(t).$$

Therefore, the line integral over the vector field $\mathbf{F}$ along the line segment C is

$$\begin{aligned} \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}(t)\cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^1 (1)\cdot(2t) + (-1)\cdot(-2t) + (1)\cdot(-2+t)\,\mathrm{d}t \\ &= \int_0^1 2t + 2t - 2 + t\,\mathrm{d}t \\ &= \int_0^1 5t - 2\,\mathrm{d}t = \left(\frac{5}{2} - 2\right) = \frac{1}{2}. \end{aligned}$$

□

Solution to (ii). Let C be the curve $\mathbf{r}(t) = \langle t-1, t^2-2t, t^3+1\rangle$ where $0 \le t \le 1$. Then, we have

$$\mathbf{r}'(t) = \left\langle 1, 2t-2, 3t^2\right\rangle.$$

Converting $\mathbf{F}$ from a vector function of x and y to a vector function of t, we have

$$\begin{aligned} \mathbf{F}(x,y) = \langle x+z, 2y, y+2x\rangle &= \left\langle (t-1) + \left(t^3+1\right), 2\left(t^2-2t\right), \left(t^2-2t\right) + 2(t-1)\right\rangle \\ &= \left\langle t^3+t, 2t^2-4t, t^2-2\right\rangle. \end{aligned}$$

Evaluating the dot product of $\mathbf{F}$ and $\mathbf{r}'$, we have

$$\begin{aligned} \mathbf{F}(t)\cdot\mathbf{r}'(t) &= \left\langle t^3+t, 2t^2-4t, t^2-2\right\rangle \cdot \left\langle 1, 2t-2, 3t^2\right\rangle \\ &= \left(t^3+t\right)\cdot(1) + \left(2t^2-4t\right)\cdot(2t-2) + \left(t^2-2\right)\cdot\left(3t^2\right) \\ &= t^3 + t + 4t^3 - 4t^2 - 8t^2 + 8t + 3t^4 - 6t^2 \\ &= 3t^4 + 5t^3 - 18t^2 + 9t. \end{aligned}$$

Therefore, the line integral over the vector field $\mathbf{F}$ along the curve C is

$$\begin{aligned} \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^1 3t^4 + 5t^3 - 18t^2 + 9t\,\mathrm{d}t \\ &= \frac{3}{5} + \frac{5}{4} - \frac{18}{3} + \frac{9}{2} = \frac{7}{20}. \end{aligned}$$

□

Problem 2. Compute the amount of work done by the vector field $\mathbf{F} = \langle -y, x, x-z\rangle$ moving a particle along the curve of intersection of the surfaces $x^2+y^2+z^2 = 9$ and $y - x + z = 1$ oriented in the counterclockwise direction about the cylinder.

Note: Do not use a method we haven't discussed yet. The use of anything other than directly evaluating the line integral will not be accepted.

Solution. Let C be the curve of intersection of the surfaces $x^2+y^2+z^2=9$ and $y-x+z=1$. Converting to cylindrical coordinates, we have

$$\begin{cases} x^2+y^2+z^2=9 \\ y-x+z=1 \end{cases} \Rightarrow \begin{cases} r^2+z^2=9 \\ r\sin\theta - r\cos\theta + z = 1 \end{cases} \Rightarrow \begin{cases} r^2+z^2=9 \\ r(\sin\theta-\cos\theta)+z=1 \end{cases}$$
$$\Rightarrow z = 1 - r(\sin(\theta)-\cos(\theta)).$$

To parameterize C, notice that $x^2+y^2+z^2=9$ is just a sphere. Therefore, we have $r=3$, $x=3\cos(\theta)$, and $y=3\sin(\theta)$. Using $r=3$, we have the following parameterization for C

$$\mathbf{r}(t) = \langle 3\cos(\theta), 3\sin(\theta), 1-3(\sin(\theta)-\cos(\theta))\rangle\,.$$

Evaluating $\mathbf{r}'(t)$ gives us

$$\mathbf{r}'(t) = \langle -3\sin(\theta), 3\cos(\theta), 3(\cos(\theta)+\sin(\theta))\rangle.$$

The bounds for θ is $0 \le \theta \le 2\pi$. Converting $\mathbf{F}$ from a vector function of x and y to a vector function of t, we have

$$\begin{aligned}\mathbf{F}(t) = \langle -y(t), x(t), x(t)-z(t)\rangle &= \langle -3\sin(\theta), 3\cos(\theta), 3\cos(\theta) - (1-3\,(\sin(\theta)-\cos(\theta)))\\ &= \langle -3\sin(\theta), 3\cos(\theta), 3\sin(\theta)-1\rangle.\end{aligned}$$

Evaluating the dot product of $\mathbf{F}$ and $\mathbf{r}'$, we have

$$\begin{aligned}\mathbf{F}(t)\cdot\mathbf{r}'(t) &= \langle -3\sin(\theta), 3\cos(\theta), 3\sin(\theta)-1\rangle \cdot \langle -3\sin(\theta), 3\cos(\theta), 3(\cos(\theta)+\sin(\theta))\rangle \\ &= (-3\sin(\theta))\cdot(-3\sin(\theta)) + (3\cos(\theta))\cdot(3\cos(\theta)) + (3\sin(\theta)-1)\cdot(3(\cos(\theta)+\sin(\theta))) \\ &= 9(\sin^2(\theta)+\cos^2(\theta)) + (-9\sin^2(\theta) - 9\sin(\theta)\cos(\theta) + 3\cos(\theta) + 3\sin(\theta)) \\ &= 9 - 9\sin^2(\theta) - 9\sin(\theta)\cos(\theta) + 3\cos(\theta) + 3\sin(\theta).\end{aligned}$$

Computing the work integral over the vector field $\mathbf{F}$ along the curve C, we have

$$\begin{aligned} W = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} &\mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} 9 - 9\sin^2(\theta) - 9\sin(\theta)\cos(\theta) + 3\cos(\theta) + 3\sin(\theta)\,\mathrm{d}t \\ &= \int 9\,\mathrm{d}\theta - 9\int \sin^2(\theta)\,\mathrm{d}\theta - 9\int \sin(\theta)\cos(\theta)\,\mathrm{d}\theta + 3\int \cos(\theta)\,\mathrm{d}\theta + 3\int \sin(\theta)\,\mathrm{d}\theta\Big|_0^{2\pi} \\ &= 9\theta - \frac{9}{2}(\theta - \sin(\theta)\cos(\theta)) - \frac{9}{2}\cos^2(\theta) + 3\sin(\theta) - 3\cos(\theta)\Big|_0^{2\pi} \\ &= \left[9(2\pi) - \frac{9}{2}(2\pi - \sin(2\pi)\cos(2\pi)) - \frac{9}{2}\cos^2(2\pi) + 3\sin(2\pi) - 3\cos(2\pi)\right] \\ &\quad - \left[9(0) - \frac{9}{2}(0-\sin(0)\cos(0)) - \frac{9}{2}\cos^2(0) + 3\sin(0) - 3\cos(0)\right] \\ &= \left[18\pi - \frac{9}{2}(2\pi - (0)(-1)) - \frac{9}{2}(1) + 3(0) - 3(1)\right] \\ &\quad - \left[0 - \frac{9}{2}(0-(0)(1)) - \frac{9}{2}(1) + 3(0) - 3(1)\right] \\ &= 18\pi - 9\pi - \frac{9}{2} - 3 + \frac{9}{2} + 3 = 9\pi. \end{aligned}$$

□

Problem 3. Evaluate $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$. Use the fundamental theorem for line integrals whenever it applies.

(i) $\mathbf{F} = \langle 2xe^{xy} + x^2ye^{xy} + 3x^2, x^3e^{xy} + 2\sin(y)\rangle$, C is the line segment from $(-1, 0)$ to $(0, 3)$.

(ii) $\mathbf{F} = \langle y^3 - 2x, 3xy^2 + \sin(\pi y)\rangle$, C is the path $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.

(iii) $\mathbf{F} = \langle 6xy - z^2, 3x^2 + 6y^2, 1 - 2xz\rangle$, C is the circular helix $\mathbf{r}(t) = \langle t, 2\cos(t), 2\sin(t)\rangle$, $0 \leq t \leq \pi$.

(iv) $\mathbf{F} = \langle y + z, x - 2z, x + 2y\rangle$, C is the intersection of sphere $x^2 + y^2 + z^2 = 4$ and the plane $x = 1$ in the first octant oriented upward.

Solution to (i). A vector field $\mathbf{F} = \langle P, Q\rangle$ is conservative if there exists a potential function $f(x, y)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Evaluating each partial derivative, we have

$$\begin{aligned}
\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left[2xe^{xy} + x^2ye^{xy} + 3x^2\right] \\
&= 2x \cdot \frac{\partial}{\partial y}\left[e^{xy}\right] + x^2e^{xy} \cdot \frac{\partial}{\partial y}\left[y\right] + x^2y \cdot \frac{\partial}{\partial y}\left[e^{xy}\right] + \frac{\partial}{\partial y}\left[3x^2\right] \\
&= 2x^2e^{xy} + x^2e^{xy} + x^3ye^{xy} + 0 \\
&= 3x^2e^{xy} + x^3ye^{xy} \\
\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}\left[x^3e^{xy} + 2\sin(y)\right] \\
&= e^{xy} \cdot \frac{\partial}{\partial x}\left[x^3\right] + x^3 \cdot \frac{\partial}{\partial x}\left[e^{xy}\right] + \frac{\partial}{\partial x}\left[2\sin(y)\right] \\
&= 3x^2e^{xy} + x^3ye^{xy} + 0 \\
&= 3x^2e^{xy} + x^3ye^{xy}.
\end{aligned}$$

Clearly, they are the same. Also, note that $\mathbf{F}$ is simply connected, as there are no singularities. Therefore, $\mathbf{F}$ is conservative.

Therefore, we have the following system of equations

$$f_x = 2xe^{xy} + x^2ye^{xy} + 3x^2 \tag{1}$$
$$f_y = x^3e^{xy} + 2\sin(y) \tag{2}$$

Integrating equation 1 with respect to x, we have

$$\begin{aligned}
f = \int f_x \,\mathrm{d}x &= \int 2xe^{xy} + x^2ye^{xy} + 3x^2 \,\mathrm{d}x \\
&= \text{TODO} \\
&= x^2e^{xy} + x^3 + g(y).
\end{aligned}$$

Plugging this into equation 2, we have

$$f_y = x^3e^{xy} + h'(y) = Q \Rightarrow h'(y) = 2\sin(y) \Rightarrow h(y) = -2\cos(y).$$

Therefore, the potential function is

$$f(x, y) = x^2e^{xy} + x^3 - 2\cos(y).$$

Therefore, the line integral over the vector field $\mathbf{F}$ along the line segment C is

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= f(0, 3) - f(-1, 0) \\
&= [-2\cos(3)] - [1 - 1 - 2] \\
&= -2\cos(3) + 2.
\end{aligned}$$

□

Solution to (ii). A vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative if there exists a potential function $f(x, y)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Evaluating each partial derivative, we have

$$\begin{aligned}
\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left[y^3 - 2x\right] \\
&= 3y^2 - 0 \\
&= 3y^2 \\
\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}\left[3xy^2 + \sin(\pi y)\right] \\
&= 3y^2 + 0 \\
&= 3y^2.
\end{aligned}$$

Clearly, they are the same. Also, note that $\mathbf{F}$ is simply connected, as there are no singularities. Therefore, $\mathbf{F}$ is conservative.

Therefore, we have the following system of equations

$$f_x = y^3 - 2x \tag{3}$$

$$f_y = 3xy^2 + \sin(\pi y) \tag{4}$$

Integrating equation 3 with respect to x, we have

$$\begin{aligned}
f = \int f_x \, \mathrm{d}x = \int y^3 - 2x \, \mathrm{d}x \\
= y^3 x - x^2 + g(y).
\end{aligned}$$

Plugging this into equation 4, we have

$$f_y = 3xy^2 + h'(y) = Q \Rightarrow h'(y) = \sin(\pi y) \Rightarrow h(y) = -\frac{1}{\pi}\cos(\pi y).$$

Therefore, the potential function is

$$f(x, y) = y^3 x - x^2 - \frac{1}{\pi}\cos(\pi y).$$

Therefore, the line integral over the vector field $\mathbf{F}$ along the path C is

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= f(4, 2) - f(1, 1) \\
&= \left[32 - 16 - \frac{1}{\pi}\cos(2\pi)\right] - \left[1 - 1 - \frac{1}{\pi}\cos(\pi)\right] \\
&= 16 - \frac{2}{\pi}.
\end{aligned}$$

□

Solution to (iii). A vector field $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if there exists a potential function $f(x, y, z)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Evaluating each partial derivative, we have

$$\frac{\partial P}{\partial y} = 6x = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial z} = -2z = \frac{\partial P}{\partial x}.$$

Clearly, they are the same. Also, note that $\mathbf{F}$ is simply connected, as there are no singularities. Therefore, $\mathbf{F}$ is conservative.

Therefore, we have the following system of equations

$$f_x = 6xy - z^2 \tag{5}$$
$$f_y = 3x^2 + 6y^2 \tag{6}$$
$$f_z = 1 - 2xz. \tag{7}$$

Integrating equation 5 with respect to x, we have

$$f = \int 6xy - z^2 \, \mathrm{d}x = 3x^2y - xz^2 + g(y, z).$$

Plugging this into equation 6, we have

$$f_y = 3x^2 + g_y(y, z) = Q \Rightarrow g_y(y, z) = 6y^2 \Rightarrow g(y, z) = 2y^3 + h(z).$$

Plugging this into equation 7, we have

$$f_z = 1 - 2xz + h'(z) = R \Rightarrow h'(z) = 1 \Rightarrow h(z) = z.$$

Therefore, the potential function is

$$f(x, y, z) = 3x^2y - xz^2 + 2y^3 + z.$$

The starting and ending points are

$$A = \mathbf{r}(0) = \langle 0, 2, 0 \rangle \quad \text{and} \quad B = \langle \pi, -2, 0 \rangle.$$

Therefore, the line integral over the vector field $\mathbf{F}$ along the circular helix C is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= f(\pi, -2, 0) - f(0, 2, 0) \\ &= \left[3\pi^2(-2) - \pi(0)^2 + 2(-2)^3 + 0\right] - \left[3(0)^2(2) - 0(0)^2 + 2(2)^3 + 0\right] \\ &= -6\pi^2 - 32. \end{aligned}$$

□

Solution to (iv). A vector field $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if there exists a potential function $f(x, y, z)$ such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Evaluating each partial derivative, we have

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = -2 \neq 2 = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = 1 = \frac{\partial P}{\partial z}.$$

Clearly, they are not the same. Therefore, $\mathbf{F}$ is not conservative.

Since C is the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $x = 1$ in the first octant, we get $x = 1$, $y^2 + z^2 = 3$, and $y, z \geq 0$. Therefore, we have

$$\mathbf{r}(t) = \langle 1, \sqrt{3}\cos(t), \sqrt{3}\sin(t) \rangle \Rightarrow \mathbf{r}'(t) = \langle 0, -\sqrt{3}\sin(t), \sqrt{3}\cos(t) \rangle.$$

The bounds for t are $0 \leq t \leq \frac{\pi}{2}$. Converting $\mathbf{F}$ from a vector function of x and y to a vector function of t, we have

$$\mathbf{F}(x, y) = \langle y + z, x - 2z, x + 2y \rangle = \langle \sqrt{3}\cos(t) + \sqrt{3}\sin(t), 1 - 2\sqrt{3}\sin(t), 1 + 2\sqrt{3}\cos(t) \rangle = \mathbf{F}(t).$$

Evaluating the dot product of $\mathbf{F}$ and $\mathbf{r}'$, we have

$$\begin{aligned}\mathbf{F}(t)\cdot\mathbf{r}'(t) &= \left\langle \sqrt{3}\cos(t)+\sqrt{3}\sin(t), 1-2\sqrt{3}\sin(t), 1+2\sqrt{3}\cos(t)\right\rangle \cdot \left\langle 0, -\sqrt{3}\sin(t), \sqrt{3}\cos(t)\right\rangle \\ &= (\sqrt{3}\cos(t)+\sqrt{3}\sin(t))(0) + (1-2\sqrt{3}\sin(t))(-\sqrt{3}\sin(t)) + (1+2\sqrt{3}\cos(t))(\sqrt{3}\cos(t)) \\ &= -\sqrt{3}\sin(t) + 6\sin^2(t) + \sqrt{3}\cos(t) + 6\cos^2(t) \\ &= \sqrt{3}\cos(t) - \sqrt{3}\sin(t) + 6.\end{aligned}$$

Therefore, the line integral over the vector field $\mathbf{F}$ along the curve C is

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{\frac{\pi}{2}} \mathbf{F}(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \sqrt{3}\cos(t) - \sqrt{3}\sin(t) + 6\,\mathrm{d}t \\ &= \left[\sqrt{3}\sin(t) + \sqrt{3}\cos(t) + 6t\right]\Big|_0^{\frac{\pi}{2}} \\ &= \left[\sqrt{3} + 3\pi\right] - \left[\sqrt{3}\right] = 3\pi.\end{aligned}$$

□

Problem 4. Consider $\mathbf{F} = \langle P, Q\rangle$ where $P(x,y) = \dfrac{-y}{x^2+y^2}$ and $Q(x,y) = \dfrac{x}{x^2+y^2}$.

(i) Show $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on the domain of $\mathbf{F}$.

(ii) Use the definition of the line integral to show that $\displaystyle\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 2\pi$ where C is the circle $x^2+y^2=a^2$, counterclockwise orientation, for any constant $a > 0$. Is $\mathbf{F}$ conservative?

Solution to (i). We have

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.\end{aligned}$$

Therefore, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on the domain of $\mathbf{F}$. □

Solution to (ii). We parameterize the circle C

$$\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle.$$

Evaluating $\mathbf{r}'(t)$ gives us

$$\mathbf{r}'(t) = \langle -a\sin(t), a\cos(t)\rangle,$$

where $0 \le t \le 2\pi$. Converting $\mathbf{F}$ from a vector function of x and y to a vector function of t, we have

$$\mathbf{F}(t) = \left\langle \frac{-a\sin(t)}{a^2}, \frac{a\cos(t)}{a^2}\right\rangle = \left\langle \frac{-\sin(t)}{a}, \frac{\cos(t)}{a}\right\rangle.$$

Evaluating the dot product of $\mathbf{F}$ and $\mathbf{r}'$, we have

$$\mathbf{F}(t)\cdot\mathbf{r}'(t) = \left\langle \frac{-\sin(t)}{a}, \frac{\cos(t)}{a}\right\rangle \cdot \langle -a\sin(t), a\cos(t)\rangle = \frac{-\sin(t)}{a}\cdot(-a\sin(t)) + \frac{\cos(t)}{a}\cdot(a\cos(t)) = 1.$$

Thus, the line integral evaluates to

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} 1 \, \mathrm{d}t = 2\pi.$$

A vector field $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if there exists a function $f(x, y)$ such that $\nabla f = \langle P, Q \rangle$. A necessary condition for $\mathbf{F}$ to be conservative is that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, which we verified in part (i).

However, $\mathbf{F}$ is not defined at $(0, 0)$, and the domain $\mathbb{R}^2 - \{(0, 0)\}$ is not simply connected, since any loop enclosing the origin cannot be continuously shrunk to a point without leaving the domain. A conservative vector field must be path-independent, meaning that the line integral around any closed curve should be zero.

Since we computed

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi \neq 0,$$

we conclude that $\mathbf{F}$ is not conservative. □