Introduction to Statistics I: Homework 7

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Problem 1. The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} c(y^2 - x^2)e^{-y} & -y \leq x \leq y, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find c.

(ii) Find the marginal densities of X and Y.

(iii) Find $E[X]$.

Solution to (i). In order to find c, we use the fact that the total probability must equal 1. Thus, we have

$$\begin{aligned} 1 &= \iint_{-y \leq x \leq y,\, 0<y<\infty} c(y^2 - x^2)e^{-y} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^\infty ce^{-y} \int_{x=-y}^{y} (y^2 - x^2) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^\infty ce^{-y} \left(y^2 x - \frac{x^3}{3} \right)\Bigg|_{x=-y}^{x=y} \mathrm{d}y \\ &= \frac{4c}{3} \int_0^\infty y^3 e^{-y} \, \mathrm{d}y \\ &= \frac{4c}{3} \cdot \Gamma(4) = 8c. \end{aligned}$$

Thus, we have $c = 1/8$. □

Solution to (ii). The marginal densities of X and Y are given by

$$\begin{aligned} f_X(x) &= \int_{y=|x|}^{\infty} f(x, y) \, \mathrm{d}y \\ &= \int_{|x|}^{\infty} \frac{1}{8}(y^2 - x^2)e^{-y} \, \mathrm{d}y \\ &= \frac{1}{8}e^{-|x|} \left((|x|^2 - x^2) + 2|x| + 2 \right) \\ &= \frac{1}{4}(|x| + 1)e^{-|x|}, \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{x=-y}^{y} f(x, y) \, \mathrm{d}x \\ &= \int_{-y}^{y} \frac{1}{8}(y^2 - x^2)e^{-y} \, \mathrm{d}x \\ &= \frac{1}{8}e^{-y} \left(y^3 - \frac{y^3}{3} - (-y^3 + \frac{y^3}{3}) \right) \\ &= \frac{1}{3}y^3 e^{-y}. \end{aligned}$$

□

Solution to (iii). To find $E[X]$, we compute

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} x \cdot \frac{1}{4}(|x|+1)e^{-|x|} \, \mathrm{d}x \\ &= 0, \end{aligned}$$

since the integrand is an odd function. □

Problem 2. The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if its joint probability density is

$$f(x, y) = \begin{cases} \frac{1}{A} & (x, y) \in R, \\ 0 & \text{otherwise,} \end{cases}$$

where A is the area of region R. Therefore if B is any subset of R with area a, then $P\{(X, Y) \in B\} = a/A$ and all regions of equal area are equally likely to contain a randomly selected point, (x, y)

Suppose (X, Y) is uniformly distributed over the square centered at $(0, 0)$ and with sides of length 2.

(i) Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.

(ii) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \leq 1\}$.

Solution to (i). The square is centered at $(0, 0)$ with side length 2, so the region R is

$$R = \{(x, y) \mid -1 \leq x \leq 1,\ -1 \leq y \leq 1\}.$$

Its area is $A = 2 \cdot 2 = 4$. Therefore the joint density of (X, Y) is

$$f(x, y) = \frac{1}{4},$$

for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, and $f(x, y) = 0$ otherwise.

The marginal densities of X and Y are given by

$$f_X(x) = \int_{-1}^{1} f(x, y)\,\mathrm{d}y = \int_{-1}^{1} \frac{1}{4}\,\mathrm{d}y = \frac{1}{2}$$
$$f_Y(y) = \int_{-1}^{1} f(x, y)\,\mathrm{d}x = \int_{-1}^{1} \frac{1}{4}\,\mathrm{d}x = \frac{1}{2}.$$

Since

$$f(x, y) = \frac{1}{4} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = f_X(x)\, f_Y(y),$$

the joint density factors as the product of the marginals. Therefore X and Y are independent, each uniformly distributed over $(-1, 1)$. □

Solution to (ii). We want the probability that (X, Y) lies inside the circle of radius 1 centered at the origin. Since (X, Y) is uniformly distributed over the square of area 4, the probability that it falls in any region is the area of that region divided by 4. The set $\{(x, y) \mid X^2 + Y^2 \leq 1\}$ is the disk of radius 1, whose area is $\pi(1)^2 = \pi$. □

Problem 3. Let $f(x, y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$

(i) Show that $f(x, y)$ is a joint probability density function.

(ii) Find $E[Y]$.

Solution to (i). For the first condition, clearly $f(x, y) \geq 0$ over the given bounds.

For the second condition, we need to verify that the total integral of $f(x, y)$ over its bounds, R, equals 1

$$\begin{aligned} \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{y=0}^{1} \int_{x=0}^{1-y} 24xy\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 12(1-y)^2 y\,\mathrm{d}y \\ &= 12 \int_0^1 (y - 2y^2 + y^3)\,\mathrm{d}y \\ &= 12\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1 = 12\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1. \end{aligned}$$

Therefore, $f(x, y)$ is a valid joint probability density function. □

Solution to (ii). First, we need to find the marginal density of Y

$$f_Y(y) = \int_0^{1-y} 24xy \,\mathrm{d}x = 12(1-y)^2 y.$$

Then, we can compute $E[Y]$ as follows

$$\begin{aligned}
E[Y] &= \int_0^1 y f_Y(y) \,\mathrm{d}y \\
&= \int_0^1 y \cdot 12(1-y)^2 y \,\mathrm{d}y \\
&= 12 \int_0^1 (y^2 - 2y^3 + y^4) \,\mathrm{d}y \\
&= 12 \left[\frac{y^3}{3} - \frac{2y^4}{4} + \frac{y^5}{5} \right]_0^1 = 12 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{2}{5}.
\end{aligned}$$

□

Problem 4. The joint density function of X and Y is $f(x,y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

(i) Are X and Y independent?

(ii) Find $P\{X + Y < 1\}$.

Solution to (i). To determine if X and Y are independent, we need to find their marginal densities and see if the joint density factors as the product of the marginals.

Their marginal densities are given by

$$\begin{aligned}
f_X(x) &= \int_0^1 (x+y) \,\mathrm{d}y = \left[xy + \frac{y^2}{2} \right]_0^1 = x + \frac{1}{2} \\
f_Y(y) &= \int_0^1 (x+y) \,\mathrm{d}x = \left[\frac{x^2}{2} + yx \right]_0^1 = \frac{1}{2} + y.
\end{aligned}$$

Notice that

$$f_X(x) f_Y(y) = \left(x + \frac{1}{2} \right) \left(\frac{1}{2} + y \right) = xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4} \neq f(x,y).$$

Therefore, X and Y are not independent. □

Solution to (ii). The region $0 < x < 1$, $0 < y < 1$, and $x + y < 1$ is the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Integrate in x first, we have

$$\begin{aligned}
\int_{y=0}^1 \int_{x=0}^{1-y} (x+y) \,\mathrm{d}x \,\mathrm{d}y &= \int_0^1 \left[\frac{x^2}{2} + yx \right]_{x=0}^{x=1-y} \mathrm{d}y \\
&= \int_0^1 \frac{1-y^2}{2} \,\mathrm{d}y \\
&= \left[\frac{y}{2} - \frac{y^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.
\end{aligned}$$

□

Problem 5. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Then, compute $P\{X_1 < X_2\}$.

Solution. Let X_1 and X_2 be independent exponential random variables with parameters λ_1 and λ_2, respectively. We define $Z = X_1/X_2$ and first find its cumulative distribution function. For $z \geq 0$, we have

$$F_Z(z) = P(Z \leq z) = P\left(\frac{X_1}{X_2} \leq z\right) = P(X_1 \leq zX_2).$$

Using independence of X_1 and X_2, this probability can be written as

$$F_Z(z) = \int_0^\infty P(X_1 \leq zx_2) f_{X_2}(x_2)\,\mathrm{d}x_2.$$

The CDF of X_1 is $P(X_1 \leq t) = 1 - e^{-\lambda_1 t}$, so

$$F_Z(z) = \int_0^\infty (1 - e^{-\lambda_1 z x_2})\lambda_2 e^{-\lambda_2 x_2}\,\mathrm{d}x_2 = \int_0^\infty \lambda_2 e^{-\lambda_2 x_2}\,\mathrm{d}x_2 - \int_0^\infty \lambda_2 e^{-(\lambda_2 + \lambda_1 z)x_2}\,\mathrm{d}x_2.$$

Evaluating these integrals gives

$$\int_0^\infty \lambda_2 e^{-\lambda_2 x_2}\,\mathrm{d}x_2 = 1 \quad \text{and} \quad \int_0^\infty \lambda_2 e^{-(\lambda_2 + \lambda_1 z)x_2}\,\mathrm{d}x_2 = \frac{\lambda_2}{\lambda_2 + \lambda_1 z}.$$

Therefore, the CDF of Z is

$$F_Z(z) = 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 z} = \frac{\lambda_1 z}{\lambda_2 + \lambda_1 z},$$

where $z \geq 0$. Differentiating this CDF with respect to z gives the probability density function of Z:

$$f_Z(z) = \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_1 z)^2},$$

where $z \geq 0$. To compute $P(X_1 < X_2)$, observe that

$$P(X_1 < X_2) = P\left(\frac{X_1}{X_2} < 1\right) = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

In summary, the distribution of Z has PDF $f_Z(z) = (\lambda_1\lambda_2)/(\lambda_2 + \lambda_1 z)^2$ for $z \geq 0$, and the probability that X_1 is less than X_2 is $\lambda_1/(\lambda_1 + \lambda_2)$. □

Problem 6. The gross weekly sales at a certain restaurant area normal random variable with mean \$2200 and standard deviation \$230. What is the probability the total gross sales over the next 2 weeks exceeds \$5000?

Solution. Let X_1 and X_2 denote the gross weekly sales for the next two weeks. Since each week is normally distributed with mean 2200 and standard deviation 230, we have

$$X_1, X_2 \sim N(2200, 230^2) \quad \text{and} \quad X_1 + X_2 \sim N(4400,\ 2 \cdot 230^2).$$

The total sales over two weeks is therefore

$$T = X_1 + X_2 \sim N(4400,\ (230\sqrt{2})^2).$$

We compute

$$P(T > 5000) = P\left(Z > \frac{5000 - 4400}{230\sqrt{2}}\right) = P\left(Z > \frac{600}{230\sqrt{2}}\right) = P(Z > 1.844\ldots).$$

Using standard normal tables,

$$P(Z > 1.844) \approx 0.033.$$

Therefore, the probability that the total gross sales exceed \$5000 is approximately 0.033. □

Problem 7. The monthly worldwide average number of airplane crashes of commercial airlines is 2.2 What is the probability that there will be

(i) more than 2 such accidents in the next month?

(ii) more than 4 such accidents in the next 2 months?

Solution to (i). The number of airplane crashes in a month is modeled as a Poisson random variable with parameter $\lambda = 2.2$. We want to find

$$P(X > 2) = 1 - P(X \leq 2),$$

where $X \sim \text{Poisson}(2.2)$. Compute the partial sum

$$P(X \leq 2) = e^{-2.2}\left(\frac{2.2^0}{0!} + \frac{2.2^1}{1!} + \frac{2.2^2}{2!}\right) = e^{-2.2}(5.62) \approx 0.1108.$$

Therefore, $P(X > 2) = 1 - 0.6227 \approx 0.3773$. $\square$

Solution to (ii). Over 2 months, the Poisson parameter doubles, so the number of crashes in two months is $Y \sim \text{Poisson}(4.4)$. We want $P(Y > 4) = 1 - P(Y \leq 4)$. We compute

$$P(Y \leq 4) = e^{-4.4}\left(\frac{4.4^0}{0!} + \frac{4.4^1}{1!} + \frac{4.4^2}{2!} + \frac{4.4^3}{3!} + \frac{4.4^4}{4!}\right) = 44.894 \cdot e^{-4.4} \approx 0.5522.$$

Therefore, $P(Y > 4) = 1 - 0.5522 \approx 0.4478$. $\square$