Math 432/532: Introduction to Topology II

Solutions to HW #2

1. Describe a simplicial complex in $\mathbb{R}^{n+1}$ whose realization is homeomorphic to the n-dimensional sphere S^n.

Solution: Let $v = (v_0, \ldots, v_{n+1})$ be any $(n+2)$-tuple of vectors in general position in $\mathbb{R}^{n+1}$. For example, we can take v_0 to be 0 and v_i to be the i^{th} basis vector. Then Δ_v is an $(n+1)$-simplex. Let K be the complex consisting of all proper faces of Δ_v. Then $|K| \cong S^n$.

2. Describe a simplicial complex whose realization is homeomorphic to the projective plane $\mathbb{R}P^2$. You can be loosey goosey about where it lives, so that your answer is in the same form as the triangulation of the torus given in Section 6.1 of the text book.

Solution: Start with the picture like Figure 6.4, with the direction of the arrows on the left and the bottom flipped. This almost works, but the problem is that the top-left and bottom-right triangles now intersect along two different edges, which is no good. One way to fix this is by replacing the diagonal of the bottom-right square with the opposite diagonal.

3. Let X be any topological space, and let $f : X \to S^n$ be a map that is not surjective. Prove that f is homotopic to a map that takes all of X to a single point.

Solution: Let $p \in S^n$ be a point that is not in the image of f, and choose a homeomorphism $g : \mathbb{R}^n \to S^n \setminus \{p\}$. Then there exists a (unique) map $h : f \to \mathbb{R}^n$ such that $f = g \circ h$. Define $F : X \times [0,1] \to S^n$ by the formula $F(x,t) = g\,(th(x))$. Then $F(x,1) = g(h(x)) = f(x)$ for all x, while $F(x,0) = g(0)$ for all x.

4. Let $f : S^4 \to S^7$ be any map. Use simplicial approximation to prove that f is homotopic to a map that takes all of S^4 to a single point.

Solution: Using Problem 1, we can find a 4-dimensional simplicial complex K with $|K| \cong S^4$ and a 7-dimensional simplical complex L with $|L| \cong S^7$. By the simplicial approximation theorem, there exists a natural number m and a simplicial map $s : |K^m| \to |L|$ that is a simplicial approximation of f, and in particular is homotopic to f. Since s is simplicial, all of the simplices in the image of s have dimension 4 or smaller. In particular, s is not surjective, so we may apply Problem 3 to see that s is homotopic to a constant map. Since homotopy is transitive, f is also homotopic to a constant map.

5. Let K and L be simplicial complexes. Use the simplicial approximation theorem to show that the set of homotopy classes of maps from $|K|$ to $|L|$ is countable.

Solution: For any natural number m, there are only finitely many morphisms from K^m to L, and therefore only finitely many simplicial maps from K^m to L. Since any map from $|K|$ to $|L|$ is homotopic to such a map, there are only countably many such maps.