Math 432/532: Introduction to Topology II

HW #1

1. Show that, if X has finitely many connected components, then each component is both open and closed. On the other hand, find an example of a space X none of whose connected components are open sets.

2. Fix real numbers $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Use connectedness of the interval $[a, b]$ to prove the intermediate value theorem, which says that there exists an element $c \in (a, b)$ with $f(c) = 0$.

3. Prove that, if $f : X \to Y$ is surjective and X is path-connected, then so is Y.

4. Prove that, if X and Y are path-connected, then so is $X \times Y$.

5. Suppose that $X = A \cup B$ with A, B, and $A \cap B$ all path-connected. Show that, if $A \cap B$ is nonempty, then X is also path-connected.

6. For each description below, name a familiar space that is homeomorphic to the corresponding identification space (no proofs required).

   (i) The cylinder $S^1 \times [0, 1]$ with each of its boundary circles collapsed to a point. (That is, $(x, s) \sim (y, t)$ if and only if $s = t \in \{0, 1\}$.)

   (ii) The torus $S^1 \times S^1$ with both a longitude $(1, 0) \times S^1$ and a meridian $S^1 \times (1, 0)$ collapsed to a point.

   (iii) The Möbius strip M with its boundary circle collapsed to a point.

7. Give an example of an identification map $f : X \to Y$ and a subspace $A \subset X$ such that the surjection $f : A \to f(A)$ is not an identification map.

8. Define $f : S^2 \to \mathbb{R}^4$ by the formula $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. You can convince yourself (and you may assume) that $f(x, y, z) = f(a, b, c)$ only if $(a, b, c) = (x, y, z)$ or $(a, b, c) = (-x, -y, -z)$. Show that f descends to a map $g : \mathbb{R}P^2 \to \mathbb{R}^4$, and that g is a homeomorphism from $\mathbb{R}P^2$ onto its image.