

Fundamentals of Analysis I: Homework 4

Due on October 30, 2024 at 13:00

Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

SECTION 2.3

Exercise 2.3.5 Let (x_n) and (y_n) be given, and define (z_n) to be “shuffled” sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$.

Solution.

\Rightarrow Suppose (z_n) is convergent. Let $\lim_{n \rightarrow \infty} z_n = L$ for some real number L .

The sequence (z_n) is constructed by alternating elements from (x_n) and (y_n) , so

$$(\forall n \in \mathbb{N})[z_{2n-1} = x_n \wedge z_{2n} = y_n].$$

Since (z_n) converges to L , both subsequences (z_{2n-1}) and (z_{2n}) must also converge to L (by the property that every subsequence of a convergent sequence converges to the same limit).

(a) *Convergence of (x_n) :* Since $z_{2n-1} = x_n$ for each n , the sequence (x_n) is the subsequence (z_{2n-1}) . Thus, (x_n) converges to L .

(b) *Convergence of (y_n) :* Similarly, since $z_{2n} = y_n$ for each n , the sequence (y_n) is the subsequence (z_{2n}) . Thus, (y_n) also converges to L .

Therefore, both (x_n) and (y_n) are convergent, and we have $\lim_{n \rightarrow \infty} x_n = L = \lim_{n \rightarrow \infty} y_n$.

\Leftarrow Suppose (x_n) and (y_n) are both convergent with

$$\lim_{n \rightarrow \infty} x_n = L \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = L.$$

We need to show that (z_n) converges to L .

For any $\varepsilon > 0$, since $x_n \rightarrow L$ and $y_n \rightarrow L$, there exists an integer N such that for all $n \geq N$,

$$|x_n - L| < \varepsilon \quad \text{and} \quad |y_n - L| < \varepsilon.$$

In the sequence (z_n) , every x_n and y_n appears as an element, specifically

$$z_{2n-1} = x_n \quad \text{and} \quad z_{2n} = y_n.$$

Thus, for all $m \geq 2N$, each term z_m is either x_n or y_n for some $n \geq N$. Therefore, for $m \geq 2N$,

$$|z_m - L| < \varepsilon.$$

This shows that $z_n \rightarrow L$ as $n \rightarrow \infty$, so (z_n) converges to L .

Therefore, we conclude that

$$\lim_{n \rightarrow \infty} z_n = L \Leftrightarrow \lim_{n \rightarrow \infty} x_n = L = \lim_{n \rightarrow \infty} y_n.$$

□

Exercise 2.3.9

- (i) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim_{n \rightarrow \infty} b_n = 0$. Show that $\lim_{n \rightarrow \infty} (a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

- (ii) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b ?
(iii) Use (i) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution to (i). Let $\varepsilon > 0$ be arbitrary. Let $|b_n| < \varepsilon/M$. We can't use the ALT since a_n might not converge. However, since a_n is bounded, we have

$$(\exists M \in \mathbb{N})(\forall n \in \mathbb{N})[|a_n b_n| \leq M|b_n| < \varepsilon]. \quad \square$$

Solution to (ii). No, since all we know about a_n is that it's bounded, not necessarily convergent. \square

Solution to (iii). In part (i), we showed that if (a_n) is bounded and $(b_n) \rightarrow 0$, then $(a_n b_n) \rightarrow 0$. If $a = 0$, then $(a_n) \rightarrow 0$, so we can apply the Algebraic Limit Theorem to conclude that $(a_n b_n) \rightarrow 0$. \square

Exercise 2.3.10 Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

- (i) If $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$, then $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.
- (ii) If $(b_n) \rightarrow b$, then $|b_n| \rightarrow |b|$.
- (iii) If $(a_n) \rightarrow a$ and $(b_n - a_n) \rightarrow 0$, then $(b_n) \rightarrow a$.
- (iv) If $(a_n) \rightarrow 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \rightarrow b$.

Solution to (i). False. Counterexample, let $a_n = n$ and $b_n = -n$. \square

Solution to (ii). True, since if $|b_n - b| < \varepsilon$, then $||b_n| - |b|| \leq |b_n - b| < \varepsilon$. \square

Solution to (iii). True by ALT since $\lim_{n \rightarrow \infty} (b_n - a_n) + \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a$. \square

Solution to (iv). True, since $0 \leq |b_n - b| \leq a_n$, we have $a_n \geq 0$. Let $\varepsilon > 0$. Choose N such that $(\forall n \geq N)[a_n < \varepsilon]$. Therefore, $|b_n - b| \leq a_n < \varepsilon$. \square

SECTION 2.4

Exercise 2.4.1

- (i) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

- (ii) Now that we know $\lim_{n \rightarrow \infty} x_n$ exists, explain why $\lim_{n \rightarrow \infty} x_{n+1}$ must also exist and equal the same value.
(iii) Take the limit of each side of the recursive equation in part (i) to explicitly compute $\lim_{n \rightarrow \infty} x_n$.

Solution to (i). Let L be the limit of the sequence, assuming it exists. If $x_n \rightarrow L$ as $n \rightarrow \infty$, then $x_{n+1} \rightarrow L$ as well, and we can take limits on both sides of the recurrence relation

$$L = \frac{1}{4-L} \Leftrightarrow L(4-L) = 1 \Leftrightarrow L^2 - 4L + 1 = 0 \Rightarrow x = 2 \pm \sqrt{3}.$$

Base Case: For $n = 1$, we have $x_1 = 3$, and indeed $2 - \sqrt{3} \approx 0.27 < 3 < 2 + \sqrt{3} \approx 4.73$.

Inductive Step: Suppose $2 - \sqrt{3} < x_n < 2 + \sqrt{3}$ for some $n \geq 1$. Then

$$4 - (2 + \sqrt{3}) < 4 - x_n < 4 - (2 - \sqrt{3}),$$

which simplifies to

$$2 - \sqrt{3} < 4 - x_n < 2 + \sqrt{3}.$$

Taking reciprocals, we get

$$\frac{1}{2 + \sqrt{3}} < \frac{1}{4 - x_n} < \frac{1}{2 - \sqrt{3}}.$$

Since $x_{n+1} = \frac{1}{4-x_n}$, it follows that $2 - \sqrt{3} < x_{n+1} < 2 + \sqrt{3}$.

By induction, $2 - \sqrt{3} < x_n < 2 + \sqrt{3}$ for all $n \geq 1$, so the sequence is bounded.

Next, we analyze whether the sequence (x_n) is increasing or decreasing. Consider the difference $x_{n+1} - x_n$

$$x_{n+1} - x_n = \frac{1}{4 - x_n} - x_n.$$

Since the sequence is bounded and has only one possible limit within the bounds (namely $2 - \sqrt{3}$), any accumulation point must be $2 - \sqrt{3}$. \square

Solution to (ii). Skipping a single term does not change the limit of the sequence. \square

Solution to (iii). Since $x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1}$, we get

$$x = \frac{1}{4 - x} \Leftrightarrow x^2 - 4x + 1 = 0 \Leftrightarrow (x - 2)^2 = 3 \Leftrightarrow x = 2 \pm \sqrt{3}.$$

Since $2 + \sqrt{3} > 3$ is impossible, we conclude $\lim_{n \rightarrow \infty} x_n = 2 - \sqrt{3}$. \square

Exercise 2.4.2

(i) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim_{n \rightarrow \infty} y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y , we conclude $\lim_{n \rightarrow \infty} y_n = 3/2$. What is wrong with this argument?

(ii) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (i) be applied to compute the limit of this sequence?

Solution to (i). The sequence $y_n = (1, 2, 1, 2, 1, \dots)$ is not convergent, so the argument is invalid. \square

Solution to (ii). Yes, y_n converges by the monotone convergence theorem, since (y_n) is bounded between $0 < y_n < 3$ and y_n is increasing. \square

Exercise 2.4.6

- (i) Explain why $\sqrt{xy} \leq \frac{x+y}{2}$ for any two positive real numbers x and y . (The geometric mean is always less than the arithmetic mean.)
- (ii) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim_{n \rightarrow \infty} x_n$ and $\lim_{n \rightarrow \infty} y_n$ both exist and are equal.

Solution to (i). Since $x, y > 0$, we have

$$\sqrt{xy} \leq \frac{x+y}{2} \Leftrightarrow 2\sqrt{xy} \leq x+y \Leftrightarrow 4xy \leq x^2 + 2xy + y^2 \Leftrightarrow 0 \leq (x-y)^2.$$

The last inequality is always true, so $\sqrt{xy} \leq x+y/2$. □

Solution to (ii). The inequality $x_1 \leq y_1$ is always true, since

$$\sqrt{x_n y_n} \leq \frac{x_n + y_n}{2} \Rightarrow x_{n+1} \leq y_{n+1}.$$

Also $x_n \leq y_n$ implies $\frac{x_n + y_n}{2} = y_{n+1} \leq y_n$. Similarly, $\sqrt{x_n y_n} = x_{n+1} \geq x_n$. By MCT, both converge since they are bounded and monotone. □