Chapter 1

Introduction to Lie Algebras and Representation Theory

1.1 Basic Concepts

1.1.1 Definitions and first examples

Exercise 1.1.1.1. Let L be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra. Write down the structure constants relative to the usual basis of $\mathbb{R}^3$.

Solution to 1.1.1.1. □

Exercise 1.1.1.2. Verify that the following equations and those implied by (L1) (L2) define a Lie algebra structure on a three dimensional vector space with basis (x, y, z) : $[xy] = z, [xz] = y, [yz] = 0$.

Solution to 1.1.1.2. □

Exercise 1.1.1.3. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}(2, \mathrm{F})$. Compute the matrices of $\operatorname{ad} x$, $\operatorname{ad} h$, $\operatorname{ad} y$ relative to this basis.

Solution to 1.1.1.3. □

Exercise 1.1.1.4. Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in (1.4). [Hint: Look at the adjoint representation.]

Solution to 1.1.1.4. □

Exercise 1.1.1.5. Verify the assertions made in (1.2) about $\mathfrak{t}(n, \mathrm{F}), \mathfrak{d}(n, \mathrm{F}), \mathfrak{n}(n, \mathrm{F})$, and compute the dimension of each algebra, by exhibiting bases.

Solution to 1.1.1.5. □

Exercise 1.1.1.6. Let $x \in \mathfrak{gl}(n, \mathrm{F})$ have n distinct eigenvalues $a_1, \ldots, a_n$ in F. Prove that the eigenvalues of $\operatorname{ad} x$ are precisely the n^2 scalars $a_i - a_j$ $(1 \leq i, j \leq n)$, which of course need not be distinct.

Solution to 1.1.1.6. □

Exercise 1.1.1.7. Let $\mathfrak{s}(n, \mathrm{F})$ denote the **scalar matrices** (= scalar multiples of the identity) in $\mathfrak{gl}(n, \mathrm{F})$. If char F is 0 or else a prime not dividing n, prove that $\mathfrak{gl}(n, \mathrm{F}) = \mathfrak{sl}(n, \mathrm{F}) + \mathfrak{s}(n, \mathrm{F})$ (direct sum of vector spaces), with $[\mathfrak{s}(n, \mathrm{F}), \mathfrak{gl}[(n, \mathrm{F})] = 0$.

Solution to 1.1.1.7. □

Exercise 1.1.1.8. Verify the stated dimension of D_ℓ.

Solution to 1.1.1.8. □

Exercise 1.1.1.9. When char $\mathrm{F}=0$, show that each classical algebra $L = \mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell$, or D_ℓ is equal to $[LL]$. (This shows again that each algebra consists of trace 0 matrices.)

Solution to 1.1.1.9. □

Exercise 1.1.1.10. For small values of ℓ, isomorphisms occur among certain of the classical algebras. Show that $\mathrm{A}_1, \mathrm{B}_1, \mathrm{C}_1$ are all isomorphic, while D_1 is the one dimensional Lie algebra. Show that B_2 is isomorphic to $\mathrm{C}_2, \mathrm{D}_3$ to A_3. What can you say about D_2 ?

Solution to 1.1.1.10. □

Exercise 1.1.1.11. Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.

Solution to 1.1.1.11. □

Exercise 1.1.1.12. Let L be a Lie algebra and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of $\operatorname{ad} x$ is a subalgebra.

Solution to 1.1.1.12. □

1.1.2 Ideals and homomorphisms

Exercise 1.1.2.1. Prove that the set of all inner derivations $\operatorname{ad} x, x \in L$, is an ideal of $\operatorname{Der} L$.

Solution to 1.1.2.1. □

Exercise 1.1.2.2. Show that $\mathfrak{sl}(n, \mathrm{F})$ is precisely the derived algebra of $\mathfrak{gl}(n, \mathrm{F})$ (cf. Exercise 1.1.1.9).

Solution to 1.1.2.2. □

Exercise 1.1.2.3. Prove that the center of $\mathfrak{gl}(n, \mathrm{F})$ equals $\mathfrak{s}(n, \mathrm{F})$ (the scalar matrices). Prove that $\mathfrak{sl}(n, \mathrm{F})$ has center 0, unless char F divides n, in which case the center is $\mathfrak{s}(n, \mathrm{F})$.

Solution to 1.1.2.3. □

Exercise 1.1.2.4. Show that (up to isomorphism) there is a unique Lie algebra over F of dimension 3 whose derived algebra has dimension 1 and lies in $Z(L)$.

Solution to 1.1.2.4. □

Exercise 1.1.2.5. Suppose $\dim L = 3$, $L = [LL]$. Prove that L must be simple. [Observe first that any homomorphic image of L also equals its derived algebra.] Recover the simplicity of $\mathfrak{sl}(2, \mathrm{F})$, char $\mathrm{F} \neq 2$.

Solution to 1.1.2.5. □

Exercise 1.1.2.6. Prove that $\mathfrak{sl}(3, \mathrm{F})$ is simple, unless char $\mathrm{F} = 3$ (cf. Exercise 1.1.2.3). [Use the standard basis h_1, h_2, e_{ij} $(i \neq j)$. If $I \neq 0$ is an ideal, then I is the direct sum of eigenspaces for $\operatorname{ad} h_1$ or $\operatorname{ad} h_2$; compare the eigenvalues of $\operatorname{ad} h_1$, ad h_2 acting on the e_{ij}.]

Solution to 1.1.2.6. □

Exercise 1.1.2.7. Prove that $\mathfrak{t}(n, \mathrm{F})$ and $\mathfrak{d}(n, \mathrm{F})$ are self-normalizing subalgebras of $\mathfrak{gl}(n, \mathrm{F})$, whereas $\mathfrak{n}(n, \mathrm{F})$ has normalizer $\mathfrak{t}(n, \mathrm{F})$.

Solution to 1.1.2.7. □

Exercise 1.1.2.8. Prove that in each classical linear Lie algebra (1.2), the set of diagonal matrices is a self-normalizing subalgebra, when char $\mathrm{F} = 0$.

Solution to 1.1.2.8. □

Exercise 1.1.2.9. Prove Proposition 2.2.

Solution to 1.1.2.9. □

Exercise 1.1.2.10. Let σ be the automorphism of $\mathfrak{sl}(2, \mathrm{F})$ defined in (2.3). Verify that $\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$.

Solution to 1.1.2.10. □

Exercise 1.1.2.11. If $L = \mathfrak{sl}(n, \mathrm{F})$, $g \in \mathrm{GL}(n, \mathrm{F})$, prove that the map of L to itself defined by $x \mapsto -gx^t g^{-1}$ (x^t = transpose of x) belongs to $\mathrm{Aut}\, L$. When $n = 2$, g = identity matrix, prove that this automorphism is inner.

Solution to 1.1.2.11. □

Exercise 1.1.2.12. Let L be an orthogonal Lie algebra (type B_ℓ or D_ℓ). If g is an **orthogonal** matrix, in the sense that g is invertible and $g^t s g = s$, prove that $x \mapsto gxg^{-1}$ defines an automorphism of L.

Solution to 1.1.2.12. □

1.1.3 Solvable and nilpotent Lie algebras

Exercise 1.1.3.1. Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.

Solution to 1.1.3.1. □

Exercise 1.1.3.2. Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.

Solution to 1.1.3.2. □

Exercise 1.1.3.3. Let $\mathrm{char}\, \mathrm{F} = 2$. Prove that $\mathfrak{sl}(2, \mathrm{F})$ is nilpotent.

Solution to 1.1.3.3. □

Exercise 1.1.3.4. Prove that L is solvable (resp. nilpotent) if and only if $\mathrm{ad}\, L$ is solvable (resp. nilpotent).

Solution to 1.1.3.4. □

Exercise 1.1.3.5. Prove that the nonabelian two dimensional algebra constructed in (1.4) is solvable but not nilpotent. Do the same for the algebra in Exercise 1.1.1.2.

Solution to 1.1.3.5. □

Exercise 1.1.3.6. Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal. Determine this ideal for each algebra in Exercise 1.1.3.5.

Solution to 1.1.3.6. □

Exercise 1.1.3.7. Let L be nilpotent, K a proper subalgebra of L. Prove that $N_L(K)$ includes K properly.

Solution to 1.1.3.7. □

Exercise 1.1.3.8. Let $L \neq 0$ be nilpotent. Prove that L has an ideal of codimension 1 .

Solution to 1.1.3.8. □

Exercise 1.1.3.9. Prove that every nilpotent Lie algebra $L \neq 0$ has an outer derivation (see (1.3)), as follows: Write $L = K + \mathrm{F}x$ for some ideal K of codimension one (Exercise 1.1.3.8). Then $C_L(K) \neq 0$ (why?). Choose n so that $C_L(K) \subset L^n$, $C_L(K) \not\subseteq L^{n+1}$, and let $z \in C_L(K) - L^{n+1}$. Then the linear map δ sending K to $0, x$ to z, is an outer derivation.

Solution to 1.1.3.9. □

Exercise 1.1.3.10. Let L be a Lie algebra, K an ideal of L such that L/K is nilpotent and such that $\mathrm{ad}\, x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.

Solution to 1.1.3.10. □

1.2 Semisimple Lie Algebras

1.2.1 Theorems of Lie and Cartan

Exercise 1.2.1.1. Let $L = \mathfrak{sl}(V)$. Use Lie's Theorem to prove that $\operatorname{Rad} L = Z(L)$; conclude that L is semisimple (cf. Exercise 1.1.2.3). [Observe that $\operatorname{Rad} L$ lies in each maximal solvable subalgebra B of L. Select a basis of V so that $B = L \cap \mathfrak{t}(n, \mathrm{F})$, and notice that the transpose of B is also a maximal solvable subalgebra of L. Conclude that $\operatorname{Rad} L \subset L \cap \mathfrak{d}(n, \mathrm{F})$, then that $\operatorname{Rad} L = Z(L)$.]

Solution to 1.2.1.1. □

Exercise 1.2.1.2. Show that the proof of Theorem 4.1 still goes through in prime characteristic, provided $\dim V$ is less than $\operatorname{char} \mathrm{F}$.

Solution to 1.2.1.2. □

Exercise 1.2.1.3. This exercise illustrates the failure of Lie's Theorem when F is allowed to have prime characteristic p. Consider the $p \times p$ matrices:

$$x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad y = \operatorname{diag}(0, 1, 2, 3, \cdots, p-1)..$$

Check that $[x, y] = x$, hence that x and y span a two dimensional solvable subalgebra L of $\mathfrak{gl}(p, \mathrm{F})$. Verify that x, y have no common eigenvector.

Solution to 1.2.1.3. □

Exercise 1.2.1.4. Exercise 1.2.1.3 shows that a solvable Lie algebra of endomorphisms over a field of prime characteristic p need not have derived algebra consisting of nilpotent endomorphisms. For arbitrary p, construct a counterexample to Corollary C of Theorem 4.1 as follows: Start with $L \subset \mathfrak{gl}(p, \mathrm{F})$ as in Exercise 1.2.1.3. Form the vector space direct sum $M = L + \mathrm{F}^p$, and make M a Lie algebra by decreeing that F^p is abelian, while L has its usual product and acts on F^p in the given way. Verify that M is solvable, but that its derived algebra ($= \mathrm{F}x + \mathrm{F}^p$) fails to be nilpotent.

Solution to 1.2.1.4. □

Exercise 1.2.1.5. If $x, y \in \operatorname{End} V$ commute, prove that $(x+y)_s = x_s + y_s$, and $(x+y)_n = x_n + y_n$. Show by example that this can fail if x, y fail to commute. [Show first that x, y semisimple (resp. nilpotent) implies $x + y$ semisimple (resp. nilpotent).]

Solution to 1.2.1.5. □

Exercise 1.2.1.6. Check formula (*) at the end of (4.2).

Solution to 1.2.1.6. □

Exercise 1.2.1.7. Prove the converse of Theorem 4.3.

Solution to 1.2.1.7. □

Exercise 1.2.1.8. Note that it suffices to check the hypothesis of Theorem 4.3 (or its corollary) for x, y ranging over a basis of $[LL]$, resp. L. For the example given in Exercise 1.1.1.2, verify solvability by using Cartan's Criterion.

Solution to 1.2.1.8. □

1.2.2 Killing form

Exercise 1.2.2.1. Prove that if L is nilpotent, the Killing form of L is identically zero.

Solution to 1.2.2.1. □

Exercise 1.2.2.2. Prove that L is solvable if and only if $[LL]$ lies in the radical of the Killing form.

Solution to 1.2.2.2. □

Exercise 1.2.2.3. Let L be the two dimensional nonabelian Lie algebra (1.4), which is solvable. Prove that L has nontrivial Killing form.

Solution to 1.2.2.3. □

Exercise 1.2.2.4. Let L be the three dimensional solvable Lie algebra of Exercise 1.1.1.2. Compute the radical of its Killing form.

Solution to 1.2.2.4. □

Exercise 1.2.2.5. Let $L = \mathfrak{sl}(2, \mathrm{F})$. Compute the basis of L dual to the standard basis, relative to the Killing form.

Solution to 1.2.2.5. □

Exercise 1.2.2.6. Let char $\mathrm{F} = p \neq 0$. Prove that L is semisimple if its Killing form is nondegenerate. Show by example that the converse fails. [Look at $\mathfrak{sl}[(3, \mathrm{F})$ modulo its center, when char $\mathrm{F} = 3$.]

Solution to 1.2.2.6. □

Exercise 1.2.2.7. Relative to the standard basis of $\mathfrak{sl}(3, \mathrm{F})$, compute the determinant of κ. Which primes divide it?

Solution to 1.2.2.7. □

Exercise 1.2.2.8. Let $L = L_1 \oplus \cdots \oplus L_t$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the various L_i of the components of x.

Solution to 1.2.2.8. □

1.2.3 Complete reducibility of representations

Exercise 1.2.3.1. Using the standard basis for $L = \mathfrak{sl}(2, \mathrm{F})$, write down the Casimir element of the adjoint representation of L (cf. Exercise 1.2.2.5). Do the same thing for the usual (3-dimensional) representation of $\mathfrak{sl}(3, \mathrm{F})$, first computing dual bases relative to the trace form.

Solution to 1.2.3.1. □

Exercise 1.2.3.2. Let V be an L-module. Prove that V is a direct sum of irreducible submodules if and only if each L-submodule of V possesses a complement.

Solution to 1.2.3.2. □

Exercise 1.2.3.3. If L is solvable, every irreducible representation of L is one dimensional.

Solution to 1.2.3.3. □

Exercise 1.2.3.4. Use Weyl's Theorem to give another proof that for L semisimple, $\operatorname{ad} L = \operatorname{Der} L$ (Theorem 5.3). [If $\delta \in \operatorname{Der} L$, make the direct sum $\mathrm{F} + L$ into an L-module via the rule $x.(a, y) = (0, a\delta(x) + [xy])$. Then consider a complement to the submodule L.]

Solution to 1.2.3.4. □

Exercise 1.2.3.5. A Lie algebra L for which $\operatorname{Rad} L = Z(L)$ is called reductive. (Examples: L abelian, L semisimple, $L = \mathfrak{gl}(n, \mathrm{F})$.)

(i) If L is reductive, then L is a completely reducible $\operatorname{ad} L$-module. [If $\operatorname{ad} L \neq 0$, use Weyl's Theorem.] In particular, L is the direct sum of $Z(L)$ and $[LL]$, with $[LL]$ semisimple.

(ii) If L is a classical linear Lie algebra (1.2), then L is semisimple. [Cf. Exercise 1.1.1.9.]

(iii) If L is a completely reducible $\operatorname{ad} L$-module, then L is reductive.

(iv) If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphisms are completely reducible.

Solution to 1.2.3.5(i). □

Solution to 1.2.3.5(ii). □

Solution to 1.2.3.5(iii). □

Solution to 1.2.3.5(iv). □

Exercise 1.2.3.6. Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on L. If β, γ are nondegenerate, prove that β and γ are proportional. [Use Schur's Lemma.]

Solution to 1.2.3.6. □

Exercise 1.2.3.7. It will be seen later on that $\mathfrak{sl}(n, \mathrm{F})$ is actually simple. Assuming this and using Exercise 1.2.3.6, prove that the Killing form κ on $\mathfrak{sl}(n, \mathrm{F})$ is related to the ordinary trace form by $\kappa(x, y) = 2n \operatorname{Tr}(xy)$.

Solution to 1.2.3.7. □

Exercise 1.2.3.8. If L is a Lie algebra, then L acts (via ad) on $(L \otimes L)^*$, which may be identified with the space of all bilinear forms β on L. Prove that β is associative if and only if $L.\beta = 0$.

Solution to 1.2.3.8. □

Exercise 1.2.3.9. Let L' be a semisimple subalgebra of a semisimple Lie algebra L. If $x \in L'$, its Jordan decomposition in L' is also its Jordan decomposition in L.

Solution to 1.2.3.9. □

1.2.4 Representations of $\mathfrak{sl}(2, \mathrm{F})$

Exercise 1.2.4.1. Use Lie's Theorem to prove the existence of a maximal vector in an arbitrary finite dimensional L-module. [Look at the subalgebra B spanned by h and x.]

Solution to 1.2.4.1. □

Exercise 1.2.4.2. $M = \mathfrak{sl}(3, \mathrm{F})$ contains a copy of L in its upper left-hand 2×2 position. Write M as direct sum of irreducible L-submodules (M viewed as L module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Solution to 1.2.4.2. □

Exercise 1.2.4.3. Verify that formulas (a)-(c) of Lemma 7.2 do define an irreducible representation of L. [To show that they define a representation, it suffices to show that the matrices corresponding to x, y, h satisfy the same structural equations as x, y, h.]

Solution to 1.2.4.3. □

Exercise 1.2.4.4. The irreducible representation of L of highest weight m can also be realized "naturally", as follows. Let X, Y be a basis for the two dimensional vector space F^2, on which L acts as usual. Let $\mathscr{R} = \mathrm{F}[X, Y]$ be the polynomial algebra in two variables, and extend the action of L to $\mathscr{R}$ by the derivation rule: $z.fg = (z.f)g + f(z.g)$, for $z \in L, f, g \in \mathscr{R}$. Show that this extension is well defined and that $\mathscr{R}$ becomes an L-module. Then show that the subspace of homogeneous polynomials of degree m, with basis X^m, $X^{m-1}Y, \cdots, XY^{m-1}, Y^m$, is invariant under L and irreducible of highest weight m.

Solution to 1.2.4.4. □

Exercise 1.2.4.5. Suppose char $\mathrm{F} = p > 0, L = \mathfrak{sl}(2, \mathrm{F})$. Prove that the representation $V(m)$ of L constructed as in Exercise 1.2.4.3 or 1.2.4.4 is irreducible so long as the highest weight m is strictly less than p, but reducible when $m = p$.

Solution to 1.2.4.5. □

Exercise 1.2.4.6. Decompose the tensor product of the two L-modules $V(3), V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

Solution to 1.2.4.6. $\square$

Exercise 1.2.4.7. In this exercise we construct certain infinite dimensional L-modules. Let $\lambda \in \mathrm{F}$ be an arbitrary scalar. Let $Z(\lambda)$ be a vector space over F with countably infinite basis $(v_0, v_1, v_2, \cdots)$.

(i) Prove that formulas (a) – (c) of Lemma 7.2 define an L-module structure on $Z(\lambda)$, and that every nonzero L-submodule of $Z(\lambda)$ contains at least one maximal vector.

(ii) Suppose $\lambda + 1 = i$ is a positive integer. Prove that v_i is a maximal vector. This induces an L-module homomorphism $Z(\mu) \xrightarrow{\phi} Z(\lambda), \mu = \lambda - 2i$, sending v_0 to v_i. Show that ϕ is a monomorphism, and that $\operatorname{Im}\phi$, $Z(\lambda)/\operatorname{Im}\phi$ are both irreducible L-modules (but $Z(\lambda)$ fails to be completely reducible).

(iii) Suppose $\lambda + 1$ is not a positive integer. Prove that $Z(\lambda)$ is irreducible.

Solution to 1.2.4.7(i). $\square$

Solution to 1.2.4.7(ii). $\square$

Solution to 1.2.4.7(iii). $\square$

1.2.5 Root space decomposition

Exercise 1.2.5.1. If L is a classical linear Lie algebra of type $\mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell$, or D_ℓ (see (1.2)), prove that the set of all diagonal matrices in L is a maximal toral subalgebra, of dimension ℓ. (Cf. Exercise 1.1.2.8.)

Solution to 1.2.5.1. $\square$

Exercise 1.2.5.2. For each algebra in Exercise 1.2.5.1, determine the roots and root spaces. How are the various h_α expressed in terms of the basis for H given in (1.2)?

Solution to 1.2.5.2. $\square$

Exercise 1.2.5.3. If L is of classical type, compute explicitly the restriction of the Killing form to the maximal toral subalgebra described in Exercise 1.2.5.1.

Solution to 1.2.5.3. $\square$

Exercise 1.2.5.4. If $L = \mathfrak{sl}(2, \mathrm{F})$, prove that each maximal toral subalgebra is one dimensional.

Solution to 1.2.5.4. $\square$

Exercise 1.2.5.5. If L is semisimple, H a maximal toral subalgebra, prove that H is selfnormalizing (i.e., $H = N_L(H)$).

Solution to 1.2.5.5. $\square$

Exercise 1.2.5.6. Compute the basis of $\mathfrak{sl}(n, \mathrm{F})$ which is dual (via the Killing form) to the standard basis. (Cf. Exercise 1.2.3.5.)

Solution to 1.2.5.6. $\square$

Exercise 1.2.5.7. Let L be semisimple, H a maximal toral subalgebra. If $h \in H$, prove that $C_L(h)$ is reductive (in the sense of Exercise 1.2.4.5). Prove that H contains elements h for which $C_L(h) = H$; for which h in $\mathfrak{sl}(n, \mathrm{F})$ is this true?

Solution to 1.2.5.7. $\square$

Exercise 1.2.5.8. For $\mathfrak{sl}(n, \mathrm{F})$ (and other classical algebras), calculate explicitly the root strings and Cartan integers. In particular, prove that all Cartan integers $2(\alpha, \beta)/(\beta, \beta), \alpha \neq \pm\beta$, for $\mathfrak{sl}(n, \mathrm{F})$ are $0, \pm 1$.

Solution to 1.2.5.8. $\square$

Exercise 1.2.5.9. Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}(2, \mathrm{F})$, hence is isomorphic to $\mathfrak{sl}(2, \mathrm{F})$.

Solution to 1.2.5.9. □

Exercise 1.2.5.10. Prove that no four, five or seven dimensional semisimple Lie algebras exist.

Solution to 1.2.5.10. □

Exercise 1.2.5.11. If $(\alpha, \beta) > 0$, prove that $\alpha - \beta \in \Phi(\alpha, \beta \in \Phi)$. Is the converse true?

Solution to 1.2.5.11. □

1.3 Root Systems

1.3.1 Axiomatics

Note. Unless otherwise specified, Φ denotes a root system in E, with Weyl group $\mathscr{W}$.

Exercise 1.3.1.1. Let E' be a subspace of E. If a reflection σ_α leaves E' invariant, prove that either $\alpha \in \mathrm{E}'$ or else $\mathrm{E}' \in P_\alpha$.

Solution to 1.3.1.1. □

Exercise 1.3.1.2. Prove that $\Phi^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathscr{W}$; show also that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$, and draw a picture of $\Phi^\vee$ in the cases $\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_2, \mathrm{G}_2$.

Solution to 1.3.1.2. □

Exercise 1.3.1.3. In Table 1, show that the order of $\sigma_\alpha \sigma_\beta$ in $\mathscr{W}$ is (respectively) 2, 3, 4, 6 when $\theta = \pi/2$, $\pi/3$ (or $2\pi/3$), $\pi/4$ (or $3\pi/4$), $\pi/6$ (or $5\pi/6$). [Note that $\sigma_\alpha \sigma_\beta =$ rotation through 2θ.]

Solution to 1.3.1.3. □

Exercise 1.3.1.4. Prove that the respective Weyl groups of $\mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_2, \mathrm{G}_2$ are dihedral of order 4, 6, 8, 12. If Φ is any root system of rank 2, prove that its Weyl group must be one of these.

Solution to 1.3.1.4. □

Exercise 1.3.1.5. Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) \leq 0$ (cf. Lemma 9.4).

Solution to 1.3.1.5. □

Exercise 1.3.1.6. Prove that $\mathscr{W}$ is a normal subgroup of $\operatorname{Aut}\Phi$ (= group of all isomorphisms of Φ onto itself).

Solution to 1.3.1.6. □

Exercise 1.3.1.7. Let $\alpha, \beta \in \Phi$ span a subspace E' of E. Prove that $\mathrm{E}' \cap \Phi$ is a root system in E'. Prove similarly that $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is a root system in E' (must this coincide with $\mathrm{E}' \cap \Phi$?). More generally, let Φ' be a nonempty subset of Φ such that $\Phi' = -\Phi'$, and such that $\alpha, \beta \in \Phi', \alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E it spans. [Use Table 1].

Solution to 1.3.1.7. □

Exercise 1.3.1.8. Compute root strings in G_2 to verify the relation $r - q = \langle \beta, \alpha \rangle$.

Solution to 1.3.1.8. □

Exercise 1.3.1.9. Let Φ be a set of vectors in a euclidean space E, satisfying only (R1), (R3), (R4). Prove that the only possible multiples of $\alpha \in \Phi$ which can be in Φ are $\pm 1/2\alpha, \pm\alpha, \pm 2\alpha$. Verify that $\{\alpha \in \Phi \mid 2\alpha \notin \Phi\}$ is a root system.

Solution to 1.3.1.9. □

1.3.2 Simple roots and Weyl group

Exercise 1.3.2.1. Let $\Phi^\vee$ be the dual system of Φ, $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. Prove that $\Delta^\vee$ is a base of $\Phi^\vee$. [Compare Weyl chambers of Φ and $\Phi^\vee$.]

Solution to 1.3.2.1. □

Exercise 1.3.2.2. If Δ is a base of Φ, prove that the set $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$ ($\alpha \neq \beta$ in Δ) is a root system of rank 2 in the subspace of E spanned by α, β (cf. Exercise 1.3.1.7). Generalize to an arbitrary subset of Δ.

Solution to 1.3.2.2. □

Exercise 1.3.2.3. Prove that each root system of rank 2 is isomorphic to one of those listed in (9.3).

Solution to 1.3.2.3. □

Exercise 1.3.2.4. Verify the Corollary of Lemma 10.2A directly for G_2.

Solution to 1.3.2.4. □

Exercise 1.3.2.5. If $\sigma \in \mathscr{W}$ can be written as a product of t simple reflections, prove that t has the same parity as $\ell(\sigma)$.

Solution to 1.3.2.5. □

Exercise 1.3.2.6. Define a function $sn : \mathscr{W} \to \{\pm 1\}$ by $sn(\sigma) = (-1)^{\ell(\sigma)}$. Prove that sn is a homomorphism (cf. the case A_2, where $\mathscr{W}$ is isomorphic to the symmetric group $\mathscr{S}_3$).

Solution to 1.3.2.6. □

Exercise 1.3.2.7. Prove that the intersection of "positive" open half-spaces associated with any basis $\gamma_1, \cdots, \gamma_l$ of E is nonvoid. [If δ_i is the projection of γ_i on the orthogonal complement of the subspace spanned by all basis vectors except γ_i, consider $\gamma = \sum r_i \delta_i$ when all $r_i > 0$.]

Solution to 1.3.2.7. □

Exercise 1.3.2.8. Let Δ be a base of Φ, $\alpha \neq \beta$, simple roots, $\Phi_{\alpha\beta}$ the rank 2 root system in $\mathrm{E}_{\alpha\beta} = \mathbb{R}\alpha + \mathbb{R}\beta$ (see Exercise refexc:1.3.2.2 above). The Weyl group $\mathscr{W}_{\alpha\beta}$ of $\Phi_{\alpha\beta}$ is generated by the restrictions τ_α, τ_β to $\mathrm{E}_{\alpha\beta}$ of $\sigma_\alpha, \sigma_\beta$, and $\mathscr{W}_{a\beta}$ may be viewed as a subgroup of $\mathscr{W}$. Prove that the "length" of an element of $\mathscr{W}_{\alpha\beta}$ (relative to τ_α, τ_β) coincides with the length of the corresponding clement of $\mathscr{W}$.

Solution to 1.3.2.8. □

Exercise 1.3.2.9. Prove that there is a unique element σ in $\mathscr{W}$ sending Ψ^+ to Φ^-(relative to Δ). Prove that any reduced expression for σ must involve all σ_α ($\alpha \in \Delta$). Discuss $\ell(\sigma)$.

Solution to 1.3.2.9. □

Exercise 1.3.2.10. Given $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ in Φ, let $\lambda = \sum_{i=1}^{l} k_i \alpha_i$ ($k_i \in \mathbb{Z}$, for all $k_i \geq 0$ or all $k_i \leq 0$). Prove that either λ is a multiple (possibly 0) of a root, or else there exists $\sigma \in \mathscr{W}$ such that $\sigma\lambda = \sum_{i=1}^{l} k_i' \alpha_i$, with some $k_i' > 0$ and some $k_i' < 0$. [Sketch of proof: If λ is not a multiple of any root, then the hyperplane P_λ orthogonal to λ is not included in $\bigcup_{\alpha \in \Phi} P_\alpha$. Take $\mu \in P_\lambda - \bigcup_{\alpha \in \Phi} P_\alpha$. Then find $\sigma \in \mathscr{W}$ for which all $(\alpha_i, \sigma\mu) > 0$. It follows that $0 = (\lambda, \mu) = (\sigma\lambda, \sigma\mu) = \sum k(\alpha_i, \sigma\mu)$.]

Solution to 1.3.2.10. □

Exercise 1.3.2.11. Let Φ be irreducible. Prove that $\Phi^\vee$ is also irreducible. If Φ has all roots of equal length, so does $\Phi^\vee$ (and then $\Phi^\vee$ is isomorphic to Φ). On the other hand, if Φ has two root lengths, then so does $\Phi^\vee$; but if α is long, then $\alpha^\vee$ is short (and vice versa). Use this fact to prove that Φ has a unique maximal short root (relative to the partial order $\prec$ defined by Δ).

Solution to 1.3.2.11. □

Exercise 1.3.2.12. Let $\lambda \in \mathbb{C}(\Delta)$. If $\sigma\lambda = \lambda$ for some $\sigma \in \mathscr{W}$, then $\sigma = 1$.

Solution to 1.3.2.12. □

Exercise 1.3.2.13. The only reflections in $\mathscr{W}$ are those of the form σ_α ($\alpha \in \Phi$). [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in $\mathscr{W}$.]

Solution to 1.3.2.13. □

Exercise 1.3.2.14. Prove that each point of E is $\mathscr{W}$-conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. [Enlarge the partial order on E by defining $\mu \prec \lambda$ iff $\lambda - \mu$ is a nonnegative $\mathbb{R}$-linear combination of simple roots. If $\mu \in \mathrm{E}$, choose $\sigma \in \mathscr{W}$ for which $\lambda = \sigma\mu$ is maximal in this partial order.]

Solution to 1.3.2.14. □

1.3.3 Classification

Exercise 1.3.3.1. Verify the Cartan matrices (Table 1).

Solution to 1.3.3.1. □

Exercise 1.3.3.2. Calculate the determinants of the Cartan matrices (using induction on ℓ for types A_ℓ–D_ℓ), which are as follows:

$$\mathrm{A}_\ell : \ell+1;\ \mathrm{B}_\ell : 2;\ \mathrm{C}_\ell : 2;\ \mathrm{D}_\ell : 4;\ \mathrm{E}_6 : 3;\ \mathrm{E}_7 : 2;\ \mathrm{E}_8,\ \mathrm{F}_4 \text{ and } \mathrm{G}_2 : 1.$$

Solution to 1.3.3.2. □

Exercise 1.3.3.3. Use the algorithm of (11.1) to write down all roots for G_2. Do the same for $\mathcal{C}_3$: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.

Solution to 1.3.3.3. □

Exercise 1.3.3.4. Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.

Solution to 1.3.3.4. □

Exercise 1.3.3.5. Prove that each irreducible root system is isomorphic to its dual, except that B_ℓ, C_ℓ are dual to each other.

Solution to 1.3.3.5. □

Exercise 1.3.3.6. Prove that an inclusion of one Dynkin diagram in another (e.g., E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.

Solution to 1.3.3.6. □

1.3.4 Construction of root systems and automorphisms

Exercise 1.3.4.1. Verify the details of the constructions in (12.1).

Solution to 1.3.4.1. □

Exercise 1.3.4.2. Verify Table 2.

Solution to 1.3.4.2. □

Exercise 1.3.4.3. Let $\Phi \subset E$ satisfy (R1), (R3), (R4), but not (R2), cf. Exercise 1.3.2.9. Suppose moreover that Φ is irreducible, in the sense of §11. Prove that Φ is the union of root systems of type $\mathrm{B}_n, \mathbb{C}_n$ in E (if $\dim \mathrm{E} = n > 1$), where the long roots of B_n are also the short roots of $\mathbb{C}_n$. (This is called the non-reduced root system of type BC_n in the literature). See table 1.1.

Solution to 1.3.4.3. □

Exercise 1.3.4.4. Prove that the long roots in G_2 form a root system in E of type A_2.

Solution to 1.3.4.4. □

Exercise 1.3.4.5. In constructing C_ℓ, would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.

Solution to 1.3.4.5. □

Exercise 1.3.4.6. Prove that the map $\alpha \mapsto -\alpha$ is an automorphism of Φ. Try to decide for which irreducible Φ this belongs to the Weyl group.

Solution to 1.3.4.6. □

Table 1.1: Highest long and short roots

Type	Long	Short
A_ℓ	$\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$	
B_ℓ	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_\ell$	$\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$
C_ℓ	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$
D_ℓ	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$	
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$	$2\alpha_1 + \alpha_2$

Exercise 1.3.4.7. Describe $\operatorname{Aut}\Phi$ when Φ is not irreducible.

Solution to 1.3.4.7. □

1.3.5 Abstract theory of weights

Exercise 1.3.5.1. Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ be the decomposition of Φ into its irreducible components, with $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$. Prove that Λ decomposes into a direct sum $\Lambda_1 \oplus \cdots \oplus \Lambda_t$; what about Λ^+?

Solution to 1.3.5.1. □

Exercise 1.3.5.2. Show by example (e.g., for A_2) that $\lambda \notin \Lambda^+$, $\alpha \in \Delta$, $\lambda - \alpha \in \Lambda^+$ is possible.

Solution to 1.3.5.2. □

Exercise 1.3.5.3. Verify some of the data in Table 1, e.g., for F_4.

Solution to 1.3.5.3. □

Exercise 1.3.5.4. Using Table 1, show that the fundamental group of A_ℓ is cyclic of order $\ell+1$, while that of D_ℓ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (ℓ odd), or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (ℓ even). (It is easy to remember which is which, since $\mathrm{A}_3 = \mathrm{D}_3$.)

Solution to 1.3.5.4. □

Exercise 1.3.5.5. If Λ' is any subgroup of Λ which includes Λ_r, prove that Λ' is $\mathscr{W}$-invariant. Therefore, we obtain a homomorphism $\phi : \operatorname{Aut}\Phi/\mathscr{W} \to \operatorname{Aut}(\Lambda/\Lambda_r)$. Prove that ϕ is injective, then deduce that $-1 \in \mathscr{W}$ if and only if $\Lambda_r \supset 2\Lambda$ (cf. Exercise 1.3.4.6). Show that $-1 \in \mathscr{W}$ for precisely the irreducible root systems $\mathrm{A}_1, \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell$ (ℓ even), $\mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$.

Solution to 1.3.5.5. □

Exercise 1.3.5.6. Prove that the roots in Φ which are dominant weights are precisely the highest long root and (if two root lengths occur) the highest short root (cf. (10.4) and Exercise 1.3.2.11), when Φ is irreducible.

Solution to 1.3.5.6. □

Exercise 1.3.5.7. If $\epsilon_1, \cdots, \epsilon_\ell$ is an *obtuse* basis of the euclidean space E (i.e., all $(\epsilon_i, \epsilon_j) \leq 0$ for $i \neq j$), prove that the dual basis is acute (i.e., all $\left(\epsilon_i^*, \epsilon_j^*\right) \geq 0$ for $i \neq j$). [Reduce to the case $\ell = 2$.]

Solution to 1.3.5.7. □

Exercise 1.3.5.8. Let Φ be irreducible. Without using the data in Table 1, prove that each λ_i is of the form $\sum_j q_{ij}\alpha_j$, where all q_{ij} are positive rational numbers. [Deduce from Exercise 1.3.5.7 that all q_{ij} are nonnegative. From $(\lambda_i, \lambda_i) > 0$ obtain $q_{ii} > 0$. Then show that if $q_{ij} > 0$ and $(\alpha_j, \alpha_k) < 0$, then $q_{ik} > 0$.]

Solution to 1.3.5.8. □

Exercise 1.3.5.9. Let $\lambda \in \Lambda^+$. Prove that $\sigma(\lambda + \delta) - \delta$ is dominant only for $\sigma = 1$.

Solution to 1.3.5.9. □

Exercise 1.3.5.10. If $\lambda \in \Lambda^+$, prove that the set Π consisting of all dominant weights $\mu \prec \lambda$ and their $\mathscr{W}$-conjugates is saturated, as asserted in (13.4).

Solution to 1.3.5.10. $\square$

Exercise 1.3.5.11. Prove that each subset of Λ is contained in a unique smallest saturated set, which is finite if the subset in question is finite.

Solution to 1.3.5.11. $\square$

Exercise 1.3.5.12. For the root system of type A_2, write down the effect of each element of the Weyl group on each of λ_1, λ_2. Using this data, determine which weights belong to the saturated set having highest weight $\lambda_1 + 3\lambda_2$. Do the same for type G_2 and highest weight $\lambda_1 + 2\lambda_2$.

Solution to 1.3.5.12. $\square$

Exercise 1.3.5.13. Call $\lambda \in \Lambda^+$ **minimal** if $\mu \in \Lambda^+$, $\mu \prec \lambda$ implies that $\mu = \lambda$. Show that each coset of Λ_r in Λ contains precisely one minimal λ. Prove that λ is minimal if and only if the $\mathscr{W}$-orbit of λ is saturated (with highest weight λ), if and only if $\lambda \in \Lambda^+$ and $\langle \lambda, \alpha \rangle = 0, 1, -1$ for all roots α. Determine (using Table 1) the nonzero minimal λ for each irreducible Φ, as follows:

$$\begin{aligned}
&\mathrm{A}_\ell : \lambda_1, \cdots, \lambda_l \\
&\mathrm{B}_\ell : \lambda_\ell \\
&\mathrm{C}_\ell : \lambda_1 \\
&\mathrm{D}_\ell : \lambda_1, \lambda_{\ell-1}, \lambda_\ell{}^{\cdot} \\
&\mathrm{E}_6 : \lambda_1, \lambda_6 \\
&\mathrm{E}_7 : \lambda_7
\end{aligned}$$

Solution to 1.3.5.13. $\square$

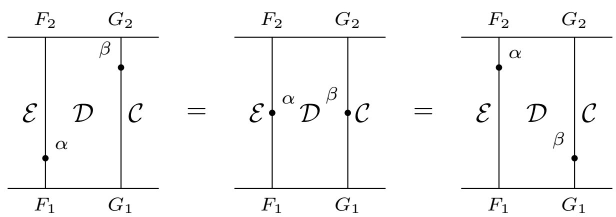
Chapter 2

Introduction to Soergel Bimodules

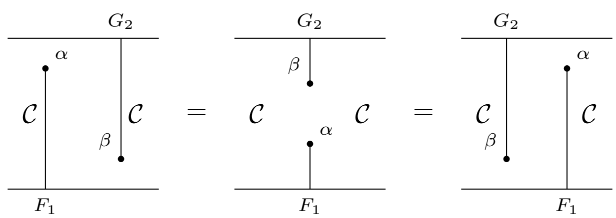
2.7 How to Draw Monoidal Categories

2.7.2 Planar Diagrams for 2-Categories

Exercise 2.7.2.8. Show that the axioms of a 2-category imply the following equalities.



(2.1)



(2.2)

Solution to 2.7.2.8. □

2.7.4 The Temperley–Lieb Category

Exercise 2.7.4.16. We can view the algebra $A = \mathbb{R}[x]/(x^2)$ as an object in the monoidal category of $\mathbb{R}$-vector spaces. Let $\cap : A \otimes A \to \mathbb{R}$ denote the linear map which sends $f \otimes g$ to the coefficient of x in fg. Let $\cup : \mathbb{R} \to A \otimes A$ denote the map which sends 1 to $x \otimes 1 + 1 \otimes x$.

(i) We wish to encode these maps diagrammatically, drawing $\cap$ as a cap and $\cup$ as a cup. Justify this diagrammatic notation, by checking the isotopy relations.

(ii) Draw a sequence of nested circles, as in an archery target. Evaluate this morphism.

Solution to 2.7.4.16(i). □

Solution to 2.7.4.16(ii). □

Exercise 2.7.4.17. This question is about the Temperley–Lieb category.

(i) Finish the proof that the isotopy relation holds in vector spaces.

(ii) There is a map $V \otimes V \to V \otimes V$ which sends $x \otimes y \mapsto y \otimes x$. Draw this as an element of the Temperley–Lieb category (a linear combination of diagrams).

(iii) Find an endomorphism of 2 strands which is killed by placing a cap on top. Can you find one which is an idempotent? Also find an endomorphism killed by putting a cup on bottom.

(iv) (Harder) Find an idempotent endomorphism of 3 strands which is killed by a cap on top (for either of the two placements of the cap).

Solution to 2.7.4.17(i). ☐

Solution to 2.7.4.17(ii). ☐

Solution to 2.7.4.17(iii). ☐

2.7.5 More About Isotopy

Exercise 2.7.5.19. One can think about the right mate and the left mate as "twisting" or "rotating" α by 180° to the right or to the left. Visualize what it would mean to twist α by 360° to the right, yielding another 2-morphism $\alpha^{\vee\vee} : E \to F$. Verify that ${}^{\vee}\alpha = \alpha^{\vee}$, if and only if $\alpha = \alpha^{\vee\vee}$. Thus cyclicity is the same as "360 degree rotation invariance," which one might expect from any planar picture.

Solution to 2.7.5.19. ☐

Exercise 2.7.5.20. Suppose that B is an object in a monoidal category with biadjoints, and $\Phi : B \otimes B \otimes B \to \mathbb{1}$ is a cyclic morphism. What should it mean to "rotate" Φ by 120°? Suppose that $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{C}$. What can you say about the 120° rotation of Φ, vis a vis Φ? What if $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{R}$?

Solution to 2.7.5.20. ☐

2.9 The Dihedral Cathedral

Exercise 9.25. Let our base ring be some specialization of $\mathbb{Z}[\delta]$. Inside $\mathrm{TL}_{n,\delta}$ let T be the vector space of elements which are killed by all the $(n-1)$ caps on top, and let B be the space killed by cups on the bottom. For an element $x \in \mathrm{TL}_{n,\delta}$ let $\bar{x}$ denote the same element with each diagram flipped upside down. Thus, for example, $x \in T$ if and only if $\bar{x} \in B$.

(i) Show that any crossingless matching is either the identity diagram, or has both a cap on bottom and a cup on top.

(ii) We now make the following assumption:

$$\text{There exist some } f \in T \text{ for which the coefficients of the identity diagram is invertible.} \tag{2.3}$$

Why is this equivalent to the analogous assumption for B?

(iii) Let $f \in T$, with invertible coefficient c for the identity diagram. Let $g \in B$, with invertible coefficient d for the identity diagram. Compute the composition fg in two ways and deduce that f and g are colinear.

(iv) Assuming 2.3 deduce that $T = B$, that this space is one-dimensional, and that $f = \bar{f}$ for $f \in T$.

(v) Thus, assuming 2.3, there is a unique element $\mathrm{JW}_n \in T$ whose identity coefficient is 1. Prove that JW_n is idempotent. (If we construct JW_n in some other way, this proves 2.3.)

Solution to 9.25(i). ☐

Solution to 9.25(ii). □

Solution to 9.25(iii). □

Solution to 9.25(iv). □

Solution to 9.25(v). □

Exercise 9.26. Let TL_n be the Temperley–Lieb algebra with n-strands where the bubble evaluates to $-[2] = q + q^{-1} \in \mathbb{Q}(q)$. Clearly, JW_1 is just the identity element, where the condition of being killed by caps and cups is vacuous. Verify the following recursive formula:

(2.4)

In this last diagram, the cup on top matches the a-th and $(a+1)$-st boundary points, counting from the left.

Solution to 9.26. □

Exercise 9.27. Prove the following recursive formula.

Solution to 9.27. □

Exercise 9.28. The trace of an element $a \in \mathrm{TL}_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the following closed diagram:

(2.5)

(i) Calculate the trace of JW_n.

(ii) Suppose that q is a primitive $2m$-th root of unity. What is the trace of JW_{m-1}? What do you get when you rotate JW_{m-1} by one strand?

Solution to 9.28(i). □

Solution to 9.28(ii). □

Exercise 9.34.

(i) Write down the two-color relations when $m = 2$. Prove that $B_s B_t \simeq B_t B_s$ by constructing inverse isomorphisms.

(ii) Write down the two-color relations when $m = 3$. Prove that $B_s B_t B_S \simeq X \oplus B_s$, where X is the image of an idempotent constructed using two 6-valent vertices, by following the rubric of Exercise 8.39.

(iii) (Still for m = 3) Similarly, one has $B_t B_s B_t \simeq Y \oplus B_t$. Prove that X is isomorphic to Y. Extend the rubric of Exercise 8.39 to a rubric which describes when two summands of different objects are isomorphic.

Solution to 9.34(i). □

Solution to 9.34(ii). □

Solution to 9.34(iii). □

Exercise 9.35. Prove that there is an autoequivalence of $\mathcal{H}_{\mathrm{BS}}$ which flips each diagram vertically (resp. horizontally). See Exercise 8.10 for inspiration.

Solution to 9.35. □

Exercise 9.36. Show that the diagram obtained by attaching a “handle” to the left or right of a Jones–Wenzl projector equals 0. For example,

(2.6)

(Hint: use (9.16).)

Solution to 9.36. □

Exercise 9.37.

(i) A *pitchfork* is a diagram of the form

(2.7)

(or its color swap). The death by pitchfork relation states that the diagram obtained by placing a pitchfork anywhere on top or bottom of a Jones–Wenzl projector equals 0. For example:

(2.8)

Why is death by pitchfork implied by the defining property of the Jones–Wenzl projector?

(ii) Use (9.28) and (9.31) to prove that the diagram obtained by placing a pitchfork anywhere on top or bottom of the $2m$-valent vertex equals 0. We also call this *death by pitchfork*.

Solution to 9.37(i). $\square$

Solution to 9.37(ii). $\square$

Exercise 9.38.

(i) Prove (9.29) and (9.30) using the relations in (9.27). (Hint: each relation follows from two careful applications of (9.27c). Alternatively, (9.29) can be proved by repeatedly applying (9.30).

(ii) Prove (9.27b) using (9.28) and the other relations in (9.27). (Hint: first use (8.12) to create a dot and a trivalent vertex on the left hand side, and then dispose of the trivalent vertex with two-color associativity.)

The following exercise is harder, but very worthwhile.

Solution to 9.38(i). $\square$

Solution to 9.38(ii). $\square$

Solution to 9.38(iii). $\square$

Exercise 9.39.

(i) Prove (9.28) using the relations in (9.27).

(ii) Prove that the Elias–Jones–Wenzl relation (9.27b) follows from two-color associativity (9.27c) and two-color dot contraction (9.28).

Solution to 9.39(i). $\square$

Solution to 9.39(ii). $\square$

Chapter 3

Research Papers

Exercise 3.1. Compute the value of a bigon at $q = 1$ or at general q.

Solution to 3.1. □

Exercise 3.2. Look at (2.9). Can you find associativity and coassociativity inside? Use only these relations and (2.4) to prove (2.9).

Solution to 3.2. □

Exercise 3.3. Write down what (2.10) means explicitly for some small values of k, l, r, s, until you get a feeling for how it works. You'll definitely want an exampel where k-l+r-s is at least 2 eventually. Then try to verify it using vectors for small values.

Solution to 3.3. □

Exercise 3.4. Try to prove Lemma 2.9 from Light Ladders and Clasp Conjectures

Solution to 3.4. □

Exercise 3.5. Remember how for the Temperley-Lieb algebra you described the "Crossing" $v \otimes w \mapsto w \otimes v$ as a linear combination of other maps. Let's do this again, but with webs this time. You're going to have to use $q = 1$ do this exercise, so forget about the q-deformation.

Consider the map $\Lambda^1 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^1 V$ which just swaps the tensor factors. This is a linear combination of:

(i) The web which merges $1, 2$ into 3 and then splits 3 into $2, 1$.

(ii) The web which splits $1, 2$ into $1, 1, 1$ and then merges $1, 1, 1$ into $2, 1$.

Find the linear combo.

Solution to 3.5(i). □

Solution to 3.5(ii). □

Exercise 3.6. Consider the map $\Lambda^2 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^2 V$ which just swaps the tensor factors. This is a linear combination of:

(i) the web which merges $2, 2$ into 4 and then splits 4 into $2, 2$.

(ii) the web which splits $2, 2$ into $2, 1, 1$ and then merges $2, 1, 1$ into $3, 1$ and then splits back to $2, 1, 1$ and marges back to $2, 2$.

(iii) the identity of $2, 2$.

Find the linear combo.

Solution to 3.6(i). □

Solution to 3.6(ii). □

Solution to 3.6(iii). □

Chapter 4

Misc

Exercise 4.1. Find a formula for the product $[n][3]$ when $n \geq 3$ and $[n][4]$ when $n \geq 4$. Generalize this.

Solution to 4.1. $\square$

Exercise 4.2. What is $[n][n] - [n+1][n-1]$?

Solution to 4.2. $\square$

Exercise 4.3. What is $[n][k] - [n+1][k-1]$ for $k < n$?

Solution to 4.3. $\square$