

# Fundamentals of Analysis II: Homework 1

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**Exercise 4.3.3.** Supply a proof for Theorem 4.3.9 (Composition of continuous functions) using  $\varepsilon - \delta$  characterization of continuity.

*Solution.* Let  $f$  be continuous at  $c$ , and let  $g$  be continuous at  $f(c)$ . Let  $\varepsilon > 0$  be arbitrary. Since  $g$  is continuous at  $f(c)$ , there exists  $\delta_1 > 0$  such that  $|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$ . Since  $f$  is continuous at  $c$ , there exists  $\delta_2 > 0$  such that  $|x - c| < \delta_2 \Rightarrow |f(x) - f(c)| < \delta_1$ . Now, consider  $\delta = \delta_2$ . If  $|x - c| < \delta$ , then by the continuity of  $f$ , we have  $|f(x) - f(c)| < \delta_1$ . Using the continuity of  $g$ , this implies  $|g(f(x)) - g(f(c))| < \varepsilon$ . Thus,  $|x - c| < \delta$  implies  $|g(f(x)) - g(f(c))| < \varepsilon$ . Since  $\varepsilon > 0$  was arbitrary, we conclude that  $g \circ f$  is continuous at  $c$ .  $\square$

**Exercise 4.3.4.** Assume  $f$  and  $g$  are defined on all of  $\mathbb{R}$  and that  $\lim_{x \rightarrow p} f(x) = q$  and  $\lim_{x \rightarrow q} g(x) = r$ .

(i) Give an example to show that it may not be true that

$$\lim_{x \rightarrow p} g(f(x)) = r.$$

(ii) Show that the result in (i) does follow if we assume  $f$  and  $g$  are continuous.

(iii) Does the result in (i) hold if we only assume  $f$  is continuous? How about if we only assume that  $g$  is continuous?

*Solution to (i).* Define  $f(x)$  and  $g(x)$  as follows

$$f(x) = q \quad \text{and} \quad g(x) = \begin{cases} \frac{xr}{q} & \text{if } x \neq q \\ 0 & \text{if } x = q \end{cases}.$$

Observe that  $\lim_{x \rightarrow p} f(x) = q$  holds trivially, since  $f(x) = q$  for all  $x$ . For  $g(x)$ ,

$$\lim_{x \rightarrow q} g(x) = \lim_{x \rightarrow q} \frac{xr}{q} = \frac{qr}{q} = r.$$

Thus,  $\lim_{x \rightarrow q} g(x) = r$ . However, for the composition

$$\lim_{x \rightarrow p} g(f(x)) = g(f(p)) = g(q) = 0.$$

Therefore,  $\lim_{x \rightarrow p} g(f(x)) = 0 \neq r$ , even though  $\lim_{x \rightarrow p} f(x) = q$  and  $\lim_{x \rightarrow q} g(x) = r$ .  $\square$

*Solution to (ii).* If  $f$  and  $g$  are both continuous, then we use Theorem 4.3.9, which is proven in Exercise 4.3.3.  $\square$

*Solution to (iii).* Not if  $f$  is continuous, as in the example I provided,  $f$  is continuous. But if  $g$  is continuous, then yes.  $\square$

**Exercise 4.3.6.** Provide an example of each or explain why the request is impossible.

- (i) Two functions  $f$  and  $g$ , neither of which is continuous at 0 but such that  $f(x)g(x)$  and  $f(x) + g(x)$  are continuous at 0.
- (ii) A function  $f(x)$  continuous at 0 and  $g(x)$  not continuous at 0 such that  $f(x) + g(x)$  is continuous at 0.
- (iii) A function  $f(x)$  continuous at 0 and  $g(x)$  not continuous at 0 such that  $f(x)g(x)$  is continuous at 0.

*Solution to (i).* Define  $f(x)$  and  $g(x)$  as follows

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}.$$

Notice that  $f(x)$  and  $g(x)$  are both not continuous at 0. Then, we get the constant function  $f(x) + g(x) = 0$  no matter the  $x$ -value, which is continuous at 0. Same thing for their product, we get a constant function  $f(x) \cdot g(x) = -1$  no matter the  $x$ -value, making it continuous at 0.  $\square$

*Solution to (ii).* Impossible, because the sum of a continuous function and a discontinuous function is always discontinuous.  $\square$

*Solution to (iii).* Let  $f(x)$  be a constant function at 0. Then, as long as  $g(x)$  is bounded, regardless if it's continuous or not, then by exercise 4.2.7,  $f(x)g(x) = 0$ .  $\square$

**Exercise 4.3.8.** Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that  $g$  is defined and continuous on all of  $\mathbb{R}$ .

(i) If  $g(x) \geq 0$  for all  $x < 1$ , then  $g(1) \geq 0$  as well.

(ii) If  $g(r) = 0$  for all  $r \in \mathbb{Q}$ , then  $g(x) = 0$  for all  $x \in \mathbb{R}$ .

(iii) If  $g(x_0) > 0$  for a single point  $x_0 \in \mathbb{R}$ , then  $g(x)$  is in fact strictly positive for uncountably many points.

*Solution to (i).* True, using the Sequential Definition for Functional Limits, letting  $x_n \rightarrow 1$ , we have  $g(x_n) \geq 0$  and  $g(x_n) \rightarrow g(1)$ . Then, by the Order Limit Theorem,  $g(1) \geq 0$ .  $\square$

*Solution to (ii).* True. Assume  $(\exists c \in \mathbb{R} - \mathbb{Q})[g(c) \neq 0]$ . That would cause  $g$  to not be continuous at  $x$  because we can't make  $\varepsilon$  smaller than  $|g(x)|$  because we can find a rational number  $r$  such that  $g(r) = 0$  inside any  $\delta$ -neighborhood.  $\square$

*Solution to (iii).* True, since  $g$  is continuous on all of  $\mathbb{R}$ , the positivity of  $g(x_0)$  implies that there exists an open interval  $I = (x_0 - \varepsilon, x_0 + \varepsilon)$  around  $x_0$ , where  $g(x) > 0$  for all  $x \in I$ . Then, using the fact that any non-empty interval  $I \subset \mathbb{R}$  contains uncountably many points. Since  $g(x) > 0$  for all  $x \in I$ , this means  $g(x)$  is strictly positive for uncountably many points.  $\square$

**Exercise 4.3.9.** Assume  $h : \mathbb{R} \rightarrow \mathbb{R}$  is continuous on  $\mathbb{R}$  and let  $K = \{x \mid h(x) = 0\}$ . Show that  $K$  is a closed set.

*Solution.* If  $K = \emptyset$ , then  $K$  is closed since  $\emptyset^c = \mathbb{R}$ , which is open. Let  $K'$  be the collection of limit points of  $K$ . Let  $a \in K'$ . Then, there exists a sequence in  $K$  such that  $x_n \rightarrow a$ . Since  $h$  is continuous on  $\mathbb{R}$ , then  $h(x_n) \rightarrow h(a)$ . But  $h(x_n) = 0$ , for all  $n$ . So,  $h(a) = 0$ , implying  $a \in K$ . Therefore,  $K' \subseteq K$ , meaning  $K$  is closed.  $\square$

**Exercise 4.5.3.** A function  $f$  is increasing on  $A$  if  $f(x) \leq f(y)$  for all  $x < y$  in  $A$ . Show that if  $f$  is increasing on  $[a, b]$  and satisfies the intermediate value property (Definition 4.5.3), then  $f$  is continuous on  $[a, b]$ .

*Solution.* Let  $x \in [a, b]$  and choose  $\varepsilon > 0$ . Let  $\ell_1 \in (f(a), f(x)) \cap (f(x) - \varepsilon, f(x))$ , as this ensures that  $\ell_1$  is both within range of  $[a, x]$  and close to  $f(x)$ . Since  $f$  satisfies the intermediate value property, there must exist a  $c_1 \in (a, x)$  such that  $f(c_1) = \ell_1$ . Since  $\ell_1$  was chosen to satisfy  $|f(x) - \ell_1| < \varepsilon$ , we get  $|f(x) - f(c_1)| < \varepsilon$ . Similarly, choose an  $\ell_2 \in (f(x), f(b)) \cap (f(x) + \varepsilon, f(b))$ . Like before, there must exist a  $c_2 \in (x, b)$  such that  $f(c_2) = \ell_2$ . Then, we get  $|f(x) - f(c_2)| < \varepsilon$ .

Since  $f$  is increasing, for all  $y \in [c_1, c_2]$ , we have  $f(c_1) \leq f(y) \leq f(c_2)$ . Thus,

$$|f(x) - f(y)| \leq \max\{|f(x) - f(c_1)|, |f(x) - f(c_2)|\}.$$

From earlier, we know that  $|f(x) - f(c_1)| < \varepsilon$  and  $|f(x) - f(c_2)| < \varepsilon$ , so this implies  $|f(x) - f(y)| < \varepsilon$  for all  $y \in [c_1, c_2]$ . To ensure  $y \in [c_1, c_2]$ , set  $\delta = \min\{x - c_1, c_2 - x\}$ . Then, for all  $y \in [a, b]$  with  $|x - y| < \delta$ , it follows that  $y \in [c_1, c_2]$ , and hence  $|f(x) - f(y)| < \varepsilon$ .  $\square$

**Exercise 4.5.6a.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous with  $f(0) = f(1)$ . Show that there must exist  $x, y \in [0, 1]$  satisfying  $|x - y| = 1/2$  and  $f(x) = f(y)$ .

*Solution.* Using the hint, this reduces to finding  $x \in [0, 1/2]$  such that  $F(x) = 0$ , where

$$F(x) = f\left(x + \frac{1}{2}\right) - f(x).$$

Since  $f$  is continuous on  $[0, 1]$ ,  $F(x)$  is also continuous on the interval where it is defined, which is  $[0, 1/2]$ , because the argument  $x + 1/2$  remains within  $[0, 1]$  when  $x \in [0, 1/2]$ . Evaluating  $F(x)$  at certain points gives us

$$\text{At } x = 0 \Rightarrow F(0) = f\left(0 + \frac{1}{2}\right) - f(0)$$

$$\text{At } x = \frac{1}{2} \Rightarrow F(0) = f(1) - f(0).$$

Since  $f(0) = f(1)$ , it follows that

$$F\left(\frac{1}{2}\right) = f(0) - f\left(\frac{1}{2}\right).$$

Therefore,

$$F(0) = f\left(\frac{1}{2}\right) - f(0) \quad \text{and} \quad F\left(\frac{1}{2}\right) = f(0) - f\left(\frac{1}{2}\right).$$

From the previous expression, notice that  $F(0) = -F(1/2)$ , meaning  $F(0)$  and  $F(1/2)$  have opposite signs or at least one of them is zero. Since  $F(x)$  is continuous on  $[0, 1/2]$ , by the Intermediate Value Theorem, there exists  $c \in [0, 1/2]$  such that  $F(c) = 0$ . Therefore, we have

$$F(c) = f\left(c + \frac{1}{2}\right) - f(c) = 0 \Rightarrow f\left(c + \frac{1}{2}\right) = f(c).$$

Setting  $x = c$  and  $y = c + 1/2$ , we found  $|x - y| = 1/2$  and  $f(x) = f(y)$ .  $\square$

**Exercise 4.5.7.** Let  $f$  be a continuous function on the closed interval  $[0, 1]$  with range also contained in  $[0, 1]$ . Prove that  $f$  must have a fixed point; that is, show  $f(x) = x$  for at least one value of  $x \in [0, 1]$ .

*Solution.* Define  $g(x) = f(x) - x$ , which is continuous on  $[0, 1]$ , since it's the sum of two continuous functions. At  $x = 0$ ,  $g(0) = f(0)$ . Since the range of  $f$  is contained in  $[0, 1]$ , we know  $f(0) \geq 0$ , so  $g(0) \geq 0$ . At  $x = 1$ ,  $g(1) = f(1) - 1$ , so  $g(1) \leq 0$ . Then, since  $g(x)$  is continuous on  $[0, 1]$ ,  $g(0) \geq 0$ , and  $g(1) \leq 0$ , the intermediate value theorem guarantees us that there exist a  $c \in [0, 1]$  such that  $g(c) = 0$ . Notice that  $f(c) - c = 0 \Rightarrow f(c) = c$ , meaning at we have a fixed point at  $x = c$ .  $\square$