MATH 410 - WINTER 2026 - HOMEWORK 2

1. (Gaussian integral computation.) Let $a > 0$. Compute the value of
$$\int_{\mathbb{R}^n} e^{-a|x|^2}\,dx.$$

Hint. First reduce matters to computing a one-dimensional integral by writing $|x|^2 = x_1^2 + \cdots + x_n^2$. To compute the one-dimensional case, start with
$$\left(\int_{\mathbb{R}} e^{-ax^2}\,dx\right)^2$$
View this as a two-dimensional integral and then use polar coordinates.

2. (Radon transform of a Gaussian.) Let $f(x) = e^{-a|x|^2}$. Compute the Radon transform of f. (You will need your solution from Problem 1.)

3. (Back-projection) Let f be the characteristic function of the unit ball in $\mathbb{R}^2$. First verify that the Radon transform is given by
$$\mathcal{R}\chi_B(t,\omega) = \begin{cases} 2\sqrt{1-t^2} & |t| \le 1, \\ 0 & |t| > 1. \end{cases}$$
Now consider the back-projection defined by
$$\tilde{f}(x) = \tfrac{1}{2\pi}\int_0^{2\pi} \mathcal{R}f(\langle x,\omega\rangle,\omega)\,d\theta, \quad \omega = (\cos\theta,\sin\theta).$$
Fix $x = (r,0)$ for some $r \ge 0$, and establish the following properties:

- For $0 \le r \le 1$, we have
$$\tilde{f}((r,0)) = \tfrac{1}{\pi}\int_0^{2\pi} \sqrt{1 - r^2\cos^2\theta}\,d\theta.$$
- For $r > 1$, we have a bound of the form
$$|\tilde{f}((r,0))| \le \tfrac{C}{r}.$$

4. (Projection onto a hyperplane) Let $p \in \mathbb{R}$ and $r \in \mathbb{R}^J$. Show that the projection of a vector $y \in \mathbb{R}^J$ onto the hyperplane $\{x \in \mathbb{R}^J : x \cdot r = p\}$ is given by
$$y \mapsto y - \left[\tfrac{y\cdot r - p}{r\cdot r}\right] r.$$

5. ($1d$ version of pixel basis) For each $N = 1, 2, \ldots$ define the intervals
$$I_j^N = [\tfrac{j}{N}, \tfrac{j+1}{N}), \quad j = 0, \ldots, N-2 \quad \text{and} \quad I_{N-1}^N = [\tfrac{N-1}{N}, 1].$$
Suppose $f : [0,1] \to \mathbb{R}$ is continuous and for each N define
$$f_N(x) = N\sum_{j=0}^{N-1}\left[\int_{I_j^N} f(y)\,dy\right]\chi_{I_j^N}(x),$$
where $\chi_{I_j^N}$ is the characteristic function of I_j^N. Show that f_N converges to f uniformly on $[0,1]$ as $N \to \infty$.