1. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let T be a linear transformation on V. Prove that T is self-adjoint if and only if $(\mathbf{x}, T\mathbf{x})$ is real a real number for all $\mathbf{x} \in V$.

2. Let $A \in \mathbb{C}^{n \times n}$. Prove that A is normal if and only if A can be written in the form of $A = A_1 + iA_2$ where A_1 and A_2 are Hermitian and $A_1 A_2 = A_2 A_1$.

(Hint: Let $A_1 = \frac{1}{2}(A + A^*)$ and $A_2 = \frac{1}{2i}(A - A^*)$.)

3. Prove that a normal and nilpotent linear transformation is the zero linear transformation.
(Note: A linear transformation T is nilpotent if there exists a positive integer r such that $T^r = 0$.)

4. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Prove that A is positive definite if and only if all the eigenvalues of A are positive.

5. Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix. Prove that A is unitary if and only if every eigenvalue of A has magnitude (or absolute value) 1.

6. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Suppose A is both positive definite and unitary. Prove that $A = I$. *Hint: You may use conclusions from the above two problems.*

7. Consider $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

   1). Find the singular value decomposition of A. (Show your computation work, do not use technology.)

   2). Find the generalized inverse A^+.

8. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and indefinite (i.e. not positive definite or positive semi-definite). Suppose $A = PDP^*$ for some unitary matrix P and some diagonal matrix $D \in \mathbb{R}^{n \times n}$. Find the singular value decomposition of $A = V\Sigma U^*$ by constructing V, Σ and U using some possible variations of the columns or entries of P and D.

9. Let A be an $m \times n$ matrix, and let P and Q be $m \times m$ and $n \times n$ unitary matrices. Show that A and PAQ have the same singular values.