Introduction to Toplogy I: Homework 6

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Exercise 6.1.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f(A)$ for the following subsets $A \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

(ii) Now let $f : X \to Y$ be arbitrary, and let $A, B \subset X$. Prove that if $A \subset B$ then $f(A) \subset f(B)$. Prove that $f(A \cup B) = f(A) \cup f(B)$. Prove that $f(A \cap B) \subset f(A) \cap f(B)$, but give an example where they are not equal.

Solution to (i). For $f(x) = x^2$ we have:

$$f([-1, 1]) = [0, 1], \quad f([-1, 1)) = [0, 1], \quad f((-1, 1)) = [0, 1), \quad f([0, 1]) = [0, 1]$$
$$f([0, 1)) = [0, 1), \quad f((0, 1)) = (0, 1), \quad f(\{-1\}) = \{1\}, \quad f(\{0\}) = \{0\}, \quad f(\{1\}) = \{1\}. \qquad \square$$

Solution to (ii). Let $f : X \to Y$ and let $A, B \subset X$.

If $A \subset B$, then for every $x \in A$ we have $x \in B$. Hence $f(x) \in f(B)$ for all $x \in A$, so $f(A) \subset f(B)$.

For the union,

$$f(A \cup B) = \{f(x) \mid x \in A \cup B\} = \{f(x) \mid x \in A\} \cup \{f(x) : x \in B\} = f(A) \cup f(B).$$

For the intersection, if $x \in A \cap B$ then $x \in A$ and $x \in B$, hence $f(x) \in f(A)$ and $f(x) \in f(B)$. Therefore $f(A \cap B) \subset f(A) \cap f(B)$.

Equality doesn't have to hold. Take, for instance, $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, and $A = \{-1\}$, $B = \{1\}$. Then, we have that $A \cap B = \emptyset$. Therefore, $f(A \cap B) = \emptyset$, but $f(A) = \{1\} = f(B)$, so

$$f(A) \cap f(B) = \{1\} \neq \emptyset = f(A \cap B).$$

Thus, in this case, we have $f(A \cap B) \neq f(A) \cap f(B)$. $\square$

Exercise 6.2.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f^{-1}(B)$ for the following subsets $B \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

(ii) Now let $f : X \to Y$ be arbitrary, and let $A, B \subset Y$. Prove that if $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Solution to (i). For $f(x) = x^2$ we have:

$$f^{-1}([-1, 1]) = [-1, 1], \quad f^{-1}([-1, 1)) = (-1, 1), \quad f^{-1}((-1, 1)) = (-1, 1)$$
$$f^{-1}([0, 1]) = [-1, 1], \quad f^{-1}([0, 1)) = (-1, 1), \quad f^{-1}((0, 1)) = (-1, 0) \cup (0, 1)$$
$$f^{-1}(\{-1\}) = \emptyset, \quad f^{-1}(\{0\}) = \{0\}, \quad f^{-1}(\{1\}) = \{-1, 1\}. \qquad \square$$

Solution to (ii). Let $f : X \to Y$ and let $A, B \subset Y$.

If $A \subset B$ and $x \in f^{-1}(A)$, then $f(x) \in A \subset B$, so $x \in f^{-1}(B)$. Thus $f^{-1}(A) \subset f^{-1}(B)$. For the union, we have

$$f^{-1}(A \cup B) = \{x \in X \mid f(x) \in A \cup B\} = f^{-1}(A) \cup f^{-1}(B),$$

and for the intersection, we have

$$f^{-1}(A \cap B) = \{x \mid f(x) \in A \cap B\} = \{x \mid f(x) \in A \text{ and } f(x) \in B\} = f^{-1}(A) \cap f^{-1}(B). \qquad \square$$

Exercise 6.5. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases},$$

is discontinuous (with respect to the usual metric). Find an open set $V \subset \mathbb{R}$ such that $f^{-1}(V)$ is not open.

Solution. Take the open set $V = (-1/2, 1/2) \subset \mathbb{R}$. Now, we compute $f^{-1}(V)$. If $x \leq 0$ then $f(x) = x$, so $x \in f^{-1}(V)$ precisely when $-1/2 < x \leq 0$, i.e. this contribution is $(-1/2, 0]$. If $x > 0$ then $f(x) = x + 1$, and $x + 1 \in (-1/2, 1/2)$ would force $-3/2 < x < -1/2$, which is impossible for $x > 0$. Hence the $x > 0$ part contributes nothing. Therefore

$$f^{-1}(V) = \left(-\frac{1}{2}, 0\right].$$

The set $(-1/2, 0]$ is not open in $\mathbb{R}$ because it contains the point 0 but no open interval around 0 is contained. Thus $f^{-1}(V)$ is not open. □

Exercise 6.6. Use Proposition 6.6 to prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous with respect to the usual metric, as follows.

(i) Let A be an open interval of the form $(a, \infty) \subset \mathbb{R}$. Find $f^{-1}(A)$ and observe that it is open.

(ii) Let B be an open interval of the form $(-\infty, b) \subset \mathbb{R}$. Find $f^{-1}(B)$ and observe that it is open.

(iii) Let C be an open interval of the form $(a, b) \subset \mathbb{R}$. Prove from the previous two parts that $f^{-1}(C)$ is open.

(iv) Let $V \subset \mathbb{R}$ be an arbitrary open set. Prove that V is a union of open intervals. Conclude that $f^{-1}(V)$ is open.

(v) Look up a δ-ε proof that f is continuous and reproduce it here, or write one yourself. Which proof would you say is more straightforward?

Solution to (i). Let $A = (a, \infty) \subset \mathbb{R}$. Then $f^{-1}(A) = \{x \in \mathbb{R} \mid x^2 > a\}$. If $a < 0$, then $x^2 > a$ for all $x \in \mathbb{R}$, so $f^{-1}(A) = \mathbb{R}$. If $a = 0$, then $f^{-1}(A) = \{x \mid x^2 > 0\} = \mathbb{R} \setminus \{0\}$. If $a > 0$, then $f^{-1}(A) = \{x \mid |x| > \sqrt{a}\} = (-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$. In each case, $f^{-1}(A)$ is open. □

Solution to (ii). Let $B = (-\infty, b) \subset \mathbb{R}$. Then $f^{-1}(B) = \{x \in \mathbb{R} \mid x^2 < b\}$. If $b \leq 0$, then there is no $x \in \mathbb{R}$ with $x^2 < b$, so $f^{-1}(B) = \emptyset$. If $b > 0$, then $f^{-1}(B) = \{x \mid |x| < \sqrt{b}\} = (-\sqrt{b}, \sqrt{b})$. In both cases, $f^{-1}(B)$ is open. □

Solution to (iii). Let $C = (a, b) \subset \mathbb{R}$. Then

$$f^{-1}(C) = \{x \in \mathbb{R} \mid a < x^2 < b\} = \{x \mid x^2 > a\} \cap \{x \mid x^2 < b\} = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b)).$$

By parts (i) and (ii), both $f^{-1}((a, \infty))$ and $f^{-1}((-\infty, b))$ are open, so their intersection $f^{-1}(C)$ is also open. □

Solution to (iv). Let $V \subset \mathbb{R}$ be an arbitrary open set. By definition of the usual topology on $\mathbb{R}$, for each $x \in V$ there exists an open interval (a_x, b_x) such that $x \in (a_x, b_x) \subset V$. Therefore,

$$V = \bigcup_{x \in V} (a_x, b_x).$$

Since each (a_x, b_x) is an open interval, by part (iii) each $f^{-1}((a_x, b_x))$ is open. Hence,

$$f^{-1}(V) = f^{-1}\left(\bigcup_{x \in V} (a_x, b_x)\right) = \bigcup_{x \in V} f^{-1}((a_x, b_x)),$$

which is a union of open sets, and therefore open. By Proposition 6.6, it follows that f is continuous. □

Solution to (v). To prove that f is continuous using the δ-ε definition, let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$. We want to find a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Notice that

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0|.$$

To bound $|x + x_0|$, choose $\delta \leq 1$. Then $|x - x_0| < \delta \leq 1$ implies $|x| < |x_0| + 1$, so

$$|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1.$$

Thus,

$$|f(x) - f(x_0)| < \delta(2|x_0| + 1).$$

To ensure that this is less than ε, choose

$$\delta = \min\left\{1, \frac{\varepsilon}{2|x_0| + 1}\right\}.$$

Then whenever $|x - x_0| < \delta$, it follows that $|f(x) - f(x_0)| < \varepsilon$. Therefore, f is continuous at every point $x_0 \in \mathbb{R}$.

Between the two proofs, the one using preimages of open sets is more straightforward, because it avoids all the technical details of choosing δ in terms of ε and just deals with open sets directly. □

Exercise 7.1. Prove that each of the four topologies on $\mathbb{R}$ given in Example 7.3 is a topology, that is, it satisfies the three conditions in Definition 7.1.

Solution. Let $\mathcal{T}_1 = \{U \subseteq \mathbb{R} \mid U^c \text{ is finite}\} \cup \{\emptyset\}$. Then, $\emptyset, \mathbb{R} \in \mathcal{T}_1$. If $U, V \in \mathcal{T}_1$, then notice that $(U \cap V)^c = U^c \cup V^c$, which is a union of finite sets. Hence, $U \cap V \in \mathcal{T}_1$. For any arbitrary collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}_1$, for any arbitrary index set I, we have

$$\left(\bigcup_{i \in I} U_i\right)^c = \bigcap_{i \in I} U_i^c,$$

which is an intersection of finite sets, and hence finite. Therefore, $\bigcup_{i \in I} U_i \in \mathcal{T}_1$. Thus, the finite complement topology is indeed a topology.

Fix $p \in \mathbb{R}$, and let $\mathcal{T}_2 = \{U \subseteq \mathbb{R} \mid p \in U\} \cup \{\emptyset\}$. Then $\emptyset, \mathbb{R} \in \mathcal{T}_2$. If $U, V \in \mathcal{T}_2$, then $p \in U$ and $p \in V$, so $p \in U \cap V$, hence $U \cap V \in \mathcal{T}_2$. For any collection $\{U_i\}_{i \in I} \subseteq \mathcal{T}_2$, we have $p \in U_i$ for all i, so $p \in \bigcup_{i \in I} U_i$, hence $\bigcup_{i \in I} U_i \in \mathcal{T}_2$. Thus $\mathcal{T}_2$ is a topology.

Let $\mathcal{S} = \{(-\infty, a) \mid a \in \mathbb{R}\}$ and let $\mathcal{T}_3$ be the topology generated by $\mathcal{S}$, i.e., the smallest topology containing all sets of the form $(-\infty, a)$. Since $\emptyset, \mathbb{R}$ are unions of sets in $\mathcal{S}$, they lie in $\mathcal{T}_3$. Let $U, V \in \mathcal{T}_3$. Then U and V are unions of sets from $\mathcal{S}$, and for any $a, b \in \mathbb{R}$,

$$(-\infty, a) \cap (-\infty, b) = (-\infty, \min\{a, b\}),$$

which is again in $\mathcal{S}$. Thus $U \cap V$ is a union of sets in $\mathcal{S}$, hence in $\mathcal{T}_3$. Arbitrary unions of unions of sets in $\mathcal{S}$ remain unions of sets in $\mathcal{S}$, so $\mathcal{T}_3$ is a topology.

Let $\mathcal{B} = \{[a, b) \mid a < b, a, b \in \mathbb{R}\}$, and let $\mathcal{T}_4$ be the topology generated by $\mathcal{B}$. Then $\emptyset, \mathbb{R}$ are unions of sets in $\mathcal{B}$, so they belong to $\mathcal{T}_4$. For $[a, b), [c, d) \in \mathcal{B}$, we have

$$[a, b) \cap [c, d) = [\max\{a, c\}, \min\{b, d\}),$$

which is either empty or again in $\mathcal{B}$. Hence the intersection of any two basis elements is a union of basis elements. Arbitrary unions of basis elements remain in $\mathcal{T}_4$, so $\mathcal{T}_4$ is a topology. □

Exercise 7.2. Find the interiors, closures, and boundaries of the following subsets $A \subset \mathbb{R}$ in the topologies from Example 7.3:

(i) $\mathbb{Z} \subset \mathbb{R}$ in the finite complement topology.

(ii) $\{0\} \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ in the particular point topology.

(iii) $(0, 1) \subset \mathbb{R}$ in the lower semi-continuous topology.

(iv) $(0, 1) \subset \mathbb{R}$ in the lower limit topology.

Solution to (i). The interior of $\mathbb{Z}$ is the largest open set contained in $\mathbb{Z}$. Any nonempty open set is cofinite and thus contains non-integer real numbers, so no nonempty open set is contained in $\mathbb{Z}$. Hence $\operatorname{int}(\mathbb{Z}) = \emptyset$.

Closed sets are exactly $\mathbb{R}$ and all finite subsets. Since $\mathbb{Z}$ is infinite, the only closed set containing $\mathbb{Z}$ is $\mathbb{R}$. Thus $\overline{\mathbb{Z}} = \mathbb{R}$.

The boundary is just given by $\partial\mathbb{Z} = \overline{\mathbb{Z}} \setminus \operatorname{int}(\mathbb{Z}) = \mathbb{R} \setminus \emptyset = \mathbb{R}$. $\square$

Solution to (ii). First consider $\{0\}$. If $p = 0$, then $\{0\}$ contains p, so it is open. Therefore $\operatorname{int}(\{0\}) = \{0\}$. To compute the closure, note that a closed set contains $\{0\}$ only if it is $\mathbb{R}$ (since any closed set other than $\mathbb{R}$ must omit p, but here $p = 0$). Hence $\overline{\{0\}} = \mathbb{R}$, and $\partial\{0\} = \mathbb{R} \setminus \{0\}$.

Now assume $p \neq 0$. Then $\{0\}$ does not contain p, so it is closed. The largest open subset of $\{0\}$ is $\emptyset$, so $\operatorname{int}(\{0\}) = \emptyset$. Since $\{0\}$ is closed, $\overline{\{0\}} = \{0\}$, and therefore $\partial\{0\} = \{0\}$.

Next consider $\{1\}$. If $p = 1$, then $\{1\}$ is open, so $\operatorname{int}(\{1\}) = \{1\}$. The only closed set containing $\{1\}$ is $\mathbb{R}$, so $\overline{\{1\}} = \mathbb{R}$, and thus $\partial\{1\} = \mathbb{R} \setminus \{1\}$.

Finally, if $p \neq 1$, then $\{1\}$ does not contain p, hence it is closed. Its interior is $\emptyset$, its closure is $\{1\}$, and therefore $\partial\{1\} = \{1\}$. $\square$

Solution to (iii). To find the interior, note that any basic open set $(-\infty, a)$ that is contained in $(0, 1)$ must satisfy $a \leq 0$, which forces $(-\infty, a) \subseteq (-\infty, 0]$. But no such set lies inside $(0, 1)$ except the empty set. Hence the interior of $(0, 1)$ is empty, so $\operatorname{int}(A) = \emptyset$.

To find the closure, suppose $x \in \mathbb{R}$. Then every basic open neighborhood of x is of the form $(-\infty, a)$ with $a > x$, and such a neighborhood always intersects $(0, 1)$ whenever $x < 1$, since $(0, 1) \subset (-\infty, a)$ for all $a > 1$. If $x \geq 1$, then every basic neighborhood $(-\infty, a)$ of x with $a > x \geq 1$ also contains $(0, 1)$. Thus every point $x \in \mathbb{R}$ satisfies the condition for being in the closure, and therefore $\overline{A} = \mathbb{R}$.

Since $\partial A = \overline{A} \setminus \operatorname{int}(A)$, we conclude that $\partial(0, 1) = \mathbb{R}$. $\square$

Solution to (iv). To find the interior, observe that for any $x \in (0, 1)$, we may choose a basis element $[x, x+\varepsilon)$ for some $\varepsilon > 0$, and this set is contained in $(0, 1)$ provided $x + \varepsilon \leq 1$. Therefore each $x \in (0, 1)$ has a basis neighborhood contained in $(0, 1)$, and hence $\operatorname{int}(A) = (0, 1)$.

To find the closure, let $x < 0$. Then every basis neighborhood $[x, x+\varepsilon)$ contains negative points only, and thus cannot intersect $(0, 1)$. Hence such x are not in the closure. If $0 \leq x \leq 1$, then every basis neighborhood $[x, x + \varepsilon)$ intersects $(0, 1)$, since it contains points in $(0, 1)$ whenever $\varepsilon > 0$. If $x > 1$, then we may take ε small enough so that $[x, x+\varepsilon)$ lies entirely to the right of 1, hence such neighborhoods do not intersect $(0, 1)$. Therefore $\overline{A} = [0, 1]$.

Finally, since $\partial A = \overline{A} \setminus \operatorname{int}(A)$, we conclude that $\partial(0, 1) = [0, 1] \setminus (0, 1) = \{0, 1\}$. $\square$