CS 410/510 HW1

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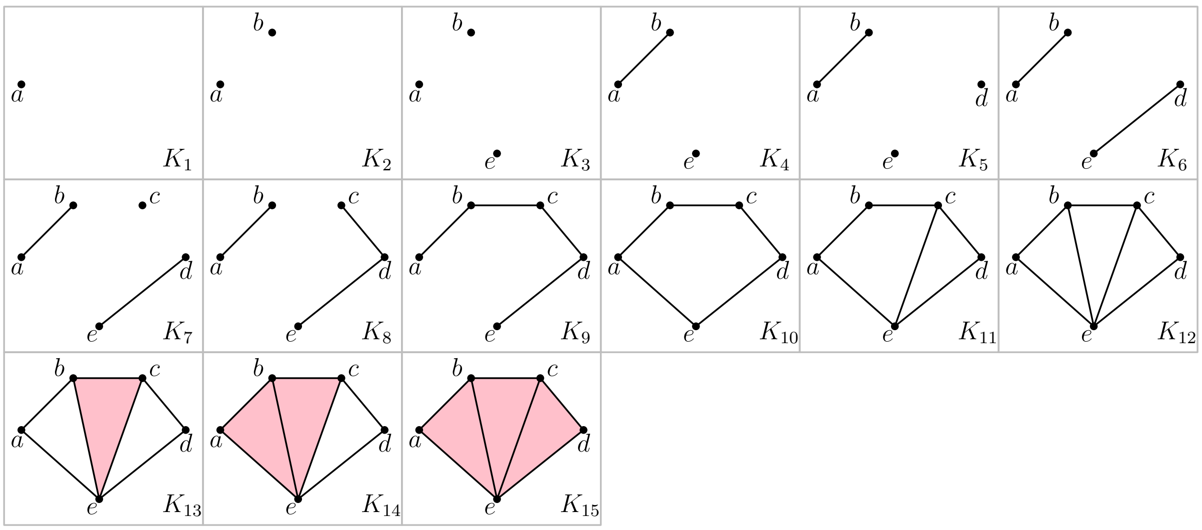
Note: You should submit a single pdf to Canvas. If you are handwriting your answer, I would recommend you use a scanner app such as Adobe Scan to convert you writing into pdf (it gives you a much cleaner scan than simply using a camera).

1. (20 points) Using the algorithm described in class, compute the PD of the following simplex-wise filtration:



Specifically, provide intervals in the PD of each dimension and also for each finite interval (not ending with $+\infty$), provide its representative (aka. its entry in the "ζ" table as demonstrated in class). Notice that a representative (chain) should be denoted as a sum of σ_i's as done in class, where i is the index of the simplex added in the filtration. For example, the chain $a + b$ should be denoted as $\sigma_1 + \sigma_2$ where $\sigma_1 = a$ and $\sigma_2 = b$.

Remark: There are several ways to quickly check whether your results make sense:

- Each index should occur exactly once in an interval, either as the start or the end.
- The representative for an interval $[b, d)$ should be a cycle created in K_b and becoming a boundary in K_d.

Solution:

- $\sigma_1 = a$: $z = \partial(a) = 0$
- $\sigma_2 = b$: $z = \partial(b) = 0$
- $\sigma_3 = e$: $z = \partial(e) = 0$

- $\sigma_4 = ab$:

  $z = \partial(ab) = a + b = \sigma_1 + \sigma_2$

  $\max(z) = \sigma_2$ (unpaired)

  add interval $[2, 4)$ to PD_0

  $\zeta(\sigma_2) = \sigma_1 + \sigma_2$

- $\sigma_5 = d$: $z = \partial(d) = 0$

- $\sigma_6 = de$:

  $z = \partial(de) = d + e = \sigma_3 + \sigma_5$

  $\max(z) = \sigma_5$ (unpaired)

  add interval $[5, 6)$ to PD_0

  $\zeta(\sigma_5) = \sigma_3 + \sigma_5$

- $\sigma_7 = c$: $z = \partial(c) = 0$

- $\sigma_8 = cd$:

  $z = \partial(cd) = c + d = \sigma_5 + \sigma_7$

  $\max(z) = \sigma_7$ (unpaired)

  add interval $[7, 8)$ to PD_0

  $\zeta(\sigma_7) = \sigma_5 + \sigma_7$

- $\sigma_9 = bc$:

  $z = \partial(bc) = b + c = \sigma_2 + \sigma_7$

  $\max(z) = \sigma_7$ (paired)

  $z = z + \zeta(\sigma_7) = \sigma_2 + \sigma_7 + \sigma_5 + \sigma_7 = \sigma_2 + \sigma_5$

  $\max(z) = \sigma_5$ (paired)

  $z = z + \zeta(\sigma_5) = \sigma_2 + \sigma_5 + \sigma_3 + \sigma_5 = \sigma_2 + \sigma_3$

  $\max(z) = \sigma_3$ (unpaired)

  add interval $[3, 9)$ to PD_0

  $\zeta(\sigma_3) = \sigma_2 + \sigma_3$

- $\sigma_{10} = ae$:

  ... z eventually reduces to 0

- $\sigma_{11} = ce$:

  ... z eventually reduces to 0

- $\sigma_{12} = be$:

  ... z eventually reduces to 0

- $\sigma_{13} = bce$:

  $z = \partial(bce) = bc + be + ce = \sigma_9 + \sigma_{11} + \sigma_{12}$

  $\max(z) = \sigma_{12}$ (unpaired)

  add interval $[12, 13)$ to PD_1

  $\zeta(\sigma_{12}) = \sigma_9 + \sigma_{11} + \sigma_{12}$

- $\sigma_{14} = abe$:

  $z = \partial(abe) = ab + ae + be = \sigma_4 + \sigma_{10} + \sigma_{12}$

  $\max(z) = \sigma_{12}$ (paired)

  $z = z + \zeta(\sigma_{12}) = \sigma_4 + \sigma_{10} + \sigma_{12} + \sigma_9 + \sigma_{11} + \sigma_{12} = \sigma_4 + \sigma_9 + \sigma_{10} + \sigma_{11}$

  $\max(z) = \sigma_{11}$ (unpaired)

  add interval $[11, 14)$ to PD_1

  $\zeta(\sigma_{11}) = \sigma_4 + \sigma_9 + \sigma_{10} + \sigma_{11}$

- $\sigma_{15} = cde$:

  $z = \partial(cde) = cd + ce + de = \sigma_6 + \sigma_8 + \sigma_{11}$

  $\max(z) = \sigma_{11}$ (paired)

  $z = z + \zeta(\sigma_{11}) = \sigma_6 + \sigma_8 + \sigma_{11} + \sigma_4 + \sigma_9 + \sigma_{10} + \sigma_{11} = \sigma_4 + \sigma_6 + \sigma_8 + \sigma_9 + \sigma_{10}$

  $\max(z) = \sigma_{10}$ (unpaired)

  add interval $[10, 15)$ to PD_1

  $\zeta(\sigma_{10}) = \sigma_4 + \sigma_6 + \sigma_8 + \sigma_9 + \sigma_{10}$

ζ array:

- σ_1:
- σ_2: $\sigma_1 + \sigma_2$
- σ_3: $\sigma_2 + \sigma_3$
- σ_4:
- σ_5: $\sigma_3 + \sigma_5$
- σ_6:
- σ_7: $\sigma_5 + \sigma_7$
- σ_8:
- σ_9:
- σ_{10}: $\sigma_4 + \sigma_6 + \sigma_8 + \sigma_9 + \sigma_{10}$

- σ_{11}: $\sigma_4 + \sigma_9 + \sigma_{10} + \sigma_{11}$
- σ_{12}: $\sigma_9 + \sigma_{11} + \sigma_{12}$
- σ_{13}:
- σ_{14}:
- σ_{15}:

PD_0:

- $[2, 4)$ (Rep: $\sigma_1 + \sigma_2$)
- $[5, 6)$ (Rep: $\sigma_3 + \sigma_5$)
- $[7, 8)$ (Rep: $\sigma_5 + \sigma_7$)
- $[3, 9)$ (Rep: $\sigma_2 + \sigma_3$)
- $[1, +\infty)$

PD_1:

- $[12, 13)$ (Rep: $\sigma_9 + \sigma_{11} + \sigma_{12}$)
- $[11, 14)$ (Rep: $\sigma_4 + \sigma_9 + \sigma_{10} + \sigma_{11}$)
- $[10, 15)$ (Rep: $\sigma_4 + \sigma_6 + \sigma_8 + \sigma_9 + \sigma_{10}$)

2. (6 points) Given a filtration $\mathcal{F}$, its *p-th Betti curve* $B_p : I \to \mathbb{N}$ is a map from the indices I of the filtration $\mathcal{F}$ to natural numbers $\mathbb{N}$ such that $B_p(i)$ is the p-th Betti number of the complex K_i in $\mathcal{F}$. Compute the 0th and 1st Betti curve for the filtration in Question 1. Specifically, fill out a table of the following form:

Index	1	2	...	15
B_0				
B_1				

Hint: Recall that there is a way mentioned in class about reading off the Betti numbers for a complex in a filtration using the persistence diagram (barcode).

Solution:

Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
B_0	1	2	3	2	3	2	3	2	1	1	1	1	1	1	1
B_1	0	0	0	0	0	0	0	0	0	1	2	3	2	1	0

3. (6 points) Consider the following non-simplex-wise filtration $\mathcal{F}$ and its corresponding simplex-wise filtration $\mathcal{F}'$ where the dashed lines indicate equality of complexes:

$$\mathcal{F} : K_0 \dashrightarrow K_1 \dashrightarrow K_2 \dashrightarrow K_3 \dashrightarrow K_4 \dashrightarrow K_5 \dashrightarrow K_6$$

$$\mathcal{F}' : K'_0 \hookrightarrow K'_1 \hookrightarrow K'_2 \hookrightarrow K'_3 \hookrightarrow K'_4 \hookrightarrow K'_5 \hookrightarrow K'_6 \hookrightarrow K'_7 \hookrightarrow K'_8 \hookrightarrow K'_9 \hookrightarrow K'_{10} \hookrightarrow K'_{11} \hookrightarrow K'_{12} \hookrightarrow K'_{13}$$

Convert the following intervals in the PD of $\mathcal{F}'$ to intervals in the PD of $\mathcal{F}$ (if an interval does not have correspondence in $\mathcal{F}$, just say "none"):

- $[2, 5)$
- $[7, 13)$
- $[2, 4)$

Solution:

- $[2, 5)$: $[1, 2)$
- $[7, 12)$: $[3, 6)$
- $[2, 4)$: none

4. (10 points) Consider four points a, b, c, d with the following pair-wise distance:

	a	b	c	d
a		1.5	0.1	0.5
b			1.0	0.3
c				0.9
d				

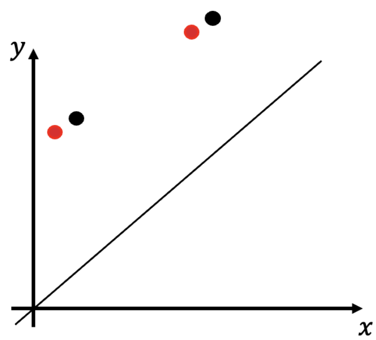
Construct the Vietoris-Rips filtration for the four points. You only need to list the different complexes in the filtration, by denoting each simplex as, say, *abcd*. (Recall that to build VR filtration, you only need to first figure out the order of the edges being introduced, and then you just take the clique complexes once you figure out the edges.)

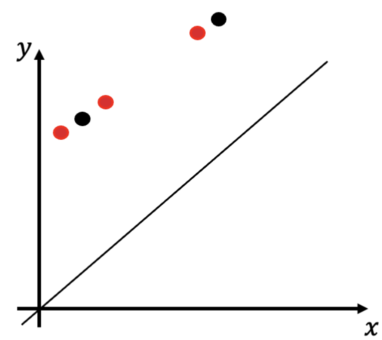
Solution: Complexes after adding each edge one by one based on the length (vertices omitted):

- add ac: $\{ac\}$
- add bd: $\{ac, bd\}$
- add ad: $\{ac, ad, bd\}$

- add cd: $\{ac, ad, bd, cd, acd\}$
- add bc: $\{ac, ad, bc, bd, cd, acd, bcd\}$
- add ab: $\{ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd\}$

5. (6 points)





Consider the two PDs on the left $\mathbb{R}^2$ plane where one PD contains red points and another PD contains black points. Assume the following:

- Each red or black point (interval) has the same length (aka. $d - b$) of 1.0.
- The first red point and the first black point are 0.2 distance away, and the second red point and the second black point are 0.2 distance away.
- The first black point and the second red points are 1.5 distance away.

We then have that, the two PDs (on the left $\mathbb{R}^2$ plane) have a bottleneck distance of 0.2, with the two closest pairs of red and black points perfectly matched.

Now, add another red point whose length is also 1.0 and whose distance to the first black point is 0.2, as indicated in right figure. What is the bottleneck distance of the two PDs in this case? You should also provide brief justifications for your answer.

Solution: The bottleneck distance becomes 1.0. With a new red point added, the two PDs cannot be perfectly matched, meaning that at least one of the points needs to be matched to the diagonal, so that the bottleneck distance is at least 1.0. Matching the new red point to diagonal, we achieve a distance of 1.0.