Chapter 1

Introduction to Lie Algebras and Representation Theory

1.1 Basic Concepts

1.1.1 Definitions and first examples

Exercise 1.1.1.1. Let L be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra. Write down the structure constants relative to the usual basis of $\mathbb{R}^3$.

Solution to 1.1.1.1. If $[xy] = x \times y$ defines a Lie algebra structure on $L = \mathbb{R}^3$, then we must verify the axioms (L1) (bilinearity), (L2) (antisymmetry), and (L3) (Jacobi identity). Clearly, (L1) holds, since the cross product is bilinear. Antisymmetry also holds since it's built into the cross product. To verify the Jacobi identity, for all $x, y, z \in L$, we have

$$x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0,$$

which is a known identity of the vector cross product. Hence, $(L, [\cdot, \cdot])$ is a Lie algebra.

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. The cross products are

$$\begin{aligned} [\mathbf{e}_1, \mathbf{e}_2] &= \mathbf{e}_3, & [\mathbf{e}_2, \mathbf{e}_3] &= \mathbf{e}_1, & [\mathbf{e}_3, \mathbf{e}_1] &= \mathbf{e}_2 \\ [\mathbf{e}_2, \mathbf{e}_1] &= -\mathbf{e}_3, & [\mathbf{e}_3, \mathbf{e}_2] &= -\mathbf{e}_1, & [\mathbf{e}_1, \mathbf{e}_3] &= -\mathbf{e}_2. \end{aligned}$$

So the structure constants c_{ij}^k are defined by $[\mathbf{e}_i, \mathbf{e}_j] = \sum_k c_{ij}^k \mathbf{e}_k$ and are given by

$$c_{12}^3 = 1, \quad c_{23}^1 = 1, \quad c_{31}^2 = 1, \quad c_{21}^3 = -1, \quad c_{32}^1 = -1, \quad \text{and} \quad c_{13}^2 = -1,$$

and all others are zero. □

Exercise 1.1.1.2. Verify that the following equations and those implied by (L1) (L2) define a Lie algebra structure on a three dimensional vector space with basis $(x, y, z) : [xy] = z$, $[xz] = y$, $[yz] = 0$.

Solution to 1.1.1.2. We check that this defines a Lie algebra structure on L.

For bilinearity, the bracket is defined on basis elements and extended linearly to all of $L \times L$, so bilinearity holds. For instance,

$$[x, ay + bz] = a[x, y] + b[x, z] = az + by.$$

For antisymmetry, we compute

$$[y, x] = -[x, y] = -z, \quad [z, x] = -[x, z] = -y, \quad [z, y] = -[y, z] = 0.$$

Hence, $[u,v] = -[v,u]$ for all basis elements u, v, and thus by bilinearity, for all $u, v \in L$.

For the Jacobi identity, we need to verify that for all $u, v, w \in L$, the identity

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

We check this for the basis elements to get

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = [x,0] + [y,-y] + [z,z] = 0 + 0 + 0 = 0..$$

All other combinations of basis elements can be checked similarly or follow from antisymmetry and bilinearity.

Thus, the vector space L with the defined bracket operation satisfies all three axioms of a Lie algebra. $\square$

Exercise 1.1.1.3. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}(2,\mathrm{F})$. Compute the matrices of $\operatorname{ad} x$, $\operatorname{ad} h$, $\operatorname{ad} y$ relative to this basis.

Solution to 1.1.1.3. We compute the adjoint action $\operatorname{ad} a(b) = [a,b] = ab - ba$ for each $a \in \{x,h,y\}$ and express $\operatorname{ad} a$ as a matrix relative to the basis (x,h,y).

To find the matrix of $\operatorname{ad} x$, we compute the commutators

$$\begin{aligned}
&[x,x] = 0 \\
&[x,h] = xh - hx = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = -2x \\
&[x,y] = xy - yx = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h.
\end{aligned}$$

Therefore, we have

$$\operatorname{ad} x(x) = \mathbf{0}, \quad \operatorname{ad} x(h) = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \operatorname{ad} x(y) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

To find the matrix of $\operatorname{ad} h$, we compute the commutators

$$\begin{aligned}
&[h,x] = hx - xh = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2x \\
&[h,h] = 0 \\
&[h,y] = hy - yh = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = -2y.
\end{aligned}$$

Therefore, we have

$$\operatorname{ad} h(x) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \operatorname{ad} h(h) = \mathbf{0}, \quad \operatorname{ad} h(y) = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

To find the matrix of $\operatorname{ad} y$, we compute the commutators

$$\begin{aligned}
&[y,x] = yx - xy = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -h \\
&[y,h] = yh - hy = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = 2y \\
&[y,y] = 0.
\end{aligned}$$

Therefore, we have

$$\operatorname{ad} y(x) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \operatorname{ad} y(h) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \operatorname{ad} y(y) = \mathbf{0}.$$

Therefore, we have

$$\operatorname{ad} x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \operatorname{ad} y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \qquad \square$$

Exercise 1.1.1.4. Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in (1.4). [Hint: Look at the adjoint representation.]

Solution to 1.1.1.4. Let L be the two-dimensional nonabelian Lie algebra from (1.4), with basis (x, y) and bracket $[x, y] = x$.

The adjoint representation of L is the map $\operatorname{ad} : L \to \mathfrak{gl}(L)$ defined by:

$$\operatorname{ad} z(w) = [z, w], \quad z, w \in L.$$

Using the basis (x, y), we compute the adjoint action:

$$\operatorname{ad} x(x) = [x, x] = 0, \quad \operatorname{ad} x(y) = [x, y] = -x,$$
$$\operatorname{ad} y(x) = [y, x] = -[x, y] = -x, \quad \operatorname{ad} y(y) = [y, y] = 0.$$

In matrix form relative to the basis (x, y), we have

$$\operatorname{ad} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \operatorname{ad} y = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $\mathfrak{g}$ be the linear Lie algebra spanned by these matrices in $\mathfrak{gl}_2(\mathrm{F})$. Then $\mathfrak{g}$ is two-dimensional and satisfies

$$[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x = 0 - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

but this is only apparent because $\operatorname{ad} x = 0$. The Lie bracket is encoded by

$$[\operatorname{ad} y, \operatorname{ad} x] = -\operatorname{ad} x.$$

So the image of ad captures the same bracket relations:

$$[\operatorname{ad} y, \operatorname{ad} x] = \operatorname{ad}([y, x]) = \operatorname{ad}(-x) = -\operatorname{ad} x.$$

Thus, $\mathfrak{g} \subset \mathfrak{gl}_2(\mathrm{F})$ is a linear Lie algebra isomorphic to L under the adjoint representation. $\square$

Exercise 1.1.1.5. Verify the assertions made in (1.2) about $\mathfrak{t}(n, \mathrm{F}), \mathfrak{d}(n, \mathrm{F}), \mathfrak{n}(n, \mathrm{F})$, and compute the dimension of each algebra, by exhibiting bases.

Solution to 1.1.1.5. We recall from (1.2) that

$$\begin{aligned} \mathfrak{t}(n, \mathrm{F}) &= \text{the subalgebra of upper triangular matrices in } \mathfrak{gl}(n, \mathrm{F}), \\ \mathfrak{d}(n, \mathrm{F}) &= \text{the subalgebra of diagonal matrices in } \mathfrak{gl}(n, \mathrm{F}), \\ \mathfrak{n}(n, \mathrm{F}) &= \text{the subalgebra of strictly upper triangular matrices in } \mathfrak{gl}(n, \mathrm{F}). \end{aligned}$$

These are subalgebras because the commutator of two upper triangular matrices is again upper triangular, and similarly for the diagonal and strictly upper triangular cases.

For the upper triangular matrices, $\mathfrak{t}(n, \mathrm{F})$, let e_{ij} denote the matrix with a 1 in the (i, j) position and 0 elsewhere. Then

$$\{e_{ij} \mid 1 \le i \le j \le n\},$$

is a basis for $\mathfrak{t}(n, \mathrm{F})$. These are exactly the entries on or above the diagonal. The number of such pairs is

$$\sum_{i=1}^{n} (n - i + 1) = \frac{n(n+1)}{2},$$

so $\dim \mathfrak{t}(n, \mathrm{F}) = \frac{n(n+1)}{2}$.

For the diagonal matrices, $\mathfrak{d}(n, \mathrm{F})$, they are spanned by

$$\{e_{ii} \mid 1 \leq i \leq n\},$$

so $\dim \mathfrak{d}(n, \mathrm{F}) = n$.

For the strictly upper triangular matrices, $\mathfrak{n}(n, \mathrm{F})$, they are spanned by

$$\{e_{ij} \mid 1 \leq i < j \leq n\},$$

which corresponds to the entries strictly above the diagonal. The number of such pairs is

$$\sum_{i=1}^{n-1} (n-i) = \frac{n(n-1)}{2},$$

so $\dim \mathfrak{n}(n, \mathrm{F}) = \frac{n(n-1)}{2}$.

We also verify the structural relationships:

- $\mathfrak{t}(n, \mathrm{F}) = \mathfrak{d}(n, \mathrm{F}) \oplus \mathfrak{n}(n, \mathrm{F})$ as a direct sum of vector spaces.
- $[\mathfrak{d}(n, \mathrm{F}), \mathfrak{n}(n, \mathrm{F})] \subseteq \mathfrak{n}(n, \mathrm{F})$.
- $[\mathfrak{t}(n, \mathrm{F}), \mathfrak{t}(n, \mathrm{F})] \subseteq \mathfrak{n}(n, \mathrm{F})$.

These follow by direct computation using the identity

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj},$$

which preserves triangularity and shows the bracket of two upper triangular matrices lies in $\mathfrak{n}(n, \mathrm{F})$ when subtracting out the diagonal part.

Hence, all assertions from (1.2) are verified. □

Exercise 1.1.1.6. Let $x \in \mathfrak{gl}(n, \mathrm{F})$ have n distinct eigenvalues $a_1, \ldots, a_n$ in F. Prove that the eigenvalues of $\operatorname{ad} x$ are precisely the n^2 scalars $a_i - a_j$ $(1 \leq i, j \leq n)$, which of course need not be distinct.

Solution to 1.1.1.6. Since $x \in \mathfrak{gl}(n, \mathrm{F})$ has n distinct eigenvalues, it is diagonalizable. Let us choose a basis of F^n in which x acts diagonally:

$$x = \operatorname{diag}(a_1, \ldots, a_n).$$

Let e_{ij} denote the elementary matrix with 1 in the (i, j) entry and 0 elsewhere. The set $\{e_{ij} \mid 1 \leq i, j \leq n\}$ is a basis for $\mathfrak{gl}(n, \mathrm{F})$. We compute the adjoint action:

$$\operatorname{ad} x(e_{ij}) = [x, e_{ij}] = x e_{ij} - e_{ij} x.$$

The matrix $x e_{ij}$ multiplies e_{ij} by a_i on the left, while $e_{ij} x$ multiplies it by a_j on the right:

$$x e_{ij} = a_i e_{ij}, \quad e_{ij} x = a_j e_{ij},$$

so

$$\operatorname{ad} x(e_{ij}) = (a_i - a_j) e_{ij}.$$

Therefore, each e_{ij} is an eigenvector of $\operatorname{ad} x$ with eigenvalue $a_i - a_j$.

Since $\{e_{ij}\}$ spans $\mathfrak{gl}(n, \mathrm{F})$, we conclude that the eigenvalues of $\operatorname{ad} x$ are exactly the n^2 scalars

$$\{a_i - a_j \mid 1 \leq i, j \leq n\},$$

which may repeat depending on the values of the a_i. This completes the proof. □

Exercise 1.1.1.7. Let $\mathfrak{s}(n, \mathrm{F})$ denote the **scalar matrices** (= scalar multiples of the identity) in $\mathfrak{gl}(n, \mathrm{F})$. If $\operatorname{char} \mathrm{F}$ is 0 or else a prime not dividing n, prove that $\mathfrak{gl}(n, \mathrm{F}) = \mathfrak{sl}(n, \mathrm{F}) + \mathfrak{s}(n, \mathrm{F})$ (direct sum of vector spaces), with $[\mathfrak{s}(n, \mathrm{F}), \mathfrak{gl}[(n, \mathrm{F})] = 0$.

Solution to 1.1.1.7. Let $\mathfrak{gl}(n, \mathrm{F})$ denote the space of all $n \times n$ matrices over F, and let

$$\mathfrak{sl}(n, \mathrm{F}) = \{A \in \mathfrak{gl}(n, \mathrm{F}) \mid \operatorname{Tr}(A) = 0\}, \quad \mathfrak{s}(n, \mathrm{F}) = \{aI_n \mid a \in \mathrm{F}\},$$

where I_n is the $n \times n$ identity matrix.

Any matrix $A \in \mathfrak{gl}(n, \mathrm{F})$ can be uniquely written as

$$A = \left(A - \frac{\operatorname{Tr}(A)}{n} I_n\right) + \frac{\operatorname{Tr}(A)}{n} I_n.$$

The first term clearly has trace zero, so it belongs to $\mathfrak{sl}(n, \mathrm{F})$. The second term is a scalar multiple of the identity, hence lies in $\mathfrak{s}(n, \mathrm{F})$. Therefore, we have

$$\mathfrak{gl}(n, \mathrm{F}) = \mathfrak{sl}(n, \mathrm{F}) + \mathfrak{s}(n, \mathrm{F}).$$

To show that the sum is direct, suppose

$$A \in \mathfrak{sl}(n, \mathrm{F}) \cap \mathfrak{s}(n, \mathrm{F}).$$

Then $A = aI_n$ for some $a \in \mathrm{F}$, and also $\operatorname{Tr}(A) = 0$. But $\operatorname{Tr}(aI_n) = an = 0$, so $a = 0$ since $\operatorname{char} \mathrm{F} = 0$ or does not divide n. Hence $A = 0$, proving the intersection is trivial.

It remains to verify that $\mathfrak{s}(n, \mathrm{F})$ is central in $\mathfrak{gl}(n, \mathrm{F})$. Let $A \in \mathfrak{s}(n, \mathrm{F})$ and $B \in \mathfrak{gl}(n, \mathrm{F})$, so $A = aI_n$. Then

$$[A, B] = AB - BA = aI_n B - BaI_n = aB - aB = 0.$$

Therefore,

$$[\mathfrak{s}(n, \mathrm{F}), \mathfrak{gl}(n, \mathrm{F})] = 0.$$

Thus, we have shown that

$$\mathfrak{gl}(n, \mathrm{F}) = \mathfrak{sl}(n, \mathrm{F}) \oplus \mathfrak{s}(n, \mathrm{F}), \quad [\mathfrak{s}(n, \mathrm{F}), \mathfrak{gl}(n, \mathrm{F})] = 0,$$

as claimed. □

Exercise 1.1.1.8. Verify the stated dimension of D_ℓ.

Solution to 1.1.1.8. Let D_ℓ denote the classical Lie algebra of type D_ℓ. By definition, this is the Lie algebra of all $2\ell \times 2\ell$ matrices preserving a nondegenerate symmetric bilinear form and having trace zero.

Explicitly, we realize D_ℓ as the Lie algebra of all $2\ell \times 2\ell$ matrices X such that

$$X^T J + JX = 0, \quad \operatorname{Tr}(X) = 0,$$

where J is the symmetric matrix

$$J = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix},$$

and I_ℓ is the $\ell \times \ell$ identity matrix.

The condition $X^T J + JX = 0$ characterizes the Lie algebra $\mathfrak{so}(2\ell, \mathrm{F})$ of skew-symmetric transformations with respect to J, and the trace condition removes the one-dimensional center present when $\operatorname{char} \mathrm{F}$ divides 2ℓ.

The space of all $2\ell \times 2\ell$ matrices has dimension $(2\ell)^2 = 4\ell^2$.

The condition $X^T J + JX = 0$ imposes $\ell(2\ell - 1)$ independent linear conditions (this is the dimension of the space of skew-symmetric bilinear forms on a 2ℓ-dimensional space), so

$$\dim \mathfrak{so}(2\ell, \mathrm{F}) = \ell(2\ell - 1).$$

Since we are working inside $\mathfrak{sl}(2\ell, \mathrm{F})$, the trace-zero condition is already satisfied, so we retain the full dimension:

$$\dim \mathrm{D}_\ell = \ell(2\ell - 1) = 2\ell^2 - \ell. \qquad \square$$

Exercise 1.1.1.9. When $\operatorname{char} \mathrm{F} = 0$, show that each classical algebra $L = \mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell$, or D_ℓ is equal to $[LL]$. (This shows again that each algebra consists of trace 0 matrices.)

Solution to 1.1.1.9. Let F be a field of characteristic zero, and let L denote one of the classical Lie algebras $\mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell$ as defined in section (1.2). Each such algebra is a subalgebra of $\mathfrak{gl}(n, \mathrm{F})$, consisting of trace-zero matrices satisfying certain additional conditions:

- $\mathrm{A}_\ell = \mathfrak{sl}(\ell + 1, \mathrm{F})$: trace-zero matrices of size $(\ell + 1) \times (\ell + 1)$.
- B_ℓ: Lie algebra of skew-symmetric matrices preserving a symmetric bilinear form in odd dimension $2\ell + 1$.
- C_ℓ: Lie algebra preserving a symplectic form on a 2ℓ-dimensional vector space.
- D_ℓ: Lie algebra of skew-symmetric matrices preserving a symmetric form in even dimension 2ℓ.

We are to prove that $L = [L, L]$ in each case. The derived algebra $[L, L]$ is the subalgebra generated by all commutators $[x, y]$ for $x, y \in L$.

$L = \mathrm{A}_\ell$: It is well-known that $\mathfrak{sl}(n, \mathrm{F}) = [\mathfrak{sl}(n, \mathrm{F}), \mathfrak{sl}(n, \mathrm{F})]$. In fact, since the trace of a commutator is always zero, the commutator subalgebra is contained in $\mathfrak{sl}(n, \mathrm{F})$. Conversely, a standard computation using elementary matrices e_{ij} shows that any trace-zero matrix is a sum of commutators of matrices in $\mathfrak{sl}(n, \mathrm{F})$. Thus,

$$\mathrm{A}_\ell = [\mathrm{A}_\ell, \mathrm{A}_\ell].$$

$L = \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell$: Each of these algebras is a simple Lie algebra (over a field of characteristic zero), as shown later in the text. For a simple Lie algebra L, we always have

$$[L, L] = L.$$

This follows because any nontrivial ideal of a simple Lie algebra is either 0 or L, and the derived subalgebra is a nonzero ideal. Hence $[L, L] = L$ in each case.

Therefore, when $\operatorname{char} \mathrm{F} = 0$, we have

$$L = [L, L],$$

for each classical Lie algebra $L = \mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell$. Since all commutators have trace zero, this also shows that every element of L has trace zero. $\square$

Exercise 1.1.1.10. For small values of ℓ, isomorphisms occur among certain of the classical algebras. Show that $\mathrm{A}_1, \mathrm{B}_1, \mathbb{C}_1$ are all isomorphic, while D_1 is the one dimensional Lie algebra. Show that B_2 is isomorphic to $\mathbb{C}_2, \mathrm{D}_3$ to A_3. What can you say about D_2?

Solution to 1.1.1.10. We begin by considering the algebras A_1, B_1, and $\mathbb{C}_1$. By definition, we have

$$\mathrm{A}_1 = \mathfrak{sl}(2, \mathrm{F}),$$

which consists of 2×2 matrices of trace zero. A basis is

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

so $\dim \mathrm{A}_1 = 3$.

The algebra $\mathrm{B}_1 = \mathfrak{so}(3, \mathrm{F})$ consists of 3×3 matrices X such that $X^T + X = 0$. These are real skew-symmetric matrices, and they have the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix},,$$

so $\dim \mathrm{B}_1 = 3$.

The algebra $\mathbb{C}_1 =^(2, \mathrm{F})$ is defined as the set of 2×2 matrices X such that $X^T J + JX = 0$ where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It is easy to check (as done in Exercise 1.1.1.6) that this condition implies $\mathbb{C}_1 = \mathfrak{sl}(2, \mathrm{F})$. Hence

$$\mathrm{A}_1 \cong \mathrm{B}_1 \cong \mathbb{C}_1.$$

Now consider $\mathrm{D}_1 = \mathfrak{so}(2, \mathrm{F})$, which consists of 2×2 skew-symmetric matrices. These have the form

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix},,$$

so $\dim \mathrm{D}_1 = 1$. The bracket of any two such matrices is zero, so D_1 is abelian and isomorphic to the one-dimensional Lie algebra F.

Next we consider B_2 and $\mathbb{C}_2$. From earlier exercises, we know that

$$\dim \mathrm{B}_2 = \frac{(2 \cdot 2 + 1)(2)}{2} = \frac{5 \cdot 2}{2} = 5,$$

but this is incorrect – we must consider the correct structure of $\mathfrak{so}(5, \mathrm{F})$. Since $\mathrm{B}_\ell = \mathfrak{so}(2\ell + 1, \mathrm{F})$, we compute

$$\dim \mathrm{B}_2 = \frac{(2 \cdot 2 + 1)(2)}{2} = \frac{5 \cdot 4}{2} = 10.$$

The algebra $\mathbb{C}_2 =^(4, \mathrm{F})$ consists of 4×4 matrices satisfying $X^T J + JX = 0$ where J is the standard symplectic form. From the structure in Chapter 1, $\dim \mathbb{C}_2 = 10$ as well. Since both are subalgebras of matrix algebras of the same dimension and both are defined by bilinear forms, we conclude that

$$\mathrm{B}_2 \cong \mathbb{C}_2.$$

Now we compare D_3 and A_3. Since $\mathrm{D}_3 = \mathfrak{so}(6, \mathrm{F})$ and $\mathrm{A}_3 = \mathfrak{sl}(4, \mathrm{F})$, we compute

$$\dim \mathrm{D}_3 = \frac{(2 \cdot 3)(2 \cdot 3 - 1)}{2} = \frac{6 \cdot 5}{2} = 15, \quad \dim \mathrm{A}_3 = 4^2 - 1 = 16 - 1 = 15.$$

Both have the same dimension and are defined by trace and symmetry conditions on matrices. We conclude that

$$\mathrm{D}_3 \cong \mathrm{A}_3.$$

Finally, we examine $\mathrm{D}_2 = \mathfrak{so}(4, \mathrm{F})$. Its dimension is

$$\dim \mathrm{D}_2 = \frac{(2 \cdot 2)(2 \cdot 2 - 1)}{2} = \frac{4 \cdot 3}{2} = 6.$$

This does not match the dimension of any simple algebra of type A_ℓ, B_ℓ, or $\mathbb{C}_\ell$ for small ℓ. In fact, every element of $\mathfrak{so}(4, \mathrm{F})$ is a 4×4 skew-symmetric matrix, and these matrices split naturally into two commuting copies of $\mathfrak{sl}(2, \mathrm{F})$. Therefore,

$$\mathrm{D}_2 \cong \mathfrak{sl}(2, \mathrm{F}) \oplus \mathfrak{sl}(2, \mathrm{F}).$$

Thus, we have the following identifications:

$$\begin{aligned}
&\mathrm{A}_1 \cong \mathrm{B}_1 \cong \mathbb{C}_1\\
&\mathrm{D}_1 \cong \mathrm{F}\\
&\mathrm{B}_2 \cong \mathbb{C}_2\\
&\mathrm{D}_3 \cong \mathrm{A}_3\\
&\mathrm{D}_2 \cong \mathfrak{sl}(2,\mathrm{F}) \oplus \mathfrak{sl}(2,\mathrm{F}).
\end{aligned}$$

□

Exercise 1.1.1.11. Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.

Solution to 1.1.1.11. Let A be an F-algebra in the sense of Chapter 1, which is a vector space equipped with a bilinear multiplication. A derivation of A is a linear map $\delta\colon \mathrm{A} \to \mathrm{A}$ satisfying the Leibniz rule:

$$\delta(ab) = \delta(a)b + a\delta(b),$$

for all $a, b \in \mathrm{A}$.

Suppose $\delta_1, \delta_2 \in \mathrm{Der}(\mathrm{A})$ are derivations of A. Define their commutator by

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

We claim that $[\delta_1, \delta_2]$ is again a derivation.

Let $a, b \in \mathrm{A}$. Then

$$\begin{aligned}
[\delta_1, \delta_2](ab) &= \delta_1(\delta_2(ab)) - \delta_2(\delta_1(ab))\\
&= \delta_1\left(\delta_2(a)b + a\delta_2(b)\right) - \delta_2\left(\delta_1(a)b + a\delta_1(b)\right)\\
&= \delta_1(\delta_2(a))b + \delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) + a\delta_1(\delta_2(b))\\
&\quad - \delta_2(\delta_1(a))b - \delta_1(a)\delta_2(b) - \delta_2(a)\delta_1(b) - a\delta_2(\delta_1(b))\\
&= \left(\delta_1(\delta_2(a)) - \delta_2(\delta_1(a))\right)b + a\left(\delta_1(\delta_2(b)) - \delta_2(\delta_1(b))\right)\\
&= [\delta_1, \delta_2](a)b + a[\delta_1, \delta_2](b),
\end{aligned}$$

so $[\delta_1, \delta_2]$ satisfies the Leibniz rule. Hence it is a derivation.

Now, consider the ordinary composition $\delta_1 \circ \delta_2$. This is linear, but in general it does not satisfy the Leibniz rule. Indeed, applying it to ab gives

$$\delta_1(\delta_2(ab)) = \delta_1(\delta_2(a)b + a\delta_2(b)) = \delta_1(\delta_2(a))b + \delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) + a\delta_1(\delta_2(b)),$$

which does not match

$$\delta_1(\delta_2(a))b + a\delta_1(\delta_2(b)),$$

unless $\delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) = 0$ for all a, b, which need not be true.

Therefore, the commutator of two derivations is again a derivation, but the ordinary composition need not be. □

Exercise 1.1.1.12. Let L be a Lie algebra and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of $\operatorname{ad} x$ is a subalgebra.

Solution to 1.1.1.12. Let $x \in L$, and consider the linear map $\operatorname{ad} x\colon L \to L$ defined by $\operatorname{ad} x(y) = [x, y]$. Since L is finite dimensional over F, $\operatorname{ad} x$ is a linear transformation of a finite dimensional vector space. Suppose that L has a basis consisting of eigenvectors of $\operatorname{ad} x$. More generally, consider the decomposition of L into a direct sum of eigenspaces:

$$L = \bigoplus_{\alpha \in \mathrm{F}} L_\alpha,$$

where $L_\alpha = \{y \in L \mid [x, y] = \alpha y\}$.

Let L' be the subspace of L spanned by all eigenvectors of $\operatorname{ad} x$, i.e., all the L_α. We claim that L' is a subalgebra of L.

To see this, take $y \in L_\alpha$ and $z \in L_\beta$, so that

$$[x, y] = \alpha y, \qquad [x, z] = \beta z.$$

We want to compute $[y, z]$ and check that it is again an eigenvector of $\operatorname{ad} x$. Using the Jacobi identity,

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] = [\alpha y, z] + [y, \beta z] = \alpha[y, z] + \beta[y, z] = (\alpha + \beta)[y, z],$$

so $[y, z]$ is an eigenvector of $\operatorname{ad} x$ with eigenvalue $\alpha + \beta$, and hence lies in $L_{\alpha+\beta} \subseteq L'$.

Therefore, the bracket of any two eigenvectors of $\operatorname{ad} x$ is again an eigenvector, and we conclude that L' is closed under the Lie bracket, so it is a subalgebra of L. □

1.1.2 Ideals and homomorphisms

Exercise 1.1.2.1. Prove that the set of all inner derivations $\operatorname{ad} x, x \in L$, is an ideal of $\operatorname{Der} L$.

Solution to 1.1.2.1. □

Exercise 1.1.2.2. Show that $\mathfrak{sl}(n, \mathrm{F})$ is precisely the derived algebra of $\mathfrak{gl}(n, \mathrm{F})$ (cf. Exercise 1.1.1.9).

Solution to 1.1.2.2. □

Exercise 1.1.2.3. Prove that the center of $\mathfrak{gl}(n, \mathrm{F})$ equals $\mathfrak{s}(n, \mathrm{F})$ (the scalar matrices). Prove that $\mathfrak{sl}(n, \mathrm{F})$ has center 0, unless $\operatorname{char} \mathrm{F}$ divides n, in which case the center is $\mathfrak{s}(n, \mathrm{F})$.

Solution to 1.1.2.3. □

Exercise 1.1.2.4. Show that (up to isomorphism) there is a unique Lie algebra over F of dimension 3 whose derived algebra has dimension 1 and lies in $Z(L)$.

Solution to 1.1.2.4. □

Exercise 1.1.2.5. Suppose $\dim L = 3$, $L = [LL]$. Prove that L must be simple. [Observe first that any homomorphic image of L also equals its derived algebra.] Recover the simplicity of $\mathfrak{sl}(2, \mathrm{F})$, $\operatorname{char} \mathrm{F} \neq 2$.

Solution to 1.1.2.5. □

Exercise 1.1.2.6. Prove that $\mathfrak{sl}(3, \mathrm{F})$ is simple, unless $\operatorname{char} \mathrm{F} = 3$ (cf. Exercise 1.1.2.3). [Use the standard basis h_1, h_2, e_{ij} $(i \neq j)$. If $I \neq 0$ is an ideal, then I is the direct sum of eigenspaces for $\operatorname{ad} h_1$ or $\operatorname{ad} h_2$; compare the eigenvalues of $\operatorname{ad} h_1$, $\operatorname{ad} h_2$ acting on the e_{ij}.]

Solution to 1.1.2.6. □

Exercise 1.1.2.7. Prove that $\mathfrak{t}(n, \mathrm{F})$ and $\mathfrak{d}(n, \mathrm{F})$ are self-normalizing subalgebras of $\mathfrak{gl}(n, \mathrm{F})$, whereas $\mathfrak{n}(n, \mathrm{F})$ has normalizer $\mathfrak{t}(n, \mathrm{F})$.

Solution to 1.1.2.7. □

Exercise 1.1.2.8. Prove that in each classical linear Lie algebra (1.2), the set of diagonal matrices is a self-normalizing subalgebra, when $\operatorname{char} \mathrm{F} = 0$.

Solution to 1.1.2.8. □

Exercise 1.1.2.9. Prove Proposition 2.2.

Solution to 1.1.2.9. □

Exercise 1.1.2.10. Let σ be the automorphism of $\mathfrak{sl}(2, \mathrm{F})$ defined in (2.3). Verify that $\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$.

Solution to 1.1.2.10. □

Exercise 1.1.2.11. If $L = \mathfrak{sl}(n, \mathrm{F})$, $g \in \mathrm{GL}(n, \mathrm{F})$, prove that the map of L to itself defined by $x \mapsto -gx^tg^{-1}$ $(x^t = \text{transpose of } x)$ belongs to $\operatorname{Aut} L$. When $n = 2$, $g =$ identity matrix, prove that this automorphism is inner.

Solution to 1.1.2.11. □

Exercise 1.1.2.12. Let L be an orthogonal Lie algebra (type B_ℓ or D_ℓ). If g is an **orthogonal** matrix, in the sense that g is invertible and $g^t s g = s$, prove that $x \mapsto gxg^{-1}$ defines an automorphism of L.

Solution to 1.1.2.12. □

1.1.3 Solvable and nilpotent Lie algebras

Exercise 1.1.3.1. Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.

Solution to 1.1.3.1. □

Exercise 1.1.3.2. Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.

Solution to 1.1.3.2. □

Exercise 1.1.3.3. Let $\operatorname{char}\mathrm{F} = 2$. Prove that $\mathfrak{sl}(2, \mathrm{F})$ is nilpotent.

Solution to 1.1.3.3. □

Exercise 1.1.3.4. Prove that L is solvable (resp. nilpotent) if and only if $\operatorname{ad} L$ is solvable (resp. nilpotent).

Solution to 1.1.3.4. □

Exercise 1.1.3.5. Prove that the nonabelian two dimensional algebra constructed in (1.4) is solvable but not nilpotent. Do the same for the algebra in Exercise 1.1.1.2.

Solution to 1.1.3.5. □

Exercise 1.1.3.6. Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal. Determine this ideal for each algebra in Exercise 1.1.3.5.

Solution to 1.1.3.6. □

Exercise 1.1.3.7. Let L be nilpotent, K a proper subalgebra of L. Prove that $N_L(K)$ includes K properly.

Solution to 1.1.3.7. □

Exercise 1.1.3.8. Let $L \neq 0$ be nilpotent. Prove that L has an ideal of codimension 1 .

Solution to 1.1.3.8. □

Exercise 1.1.3.9. Prove that every nilpotent Lie algebra $L \neq 0$ has an outer derivation (see (1.3)), as follows: Write $L = K + \mathrm{F}x$ for some ideal K of codimension one (Exercise 1.1.3.8). Then $C_L(K) \neq 0$ (why?). Choose n so that $C_L(K) \subset L^n$, $C_L(K) \not\subseteq L^{n+1}$, and let $z \in C_L(K) - L^{n+1}$. Then the linear map δ sending K to $0, x$ to z, is an outer derivation.

Solution to 1.1.3.9. □

Exercise 1.1.3.10. Let L be a Lie algebra, K an ideal of L such that L/K is nilpotent and such that $\operatorname{ad} x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.

Solution to 1.1.3.10. □

1.2 Semisimple Lie Algebras

1.2.1 Theorems of Lie and Cartan

Exercise 1.2.1.1. Let $L = \mathfrak{sl}(V)$. Use Lie's Theorem to prove that $\operatorname{Rad} L = Z(L)$; conclude that L is semisimple (cf. Exercise 1.1.2.3). [Observe that $\operatorname{Rad} L$ lies in each maximal solvable subalgebra B of L. Select a basis of V so that $B = L \cap \mathfrak{t}(n, \mathrm{F})$, and notice that the transpose of B is also a maximal solvable subalgebra of L. Conclude that $\operatorname{Rad} L \subset L \cap \mathfrak{d}(n, \mathrm{F})$, then that $\operatorname{Rad} L = Z(L)$.]

Solution to 1.2.1.1. ☐

Exercise 1.2.1.2. Show that the proof of Theorem 4.1 still goes through in prime characteristic, provided $\dim V$ is less than char F.

Solution to 1.2.1.2. ☐

Exercise 1.2.1.3. This exercise illustrates the failure of Lie's Theorem when F is allowed to have prime characteristic p. Consider the $p \times p$ matrices:

$$x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad y = \mathrm{diag}(0, 1, 2, 3, \cdots, p-1)..$$

Check that $[x, y] = x$, hence that x and y span a two dimensional solvable subalgebra L of $\mathfrak{gl}(p, \mathrm{F})$. Verify that x, y have no common eigenvector.

Solution to 1.2.1.3. ☐

Exercise 1.2.1.4. Exercise 1.2.1.3 shows that a solvable Lie algebra of endomorphisms over a field of prime characteristic p need not have derived algebra consisting of nilpotent endomorphisms. For arbitrary p, construct a counterexample to Corollary C of Theorem 4.1 as follows: Start with $L \subset \mathfrak{gl}(p, \mathrm{F})$ as in Exercise 1.2.1.3. Form the vector space direct sum $M = L + \mathrm{F}^p$, and make M a Lie algebra by decreeing that F^p is abelian, while L has its usual product and acts on F^p in the given way. Verify that M is solvable, but that its derived algebra ($= \mathrm{F}x + \mathrm{F}^p$) fails to be nilpotent.

Solution to 1.2.1.4. ☐

Exercise 1.2.1.5. If $x, y \in \mathrm{End}\, V$ commute, prove that $(x+y)_s = x_s + y_s$, and $(x+y)_n = x_n + y_n$. Show by example that this can fail if x, y fail to commute. [Show first that x, y semisimple (resp. nilpotent) implies $x + y$ semisimple (resp. nilpotent).]

Solution to 1.2.1.5. ☐

Exercise 1.2.1.6. Check formula (*) at the end of (4.2).

Solution to 1.2.1.6. ☐

Exercise 1.2.1.7. Prove the converse of Theorem 4.3.

Solution to 1.2.1.7. ☐

Exercise 1.2.1.8. Note that it suffices to check the hypothesis of Theorem 4.3 (or its corollary) for x, y ranging over a basis of $[LL]$, resp. L. For the example given in Exercise 1.1.1.2, verify solvability by using Cartan's Criterion.

Solution to 1.2.1.8. ☐

1.2.2 Killing form

Exercise 1.2.2.1. Prove that if L is nilpotent, the Killing form of L is identically zero.

Solution to 1.2.2.1. ☐

Exercise 1.2.2.2. Prove that L is solvable if and only if $[LL]$ lies in the radical of the Killing form.

Solution to 1.2.2.2. ☐

Exercise 1.2.2.3. Let L be the two dimensional nonabelian Lie algebra (1.4), which is solvable. Prove that L has nontrivial Killing form.

Solution to 1.2.2.3. ☐

Exercise 1.2.2.4. Let L be the three dimensional solvable Lie algebra of Exercise 1.1.1.2. Compute the radical of its Killing form.

Solution to 1.2.2.4. □

Exercise 1.2.2.5. Let $L = \mathfrak{sl}(2, \mathrm{F})$. Compute the basis of L dual to the standard basis, relative to the Killing form.

Solution to 1.2.2.5. □

Exercise 1.2.2.6. Let char $\mathrm{F} = p \neq 0$. Prove that L is semisimple if its Killing form is nondegenerate. Show by example that the converse fails. [Look at $\mathfrak{sl}[(3, \mathrm{F})$ modulo its center, when char $\mathrm{F} = 3$.]

Solution to 1.2.2.6. □

Exercise 1.2.2.7. Relative to the standard basis of $\mathfrak{sl}(3, \mathrm{F})$, compute the determinant of κ. Which primes divide it?

Solution to 1.2.2.7. □

Exercise 1.2.2.8. Let $L = L_1 \oplus \cdots \oplus L_t$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the various L_i of the components of x.

Solution to 1.2.2.8. □

1.2.3 Complete reducibility of representations

Exercise 1.2.3.1. Using the standard basis for $L = \mathfrak{sl}(2, \mathrm{F})$, write down the Casimir element of the adjoint representation of L (cf. Exercise 1.2.2.5). Do the same thing for the usual (3-dimensional) representation of $\mathfrak{sl}(3, \mathrm{F})$, first computing dual bases relative to the trace form.

Solution to 1.2.3.1. □

Exercise 1.2.3.2. Let V be an L-module. Prove that V is a direct sum of irreducible submodules if and only if each L-submodule of V possesses a complement.

Solution to 1.2.3.2. □

Exercise 1.2.3.3. If L is solvable, every irreducible representation of L is one dimensional.

Solution to 1.2.3.3. □

Exercise 1.2.3.4. Use Weyl's Theorem to give another proof that for L semisimple, $\operatorname{ad} L = \operatorname{Der} L$ (Theorem 5.3). [If $\delta \in \operatorname{Der} L$, make the direct sum $\mathrm{F} + L$ into an L-module via the rule $x.(a, y) = (0, a\delta(x) + [xy])$. Then consider a complement to the submodule L.]

Solution to 1.2.3.4. □

Exercise 1.2.3.5. A Lie algebra L for which $\operatorname{Rad} L = Z(L)$ is called reductive. (Examples: L abelian, L semisimple, $L = \mathfrak{gl}(n, \mathrm{F})$.)

(i) If L is reductive, then L is a completely reducible $\operatorname{ad} L$-module. [If $\operatorname{ad} L \neq 0$, use Weyl's Theorem.] In particular, L is the direct sum of $Z(L)$ and $[LL]$, with $[LL]$ semisimple.

(ii) If L is a classical linear Lie algebra (1.2), then L is semisimple. [Cf. Exercise 1.1.1.9.]

(iii) If L is a completely reducible $\operatorname{ad} L$-module, then L is reductive.

(iv) If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphisms are completely reducible.

Solution to 1.2.3.5(i). □

Solution to 1.2.3.5(ii). □

Solution to 1.2.3.5(iii). □

Solution to 1.2.3.5(iv). □

Exercise 1.2.3.6. Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on L. If β, γ are nondegenerate, prove that β and γ are proportional. [Use Schur's Lemma.]

Solution to 1.2.3.6. □

Exercise 1.2.3.7. It will be seen later on that $\mathfrak{sl}(n, \mathrm{F})$ is actually simple. Assuming this and using Exercise 1.2.3.6, prove that the Killing form κ on $\mathfrak{sl}(n, \mathrm{F})$ is related to the ordinary trace form by $\kappa(x, y) = 2n \operatorname{Tr}(xy)$.

Solution to 1.2.3.7. □

Exercise 1.2.3.8. If L is a Lie algebra, then L acts (via ad) on $(L \otimes L)^*$, which may be identified with the space of all bilinear forms β on L. Prove that β is associative if and only if $L.\beta = 0$.

Solution to 1.2.3.8. □

Exercise 1.2.3.9. Let L' be a semisimple subalgebra of a semisimple Lie algebra L. If $x \in L'$, its Jordan decomposition in L' is also its Jordan decomposition in L.

Solution to 1.2.3.9. □

1.2.4 Representations of $\mathfrak{sl}(2, \mathrm{F})$

Exercise 1.2.4.1. Use Lie's Theorem to prove the existence of a maximal vector in an arbitrary finite dimensional L-module. [Look at the subalgebra B spanned by h and x.]

Solution to 1.2.4.1. □

Exercise 1.2.4.2. $M = \mathfrak{sl}(3, \mathrm{F})$ contains a copy of L in its upper left-hand 2×2 position. Write M as direct sum of irreducible L-submodules (M viewed as L module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Solution to 1.2.4.2. □

Exercise 1.2.4.3. Verify that formulas (a)-(c) of Lemma 7.2 do define an irreducible representation of L. [To show that they define a representation, it suffices to show that the matrices corresponding to x, y, h satisfy the same structural equations as x, y, h.]

Solution to 1.2.4.3. □

Exercise 1.2.4.4. The irreducible representation of L of highest weight m can also be realized "naturally", as follows. Let X, Y be a basis for the two dimensional vector space F^2, on which L acts as usual. Let $\mathscr{R} = \mathrm{F}[X, Y]$ be the polynomial algebra in two variables, and extend the action of L to $\mathscr{R}$ by the derivation rule: $z.fg = (z.f)g + f(z.g)$, for $z \in L, f, g \in \mathscr{R}$. Show that this extension is well defined and that $\mathscr{R}$ becomes an L-module. Then show that the subspace of homogeneous polynomials of degree m, with basis X^m, $X^{m-1}Y, \cdots, XY^{m-1}, Y^m$, is invariant under L and irreducible of highest weight m.

Solution to 1.2.4.4. □

Exercise 1.2.4.5. Suppose char $\mathrm{F} = p > 0, L = \mathfrak{sl}(2, \mathrm{F})$. Prove that the representation $V(m)$ of L constructed as in Exercise 1.2.4.3 or 1.2.4.4 is irreducible so long as the highest weight m is strictly less than p, but reducible when $m = p$.

Solution to 1.2.4.5. □

Exercise 1.2.4.6. Decompose the tensor product of the two L-modules $V(3), V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

Solution to 1.2.4.6. □

Exercise 1.2.4.7. In this exercise we construct certain infinite dimensional L-modules. Let $\lambda \in \mathrm{F}$ be an arbitrary scalar. Let $Z(\lambda)$ be a vector space over F with countably infinite basis $(v_0, v_1, v_2, \cdots)$.

(i) Prove that formulas (a) – (c) of Lemma 7.2 define an L-module structure on $Z(\lambda)$, and that every nonzero L-submodule of $Z(\lambda)$ contains at least one maximal vector.

(ii) Suppose $\lambda+1=i$ is a positive integer. Prove that v_i is a maximal vector. This induces an L-module homomorphism $Z(\mu) \xrightarrow{\phi} Z(\lambda), \mu=\lambda-2i$, sending v_0 to v_i. Show that ϕ is a monomorphism, and that $\operatorname{Im}\phi$, $Z(\lambda)/\operatorname{Im}\phi$ are both irreducible L-modules (but $Z(\lambda)$ fails to be completely reducible).

(iii) Suppose $\lambda+1$ is not a positive integer. Prove that $Z(\lambda)$ is irreducible.

Solution to 1.2.4.7(i). ☐

Solution to 1.2.4.7(ii). ☐

Solution to 1.2.4.7(iii). ☐

1.2.5 Root space decomposition

Exercise 1.2.5.1. If L is a classical linear Lie algebra of type $\mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell$, or D_ℓ (see (1.2)), prove that the set of all diagonal matrices in L is a maximal toral subalgebra, of dimension ℓ. (Cf. Exercise 1.1.2.8.)

Solution to 1.2.5.1. ☐

Exercise 1.2.5.2. For each algebra in Exercise 1.2.5.1, determine the roots and root spaces. How are the various h_α expressed in terms of the basis for H given in (1.2)?

Solution to 1.2.5.2. ☐

Exercise 1.2.5.3. If L is of classical type, compute explicitly the restriction of the Killing form to the maximal toral subalgebra described in Exercise 1.2.5.1.

Solution to 1.2.5.3. ☐

Exercise 1.2.5.4. If $L=\mathfrak{sl}(2,\mathrm{F})$, prove that each maximal toral subalgebra is one dimensional.

Solution to 1.2.5.4. ☐

Exercise 1.2.5.5. If L is semisimple, H a maximal toral subalgebra, prove that H is selfnormalizing (i.e., $H=N_L(H)$).

Solution to 1.2.5.5. ☐

Exercise 1.2.5.6. Compute the basis of $\mathfrak{sl}(n,\mathrm{F})$ which is dual (via the Killing form) to the standard basis. (Cf. Exercise 1.2.3.5.)

Solution to 1.2.5.6. ☐

Exercise 1.2.5.7. Let L be semisimple, H a maximal toral subalgebra. If $h\in H$, prove that $C_L(h)$ is reductive (in the sense of Exercise 1.2.4.5). Prove that H contains elements h for which $C_L(h)=H$; for which h in $\mathfrak{sl}(n,\mathrm{F})$ is this true?

Solution to 1.2.5.7. ☐

Exercise 1.2.5.8. For $\mathfrak{sl}(n,\mathrm{F})$ (and other classical algebras), calculate explicitly the root strings and Cartan integers. In particular, prove that all Cartan integers $2(\alpha,\beta)/(\beta,\beta), \alpha\neq\pm\beta$, for $\mathfrak{sl}(n,\mathrm{F})$ are $0,\pm1$.

Solution to 1.2.5.8. ☐

Exercise 1.2.5.9. Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}(2,\mathrm{F})$, hence is isomorphic to $\mathfrak{sl}(2,\mathrm{F})$.

Solution to 1.2.5.9. ☐

Exercise 1.2.5.10. Prove that no four, five or seven dimensional semisimple Lie algebras exist.

Solution to 1.2.5.10. ☐

Exercise 1.2.5.11. If $(\alpha,\beta)>0$, prove that $\alpha-\beta\in\Phi(\alpha,\beta\in\Phi)$. Is the converse true?

Solution to 1.2.5.11. ☐

1.3 Root Systems

1.3.1 Axiomatics

Note. Unless otherwise specified, Φ denotes a root system in E, with Weyl group $\mathscr{W}$.

Exercise 1.3.1.1. Let E′ be a subspace of E. If a reflection σ_α leaves E′ invariant, prove that either $\alpha \in \mathrm{E}'$ or else $\mathrm{E}' \in P_\alpha$.

Solution to 1.3.1.1. □

Exercise 1.3.1.2. Prove that $\Phi^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathscr{W}$; show also that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$, and draw a picture of $\Phi^\vee$ in the cases $\mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_2, \mathrm{G}_2$.

Solution to 1.3.1.2. □

Exercise 1.3.1.3. In Table 1, show that the order of $\sigma_\alpha \sigma_\beta$ in $\mathscr{W}$ is (respectively) 2, 3, 4, 6 when $\theta = \pi/2$, $\pi/3$ (or $2\pi/3$), $\pi/4$ (or $3\pi/4$), $\pi/6$ (or $5\pi/6$). [Note that $\sigma_\alpha \sigma_\beta =$ rotation through 2θ.]

Solution to 1.3.1.3. □

Exercise 1.3.1.4. Prove that the respective Weyl groups of $\mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_2, \mathrm{G}_2$ are dihedral of order 4, 6, 8, 12. If Φ is any root system of rank 2, prove that its Weyl group must be one of these.

Solution to 1.3.1.4. □

Exercise 1.3.1.5. Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) \leq 0$ (cf. Lemma 9.4).

Solution to 1.3.1.5. □

Exercise 1.3.1.6. Prove that $\mathscr{W}$ is a normal subgroup of $\operatorname{Aut}\Phi$ (= group of all isomorphisms of Φ onto itself).

Solution to 1.3.1.6. □

Exercise 1.3.1.7. Let $\alpha, \beta \in \Phi$ span a subspace E′ of E. Prove that $\mathrm{E}' \cap \Phi$ is a root system in E′. Prove similarly that $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is a root system in E′ (must this coincide with $\mathrm{E}' \cap \Phi$?). More generally, let Φ' be a nonempty subset of Φ such that $\Phi' = -\Phi'$, and such that $\alpha, \beta \in \Phi', \alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E it spans. [Use Table 1].

Solution to 1.3.1.7. □

Exercise 1.3.1.8. Compute root strings in G_2 to verify the relation $r - q = \langle \beta, \alpha \rangle$.

Solution to 1.3.1.8. □

Exercise 1.3.1.9. Let Φ be a set of vectors in a euclidean space E, satisfying only (R1), (R3), (R4). Prove that the only possible multiples of $\alpha \in \Phi$ which can be in Φ are $\pm 1/2\alpha, \pm\alpha, \pm 2\alpha$. Verify that $\{\alpha \in \Phi \mid 2\alpha \notin \Phi\}$ is a root system.

Solution to 1.3.1.9. □

1.3.2 Simple roots and Weyl group

Exercise 1.3.2.1. Let $\Phi^\vee$ be the dual system of Φ, $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. Prove that $\Delta^\vee$ is a base of $\Phi^\vee$. [Compare Weyl chambers of Φ and $\Phi^\vee$.]

Solution to 1.3.2.1. □

Exercise 1.3.2.2. If Δ is a base of Φ, prove that the set $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi$ ($\alpha \neq \beta$ in Δ) is a root system of rank 2 in the subspace of E spanned by α, β (cf. Exercise 1.3.1.7). Generalize to an arbitrary subset of Δ.

Solution to 1.3.2.2. □

Exercise 1.3.2.3. Prove that each root system of rank 2 is isomorphic to one of those listed in (9.3).

Solution to 1.3.2.3. □

Exercise 1.3.2.4. Verify the Corollary of Lemma 10.2A directly for G_2.

Solution to 1.3.2.4. □

Exercise 1.3.2.5. If $\sigma \in \mathscr{W}$ can be written as a product of t simple reflections, prove that t has the same parity as $\ell(\sigma)$.

Solution to 1.3.2.5. □

Exercise 1.3.2.6. Define a function $sn : \mathscr{W} \to \{\pm 1\}$ by $sn(\sigma) = (-1)^{\ell(\sigma)}$. Prove that sn is a homomorphism (cf. the case A_2, where $\mathscr{W}$ is isomorphic to the symmetric group $\mathscr{S}_3$).

Solution to 1.3.2.6. □

Exercise 1.3.2.7. Prove that the intersection of "positive" open half-spaces associated with any basis $\gamma_1, \cdots, \gamma_l$ of E is nonvoid. [If δ_i is the projection of γ_i on the orthogonal complement of the subspace spanned by all basis vectors except γ_i, consider $\gamma = \sum r_i\delta_i$ when all $r_i > 0$.]

Solution to 1.3.2.7. □

Exercise 1.3.2.8. Let Δ be a base of Φ, $\alpha \neq \beta$, simple roots, $\Phi_{\alpha\beta}$ the rank 2 root system in $E_{\alpha\beta} = \mathbb{R}\alpha + \mathbb{R}\beta$ (see Exercise refexc:1.3.2.2 above). The Weyl group $\mathscr{W}_{\alpha\beta}$ of $\Phi_{\alpha\beta}$ is generated by the restrictions τ_α, τ_β to $E_{\alpha\beta}$ of $\sigma_\alpha, \sigma_\beta$, and $\mathscr{W}_{a\beta}$ may be viewed as a subgroup of $\mathscr{W}$. Prove that the "length" of an element of $\mathscr{W}_{\alpha\beta}$ (relative to τ_α, τ_β) coincides with the length of the corresponding clement of $\mathscr{W}$.

Solution to 1.3.2.8. □

Exercise 1.3.2.9. Prove that there is a unique element σ in $\mathscr{W}$ sending Ψ^+ to Φ^-(relative to Δ). Prove that any reduced expression for σ must involve all σ_α $(\alpha \in \Delta)$. Discuss $\ell(\sigma)$.

Solution to 1.3.2.9. □

Exercise 1.3.2.10. Given $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ in Φ, let $\lambda = \sum_{i=1}^{l} k_i\alpha_i$ ($k_i \in \mathbb{Z}$, for all $k_i \geq 0$ or all $k_i \leq 0$). Prove that either λ is a multiple (possibly 0) of a root, or else there exists $\sigma \in \mathscr{W}$ such that $\sigma\lambda = \sum_{i=1}^{l} k_i'\alpha_i$, with some $k_i' > 0$ and some $k_i' < 0$. [Sketch of proof: If λ is not a multiple of any root, then the hyperplane P_λ orthogonal to λ is not included in $\bigcup_{\alpha\in\Phi} P_\alpha$. Take $\mu \in P_\lambda - \bigcup_{\alpha\in\Phi} P_\alpha$. Then find $\sigma \in \mathscr{W}$ for which all $(\alpha_i, \sigma\mu) > 0$. It follows that $0 = (\lambda, \mu) = (\sigma\lambda, \sigma\mu) = \sum k(\alpha_i, \sigma\mu)$.]

Solution to 1.3.2.10. □

Exercise 1.3.2.11. Let Φ be irreducible. Prove that $\Phi^\vee$ is also irreducible. If Φ has all roots of equal length, so does $\Phi^\vee$ (and then $\Phi^\vee$ is isomorphic to Φ). On the other hand, if Φ has two root lengths, then so does $\Phi^\vee$; but if α is long, then $\alpha^\vee$ is short (and vice versa). Use this fact to prove that Φ has a unique maximal short root (relative to the partial order $\prec$ defined by Δ).

Solution to 1.3.2.11. □

Exercise 1.3.2.12. Let $\lambda \in \mathbb{C}(\Delta)$. If $\sigma\lambda = \lambda$ for some $\sigma \in \mathscr{W}$, then $\sigma = 1$.

Solution to 1.3.2.12. □

Exercise 1.3.2.13. The only reflections in $\mathscr{W}$ are those of the form σ_α $(\alpha \in \Phi)$. [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in $\mathscr{W}$.]

Solution to 1.3.2.13. □

Exercise 1.3.2.14. Prove that each point of E is $\mathscr{W}$-conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. [Enlarge the partial order on E by defining $\mu \prec \lambda$ iff $\lambda - \mu$ is a nonnegative $\mathbb{R}$-linear combination of simple roots. If $\mu \in$ E, choose $\sigma \in \mathscr{W}$ for which $\lambda = \sigma\mu$ is maximal in this partial order.]

Solution to 1.3.2.14. □

1.3.3 Classification

Exercise 1.3.3.1. Verify the Cartan matrices (Table 1).

Solution to 1.3.3.1. □

Table 1.1: Highest long and short roots

Type	Long	Short
A_ℓ	$\alpha_1+\alpha_2+\cdots+\alpha_\ell$	
B_ℓ	$\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_\ell$	$\alpha_1+\alpha_2+\cdots+\alpha_\ell$
C_ℓ	$2\alpha_1+2\alpha_2+\cdots+2\alpha_{\ell-1}+\alpha_\ell$	$\alpha_1+2\alpha_2+\cdots+2\alpha_{\ell-1}+\alpha_\ell$
D_ℓ	$\alpha_1+2\alpha_2+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$	
E_6	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	
E_7	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	
E_8	$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8$	
F_4	$2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$
G_2	$3\alpha_1+2\alpha_2$	$2\alpha_1+\alpha_2$

Exercise 1.3.3.2. Calculate the determinants of the Cartan matrices (using induction on ℓ for types A_ℓ–D_ℓ), which are as follows:

$$A_\ell : \ell+1;\ B_\ell : 2;\ C_\ell : 2;\ D_\ell : 4;\ E_6 : 3;\ E_7 : 2;\ E_8,\ F_4 \text{ and } G_2 : 1.$$

Solution to 1.3.3.2. □

Exercise 1.3.3.3. Use the algorithm of (11.1) to write down all roots for G_2. Do the same for $\mathcal{C}_3$: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.

Solution to 1.3.3.3. □

Exercise 1.3.3.4. Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.

Solution to 1.3.3.4. □

Exercise 1.3.3.5. Prove that each irreducible root system is isomorphic to its dual, except that B_ℓ, C_ℓ are dual to each other.

Solution to 1.3.3.5. □

Exercise 1.3.3.6. Prove that an inclusion of one Dynkin diagram in another (e.g., E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.

Solution to 1.3.3.6. □

1.3.4 Construction of root systems and automorphisms

Exercise 1.3.4.1. Verify the details of the constructions in (12.1).

Solution to 1.3.4.1. □

Exercise 1.3.4.2. Verify Table 2.

Solution to 1.3.4.2. □

Exercise 1.3.4.3. Let $\Phi \subset E$ satisfy (R1), (R3), (R4), but not (R2), cf. Exercise 1.3.2.9. Suppose moreover that Φ is irreducible, in the sense of §11. Prove that Φ is the union of root systems of type $B_n, \mathbb{C}_n$ in E (if $\dim E = n > 1$), where the long roots of B_n are also the short roots of $\mathbb{C}_n$. (This is called the non-reduced root system of type BC_n in the literature). See table 1.1.

Solution to 1.3.4.3. □

Exercise 1.3.4.4. Prove that the long roots in G_2 form a root system in E of type A_2.

Solution to 1.3.4.4. □

Exercise 1.3.4.5. In constructing C_ℓ, would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.

Solution to 1.3.4.5. □

Exercise 1.3.4.6. Prove that the map $\alpha \mapsto -\alpha$ is an automorphism of Φ. Try to decide for which irreducible Φ this belongs to the Weyl group.

Solution to 1.3.4.6. □

Exercise 1.3.4.7. Describe $\operatorname{Aut}\Phi$ when Φ is not irreducible.

Solution to 1.3.4.7. □

1.3.5 Abstract theory of weights

Exercise 1.3.5.1. Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ be the decomposition of Φ into its irreducible components, with $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$. Prove that Λ decomposes into a direct sum $\Lambda_1 \oplus \cdots \oplus \Lambda_t$; what about Λ^+?

Solution to 1.3.5.1. □

Exercise 1.3.5.2. Show by example (e.g., for A_2) that $\lambda \notin \Lambda^+$, $\alpha \in \Delta$, $\lambda - \alpha \in \Lambda^+$ is possible.

Solution to 1.3.5.2. □

Exercise 1.3.5.3. Verify some of the data in Table 1, e.g., for F_4.

Solution to 1.3.5.3. □

Exercise 1.3.5.4. Using Table 1, show that the fundamental group of A_ℓ is cyclic of order $\ell+1$, while that of D_ℓ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (ℓ odd), or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (ℓ even). (It is easy to remember which is which, since $\mathrm{A}_3 = \mathrm{D}_3$.)

Solution to 1.3.5.4. □

Exercise 1.3.5.5. If Λ' is any subgroup of Λ which includes Λ_r, prove that Λ' is $\mathscr{W}$-invariant. Therefore, we obtain a homomorphism $\phi : \operatorname{Aut}\Phi/\mathscr{W} \to \operatorname{Aut}(\Lambda/\Lambda_r)$. Prove that ϕ is injective, then deduce that $-1 \in \mathscr{W}$ if and only if $\Lambda_r \supset 2\Lambda$ (cf. Exercise 1.3.4.6). Show that $-1 \in \mathscr{W}$ for precisely the irreducible root systems $\mathrm{A}_1, \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell$ (ℓ even), $\mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$.

Solution to 1.3.5.5. □

Exercise 1.3.5.6. Prove that the roots in Φ which are dominant weights are precisely the highest long root and (if two root lengths occur) the highest short root (cf. (10.4) and Exercise 1.3.2.11), when Φ is irreducible.

Solution to 1.3.5.6. □

Exercise 1.3.5.7. If $\epsilon_1, \cdots, \epsilon_\ell$ is an *obtuse* basis of the euclidean space E (i.e., all $(\epsilon_i, \epsilon_j) \leq 0$ for $i \neq j$), prove that the dual basis is acute (i.e., all $\left(\epsilon_i^*, \epsilon_j^*\right) \geq 0$ for $i \neq j$). [Reduce to the case $\ell = 2$.]

Solution to 1.3.5.7. □

Exercise 1.3.5.8. Let Φ be irreducible. Without using the data in Table 1, prove that each λ_i is of the form $\sum_j q_{ij}\alpha_j$, where all q_{ij} are positive rational numbers. [Deduce from Exercise 1.3.5.7 that all q_{ij} are nonnegative. From $(\lambda_i, \lambda_i) > 0$ obtain $q_{ii} > 0$. Then show that if $q_{ij} > 0$ and $(\alpha_j, \alpha_k) < 0$, then $q_{ik} > 0$.]

Solution to 1.3.5.8. □

Exercise 1.3.5.9. Let $\lambda \in \Lambda^+$. Prove that $\sigma(\lambda + \delta) - \delta$ is dominant only for $\sigma = 1$.

Solution to 1.3.5.9. □

Exercise 1.3.5.10. If $\lambda \in \Lambda^+$, prove that the set Π consisting of all dominant weights $\mu \prec \lambda$ and their $\mathscr{W}$-conjugates is saturated, as asserted in (13.4).

Solution to 1.3.5.10. □

Exercise 1.3.5.11. Prove that each subset of Λ is contained in a unique smallest saturated set, which is finite if the subset in question is finite.

Solution to 1.3.5.11. □

Exercise 1.3.5.12. For the root system of type A_2, write down the effect of each element of the Weyl group on each of λ_1, λ_2. Using this data, determine which weights belong to the saturated set having highest weight $\lambda_1 + 3\lambda_2$. Do the same for type G_2 and highest weight $\lambda_1 + 2\lambda_2$.

Solution to 1.3.5.12. $\square$

Exercise 1.3.5.13. Call $\lambda \in \Lambda^+$ **minimal** if $\mu \in \Lambda^+$, $\mu \prec \lambda$ implies that $\mu = \lambda$. Show that each coset of Λ_r in Λ contains precisely one minimal λ. Prove that λ is minimal if and only if the $\mathscr{W}$-orbit of λ is saturated (with highest weight λ), if and only if $\lambda \in \Lambda^+$ and $\langle \lambda, \alpha \rangle = 0, 1, -1$ for all roots α. Determine (using Table 1) the nonzero minimal λ for each irreducible Φ, as follows:

$$\begin{aligned}
&\mathrm{A}_\ell : \lambda_1, \cdots, \lambda_l \\
&\mathrm{B}_\ell : \lambda_\ell \\
&\mathrm{C}_\ell : \lambda_1 \\
&\mathrm{D}_\ell : \lambda_1, \lambda_{\ell-1}, \lambda_\ell \\
&\mathrm{E}_6 : \lambda_1, \lambda_6 \\
&\mathrm{E}_7 : \lambda_7
\end{aligned}$$

Solution to 1.3.5.13. $\square$

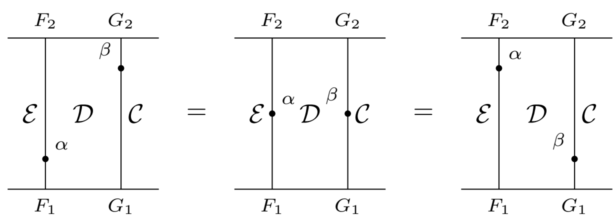
Chapter 2

Introduction to Soergel Bimodules

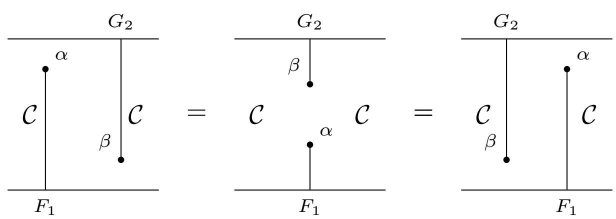
2.7 How to Draw Monoidal Categories

2.7.2 Planar Diagrams for 2-Categories

Exercise 2.7.2.8. Show that the axioms of a 2-category imply the following equalities.



(2.1)



(2.2)

Solution to 2.7.2.8. The diagram 2.1 follows from the axioms of a 2-category. In particular, the interchange law states that horizontal and vertical compositions of 2-morphisms are compatible. More precisely, given 2-morphisms $\alpha : F_1 \implies F_2$ and $\beta : G_1 \implies G_2$, the horizontal composition $\beta * \alpha : G_1 \circ F_1 \implies G_2 \circ F_2$ is defined, and satisfies:

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha).$$

In 2.1, each equality reflects this interchange identity. That is, we may either first compose vertically and then horizontally, or compose horizontally first and then vertically, and the result is the same. This coherence condition is one of the core structural axioms of any strict 2-category.

For diagram 2.2, this identity expresses the fact that the horizontal composition of 2-morphisms with identity 1-morphisms on either side is strictly associative, and that such composition is natural. Specifically, suppose:

$$\alpha : F_1 \implies G_2 \quad \text{and} \quad \beta : \mathrm{id}_{\mathcal{C}} \implies \mathrm{id}_{\mathcal{C}},$$

are 2-morphisms in the hom-category $\mathcal{C}(\mathcal{C}, \mathcal{C})$. Then, under horizontal composition, we form:

$$(\mathrm{id}_{\mathcal{C}} * \alpha) \circ (\beta * \mathrm{id}_{F_1}) = (\beta \circ \mathrm{id}_{\mathcal{C}}) * (\mathrm{id}_{\mathcal{C}} \circ \alpha) = \beta * \alpha.$$

This follows from the interchange law and the unit laws of 2-categories. In particular, composing with identity 2-morphisms has no effect, and the strict associativity of composition allows us to freely rebracket the diagram. This justifies the equality of all three diagrams above. $\square$

2.7.4 The Temperley–Lieb Category

Exercise 2.7.4.16. We can view the algebra $A = \mathbb{R}[x]/(x^2)$ as an object in the monoidal category of $\mathbb{R}$-vector spaces. Let $\cap : A \otimes A \to \mathbb{R}$ denote the linear map which sends $f \otimes g$ to the coefficient of x in fg. Let $\cup : \mathbb{R} \to A \otimes A$ denote the map which sends 1 to $x \otimes 1 + 1 \otimes x$.

(i) We wish to encode these maps diagrammatically, drawing $\cap$ as a cap and $\cup$ as a cup. Justify this diagrammatic notation, by checking the isotopy relations.

(ii) Draw a sequence of nested circles, as in an archery target. Evaluate this morphism.

Solution to 2.7.4.16(i). Let $v = ax + b \in A$. To check isotopy relations, we need to check the following compositions: $(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax+b)$ and $(\mathrm{id}_A \otimes \cup) \circ (\cap \otimes \mathrm{id}_A)(ax+b)$. To keep notation simple, recall that $\mathbb{R} \otimes A \cong A \cong A \otimes \mathbb{R}$.

We deal with the first composition first. We know that $\mathrm{id}_A \otimes \cup : A \to A \otimes A \otimes A$, we get

$$(\mathrm{id}_A \otimes \cup)(ax + b) = (ax + b) \otimes (x \otimes 1 + 1 \otimes x) = (ax + b) \otimes x \otimes 1 + (ax + b) \otimes 1 \otimes x.$$

Now, we know that $\cap \otimes \mathrm{id}_A : A \otimes A \otimes A \to A$, applying this to the above, we get

$$(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax + b) = (\cap \otimes \mathrm{id}_A)((ax + b) \otimes x \otimes 1 + (ax + b) \otimes 1 \otimes x).$$

Computing $\cap((ax + b) \otimes x)$, we compute fg to get $(ax + b) \cdot x = ax^2 + bx$. Therefore, $\cap((ax + b) \otimes x) = b$. Similarly, we get $\cap((ax + b) \otimes 1) = a$. Therefore, we have

$$(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax + b) = b + ax = ax + b.$$

Therefore, we have shown that the first composition is the identity.

We now deal with the second composition, $(\mathrm{id}_A \otimes \cup) \circ (\cap \otimes \mathrm{id}_A)(ax+b)$. Computing the inner composition, $\cup \otimes \mathrm{id}_A : A \otimes A \to A \otimes A \otimes A$, we get

$$(\cup \otimes \mathrm{id}_A)(ax + b) = (x \otimes 1 + 1 \otimes x) \otimes (ax + b) = x \otimes 1 \otimes (ax + b) + 1 \otimes x \otimes (ax + b).$$

Now, we know that $\mathrm{id}_A \otimes \cap : A \otimes A \otimes A \to A$, so applying this to the above, we get

$$(\mathrm{id}_A \otimes \cap)(x \otimes 1 \otimes (ax + b) + 1 \otimes x \otimes (ax + b)) = ax + b.$$

Therefore, $(\mathrm{id}_A \otimes \cap) \circ (\cup \otimes \mathrm{id}_A)(ax + b) = ax + b$, as desired.

Thus, we have shown that the isotopy relations hold in vector spaces. $\square$

Solution to 2.7.4.16(ii). Let f be the entire morphism. For simplicity, I use the notation, $\cup^{\otimes k} : \mathbb{R} \to A^{\otimes k}$, $\cap^{\otimes k} : A^{\otimes k} \to \mathbb{R}$, and $f^k = \cap^{\otimes k} \circ \cup^{\otimes k} : \mathbb{R} \to \mathbb{R}$, to denote k-circles.

For the 1-circle case, we have the following morphism

$$f(1) = (\cap \circ \cup)(1) = \cap(\cup(1)) = \cap(x \otimes 1 + 1 \otimes x) = 1 + 1 = 2.$$

For the 2-circle case, we have the following morphism

$$f^2(1) = \left(\cap^{\otimes 2} \circ \cup^{\otimes 2}\right)(1) = \cap^{\otimes 2}\left(\cup^{\otimes 2}(1)\right) = \cap^{\otimes 2}(x \otimes 1 + 1 \otimes x \otimes x \otimes 1 + 1 \otimes x) = 4.$$

For the k-circle case, we have the following morphism

$$f^k(1) = \left(\cap^{\otimes k} \circ \cup^{\otimes k}\right)(1).$$

Since $\cup(1) = x \otimes 1 + 1 \otimes x$, then $\cup^{\otimes k}(1)$ is the sum of 2^k simple tensors, each term of the form $v_1 \otimes w_1 \otimes \cdots \otimes v_k \otimes w_k$, where each pair $(v_i \otimes w_i)$ is either $x \otimes 1$ or $1 \otimes x$. Applying $\cap^{\otimes k}$ to each such term gives 1 for every pair, since $\cap(x \otimes 1) = \cap(1 \otimes x) = 1$. Therefore, we get $f^k(1) = 2^k$. $\square$

Exercise 2.7.4.17. This question is about the Temperley–Lieb category.

(i) Finish the proof that the isotopy relation holds in vector spaces.

(ii) There is a map $V \otimes V \to V \otimes V$ which sends $x \otimes y \mapsto y \otimes x$. Draw this as an element of the Temperley–Lieb category (a linear combination of diagrams).

(iii) Find an endomorphism of 2 strands which is killed by placing a cap on top. Can you find one which is an idempotent? Also find an endomorphism killed by putting a cup on bottom.

(iv) (Harder) Find an idempotent endomorphism of 3 strands which is killed by a cap on top (for either of the two placements of the cap).

Solution to 2.7.4.17(i). Define the following swap function:

$$\tau^k \left(\bigotimes_{n=1}^{k} v_n \right) = \bigotimes_{n=1}^{k} v_{k-n+1}.$$

I'll be using the notation $\lfloor f^k(x) \rfloor$ to denote the coefficient of x^k in the polynomial $f(x)$.

To prove isotopy relations hold in vector spaces, we need to prove the following: the following identities,

$$(\cap \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \cup) = \mathrm{id}_V$$
$$(\mathrm{id}_V \otimes \cap) \circ (\cup \otimes \mathrm{id}_V) = \mathrm{id}_V,$$

loop evaluation, symmetry of cups and caps, and sliding. I've already shown the snake identities in 2.7.4.16(i) and the loop evaluation in 2.7.4.16(ii). □

Solution to 2.7.4.17(ii). □

Solution to 2.7.4.17(iii). □

2.7.5 More About Isotopy

Exercise 2.7.5.19. One can think about the right mate and the left mate as "twisting" or "rotating" α by 180° to the right or to the left. Visualize what it would mean to twist α by 360° to the right, yielding another 2-morphism $\alpha^{\vee\vee} : E \to F$. Verify that ${}^{\vee}\alpha = \alpha^{\vee}$, if and only if $\alpha = \alpha^{\vee\vee}$. Thus cyclicity is the same as "360 degree rotation invariance," which one might expect from any planar picture.

Solution to 2.7.5.19. □

Exercise 2.7.5.20. Suppose that B is an object in a monoidal category with biadjoints, and $\Phi : B \otimes B \otimes B \to \mathbb{1}$ is a cyclic morphism. What should it mean to "rotate" Φ by 120°? Suppose that $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{C}$. What can you say about the 120° rotation of Φ, vis a vis Φ? What if $\mathrm{Hom}(B \otimes B \otimes B, \mathbb{1})$ is one-dimensional over $\mathbb{R}$?

Solution to 2.7.5.20. □

2.9 The Dihedral Cathedral

Exercise 9.25. Let our base ring be some specialization of $\mathbb{Z}[\delta]$. Inside $\mathrm{TL}_{n,\delta}$ let T be the vector space of elements which are killed by all the $(n-1)$ caps on top, and let B be the space killed by cups on the bottom. For an element $x \in \mathrm{TL}_{n,\delta}$ let $\bar{x}$ denote the same element with each diagram flipped upside down. Thus, for example, $x \in T$ if and only if $\bar{x} \in B$.

(i) Show that any crossingless matching is either the identity diagram, or has both a cap on bottom and a cup on top.

(ii) We now make the following assumption:

$$\text{There exist some } f \in T \text{ for which the coefficients of the identity diagram is invertible.} \tag{2.3}$$

Why is this equivalent to the analogous assumption for B?

(iii) Let $f \in T$, with invertible coefficient c for the identity diagram. Let $g \in B$, with invertible coefficient d for the identity diagram. Compute the composition fg in two ways and deduce that f and g are colinear.

(iv) Assuming 2.3 deduce that $T = B$, that this space is one-dimensional, and that $f = \bar{f}$ for $f \in T$.

(v) Thus, assuming 2.3, there is a unique element $\mathrm{JW}_n \in T$ whose identity coefficient is 1. Prove that JW_n is idempotent. (If we construct JW_n in some other way, this proves 2.3.)

Solution to 9.25(i). □

Solution to 9.25(ii). □

Solution to 9.25(iii). □

Solution to 9.25(iv). □

Solution to 9.25(v). □

Exercise 9.26. Let TL_n be the Temperley–Lieb algebra with n-strands where the bubble evaluates to $-[2] = q + q^{-1} \in \mathbb{Q}(q)$. Clearly, JW_1 is just the identity element, where the condition of being killed by caps and cups is vacuous. Verify the following recursive formula:

(2.4)

In this last diagram, the cup on top matches the a-th and $(a+1)$-st boundary points, counting from the left.

Solution to 9.26. □

Exercise 9.27. Prove the following recursive formula.

Solution to 9.27. □

Exercise 9.28. The trace of an element $a \in \mathrm{TL}_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the following closed diagram:

(2.5)

(i) Calculate the trace of JW_n.

(ii) Suppose that q is a primitive $2m$-th root of unity. What is the trace of JW_{m-1}? What do you get when you rotate JW_{m-1} by one strand?

Solution to 9.28(i). □

Solution to 9.28(ii). □

Exercise 9.34.

(i) Write down the two-color relations when $m = 2$. Prove that $B_s B_t \simeq B_t B_s$ by constructing inverse isomorphisms.

(ii) Write down the two-color relations when $m = 3$. Prove that $B_s B_t B_S \simeq X \oplus B_s$, where X is the image of an idempotent constructed using two 6-valent vertices, by following the rubric of Exercise 8.39.

(iii) (Still for m = 3) Similarly, one has $B_t B_s B_t \simeq Y \oplus B_t$. Prove that X is isomorphic to Y. Extend the rubric of Exercise 8.39 to a rubric which describes when two summands of different objects are isomorphic.

Solution to 9.34(i). □

Solution to 9.34(ii). □

Solution to 9.34(iii). □

Exercise 9.35. Prove that there is an autoequivalence of $\mathcal{H}_{\mathrm{BS}}$ which flips each diagram vertically (resp. horizontally). See Exercise 8.10 for inspiration.

Solution to 9.35. □

Exercise 9.36. Show that the diagram obtained by attaching a "handle" to the left or right of a Jones–Wenzl projector equals 0. For example,

$$\tag{2.6}$$

(Hint: use (9.16).)

Solution to 9.36. □

Exercise 9.37.

(i) A *pitchfork* is a diagram of the form

$$\tag{2.7}$$

(or its color swap). The death by pitchfork relation states that the diagram obtained by placing a pitchfork anywhere on top or bottom of a Jones–Wenzl projector equals 0. For example:

$$\tag{2.8}$$

Why is death by pitchfork implied by the defining property of the Jones–Wenzl projector?

(ii) Use (9.28) and (9.31) to prove that the diagram obtained by placing a pitchfork anywhere on top or bottom of the $2m$-valent vertex equals 0. We also call this *death by pitchfork*.

Solution to 9.37(i). □

Solution to 9.37(ii). □

Exercise 9.38.

(i) Prove (9.29) and (9.30) using the relations in (9.27). (Hint: each relation follows from two careful applications of (9.27c). Alternatively, (9.29) can be proved by repeatedly applying (9.30).

(ii) Prove (9.27b) using (9.28) and the other relations in (9.27). (Hint: first use (8.12) to create a dot and a trivalent vertex on the left hand side, and then dispose of the trivalent vertex with two-color associativity.)

The following exercise is harder, but very worthwhile.

Solution to 9.38(i). □

Solution to 9.38(ii). □

Solution to 9.38(iii). □

Exercise 9.39.

(i) Prove (9.28) using the relations in (9.27).

(ii) Prove that the Elias–Jones–Wenzl relation (9.27b) follows from two-color associativity (9.27c) and two-color dot contraction (9.28).

Solution to 9.39(i). □

Solution to 9.39(ii). □

Chapter 3

Research Papers

Exercise 3.1. Compute the value of a bigon at $q = 1$ or at general q.

Solution to 3.1. □

Exercise 3.2. Look at (2.9). Can you find associativity and coassociativity inside? Use only these relations and (2.4) to prove (2.9).

Solution to 3.2. □

Exercise 3.3. Write down what (2.10) means explicitly for some small values of k, l, r, s, until you get a feeling for how it works. You'll definitely want an exampel where k-l+r-s is at least 2 eventually. Then try to verify it using vectors for small values.

Solution to 3.3. □

Exercise 3.4. Try to prove Lemma 2.9 from Light Ladders and Clasp Conjectures

Solution to 3.4. □

Exercise 3.5. Remember how for the Temperley-Lieb algebra you described the "Crossing" $v \otimes w \mapsto w \otimes v$ as a linear combination of other maps. Let's do this again, but with webs this time. You're going to have to use $q = 1$ do this exercise, so forget about the q-deformation.

Consider the map $\Lambda^1 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^1 V$ which just swaps the tensor factors. This is a linear combination of:

(i) The web which merges $1, 2$ into 3 and then splits 3 into $2, 1$.

(ii) The web which splits $1, 2$ into $1, 1, 1$ and then merges $1, 1, 1$ into $2, 1$.

Find the linear combo.

Solution to 3.5(i). □

Solution to 3.5(ii). □

Exercise 3.6. Consider the map $\Lambda^2 V \otimes \Lambda^2 V \to \Lambda^2 V \otimes \Lambda^2 V$ which just swaps the tensor factors. This is a linear combination of:

(i) the web which merges $2, 2$ into 4 and then splits 4 into $2, 2$.

(ii) the web which splits $2, 2$ into $2, 1, 1$ and then merges $2, 1, 1$ into $3, 1$ and then splits back to $2, 1, 1$ and marges back to $2, 2$.

(iii) the identity of $2, 2$.

Find the linear combo.

Solution to 3.6(i). □

Solution to 3.6(ii). $\square$

Solution to 3.6(iii). $\square$

Chapter 4

Misc

Exercise 4.1. Find a formula for the product $[n][3]$ when $n \geq 3$ and $[n][4]$ when $n \geq 4$. Generalize this.

Solution to 4.1. $\square$

Exercise 4.2. What is $[n][n] - [n+1][n-1]$?

Solution to 4.2. $\square$

Exercise 4.3. What is $[n][k] - [n+1][k-1]$ for $k < n$?

Solution to 4.3. $\square$