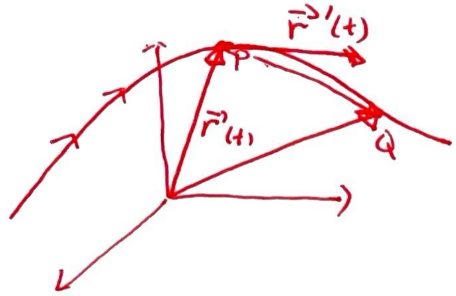
§ 13.2: Derivatives

Consider a vector function $\vec{r}(t) = \langle f(t), g(t), K(t) \rangle$



$$\frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \qquad \vec{PQ} = \vec{r}(t+h) - \vec{r}(t)$$

$\frac{d\vec{r}}{dt} = \vec{r}'(t)$ is tangent to the space curve at $P(x, y, z)$ where $\vec{r}(t) = \vec{OP}$. It determines the direction of the space curve for increasing t.

Suppose $f(t)$, $g(t)$, $K(t)$ are differentiable, then

$$\frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h \to 0} \frac{\langle f(t+h), g(t+h), K(t+h) \rangle - \langle f(t), g(t), K(t) \rangle}{h}$$

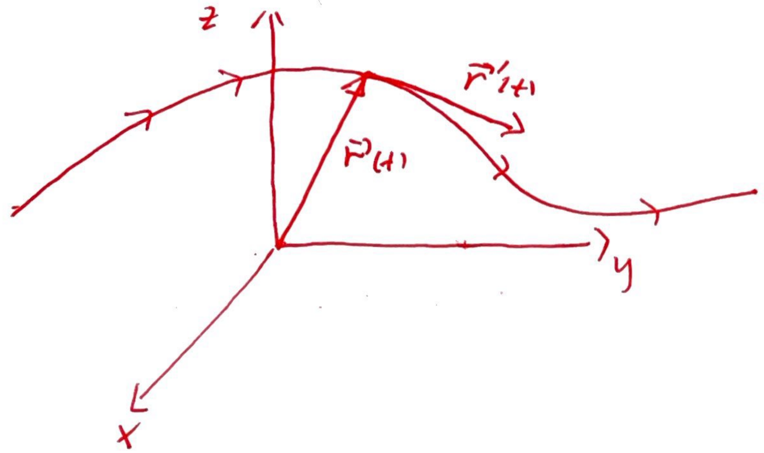
$$= \lim_{h \to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{K(t+h) - K(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \to 0} \frac{K(t+h) - K(t)}{h} \right\rangle$$

$$= \langle f'(t), g'(t), K'(t) \rangle$$

Ex! Given $\vec{r}(t) = \langle e^{t^2}, t^4 - t, \sin(3t) \rangle$

then $\vec{r}'(t) = \langle 2te^{t^2}, 4t^3 - 1, 3\cos(3t) \rangle$



$\vec{r}(t)$ is a position vector that terminates at points along a space curve.

$\vec{r}'(t)$ is tangent to curve and determines its direction for increasing t.

If there exists t_0 such that $\vec{r}(t_0) = \overrightarrow{OP} = \langle x_0, y_0, z_0 \rangle$, then the tangent line to the curve at $P(x_0, y_0, z_0)$ has direction vector $\vec{r}'(t_0)$

and the line is

$$\vec{L}(t) = \vec{r}(t_0) + t\,\vec{r}'(t_0)$$

Ex: Find the tangent line to $\vec{r}(t) = \langle t^2+1, 4\sqrt{t}, e^{t^2-t} \rangle$

at $P(2, 4, 1)$.

Find t such that $\vec{r}(t) = \overrightarrow{OP}$.

$x = t^2 + 1 = 2$ If $t=1$, $x = 1+1 = 2$

$y = 4\sqrt{t} = 4 \Rightarrow \sqrt{t} = 1 \Rightarrow t = 1$

$z = e^{t^2-t} = 1$ If $t=1$, $z = e^{1-1} = e^0 = 1$

$\vec{r}'(t) = \langle 2t, \frac{2}{\sqrt{t}}, (2t-1)e^{t^2-t} \rangle$

$\vec{r}'(1) = \langle 2, 2, 1 \rangle$

The tangent line to curve at P is

$\vec{L}(t) = \langle 2, 4, 1 \rangle + t\langle 2, 2, 1 \rangle$

Theorem: Suppose $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vectors functions, $f(t)$ is a differentiable scalar function, and c is a constant.

1) $\frac{d}{dt}\left(\vec{u}(t) + c\vec{v}(t)\right) = \vec{u}'(t) + c\vec{v}'(t)$

2) $\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f(t)\vec{u}'(t) + f'(t)\vec{u}(t)$

3) $\frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}(t) \cdot \vec{v}'(t) + \vec{u}'(t) \cdot \vec{v}(t)$

4) $\frac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}(t) \times \vec{v}'(t) + \vec{u}'(t) \times \vec{v}(t)$

5) $\frac{d}{dt}\left(\vec{u}(f(t))\right) = \vec{u}'(f(t))\, f'(t)$

For 5, if $\vec{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$, then

$$\vec{u}(f(t)) = \langle u_1(f(t)),\ u_2(f(t)),\ u_3(f(t)) \rangle$$

$$\begin{aligned}
\frac{d}{dt}\vec{u}(f(t)) &= \langle u_1'(f(t))f'(t),\ u_2'(f(t))f'(t),\ u_3'(f(t))f'(t) \rangle \\
&= \langle u_1'(f(t)),\ u_2'(f(t)),\ u_3'(f(t)) \rangle f'(t) \\
&= \vec{u}'(f(t))\, f'(t)
\end{aligned}$$