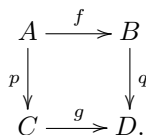Math 307, Homework #8
Due Wednesday, November 27

1. In this problem we have four functions, as indicated in the diagram below:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
C & \xrightarrow{\ g\ } & D.
\end{array}
$$

You are given that $q \circ f = g \circ p$. Let $X \subseteq C$.

(a) Prove that $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$.

(b) If p is onto and q is one-to-one, prove $f(p^{-1}(X)) = q^{-1}(g(X))$.

2. For all $n \geq 2$, $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{(n-1)(3n+2)}{4n(n+1)}$.

3. Let $a_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1} n$. Prove by induction that $a_{2n} = -n$ for all $n \geq 1$.

4. Suppose $f \colon \mathbb{Z} \to \mathbb{Z}$ is a function with the property that $(\forall x, y \in \mathbb{Z})[f(x+y) = f(x) + f(y)]$.

(a) Prove by induction that $(\forall n \in \mathbb{N})[n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]]$.

(b) Prove that $(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k]$.

5. For all $n \geq 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.

6. For all $n \geq 2$, $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n^2}\right) = \frac{n+1}{2n}$.

7. For all $n \geq 2$, $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$.

8. You have a huge collection of "trionimo" tiles that look like this:
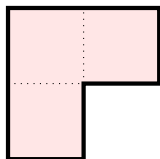
Prove by induction that for all $k \in \mathbb{N}$ such that $k \geq 1$, a $2^k \times 2^k$ checkerboard with the upper-right corner square removed can be tiled using trionimos. [Hint to get started: As scratchwork, do the cases $k = 1$, $k = 2$, and $k = 3$ by hand. Look for a link between the $2^{k+1} \times 2^{k+1}$ case and the $2^k \times 2^k$ case.]

9. There is a famous proof that all horses are the same color. Let $P(n)$ be the statement "for all sets of n horses, all the horses in the set have the same color". We will prove this by induction. The base case $n = 1$ is clear, since in a set consisting of exactly 1 horse all the horses have the same color. Now assume that $P(n)$ is true, and let S be a set of $n + 1$ horses. Label the horses $1, 2, \ldots, n + 1$. Then the first n horses constitute a set of n horses, so by the induction hypothesis they all have the same color. Likewise, the last n horses are a set of n horses; so by induction *they* all have the same color. But if the first n horses all have the same color, and the last n horses all have the same color, then since these two sets overlap the two colors must be identical. So all the horses in S have the same color, and we are done by induction.

Find the mistake in the above proof.