

1. **Classify** by Jordan Canonical form (i.e. up to similarity and up to the order of the Jordan blocks) for all  $6 \times 6$  matrices which have characteristic polynomial  $(x - 2)^2(x + 3)^4$ .

Possible Jordan blocks for  $\lambda=2$  with multiplicity 2 are:  $J_1=2$ .  $J_2=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Possible Jordan blocks for  $\lambda=-3$  with multiplicity 4 are:

$$J_3 = -3, \quad J_4 = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}, \quad J_5 = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad J_6 = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

$\Rightarrow$  Possible Jordan Canonical form of the given  $6 \times 6$  matrix:

$$\left( \begin{array}{cccccc} J_1 & J_1 & & & & \\ & J_1 & & & & \\ & & J_3 & J_3 & J_3 & J_3 \\ & & & J_3 & J_3 & J_3 \\ & & & & J_3 & J_3 \\ & & & & & J_4 \end{array} \right), \quad \left( \begin{array}{cccccc} J_1 & J_1 & & & & \\ & J_1 & & & & \\ & & J_3 \rightarrow 1 \times 1 & & & \\ & & & J_3 \rightarrow 1 \times 1 & & \\ & & & & J_4 \rightarrow 1 \times 1 & \\ & & & & & J_5 \rightarrow 3 \times 3 \end{array} \right), \quad \left( \begin{array}{cccccc} J_1 & J_1 & & & & \\ & J_1 & & & & \\ & & J_4 \rightarrow 2 \times 2 & & & \\ & & & J_4 \rightarrow 2 \times 2 & & \\ & & & & J_5 \rightarrow 3 \times 3 & \\ & & & & & J_6 \end{array} \right),$$

$$\left( \begin{array}{cccccc} J_2 & & & & & \\ & J_3 & & & & \\ & & J_3 & J_3 & J_3 & J_3 \\ & & & J_4 & & \\ & & & & J_5 & \\ & & & & & J_6 \end{array} \right), \quad \left( \begin{array}{cccccc} J_2 & & & & & \\ & J_3 & & & & \\ & & J_3 & J_3 & & \\ & & & J_4 & & \\ & & & & J_5 & \\ & & & & & J_6 \end{array} \right), \quad \left( \begin{array}{cccccc} J_2 & & & & & \\ & J_4 & & & & \\ & & J_4 & J_4 & & \\ & & & J_4 & & \\ & & & & J_5 & \\ & & & & & J_6 \end{array} \right), \quad \left( \begin{array}{cccccc} J_2 & & & & & \\ & J_3 & & & & \\ & & J_3 & J_5 & & \\ & & & J_5 & & \\ & & & & J_6 & \\ & & & & & J_6 \end{array} \right), \quad \left( \begin{array}{cccccc} J_2 & & & & & \\ & J_6 & & & & \\ & & J_6 & & & \\ & & & J_6 & & \\ & & & & J_6 & \\ & & & & & J_6 \end{array} \right)$$

2. Find the Jordan canonical form of the matrix

$$A = \begin{pmatrix} a & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Step 1: Compute eigenvalue  $\lambda=a$  with multiplicity  $n$

Step 2: Compute eigenvectors:  
 (linearly independent ones).  $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow$  expect two cycles  $\hookrightarrow$  two Jordan blocks.

Step 3: Compute the two cycles.

i). For the initial vector  $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ , solve  $(A-aI)\vec{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{v}_1$ , we get  $\vec{x} = \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow$  3rd entry.

Solve  $(A-aI)\vec{x} = \vec{v}_k$ , we get  $\vec{x} = \vec{v}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow$   $(2k-1)$ -th entry

$\vdots$

Follow the pattern: if  $n=2m$ :  $\vec{v}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow$   $(2k-1)$ -th entry,  $k=1, 2, \dots, m$

if  $n=2m+1$ :  $\vec{v}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow$   $(2k-1)$ -th entry,  $k=1, 2, \dots, m, m+1$

2). For the cycle with initial vector  $\vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ , solve  $(A - aI)\vec{x} = \vec{w}_1 \Rightarrow$  we get  $\vec{x} = \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

Solve  $(A - aI)\vec{x} = \vec{w}_{k+1} \Rightarrow$  we get  $\vec{x} = \vec{w}_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

If  $n=2m$ :  $\vec{w}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \xrightarrow{(2k)\text{-th entry}}, k=1, \dots, m$

If  $n=2m+1$ :  $\vec{w}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \xrightarrow{(2k)\text{-th entry}}, k=1, \dots, m$ .

$\Rightarrow$  If  $n=2m$ :  $P = (\vec{v}_1 \vec{v}_2 \dots \vec{v}_m \vec{w}_1 \vec{w}_2 \dots \vec{w}_m)$ , where  $\vec{v}_k$  and  $\vec{w}_k$  were specified above

$$J = \begin{pmatrix} J_1 & \\ & J_1 \end{pmatrix} \text{ where } J_1 = \begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & \\ & & & a \end{pmatrix} \in \mathbb{R}^{m \times m}$$

$$\text{then } A = PJP^{-1}$$

If  $n=2m+1$ :  $P = (\vec{v}_1 \vec{v}_2 \dots \vec{v}_m \vec{v}_{m+1} \vec{w}_1 \dots \vec{w}_m)$  where  $\vec{v}_k$  and  $\vec{w}_k$  were specified above

$$J = \begin{pmatrix} J_2 & \\ & J_1 \end{pmatrix} \text{ where } J_2 = \begin{pmatrix} a & 1 & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & a \end{pmatrix} \in \mathbb{R}^{(m+1) \times (m+1)}$$

$$J_1 = \begin{pmatrix} a & 1 & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & a \end{pmatrix} \in \mathbb{R}^{m \times m}$$

3. Let  $A = \begin{bmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{bmatrix}$ .

- 1). Find a basis for each generalized eigenspace of  $A$  consisting of a union of disjoint cycles of generalized eigenvectors.
- 2). Find a Jordan canonical form  $J$  of  $A$  using your basis in Part 1).
- 3). Find the minimal polynomial of  $A$ .

i) Step 1. Find eigenvalues and eigenvectors of  $A$ :

$$\lambda=2 \text{ (multiplicity 2)}, \quad \lambda=-1 \text{ (multiplicity 1)}$$

$$\text{Solve } (A-2I)\vec{x}=\vec{0}: \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad E_2 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Solve } (A+I)\vec{x}=\vec{0}: \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad E_{-1} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

Step 2: Two cycles: ①. For the cycle w/ initial vector  $\vec{v}_1$ , the length should be 2.

$$\text{Solve } (A-2I)\vec{x}=\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ we get } \vec{x}=\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{A cycle corresponding to } \lambda=2 \text{ is: } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

② For the cycle w/ initial vector  $\vec{v}_2$ , the length should be 1

$$\Rightarrow \text{A cycle corresponding to } \lambda=-1 \text{ is: } \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

A basis of generalized eigenvectors consisting of disjoint cycles is given by:  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

2).  $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix} \text{ s.t. } A = PJP^{-1}$

3). Minimal polynomial:  $m(x)=(x-2)^2(x+1)$ .

4. Let  $D$  be the differential linear operator on the vector space  $V = \text{span}\{1, t, t^2, e^t, te^t\}$ , i.e.

$$D(f(x)) = \frac{d}{dx}(f(x)), \quad \text{for any } f(x) \in V.$$

- 1). Let  $\mathcal{B} = \{1, t, t^2, e^t, te^t\}$ . Find the matrix representation  $A = [D]_{\mathcal{B}}$ .
- 2). Find a basis for each generalized eigenspace of  $D$  consisting of a union of disjoint cycles of generalized eigenvectors.
- 3). Find a Jordan canonical form  $J$  of  $D$  using your basis in Part 2).
- 4). Find the minimal polynomial of  $D$ .

(sketch of solutions, detailed computation work was not included)

$$1) \quad [D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \leftarrow \text{(Need to show more work!)}$$

$$2) \quad \text{Eigenvalues: } \lambda_1 = 0 \quad (\text{multiplicity}=3) \quad \text{Solve } A\vec{x} = \vec{0} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow E_0 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\text{Eigenvalues } \lambda_2 = 1 \quad (\text{multiplicity}=2) \quad \text{Solve } (A - I)\vec{x} = \vec{0} \Rightarrow \vec{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow E_1 = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

For  $\lambda = 0$ : one cycle corresponding to  $\lambda=0$  with length 3, and the Jordan block should be a  $3 \times 3$  block.

$$\text{Solve } (A - 0I)\vec{x} = \vec{v}_1 \Rightarrow \vec{x} = \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Solve } (A - 0I)\vec{x} = \vec{v}_2 \Rightarrow \vec{x} = \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\text{the Cycles Corresponding to } \lambda=0 : \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right\}$$

For  $\lambda = 1$ : one cycle corresponding to  $\lambda=1$  with length 2 and the Jordan block should be a  $2 \times 2$  block

$$\text{Solve } (A - I)\vec{x} = \vec{w}_1 : \quad \vec{x} = \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The cycle corresponding to  $\lambda=1$  :  $\left\{ \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ .

3).  $[D]_B = P \bar{J} P^{-1}$  where  $\bar{J} = \begin{pmatrix} 0 & 1 & 0 & | & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 \\ \hline 0 & 0 & 0 & | & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4). minimal polynomial for  $D$ :  $m(x) = x^3 (x-1)^2$

5. Let  $A \in \mathbb{C}^{n \times n}$ . Assume  $(A + I_n)^m = 0$ . Prove that  $A$  is invertible and find  $\det(A)$ .

Proof: Let  $g(x) = (x + 1)^m$ . Since  $g(A) = (A + I)^m = 0$ ,  $g(x)$  annihilate  $A$ . Let  $m(x)$  be the minimal polynomial and let  $f(x)$  be the characteristic polynomial of  $A$ . Then  $m(x)|g(x)$ . Thus  $m(x) = (x + 1)^m$  for some  $m \leq n$ . Thus the only root  $m(x)$  has is  $-1$ . Since  $m(x)$  and  $f(x)$  have the same roots (may not have the same multiplicity for each root), thus  $f(x)$  also has only  $-1$  as its roots. Thus  $f(x) = (x + 1)^n$ . Therefore all the eigenvalue of  $A$  are  $-1$ . Thus  $\det(A) = (-1)^n \neq 0$ . Thus  $A$  is invertible.

6. Let  $V$  be the vector space of all polynomials over  $\mathbb{R}$ . Let  $D$  be the differentiation operator on the vector space  $V$ . Find the minimal polynomial of  $D$  on  $V$  or prove that  $D$  has no minimal polynomial on  $V$ .

Answer:  $D$  does not have a minimal polynomial on  $V$ .

Proof by contradiction. Suppose towards a contradiction that  $m(x)$  is the minimal polynomial of  $D$ . Then

$$m(x) = x^m + a_{m-1}x^{m-1} + \dots + a_1x + a_0 \text{ where } m \in \mathbb{Z}^+.$$

$$\text{Then } m(D) = D^m + a_{m-1}D^{m-1} + \dots + a_1D + a_0I$$

$$\text{i.e. } \forall p(x) \in V: m(D)(p(x)) = 0.$$

$$\text{On the other hand: let } p_0(t) = t^m. \text{ Then } D(p_0(x)) = mt^{m-1}$$

$$\vdots$$

$$D^k(p_0(x)) = m(m-1)\dots(m-k+1)t^{m-k}$$

$$D^m(p_0(x)) = m!$$

$$\Rightarrow 0 = m(D)(p_0(t)) = D^m(p_0(t)) + a_{m-1}D^{m-1}(p_0(t)) + \dots + a_1D(p_0(t)) + a_0p_0(t)$$

$$= m! + a_{m-1}(m-1)\dots(2)t + a_{m-2}(m-1)\dots(3)t^2 + \dots + a_1mt^{m-1} + a_0t^m$$

$\neq 0$  b/c the constant term is  $m! \neq 0$

Contradiction!

7. Suppose  $A \in \mathbb{C}^{n \times n}$  satisfies  $A^2 = A$ . Prove that  $A$  is diagonalizable.

Let  $g(x) = x^2 - x = x(x-1)$ . Then  $g(A) = A^2 - A = 0 \Rightarrow g(x)$  annihilates  $A$ .

Let  $m(x)$  be the minimal polynomial of  $A$ . Then  $m(x) | g(x)$

$$\Rightarrow m(x) = x \quad \text{or} \quad m(x) = x-1 \quad \text{or} \quad m(x) = x(x-1)$$

In any of the above three cases,  $m(x)$  has no repeated roots.

$\Rightarrow A$  is diagonalizable. (Note: This is using the result of:  $A$  is diagonalizable if and only if its minimal polynomial has no repeated roots).

8. Give an example of two matrices who have the same characteristic polynomial but distinct minimal polynomials.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Characteristic polynomials:  $f_A(x) = f_B(x) = (x-1)^2$

minimal polynomials: A:  $m_A(x) = (x-1)^2$

B:  $m_B(x) = (x-1)$ .

(You may use many of other examples )