Introduction to Toplogy I: Homework 8

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Exercise 9.4. Let X, Y, and Z be topological spaces, let $f : X \times Y \to Z$ be continuous, and let $c \in X$. Prove that the map $h : Y \to Z$ given by $h(y) = f(c, y)$ is continuous

Solution. Let $U \subset Z$ be an open set. Since f is continuous, the preimage $f^{-1}(U)$ is open in $X \times Y$. By the definition of the product topology, there exist open sets $A \subset X$ and $B \subset Y$ such that $(c, y) \in A \times B \subset f^{-1}(U)$. Since $c \in A$, we have

$$h^{-1}(U) = \{y \in Y \mid h(y) \in U\} = \{y \in Y \mid f(c, y) \in U\} = \{y \in Y \mid (c, y) \in f^{-1}(U)\} \supset B.$$

Therefore, for every open set U in Z, the preimage $h^{-1}(U)$ contains an open set B in Y. This shows that $h^{-1}(U)$ is open in Y, and thus h is continuous. □

Exercise 9.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the discontinuous function given in (9.2). Find an open set $V \subset \mathbb{R}$ such that $f^{-1}(V)$ is not open in $\mathbb{R}^2$. But notice that the intersection of $f^{-1}(V)$ with line of the form $\{c\} \times \mathbb{R}$ or $\mathbb{R} \times \{c'\}$ is open (in that line), reflecting the fact that $f(c, y)$ is continuous as a function of y and $f(x, c')$ is continuous as a function of x.

Solution. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Take $U = (-0.1, 0.1)$. Then, we have

$$\begin{aligned} f^{-1}(U) &= \{(x, y) \in \mathbb{R}^2 \mid -0.1 < f(x, y) < 0.1\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, -0.1 < \frac{xy}{x^2 + y^2} < 0.1\right\}\bigg| \\ &= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, -0.1(x^2 + y^2) < xy < 0.1(x^2 + y^2)\right\}\big| \\ &= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0.1x^2 - xy + 0.1y^2 > 0 \text{ and } 0.1x^2 + xy + 0.1y^2 > 0\right\}\big| \\ &= \mathbb{R}^2 \setminus \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0.1x^2 - xy + 0.1y^2 \leq 0 \text{ or } 0.1x^2 + xy + 0.1y^2 \leq 0\right\}\big| \\ &= \mathbb{R}^2 \setminus \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, (x - 5y)^2 \leq 0 \text{ or } (x + 5y)^2 \leq 0\right\}\big| \\ &= \mathbb{R}^2 \setminus (\{(5y, y) \mid y \in \mathbb{R}\} \cup \{(-5y, y) \mid y \in \mathbb{R}\}) \\ &= \mathbb{R}^2 \setminus (L_1 \cup L_2), \end{aligned}$$

where L_1 and L_2 are the lines defined by $y = \frac{1}{5}x$ and $y = -\frac{1}{5}x$, respectively. □

Exercise 10.4. Given a map $f : X \to Y$, we can consider its graph

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}.$$

(i) Prove that if X and Y are topological spaces, Y is Hausdorff, and f is continuous, then Γ_f is closed.

Hint: You could do this by hand, or you could consider the pre-image of the diagonal $\Delta \subset Y \times Y$ and under the map $X \times Y \to Y \times Y$ that sends (x, y) to $(f(x), y)$.

(ii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not continuous (in the usual topology) but whose graph is nonetheless closed.

Hint: It won't work if f is bounded.

Solution to (i). Define the diagonal subset of $Y \times Y$ as $\Delta = \{(y, y) \mid y \in Y\}$. Since Y is Hausdorff, by Proposition 10.4, the diagonal, Δ, is closed in $Y \times Y$. Define the function $F : X \times Y \to Y \times Y$, where $(x, y) \mapsto (f(x), y)$. Notice that we can re-write F as

$$F(x, y) = (f \circ \pi_X(x, y), \pi_Y(x, y)),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projection maps. Since f and the projection maps are continuous, F is continuous as well. Then, we have

$$(x, y) \in F^{-1}(\Delta) \Leftrightarrow (f(x), y) \in \Delta \Leftrightarrow f(x) = y \Leftrightarrow (x, y) \in \Gamma_f,$$

so $\Gamma_f = F^{-1}(\Delta)$. Since Δ is closed in $Y \times Y$ and F is continuous, $F^{-1}(\Delta)$ is closed in $X \times Y$. Therefore, Γ_f is closed in $X \times Y$. □

Solution to (ii). Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is not continuous at $x = 0$, since $\lim\limits_{x\to 0} f(x)$ does not exist. However, we can show that its graph Γ_f is closed in $\mathbb{R}^2$. Let $(x_n, f(x_n))$ be a sequence in Γ_f that converges to some point $(x, y) \in \mathbb{R}^2$. We need to show that $(x, y) \in \Gamma_f$.

If $x \neq 0$, then for sufficiently large n, $x_n \neq 0$ and $f(x_n) = 1/x_n$. Since $(x_n, f(x_n)) \to (x, y)$, it follows that $y = 1/x$, so $(x, y) \in \Gamma_f$.

If $x = 0$, then for the sequence (x_n) to converge to 0, infinitely many x_n must be nonzero. But for $x_n \neq 0$, we have $f(x_n) = 1/x_n$, which becomes arbitrarily large in magnitude. Therefore, in order for $(f(x_n))$ to converge to a finite y, eventually $x_n = 0$. Then $f(x_n) = 0$, so $y = 0$. Hence, $(0, 0) \in \Gamma_f$.

In both cases, any limit of a sequence in Γ_f lies in Γ_f, proving that Γ_f is closed in $\mathbb{R}^2$. □

Exercise 10.5.

(i) Let X and Y be topological spaces, and suppose that Y is Hausdorff. Prove that if two continuous maps $f, g : X \to Y$ agree on a dense subset $D \subset X$, then $f = g$.

Hint: Let $E = \{x \in X \mid f(x) = g(x)\}$, and prove that it's closed.

(ii) Give a counterexample when Y is not Hausdorff.

Solution to (i). Let $E = \{x \in X \mid f(x) = g(x)\}$. Define the following mapping $h : X \to Y \times Y$ by $h(x) = (f(x), g(x))$. Since f and g are continuous, the map h is also continuous. Now, take the diagonal subset $\Delta = \{(y_1, y_2) \in Y \times Y \mid y_1 = y_2\}$. Since Y is Hausdorff, by Proposition 10.4, the diagonal Δ is closed in $Y \times Y$. Notice that $x \in E$ if and only if $f(x) = g(x)$. This is equivalent to saying $(f(x), g(x)) \in \Delta$, which is also equivalent to saying $h(x) \in \Delta$. Thus, we have $E = h^{-1}(\Delta)$. Since Δ is closed in $Y \times Y$ and h is continuous, $h^{-1}(\Delta)$ is closed in X. Therefore, E is closed in X. Since f and g agree on the dense subset $D \subset X$, we have $D \subset E$. Then, we have $\overline{D} \subseteq \overline{E}$, but $\overline{D} = X$, so $X \subseteq \overline{E}$. This implies that $E = X$. Hence, $f(x) = g(x)$ for all $x \in X$, and thus $f = g$. □

Solution to (ii). Consider the topological space $Y = \{a, b\}$ with the trivial topology $\{\emptyset, Y\}$. Define the functions $f, g : \mathbb{R} \to Y$ by $f(x) = a$, for all $x \in \mathbb{R}$, and

$$g(x) = \begin{cases} a, & x \neq 0, \\ b, & x = 0. \end{cases}$$

Both f and g are continuous since the preimage of any open set in Y is either $\emptyset$ or $\mathbb{R}$, both of which are open in $\mathbb{R}$. The functions f and g agree on the dense subset $D = \mathbb{R} \setminus \{0\}$, but they do not agree at $x = 0$, where $f(0) = a$ and $g(0) = b$. □