Abstract Linear Algebra: Homework 9

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Problem 1. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let T be a linear transformation on V. Prove that T is self-adjoint if and only if $\langle \mathbf{x}, T\mathbf{x} \rangle$ is real a real number for all $\mathbf{x} \in V$.

Solution. Assume T is self-adjoint. We need to show that $\langle \mathbf{x}, T\mathbf{x} \rangle$ is real for all $\mathbf{x} \in V$. Since T is self-adjoint, we have

$$\langle T\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, T\mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

Setting $\mathbf{y} = \mathbf{x}$, we obtain $\langle T\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, T\mathbf{x} \rangle$. By the conjugate symmetry of the inner product, we know that

$$\langle \mathbf{x}, T\mathbf{x} \rangle = \overline{\langle T\mathbf{x}, \mathbf{x} \rangle}.$$

Thus, we have

$$\langle \mathbf{x}, T\mathbf{x} \rangle = \overline{\langle \mathbf{x}, T\mathbf{x} \rangle}.$$

This means that $\langle \mathbf{x}, T\mathbf{x} \rangle$ is equal to its own complex conjugate, which implies that it is a real number.

Assume $\langle \mathbf{x}, T\mathbf{x} \rangle$ is real for all $\mathbf{x} \in V$. We need to show that T satisfies $\langle T\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, T\mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$. Define the function $f : V \times V \to \mathbb{C}$ by

$$f(\mathbf{x}, \mathbf{y}) = \langle T\mathbf{x}, \mathbf{y} \rangle.$$

We need to show that $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$, i.e.,

$$\langle T\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, T\mathbf{y} \rangle$$

Define the function $g(\mathbf{x}, \mathbf{y})$ as

$$g(\mathbf{x}, \mathbf{y}) = \langle T\mathbf{x}, \mathbf{y} \rangle - \overline{\langle T\mathbf{y}, \mathbf{x} \rangle}.$$

Since the inner product satisfies conjugate symmetry, we get

$$\overline{\langle T\mathbf{y}, \mathbf{x} \rangle} = \langle \mathbf{x}, T\mathbf{y} \rangle.$$

Thus, we can rewrite g as

$$g(\mathbf{x}, \mathbf{y}) = \langle T\mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, T\mathbf{y} \rangle.$$

We need to show that $g(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{x}, \mathbf{y} \in V$.

For any $\mathbf{x}$, define $h(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})$. Notice that h is linear in $\mathbf{y}$. Taking $\mathbf{y} = \mathbf{x}$, we get

$$g(\mathbf{x}, \mathbf{x}) = \langle T\mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, T\mathbf{x} \rangle = 0,$$

since we assumed that $\langle \mathbf{x}, T\mathbf{x} \rangle$ is real.

For any $\mathbf{x}, \mathbf{y}$, consider

$$g(\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) = \langle T(\mathbf{x} + \mathbf{y}), \mathbf{x} + \mathbf{y} \rangle - \langle \mathbf{x} + \mathbf{y}, T(\mathbf{x} + \mathbf{y}) \rangle.$$

Since each term is real, we get

$$\langle T\mathbf{x}, \mathbf{x} \rangle + \langle T\mathbf{x}, \mathbf{y} \rangle + \langle T\mathbf{y}, \mathbf{x} \rangle + \langle T\mathbf{y}, \mathbf{y} \rangle - (\langle \mathbf{x}, T\mathbf{x} \rangle + \langle \mathbf{x}, T\mathbf{y} \rangle + \langle \mathbf{y}, T\mathbf{x} \rangle + \langle \mathbf{y}, T\mathbf{y} \rangle) = 0.$$

Canceling out terms, we obtain

$$\langle T\mathbf{x}, \mathbf{y} \rangle + \langle T\mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{x}, T\mathbf{y} \rangle - \langle \mathbf{y}, T\mathbf{x} \rangle = 0.$$

Thus, $\langle T\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, T\mathbf{y} \rangle$. Since $\mathbf{x}$ and $\mathbf{y}$ were arbitrary, T is self-adjoint.

Thus, T is self-adjoint if and only if $\langle \mathbf{x}, T\mathbf{x} \rangle$ is real for all $\mathbf{x} \in V$. □

Problem 2. Let $A \in \mathbb{C}^{n \times n}$. Prove that A is normal if and only if A can be written in the form of $A = A_1 + iA_2$ where A_1 and A_2 are Hermitian and $A_1 A_2 = A_2 A_1$.

Solution. Assume A is normal. We want to show that A can be written $as A_1 + iA_2$ with the given properties. Define

$$A_1 = \frac{A + A^*}{2} \quad \text{and} \quad A_2 = \frac{A - A^*}{2i}.$$

We first need to verify that A_1 and A_2 are Hermitian. The conjugate transpose of A_1 is

$$A_1^* = \left(\frac{A + A^*}{2}\right)^* = \frac{A^* + (A^*)^*}{2} = \frac{A^* + A}{2} = A_1 \quad \text{and} \quad A_2^* = \left(\frac{A - A^*}{2i}\right)^* = \frac{A^* - A}{2i} = A_2.$$

Therefore, A_1 and A_2 are Hermitian. Since A_1 and A_2 are defined in terms of A and A^* respectively, we see that $A = A_1 + iA_2$.

Now, we need to show that A_1 and A_2 commute, i.e., $A_1A_2 = A_2A_1$. Since A is normal, we have $AA^* = A^*A$. Substituting $A = A_1 + iA_2$ and $A^* = A_1 - iA_2$, we compute

$$(A_1 + iA_2)(A_1 - iA_2) = A_1^2 - iA_1A_2 + iA_2A_1 + A_2^2 = A_1^2 + A_2^2 + i(A_2A_1 - A_1A_2).$$

Similarly,

$$(A_1 - iA_2)(A_1 + iA_2) = A_1^2 - iA_1A_2 + iA_2A_1 + A_2^2 = A_1^2 + A_2^2 + i(A_2A_1 - A_1A_2).$$

Since $AA^* = A^*A$, it follows that $i(A_2A_1 - A_1A_2) = 0$. Thus, $A_1A_2 = A_2A_1$. Hence, if A is normal, then it can be decomposed as $A = A_1 + iA_2$, where A_1, A_2 are Hermitian and commute.

Assume $A = A_1 + iA_2$ where A_1 and A_2 are Hermitian and commute. We need to show that $AA^* = A^*A$. We get $A^* = (A_1 + iA_2)^* = A_1^* + iA_2^* = A_1 - iA_2$. Now, compute AA^* and A^*A to get

$$\begin{aligned} AA^* &= A_1^2 - iA_1A_2 + iA_2A_1 + A_2^2 = A_1^2 + A_2^2 + i(A_2A_1 - A_1A_2) \\ \text{and} \quad A^*A &= (A_1 - iA_2)(A_1 + iA_2) = A_1^2 - iA_1A_2 + iA_2A_1 + A_2^2 = A_1^2 + A_2^2 + i(A_2A_1 - A_1A_2). \end{aligned}$$

Since, by assumption, that $A_1A_2 = A_2A_1$, the term $A_2A_1 - A_1A_2$ cancel, leaving $AA^* = A_1^2 + A_2^2 = A^*A$. Thus, A is normal.

Therefore, A is normal if and only if it can be decomposed in this form. □

Problem 3. Prove that a normal and nilpotent linear transformation is the zero linear transformation. (Note: A linear transformation T is nilpotent if there exists a positive integer r such that $T^r = 0$.)

Solution. Assume T is a normal and nilpotent linear transformation. We need to show that $T = 0$. Since T is normal, by the Spectral Theorem, we know that T is diagonalizable, i.e., $T = UDU^*$, where D is a diagonal matrix of eigenvalues and U is a unitary matrix. Since it is diagonalizable, then there exists an eigenbasis, say $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, with corresponding eigenvalues, say $\lambda_1, \cdots, \lambda_n$, so that $T\mathbf{v}_i = \lambda_i\mathbf{v}_i$, for $i = 1, \cdots, n$. Since T is nilpotent, all it's eigenvalues are zero. To show this, suppose $T^k = 0$, for some k. Applying this to an eigenvector, we have

$$T^k\mathbf{v}_i = \lambda_i^k\mathbf{v}_i \Rightarrow \mathbf{0} = \lambda_i^k\mathbf{v}_i = \mathbf{0} \Rightarrow \lambda_i^k = 0 \Rightarrow \lambda_i = 0.$$

Since T is normal, it is diagonalizable, and all its eigenvalues are zero, giving us

$$D = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \Rightarrow T = UDU^* = 0.$$

Therefore, a normal and nilpotent linear transformation is the zero linear transformation. □

Problem 4. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Prove that A is positive definite if and only if all the eigenvalues of A are positive.

Solution. Assume that A is a positive definite Hermitian matrix. We need to show that all the eigenvalues of A are positive. Since A is Hermitian, the spectral theorem states that A has an orthonormal basis of eigenvectors, and all its eigenvalues are real. Let $\mathbf{v}$ be an eigenvector of A corresponding to eigenvalue λ, i.e., $A\mathbf{v} = \lambda\mathbf{v}$. Consider the quadratic form for this eigenvector $\mathbf{v}^* A\mathbf{v} = \mathbf{v}^*(\lambda\mathbf{v}) = \lambda(\mathbf{v}^*\mathbf{v})$. Since $\mathbf{v}^*\mathbf{v} = \|\mathbf{v}\|^2 > 0$ for any nonzero eigenvector, we have that $\mathbf{v}^* A\mathbf{v} > 0$. It follows that $\lambda > 0$. Thus, if A is positive definite, then all its eigenvalues are positive.

Assume that all the eigenvalues of A are positive and A is Hermitian. We need to show that A is positive definite. Again, by the spectral theorem, we can diagonalize A as $A = UDU^*$, where D is a diagonal matrix of eigenvalues, $\lambda_1, \cdots, \lambda_n$, and U is a unitary matrix. For any nonzero vector, $\mathbf{x}$, write $\mathbf{y} = U^*\mathbf{x}$, so that $\mathbf{x}^* A\mathbf{x}$ can be re-written as

$$\mathbf{x}^* A\mathbf{x} = \mathbf{x}^* UDU^*\mathbf{x} = (U^*\mathbf{x})^* D(U^*\mathbf{x})^* = \mathbf{y}^* D\mathbf{y}.$$

Since D is diagonal, this simplifies to

$$\mathbf{y}^* D\mathbf{y} = \sum_{i=1}^{n} \lambda_i \|\mathbf{y}_i\|^2.$$

If all $\lambda_i > 0$, then each term in the sum is strictly positive for any nonzero $\mathbf{y}$, meaning

$$\mathbf{x}^* A\mathbf{x} = \sum_{i=1}^{n} \lambda_i \|\mathbf{y}_i\|^2 > 0.$$

Thus, A is positive definite.

Therefore, A is positive definite if and only if all its eigenvalues are positive. □

Problem 5. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix. Prove that A is unitary if and only if every eigenvalue of A has magnitude (or absolute value) 1.

Solution. Assume A is a unitary square normal matrix. We want to show that every eigenvalue of A has magnitude 1. Since A is unitary, by definition, we have $A^*A = I = AA^*$. Let (λ, v) be an eigenpair of A, meaning that $A\mathbf{v} = \lambda\mathbf{v}$. Taking norms on both sides, we get

$$\|Av\| = \|\lambda v\| = |\lambda|\|v\|.$$

Since A is unitary, it preserves norms, so $\|Av\| = \|v\|$. Thus, we obtain $|\lambda|\|v\| = \|v\|$. Since $v \neq 0$, it follows that $|\lambda| = 1$. Therefore, if A is unitary, then every eigenvalue of A has magnitude 1.

Conversely, assume that every eigenvalue of A has magnitude 1 and that A is normal. We need to show that A is unitary. Since A is normal, it is unitarily diagonalizable, meaning that there exists a unitary matrix U such that $U^*AU = D$, where D is a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. By assumption, each eigenvalue satisfies $\|\lambda_i\| = 1$. We compute

$$D^*D = \operatorname{diag}(\overline{\lambda_1}, \ldots, \overline{\lambda_n})\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \operatorname{diag}(|\lambda_1|^2, \ldots, |\lambda_n|^2) = I.$$

Since unitary similarity preserves this property, we compute

$$\begin{aligned} A^*A &= (UDU^*)^*(UDU^*) \\ &= UD^*U^*UDU^* \\ &= UD^*DU^* \\ &= UIU^* \\ &= I. \end{aligned}$$

Similarly, we compute

$$\begin{aligned} AA^* &= (UDU^*)(UDU^*)^* \\ &= UDU^*UD^*U^* \\ &= UDD^*U^* \\ &= UIU^* \\ &= I. \end{aligned}$$

Thus, if every eigenvalue of A has magnitude 1, then A is unitary.

Therefore, A is unitary if and only if every eigenvalue of A has magnitude 1. $\square$

Problem 6. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Suppose A is both positive definite and unitary. Prove that $A = I$. *Hint: You may use conclusions from the above two problems.*

Solution. Let λ be an eigenvalue of A. By problem 4 and 5, $\lambda > 0$ and $|\lambda| = 1$. By the Spectral Theorem, A is diagonalizable with an orthonormal basis of eigenvectors. Because all of its eigenvalues are 1, the diagonalized form of A is simply the identity matrix I. $\square$

Problem 7. Consider $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

(i) Find the singular value decomposition of A. (Show your computation work, do not use technology.)

(ii) Find the generalized inverse A^+.

Solution to (i). Computing A^TA, we get

$$A^TA = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} (-1)(-1)+(1)(1) & (1)(-1) & (-1)(1) \\ (-1)(1) & (-1)(-1) & 0 \\ (1)(-1) & 0 & (1)(1) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Now, we need to find the eigenvalues and eigenvectors of A^TA. The characteristic polynomial of A^TA is given by

$$\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = -\lambda(\lambda-3)(\lambda-1) = 0,$$

giving us the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = 0$. Next, we need to find the eigenvectors corresponding to each eigenvalue.

$$\underline{\lambda = 3} : A^TA - 3I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{\lambda = 1}: A^T A - I = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{\lambda = 0}: A^T A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The matrix Σ is made up of the square roots of the eigenvalues of $A^T A$ along the diagonal, giving us

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The columns of the matrix V are the normalized vectors

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Normalizing each eigenvector gives us

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{6}} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{pmatrix}$$

$$\hat{\mathbf{v}}_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}.$$

This gives us the matrix V as

$$V = \begin{pmatrix} -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{pmatrix}.$$

The columns of the matrix U are the left singular vectors of the original matrix. Finding each $\mathbf{u}_i$ gives us

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\hat{\mathbf{v}}_2 = \frac{1}{\sqrt{1}} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

This gives us the matrix U as

$$U = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Finally, we have the singular value decomposition of A as

$$A = U\Sigma V^* = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}. \qquad \square$$

Solution to (ii). Using the general formula for the general inverse, we have $A^+ = A^T \cdot \left(AA^T\right)^{-1}$. Computing AA^T, we get

$$A \cdot A^T = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The inverse of this matrix is

$$(AA^T)^{-1} = \frac{1}{\det(AA^T)} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Lastly, we multiply this by A^T to get

$$A^T \cdot (AA^T)^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Therefore, the generalized inverse of A is

$$A^+ = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

□

Problem 8. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian and indefinite (i.e. not positive definite or positive semi-definite). Suppose $A = PDP^*$ for some unitary matrix P and some diagonal matrix $D \in \mathbb{R}^{n\times n}$. Find the singular value decomposition of $A = V\Sigma U^*$ by constructing V, Σ, and U using some possible variations of the columns or entries of P and D.

Solution. The singular values of A are the absolute values of its eigenvalues, since A is Hermitian and diagonalizable. That is, if $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, then the singular values of A are

$$\sigma_i = |\lambda_i|, \quad i = 1, \ldots, n.$$

Thus, the singular value matrix Σ is given by

$$\Sigma = \operatorname{diag}(|\lambda_1|, \cdots, |\lambda_n|).$$

Since $A = PDP^*$, we consider how to transform this into an SVD. Firstly, choose U to align with the eigenvectors of A. Since the right singular vectors (columns of U) correspond to the eigenvectors of $A^*A = A^2$, we observe that $A^*A = A^2 = PD^2P^*$.

Since P is unitary, it diagonalizes A^*A with eigenvalues λ_i^2, meaning the right singular vectors are also given by P. Thus, we set $U = P$.

The left singular vectors (columns of V) are eigenvectors of $AA^* = A^2$, which is the same computation as A^*A, meaning we can also use P up to sign adjustments. Define a signature matrix S as $S = \operatorname{diag}(\operatorname{sgn}(\lambda_1), \operatorname{sgn}(\lambda_2), \cdots, \operatorname{sgn}(\lambda_n))$ where

$$\operatorname{sgn}(\lambda_i) = \begin{cases} +1, & \lambda_i > 0 \\ -1, & \lambda_i < 0 \end{cases}.$$

Then, setting $V = PS$ ensures that $A = V\Sigma U^*$.

Therefore, the singular value decomposition of A is $A = V\Sigma U^*$ where $U = P$, $V = PS$, where S is the signature matrix, and Σ is the diagonal matrix of singular values. □

Problem 9. Let A be an $m \times n$ matrix, and let P and Q be $m \times m$ and $n \times n$ unitary matrices. Show that A and PAQ have the same singular values.

Solution. Consider the transformed matrix $B = PAQ$. Its Gram matrix is given by $B^*B = (PAQ)^*(PAQ)$. Using the properties of the conjugate transpose, $B^* = (PAQ)^* = Q^*A^*P^*$. Thus, $B^*B = Q^*A^*P^*PAQ$. Since P is unitary, we have $P^*P = I_m$, so this simplifies to $B^*B = Q^*A^*AQ$. Since Q is unitary, it preserves eigenvalues. That is, A^*A and Q^*A^*AQ have the same eigenvalues. Therefore, the eigenvalues of B^*B are the same as those of A^*A. Since the singular values are the square roots of the eigenvalues of A^*A (or equivalently B^*B), we conclude that A and PAQ have the same singular values,

$$\sigma_i(A) = \sigma_i(PAQ) = \sigma_i(B), \quad \text{for all } i.$$

□