1. The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is defined as the sum of the diagonal elements of A, i.e. $\mathrm{Tr}(A) = \sum_{j=1}^{n} a_{jj}$. Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any $n \times n$ matrices A and B.

2. State the replacement theorem.

3. Let V be a vector space. Prove that the zero vector in V is unique.

4. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{C}\}$. Define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \text{ and } c(a_1, a_2) = (a_1, 0).$$

Determine whether or not V is a vector space over $\mathbb{C}$ with these operations. Justify your answer.

5. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{C}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{C}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Determine whether or not V is a vector space over $\mathbb{C}$ with these operations. Justify your answer.

6. If W_1 and W_2 are subspaces of a vector space V, prove that $W_1 \cap W_2$ is a subspace of V.

7. Consider the following subsets in $\mathbb{C}^n$:

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n : a_1 + \cdots + a_n = 0 \right\}, \; W_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n : a_1 + \cdots + a_n = c, \text{ where } c \neq 0 \right\}$$

Prove that W_1 is a subspace of $\mathbb{C}^n$, but W_2 is not a subspace of $\mathbb{C}^n$.

8. Let S be the subset of all symmetric matrices in $\mathbb{R}^{n\times n}$, i.e. $S = \{A \in \mathbb{R}^{n\times n} : A^T = A\}$. Prove that S is a subspace of $\mathbb{R}^{n\times n}$.

9. Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ be an ordered basis for V. Prove that for any $\mathbf{x} \in V$, there exists a unique set of scalars $\{a_1, a_2, \cdots, a_n\}$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n.$$

10. True or False. (No explanation needed).

1). A vector space may have more than one zero vector.

2). If f and g polynomials of degree n, then $f + g$ is a polynomial of degree n.

3). If V is a vector space and W is a subset of V that is a vector space, then W is a subspace.

4). If W and U are subspaces of V, then $W \cup U$ is a subspace of V.