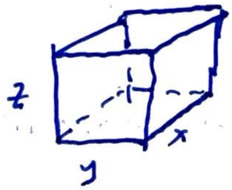
Ex: Find the dimensions of the rectangular box with largest volume if the surface area is 64 cm^2.



Volume of box: $V(x,y,z) = xyz$

Surface area of box: $A(x,y,z) = 2xy + \cancel{2xy}\,2xz + 2yz = 64$

$$2(x+y)z = 64 - 2xy$$

$$z = \frac{32 - xy}{x+y}$$

Minimize $V(x,y) = xyz = \dfrac{32xy - x^2y^2}{x+y}$

$\nabla V = \vec{0}$

$$\nabla V = \left\langle \frac{(x+y)(32y - 2xy^2) - (32xy - x^2y^2)}{(x+y)^2}, \frac{(x+y)(32x - 2x^2y) - (32xy - x^2y^2)}{(x+y)^2} \right\rangle$$

$$= \left\langle \frac{32xy + 32y^2 - 2x^2y^2 - 2xy^3 - 32xy + x^2y^2}{(x+y)^2}, \frac{32x^2 + 32xy - 2x^3y - 2x^2y^2 - 32xy + x^2y^2}{(x+y)^2} \right\rangle$$

$$= \frac{1}{(x+y)^2} \langle 32y^2 - x^2y^2 - 2xy^3, 32x^2 - 2x^3y - x^2y^2 \rangle = \vec{0}$$

1) $32y^2 - x^2y^2 - 2xy^3 = 0$

2) $32x^2 - 2x^3y - x^2y^2 = 0$

Factor System

1) $y^2(32 - x^2 - 2xy) = 0 \quad \Rightarrow y = 0$ or $32 - x^2 - 2xy = 0$

2) $x^2(32 - 2xy - y^2) = 0 \quad \Rightarrow x = 0$ or $32 - 2xy - y^2 = 0$

If $x = 0$ or $y = 0$, then $V = 0$ (minimum volume)
Disregard those cases.

Solve

3) $32 - x^2 - 2xy = 0 \quad \Rightarrow \quad x^2 = 32 - 2xy$

4) $32 - 2xy - y^2 = 0 \quad \Rightarrow \quad y^2 = 32 - 2xy$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$

(length, $x, y > 0$)

If $y = x$, then 3) becomes $32 - 3x^2 = 0$ and $x = \sqrt{\frac{32}{3}}$

Critical point is $\left(\sqrt{\frac{32}{3}}, \sqrt{\frac{32}{3}}\right)$

Only critical point that produces a ~~max~~ positive volume,

and therefore the max volume occurs at this point.

From surface area, $z = \dfrac{32 - xy}{x+y} = \dfrac{32 - x^2}{2x} = \dfrac{32 - \frac{32}{3}}{2\sqrt{\frac{32}{3}}} = \sqrt{\frac{32}{3}}$

The dimensions are $\sqrt{\frac{32}{3}} \times \sqrt{\frac{32}{3}} \times \sqrt{\frac{32}{3}}$.

§14.8: Lagrange Multipliers

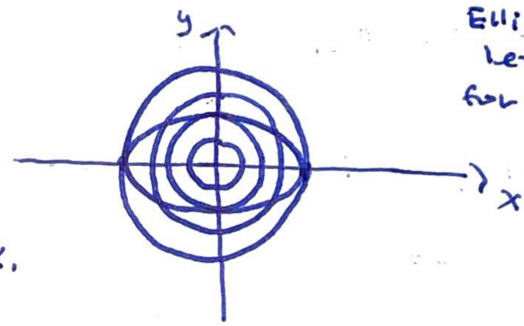
Find maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$.

Ex: Find max and min values of $f(x,y) = \sqrt{x^2+y^2}$ subject to $\frac{x^2}{4} + y^2 = 1$.

Constraint: $\frac{x^2}{4} + y^2 = 1$ is an ellipse. It is a level curve of $g(x,y) = \frac{x^2}{4} + y^2$

Level curves of $f(x,y)$ satisfy $\sqrt{x^2+y^2} = k$ for any $k \geq 0$

$x^2+y^2 = k^2$ Circle of radius k.



Ellipse and level curves for $k = 1/2, 1, 3/2, 2$

$k=0$: Point $(0,0)$: Not on constraint

$k=\frac{1}{2}$: $x^2+y^2=\frac{1}{4}$: Not on constraint

$k=1$: $x^2+y^2=1$: First time level curve touches constraint since radius of circle is minor radius of ellipse. Circle and ellipse are tangent when $x=0$ producing min $f=1$.

$k=\frac{3}{2}$: $x^2+y^2=\frac{9}{4}$: Intersects constraint at 4 points but not producing a max since we can see a larger circle will still intersect constraint.

$k=2$: $x^2+y^2=4$: Tangent to ellipse at $x=\pm 2$. Must produce max of $f=2$ since any larger circle will no longer intersect ellipse.

At a max (or min) value of f subject to $g=1$, the level curve of f is tangent to $g=1$. Otherwise there would be another level curve of f corresponding to a larger value of f that intersects the constraint curve.

At each point along the level curve $f=k$, ∇f is orthogonal to the curve and points in direction of greatest rate of increase. Similarly, ∇g is orthogonal to $g=1$ at each point. Since $f=k$ and $g=1$ are tangent at extrema, ∇f and ∇g are parallel.

To solve, find scalar λ and points (x,y) such that

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$$

λ is called the Lagrange multiplier

$$\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \lambda \left\langle \frac{x}{2}, 2y \right\rangle$$

① $\dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{\lambda}{2}x \implies x = \dfrac{\lambda}{2}x\sqrt{x^2+y^2} \implies x\left(1-\dfrac{\lambda}{2}\sqrt{x^2+y^2}\right)=0$

② $\dfrac{y}{\sqrt{x^2+y^2}} = 2\lambda y \implies y = 2\lambda y\sqrt{x^2+y^2} \implies y\left(1-2\lambda\sqrt{x^2+y^2}\right)=0$

③ $g = \dfrac{x^2}{4} + y^2 = 1$

From ①, either $x=0$ or $\lambda = \dfrac{2}{\sqrt{x^2+y^2}}$

If $x=0$, then constraint ③ implies $y^2 = 1 \implies y = \pm 1$

$(0,1)$ and $(0,-1)$ are solutions

If $\lambda = \dfrac{2}{\sqrt{x^2+y^2}}$, then ② implies $y(1-4)=0$

$-3y = 0 \implies y=0$

From ②, either $y=0$ or $\lambda = \dfrac{1}{2\sqrt{x^2+y^2}}$

If $y=0$, then constraint ③ implies $\dfrac{x^2}{4} = 1 \implies x = \pm 2$

$(2,0)$ and $(-2,0)$ are solutions

If $\lambda = \dfrac{1}{2\sqrt{x^2+y^2}}$ then ① implies $x\left(1-\dfrac{1}{4}\right)=0$

$\dfrac{3}{4}x = 0 \implies x=0$

Only solutions are $(-2,0)$, $(2,0)$, $(0,-1)$, $(0,1)$

Note: The level curve $x^2+y^2=1$ is tangent to ellipse

$\frac{x^2}{4}+y^2=1$ at $(0,-1)$ and $(0,1)$.

These produce min value $f(0,\pm 1)=1$

The level curve $x^2+y^2=4$ is tangent to

ellipse $\frac{x^2}{4}+y^2=1$ at $(-2,0)$ and $(2,0)$

These produce max value $f(\pm 2,0)=2$

Lagrange Multipliers: To find maximum and minimum values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$, assuming they exist and $\nabla g \neq \vec{0}$ on $g=k$,

① Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$, System of 4 equations

② Evaluate f at all solutions of system. Largest value is max while smallest value is the min subject to constraint $g=k$.