Introduction to Proof: Homework 5

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Problem 1

Show that $Q \Rightarrow S,\ R \Rightarrow T, \vdash (Q \vee R) \Rightarrow (S \vee T)$.

Solution 1

1.	$Q \Rightarrow S$	Hypothesis
2.	$R \Rightarrow T$	Hypothesis
3.	Assume $Q \vee R$	Dischargeable Hypothesis
4.	Case 1: Q is true	
5.	S	MP, for 4, for 1
6.	$S \vee T$	
7.	Case 2: R is true	
8.	T	MP, for 7, for 2
9.	$S \vee T$	
10.	$(Q \vee R) \Rightarrow (S \vee T)$	DT, from 3 [3 - 9 unusable]

Problem 2

Show that $(Q \wedge \neg T) \Rightarrow (Y \vee \neg P), Y \Rightarrow (V \vee \neg X) \vdash (P \wedge X) \Rightarrow [Q \Rightarrow (T \vee V)]$.

Solution 2

1.	$(Q \wedge \neg T) \Rightarrow (Y \vee \neg P)$	Hypothesis
2.	$Y \Rightarrow (V \vee \neg X)$	Hypothesis
3.	Assume $P \wedge X$	Dischargeable Hypothesis
4.	Assume Q	Dischargeable Hypothesis
5.	P	LCS for 3
6.	$\neg(Q \wedge \neg T)$	MT, for 5, for 1
7.	$\neg(Q \wedge \neg T) \Leftrightarrow \neg Q \vee T$	Tautology
8.	$\neg Q \vee T$	MPB, for 6, for 7
9.	T	DI, for 4, for 8
10.	$T \vee V$	
11.	$Q \Rightarrow (T \vee V)$	DT, from 4 [4 - 10 unusable]
12.	$(P \wedge X) \Rightarrow [Q \Rightarrow (T \vee V)]$	DT, from 3 [3 - 11 unusable]

Problem 3

Give a line proof that $(A \subseteq X \wedge B \subseteq X) \Rightarrow (A \cup B \subseteq X)$.

Solution 3

1. Assume $A \subseteq X \wedge B \subseteq X$.
2. Since $A \subseteq X$, then $(\forall x \in A)[x \in X]$.
3. Since $B \subseteq X$, then $(\forall x \in B)[x \in X]$.
4. Then, if $x \in A$ or $x \in B$, then $x \in X$.
5. Therefore, $x \in A \cup B \Rightarrow x \in X$.

Problem 4

Give a line proof that $(X \subseteq A \land X \subseteq B) \Rightarrow (X \subseteq A \cap B)$.

Solution 4

1. Assume $X \subseteq A \land X \subseteq B$.
2. Since $X \subseteq A$, then $(\forall x \in X)[x \in A]$.
3. Since $X \subseteq B$, then $(\forall x \in X)[x \in B]$.
4. Then, if $x \in X$, then $x \in A$ and $x \in B$.
5. Then, $x \in A \cap B$.
6. Therefore, $X \subseteq A \cap B$.

Problem 5

Prove or disprove: If $a, b \in \mathbb{N}$ and $a \leq b$ then $M_a \subseteq M_b$.

Solution 5

The statement is false. For example, $a = 2$ and $b = 3$. Then $M_2 = \{1, 2, 4, 6, 8, \ldots\}$ and $M_3 = \{1, 3, 6, 9, 12, \ldots\}$. But, $x = 2 \notin M_3$. Therefore, $M_a \not\subseteq M_b$.

Problem 6

Prove or disprove: $M_4 \cap M_6 = M_{24}$.

Solution 6

1. M_4 is the set of all multiples of 4.
2. M_6 is the set of all multiples of 6.
3. $\operatorname{lcm}(4, 6) = 12$.
4. Therefore, $M_4 \cap M_6$ is the set of all multiples of 12.
5. Clearly, every multiple of 24 is a multiple of 12.
6. So, $M_{24} \subseteq M_4 \cap M_6$.
7. However, $M_4 \cap M_6$ includes all multiples of 12, not just multiples of 24.
8. For example, $12 \in M_4 \cap M_6$ but $12 \notin M_{24}$.
9. This means that $M_4 \cap M_6 \not\subseteq M_{24}$.
10. Therefore, $M_{24} \subseteq M_4 \cap M_6$.

Problem 7

Prove or disprove: $M_4 \cap M_9 = M_{36}$.

Solution 7

1. M_4 is the set of all multiples of 4.
2. M_9 is the set of all multiples of 9.
3. $\operatorname{lcm}(4, 9) = 36$.
4. Therefore, $M_4 \cap M_9$ is the set of all multiples of 36.
5. So, $M_{36} \subseteq M_4 \cap M_9$.
6. M_{36} is the set of all multiples of 36.
7. Clearly, every multiple of 36 is a multiple of 4 and 9.
8. So, $M_{36} \subseteq M_4$ and $M_{36} \subseteq M_9$.
9. Then, $M_{36} \subseteq M_4 \cap M_9$.
10. Therefore, $M_{36} = M_4 \cap M_9$.

Problem 8

Give a line proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution 8

1. Assume $x \in A \cup (B \cap C)$.
2. Then, $x \in A \vee (x \in B \wedge x \in C)$.
3. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$.
4. If $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$.
5. Then, $x \in (A \cup B) \cap (A \cup C)$.
6. Therefore, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
7. Assume $x \in (A \cup B) \cap (A \cup C)$.
8. Then, $x \in A \cup B$ and $x \in A \cup C$.
9. If $x \in A$, then $x \in A \cup (B \cap C)$.
10. If $x \notin A$ and $x \in B$ and $x \in C$, then $x \in A \cup (B \cap C)$.
11. Then, $x \in A \cup (B \cap C)$.
12. Therefore, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
13. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 9

Give a line proof showing that $X - (A \cup B) = (X - A) \cap (X - B)$, for all sets A, B, and X.

Solution 9

1. Assume $x \in X - (A \cup B)$.
2. Then, $x \in X$ and $x \notin A \cup B$.
3. Then, $x \in X$ and $x \notin A$ and $x \notin B$.
4. Then, $x \in X - A$ and $x \in X - B$.
5. Then, $x \in (X - A) \cap (X - B)$.
6. Therefore, $X - (A \cup B) \subseteq (X - A) \cap (X - B)$.
7. Assume $x \in (X - A) \cap (X - B)$.
8. Then, $x \in X - A$ and $x \in X - B$.
9. Then, $x \in X$ and $x \notin A$ and $x \notin B$.
10. Then, $x \in X$ and $x \notin A \cup B$.
11. Then, $x \in X - (A \cup B)$.
12. Therefore, $(X - A) \cap (X - B) \subseteq X - (A \cup B)$.
13. Therefore, $X - (A \cup B) = (X - A) \cap (X - B)$.

Problem 10

Give a line proof showing that $X - (A \cap B) = (X - A) \cup (X - B)$, for all sets A, B, and X.

Solution 10

1. Assume $x \in X - (A \cap B)$.
2. Then, $x \in X$ and $x \notin A \cap B$.
3. Then, $x \in X$ and $x \notin A$ or $x \notin B$.
4. Then, $x \in X - A$ or $x \in X - B$.
5. Then, $x \in (X - A) \cup (X - B)$.
6. Therefore, $X - (A \cap B) \subseteq (X - A) \cup (X - B)$.
7. Assume $x \in (X - A) \cup (X - B)$.
8. Then, $x \in X - A$ or $x \in X - B$.
9. Then, $x \in X$ and $x \notin A$ or $x \notin B$.
10. Then, $x \in X$ and $x \notin A \cap B$.
11. Then, $x \in X - (A \cap B)$.
12. Therefore, $(X - A) \cup (X - B) \subseteq X - (A \cap B)$.
13. Therefore, $X - (A \cap B) = (X - A) \cup (X - B)$.

Problem 11

If f is a function from S to T and $A \subseteq S$, define $f(A) = \{x \mid (\exists y \in A)[x = f(y)]\}$. This is called the image of A under f.

(i) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ where $f(M_3 \cap \mathbb{N}) = \mathbb{N}$.

(ii) Suppose $f : S \to T$ and $A \subseteq S$, $B \subseteq S$.

- (a) Prove that $f(A \cup B) = f(A) \cup f(B)$
- (b) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$
- (c) Give an example of sets S, T, A, B, and a function f for which $f(A) \cap f(B) \not\subseteq f(A \cap B)$. [Hint: Start by trying to prove that $f(A) \cap f(B) \subseteq f(A \cap B)$, and see where you get stuck.]

Solution 11

(i) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = n/3$. Then, $f(M_3 \cap \mathbb{N}) = \mathbb{N}$. With this definition, for any $m \in \mathbb{N}$, there exists a multiple of 3, specifically $3m$, such that $f(3m) = m$. Therefore, $f(M_3 \cap \mathbb{N}) = \mathbb{N}$.

(ii) Suppose $f : S \to T$ and $A \subseteq S$, $B \subseteq S$.

(a) Here's the line proof for $f(A \cup B) = f(A) \cup f(B)$.

1. Assume $x \in f(A \cup B)$.
2. Then, $x = f(y)$ for some $y \in A \cup B$.
3. Then, $y \in A$ or $y \in B$.
4. Then, $x = f(y) \in f(A)$ or $x = f(y) \in f(B)$.
5. Then, $x \in f(A) \cup f(B)$.
6. Therefore, $f(A \cup B) \subseteq f(A) \cup f(B)$.
7. Assume $x \in f(A) \cup f(B)$.
8. Then, $x \in f(A)$ or $x \in f(B)$.
9. Then, $x = f(y)$ for some $y \in A$ or $y \in B$.
10. Then, $y \in A \cup B$.
11. Then, $x = f(y) \in f(A \cup B)$.
12. Therefore, $f(A) \cup f(B) \subseteq f(A \cup B)$.
13. Therefore, $f(A \cup B) = f(A) \cup f(B)$.

(b) Here's the line proof for $f(A \cap B) \subseteq f(A) \cap f(B)$.

1. Assume $x \in f(A \cap B)$.
2. Then, $x = f(y)$ for some $y \in A \cap B$.
3. Then, $y \in A$ and $y \in B$.
4. Then, $x = f(y) \in f(A)$ and $x = f(y) \in f(B)$.
5. Then, $x \in f(A) \cap f(B)$.
6. Therefore, $f(A \cap B) \subseteq f(A) \cap f(B)$.

(c) Let $S = \{1, 2\}$, $T = \{0, 1\}$, $A = \{1\}$, and $B = \{2\}$. Define $f : S \to T$ by $f(1) = 0$ and $f(2) = 0$. Then $f(A) = \{f(1)\} = \{0\}$, $f(B) = \{f(2)\} = \{0\}$, $f(A) \cap f(B) = \{0\}$, but $A \cap B = \emptyset$. So, $f(A) \cap f(B) = \emptyset \not\subseteq f(A \cap B) = \{0\}$.

Problem 12

Yoda has a bunch of eggs, and you have to figure out how many.

(i) He tells you two facts: When you separate the eggs into groups of 11, there are 3 left over. When you separate the eggs into groups of 8, there are 4 left over.

Given these facts, what is the least number of eggs that Yoda could have?

(ii) Suppose Yoda also tells you that he has between 100 and 200 eggs. Given this additional piece of information, you can determine exactly how many eggs Yoda has. How many?

(iii) The next day Yoda comes back with a lot more eggs. This time he says: When I separate the eggs into groups of 11, there are 3 left over. When I separate them into groups of 300, there are 51 left over. What is the least number of eggs that Yoda could have?

Solution 12

(i) We need to find the smallest positive integer n such that:

$$n \equiv_{11} 3$$
$$n \equiv_{8} 4$$

From $n \equiv_{11} 3$, we can write $n = 11k + 3$ for some integer k. Substituting $n = 11k + 3$ into $n \equiv_8 4$

$$11k + 3 \equiv_8 4 \Rightarrow 11k \equiv_8 1.$$

Since $11 \equiv_8 3$, we have

$$3k \equiv_8 1.$$

To solve $3k \equiv_8 1$, we need to find an integer k such that $3k$ gives a remainder of 1 when divided by 8. If $k = 3$, then $3 \cdot 3 = 9 \equiv_8 1$. So $k = 3$ is a solution, giving us

$$n = 11k + 3 = 11 \cdot 3 + 3 = 36.$$

Thus, the smallest positive integer n that satisfies both conditions is 36.

(ii) Since we know from (i) that $n = 36$ is a solution, any solution can be written in the form

$$n = 36 + 88m,$$

where 88 is the least common multiple of 11 and 8, and m is an integer.

Now, we need n to be between 100 and 200

$$100 \leq 36 + 88m \leq 200 \Rightarrow 0.727 \leq m \leq 1.864.$$

Since m is an integer, the only possible value for m is $m = 1$, giving us

$$n = 36 + 88 \cdot 1 = 124.$$

So, with the additional information, the exact number of eggs Yoda has is 124.

Problem 12 continued on next page . . .

(iii) Now Yoda has more eggs and gives us new conditions

$$n \equiv_{11} 3$$
$$n \equiv_{300} 51$$

From $n \equiv_{300} 51$, we can write $n = 300k + 51$ for some integer k. Substitute $n = 300k + 51$ into $n \equiv_{11} 3$

$$300k + 51 \equiv 3 \pmod{11}.$$

Since $300 \equiv_{11} 3$ and $51 \equiv_{11} 7$, we have

$$3k + 7 \equiv_{11} 3 \Rightarrow 3k \equiv_{11} -4 \Rightarrow 3k \equiv_{11} 7.$$

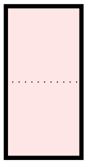
To solve $3k \equiv_{11} 7$, we need to find an integer k such that $3k$ gives a remainder of 7 when divided by 11. If $k = 9$, then $3 \cdot 9 = 27 \equiv_{11} 7 \pmod{11}$. So $k = 9$ is a solution. This gives us

$$n = 300k + 51 = 300 \cdot 9 + 51 = 2751.$$

The smallest number of eggs that Yoda could have is 2751.

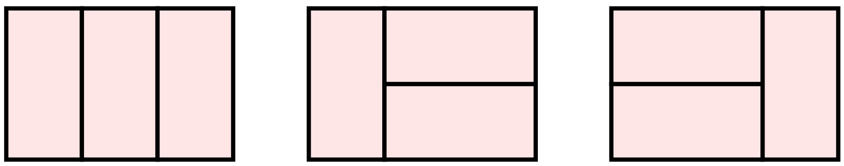
Problem 13

You have a huge collection of 1×2 dominoes (or tiles), and a 2×11 checkerboard:



Your goal in this problem is to determine how many different ways there are to tile the checkerboard using your dominoes.

To solve this problem, it is best to solve some smaller versions first. Let S_n denote the number of different ways to tile a $2 \times n$ checkerboard. For example, $S_3 = 3$ as we see below:



Make a table showing the values of S_n for $1 \leq n \leq 11$. As you are doing this, try to find a systematic way of finding all the tilings. Note any patterns that you find, and see if you can explain them. As a check, you should get $S_6 = 13$.

Solution 13

To solve the problem, let S_n represent the number of ways to tile a $2 \times n$ checkerboard with 1×2 dominoes.

We start by analyzing smaller cases to build a table of values for S_n and look for patterns.

(i) For $n = 1$:

  (a) We can only place one domino vertically, so $S_1 = 1$.

(ii) For $n = 2$:

  (a) We can either place two dominoes vertically or two horizontally, giving two possible configurations. So, $S_2 = 2$.

(iii) For $n = 3$: We can either:

  (a) Place three vertical dominoes, or
  (b) Place two horizontal dominoes on the bottom row and one vertical domino on top, or
  (c) Place one vertical domino on the left and two horizontal dominoes on the right.

  Therefore, $S_3 = 3$.

(iv) For $n = 4$: We can use:

  (a) Four vertical dominoes,
  (b) Two horizontal pairs stacked,
  (c) Two vertical dominoes on the left and two horizontal on the right, or
  (d) Two horizontal dominoes on top and two vertical on the bottom.

Problem 13 continued on next page . . .

So, $S_4 = 5$.

By examining these small cases, we observe that to cover a $2 \times n$ board, we can either

(i) Place one vertical domino at the left end, leaving a $2 \times (n-1)$ board to cover, or

(ii) Place two horizontal dominoes at the left end, leaving a $2 \times (n-2)$ board.

This leads to the recurrence relation:

$$S_n = S_{n-1} + S_{n-2}.$$

with initial conditions $S_1 = 1$ and $S_2 = 2$. This relation is identical to the Fibonacci sequence shifted by one index.

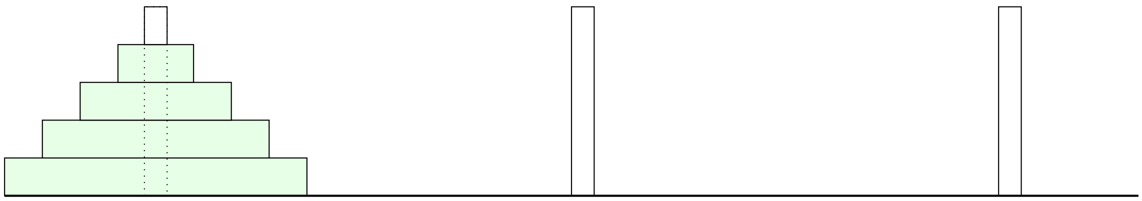
Using this recurrence, we can compute values up to S_{11}:

n	1	2	3	4	5	6	7	8	9	10	11
S_n	1	2	3	5	8	13	21	34	55	89	144

Therefore, the number of ways to tile a 2×11 checkerboard is $S_{11} = 144$.

Problem 14

A certain puzzle has three pegs, the leftmost peg starting out with a tower of n disks. The object of the puzzle is to move this tower to the rightmost peg. The rules are that you can only move one disk at a time, the disk has to be moved from one peg to another, and you can never place a larger disk on top of a smaller disk. The following picture shows the starting position of the puzzle when $n = 4$:



Your goal in this problem is to determine the minimum number of moves needed to solve the puzzle when there are 11 disks. Approach this in a similar way to what we did in problem #13, by making a table showing the minimum number of moves to solve the n-disk game for $1 \leq n \leq 11$.

Solution 14

Let $M(n)$ represent the minimum number of moves required to solve the puzzle with n disks. To understand this, we analyze smaller cases and observe the recursive pattern.

(i) For $n = 1$: We can move the single disk directly to the target peg in one move. Thus, $M(1) = 1$.

(ii) For $n = 2$: Move the first disk to an intermediate peg, the second disk to the target peg, and then the first disk from the intermediate peg to the target peg. Therefore, $M(2) = 3$.

(iii) For $n = 3$: Move the first two disks to the intermediate peg (using $M(2) = 3$ moves), the third disk to the target peg, and then the two disks from the intermediate peg to the target peg (again using $M(2) = 3$ moves). This gives $M(3) = M(2) + 1 + M(2) = 7$.

By following this pattern, we observe that to solve the n-disk problem, we need to:

(i) Move the top $n - 1$ disks to an intermediate peg (requiring $M(n - 1)$ moves),

(ii) Move the n-th disk to the target peg (1 move), and

(iii) Move the $n - 1$ disks from the intermediate peg to the target peg (again requiring $M(n - 1)$ moves).

This leads to the recurrence relation:

$$M(n) = 2M(n - 1) + 1,$$

with the initial condition $M(1) = 1$.

Using this recurrence, we can calculate $M(n)$ for $1 \leq n \leq 11$:

n	1	2	3	4	5	6	7	8	9	10	11
S_n	1	3	7	15	31	63	127	255	511	1023	2047

Thus, the minimum number of moves required to solve the Tower of Hanoi puzzle with 11 disks is $M(11) = 2047$.