Maximize and minimize $f(x,y,z)$ subject to $g(x,y,z) = K$.

Suppose $f(x,y,z)$ has an extreme value at $P(x_0, y_0, z_0)$ on the surface $g = K$.

Let C be a curve on the surface $g = K$ that passes through P.

The curve is defined by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and there exists t_0 such that $\vec{r}(t_0) = \vec{OP}$.

Let $h(t) = f(x(t), y(t), z(t))$ which determines the values of f restricted to C.

Since f has an extreme value at P, h has an extreme value when $t = t_0$.

By Fermat's Thm, $h'(t_0) = 0$

By Chain Rule, $h'(t) = f_x x'(t) + f_y y'(t) + f_z z'(t)$

$$= \langle f_x, f_y, f_z \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle$$

$$= \nabla f \cdot \vec{r}\,'(t).$$

At the point P, $h'(t_0) = \nabla f(P) \cdot \vec{r}\,'(t_0) = 0$

Therefore at P, ∇f is orthogonal to $\vec{r}'(t_0)$.

Recall, $\vec{r}'(t_0)$ is tangent to C at P.

At all points along C, ∇g is orthogonal to $\vec{r}'(t)$.

Since this is true for all curves C that pass through the point P on the surface $g = k$, then ∇f and ∇g are parallel at extreme values.

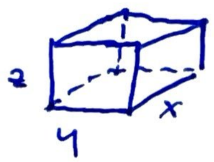
There exists a scalar λ such that

$$\nabla f = \lambda \nabla g$$

$$g = k$$

λ is the Lagrange multiplier

Ex: Find the dimensions of the rectangular box with largest volume such that the surface area is 64 cm^2.



Volume: $V(x,y,z) = xyz$

Constraint: $A(x,y,z) = 2xy + 2xz + 2yz = 64$

Find (x,y,z) and λ such that

$$\begin{cases} \nabla V = \lambda \nabla A \\ A = 64 \end{cases} \qquad \langle yz, xz, xy\rangle = \lambda \langle 2y+2z, 2x+2z, 2x+2y\rangle$$

① $yz = 2\lambda(y+z)$

② $xz = 2\lambda(x+z)$

③ $xy = 2\lambda(x+y)$

④ $A = 64$

If $x=0$, or $y=0$, or $z=0$, then $V=0$

Suppose $x,y,z>0$ and therefore

$y+z>0$

Therefore ① $\Rightarrow$ $\lambda = \dfrac{yz}{2(y+z)}$

Substitute λ into ②

$$xz = \frac{yz}{y+z}(x+z)$$

$$xz(y+z) = yz(x+z)$$

Since $z>0$, then $xy + xz = xy + yz$

$$z(x-y) = 0 \Rightarrow x = y$$

Substitute λ into ③

$$xy = \frac{yz}{y+z}(x+y)$$

$$xy(y+z) = yz(x+y)$$

Since $y>0$, then $xy + xz = xz + yz$

$$y(x-z)=0 \Rightarrow x=z$$

For a nonzero volume, equations ①, ②, and ③ imply

$x=y=z$ Rectangular box is a cube

From ④ $A = 2xy + 2xz + 2yz = 64$

$$2x^2 + 2x^2 + 2x^2 = 64$$

$$6x^2 = 64$$

$$x = \sqrt{\frac{32}{3}} = y = z$$

The dimensions that maximize volume

subject to $A=64$ are

$$\sqrt{\frac{32}{3}} \times \sqrt{\frac{32}{3}} \times \sqrt{\frac{32}{3}} \text{ cm}$$

Ex: Find the points on the surface $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Let (x, y, z) be a point in $\mathbb{R}^3$.

The distance from $(4,2,0)$ to (x,y,z) is

$$d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

minimize $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$

subject to $g(x,y,z) = x^2 + y^2 - z^2 = 0$

$\nabla f = \lambda \nabla g$ $\qquad \langle 2(x-4), 2(y-2), 2z \rangle = \lambda \langle 2x, 2y, -2z \rangle$

$g = 0$

① $2(x-4) = 2\lambda x$

② $2(y-2) = 2\lambda y$

③ $2z = -2\lambda z$ $\qquad \Rightarrow z + \lambda z = 0$

④ $g = 0$ $\qquad z(1+\lambda) = 0$ $\quad$ Either $z = 0$ or $\lambda = -1$

If $z = 0$, then from constraint $g = x^2 + y^2 - z^2$

$= x^2 + y^2 = 0$

$\Rightarrow x = 0$ and $y = 0$

If $x=0$, then ① $x-4 = \lambda x$

$-4 = 0$ Problem

If $y=0$, then ② $y-2 = \lambda y$

$-2 = 0$ Problem

$(0,0,0)$ is not a solution of the system

If $\lambda = -1$, then sub into ① and ②

① $x-4 = -x \Rightarrow x=2$

② $y-2 = -y \Rightarrow y=1$

Sub $x=2$ and $y=1$ into ④

$$g = x^2+y^2-z^2=0$$

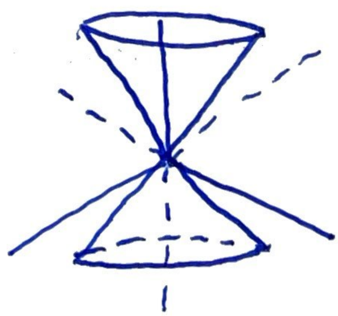
$$z^2 = 4+1 = 5$$

$$z = \pm\sqrt{5}$$

Solutions $(2,1,\sqrt{5})$ and $(2,1,-\sqrt{5})$

Distance $d = \sqrt{4+1+5} = \sqrt{10}$

$z^2=x^2+y^2$ is a cone



$(4,2,0)$ is in xy-plane.

There are two points that minimize distance. One on upper half cone and one on lower half cone