1. Verify that the determinant of $\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{bmatrix}$ is $\prod_{1\le i<j\le 3}(t_j - t_i)$.

2. Use the method introduced in the class to find a polynomial $p(x)$ in $\mathbb{P}^3(\mathbb{R})$ such that $p(1) = 1$, $p(2) = 3$, $p(3) = -1$, and $p(4) = 2$.

3. Let f be the linear functional on $\mathbb{R}^2$ defined by $f(x_1, x_2) = 2x_1 - 3x_2$. Let T be a linear transformation defined by $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$. Let T^t be the transpose linear transformation of T on the dual space of $\mathbb{R}^2$. Find the formula for the linear functional $T^t(f) : \mathbb{R}^2 \to \mathbb{R}$.

4. Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by $f(p) = \displaystyle\int_a^b p(x)\,dx$. Let D be the differentiation operator on V, and $D^t : V^* \to V^*$ be the the transpose linear transformation of D on the dual space V^*. Find the formula for the linear functional $D^t(f) : V \to \mathbb{R}$.

5. Let $V = \mathbb{R}^{n\times n}$ and let $B \in \mathbb{R}^{n\times n}$ be a fixed matrix. Let $T : V \to V$ be the linear transformation defined by $T(A) = AB - BA$, and $f : V \to \mathbb{R}$ be the trace linear functional defined by $f(C) = \mathrm{Tr}(C)$. Let $T^t : V^* \longrightarrow V^*$ be t the transpose linear transformation of T on the dual space V^*. Find the formula for the linear functional $T^t(f) : V \to \mathbb{R}$.

6. Let $\mathbb{R}^\infty$ be a vector space of infinite sequences $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ of real numbers.

1). Define a linear transformation $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$T(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, \alpha_1, \alpha_2, \alpha_3, \cdots).$$

Find the eigenvalue(s) and eigenvectors of T or prove there are no eigenvalues or eigenvectors for T.

2). Define a linear transformation $U : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$U(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\alpha_2, \alpha_3, \alpha_4, \cdots).$$

Find the eigenvalue(s) and eigenvectors of U or prove there are no eigenvalues or eigenvectors for U.

7. Let $A \in \mathbb{C}^{n\times n}$. Let $\lambda_1, \cdots, \lambda_n$ be all the eigenvalues of A.

1). Prove that the determinant of A equals to the product of all eigenvalues of A, i.e.

$$\det(A) = \lambda_1 \cdot \cdots \cdot \lambda_n.$$

2). Use Part 1) to prove that A is invertible if and only if $\lambda_i \neq 0$ for all $i = 1, \cdots, n$.

8. Let λ_1 and λ_2 be distinct eigenvalues of a linear transformation $T : V \to V$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors associated with λ_1 and λ_2 respectively. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

9. Let T be the linear transformation on $\mathbb{R}^4$ which is represented in standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what condition on a, b and c is T diagonalizable? Explain your answer.

10. Let T be a linear transformation on an n-dimensional vector space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.