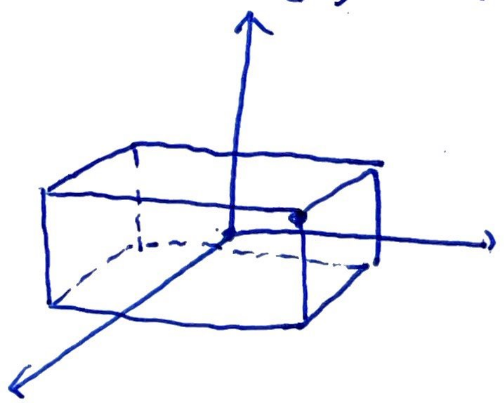
Ex: Find the volume of the largest rectangular box that can be inscribed in $4x^2+y^2+9z^2=36$

$$\left(\frac{x}{3}\right)^2+\left(\frac{y}{6}\right)^2+\left(\frac{z}{2}\right)^2=1$$



Place origin at center of box so that (x,y,z) is the vertex in octant I and on the ellipsoid

Then $w=2y$

$l=2x$

$h=2z$

Maximize $V(x,y,z)=8xyz$

Subject to $g(x,y,z)=4x^2+y^2+9z^2=36$

$\nabla V=\lambda\nabla g$

$g=36$

$$\langle 8yz,\ 8xz,\ 8xy\rangle=\lambda\langle 8x,\ 2y,\ 18z\rangle$$

① $8yz=8\lambda x$ $\quad * \ x$

② $8xz=2\lambda y$ $\quad * \ y$

③ $8xy=18\lambda z$ $\quad * \ z$

④ $g=36$

① $8xyz = 8\lambda x^2$

② $8xyz = 2\lambda y^2$

③ $8xyz = 18\lambda z^2$

$$24xyz = 2\lambda(4x^2 + y^2 + 9z^2)$$
$$= 2\lambda 9 = 2\lambda(36)$$

$$\lambda = \frac{24xyz}{72} = \frac{xyz}{3}$$

① $8xyz = \frac{8}{3}(xyz)x^2$

② $8xyz = \frac{2}{3}(xyz)y^2$

③ $8xyz = 6(xyz)z^2$

Either $x=0$ or $y=0$ or $z=0$ $(V=0)$ or

① $3 = x^2$ $\quad x = \sqrt{3}$

② $12 = y^2$ $\quad y = 2\sqrt{3}$

③ $\frac{4}{3} = z^2$ $\quad z = \frac{2}{\sqrt{3}}$

Only solution that produces positive volume and therefore max volume

$$V = 8xyz = 32\sqrt{3}$$

Ex: Maximize $f = e^{xyz}$ subject to $g = 2x^2 + y^2 + z^2 = 24$
and minimize

Find (x, y, z) and λ such that

$\nabla f = \lambda \nabla g$

$\langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle = \lambda \langle 4x, 2y, 2z \rangle$

① $yze^{xyz} = 4\lambda x$

② $xze^{xyz} = 2\lambda y$

③ $xye^{xyz} = 2\lambda z$

④ $g = 24$

From ① if $x = 0$, then $y = 0$ or $z = 0$

$x = 0$ and $y = 0$ equation ② is true
equation ③ is true with $\lambda = 0$

Constraint $z^2 = 24$ $(0, 0, \pm\sqrt{24})$

$x = 0$ and $\cancel{z} = 0$ equation ② is true with $\lambda = 0$
equation ③ is true

Constraint $y^2 = 24$ $(0, \pm\sqrt{24}, 0)$

From ① if $x \neq 0$, then $\lambda = \dfrac{yze^{xyz}}{4x}$

Sub into ② : $xze^{xyz} = \dfrac{y^2ze^{xyz}}{2x}$

$2x^2ze^{xyz} - y^2ze^{xyz} = 0$

$ze^{xyz}(2x^2 - y^2) = 0$ $z = 0$ or $y = \pm\sqrt{2}\,x$

If $z = 0$, then equation ③ implies $y = 0$

Constraint $2x^2 = 24$ $(\pm\sqrt{12}, 0, 0)$

Sub into ③ $xye^{xyz} = \dfrac{yz^2e^{xyz}}{2x}$

$2x^2ye^{xyz} - yz^2e^{xyz} = 0$

$ye^{xyz}(2x^2 - z^2) = 0$ $y = 0$ or $z = \pm\sqrt{2}\,x$

If $y = \pm\sqrt{2}x$ then (3) $\Rightarrow$ $z = \pm\sqrt{2}x$

$(+,+,+)$ $(-,+,+)$
$(+,+,-)$ $(-,+,-)$
$(+,-,+)$ $(-,-,+)$
$(+,-,-)$ $(-,-,-)$

From constraint $2x^2 + y^2 + z^2 = 24$

$$2x^2 + 2x^2 + 2\cancel{x}^2 = 24$$

$$x^2 = 4$$

$$x = \pm 2$$

Eight points $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$

For the six points $(0, \pm\sqrt{24}, 0)$, $(0, 0, \pm\sqrt{24})$, $(\pm\sqrt{12}, 0, 0)$

the function $f = e^{xyz} = 1$ since one of the coordinates is 0.

Min is $e^{-2(2\sqrt{2})(2\sqrt{2})} = e^{-16}$

Max is $e^{2(2\sqrt{2})(2\sqrt{2})} = e^{16}$