Multi-Variable Calculus I: Homework 6

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Problem 1

Find the limit, if it exists. If continuity is used, explain where the function is continuous. If the limit doesn't exist, show why not.

(i) $\displaystyle\lim_{(x,y)\to(1,0)} \ln\left(\frac{1-xy}{x^2+y^2}\right)$.

(ii) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^4-y^4}$.

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y\sin(x)}{x^2+y^2}$.

(iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin(x)}{x^2+y^2}$.

Solution 1

(i) Plugging in the coordinate $(1,0)$, we get

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1-xy}{x^2+y^2}\right) = \ln\left(\frac{1-1\cdot 0}{1^2+0^2}\right) = \ln(1) = 0.$$

The function is continuous for all $(x,y) \neq (0,0)$.

(ii) Simplifying the equation, we get

$$\frac{x^2-y^2}{x^4-y^4} = \frac{1}{x^2+y^2}.$$

Plugging in the coordinate $(1,1)$, we get

$$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^4-y^4} = \frac{1}{1^2+1^2} = \frac{1}{2}.$$

The simplified function is continuous for all $(x,y) \neq (0,0)$.

(iii) Trying two paths, we get

$$\begin{aligned}
\text{Path 1: } y = 0 \Rightarrow f(x,0) &= \frac{0\sin(x)}{x^2+0^2} = 0 & \Rightarrow \lim_{x\to 0} f(x,0) = 0 \\
\text{Path 2: } y = x \Rightarrow f(x,x) &= \frac{x\sin(x)}{2x^2} = \frac{\sin(x)}{2x} & \Rightarrow \lim_{x\to 0} f(x,x) = \frac{1}{2}.
\end{aligned}$$

Since the limit is different for different paths, the limit does not exist.

(iv) We have the following inequality

$$(\forall (x,y) \neq (0,0)) \left[y^2 \leq x^2+y^2 \Rightarrow \frac{y^2}{x^2+y^2} \leq 1\right].$$

We cannot multiply both sides by $\sin(x)$, since sometimes $\sin(x)$ is positive, and other times it's negative. So, we multiply both sides by the absolute value of $\sin(x)$ to get

$$\frac{y^2|\sin(x)|}{x^2+y^2} \leq |\sin(x)|,$$

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which is equivalent to the inequality

$$\left|\frac{y^2\sin(x)}{x^2+y^2}\right| \leq |\sin(x)|.$$

We can expand the absolute value to get

$$-|\sin(x)| \leq \frac{y^2\sin(x)}{x^2+y^2} \leq |\sin(x)|.$$

Since $\lim\limits_{x\to 0} -|\sin(x)| = 0 = \lim\limits_{x\to\infty} |\sin(x)|$, by the squeeze theorem, we have

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin(x)}{x^2+y^2} = 0.$$

Problem 2

The intersection of the ellipsoid $4x^2 + y^2 + 2z^2 = 10$ and the plane $x = 1$ is an ellipse. Find the tangent line to the ellipse at the point $(1, 2, -1)$.

Solution 2

To find the intersection of the ellipsoid and the plane, substitute $x = 1$ into the equation of the ellipsoid to obtain $y^2 + 2z^2 = 6$. This describes an ellipse in the yz-plane, with the point $(1, 2, -1)$ lying on this curve.

Next, we need the direction vector of the tangent line. Since the x-value is fixed at 1, there is no movement in the x-direction. The direction of the tangent line will therefore be determined by changes in y and z alone.

At the point $(2, -1)$ in the yz-plane, the rate of change in the z-direction relative to y is given by $\frac{\partial z}{\partial y}$. Thus, the tangent line to the ellipse at $(1, 2, -1)$ is

$$L(t) = \langle 1, 2, -1 \rangle + t \left\langle 0, 1, \left(\frac{\partial z}{\partial y} \right)_{(2,-1)} \right\rangle.$$

Problem 3

Use implicit differentiation to compute the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, $x^2z^3 = \cos(x - y^2z)$

Solution 3

For $\dfrac{\partial z}{\partial x}$, we have

$$\begin{aligned}
& \frac{\partial}{\partial x}\left(x^2z^3\right) && = \frac{\partial}{\partial x}\left(\cos(x-y^2z)\right) \\
\Rightarrow\quad & \frac{\partial}{\partial x}\left(x^2z^3\right) && = (-\sin(x-y^2z)) \cdot \left(1 - y^2\frac{\partial z}{\partial x}\right) \\
\Rightarrow\quad & 2xz^3 + 3x^2z^2\frac{\partial z}{\partial x} && = -\sin(x-y^2z) + y^2\sin(x-y^2z)\frac{\partial z}{\partial x} \\
\Rightarrow\quad & \frac{\partial z}{\partial x}\left(3x^2z^2 - y^2\sin(x-y^2z)\right) && = -\sin(x-y^2z) - 2xz^3 \\
\Rightarrow\quad & \frac{\partial z}{\partial x} && = \frac{-\sin(x-y^2z) - 2xz^3}{3x^2z^2 - y^2\sin(x-y^2z)}.
\end{aligned}$$

For $\dfrac{\partial z}{\partial y}$, we have

$$\begin{aligned}
& \frac{\partial}{\partial y}\left(x^2z^3\right) && = \frac{\partial}{\partial y}\left(\cos(x-y^2z)\right) \\
\Rightarrow\quad & 3x^3z^2\frac{\partial z}{\partial y} && = \left(-\sin(x-y^2z)\right) \cdot \left(-2yz - y^2\frac{\partial z}{\partial y}\right) \\
\Rightarrow\quad & 3x^3z^2\frac{\partial z}{\partial y} && = 2yz\sin(x-y^2z) + y^2\sin(x-y^2)\frac{\partial z}{\partial y} \\
\Rightarrow\quad & \frac{\partial z}{\partial y}\left(3x^3z^2 - y^2\sin(x-y^2z)\right) && = 2yz\sin(x-y^2z) \\
\Rightarrow\quad & \frac{\partial z}{\partial y} && = \frac{2yz\sin(x-y^2z)}{3x^3z^2 - y^2\sin(x-y^2z)}.
\end{aligned}$$

Problem 4

Let $f(x,y) = \dfrac{\sqrt{x^2 - 5y}}{3y - 2x}$.

(i) Find the tangent plane to the surface at the point $(4,3,1)$.

(ii) Find the linearization of $f(x,y)$ at $(4,3)$.

(iii) Use $L(x,y)$ to approximate $f(4.03, 2.98)$.

Solution 4

(i) We know $(x_0, y_0) = (4,3)$ and $z_0 = f(4,3) = 1$. Then, by Wolfram, we get

$$\begin{aligned} f_x(x,y) &= \frac{y(3x-10)}{\sqrt{x^2-5y}\cdot(3y-2x)^2} \\ f_y(x,y) &= \frac{-6x^2+10x+15y}{2\sqrt{x^2-5y}\cdot(3y-2x)^2}. \end{aligned}$$

Substituting $(4,3)$ into $f_x(x,y)$ and $f_y(x,y)$, we get $f_x(4,3) = 6$ and $f_y(4,3) = -{}^{11}\!/_{2}$. Now, we can write the equation of the tangent plane

$$z = 1 + 6(x-4) - \frac{11}{2}(y-3) \Rightarrow z = 6x - \frac{11}{2}y - \frac{13}{2}.$$

(ii) The linearization of $f(x,y)$ at $(4,3)$ is

$$L(x,y) = 6x - \frac{11}{2}y - \frac{13}{2}.$$

(iii) Now we use the linearization to approximate $f(4.03, 2.98)$

$$L(4.03, 2.98) = 6(4.03) - \frac{11}{2}(2.98) - \frac{13}{2} = 1.29.$$

Problem 5

Let $f(x,y) = x^2 \ln(2x-3y)$. Find the first and second Taylor polynomials for f near the point $(2,1)$.

Solution 5

Finding the partial derivatives, we get

$$f_x(x,y) = 2x\left(\frac{x}{2x-3y} + \ln(2x-3y)\right) \quad \text{and} \quad f_y(x,y) = -\frac{3x^2}{2x-3y}$$
$$f_{xx}(x,y) = 2\left(\frac{6x(x-2y)}{(2x-3y)^2} + \ln(2x-3y)\right), \quad f_{xy} = -\frac{6x(x-3y)}{(2x-3y)^2}, \quad \text{and} \quad f_{yy} = -\frac{9x^2}{(2x-3y)^2}.$$

and the points are

$$f(2,1) = 0, \quad f_x(2,1) = 8, \quad f_y(2,1) = -12, \quad f_{xx}(2,1) = 0, \quad f_{xy}(2,1) = 12, \quad \text{and} \quad f_{yy}(2,1) = -36.$$

So, the first and second-order Taylor polynomials near $(2,1)$ are

$$T_1(x,y) = 8(x-2) - 12(y-1)$$
$$T_2(x,y) = 8(x-2) - 12(y-1) + 12(x-2)(y-1) - 18(x-2)^2.$$