Differential Geometry: Homework 4

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Exercise 2.6.1. Let S be a regular surface covered by coordinate neighborhoods V_1 and V_2. Assume that $V_1 \cap V_2$ has two connected components, W_1, W_2, and that the Jacobian of the change of coordinates is positive in W_1 and negative in W_2. Prove that S is non-orientable.

Solution. Assume, for contradiction, that S is orientable. Then it is possible to assign a consistent orientation across all coordinate charts covering S, such that on any overlap of two charts, the Jacobian determinant of the transition map is positive (i.e., orientation-preserving).

Let V_1 and V_2 be two coordinate neighborhoods covering S such that their overlap consists of two connected components, W_1 and W_2. Since W_1 and W_2 are connected and the Jacobian determinant of the transition function is continuous, it must be strictly positive throughout W_1 and strictly negative throughout W_2. This implies that in W_1, the transition map preserves orientation, while in W_2, it reverses orientation.

But if S were orientable, the transition map between V_1 and V_2 would need to preserve orientation across the entire overlap. This contradiction shows that S cannot be orientable. $\square$

Exercise 2.6.2. Let S_2 be an orientable regular surface and $\Phi : S_1 \to S_2$ be a differentiable map which is a local diffeomorphism at every $p \in S_1$. Prove that S_1 is orientable.

Solution. Since S_2 is orientable, it admits an atlas of coordinate charts with consistently defined orientations, i.e., all transition maps between overlapping charts have positive Jacobian determinants.

Let $\Phi : S_1 \to S_2$ be a differentiable map that is a local diffeomorphism at every point $p \in S_1$. Then for each $p \in S_1$, there exists an open neighborhood $U \subset S_1$ such that $\Phi|_U \colon U \to \Phi(U) \subset S_2$ is a diffeomorphism.

Use the orientation of S_2 to induce an orientation on S_1 via Φ. Specifically, at each point $p \in S_1$, choose an oriented basis of the tangent space $T_{\Phi(p)}S_2$, and use the differential $d\Phi_p$ (which is an isomorphism since Φ is a local diffeomorphism) to pull back this orientation to T_pS_1.

Since the orientation on S_2 is consistent and Φ is locally a diffeomorphism, this construction gives a consistent orientation on S_1. Therefore, S_1 is orientable. $\square$

Exercise 2.6.3. Is it possible to give a meaning to the notion of area for a Möbius strip? If so, set up an integral to compute it.

Solution. Yes, it is possible to define the notion of area for a Möbius strip. The Möbius strip is a regular surface (except possibly at its boundary), and area can be defined for regular surfaces via integration of the area element induced by a parametrization.

Consider the standard parametrization of the Möbius strip:

$$\mathbf{x}(u, v) = \left(\left(1 + \frac{v}{2}\cos\left(\frac{u}{2}\right)\right)\cos(u), \left(1 + \frac{v}{2}\cos\left(\frac{u}{2}\right)\right)\sin(u), \frac{v}{2}\sin\left(\frac{u}{2}\right)\right),$$

where $u \in [0, 2\pi]$ and $v \in [-1, 1]$.

The area of the Möbius strip is given by

$$A = \int_0^{2\pi}\int_{-1}^{1} \|\mathbf{x}_u \times \mathbf{x}_v\| \, \mathrm{d}u \, \mathrm{d}y.$$

This integral is well-defined and computes the area of the Möbius strip using the standard area element for parametrized surfaces. $\square$

Exercise 2.6.7. Show that if a regular surface S contains an open set diffeomorphic to a Möbius strip, then S is non-orientable.

Solution. Suppose S is orientable and contains an open set $U \subset S$ that is diffeomorphic to an open subset of the Möbius strip M. Let $\Phi : U \to \Phi(U) \subset M$ be such a diffeomorphism.

Since S is orientable, we can assign consistent orientations to coordinate charts covering S, and in particular to U. Because Φ is a diffeomorphism, it preserves the differential structure and would transfer this orientation to $\Phi(U) \subset M$.

But this contradicts the fact that the Möbius strip is non-orientable: no open set containing a core neighborhood of M can support a consistent orientation. Therefore, $\Phi(U)$ cannot be oriented consistently, contradicting the assumption that S is orientable.

Hence, S must be non-orientable. $\square$