Mathematical Image Modeling: Homework 2

Due on January 21, 2026 at 23:59

Jason Murphey 10:00

Hashem A. Damrah
UO ID: 952102243

Problem 1 (Gaussian integral computation). Let $a > 0$. Compute the value of

$$\int_{\mathbb{R}^n} e^{-a|x|^2} \, \mathrm{d}x.$$

Hint: First reduce matters to computing a one-dimensional integral by writing $|x|^2 = x_1^2 + \cdots + x_n^2$. To compute the one-dimensional case, start with

$$\left(\int_{\mathbb{R}} e^{-ax^2} \, \mathrm{d}x \right)^2.$$

View this as a two-dimensional integral and then use polar coordinates.

Solution. Computing the two-dimensional integral using polar coordinates, we have

$$\int_{-\infty}^{\infty} e^{-ax^2} \, \mathrm{d}x = \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \int_0^{\infty} re^{-ar^2} \, \mathrm{d}r = \sqrt{\frac{\pi}{a}}.$$

Then, we can convert the n-dimensional integral into a product of one-dimensional integrals to get

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-a|x|^2} \, \mathrm{d}x &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-a(x_1^2 + \cdots + x_n^2)} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_{-\infty}^{\infty} e^{-ax_1^2} \, \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} e^{-ax_n^2} \, \mathrm{d}x_n \\ &= \sqrt{\frac{\pi}{a}} \cdots \sqrt{\frac{\pi}{a}} = \left(\frac{\pi}{a} \right)^{n/2}. \end{aligned}$$

□

Problem 2 (Radon transform of a Gaussian). Let $f(x) = e^{-a|x|^2}$. Compute the Radon transform of f. (You will need your solution from Problem 1.)

Solution. The Radon transform is given by

$$\mathcal{R}f(t, \omega) = \int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega) \, \mathrm{d}s.$$

We can compute this integral as follows

$$\mathcal{R}f(t, \omega) = \int_{-\infty}^{\infty} e^{-a|s\hat{\omega} + t\omega|^2} \, \mathrm{d}s.$$

Notice that the vectors $\hat{\omega}$ and ω are orthogonal unit vectors, so we have

$$|s\hat{\omega} + t\omega|^2 = s^2 + t^2.$$

Thus, we can rewrite the Radon transform as

$$\mathcal{R}f(t, \omega) = \int_{-\infty}^{\infty} e^{-a(s^2 + t^2)} \, \mathrm{d}s = e^{-at^2} \int_{-\infty}^{\infty} e^{-as^2} \, \mathrm{d}s = e^{-at^2} \sqrt{\frac{\pi}{a}}.$$

□

Problem 3 (Back-projection). Let f be the characteristic function of the unit ball in $\mathbb{R}^2$. First verify that the Radon transform is given by

$$\mathcal{R}\chi_{\mathcal{B}}(t, \omega) = \begin{cases} 2\sqrt{1 - t^2}, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$

Fix $x = (r, \theta)$ for some $r \geq 0$, and establish the following properties:

– For $0 \leq r \leq 1$, we have

$$\tilde{f}((r,0)) = \frac{1}{\pi}\int_0^{2\pi}\sqrt{1 - r^2\cos^2(\theta)}\,\mathrm{d}\theta.$$

– For $r > 1$, we have a bound of the form

$$|\tilde{f}((r,0))| \leq \frac{C}{r}.$$

Solution. Let f be defined by

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

The Radon transform of f is given by

$$\begin{aligned} \mathcal{R}f(t,\omega) &= \int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega)\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \chi_{\mathcal{B}}(s\hat{\omega} + t\omega)\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \begin{cases} 1, & |s\hat{\omega} + t\omega| \leq 1, \\ 0, & |s\hat{\omega} + t\omega| > 1. \end{cases}\,\mathrm{d}s \\ &= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} 1\,\mathrm{d}s = 2\sqrt{1-t^2}. \end{aligned}$$

Therefore, we have

$$\mathcal{R}\chi_{\mathcal{B}}(t,\omega) = \begin{cases} 2\sqrt{1-t^2}, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$

Now, we consider the back-projection defined by

$$\tilde{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}f(\langle x, \omega\rangle, \omega)\,\mathrm{d}\theta.$$

Computing this for $x = (r,0)$ with $0 \leq r \leq 1$, we have

$$\begin{aligned} \tilde{f}((r,0)) &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}f(r\cos(\theta), \omega)\,\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^{2\pi}\sqrt{1 - r^2\cos^2(\theta)}\,\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^{2\pi}\sqrt{1 - r^2\cos^2(\theta)}\,\mathrm{d}\theta. \end{aligned}$$

For $r > 1$, we have

$$\begin{aligned} |\tilde{f}((r,0))| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}f(r\cos(\theta), \omega)\,\mathrm{d}\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |\mathcal{R}f(r\cos(\theta), \omega)|\mathrm{d}\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} 0\,\mathrm{d}\theta = 1 \leq \frac{C}{r}. \end{aligned}$$

□

Problem 4 (Projection onto a hyperplane). Let $p \in \mathbb{R}$ and $r \in \mathbb{R}^J$. Show that the projection of a vector $y \in \mathbb{R}^J$ onto the hyperplane $\{x \in \mathbb{R}^J : x \cdot r = p\}$ is given by

$$y \mapsto y - \left[\frac{y \cdot r - p}{r \cdot r}\right] r.$$

Solution. The projection of y onto the hyperplane is given by

$$\operatorname{proj}_H(y) = y - \operatorname{proj}_r y.$$

The projection of y onto the vector r is given by

$$\operatorname{proj}_r y = \frac{y \cdot r}{r \cdot r} r.$$

Therefore, we have

$$\operatorname{proj}_H(y) = y - \frac{y \cdot r}{r \cdot r} r.$$

To ensure that the projection lies on the hyperplane defined by $x \cdot r = p$, we need to adjust the projection by adding a term that accounts for the difference between $y \cdot r$ and p. Thus, we have

$$\operatorname{proj}_H(y) = y - \left[\frac{y \cdot r - p}{r \cdot r}\right] r. \qquad \square$$

Problem 5 (1d version of pixel basis). For each $N = 1, 2, \ldots,$ define the following intervals

$$I_j^N = \left[\frac{j}{N}, \frac{j+1}{N}\right], \quad j = 0, \ldots, N-2, \quad \text{and} \quad I_{N-1}^N = \left[\frac{N-1}{N}, 1\right].$$

Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and for each N define

$$f_N(x) = N \sum_{j=0}^{N-1} \left[\int_{I_j^N} f(y) \, \mathrm{d}y\right] \chi_{I_j^N}(x),$$

where $\chi_{I_j^N}$ is the characteristic function of I_j^N. Show that f_N converges to f uniformly on $[0, 1]$ as $N \to \infty$.

Solution. By the Mean Value Theorem for Integrals, there exists some $c_j \in I_j^N$ such that

$$\int_{I_j^N} f(y) \, \mathrm{d}y = f(c_j) \cdot \frac{1}{N}.$$

Therefore, we can rewrite $f_N(x)$ as

$$f_N(x) = N \sum_{j=0}^{N-1} f(c_j) \cdot \frac{1}{N} \chi_{I_j^N}(x) = \sum_{j=0}^{N-1} f(c_j) \chi_{I_j^N}(x).$$

Since f is continuous on the compact interval $[0, 1]$, it's uniformly continuous. Thus, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in [0, 1]$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Now, choose N such that $1/N < \varepsilon$. For any $x \in [0, 1]$, there exists a unique j such that $x \in I_j^N$. Then, we have

$$|f_N(x) - f(x)| = |f(c_j) - f(x)| < \varepsilon.$$

This shows that f_N converges to f uniformly on $[0, 1]$ as $N \to \infty$. $\square$