Quiz 2 Solution **Fall 2024**

1. (4 pts) State Monotone Convergence Theorem.

   Solution. Every bounded monotone sequence converges. □

2. (7 pts) Give a sequence that does not contain $-1, 0, 1$ but contains subsequences that converge to these three points.

   Solution. For $n = 1, 2, \ldots$, let $a_{3n} = -1 + \frac{1}{n+1}$, $a_{3n+1} = \frac{1}{n+1}$, $a_{3n+2} = 1 + \frac{1}{n+1}$. Then $\{a_n\}$ satisfies the requirement. □

3. (7 pts) Prove "If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges."

   Solution. Let $\varepsilon > 0$. By the Cauchy criterion, $\sum_{n=1}^{\infty} |a_n|$ convergence implies that there is $N \in \mathbb{N}$ such that for all $n \geq m > N$,

   $$\sum_{k=m+1}^{n} |a_k| < \varepsilon.$$

   By the triangle inequality, this implies

   $$\left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| < \varepsilon,$$

   so that $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion and, hence, convergence. □

4. (7 pts) Prove "If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is a bounded sequence".

   Solution. Let a_n be a Cauchy sequence.Then, tgere is $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

   $$|a_n - a_m| < 1.$$

   Hence, taking $m = N$ and using the triangle inequality, we obtain

   $$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| \leq 1 + |a_N|, \qquad \text{for all } n \geq N.$$

   Hence, choosing

   $$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}, |a_N| + 1\},$$

   we conclude that $|a_n| \leq M$ for all $n \geq 1$. Thus, $\{a_n\}$ is bounded. □