Introduction to Proof: Homework 4

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Problem 1

Identify each of the following statements as true or false. Where you can, prove the statement by giving an example or disprove it by giving a counterexample.

(i) $(\exists x \in \mathbb{Z}_9)[x^2 \in \{5, 7\}]$.

(ii) $(\forall a \in \mathbb{Z})[a^2 \equiv_{11} 16 \Rightarrow a \equiv_{11} 4]$.

(iii) $(\exists n \in \mathbb{N})[\frac{1}{2}(n^2 + n) + 2 \text{ is prime}]$.

(iv) $(\forall n \in \mathbb{N})[\frac{1}{2}(n^2 + n) + 2 \text{ is prime}]$.

(v) $(\exists a, b \in \mathbb{Z})[12a + 20b = 4]$.

(vi) $\{x \mid x \in \mathbb{Z}_8 \wedge 4 \cdot x = 0 \pmod 8\} \cap \{4 \cdot x \pmod 8 \mid x \in \mathbb{Z}_8\} = \emptyset$.

(vii) $(\forall n \in \mathbb{N})[(n \equiv_2 1 \wedge n > 3) \Rightarrow 3 \mid n^2 - 1]$.

(viii) $(\forall a, b \in \mathbb{Z})[12 \mid ab \Rightarrow (12 \mid a \vee 12 \mid b)]$.

Solution 1

(i) True. Example: $4^2 \equiv_9 7$ and $5^2 \equiv_9 7$.

(ii) False. Counterexample: $7^2 \equiv_{11} 49 \equiv_{11} 16$ since $7 \not\equiv_{11} 4$.

(iii) True. Example: $n = 1$ and $n = 2$.

(iv) False. Counterexample: $n = 4$.

(v) True. Example: $a = 1$ and $b = -1$.

(vi) False. $\{0, 2\} \cap \{0, 4\} = \{0\} \neq \emptyset$.

(vii) True. For $n \equiv_2 1$ and $n > 3$, n is an odd integer greater than 3. For any odd n, $n^2 - 1$ can be factored as $(n-1)(n+1)$. Since n is odd, both $n-1$ and $n+1$ are even. So one of them is divisible by 3. Thus $3 \mid n^2 - 1$, for all add $n > 3$.

(viii) False. Counterexample: $a = 6$ and $b = 4$.

Problem 2

Given a line proof that $(\forall n \in \mathbb{N})[n^2 + (n+1)^2 + (n+2)^2 \equiv_3 2$.

Solution 2

1. Assume $n \in \mathbb{N}$.
2. Then, $n^2 + (n+1)^2 + (n+2)^2 = 3(n^2 + 2n + 1) - 2$.
3. $(\exists r \in \mathbb{Z})[n^2 + (n+1)^2 + (n+2)^2 - 2 = 3r]$.
4. $3 \mid (n^2 + (n+1)^2 + (n+2)^2 - 2)$.
5. $n^2 + (n+1)^2 + (n+2)^2 \equiv_3 2$.
6. $n \in \mathbb{N} \Rightarrow n^2 + (n+1)^2 + (n+2)^2 \equiv_3 2$.
7. $(\forall n \in \mathbb{N})[n^2 + (n+1)^2 + (n+2)^2 \equiv_3 2]$

Problem 3

Decide if the following statement is true or false. If it is true, give a line proof. If it is false, give a counterexample.

$$(\forall a, b, c \in \mathbb{Z})[(a > 0 \wedge a \mid (b-1) \wedge a \mid (c-1)) \Rightarrow a \mid (bc-1)].$$

Solution 3

1. Assume $a > 0$, $a \mid (b-1)$, and $a \mid (c-1)$.
2. Then, $b - 1 = ak$ and $c - 1 = aq$, for some $k, q \in \mathbb{Z}$.
3. Then, $b = ak + 1$ and $c = aq + 1$.
4. $bc = (ak+1)(aq+1) = a(akq + k + q) + 1$.
5. $(\exists r \in \mathbb{Z})[bc - 1 = ar]$.
6. $a \mid (bc-1)$.
7. $(a > 0 \wedge a \mid (b-1) \wedge a \mid (c-1)) \Rightarrow a \mid (bc-1)$
8. $(\forall a, b, c \in \mathbb{Z})[(a > 0 \wedge a \mid (b-1) \wedge a \mid (c-1)) \Rightarrow a \mid (bc-1)]$

Problem 4

Here is an important property of prime numbers. If p is prime, then

$$\text{Property (P):} \quad (\forall x, y \in \mathbb{Z})[p \mid xy \Rightarrow (p \mid x \vee p \mid y)].$$

Using this, fill in the blanks below to give a proof of the following theorem:
Theorem: $(\forall y \in \mathbb{Z})[4y^2 \equiv_7 0 \Rightarrow y \equiv_7 0]$.

1. Assume $y \in \mathbb{Z}$ and $4y^2 \equiv_7 0$.
2. Then $\boxed{}$.
3. 7 is prime, so by property (P) we know $7 \mid 4$ or $\boxed{}$.
4. But $7 \nmid 4$, so $\boxed{}$.
5. Using Property (P) again, either $7 \mid y$ or $\boxed{}$.
6. Therefore $7 \mid y$ (by using the tautology $\boxed{}$).
7. $\boxed{}$.
8. $\boxed{}$.
9. $\boxed{}$.

Solution 4

1. Assume $y \in \mathbb{Z}$ and $4y^2 \equiv_7 0$.
2. Then $7 \mid 4y^2$.
3. 7 is prime, so by property (P) we know $7 \mid 4$ or $7 \mid y^2$.
4. But $7 \nmid 4$, so $7 \mid y^2$.
5. Using Property (P) again, either $7 \mid y$ or $7 \mid y$.
6. Therefore $7 \mid y$ (by using the tautology $(P \vee P) \Rightarrow P$).
7. $4y^2 \equiv_7 0 \Rightarrow 7 \mid y$.
8. $(y \in \mathbb{Z} \wedge 4y^2 \equiv_7 0) \Rightarrow 7 \mid y$.
9. $(\forall x, y \in \mathbb{Z})[p \mid xy \Rightarrow (p \mid x \vee p \mid y)]$.

Problem 5

Give a line proof that $(\forall n \in \mathbb{N})[(3 \mid n \wedge n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12]$.

Solution 5

1. Assume $n \in \mathbb{N}$, $3 \mid n$, and $n \equiv_5 3$.
2. $(\exists r \in \mathbb{Z})[n^2 + n - 12 = 5r]$.
3. $5 \mid (n^2 + n - 12)$.
4. $n = 5k + 3$, for some $k \in \mathbb{Z}$.
5. $(5k + 3^2 + (5k + 3) = 5(5k^2 + 7k) + 12$.
6. $(\exists r \in \mathbb{Z})[n^2 + n - 12 = 5r]$.
7. $3 \mid 5k + 3$.
8. $(\exists l \in \mathbb{Z})[5k + 3 = 3l]$.
9. $5k = 3(l - 1)$.
10. $(\exists m \in \mathbb{Z})[5k = 3m]$.
11. By Property P, we get $3 \mid 5$ or $3 \mid k$.
12. Since $3 \nmid 5$, then $3 \mid k$.
13. Then $n^2 + n - 12 = 5k(5k + 7)$.
14. $(\exists n \in \mathbb{Z})[k = 3n]$.
15. $n^2 + n - 12 = 3 \cdot (5n \cdot (5 \cdot 3n + 7))$.
16. $3 \mid (n^2 + n - 12)$.
17. $15 \mid (n^2 + n - 12)$.
18. $n^2 + n \equiv_{15} 12$.
19. $(3 \mid n \wedge n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12$.
20. $(\forall n \in \mathbb{N})[(3 \mid n \wedge n \equiv_5 3) \Rightarrow n^2 + n \equiv_{15} 12]$

Problem 6

A *rational number* is a number of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. As you learned in elementary school, a rational number can always be written in the form where $\gcd(a, b) = 1$. Fill in the blanks below to give a proof of the following theorem; feel free to insert extra steps if you think it will help clarify the proof.

Theorem: $\sqrt{2}$ is not a rational number.

1. Assume that $\sqrt{2}$ is a rational number.
2. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $\gcd(a, b) = 1$.
3. So $2 = \frac{a^2}{b^2}$, and therefore $a^2 = \boxed{}$.
4. So $2 \mid a^2$.
5. Then by Property (P), $\boxed{}$.
6. $a = 2r$ for some $r \in \mathbb{Z}$.
7. $2b^2 = \boxed{}$.
8. $b^2 = \boxed{}$.
9. $2 \mid b^2$.
10. So using Property (P), $\boxed{}$.
11. $b = 2s$ for some $s \in \mathbb{Z}$.
12. This is a contradiction, because $\boxed{}$.
13. $\boxed{}$.

Solution 6

1. Assume that $\sqrt{2}$ is a rational number.
2. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $\gcd(a, b) = 1$.
3. So $2 = \frac{a^2}{b^2}$, and therefore $a^2 = 2b^2$.
4. So $2 \mid a^2$.
5. Then by Property (P), $2 \mid a$.
6. $a = 2r$ for some $r \in \mathbb{Z}$.
7. $2b^2 = (2r)^2 = 4r^2$.
8. $b^2 = 2r^2$.
9. $2 \mid b^2$.
10. So using Property (P), $2 \mid b$.
11. $b = 2s$ for some $s \in \mathbb{Z}$.
12. This is a contradiction, because $\gcd(a, b) = 1$.
13. Therefore, $\sqrt{2}$ is irrational.

Problem 7

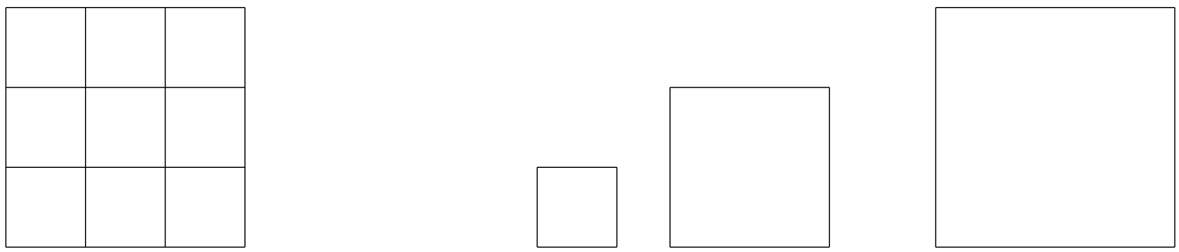
Give a line proof showing that $\sqrt{10}$ is not a rational number.

Solution 7

1. Assume that $\sqrt{10}$ is a rational number.
2. Then $\sqrt{10} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$ and $\gcd(a, b) = 1$.
3. So $10 = \frac{a^2}{b^2}$, and therefore $a^2 = 10b^2$.
4. So $10 \mid a^2$.
5. Then by Property (P), $10 \mid a$.
6. $a = 10r$ for some $r \in \mathbb{Z}$.
7. $10b^2 = (10r)^2 = 100r^2$.
8. $b^2 = 10r^2$.
9. $10 \mid b^2$.
10. So using Property (P), $10 \mid b$.
11. $b = 10s$ for some $s \in \mathbb{Z}$.
12. This is a contradiction, because $\gcd(a, b) = 1$.
13. Therefore, $\sqrt{10}$ is irrational.

Problem 8

A 3×3 grid has 14 squares in it:



There are 1×1 squares, 2×2 squares, and 3×3 squares, and if you count all the squares that you see in the above grid you should get 14.

Figure out how many squares there are in a 10×10 grid, and explain your answer. Given a exact number, not just a formula for computing it.

Hints for doing this: Get a sense of the problem by tackling smaller version. Try a 2×2 grid, you already did the 3×3 grid, maybe look at 4×4 and 5×5 grids. Analyze these smaller problems and try to find some underlying patterns.

Solution 8

2×2: There are 4 small squares and 1 square that covers the entire grid, giving us a total of 5 squares.

3×3: There are 9 small squares, 4 medium squares, and 1 large square, giving us a total of 14 squares.

4×4: There are 16 small squares, 9 medium squares, 4 large squares, and 1 extra large square, giving us a total of 30 squares.

5×5: There are 25 small squares, 16 medium squares, 9 large squares, 4 extra large squares, and 1 extra extra large square, giving us a total of 55 squares.

From these calculations, we see that for an $n \times n$ grid, the total number of squares is the sum of squares of the numbers from 1 to n

$$\text{Total squares} = \sum_{i=1}^{n} i^2.$$

Using this formula, we can calculate the total number of squares in a 10×10 grid as 385 squares.