Introduction to General Relativity: Homework 2

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Problem 1. Two inertial frames S and S' are related by a Lorentz boost with speed v in the x-direction. Units are chosen so that $c = 1$. The transformation is

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z, \end{aligned}$$

where $\gamma = (1 - v^2)^{1/2}$.

A particle moves along the x-axis in frame S with worldline

$$x(t) = at^2, \qquad y = 0 = z,$$

where a is a constant.

The four-position is $x^\mu = (t, x, y, z)$, and the four-velocity is defined as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau},$$

where τ is the particle's proper time.

(i) (5 points) Compute u^μ in frame S, expressing your answer in terms of t and a. (Hint: first compute $\mathrm{d}x^\mu/\mathrm{d}t$, then use $\mathrm{d}\tau = \sqrt{1 - v^2}$, where $v = \mathrm{d}x/\mathrm{d}t$.)

(ii) (2 points) Show explicitly that

$$u^\mu u_\mu = -1.$$

(iii) (5 points) Using the Lorentz transformation above, compute the components of u'^μ in frame S'. Verify that

$$u'^\mu u'_\mu = -1,$$

and explain briefly why this result is expected.

(iv) (5 points) The four-acceleration is defined as

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}.$$

Compute a^μ for the particle in frame S and show that

$$u^\mu a_\mu = 0.$$

(v) (2 points) Explain why differentiation with respect to proper time preserves the four-vector character of a quantity, whereas differentiation with respect to coordinate time t generally does not.

Solution to (i). Computing $\mathrm{d}x^\mu/\mathrm{d}t$, we get

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = (1, 2at, 0, 0)\,.$$

The three-velocity is $v = \mathrm{d}x/\mathrm{d}t = 2at$, so

$$\mathrm{d}\tau = \sqrt{1 - v^2}\,\mathrm{d}t = \sqrt{1 - (2at)^2}\,\mathrm{d}t.$$

Therefore,

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{(1, 2at, 0, 0)}{\sqrt{1 - (2at)^2}} = \left(\frac{1}{\sqrt{1 - 4a^2t^2}}, \frac{2at}{\sqrt{1 - 4a^2t^2}}, 0, 0\right). \qquad \square$$

Solution to (ii). Remember that

$$u^\mu u_\mu = \eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{1}{\mathrm{d}\tau^2} \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2}.$$

Then, we can use the fact that $\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 = \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$ to write

$$u^\mu u_\mu = \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = -1.$$

□

Solution to (iii). Using the Lorentz transformation, we find that

$$u'^\mu = \Lambda^\mu_\nu u^\nu \quad \text{and} \quad u'_\mu = \eta_{\mu\nu} \Lambda^\nu_\alpha u^\alpha.$$

Combining these, we get

$$u'^\mu u'_\mu = (\Lambda^\mu_\nu u^\nu)\,(\eta_{\mu\sigma} \Lambda^\sigma_\alpha u^\alpha) = (\eta_{\mu\sigma} \Lambda^\mu_\nu \Lambda^\sigma_\alpha)\, u^\nu u^\alpha = \eta_{\nu\alpha} u^\nu u^\alpha = u^\mu u_\mu = -1.$$

This result is expected because the quantity $u^\mu u_\mu$ is a Lorentz invariant, meaning it has the same value in all inertial frames. Since we have already shown that $u^\mu u_\mu = -1$ in frame S, it must also hold true in frame S'. □

Solution to (iv). Using the fact that $a^m u = \mathrm{d}u^\mu / \mathrm{d}\tau$, we have

$$u^\mu a_\mu = u^\mu \eta_{\mu\nu} a^\nu = \eta_{\mu\nu} u^\mu \frac{\mathrm{d}u^\nu}{\mathrm{d}\tau},$$

using the fact that $a_\mu = \eta_{\mu\nu} a^\nu$. Now, computing the derivative of $\mathrm{d}u^\nu / \mathrm{d}\tau$, we get

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}u^\mu}{\mathrm{d}t}.$$

Therefore,

$$u^\mu a_\mu = \eta_{\mu\nu} u^\mu \gamma \frac{\mathrm{d}u^\nu}{\mathrm{d}t} = \gamma \frac{\mathrm{d}}{\mathrm{d}t} (\eta_{\mu\nu} u^\mu u^\nu) - \gamma \eta_{\mu\nu} \frac{\mathrm{d}u^\mu}{\mathrm{d}t} u^\nu.$$

But since $\eta_{\mu\nu} u^\mu u^\nu = -1$ is a constant, its derivative is zero. Thus,

$$u^\mu a_\mu = -\gamma \eta_{\mu\nu} \frac{\mathrm{d}u^\mu}{\mathrm{d}t} u^\nu.$$

Notice that the right-hand side is just $-u^\mu a_\mu$. Therefore,

$$u^\mu a_\mu = -u^\mu a_\mu \Rightarrow 2u^\mu a_\mu = 0 \Rightarrow u^\mu a_\mu = 0.$$

□

Solution to (v). Proper time τ is the Lorentz invariant time measured by a clock moving along with the particle. Since it is invariant under Lorentz transformations, differentiating with respect to τ preserves the four-vector character of a quantity. On the other hand, coordinate time t is frame-dependent and varies between different inertial frames. Differentiating with respect to t can mix components of four-vectors in a way that does not respect their transformation properties, leading to quantities that may not transform as four-vectors. □

Problem 2 (Tachyons).

(i) (2 points) Argue that a kind of particle that always moves faster than the velocity of light would be consistent with Lorentz invariance in the sense that if its speed is greater than light in one frame, it will be greater than light in all frames. (Such a hypothetical particle is called a tachyon).

(ii) (2 points) Show that the tangent vector to the trajectory of a tachyon is spacelike and can be written $u^\alpha = \mathrm{d}x^\alpha / \mathrm{d}s$, where s is the spacelike interval along the trajectory. Show that $\mathbf{u} \cdot \mathbf{u} = 1$.

(iii) (2 points) Evaluate the components of a tachyon's four-velocity $\mathbf{u}$ in terms of the three velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$

(iv) (2 points) Define the four-momentum by $\mathbf{p} = m\mathbf{u}$ and find the relation between energy and momentum for a tachyon.

(v) (2 points) Show that there is an inertial frame where the energy of any tachyon is negative.

(vi) (2 points) Show that if tachyons interact with normal particles, a normal particle could emit a tachyon with total energy and three-momentum being conserved. This result suggests a world containing tachyons would be unstable and there is no evidence for tachyons in nature.

Solution to (i). Consider a tachyon moving with velocity $u > c$ in frame S. Using the relativistic velocity transformation, the velocity u' of the tachyon in another inertial frame S' moving at velocity v relative to S is given by

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

Since $u > c$, the denominator $1 - \frac{uv}{c^2}$ will always be negative for any $v < c$. Therefore, the magnitude of u' will also be greater than c. This shows that if a particle moves faster than light in one frame, it will do so in all inertial frames, thus preserving Lorentz invariance. □

Solution to (ii). For ordinary particles, we have

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \quad \text{and} \quad u^\alpha u_\alpha = -1.$$

We also have

$$\mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = -\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu.$$

But since tachyons are spacelike, we can define s such that $\mathrm{d}s^2 = +\,\mathrm{d}x^\alpha\,\mathrm{d}x_\alpha$. Defining

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s},$$

we check this works out by computing

$$u^\alpha u_\alpha = \eta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = \frac{1}{\mathrm{d}s^2}\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \frac{\mathrm{d}s^2}{\mathrm{d}s^2} = 1.$$

Lastly, we show that u^α is spacelike. Since s is defined such that $\mathrm{d}s^2 > 0$, we have

$$u^\alpha u_\alpha = 1 > 0.$$

Thus, u^α is indeed spacelike. □

Solution to (iii). The three-velocity $\mathbf{v}$ is defined as

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}.$$

We can express the four-velocity components in terms of $\mathbf{v}$ as

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}s} \quad \text{and} \quad u^i = \frac{\mathrm{d}x^i}{\mathrm{d}s} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = v^i\frac{\mathrm{d}t}{\mathrm{d}s}.$$

Therefore, the components of the four-velocity $\mathbf{u}$ in terms of the three-velocity $\mathbf{v}$ are

$$u^\mu = \left(\frac{1}{\sqrt{\mathbf{v}^2 - 1}}, \frac{\mathbf{v}}{\sqrt{\mathbf{v}^2 - 1}}\right).$$ □

Solution to (iv). The four-moment $\mathbf{p}$ is given as $\mathbf{p} = m\mathbf{u}$. Using the components of $\mathbf{u}$ from part (iii), we have

$$\mathbf{p} = m\left(\frac{1}{\sqrt{\mathbf{v}^2 - 1}}, \frac{\mathbf{v}}{\sqrt{\mathbf{v}^2 - 1}}\right) = (E, \mathbf{p})\,.$$

Therefore, the energy E and momentum $\mathbf{p}$ of the tachyon are

$$E = \frac{m}{\sqrt{\mathbf{v}^2 - 1}} \quad \text{and} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{\mathbf{v}^2 - 1}}.$$

To find the relation between energy and momentum, we can eliminate $\mathbf{v}$ to get

$$\begin{aligned} E^2 - \mathbf{p}^2 &= \left(\frac{m}{\sqrt{\mathbf{v}^2 - 1}}\right)^2 - \left(\frac{m\mathbf{v}}{\sqrt{\mathbf{v}^2 - 1}}\right)^2 \\ &= \frac{m^2}{\mathbf{v}^2 - 1} - \frac{m^2\mathbf{v}^2}{\mathbf{v}^2 - 1} \\ &= \frac{m^2(1 - \mathbf{v}^2)}{\mathbf{v}^2 - 1} = -m^2. \end{aligned}$$

Thus, the relation between energy and momentum for a tachyon is

$$E^2 - \mathbf{p}^2 = -m^2.$$

□

Solution to (v). Consider a Lorentz transformation to a frame moving at velocity v relative to the original frame. The energy E' in the new frame is given by

$$E' = \gamma(E - vp_x),$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$. Since tachyons have $E^2 - p^2 = -m^2$, we can choose v such that $vp_x > E$. This is always possible since tachyons can have arbitrarily large momentum for a given energy. Therefore, we can find a frame where

$$E' = \gamma(E - vp_x) < 0.$$

This shows that there exists an inertial frame where the energy of any tachyon is negative. □

Solution to (vi). Consider a normal particle with four-momentum $p^\mu = (E, \mathbf{p})$ emitting a tachyon with four-momentum $k^\mu = (E_k, \mathbf{k})$. The final four-momentum of the normal particle after emission is $p'^\mu = (E', \mathbf{p}')$. Conservation of four-momentum requires

$$p^\mu = p'^\mu + k^\mu.$$

This leads to the following conservation equations:

$$\begin{aligned} E &= E' + E_k \\ \mathbf{p} &= \mathbf{p}' + \mathbf{k}. \end{aligned}$$

Since tachyons can have negative energy in some frames (as shown in part (v)), we can choose a frame where $E_k < 0$. In this frame, it is possible for the normal particle to emit a tachyon while still satisfying the conservation of energy and momentum. Thus, a normal particle could emit a tachyon with total energy and three-momentum being conserved, suggesting that a world containing tachyons would be unstable. □

Problem 3 (Relativistic charged particle). Consider the action of a (relativistic) charge particle coupled to a background gauge field, A_μ, living in Minkowski space. It is governed by the action

$$S = -m\int \mathrm{d}\tau + q\int A_\mu(\mathbf{x})\,\mathrm{d}x^\mu,$$

where $\mathbf{A}$ is a 1-form and m, q are the mass and charge of the particle, respectively.

(i) (3 points) Suppose that the particle travels along the worldline $x^\mu(\lambda)$. Write the action in terms of x^μ and $\mathrm{d}x^\mu/\mathrm{d}\lambda$.

(ii) (5 points) Find the canonical momenta p_μ and show that

$$p_\mu = mu_\mu + qA_\mu.$$

In other words, show that the canonical momentum, which is defined by the Lagrangian procedure, is different than the mechanical momentum $p_\mu = mu_\mu$.

(iii) (3 points) Complete the Euler-Lagrange equations to show that

$$\frac{\mathrm{d}p_\gamma}{\mathrm{d}\lambda} = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

(iv) (6 points) ($\star$) Show that your answer in the previous part can be written as

$$m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau} = qF_{\gamma\mu}u^\mu,$$

where

$$F_{\gamma\mu} = \partial_\gamma A_\mu - \partial_\mu A_\gamma, \tag{1}$$

is the field strength tensor.

(v) (5 points) Show that $F_{\gamma\mu}$ is antisymmetric and has the correct transformation properties to be a tensor.

(vi) (6 points) ($\star$) By considering the acceleration of slow-moving particles, show that the components of $F_{\gamma\mu}$ agree with

$$F_{\gamma\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

(vii) (4 points) ($\star$) Write the independent components of Eq. 1 corresponding to E_x and B_x. Does this match with what you've learned in E&M of how fields are derived from a vector potential? What is the physical interpretation of the component A_0?

Solution to (i). Since $x^\mu = x^\mu(\lambda)$, we have

$$\mathrm{d}x^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau \quad \text{and} \quad \mathrm{d}\tau = \sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}.$$

Therefore, re-writing the action in terms of x^μ and $\mathrm{d}x^\mu/\mathrm{d}\lambda$, we get

$$S = -m\int \mathrm{d}\tau + q\int A_\mu(\mathbf{x})\,\mathrm{d}x^\mu = -m\int\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda + q\int A_\mu(\mathbf{x})\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda. \qquad \square$$

Solution to (ii). The canonical momenta are defined as

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu}.$$

From part (i), the Lagrangian is

$$L = -m\sqrt{-\eta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}} + qA_\nu(\mathbf{x})\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}.$$

Therefore, we compute

$$\begin{aligned} p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} &= -m \frac{\partial}{\partial \dot{x}^\mu} \sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} + qA_\mu(\mathbf{x}) \\ &= -m \frac{1}{2\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \left(-2\eta_{\mu\nu}\dot{x}^\nu\right) + qA_\mu(\mathbf{x}) \\ &= \frac{m\eta_{\mu\nu}\dot{x}^\nu}{\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} + qA_\mu(\mathbf{x}) \\ &= m\eta_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + qA_\mu(\mathbf{x}) \\ &= m\frac{\mathrm{d}x_\mu}{\mathrm{d}\lambda} + qA_\mu(\mathbf{x}) \\ &= mu_\mu + qA_\mu(\mathbf{x}). \end{aligned}$$

□

Solution to (iii). Using Euler-Lagrange equations, we have

$$\frac{\mathrm{d}p_\gamma}{\mathrm{d}\lambda} = \frac{\partial L}{\partial x^\gamma},$$

where

$$L = -m\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} + qA_\mu(\mathbf{x})\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

Computing $\partial L/\partial x^\gamma$, we have

$$\frac{\partial L}{\partial x^\gamma} = \frac{\partial}{\partial x^\gamma}\left(-m\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} + qA_\mu(\mathbf{x})\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right) = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}p_\gamma}{\mathrm{d}\lambda}.$$

□

Solution to (iv). Using the result from part (iii), we have

$$\frac{\mathrm{d}p_\gamma}{\mathrm{d}\lambda} = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

Computing the derivative on the left-hand side, we get

$$\begin{aligned} \frac{\mathrm{d}p_\gamma}{\mathrm{d}\lambda} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(mu_\gamma + qA_\gamma\right) \\ &= m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\lambda} + q\frac{\partial A_\gamma}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}. \end{aligned}$$

Therefore, we can write

$$m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\lambda} + q\frac{\partial A_\gamma}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

Rearranging terms, we have

$$m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\lambda} = q\left(\frac{\partial A_\mu}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^\mu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = qF_{\gamma\mu}u^\mu.$$

□

Solution to (v). Re-writing $F_{\gamma\mu}$, we have

$$F_{\gamma\mu} = \partial_\gamma A_\mu - \partial_\mu A_\gamma = -(\partial_\mu A_\gamma - \partial_\gamma A_\mu) = -F_{\mu\gamma}.$$

Thus, $F_{\gamma\mu}$ is antisymmetric.

To show that $F_{\gamma\mu}$ has the correct transformation properties to be a tensor, we consider a Lorentz transformation Λ. Under this transformation, the vector potential transforms as

$$A'_\mu = \Lambda^\nu_\mu A_\nu.$$

Therefore, the derivatives transform as

$$\partial'_\gamma = \Lambda_\gamma^\delta \partial_\delta.$$

Using these transformations, we can express $F'_{\gamma\mu}$ as

$$F'_{\gamma\mu} = \partial'_\gamma A'_\mu - \partial'_\mu A'_\gamma = \Lambda_\gamma^\delta \partial_\delta (\Lambda_\mu^\nu A_\nu) - \Lambda_\mu^\delta \partial_\delta (\Lambda_\gamma^\nu A_\nu).$$

Expanding this expression, we get

$$F'_{\gamma\mu} = \Lambda_\gamma^\delta \Lambda_\mu^\nu \partial_\delta A_\nu - \Lambda_\mu^\delta \Lambda_\gamma^\nu \partial_\delta A_\nu = \Lambda_\gamma^\delta \Lambda_\mu^\nu F_{\delta\nu}.$$

This shows that $F_{\gamma\mu}$ transforms as a rank-2 tensor under Lorentz transformations. □

Solution to (vi). The relativistic charge particle obeys

$$m \frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau} = q F_{\gamma\mu} u^\mu.$$

Since the particle is slow moving, we have $u^0 \approx 1$ and $u^i \ll 1$. Looking at the spatial components ($\gamma = i$), we have

$$m \frac{\mathrm{d}u_i}{\mathrm{d}\tau} = q(F_{i0} u^0 + F_{ij} u^j) \approx q(F_{i0} + F_{ij} u^j).$$

Now, define $E_i = F_{i0}$, since the term qF_{i0} since it produces acceleration even when the particle is nearly at rest, which is characteristic of an electric field. Now, define $B_k = \varepsilon_{ijk} F_{ij}$, since the term $qF_{ij}u^j$ depends on the velocity of the particle, which is characteristic of a magnetic field. Now, we can define

$$\begin{aligned} B_x &= F_{23}, \quad B_y = F_{31}, \quad B_z = F_{12} \\ E_x &= F_{10}, \quad E_y = F_{20}, \quad E_z = F_{30} \end{aligned}$$

From part (v), we have

$$F_{0i} = -F_{i0} = -E_i \quad \text{and} \quad F_{ij} = \varepsilon_{ijk} B_k.$$

Therefore,

$$m \frac{\mathrm{d}u_i}{\mathrm{d}\tau} = q(E_i + \varepsilon_{ijk} B_k u^j).$$

This matches the Lorentz force law for a charged particle in an electromagnetic field, which states that the force on a charged particle is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

The physical interpretation of the component A_0 is that it represents the scalar potential (or electric potential) in electromagnetism. It contributes to the electric field $\mathbf{E}$ and influences the energy of charged particles in the presence of an electromagnetic field. □

Solution to (vii). The independent components of Eq. 1 corresponding to B_i are

$$\begin{aligned} B_x &= F_{23} = \partial_2 A_3 - \partial_3 A_2 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ B_y &= F_{31} = \partial_3 A_1 - \partial_1 A_3 = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ B_z &= F_{12} = \partial_1 A_2 - \partial_2 A_1 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \end{aligned}$$

The independent components corresponding to E_i are

$$E_x = F_{10} = \partial_1 A_0 - \partial_0 A_1 = \frac{\partial A_0}{\partial x} - \frac{\partial A_x}{\partial t}$$

$$
\begin{aligned}
E_y &= F_{20} = \partial_2 A_0 - \partial_0 A_2 = \frac{\partial A_0}{\partial y} - \frac{\partial A_y}{\partial t} \\
E_z &= F_{30} = \partial_3 A_0 - \partial_0 A_3 = \frac{\partial A_0}{\partial z} - \frac{\partial A_z}{\partial t}.
\end{aligned}
$$

This makes sense with how fields are derived from a vector potential in E&M. They are derived as

$$
\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A},
$$

where A_0 is the scalar potential and $\mathbf{A}$ is the vector potential. The component A_0 represents the electric potential, which contributes to the electric field $\mathbf{E}$ and influences the energy of charged particles in the presence of an electromagnetic field. $\square$