Fundamentals of Analysis II: Homework 7

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Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

Exercise 6.2.1. Let

$$f_n(x) = \frac{nx}{1+nx^2}.$$

(i) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

(ii) is the convergence uniform on $(0, \infty)$?

(iii) Is the convergence uniform on $(0, 1)$?

(iv) Is the convergence uniform on $(1, \infty)$?

Solution to (i). Taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x^2} = \frac{x}{x^2} = \frac{1}{x}.$$ □

Solution to (ii). Define $x_n = 1/\sqrt{n}$. Then

$$f_n(x_n) = \frac{n \cdot \frac{1}{\sqrt{n}}}{1+n\left(\frac{1}{n}\right)} = \frac{\sqrt{n}}{1+1} = \frac{\sqrt{n}}{2}.$$

Computing the difference,

$$|f_n(x_n) - f(x_n)| = \left|\frac{\sqrt{n}}{2} - \sqrt{n}\right| = \left|\sqrt{n}\left(\frac{1}{2}-1\right)\right| = \frac{\sqrt{n}}{2}.$$

Since $\sqrt{n}/2 \to \infty$ as $n \to \infty$, for any fixed $\varepsilon > 0$, we can always find an x_n such that

$$|f_n(x_n) - f(x_n)| \geq 1.$$

Hence, the convergence is not uniform on $(0, \infty)$. □

Solution to (iii). No. Suppose for contradiction that (f_n) converges uniformly to f on $(0, 1)$. Then for any $\varepsilon > 0$, there must exist an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in (0, 1)$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Define $x_n = 1/\sqrt{n}$. Then,

$$f_n(x_n) = \frac{n \cdot (1/\sqrt{n})}{1+n\,(1/n)} = \frac{\sqrt{n}}{1+1} = \frac{\sqrt{n}}{2}.$$

Computing the difference,

$$|f_n(x_n) - f(x_n)| = \left|\frac{\sqrt{n}}{2} - \sqrt{n}\right| = \left|\sqrt{n}\left(\frac{1}{2}-1\right)\right| = \frac{\sqrt{n}}{2}.$$

Since $\sqrt{n}/2 \to \infty$ as $n \to \infty$, we conclude that for any fixed $\varepsilon > 0$, there exists an x_n such that

$$|f_n(x_n) - f(x_n)| \geq 1.$$

Hence, the convergence is not uniform on $(0, 1)$. □

Solution to (iv). Yes. Let $\varepsilon > 0$. Choose $N > 1/\varepsilon$. Then for all $n \geq N$, we have

$$|f_n(x) - f(x)| = \frac{1}{x(1+nx^2)} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$ □

Exercise 6.2.3. For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1+x^n} \quad \text{and} \quad h(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ nx & \text{if } 0 \leq x < \frac{1}{n} \end{cases}.$$

Answer the following questions for the sequences (g_n) and (h_n):

(i) Find the pointwise limit on $[0, \infty)$.

(ii) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

(iii) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution to (i). Finding the pointwise limit,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & \text{if } x \in [0,1) \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$\text{and} \quad \lim_{n\to\infty} h_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}. \qquad \square$$

Solution to (ii). It would contract Theorem 6.2.6, since both g_n and h_n are continuous, but the limit functions aren't. $\square$

Solution to (iii). For h_n: Let $\varepsilon > 0$. Choose $N > \varepsilon$. Then, for all $n \geq N$ on the interval $[1, \infty)$, we have

$$|h_n(x) - h(x)| = |1 - 1| = 0 < \varepsilon.$$

For g_n: Let $\varepsilon > 0$. Choose $N > \log_t(\varepsilon)$. Then, for all $n \geq N$ on the interval $[0, 1)$, we have

$$|g_n(x) - g(x)| = \left| \frac{x}{1+x^n} - x \right| = \left| \frac{x^{n+1}}{1+x^n} \right| < \left| t^{n+1} \right| < \varepsilon. \qquad \square$$

Exercise 7.5.8. Let

$$L(x) = \int_1^x \frac{1}{t} \, \mathrm{d}t,$$

where we consider only $x > 0$.

(i) What is $L(1)$? Explain why L is differentiable and find $L'(x)$.

(ii) Show that $L(xy) = L(x) + L(y)$. (Think of y as a constant and differentiate $g(x) = L(xy)$.)

(iii) Show $L(x/y) = L(x) - L(y)$.

Solution to (i). Using the Fundamental Theorem of Calculus, part (i), we have

$$L(1) = \int_1^1 \frac{1}{t} \, \mathrm{d}t = \ln(|t|) \bigg|_1^1 = \ln(1) - \ln(1) = 0.$$

Since the integrand $1/t$ is continuous on $(0, \infty)$, L is differentiable on $(0, \infty)$. By the Fundamental Theorem of Calculus, part (ii), we have

$$L'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_1^x \frac{1}{t} \, \mathrm{d}t = \frac{1}{x}. \qquad \square$$

Solution to (ii). Let y be a constant. Then, define

$$g(x) = L(xy) = \int_1^{xy} \frac{1}{t}\,\mathrm{d}t.$$

Differentiating,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_1^{xy} \frac{1}{t}\,\mathrm{d}t = \frac{y}{xy} = \frac{1}{x}.$$

This means $g'(x) = L'(x) \Rightarrow g(x) = L(x) + C$. To find C, let $x = 1$ and evaluate, giving us

$$g(1) = L(1) + C \Rightarrow L(y) = 0 + C \Rightarrow C = L(y).$$

Hence, we have

$$g(x) = L(xy) = L(x) + L(y).$$

□

Solution to (iii). Let y be a constant. Then, define

$$g(x) = L\left(\frac{x}{y}\right) = \int_1^{\frac{x}{y}} \frac{1}{t}\,\mathrm{d}t.$$

Differentiating,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_1^{\frac{x}{y}} \frac{1}{t}\,\mathrm{d}t = \frac{1/y}{x/y} = \frac{1}{x}.$$

This means $g'(x) = L'(x) \Rightarrow g(x) = L(x) + C$. To find C, let $x = y$ and evaluate, giving us

$$g(y) = L(1) = L(y) + C \Rightarrow 0 = L(y) + C \Rightarrow C = -L(y).$$

Hence, we have

$$g(x) = L\left(\frac{x}{y}\right) = L(x) - L(y).$$

□

Exercise S.1. Let g be a continuous function and h is a differentiable function. Show that the integral below defines a differentiable function and find the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{h(x)} g.$$

Solution. Define the function

$$F(x) = \int_a^{h(x)} g(t)\,\mathrm{d}t.$$

Since g is continuous and h is differentiable, from the Fundamental Theorem of Calculus and the Chain Rule, we have

$$F'(x) = g(h(x)) \cdot h'(x).$$

Hence, F is differentiable, since it's the product of two differentiable functions. □

Exercise S.2. Let g be a continuous function on $\mathbb{R}$. Show that each integral below is differentiable and compute the derivative

$$\text{(i)}\quad \frac{\mathrm{d}}{\mathrm{d}x}\int_{x-1}^{x+1} g \qquad\qquad \text{(ii)}\quad \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x} g(t-x)\,\mathrm{d}t.$$

Solution to (i). Define the function

$$F(x) = \int_{x-1}^{x+1} g(t)\,\mathrm{d}t.$$

Since g is continuous, from the Fundamental Theorem of Calculus, we have

$$F'(x) = g(x+1) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x+1) - g(x-1) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x-1) = g(x+1) - g(x-1).$$

Again, like the previous problem, F is differentiable, since it's the difference of two differentiable composite functions. $\square$

Solution to (ii). Define the function

$$F(x) = \int_{0}^{x} g(t-x)\,\mathrm{d}t.$$

Since g is continuous, from the Fundamental Theorem of Calculus and the Chain Rule, we have

$$F'(x) = g(x-x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x-x) - g(0-x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(0-x) = g(-x).$$

Hence, F is differentiable. $\square$