Differential Geometry: Homework 3

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Exercise 2.3.4. Construct a diffeomorphism between the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the sphere $x^2 + y^2 + z^2 = 1$.

Solution. Let S_1 be the ellipsoid and S_2 be the sphere. We can define the following parameterizations $\mathbf{x}_1 : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ and $\mathbf{x}_2 : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$, defined by

$$\mathbf{x}_1 = \langle a \sin(\varphi) \cos(\theta), b \sin(\varphi) \sin(\theta), c \cos(\varphi) \rangle \quad \text{and} \quad \mathbf{x}_2 = \langle \sin(\varphi) \cos(\theta), \sin(\varphi) \sin(\theta), \cos(\varphi) \rangle ,$$

where $\mathbf{x}_1$ is the parameterization of the ellipsoid and $\mathbf{x}_2$ is the parameterization of the sphere. The diffeomorphism $\Phi : S_1 \to S_2$ can be defined as $\Phi(x, y, z) = \left(\mathbf{x}_2 \circ \mathbf{x}_1^{-1}\right)(x, y, z)$. We're trying to go from the ellipsoid to the sphere, and both are given in terms of the same angles, just scaled differently. All we are doing is just dividing by the semi-axes, giving us

$$\Phi(x, y, z) = \left\langle \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right\rangle .$$

Now, we show that $\Phi(x, y, z)$ is a diffeomorphism between S_1 and S_2. It is clear that Φ is a smooth map since it is composed of smooth functions. The inverse $\Phi^{-1} : S_2 \to S_1$ is given by

$$\Phi(x, y, z) = \langle ax, by, cz \rangle .$$

Again, Φ^{-1} is smooth since it is composed of smooth functions.

Therefore, Φ is a diffeomorphism between the ellipsoid S_1 and the sphere S_2. □

Exercise 2.3.7. Prove that the relation "S_1 is diffeomorphism to S_2" is an equivalence relation in the set of regular surfaces.

Solution. For the relation "S_1 is diffeomorphic to S_2" to be an equivalence relation, it must satisfy three properties: reflexivity, symmetry, and transitivity.

We first prove reflexivity. For any regular surface S, we can define the identity map $I_S : S \to S$ given by $I_S(x) = x$ for all $x \in S$. The identity map is clearly a diffeomorphism since it is smooth and has a smooth inverse (itself). Thus, S is diffeomorphic to itself, satisfying the reflexivity property.

Next, we prove symmetry. If Φ is a diffeomorphic mapping from S_1 to S_2, then, there exists a smooth inverse Φ^{-1} from S_2 to S_1. By definition, Φ^{-1} is also a diffeomorphism. Therefore, if S_1 is diffeomorphic to S_2 via Φ, then S_2 is diffeomorphic to S_1 via Φ^{-1}, satisfying the symmetry property.

Finally, we prove transitivity. If S_1 is diffeomorphic to S_2 via Φ_{12} and S_2 is diffeomorphic to S_3 via Φ_{23}, then we can define a new map $\Phi_{13} = \Phi_{12} \circ \Phi_{23}$. The composition of two diffeomorphisms is also a diffeomorphism, as it is smooth and has a smooth inverse given by $(\Phi_{23})^{-1} \circ (\Phi_{12})^{-1}$. Thus, S_1 is diffeomorphic to S_3 via Φ_{13}, satisfying the transitivity property.

Since the relation satisfies reflexivity, symmetry, and transitivity, we can conclude that the relation "S_1 is diffeomorphic to S_2" is indeed an equivalence relation in the set of regular surfaces. □

Exercise 2.3.14. Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is, $A = U \cap S$, where U is an open set in $\mathbb{R}^3$.

Solution. Assume A is a regular subsurface of S, with ∂_A being the boundary of A. Suppose A is closed. Let $p \in \partial_A \subset A$. Since A is a regular surface, that means there exists a neighborhood $V_\delta(p)$, where $\delta > 0$, such that there is a differentiable, homeomorphic mapping from an open set $U \subset \mathbb{R}^2$ to $V_\delta(p) \cap A$. Since $p \in \partial_A$, there doesn't exist such a neighborhood around p that's contained in A, giving us a contradiction. Therefore, A must be open in S.

Conversely, assume A is open in S. This means that for every point $p \in A$, there exists a neighborhood $V_\delta(p)$ such that $V_\delta(p) \cap S \subset A$. Since S is a regular surface, we can find a differentiable, homeomorphic mapping from an open set $U \subset \mathbb{R}^2$ to $V_\delta(p) \cap S$. The restriction mapping $\mathbf{x} \mid_S : U \to V_\delta(p) \cap S$ is also a differentiable, homeomorphic mapping, which means that A is a regular surface.

Thus, A is a regular surface if and only if it is open in S. □

Exercise 2.4.2. Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z-axis.

Solution. The surface is given by $f(x, y, z) = x^2 + y^2 - z^2 - 1 = 0$. The gradient of f is given by

$$\nabla f(x, y, z) = \langle 2x, 2y, -2z \rangle .$$

Evaluating the normal vector at $(x, y, 0)$ gives us

$$\nabla f(x, y, 0) = \langle 2x, 2y, 0 \rangle .$$

The equation of the tangent plane at a point $(x_0, y_0, 0)$ on the surface is given by

$$\nabla f(x_0, y_0, 0) \cdot \langle x - x_0,\ y - y_0,\ z - 0 \rangle = 0,$$

which becomes

$$\langle 2x_0, 2y_0, 0 \rangle \cdot \langle x - x_0,\ y - y_0,\ z \rangle = 0.$$

Computing the dot product,

$$2x_0(x - x_0) + 2y_0(y - y_0) = 0,$$

or equivalently,

$$x_0 x + y_0 y = x_0^2 + y_0^2.$$

Since $(x_0, y_0, 0)$ lies on the surface, we know that $x_0^2 + y_0^2 = 1$, so the tangent plane becomes

$$x_0 x + y_0 y = 1.$$

This equation is independent of z, so the tangent plane is vertical and thus parallel to the z-axis. □

Exercise 2.4.8. Prove that if $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and $S \subset \mathbb{R}^3$ is a regular surface invariant under L, i.e., $L(S) \subset S$, then the restriction $L \mid_S$ is a differential map and

$$\mathrm{d}L_p(\mathbf{w}) = L(\mathbf{w}), \quad p \in S,\ \mathbf{w} \in T_p(S).$$

Solution. Assume $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and $S \subset \mathbb{R}^3$ is a regular surface invariant under L, meaning $L(S) \subset S$. Define the restriction map $L \mid_S : S \to S$. We want to show that $L \mid_S$ is a differentiable map and that its differential satisfies $\mathrm{d}L_p(\mathbf{w}) = L(\mathbf{w})$ for all $p \in S$ and $\mathbf{w} \in T_p(S)$.

Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a local parametrization of S. Then the composition $L \circ \mathbf{x} : U \to \mathbb{R}^3$ is differentiable, since L is linear (hence smooth) and $\mathbf{x}$ is smooth by definition. Therefore, $L \mid_S$ is differentiable.

Since L is linear and differentiable on all of $\mathbb{R}^3$, its differential at any point $p \in \mathbb{R}^3$ is $\mathrm{d}L_p = L$. The differential of the restriction $L \mid_S$ at a point $p \in S$ is the restriction of $\mathrm{d}L_p$ to the tangent space $T_p(S) \subset \mathbb{R}^3$. Hence, for all $\mathbf{w} \in T_p(S)$, we have

$$\mathrm{d}L_p(\mathbf{w}) = L(\mathbf{w}).$$

□

Exercise 2.4.9. Show that the parametrized surface

$$\mathbf{x}(u, v) = \langle v\cos(u), v\sin(u), au \rangle, \quad a \neq 0,$$

is regular. Compute its normal vector $\mathbf{N}(u, v)$ and show that along the coordinate line $u = u_0$, the tangent plane of $\mathbf{x}$ rotates about this line in such a way that the tangent of its angle with the z-axis is proportional to the inverse of the distance $v = \sqrt{x^2 + y^2}$ of the point $\mathbf{x}(u_0, v)$ to the z-axis.

Solution. To show that the parametrized surface

$$\mathbf{x}(u,v) = \langle v\cos(u), v\sin(u), au\rangle, \quad a \neq 0,$$

is regular, we first compute the partial derivatives

$$\mathbf{x}_u(u,v) = \langle -v\sin(u), v\cos(u), a\rangle \quad \text{and} \quad \mathbf{x}_v(u,v) = \langle \cos(u), \sin(u), 0\rangle.$$

The cross product of these two vectors gives the normal vector

$$\begin{aligned}\mathbf{N}(u,v) &= \mathbf{x}_u \times \mathbf{x}_v \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -v\sin(u) & v\cos(u) & a \\ \cos(u) & \sin(u) & 0 \end{vmatrix} \\ &= \langle -a\sin(u), -a\cos(u), -v\rangle.\end{aligned}$$

Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent for all (u,v) (note $a \neq 0$ and v arbitrary), the surface is regular. Now, fix $u = u_0$. Along the coordinate line $u = u_0$, the tangent plane is spanned by $\mathbf{x}_u(u_0,v)$ and $\mathbf{x}_v(u_0,v)$. Its normal vector is

$$\mathbf{N}(u_0,v) = \langle -a\sin(u_0), -a\cos(u_0), -v\rangle.$$

The z-axis is the vector $\hat{\mathbf{k}} = \langle 0,0,1\rangle$. Let θ be the angle between the normal vector $\mathbf{N}$ and the z-axis. Then,

$$\cos(\theta) = \frac{\mathbf{N}\cdot\hat{\mathbf{k}}}{\|\mathbf{N}\|\cdot\|\hat{\mathbf{k}}\|} = -\frac{v}{\sqrt{a^2+v^2}}.$$

Then

$$\tan(\theta) = \frac{\sqrt{a^2}}{v} = \frac{|a|}{v}.$$

Now, the distance to the z-axis is

$$\sqrt{x^2+y^2} = \sqrt{v^2\cos^2(u) + v^2\sin^2(u)} = v.$$

Therefore,

$$\tan(\theta) = \frac{|a|}{v} \propto \frac{1}{\sqrt{x^2+y^2}}.$$

Thus, the tangent plane rotates about the coordinate line $u = u_0$ in such a way that the tangent of the angle between the normal vector and the z-axis is proportional to the inverse of the distance to the z-axis. □

Exercise 2.4.15. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Solution. Let $\mathbf{x}(u,v)$ be a smooth, regular parametrized surface, and suppose that all normals to the surface pass through a fixed point $p \in \mathbb{R}^3$. That is, at every point $\mathbf{x}(u,v)$ on the surface, the vector from $\mathbf{x}(u,v)$ to p is parallel to the normal vector $\mathbf{N}(u,v)$, $p - \mathbf{x}(u,v) = \lambda(u,v)\mathbf{N}(u,v)$, for some scalar function $\lambda(u,v)$. By assumption, this vector is always normal to the surface, so we can write

$$\mathbf{F}(u,v) = \lambda(u,v)\mathbf{N}(u,v).$$

Since $\mathbf{N}(u,v)$ is normal to the surface, it is orthogonal to all tangent vectors

$$\mathbf{N}\cdot\mathbf{x}_u = 0, \qquad \mathbf{N}\cdot\mathbf{x}_v = 0.$$

Take the derivative of both sides of $\mathbf{F} = \lambda \mathbf{N}$ with respect to u

$$-\mathbf{x}_u = \lambda_u \mathbf{N} + \lambda \frac{\partial \mathbf{N}}{\partial u}.$$

Now take the dot product with $\mathbf{x}_u$

$$-\mathbf{x}_u \cdot \mathbf{x}_u = \left(\lambda_u \mathbf{N} + \lambda \frac{\partial \mathbf{N}}{\partial u} \right) \cdot \mathbf{x}_u.$$

But $\mathbf{N} \cdot \mathbf{x}_u = 0$, so

$$-\mathbf{x}_u \cdot \mathbf{x}_u = \lambda \frac{\partial \mathbf{N}}{\partial u} \cdot \mathbf{x}_u.$$

Similarly, differentiating with respect to v gives

$$-\mathbf{x}_v \cdot \mathbf{x}_v = \lambda \frac{\partial \mathbf{N}}{\partial v} \cdot \mathbf{x}_v.$$

However, we do not need to compute these expressions further. Instead, consider the squared length of the vector $\mathbf{F}(u, v)$

$$\|p - \mathbf{x}(u, v)\|^2 = \|\lambda(u, v)\mathbf{N}(u, v)\|^2 = \lambda(u, v)^2.$$

So define the scalar function $f(u, v) = \|p - \mathbf{x}(u, v)\|^2$. We then compute its partial derivatives, to get

$$f_u = 2(p - \mathbf{x}(u, v)) \cdot (-\mathbf{x}_u) = -2\mathbf{F} \cdot \mathbf{x}_u.$$

Since $\mathbf{F} = \lambda \mathbf{N}$ and $\mathbf{N} \cdot \mathbf{x}_u = 0$, we get

$$f_u = -2\lambda \mathbf{N} \cdot \mathbf{x}_u = 0.$$

Similarly, $f_v = -2\lambda \mathbf{N} \cdot \mathbf{x}_v = 0$. Since $\nabla f = 0$, then $f(u, v) = \text{const}$. So

$$\|p - \mathbf{x}(u, v)\|^2 = r^2,$$

for some constant $r > 0$, and hence the image of $\mathbf{x}(u, v)$ lies entirely on the sphere of radius r centered at p, $\mathbf{x}(u, v) \in S(p, r)$, for all (u, v) in the connected domain of definition of $\mathbf{x}$.

Therefore, if all normals to a connected surface pass through a fixed point, then the surface lies on a sphere centered at that point. □

Exercise 2.4.24. Show that if $\Phi : S_1 \to S_2$ and $\Psi : S_2 \to S_3$ are differential maps and $p \in S_1$, then

$$\mathrm{d}(\Psi \circ \Phi)_p = \mathrm{d}\Psi_{\Phi(p)} \circ \mathrm{d}\Phi_p.$$

Solution. Define $\beta : (-\varepsilon, \varepsilon) \to S_1$ to be a smooth curve such that $\beta(0) = p$ and $\beta'(0) = \mathbf{w}$. Define $\mathbf{x} = \Phi \circ \beta : (-\varepsilon, \varepsilon) \to S_2$, which is a smooth curve in S_2 with $\mathbf{x}(0) = \Phi(p)$ and $\mathbf{x}'(0) = \mathrm{d}\Phi_p(\mathbf{w})$. Now consider $\Psi \circ \mathbf{x} = \Psi \circ \Phi \circ \beta$, which is a smooth curve in S_3 passing through $\Psi(\Phi(p)) = (\Psi \circ \Phi)(p)$ at $t = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\Psi \circ \Phi \circ \beta)(t) = \mathrm{d}(\Psi \circ \Phi)_p(\mathbf{w}).$$

On the other hand, since $\mathbf{x} = \Phi \circ \beta$, and Ψ is differentiable, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\Psi \circ \mathbf{x})(t) = \mathrm{d}\Psi_{\Phi(p)}(\mathbf{x}'(0)) = \mathrm{d}\Psi_{\Phi(p)}(\mathrm{d}\Phi_p(\mathbf{w})).$$

Therefore,

$$\mathrm{d}(\Psi \circ \Phi)_p(\mathbf{w}) = \mathrm{d}\Psi_{\Phi(p)}(\mathrm{d}\Phi_p(\mathbf{w})) = (\mathrm{d}\Psi_{\Phi(p)} \circ \mathrm{d}\Phi_p)(\mathbf{w}),$$

which proves the chain rule for differentiable maps between surfaces. □

Exercise 2.5.1. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

(i) $\mathbf{x}(u,v) = \langle a\sin(u)\cos(v), b\sin(u)\sin(v), c\cos(u)\rangle$; ellipsoid.

(ii) $\mathbf{x}(u,v) = \langle au\cos(v), bu\sin(v), u^2\rangle$; elliptic paraboloid.

(iii) $\mathbf{x}(u,v) = \langle au\cosh(v), bu\sinh(v), u^2\rangle$; hyperbolic paraboloid.

(iv) $\mathbf{x}(u,v) = \langle a\sinh(u)\cos(v), b\sinh(u)\sin(v), c\cosh(u)\rangle$; hyperboloid of two sheets.

Solution to (i). Computing the partial derivatives, we have

$$\mathbf{x}_u = \langle a\cos(u)\cos(v), b\cos(u)\sin(v), -c\sin(u)\rangle \quad \text{and} \quad \mathbf{x}_v = \langle -a\sin(u)\sin(v), b\sin(u)\cos(v), 0\rangle.$$

Computing E, F, and G, we have

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle \\
&= a^2\cos^2(u)\cos^2(v) + b^2\cos^2(u)\sin^2(v) + c^2\sin^2(u) \\
&= \cos^2(u)(a^2\cos^2(v) + b^2\sin^2(v)) + c^2\sin^2(u) \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\
&= -a^2\cos(u)\cos(v)\sin(u)\sin(v) + b^2\cos(u)\sin(v)\sin(u)\cos(v) \\
&= \cos(u)\sin(u)(-a^2 + b^2)\cos(v)\sin(v) \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle \\
&= a^2\sin^2(u)\sin^2(v) + b^2\sin^2(u)\cos^2(v) \\
&= \sin^2(u)(a^2\sin^2(v) + b^2\cos^2(v)).
\end{aligned}$$

□

Solution to (ii). Computing the partial derivatives, we have

$$\mathbf{x}_u = \langle a\cos(v), b\sin(v), 2u\rangle \quad \text{and} \quad \mathbf{x}_v = \langle -au\sin(v), bu\cos(v), 0\rangle.$$

Computing E, F, and G, we have

$$\begin{aligned}
E &= a^2\cos^2(v) + b^2\sin^2(v) + 4u^2 \\
F &= -a^2u\cos(v)\sin(v) + b^2u\sin(v)\cos(v) \\
&= u(b^2 - a^2)\sin(v)\cos(v) \\
G &= a^2u^2\sin^2(v) + b^2u^2\cos^2(v) \\
&= u^2(a^2\sin^2(v) + b^2\cos^2(v)).
\end{aligned}$$

□

Solution to (iii). Computing the partial derivatives, we have

$$\mathbf{x}_u = \langle a\cosh(v), b\sinh(v), 2u\rangle \quad \text{and} \quad \mathbf{x}_v = \langle au\sinh(v), bu\cosh(v), 0\rangle.$$

Computing E, F, and G, we have

$$\begin{aligned}
E &= a^2\cosh^2(v) + b^2\sinh^2(v) + 4u^2 \\
F &= a^2u\cosh(v)\sinh(v) + b^2u\sinh(v)\cosh(v) \\
&= u(a^2 + b^2)\sinh(v)\cosh(v) \\
G &= a^2u^2\sinh^2(v) + b^2u^2\cosh^2(v) \\
&= u^2(a^2\sinh^2(v) + b^2\cosh^2(v)).
\end{aligned}$$

□

Solution to (iv). Computing the partial derivatives, we have

$$\mathbf{x}_u = \langle a\cosh(u)\cos(v), b\cosh(u)\sin(v), c\sinh(u)\rangle \quad \text{and} \quad \mathbf{x}_v = \langle -a\sinh(u)\sin(v), b\sinh(u)\cos(v), 0\rangle.$$

Computing E, F, and G, we have

$$\begin{aligned} E &= a^2\cosh^2(u)\cos^2(v) + b^2\cosh^2(u)\sin^2(v) + c^2\sinh^2(u) \\ &= \cosh^2(u)(a^2\cos^2(v) + b^2\sin^2(v)) + c^2\sinh^2(u) \\ F &= -a^2\cosh(u)\sinh(u)\cos(v)\sin(v) + b^2\cosh(u)\sinh(u)\sin(v)\cos(v) \\ &= \cosh(u)\sinh(u)(-a^2 + b^2)\cos(v)\sin(v) \\ G &= a^2\sinh^2(u)\sin^2(v) + b^2\sinh^2(u)\cos^2(v) \\ &= \sinh^2(u)(a^2\sin^2(v) + b^2\cos^2(v)). \end{aligned}$$

□

Exercise 2.5.5. Show that the area A of a bounded region R of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\,\mathrm{d}y,$$

where Q is the normal projection of R onto the xy-plane.

Solution. Let $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$ be a regular surface. Computing the cross product, we have

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle.$$

Therefore, we have

$$A = \iint_Q \|\mathbf{r}_x \wedge \mathbf{r}_y\|\,\mathrm{d}x\,\mathrm{d}y = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

□

Exercise 2.5.11. Let S be a surface of revolution and C its generating curve (cf. Example 4, Sec. 2-3). Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s.

(i) *(Pappus' Theorem.)* Show that the area of S is

$$2\pi \int_0^l \rho(s)\,\mathrm{d}s,$$

where l is the length of C.

(ii) Apply part (i) to compute the area of a torus of revolution.

Solution to (i). Every point $p \in C$ traces out a circle of radius $\rho(s)$ as it revolves around the axis of rotation. Let C be the curve parametrized by $C = \langle x(s), y(s), z(s)\rangle$. Without loss of generality, we can assume that the axis of rotation will be the z-axis. Therefore, we get the surface $\mathbf{r} : [0, l] \times [0, 2\pi] \to S$, defined by $\mathbf{r}(s, \theta) = \langle \rho(s)\cos(\theta), \rho(s)\sin(\theta), z(s)\rangle$. The surface area of a given surface is given by

$$A = \iint_S \|\mathbf{r}_s \wedge \mathbf{r}_\theta\|\,\mathrm{d}s\,\mathrm{d}\theta.$$

Computing the partial derivatives, we have

$$\mathbf{r}_s = \langle \rho'(s)\cos(\theta), \rho'(s)\sin(\theta), z'(s)\rangle \quad \text{and} \quad \mathbf{r}_\theta = \langle -\rho(s)\sin(\theta), \rho(s)\cos(\theta), 0\rangle.$$

Computing the cross product, we have

$$\|\mathbf{r}_s \times \mathbf{r}_\theta\| = \rho(s)\sqrt{(z'(s))^2 + (\rho'(s))^2} = \rho(s),,$$

since $(z')^2 + (\rho')^2 = 1$, due to arc length parametrization. Therefore, we have

$$A = \iint_S \|\mathbf{r}_s \wedge \mathbf{r}_\theta\|\,\mathrm{d}s\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^l \rho(s)\,\mathrm{d}s\,\mathrm{d}\theta = 2\pi\int_0^l \rho(s)\,\mathrm{d}s.$$

□

Solution to (ii). The generating curve of a torus is a circle of radius r centered at the origin, which is given by the parametrization, $C : [0, 2\pi] \to \mathbb{R}^2$, defined by

$$C(\theta) = \langle R + r\cos(\theta), R + r\sin(\theta)\rangle.$$

This curve is parametrized by angle, not arc length. Since it's a circle, we know its total length is $l = 2\pi r$. Without loss of generality, we can assume that the axis of rotation is the z-axis. Then, we get $\rho(\theta) = R + r\cos(\theta)$.

But we want the arc length parametrization, so we must write everything in terms of s. Fortunately, for a circle of radius r, the natural arc length parametrization is $\theta = s/r$, where $s \in [0, 2\pi r]$. Now, applying Pappus' theorem, we have

$$A = 2\pi \int_0^{2\pi r} \rho(s)\,\mathrm{d}s = 2\pi \int_0^{2\pi r} R + r\cos\left(\frac{s}{r}\right)\mathrm{d}s = 4\pi^2 Rr. \qquad \square$$