Ex: Find a vector orthogonal to the plane that contains $P(1,1,2)$, $Q(4,-1,3)$, and $R(5,3,1)$.

Three points lie in a plane. If the points do not lie along a line, then the plane containing the points is unique.

$\vec{PQ} = \langle 3,-2,1\rangle$ and $\vec{PR} = \langle 4,2,-1\rangle$

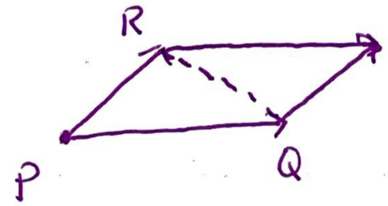
are vectors in the plane.

$\vec{PQ} \times \vec{PR}$ is orthogonal to both $\vec{PQ}$ and $\vec{PR}$ and is orthogonal to plane containing points.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$= \langle 2-2, -(-3-4), 6+8\rangle$$

$$= \langle 0, 7, 14\rangle$$



Using $\vec{PQ}$ and $\vec{PR}$,

define a parallelogram.

Then its area $A = |\vec{PQ} \times \vec{PR}|$

Or dividing along diagonal,

the area of triangle with

vertices P, Q, and R is

$\frac{1}{2} |\vec{PQ} \times \vec{PR}|$

Area of triangle is $A = \frac{1}{2}\sqrt{7^2 + 14^2}$

$= \frac{7}{2}\sqrt{5}$

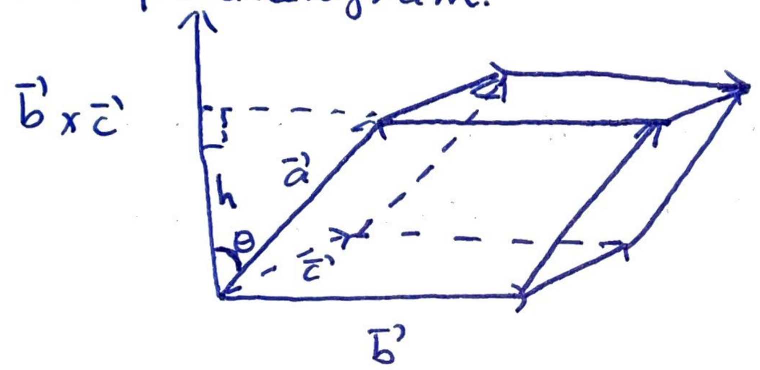
Scalar Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$.

$\vec{a} \cdot (\vec{b} \times \vec{c})$ is the scalar triple product

Consider parallelepiped determined by $\vec{a}, \vec{b}$, and $\vec{c}$

A parallelepiped is a solid where each of the six sides a parallelogram.



The volume is the area of the parallelogram base multiplied by height (distance between top & bottom).

$$V = Ah$$

Let θ be angle between $\vec{a}$ and $\vec{b} \times \vec{c}$

Then $|\cos\theta| = \dfrac{h}{|\vec{a}|}$ and $h = |\vec{a}|\,|\cos\theta|$

Area of parallelogram base is $A = |\vec{b} \times \vec{c}|$

$$V = Ah = |\vec{b} \times \vec{c}|\,|\vec{a}|\,|\cos\theta|$$

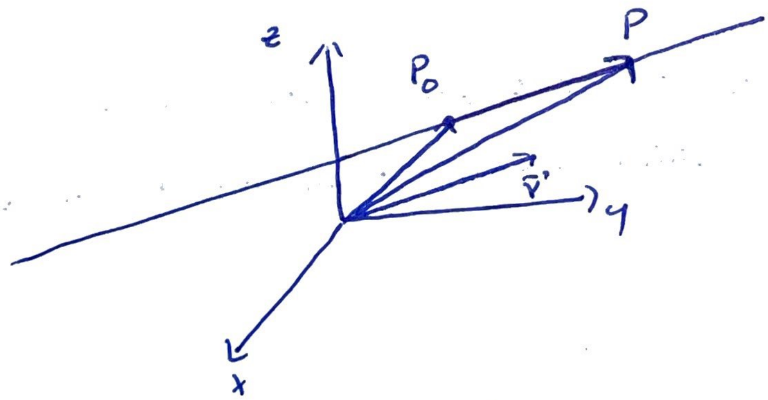
$$= \left|\, |\vec{b} \times \vec{c}|\,|\vec{a}| \cos\theta \,\right| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$ Absolute value of scalar triple product

Note: If $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, then $\vec{a}$ and $\vec{b} \times \vec{c}$ are orthogonal. Therefore $\vec{a}, \vec{b}, \vec{c}$ lie in a plane and are called <u>coplanar</u>.

§12.5: Lines and Planes

Lines

A line in $\mathbb{R}^3$ is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\vec{v} = \langle a, b, c \rangle$ parallel to the line.



Let $P(x, y, z)$ be any point on line.

$\vec{OP} = \vec{OP_0} + \vec{P_0P}$

Since $\vec{P_0P}$ and $\vec{v}$ are parallel, there exists a scalar t such that

$\vec{P_0P} = t\vec{v}$.

Let $\vec{OP} = \langle x, y, z \rangle = \vec{r}$ and $\vec{OP_0} = \langle x_0, y_0, z_0 \rangle = \vec{r_0}$

Then $\vec{OP} = \vec{OP_0} + t\vec{v}$

$\vec{r}(t) = \vec{r_0} + t\vec{v}$ is vector equation of line.