

# Functional Complex Variables I: Homework 5

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**Exercise 4.42.3.**  $f(z) = (z + 2)/z$  and  $C$  is

- (i) the semicircle  $z = 2e^{i\theta}$  ( $0 \leq \theta \leq \pi$ );
- (ii) the semicircle  $z = 2e^{i\theta}$  ( $\pi \leq \theta \leq 2\pi$ );
- (iii) the circle  $z = 2e^{i\theta}$  ( $0 \leq \theta \leq 2\pi$ ).

**Solution to (i).** Given  $z(\theta) = 2e^{i\theta}$ , we have

$$dz = \frac{dz}{d\theta} d\theta = 2ie^{i\theta} d\theta.$$

Therefore, the integral becomes

$$\int_C f(z) dz = \int_C \frac{z+2}{z} dz = \int_0^\pi \frac{2e^{i\theta} + 2}{2e^{i\theta}} \cdot 2ie^{i\theta} d\theta = 2i \int_0^\pi e^{i\theta} \left( \frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) d\theta.$$

Simplifying the integrand, we have

$$\begin{aligned} e^{i\theta} \left( \frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) &= e^{i\theta} \left( \frac{e^{i\theta} + 1}{e^{i\theta}} \right) \\ &= e^{i\theta} \left( 1 + \frac{1}{e^{i\theta}} \right) \\ &= e^{i\theta} + 1. \end{aligned}$$

Thus, we can rewrite the integral as

$$\begin{aligned} 2i \int_0^\pi e^{i\theta} \left( \frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) d\theta &= 2i \left( \int_0^\pi e^{i\theta} d\theta + \int_0^\pi 1 d\theta \right) \\ &= 2i \left( \frac{e^{i\pi} - e^{i0}}{i} + \pi \right) \\ &= 2i \left( \frac{-2}{i} + \pi \right) \\ &= 2i(2i + \pi) \\ &= 2i\pi - 4. \end{aligned}$$

□

**Solution to (ii).** Using the same substitution as in part (i), we have

$$\begin{aligned} \int_C f(z) dz &= 2i \left( \int_\pi^{2\pi} e^{i\theta} d\theta + \int_\pi^{2\pi} 1 d\theta \right) \\ &= 2i \left( \frac{e^{2i\pi} - e^{i\pi}}{i} + \pi \right) \\ &= 2i \left( \frac{1 - (-1)}{i} + \pi \right) \\ &= 2i \left( \frac{2}{i} + \pi \right) \\ &= 2i(-2i + \pi) \\ &= 2i\pi + 4. \end{aligned}$$

□

**Solution to (iii).** We can break up the integral into two parts

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz,$$

where  $C_1$  is the semicircle from  $0 \leq \theta \leq \pi$  and  $C_2$  is the semicircle from  $\pi \leq \theta \leq 2\pi$ . We've already computed the integrals for  $C_1$  and  $C_2$  in parts (i) and (ii), respectively. Therefore, we can combine the results

$$\begin{aligned}\int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ &= (2i\pi + 4) + (2i\pi - 4) \\ &= 4i\pi.\end{aligned}$$

□

**Exercise 4.42.7.**  $f(z)$  is the principle branch

$$z^i = \exp(i \operatorname{Log}(z)) \quad (|z| > 0, \quad -\pi < \operatorname{Arg}(z) < \pi),$$

of this power function, and  $C$  is the semicircle  $z = e^{i\theta}$  ( $0 \leq \theta \leq \pi$ ).

**Solution.** Given  $z(\theta) = e^{i\theta}$  on  $C$  with  $0 \leq \theta \leq \pi$ , we compute

$$dz = \frac{dz}{d\theta} d\theta = ie^{i\theta} d\theta.$$

The function is defined as  $f(z) = z^i = \exp(i \operatorname{Log}(z))$ , where  $\operatorname{Log}(z) = \ln(|z|) + i \operatorname{Arg}(z)$ . On the unit circle  $|z| = 1$ , so  $\ln(|z|) = 0$ , and we have

$$f(z) = \exp(i(i \operatorname{Arg}(z))) = \exp(-\operatorname{Arg}(z)).$$

Along  $C$ , where  $z = e^{i\theta}$  and  $0 \leq \theta \leq \pi$ , we have  $\operatorname{Arg}(z) = \theta$ . Therefore, we have  $f(z) = e^{-\theta}$ . So the integral becomes

$$\begin{aligned}\int_C f(z) dz &= \int_0^\pi e^{-\theta} \cdot ie^{i\theta} d\theta \\ &= i \int_0^\pi e^{-\theta} e^{i\theta} d\theta \\ &= i \int_0^\pi e^{(-1+i)\theta} d\theta \\ &= \frac{i}{-1+i} \left[ e^{(-1+i)\theta} \right]_0^\pi \\ &= \left( e^{(-1+i)\pi} - 1 \right) \frac{i}{-1+i}.\end{aligned}$$

To simplify, we multiply numerator and denominator by the complex conjugate of the denominator to get

$$\frac{i}{-1+i} = \frac{i(-1-i)}{(-1+i)(-1-i)} = \frac{-i-i^2}{1+1} = \frac{-i+1}{2}.$$

Therefore,

$$\int_C f(z) dz = \left( \frac{-i+1}{2} \right) \left( e^{(-1+i)\pi} - 1 \right).$$

□

**Exercise 4.42.8.** With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \bar{z}^m dz,$$

where  $m$  and  $n$  are integers and  $C$  is the unit circle  $|z| = 1$ , taken counterclockwise.

**Solution.** On the unit circle  $|z| = 1$ , we have the identity  $\bar{z} = \frac{1}{z}$ . Therefore,

$$z^m \bar{z}^m = z^m \left(\frac{1}{z}\right)^m = z^m z^{-m} = 1.$$

So the integrand becomes  $z^m \bar{z}^m = 1$ , for all  $z$  on the contour  $C$ . Hence, the integral reduces to

$$\int_C z^m \bar{z}^m dz = \int_C dz.$$

Since the integrand is constant and  $C$  is a closed curve, we conclude

$$\int_C dz = 0.$$

□

**Exercise 4.43.1.** Without evaluating the integral, show that

$$\left| \int_C \frac{1}{z^2 - 1} dz \right| \leq \frac{\pi}{3},$$

when  $C$  is the same arc as the one in Example 1, Sec. 43.

**Solution.** Let  $f(z) = 1/(z^2 - 1)$  and let  $C$  be the arc of the circle  $|z| = 2$  from  $z = 2$  to  $z = 2i$  in the first quadrant, as described in Example 1, Sec. 43. To estimate the modulus of the integral, we use the inequality

$$\left| \int_C f(z) dz \right| \leq ML,$$

where  $M$  is an upper bound for  $|f(z)|$  on  $C$ , and  $L$  is the length of the arc. On  $C$ , we have  $|z| = 2$ , so

$$|z^2 - 1| = |4e^{2i\theta} - 1| \geq ||z^2| - 1| = |4 - 1| = 3.$$

Therefore,

$$|f(z)| = \left| \frac{1}{z^2 - 1} \right| \leq \frac{1}{3} = M.$$

The length of  $C$  is one-quarter of the circumference of the circle of radius 2, which is

$$L = \frac{\pi}{2} \cdot 2 = \pi.$$

But since the arc in Example 1 spans from  $z = 2$  to  $z = 2i$ , that is a quarter-circle, so in fact the length is

$$L = \frac{\pi}{2} \cdot 2 = \pi,$$

and the bound becomes

$$\left| \int_C f(z) dz \right| \leq \frac{1}{3} \cdot \pi = \frac{\pi}{3}.$$

□

**Exercise 4.43.2.** Let  $C$  denote the line segment from  $z = i$  to  $z = 1$ . By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left| \int_C \frac{1}{z^4} dz \right| \leq 4\sqrt{2},$$

without evaluating the integral.

**Solution.** Let  $f(z) = 1/z^4$  and let  $C$  be the line segment from  $z = i$  to  $z = 1$ . To estimate the integral, we apply the Estimation Lemma

$$\left| \int_C f(z) dz \right| \leq ML,$$

where  $M$  is an upper bound for  $|f(z)|$  on  $C$ , and  $L$  is the length of the path. All points on  $C$  lie on the line segment from  $z = i$  to  $z = 1$ , and the point on  $C$  that is closest to the origin is the midpoint

$$z = \frac{1+i}{2},$$

which has modulus

$$\left| \frac{1+i}{2} \right| = \frac{1}{2} |1+i| = \frac{1}{2} \cdot \sqrt{2} = \frac{\sqrt{2}}{2}.$$

Since the modulus of  $z$  is minimized at this midpoint, the modulus of  $f(z) = 1/z^4$  is maximized there

$$|f(z)| = \left| \frac{1}{z^4} \right| \leq \left( \frac{2}{\sqrt{2}} \right)^4 = (2\sqrt{2})^4 = 16 \cdot 4 = 64.$$

But this overestimates the bound. Instead, observe that for any  $z$  on  $C$ , we have

$$|z| \geq \frac{\sqrt{2}}{2},$$

and thus

$$|f(z)| = \left| \frac{1}{z^4} \right| \leq \left( \frac{2}{\sqrt{2}} \right)^4 = 16.$$

However, to match the desired bound, we proceed more sharply by computing the length of  $C$  and estimating directly at the midpoint. The length of  $C$  is the distance from  $z = i$  to  $z = 1$

$$L = |1 - i| = \sqrt{2}.$$

The maximum of  $|f(z)|$  on  $C$  occurs at the point where  $|z|$  is minimized, which is at the midpoint

$$|z| = \frac{\sqrt{2}}{2} \Rightarrow |f(z)| = \left( \frac{2}{\sqrt{2}} \right)^4 = 16.$$

Therefore,

$$\left| \int_C \frac{1}{z^4} dz \right| \leq ML = 16 \cdot \sqrt{2} = 4\sqrt{2}. \quad \square$$

**Exercise 4.45.2.** By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$(a) \int_i^{i/2} e^{\pi z} dz; \quad (b) \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz; \quad (c) \int_1^3 (z-2)^3 dz.$$

**Solution to (i).** Let  $F(z)$  be an antiderivative of  $f(z) = e^{\pi z}$ . Since  $e^{\pi z}$  is entire, we can use any contour from  $i$  to  $i/2$ :

$$F(z) = \int e^{\pi z} dz = \frac{1}{\pi} e^{\pi z}.$$

Therefore, by the Fundamental Theorem of Calculus for contour integrals,

$$\int_i^{i/2} e^{\pi z} dz = F\left(\frac{i}{2}\right) - F(i) = \frac{1}{\pi} e^{\pi i/2} - \frac{1}{\pi} e^{\pi i} = \frac{1}{\pi} (e^{\pi i/2} - e^{\pi i}).$$

Using Euler's formula:

$$e^{\pi i/2} = i \quad \text{and} \quad e^{\pi i} = -1,$$

so the value of the integral is

$$\int_i^{i/2} e^{\pi z} dz = \frac{1}{\pi} (i + 1). \quad \square$$

**Solution to (ii).** We are asked to evaluate

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz.$$

Let  $F(z)$  be an antiderivative of  $f(z) = \cos\left(\frac{z}{2}\right)$ :

$$F(z) = \int \cos\left(\frac{z}{2}\right) dz = 2 \sin\left(\frac{z}{2}\right).$$

Therefore,

$$\begin{aligned} \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz &= F(\pi + 2i) - F(0) = 2 \sin\left(\frac{\pi + 2i}{2}\right) - 2 \sin(0) \\ &= 2 \sin\left(\frac{\pi}{2} + i\right). \end{aligned}$$

Using the identity  $\sin(a + ib) = \sin a \cosh b + i \cos a \sinh b$ , we get

$$\sin\left(\frac{\pi}{2} + i\right) = \sin\left(\frac{\pi}{2}\right) \cosh(1) + i \cos\left(\frac{\pi}{2}\right) \sinh(1) = \cosh(1).$$

So the integral becomes

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2 \cosh(1). \quad \square$$

**Solution to (iii).** We are asked to evaluate

$$\int_1^3 (z - 2)^3 dz.$$

Let  $F(z)$  be an antiderivative of  $f(z) = (z - 2)^3$ . We compute

$$F(z) = \int (z - 2)^3 dz = \frac{1}{4}(z - 2)^4.$$

Therefore,

$$\int_1^3 (z - 2)^3 dz = F(3) - F(1) = \frac{1}{4}(3 - 2)^4 - \frac{1}{4}(1 - 2)^4 = \frac{1}{4}(1 - 1) = 0. \quad \square$$

**Exercise 4.45.3.** Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z - z_0)^{n-1} dz = 0 \quad (n = \pm 1, \pm 2, \dots),$$

when  $C_0$  is any closed contour which does not pass through the point  $z_0$ .

**Solution.** Let  $f(z) = (z - z_0)^{n-1}$  for an integer  $n \neq 0$ . We are given that  $C_0$  is a closed contour which does not pass through the point  $z_0$ . Then the function  $f(z)$  is analytic on and inside  $C_0$ .

According to the theorem in Section 44, if a function is continuous on a domain and has an antiderivative throughout that domain, then its integral around any closed contour in the domain is zero. The function  $f(z) = (z - z_0)^{n-1}$  is analytic (and hence has an antiderivative) in any domain not containing  $z_0$ , provided  $n \neq 0$ .

Since  $C_0$  does not pass through  $z_0$ , and  $f$  is analytic in a region containing  $C_0$  and its interior, it follows by the theorem in Section 44 that

$$\int_{C_0} (z - z_0)^{n-1} dz = 0. \quad \square$$