Midterm 1 Solution

Fall 2024

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1. (5 pts) State the ϵ-N definition of the limit of sequence $\lim_{n\to\infty} a_n = a$.

   Solution. For every $\varepsilon > 0$, there is N in $\mathbb{N}$ such that, for all $n > N$, $|a_n - a| < \varepsilon$. $\square$

2. (5 pts each) Give an example for each statement and provide a short justification:

   (i) A set S in $\mathbb{R}$ that is bounded from below but $\inf(S) \notin S$.

   Solution. Here are two examples: $S = (0, 1)$ and $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. In both cases $\inf(S) = 0 \notin S$. $\square$

   (ii) A sequence that is bounded but does not converge.

   Solution. Here is one example: $\{(-1)^n : n \in \mathbb{N}\}$. $\square$

   (iii) A function $f : A \mapsto B$ that is not onto, where A and B are sets in $\mathbb{R}$.

   Solution. Here is an example: $f : \mathbb{N} \to \mathbb{R}$, given by $f(n) = (-1)^n$. $\square$

3. (10 pts) Let s be an upper bound of a set A in $\mathbb{R}$. If, for every $\varepsilon > 0$, there is an $a \in A$ such that $s - a < \varepsilon$, prove that s is the least upper bound of A.

   Solution. The definition that $s = \sup A$ requires: (1) s is an upper bound; (2) If b is an upper bound of A, then $s \le b$. We need to verify (2).

   Let b be an upper bound of A and assume that $s > b$. Then $s - b > 0$. Choosing $\varepsilon = s - b > 0$. By assumption, there is an $a \in A$ such that

   $$s - a < \varepsilon = s - b.$$

   This shows that

   $$b < a \quad \text{for some } a \in A,$$

   which contradicts b being an upper bound of A. Hence, $s \le b$. This proves $s = \sup A$. $\square$

4. (10 pts) Using the definition of the limit to verify that

   $$\lim_{n\to\infty} \frac{4}{3n+7} = 0.$$

   Solution. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{4}{3\varepsilon}$ (for example, $N = \lfloor \frac{4}{3\varepsilon} \rfloor + 1$), then for all $n > N$,

   $$\left| \frac{4}{3n+7} - 0 \right| = \frac{4}{3n+7} < \frac{4}{3n} < \frac{4}{3N} < \varepsilon.$$

   $\square$

5. (10 pts) Do **ONLY ONE** of the following problems.

   (a) If $\lim_{n\to\infty} b_n = b$ and $b \neq 0$. Prove that there is a positive real number $c > 0$ and an $N \in \mathbb{N}$ such that $|b_n| \ge c$ for all $n > N$.

Solution. Since $b \neq 0$, $|b| > 0$. Choose $\varepsilon > 0$ such that $\varepsilon < |b|$ (for example, $\varepsilon = |b|/2 > 0$). By the definition of limit, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$|b_n - b| < \varepsilon.$$

Let $c = |b| - \varepsilon > 0$. By the triangle inequality, for $n > N$,

$$|b_n| = |b_n - b + b| \geq |b| - |b_n - b| > |b| - \varepsilon = c.$$

□

(b) Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove, by definition, that

$$\lim_{n\to\infty} (2a_n - 3b_n) = 2a - 3b.$$

Solution. Let $\varepsilon > 0$. There is N_1 such that, for all $n > N_1$,

$$|a_n - a| < \varepsilon/4.$$

and there is N_2 such that, for all $n > N_2$,

$$|b_n - b| < \varepsilon/6.$$

Let $N = \max\{N_1, N_2\}$. Then, for $n > N$, by the triangle inequality,

$$\begin{aligned} |(2a_n - 3b_n) - (2a - 3b)| &= |2(a_n - a) - 3(b_n - b)| \\ &\leq 2|a_n - a| + 3|b_n - b| \leq 2\frac{\varepsilon}{4} + 3\frac{\varepsilon}{6} = \varepsilon. \end{aligned}$$

This proves the stated limit identity. □