§14.5: Chain Rule

Recall, if $y = f(x)$ and $x = g(t)$, then $y = f(g(t))$ is a composite function of t and by Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Theorem: Suppose that $z = f(x,y)$ is a differentiable function of x and y where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t.

Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Ordinary derivatives

Partial Derivatives

Ex: Suppose $z = x^3y^2 - 2xy^4$ where $x = \ln t$ and $y = \sin(2t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (3x^2y^2 - 2xy^4)\left(\frac{1}{t}\right) + (2x^3y - 8xy^3)(2\cos(2t))$$

Suppose $z = f(x,y)$ where $x = g(u,v)$ and $y = h(u,v)$

then $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Ex! Suppose $Z = e^{-x/y}$ where $x = u\cos v$ and $y = u^3 v^2$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \left(e^{-x/y}\frac{-1}{y}\right)(\cos v) + \left(e^{-\frac{x}{y}}\frac{x}{y^2}\right)(3u^2v^2)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \left(e^{-x/y}\frac{-1}{y}\right)(-u\sin v) + \left(e^{-\frac{x}{y}}\frac{x}{y^2}\right)(2u^3v)$$

In general, if $z = f(x_1, x_2, \dots, x_n)$ is differentiable

where $x_i = x_i(t_1, t_2, \dots, t_m)$ $i = 1, 2, \dots, n$ are differentiable,

then
$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$i = 1, 2, \dots, m$

Ex: The temperature of a metal plate at (x,y) where x and y are measured in cm, is given by

$$T(x,y) = \frac{30}{x^2+y^2+1} \quad (°C).$$ Suppose a bug walks along the path $x(t) = 0.01t^2+1$ and $y(t) = 0.001t^3 - 0.2t$ where t is measured in seconds. Find the rate of change of temperature with respect to time as the bug passes through $(2,-1)$.

$T = T(x(t), y(t))$ composite function of time.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= \frac{-60x}{(x^2+y^2+1)^2}(0.02t) + \frac{-60y}{(x^2+y^2+1)^2}(0.003t^2 - 0.2)$$

$$\frac{°C}{cm}\ \frac{cm}{s} + \frac{°C}{cm}\ \frac{cm}{s} \rightarrow \frac{°C}{s}$$

when $x = 0.01t^2+1 = 2 \Rightarrow t = 10$

$y = 0.001t^3 - 0.2t = -1$

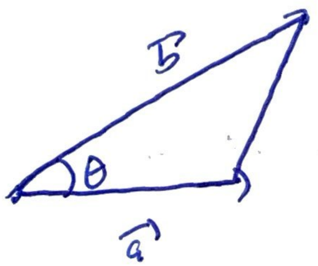
At $(2,-1)$

$$\frac{dT}{dt} = \frac{-120}{36}(0.2) + \frac{60}{36}(0.1)$$

$$= \frac{-240}{360} + \frac{60}{360} = \frac{-180}{360} = \frac{-1}{2}\ \frac{°C}{s}$$

At $(2,-1)$, the temp is decreasing at a rate of $\frac{1}{2}$ °C per second along the bug's path of travel.

Ex: One side of a triangle is increasing at a rate of $3\ \frac{cm}{s}$ and a second side is decreasing at a rate of $2\ \frac{cm}{s}$. If the area of the triangle is constant, at what rate does the angle between the sides change when the first side is 20 cm, the second side is 30 cm and the angle is $\frac{\pi}{6}$?



Let $a = |\vec{a}|$ cm

$b = |\vec{b}|$ cm

Let θ be angle between $\vec{a}$ and $\vec{b}$ radians

Let A be area of triangle cm^2

Find $\frac{d\theta}{dt}$ when $\frac{da}{dt} = 3$, $a = 20$, $\frac{db}{dt} = -2$, $b = 30$, and $\theta = \frac{\pi}{6}$ given $\frac{dA}{dt} = 0$ (constant area).

$$A = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$$

$$A = \frac{1}{2}ab\sin\theta \quad \text{where } a = a(t),\ b = b(t),\ \theta = \theta(t)$$

By Chain Rule,

$$\frac{dA}{dt} = \frac{\partial A}{\partial a}\frac{da}{dt} + \frac{\partial A}{\partial b}\frac{db}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \left(\frac{1}{2}b\sin\theta\right)\frac{da}{dt} + \left(\frac{1}{2}a\sin\theta\right)\frac{db}{dt} + \left(\frac{1}{2}ab\cos\theta\right)\frac{d\theta}{dt}$$

$$0 = \left(\frac{1}{2}\right)(30)\left(\frac{1}{2}\right)(3) + \left(\frac{1}{2}\right)(20)(-2)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(20)(30)\left(\frac{\sqrt{3}}{2}\right)\frac{d\theta}{dt}$$

$$= \frac{45}{2} - \frac{20}{2} + 150\sqrt{3}\frac{d\theta}{dt}$$

$$-\frac{25}{2} = 150\sqrt{3}\frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{-1}{12\sqrt{3}}\ \frac{rad}{s}$$