Several-Variab Calc II: Homework 2

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Jennifer Thorenson

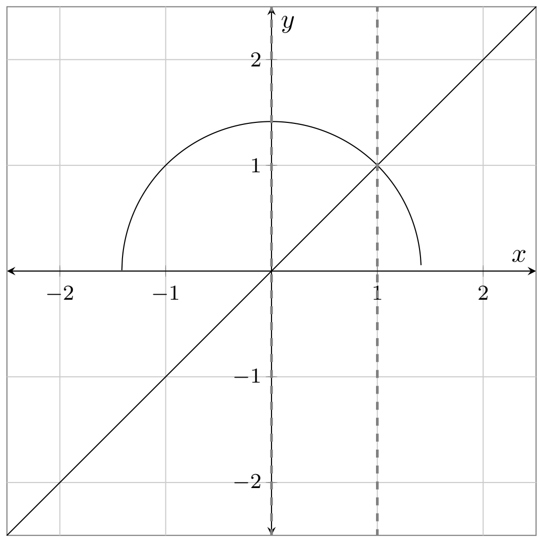
Hashem A. Damrah
UO ID: 952102243

Problem 1. Use polar coordinates to evaluate the following integrals.

(i) $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{y^2}{x^2+y^2} \,\mathrm{d}y \,\mathrm{d}x.$

(ii) $\displaystyle\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2+y^2} \,\mathrm{d}y \,\mathrm{d}x.$

Solution to (i). Graphing the bounds gives us

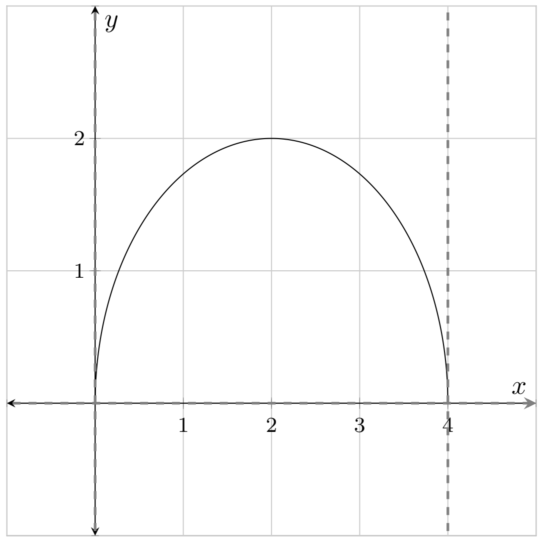


From this, we can tell what our θ bounds are $\pi/4 \leq \theta \leq \pi/2$. The bounds for r are $0 \leq r \leq \sqrt{2}$. Converting the function $f(x,y) = \frac{y^2}{x^2+y^2}$ to polar gives us $f(r,\theta) = \frac{r^2 \sin^2(\theta)}{r^2} = \sin^2(\theta)$. Expanding and evaluating the double integral gives us

$$\begin{aligned}
V = \iint_D \frac{y^2}{x^2+y^2} \,\mathrm{d}A &= \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \sin^2(\theta) \cdot r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \sin^2(\theta) \cdot r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/2} \left. \frac{r^2 \sin^2(\theta)}{2} \right|_0^{\sqrt{2}} \mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{2\sin^2(\theta)}{2} \,\mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1-\cos(2\theta)}{2} \,\mathrm{d}\theta \\
&= \left. \frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right|_{\pi/4}^{\pi/2} \\
&= \left[\frac{\pi/2}{2} - \frac{\sin(2-\pi/4)}{4} \right] - \left[\frac{\pi/4}{2} - \frac{\sin(2-\pi/4)}{4} \right] \\
&= \frac{\pi}{4} - \frac{\pi}{8} + \frac{1}{4} \\
&= \frac{\pi}{8} + \frac{1}{4}.
\end{aligned}$$

□

Solution to (ii). Graphing the bounds gives us



Clearly, from the graph, we see the bounds of θ to be $0 \leq \theta \leq {}^{\pi}/_{2}$. Converting the function $y = \sqrt{4x - x^2}$ to polar gives us

$$\begin{aligned} y^2 &= 4x - x^2 \Rightarrow 4 = (x-2)^2 + y^2 \\ 4 &= (r\cos(\theta) - 2)^2 + r^2 \sin^2(\theta) \\ 4 &= r^2\cos^2(\theta) - 4r\cos(\theta) + 4 + r^2\sin^2(\theta) \\ r^2 &= 4r\cos(\theta) \\ r &= 4\cos(\theta). \end{aligned}$$

This gives us the bounds for r as $0 \leq r \leq 4\cos(\theta)$. Converting the function $f(x, y) = \sqrt{x^2 + y^2}$ to polar gives us $f(r, \theta) = \sqrt{r^2} = r$. Expanding and evaluating the double integral gives us

$$\begin{aligned} V = \iint_D f(x, y)\,\mathrm{d}A &= \int_0^{\pi/2} \int_0^{4\cos(\theta)} r \cdot r\,\mathrm{d}r\,\mathrm{d}x \\ &= \int_0^{\pi/2} \left.\frac{r^3}{3}\right|_0^{4\cos(\theta)} \mathrm{d}\theta \\ &= \int_0^{\pi/2} \frac{64\cos^3(\theta)}{3}\,\mathrm{d}\theta \\ &= \frac{1}{3} \cdot \int_0^{\pi/2} 64\cos^2(\theta)\cos(\theta)\,\mathrm{d}\theta \\ &= \frac{1}{3} \cdot \int_0^{\pi/2} 64(1 - \sin^2(\theta))\cos(\theta)\,\mathrm{d}\theta. \end{aligned}$$

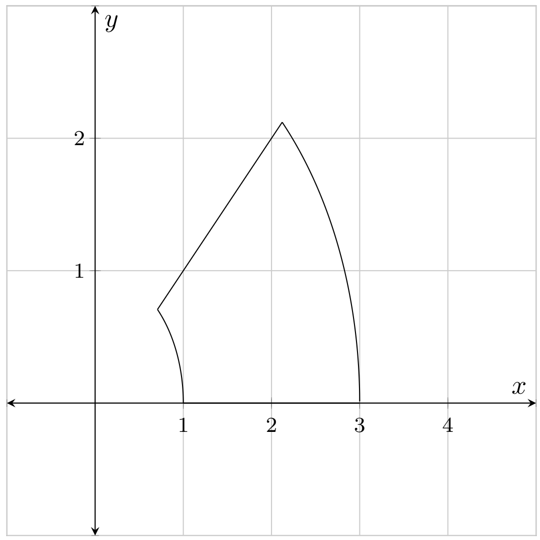
Using u-sub, let $u = \sin(\theta)$, which gives us $\mathrm{d}u = \cos(\theta)\,\mathrm{d}\theta$. Changing the bounds gives us $u(0) = 0$ and $u({}^{\pi}/_{2}) = 1$. Then, we get

$$\begin{aligned} \frac{1}{3} \cdot \int_0^{\pi/2} 64(1 - \sin^2(\theta))\cos(\theta)\,\mathrm{d}\theta &= \frac{1}{3} \cdot \int_0^1 64(1 - u^2)\,\mathrm{d}u \\ &= \frac{1}{3} \cdot \int_0^1 64 - 64u^2\,\mathrm{d}u \\ &= \frac{1}{3} \cdot \left[64u - \frac{64u^3}{3}\right]_0^1 = \frac{1}{3} \cdot \left(64 - \frac{64}{3}\right) = \frac{128}{9}. \qquad \square \end{aligned}$$

Problem 2. Use polar coordinates to rewrite the sum as a single iterated integral and then evaluate the integral.

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x}\frac{1}{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x+\int_{1}^{3/\sqrt{2}}\int_{0}^{x}\frac{1}{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x+\int_{3/\sqrt{2}}^{3}\int_{0}^{\sqrt{9-x^2}}\frac{1}{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x.$$

Solution. Graphing the bounds and removing all the redundant integrals gives us



Notice that this is just the area between the two circles $x^2+y^2=1$ and $x^2+y^2=9$. Since the left hand side is stopped by the line $y=x$, we get the angle to be from 0 to $\pi/4$. Converting the function $f(x,y)=\frac{1}{x^2+y^2}$ to polar gives us $f(r,\theta)=\frac{1}{r^2}$. Expanding and evaluating the double integral gives us

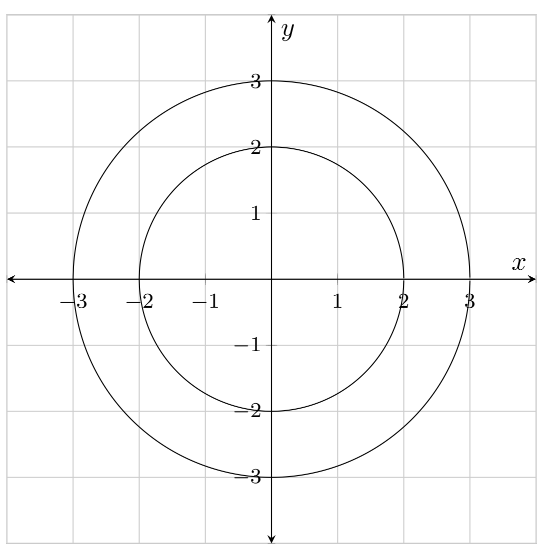
$$V=\iint_D f(x,y)\,\mathrm{d}A=\int_0^{\pi/4}\int_1^3\frac{1}{r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^{\pi/4}\mathrm{d}\theta\cdot\int_1^3\frac{1}{r}\,\mathrm{d}r=\frac{\pi}{4}\cdot(\ln(r))_1^3=\frac{\pi\ln(3)}{4}.\qquad\square$$

Problem 3. Use a double integral to find the volume of the following solids.

(i) The solid that is inside the sphere $x^2+y^2+z^2=9$ and outside the circular cylinder $x^2+y^2=4$.

(ii) The solid that is bounded by the elliptic paraboloids $z=x^2+3y^2$ and $z=16-3x^2-y^2$.

Solution to (i). Graphing the bounds gives us



Notice that the solid is symmetric about the z-axis, so I'll just solve for the top half and multiply by 2 at the end. The sphere has a radius of 3 and the circular cylinder has a radius of 2 which gives us our r bounds to be from $2 \leq r \leq 3$. We want to integrate over the entire thing. This gives us our θ angles to be $0 \leq \theta \leq 2\pi$. Solving for z gives us $z = \sqrt{9 - x^2 - y^2}$ and converting to polar coordinates gives us $z = \sqrt{9 - r^2}$. Expanding and evaluating the double integral gives us

$$V = \iint_D z \, \mathrm{d}A = 2 \int_0^{2\pi} \int_2^3 \sqrt{9 - r^2} \cdot r \, \mathrm{d}r \, \mathrm{d}\theta.$$

Using u-sub, let $u = 9 - r^2$, which gives us $\mathrm{d}u = -2r \, \mathrm{d}r$. Changing the bounds gives us $u(2) = 5$ and $u(3) = 0$. Then, we get

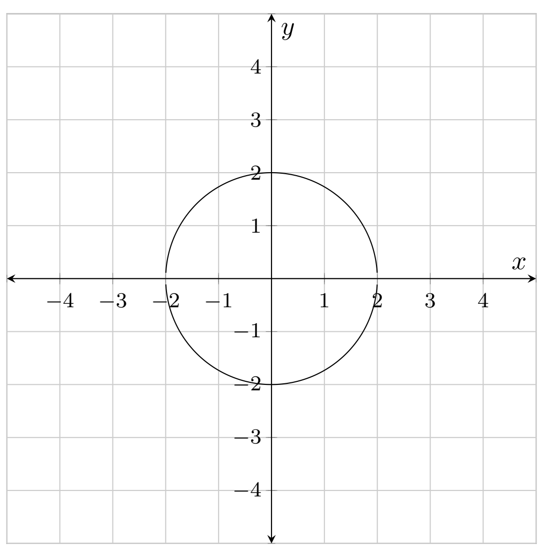
$$\begin{aligned}
2 \int_0^{2\pi} \int_2^3 \sqrt{9 - r^2} \cdot r \, \mathrm{d}r \, \mathrm{d}\theta &= 2 \int_0^{2\pi} \frac{1}{2} \int_0^5 \sqrt{u} \, \mathrm{d}u \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^5 u^{1/2} \, \mathrm{d}u \\
&= 2\pi \cdot \left(\frac{u^{3/2}}{\frac{3}{2}} \right)_0^5 \\
&= \frac{4\pi}{3} \cdot \left(5^{3/2} \right) \\
&= \frac{4\pi}{3} \cdot 5\sqrt{5} = \frac{20\sqrt{5}\pi}{3}.
\end{aligned}$$

□

Solution to (ii). First, we need to find where the two surfaces intersect

$$\begin{aligned}
& z &&= z \\
\Rightarrow \quad & x^2 + 3y^2 &&= 16 - 3x^2 - y^2 \\
\Rightarrow \quad & 4x^2 + 4y^2 &&= 16 \\
\Rightarrow \quad & x^2 + y^2 &&= 4.
\end{aligned}$$

This is just a circle of radius 2. Graphing the bounds gives us



The height of the solid is given by $z = z_T - z_B = \left(16 - 3x^2 - y^2\right) - \left(x^2 + 3^2\right) = 16 - 4x^2 - y^4$. Therefore, we get the bounds for r as $0 \leq r \leq 2$. The angle θ is from 0 to 2π. Converting the function $z = 16 - 4x^2 - y^2$

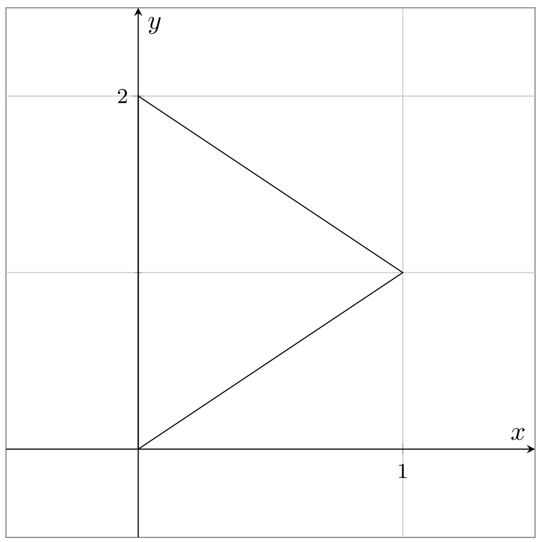
to polar gives us $z = 16 - 4r^2$. Expanding and evaluating the double integral gives us

$$\begin{aligned} V = \iint_D z \, \mathrm{d}A &= \int_0^{2\pi} \int_0^2 (16 - 4r^2) \cdot r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^2 16r - 4r^3 \, \mathrm{d}r \\ &= 2\pi \cdot \left(8r^2 - r^4\right)_0^2 \\ &= 2\pi \cdot \left(8(2)^2 - 2^4\right) \\ &= 2\pi \cdot 16 = 32\pi. \end{aligned}$$

□

Problem 4. Find the center of mass of the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,2)$ with density $\rho(x,y) = 3x + 2y$.

Solution. Graphing the bounds gives us



Then, from the graph, I'll be using a $\mathrm{d}y\,\mathrm{d}x$ integral, where $0 \leq x \leq 1$ and $x \leq y \leq -x + 2$. First, we need to find the mass

$$\begin{aligned} m = \iint_D \rho(x,y) \, \mathrm{d}A &= \int_0^1 \int_x^{-x+2} 3x + 2y \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 \left(3xy + y^2\right) \Big|_x^{-x+2} \mathrm{d}x \\ &= \int_0^1 \left[3x(-x+2) + (-x+2)^2\right] - \left[3x(x) + (x)^2\right] \mathrm{d}x \\ &= \int_0^1 -3x^2 + 6x + x^2 - 4x + 4 - 3x^2 - x^2 \, \mathrm{d}x \\ &= \int_0^1 -6x^2 + 2x + 4 \, \mathrm{d}x \\ &= \left[-2x^3 + x^2 + 4x\right]_0^1 = -2 + 1 + 4 = 3. \end{aligned}$$

Next, we need to find the center of mass by finding the x moment

$$M_x = \int_0^1 \int_x^{-x+2} 3xy + 2y^2 \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \left[\int_x^{-x+2} 3xy \, \mathrm{d}y + \int_x^{-x+2} 2y^2 \, \mathrm{d}y\right] \mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^1 \left[3x \int_x^{-x+2} y\,\mathrm{d}y + 2\int_x^{-x+2} y^2\,\mathrm{d}y\right] \mathrm{d}x \\
&= \int_0^1 \left[3x\left(\frac{(-x+2)^2}{2} - \frac{x^2}{2}\right) + 2\left(\frac{(-x+2)^3}{3} - \frac{x^3}{3}\right)\right] \mathrm{d}x \\
&= \int_0^1 \left[3x\left(\frac{x^2-4x+4}{2} - \frac{x^2}{2}\right) + 2\left(\frac{-x^3+6x^2-12x+8}{3} - \frac{x^3}{3}\right)\right] \mathrm{d}x \\
&= \int_0^1 \left[3x \cdot \frac{-4x+4}{2} + 2 \cdot \frac{-2x^3+6x^2-12x+8}{3}\right] \mathrm{d}x \\
&= \int_0^1 \left[-6x^2 + 6x + \frac{-4x^3+12x^2-24x+16}{3}\right] \mathrm{d}x \\
&= \int_0^1 \frac{-4x^3-18x^2-6x+16}{3}\,\mathrm{d}x \\
&= \frac{1}{3}\int_0^1 \left(-4x^3-18x^2-6x+16\right) \mathrm{d}x \\
&= \frac{1}{3}\left[\int_0^1 -4x^3\,\mathrm{d}x + \int_0^1 -18x^2\,\mathrm{d}x + \int_0^1 -6x\,\mathrm{d}x + \int_0^1 16\,\mathrm{d}x\right] \\
&= \frac{1}{3}\left[\left(-\frac{x^4}{1}\bigg|_0^1\right) + \left(-6x^3\bigg|_0^1\right) + \left(-3x^2\bigg|_0^1\right) + \left(16x\bigg|_0^1\right)\right] = \frac{10}{3},
\end{aligned}
$$

and the y moment

$$
\begin{aligned}
M_y = \int_0^1 \int_x^{-x+2} x(3x+2y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_x^{-x+2} (3x^2+2xy)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\int_x^{-x+2} 3x^2\,\mathrm{d}y + \int_x^{-x+2} 2xy\,\mathrm{d}y\right] \mathrm{d}x \\
&= \int_0^1 \left[3x^2 \int_x^{-x+2} \mathrm{d}y + 2x\int_x^{-x+2} y\,\mathrm{d}y\right] \mathrm{d}x \\
&= \int_0^1 \left[3x^2\left((-x+2)-x\right) + 2x\left(\frac{(-x+2)^2}{2} - \frac{x^2}{2}\right)\right] \mathrm{d}x \\
&= \int_0^1 \left[3x^2(-2x+2) + 2x\left(\frac{x^2-4x+4}{2} - \frac{x^2}{2}\right)\right] \mathrm{d}x \\
&= \int_0^1 \left[-6x^3 + 6x^2 + 2x \cdot \frac{-4x+4}{2}\right] \mathrm{d}x \\
&= \int_0^1 \left[-6x^3+6x^2-4x^2+4x\right] \mathrm{d}x \\
&= \int_0^1 \left[-6x^3+2x^2+4x\right] \mathrm{d}x \\
&= \left[-\frac{6x^4}{4} + \frac{2x^3}{3} + \frac{4x^2}{2}\right]_0^1 \\
&= \left[-\frac{3x^4}{2} + \frac{2x^3}{3} + 2x^2\right]_0^1 \\
&= \left(-\frac{3(1)^4}{2} + \frac{2(1)^3}{3} + 2(1)^2\right) - \left(-\frac{3(0)^4}{2} + \frac{2(0)^3}{3} + 2(0)^2\right) = \frac{7}{6}.
\end{aligned}
$$

Finally, we can find the center of mass

$$
(\bar{x},\bar{y}) = \left(\frac{7/6}{3}, \frac{10/3}{3}\right) = \left(\frac{7}{18}, \frac{10}{9}\right). \qquad \square
$$