Multi-Variable Calculus I: Homework 9

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Jennifer Thorenson 08:00

Hashem A. Damrah
UO ID: 952102243

Problem 1 Find the absolute maximum and minimum values of $f(x, y) = (x-1)y^2 - 2x$ on the set $D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 10\}$.

Solution. We start by finding the partial derivatives of $f(x, y)$

$$f_x(x, y) = \frac{\partial}{\partial x}\left[(x-1)y^2 - 2x\right] = y^2 - 2 \quad \text{and} \quad f_y(x, y) = \frac{\partial}{\partial y}\left[(x-1)y^2 - 2x\right] = 2(x-1)y.$$

Next, we set these partial derivatives equal to zero to find the critical points

$$f_x(x, y) = 0 \Rightarrow y^2 = 2 \Rightarrow y = \sqrt{2} \quad (\text{since } y \geq 0)$$
$$f_y(x, y) = 0 \Rightarrow 2(x-1)y = 0 \Rightarrow y = 0 \quad \text{or} \quad x = 1.$$

Combining these, the critical point inside D is $(x, y) = (1, \sqrt{2})$. Evaluating the function at this point, we find

$$f(1, \sqrt{2}) = (1-1)(\sqrt{2})^2 - 2(1) = -2.$$

Next, we examine the boundary of D, which consists of

(i) The line $x = 0$, where $0 \leq y \leq \sqrt{10}$,

On this boundary, $f(0, y) = (0-1)y^2 - 2(0) = -y^2$. Evaluating this, we find

$$f(0, 0) = 0 \quad \text{and} \quad f(0, \sqrt{10}) = -(\sqrt{10})^2 = -10.$$

(ii) The line $y = 0$, where $0 \leq x \leq \sqrt{10}$,

On this boundary, $f(x, 0) = (x-1)(0)^2 - 2x = -2x$. Evaluating this, we find

$$f(0, 0) = 0 \quad \text{and} \quad f(\sqrt{10}, 0) = -2\sqrt{10}.$$

(iii) The circular arc $x^2 + y^2 = 10$, where $x \geq 0$ and $y \geq 0$.

We parametrize the boundary using $x = \sqrt{10}\cos(\theta)$ and $y = \sqrt{10}\sin(\theta)$, where $0 \leq \theta \leq \frac{\pi}{2}$. Substituting into $f(x, y)$, we get

$$f(x, y) = \left(\sqrt{10}\cos(\theta) - 1\right)(\sqrt{10}\sin(\theta))^2 - 2\sqrt{10}\cos(\theta).$$

Simplifying, we find

$$f(x, y) = 10\left(\sqrt{10}\cos(\theta) - 1\right)\sin^2(\theta) - 2\sqrt{10}\cos(\theta).$$

To find the extrema on this arc, we examine $f(x, y)$ at key points

- At $\theta = 0$, $\cos(\theta) = 1$ and $\sin(\theta) = 0$, so

$$f(x, y) = -2\sqrt{10}.$$

- At $\theta = \frac{\pi}{2}$, $\cos(\theta) = 0$ and $\sin(\theta) = 1$, so

$$f(x, y) = -10.$$

Since $f(x, y)$ does not achieve any values larger than 0 or smaller than -10 on this arc, we conclude that no new extrema arise here.

Thus, the absolute extrema of $f(x, y)$ on the domain D are

(i) Absolute maximum: $f(0,0) = 0$,

(ii) Absolute minimum: $f(0, \sqrt{10}) = -10$. $\square$

Problem 2 Find the volume of the largest rectangular box in the first octant with three sides in the coordinate planes and one vertex in the plane $2x + 3y + z = 6$.

Complete this problem using two methods.

(i) Substitute the constraint into the volume function and apply second derivative test to verify it is a maximum.

(ii) Use the method of Lagrange Multipliers to find the maximum.

Solution to (i). Let the vertex of the rectangular box in the plane $2x + 3y + z = 6$ be (x, y, z), where $x, y, z \geq 0$. The volume of the box is

$$V(x, y, z) = xyz.$$

Since the vertex lies on the plane $2x + 3y + z = 6$, we substitute

$$z = 6 - 2x - 3y$$

into $V(x, y, z)$ to express the volume as

$$V(x, y) = xy(6 - 2x - 3y) = 6xy - 2x^2y - 3xy^2.$$

To find critical points, we compute the partial derivatives

$$\begin{aligned} V_x(x, y) &= \frac{\partial}{\partial x}\left(6xy - 2x^2y - 3xy^2\right) = 6y - 4xy - 3y^2 \\ V_y(x, y) &= \frac{\partial}{\partial y}\left(6xy - 2x^2y - 3xy^2\right) = 6x - 2x^2 - 6xy. \end{aligned}$$

Setting these equal to zero, we solve

$$\begin{aligned} V_x(x, y) = 0 &\Rightarrow y(6 - 4x - 3y) = 0 \\ V_y(x, y) = 0 &\Rightarrow x(3 - x - 3y) = 0. \end{aligned}$$

Since $x, y > 0$, we solve the nonzero factors

$$\begin{aligned} 6 - 4x - 3y = 0 &\Rightarrow y = 2 - \frac{4x}{3} \\ 3 - x - 3y = 0 &\Rightarrow x = 3 - 3y. \end{aligned}$$

Substituting $y = 2 - \frac{4x}{3}$ into $x = 3 - 3y$, we get

$$x = 3 - 3\left(2 - \frac{4x}{3}\right) = 3 - 6 + 4x = -3 + 4x \Rightarrow x = \frac{3}{4}.$$

Substituting $x = \frac{3}{4}$ into $y = 2 - \frac{4x}{3}$, we find

$$y = 2 - \frac{4\left(\frac{3}{4}\right)}{3} = 2 - \frac{4}{3} = \frac{2}{3}.$$

Finally, substituting $x = \frac{3}{4}$ and $y = \frac{2}{3}$ into $z = 6 - 2x - 3y$, we get

$$z = 6 - 2\left(\frac{3}{4}\right) - 3\left(\frac{2}{3}\right) = 6 - \frac{3}{2} - 2 = \frac{5}{2}.$$

The critical point is $(x, y, z) = \left(\frac{3}{4}, \frac{2}{3}, \frac{5}{2}\right)$. The volume is

$$V\left(\frac{3}{4}, \frac{2}{3}, \frac{5}{2}\right) = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{5}{2} = \frac{4}{3}.$$

To confirm this is a maximum, we examine the second derivatives:

$$V_{xx}(x, y) = -4y, \qquad V_{yy}(x, y) = -6x, \qquad V_{xy}(x, y) = 6 - 4x - 6y.$$

The Hessian determinant is

$$H = V_{xx}(x, y)V_{yy}(x, y) - (V_{xy}(x, y))^2 = (-4y)(-6x) - (6 - 4x - 6y)^2.$$

At $(x, y) = \left(\frac{3}{4}, \frac{2}{3}\right)$, this evaluates to

$$H = \left(-4 \cdot \frac{2}{3}\right)\left(-6 \cdot \frac{3}{4}\right) - \left(6 - 4 \cdot \frac{3}{4} - 6 \cdot \frac{2}{3}\right)^2 > 0,$$

confirming that $V(x, y)$ has a local maximum. Thus, the maximum volume is $\frac{4}{3}$ cubic units. □

Solution to (ii). Using the method of Lagrange multipliers, we maximize $V(x, y, z) = xyz$ subject to $g(x, y, z) = 2x + 3y + z - 6 = 0$. The gradients are

$$\nabla V = \langle yz, xz, xy \rangle \quad \text{and} \quad \nabla g = \langle 2, 3, 1 \rangle.$$

Setting $\nabla V = \lambda \nabla g$, we have

$$\begin{aligned} yz &= 2\lambda \\ xz &= 3\lambda \\ xy &= \lambda. \end{aligned}$$

From $yz = 2\lambda$ and $xy = \lambda$, we find

$$yz = 2xy \Rightarrow z = 2.$$

From $xz = 3\lambda$ and $xy = \lambda$, we find

$$xz = 3xy \Rightarrow z = 3y.$$

Equating $z = 2x$ and $z = 3y$, we find

$$2x = 3y \Rightarrow y = \frac{2x}{3}.$$

Substituting $y = \frac{2x}{3}$ and $z = 2x$ into $g(x, y, z) = 0$, we get

$$2x + 3\left(\frac{2x}{3}\right) + 2x = 6 \Rightarrow 6x = 6 \Rightarrow x = 1.$$

Substituting $x = 1$ into $y = \frac{2x}{3}$ and $z = 2x$, we find

$$y = \frac{2}{3}, \quad z = 2.$$

The maximum volume is

$$V(1, \frac{2}{3}, 2) = 1 \cdot \frac{2}{3} \cdot 2 = \frac{4}{3}.$$

Thus, the maximum volume is $\frac{4}{3}$ cubic units. □

Problem 3 Use the method of Lagrange Multipliers to find the maximum and minimum values of $f(x, y, z) = xy - z^2$ for all points on the ellipsoid $x^2 + 8y^2 + z^2 = 64$.

Solution. Using the method of Lagrange multipliers, we maximize and minimize $f(x, y, z) = xy - z^2$ subject to $g(x, y, z) = x^2 + 8y^2 + z^2 - 64 = 0$. The gradients are

$$\nabla f = \langle y, x, -2z \rangle \text{and} \nabla g = \langle 2x, 16y, 2z \rangle.$$

Setting $\nabla f = \lambda \nabla g$, we have the system

$$\begin{aligned} y &= 2\lambda x \\ x &= 16\lambda y \\ -2z &= 2\lambda z. \end{aligned}$$

From $-2z = 2\lambda z$, we find

$$z = 0 \quad \text{or} \quad \lambda = -1.$$

Case 1: $z = 0$. If $z = 0$, substituting into $y = 2\lambda x$ and $x = 16\lambda y$, we eliminate λ

$$y = 2\lambda x \quad \text{and} \quad x = 16\lambda y \Rightarrow y = 2\lambda(16\lambda y).$$

Simplifying, $y(1 - 32\lambda^2) = 0$. Thus, $y = 0$ or $\lambda = \pm\frac{1}{4}$.

(i) If $y = 0$, then $x = 0$, which is invalid as $x^2 + 8y^2 + z^2 = 64$ must hold.

(ii) If $\lambda = \pm\frac{1}{4}$, substituting back gives

$$y = \pm 2, \quad x = \pm 4\sqrt{2}.$$

The points are $(\pm 4\sqrt{2}, \pm 2, 0)$.
Case 2: $\lambda = -1$. If $\lambda = -1$, the equations become

$$y = -2x, \quad x = -16y.$$

Solving $y = -2x$ and $x = -16(-2x)$, we find

$$y = -2x, \quad x = \frac{1}{8}x \Rightarrow x = 0 \Rightarrow y = 0.$$

Substituting $x = 0$ and $y = 0$ into $x^2 + 8y^2 + z^2 = 64$, we find $z = \pm 8$. The points are $(0, 0, \pm 8)$.

– At $(4\sqrt{2}, 2, 0)$ and $(-4\sqrt{2}, -2, 0)$

$$f(x, y, z) = xy - z^2 = (4\sqrt{2})(2) - 0 = 8\sqrt{2}.$$

– At $(0, 0, 8)$ and $(0, 0, -8)$

$$f(x, y, z) = xy - z^2 = 0 - (8)^2 = -64.$$

Therefore, we get

(i) The maximum value of $f(x, y, z)$ is $8\sqrt{2}$, occurring at $(4\sqrt{2}, 2, 0)$ and $(-4\sqrt{2}, -2, 0)$.

(ii) The minimum value of $f(x, y, z)$ is -64, occurring at $(0, 0, 8)$ and $(0, 0, -8)$. $\square$