Fundamentals of Analysis II: Homework 3

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Exercise 5.2.4a. Show that a function $h : A \to \mathbb{R}$ is differentiable at $a \in A$ if and only if there exists a function $l : A \to \mathbb{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = l(x)(x - a) \quad \text{for all } x \in A.$$

Solution. Assume h is differentiable at a. Define

$$l(x) = \begin{cases} \frac{h(x) - h(a)}{x - a} & \text{if } x \neq a \\ h'(a) & \text{if } x = a \end{cases}.$$

Then l is continuous at a and satisfies the equation $h(x) - h(a) = l(x)(x - a)$ for all $x \in A$.

Conversely, assume there exists a function $l : A \to \mathbb{R}$ which is continuous at a and satisfies the equation $h(x) - h(a) = l(x)(x - a)$ for all $x \in A$. Rewriting the equation, we have

$$\frac{h(x) - h(a)}{x - a} = l(x),$$

where $x \neq a$. Taking the limit on both sides as $x \to a$ gives us $h'(a) = l(a)$. $\qquad\square$

Exercise 5.2.6b. Let g be defined on an interval A, and let $c \in A$. Assume A is open. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h \to 0} \frac{g(c + h) - g(c - h)}{2h}.$$

Solution. Rewriting the limit, we have

$$\lim_{h \to 0} \frac{g(c+h) - g(c) + g(c) - g(c-h)}{2h} = \frac{1}{2} \underbrace{\lim_{h \to 0} \frac{g(c+h) - g(c)}{h}}_{g'(c)} + \frac{1}{2} \underbrace{\lim_{h \to 0} \frac{g(c) - g(c-h)}{h}}_{g'(c) \text{ with } h = c - x} = \frac{1}{2} \cdot 2g'(c) = g'(c).$$

Thus, $g'(c) = \lim\limits_{h \to 0} \dfrac{g(c+h) - g(c-h)}{2h}$. $\qquad\square$

Exercise 5.2.7. Find a particular (potentially non integer) value for a so that

  (i) g_a is differentiable on $\mathbb{R}$ but such that g'_a is unbounded on $[0, 1]$.

  (ii) g_a is differentiable on $\mathbb{R}$ with g'_a continuous but not differentiable at zero.

  (iii) g_a is differentiable on $\mathbb{R}$j and g'_a is differentiable on $\mathbb{R}$, but such that g''_a is not continuous at zero.

Solution to (i). Pick $a = 1.5$. The first derivative of g_a is $ax^{a-1} \sin(1/x) - x^{a-2} \cos(1/x)$. To make g'_a unbounded on $[0, 1]$, we need the second term to be unbounded. This can be achieved by choosing any $1 < a < 2$. $\qquad\square$

Solution to (ii). Pick $a = 2.5$. The first derivative, $g'_a = 2.5x^{1.5} \sin(1/x) - x^{0.5} \cos(1/x)$, is continuous at zero, but isn't differentiable at zero, as the second term isn't differentiable at zero, and the sum of two differentiable functions is differentiable if and only if both functions are differentiable. This can be achieved by choosing any $2 < a < 3$. $\qquad\square$

Solution to (iii). Pick $a = 3.5$. The first derivative and second derivative are

$$g'_a = -x^{1.5} \cos(1/x) + 3.5x^{2.5} \sin(1/x)$$
$$g''_a = -5x^{0.5} \cos(1/x) - \frac{\sin(1/x)}{x^{0.5}} + 8.75x^{1.5} \sin(1/x).$$

Clearly, g'_a is differentiable on $\mathbb{R}$, and g''_a is not continuous at zero. This can be achieved by choosing any $3 < a < 4$. $\qquad\square$

Exercise 5.2.9. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(i) If f' exists on an interval and is not constant, then f' must take on some irrational values.

(ii) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(iii) If f is differentiable on an interval containing zero and if $\lim_{x \to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution to (i). It's true. If f' is not constant, then there must exist two points, x_1 and x_2, such that $f'(x_1) < f'(x_2)$, or $f'(x_1) > f'(x_2)$. By the intermediate value theorem, there must exist a point x_3 such that $f'(x_1) < f'(x_3) < f'(x_2)$, and $f'(x_3)$ is irrational. $\qquad\square$

Solution to (ii). It's true if f' is continuous at c. If f' is continuous at c, then there exists a δ-neighborhood $V_\delta(c)$ around c such that $f'(x) > 0$ for all $x \in V_\delta(c)$.

Otherwise, it's false. Assume f' isn't continuous. Consider the function $f(x) = x/2 + x^2 \sin(1/x)$. The derivative of f is

$$f' = 2x \sin(1/x) - \cos(1/x) + \frac{1}{2}.$$

The function f' keeps alternating between positive and negative values as $x \to 0$. Define $x_n = 1/2\pi n$, where $n \in \mathbb{N}$. Then $f'(x_n) = -1/2$, but $f'(0) = 1/2$. $\qquad\square$

Solution to (iii). It's true. Suppose $\lim_{x \to 0} f'(x) = L$ with $f'(0) \neq L$. Let

$$0 < \varepsilon = \frac{|f'(0) - L|}{2}.$$

There exists a $\delta > 0$ such that $f'(x) \in V_L(\varepsilon)$, for $x \in V_\delta(0)$. By definition, $f'(0) \in V_L(\varepsilon)$. Hence, there's a gap between $f'(x)$ and $f'(0)$, which, by Darboux's theorem, is a contradiction. $\qquad\square$

Exercise 5.3.1a. Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M,$$

for all $x \neq y$ in A.

Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Solution. Every closed interval on $\mathbb{R}$ is compact. Let (x_n) be a sequence contained entirely in $[a, b]$. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}) that converges to some limit L. Since $[a, b]$ is closed, $L \in [a, b]$. Therefore, $[a, b]$ is compact.

Using that fact, f' is continuous on a compact set $[a, b]$. Let M be the maximum of $[f'(a), f'(b)]$ (or $[f'(b), f'(a)]$, as f' is defined to be monotone). Pick $x, y \in [a, b]$ with $x < y$. Then, by the Mean Value Theorem, there exists $c \in (x, y)$ with

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq M.$$

Therefore, f is Lipschitz on $[a, b]$. $\qquad\square$

Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Solution. Assume $f'(x) \neq 0$ on A. Let $x, y \in A$ with $x \neq y$. Without loss of generality, assume $x < y$. Then, by the Mean Value Theorem, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x) \neq 0.$$

Therefore, f is one-to-one on A.

An example to show that the converse statement need not be true is the function $f_a(x) = x^{2a+1}$, where $a \in \mathbb{R}$. The derivative of f_a is $f_a'(x) = (2a + 1)x^{2a}$. The function f_a is one-to-one on $\mathbb{R}$, but $f_a'(0) = 0$. $\qquad \square$