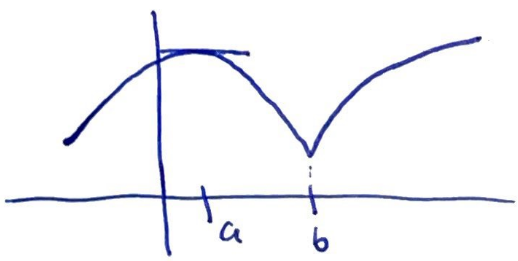
§14.7: Maximum and Minimum Values

Defn: A function $f(x,y)$ has a local maximum at $(a,b)$ if $f(x,y) \le f(a,b)$ for all $(x,y)$ near $(a,b)$. Then $f(a,b)$ is the local maximum value.

A function $f(x,y)$ has a local minimum at $(a,b)$ if $f(a,b) \le f(x,y)$ for all $(x,y)$ near $(a,b)$. Then $f(a,b)$ is the local minimum value.

Recall, for $y=f(x)$, critical numbers of $f$ are numbers $c$ in domain of $f$ such that $f'(c)=0$ or $f'(c)$ does not exist.




$f(a)$ local max $\qquad$ $f(b)$ local min

$f'(a)=0$ $\qquad$ $f'(b)$ does not exist

Theorem: If $f$ has a local maximum or local minimum at $(a,b)$ and $f_x$ and $f_y$ exist at $(a,b)$, then

$D_{\hat{u}} f(a,b) = 0$ for all directions $\hat{u}$.

$D_{\hat{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b) = 0$ for all $\hat{u}$ $\Rightarrow \nabla f(a,b) = \vec{0}$

Recall, the tangent plane to $z = f(x,y)$ at $(a, b, f(a,b))$

is $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

At a max/min, then $z = f(a,b)$ is the tangent plane (horizontal plane).

Defn: The point $(a,b)$ is a critical point of $f(x,y)$ if $\nabla f(a,b) = \vec{0}$ or $f_x$ and/or $f_y$ does not exist at $(a,b)$

Ex: Find critical ~~numbers~~ points of $f(x,y) = x^2 - 2x + y^2 + 4y + 10$

$\nabla f = \langle 2x-2, 2y+4 \rangle = \vec{0}$

$2x - 2 = 0 \Rightarrow x = 1$

$2y + 4 = 0 \Rightarrow y = -2$

Critical point is $(1,-2)$.

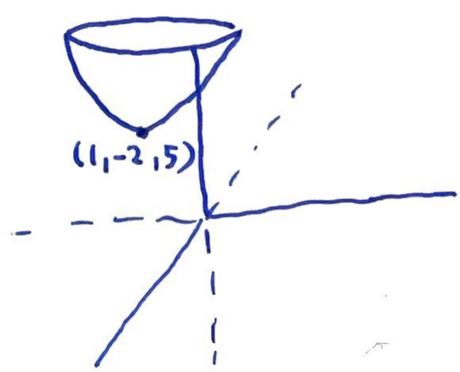
Note $f = x^2 - 2x + y^2 + 4y + 10$

$= (x-1)^2 + (y+2)^2 + 5$ ; Paraboloid

$\geq 5$ for all $(x,y)$

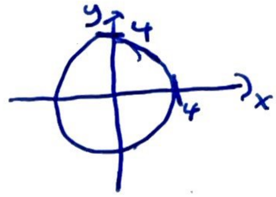
and $f(1,-2) = 5$

Local min of 5 at critical ~~number~~ point $(1,-2)$



Ex: Find the critical points

a) $f(x,y) = \sqrt{16-x^2-y^2}$




Domain $D = \{(x,y) : x^2+y^2 \le 16\}$

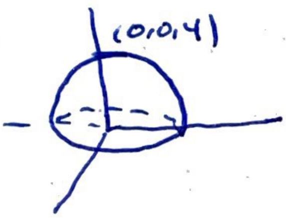
$$\nabla f = \langle -(16-x^2-y^2)^{-1/2}x,\ -(16-x^2-y^2)^{-1/2}y \rangle$$

$$= \frac{-1}{\sqrt{16-x^2-y^2}} \langle x, y \rangle$$

In $D$, $\nabla f$ is undefined on $x^2+y^2=16$

$\nabla f = \vec{0}$ at $(0,0)$ Infinitely many critical points

$z = \sqrt{16-x^2-y^2}$ : Upper half sphere of radius 4.




$0 \le f \le 4$ on $D$.

min value of $0$ on $x^2+y^2=16$

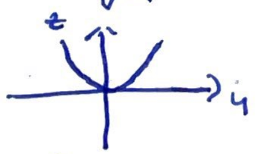
max value of $4$ at $(0,0)$

b) $f(x,y) = y^2 - x^2$ Domain $D = \mathbb{R}^2$

$\nabla f = \langle -2x, 2y \rangle = \vec{0}$ at $(0,0)$ which is the critical point.

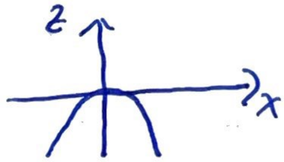
Look at traces through $(0,0)$

$x=0$: $z = y^2$




Trace has a min at $y=0$

$y=0$: $z = -x^2$



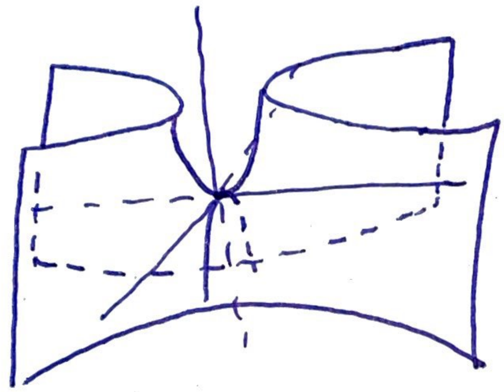

Trace has a max at $x=0$

In some directions, surface behaves like a min at $(0,0)$ while in other directions it behaves like a max.

$f(0,0) = 0$ is neither a max nor a min.

$(0,0)$ is called a <u>saddle point</u>.

$z = y^2 - x^2$ is a hyperbolic paraboloid or saddle.



The critical points are the potential location of local extrema. If $f(x,y)$ has a local extrema at $(a,b)$, $(a,b)$ is a critical point. Not all critical points produce extrema.

Second Derivative Test: Suppose $f(x,y)$ has continuous second order partial derivative in a disk centered at $(a,b)$ and

$\nabla f(a,b) = \vec{0}$ ($(a,b)$ is a critical point)

Define $D(x,y) = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$

$$= f_{xx} f_{yy} - f_{xy} f_{yx}$$

$$= f_{xx} f_{yy} - (f_{xy})^2 \quad \text{by continuity}$$

a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min value of $f$.

b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max value of $f$.

c) If $D(a,b) < 0$, then $(a,b)$ is a saddle point and $f(a,b)$ is neither a max nor a min.

Partial Proof: Let $\hat{u} = \langle h, k \rangle$ be any unit vector and suppose $f_{xx}(a,b) \neq 0$.

$$D_{\hat{u}} f = \hat{u} \cdot \nabla f = h f_x + k f_y$$

Second directional derivative

$$D^2_{\hat{u}} f = \nabla ( D_{\hat{u}} f ) \cdot \hat{u} = h \left( h f_{xx} + k f_{yx} \right) + k \left( h f_{xy} + k f_{yy} \right)$$

$$= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$

since $f$ has continuous second order partial derivatives.

$$= f_{xx} \left( h^2 + 2hk \frac{f_{xy}}{f_{xx}} + k^2 \frac{f_{yy}}{f_{xx}} \right)$$

$$= f_{xx} \left( h^2 + 2hk \frac{f_{xy}}{f_{xx}} + k^2 \frac{f_{xy}^2}{f_{xx}^2} - k^2 \frac{f_{xy}^2}{f_{xx}^2} + k^2 \frac{f_{yy}}{f_{xx}} \right)$$

$$= f_{xx} \left( \left( h + k \frac{f_{xy}}{f_{xx}} \right)^2 + \frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}^2} k^2 \right)$$

$$D_{\hat{u}}^2 f = f_{xx}\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + \frac{D}{f_{xx}} k^2$$

If $f_{xx}(a,b) > 0$ and $D(a,b) > 0$, ~~for all~~ then

$D_{\hat{u}}^2 f(a,b) > 0$ for all directions $\hat{u}$.

All curves on graph of $z=f$ at $(a,b)$ are concave up and $f$ has a local min at $(a,b)$.

If $f_{xx}(a,b) < 0$ and $D(a,b) > 0$, then

$D_{\hat{u}}^2 f(a,b) < 0$ for all directions $\hat{u}$.

All curves on graph of $z=f$ at $(a,b)$ are concave down and $f$ has a local max at $(a,b)$.

Ex: Find and classify all critical points of the function

a) $f(x,y) = 2x^3 + y^2 - 2xy + y - x - 3$

critical points: $\nabla f = \vec{0}$.

1) $f_x = 6x^2 - 2y - 1 = 0$

2) $f_y = 2y - 2x + 1 = 0 \Rightarrow 2y = 2x - 1$

Then $6x^2 - 2x + 1 - 1 = 6x^2 - 2x = 0$

$2x(3x-1) = 0$

$x = 0$ or $x = \frac{1}{3}$

If $x = 0$, then $2y = -1$ and $y = \frac{-1}{2}$

If $x = \frac{1}{3}$, then $2y = \frac{2}{3} - 1$ and $y = \frac{-1}{6}$

Critical points $(0, \frac{-1}{2})$ $(\frac{1}{3}, \frac{-1}{6})$

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x & -2 \\ -2 & 2 \end{vmatrix} = 24x - 4$$

1) $D(0, -\frac{1}{2}) = -4 < 0$

$(0, -\frac{1}{2})$ is a saddle point

2) $D(\frac{1}{3}, \frac{-1}{6}) = 8 - 4 = 4 > 0$

$f_{xx}(\frac{1}{3}, \frac{-1}{6}) = 4 > 0$

All curves on surface through $(\frac{1}{3}, \frac{-1}{6})$ are concave up.

$f$ has a local min at $(\frac{1}{3}, \frac{-1}{6})$

$$f(\tfrac{1}{3}, -\tfrac{1}{6}) = \frac{2}{27} + \frac{1}{36} + \frac{1}{9} - \frac{1}{6} - \frac{1}{3} - 3$$

$$= \frac{2}{27} + \frac{5}{36} - \frac{1}{2} - 3$$

$$= \frac{-355}{108}$$

Ex: Find and classify the critical points of $f = x^3 - 12xy + 8y^3$

$\nabla f = \vec{0}$

1) $f_x = 3x^2 - 12y = 0$ $\quad * 4$

2) $f_y = -12x + 24y^2 = 0$ $\quad * x$

Equivalent system

1) $12x^2 - 48y = 0$

2) $-12x^2 + 24xy^2 = 0$

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$24xy^2 - 48y = 0$

$24y(xy - 2) = 0$

Either $y = 0$ or $xy - 2 = 0$

If $y = 0$, then $x = 0$ $\quad (0,0)$ is a critical point.

If $xy = 2$, then $y = \frac{2}{x}$. Substitute into 1),

$$3x^2 - \frac{24}{x} = 0$$

$$\frac{3x^3 - 24}{x} = 0 \Rightarrow x = 8^{1/3} = 2 \quad \text{then } y = \frac{2}{2} = 1$$

$(2,1)$ is a critical point.

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -12 \\ -12 & 48y \end{vmatrix} = 6(48)xy - 12^2 = 144(2xy - 1)$$

1) $D(0,0) = -144 < 0$ $\quad (0,0)$ is a saddle point

2) $\left.\begin{array}{l} D(2,1) = 144(3) > 0 \\ f_{xx}(2,1) = 12 > 0 \end{array}\right\}$ $f(2,1) = 8 - 24 + 8 = -8$ is a local min value.