Solution to 1. Given the integral

$$\iint_D [(x-2)(x-2y)]^{1/2}\,\mathrm{d}A,$$

we can use the change of variables $u = x - y$ and $v = x - 2y$. Solving in terms of x and y, we have

$$\begin{aligned} & u = x - y \Rightarrow x = u + y \\ \Rightarrow \quad & v = u + y - 2y \Rightarrow y = u - v \\ \Rightarrow \quad & x = u + (u - v). \end{aligned}$$

Therefore, we have

$$\begin{aligned} x &= 2u - v \\ y &= u - v. \end{aligned}$$

The Jacobian is

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = |-1| = 1.$$

Therefore, the integral becomes

$$\iint_D [(x-2)(x-2y)]^{1/2}\,\mathrm{d}A = \iint_{D'} \sqrt{uv}\,\mathrm{d}u\,\mathrm{d}v,$$

where D' is the image of D under the transformation. To find the bounds for D', say $(0,0), (2,0), (2,1)$, we compute the following

$$\begin{aligned} (0,0) &\Rightarrow \begin{cases} u = 0 - 0 & \Rightarrow u = 0 \\ v = 0 - 2(0) & \Rightarrow v = 0 \end{cases} \\ &(0,0) \mapsto (0,0) \\ (2,0) &\Rightarrow \begin{cases} u = 2 - 0 & \Rightarrow u = 2 \\ v = 2 - 2(0) & \Rightarrow v = 2 \end{cases} \\ &(2,0) \mapsto (2,2) \\ (2,1) &\Rightarrow \begin{cases} u = 2 - 1 & \Rightarrow u = 1 \\ v = 2 - 2(1) & \Rightarrow v = 0 \end{cases} \\ &(2,1) \mapsto (1,0). \end{aligned}$$

Therefore, the new bounds for D' are $(0,0), (2,2), (1,0)$. Then, you would just graph the triangle and find the equation of the line from $(2,2)$ to $(1,0)$, and setup the bounds accordingly. $\square$

Solution to 2. The Cartesian parametrization of the surface is given by

$$\mathbf{r}(x,y) = \langle x, y, 2\sqrt{x^2+y^2}\rangle.$$

Converting $z = 2\sqrt{x^2+y^2}$ to polar coordinates, we have $z = 2r$. Using the polar parametrization, we have

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle.$$

Converting $z = 2\sqrt{x^2+y^2}$ to spherical coordinates, we have

$$z = 2\sqrt{\rho^2\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta)} = 2\rho\sin(\varphi).$$

Using the spherical parametrization, we have

$$\mathbf{r}(\rho,\varphi,\theta) = \langle \rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), 2\rho\sin(\varphi)\rangle.$$ $\square$

Solution to 3. For the FTLI, we have to state the following:

Let $\mathbf{F} = \langle y^2z + 2xz^2, 2xyz, xy^2 + 2x^2z \rangle$. We need to start by first checking if $\mathbf{F}$ is conservative by taking the curl of $\mathbf{F}$

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z + 2xz^2 & 2xyz & xy^2 + 2x^2z \end{vmatrix} = \mathbf{0}.$$

Since the curl is zero and $\mathbf{F}$ is simply connected, we can conclude that $\mathbf{F}$ is conservative. Therefore, we can find a potential function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

Now, we can start with the computation to find f.

I'm going to start by integrating Q with respect to y to get

$$f = \int 2xyz \, \mathrm{d}y = xy^2z + g(x, z).$$

Differentiating the whole thing with respect to x gives us

$$f_x = y^2z + g_x(x, z).$$

Setting it equal to P gives us

$$f_x = P \Rightarrow y^2z + g_x(x, z) = y^2z + 2xz^2 \Rightarrow g_x(x, z) = 2xz^2.$$

Now, integrating g_x with respect to x gives us

$$g(x, z) = \int 2xz^2 \, \mathrm{d}x = x^2z^2 + h(z).$$

Now, plugging everything back into f, we get

$$f = xy^2z + x^2z^2 + h(z).$$

Next, we differentiate f with respect to z and set it equal to R to solve for $h(z)$

$$f_z = xy^2 + 2x^2z + h'(z).$$

Setting it equal to R gives us

$$f_z = R \Rightarrow xy^2 + 2x^2z + h'(z) = xy^2 + 2x^2z \Rightarrow h'(z) = 0.$$

Therefore, $h(z) = C$, where C is a constant, but we can ignore it, as we don't want a specific solution, just a solution. Finally, we have

$$f(x, y, z) = xy^2z + x^2z^2.$$

Now, using the FTLI, we get

$$\boxed{\int_C \mathbf{F} \cdot \mathbf{dr} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

□

Solution to 4. For Green's theorem, you have to state the following:

Since we have a positively orientated simply closed curve C and a region D that is bounded by C, we can apply Green's theorem.

Then, you find the difference of the partials giving us

$$\boxed{\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A}$$

□

Solution to 5. The surface element is given by

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}A.$$

Solving for z in terms of x and y, we have

$$z = \frac{1 - x^2 + y - 8}{2}.$$

Therefore, we have

$$\mathrm{d}S = \sqrt{1 + 4x^2 + \frac{1}{4}} = \frac{\sqrt{16x^2+5}}{2}.$$

Graphing the triangle and finding the bounds, we have $0 \le x \le 1$ and $0 \le y \le 4x$. Therefore, the surface area is given by

$$\begin{aligned}
\iint_S \mathrm{d}S &= \iint_D \sqrt{1 + 4x^2 + \frac{1}{4}}\,\mathrm{d}A \\
&= \iint_D \frac{\sqrt{16x^2+5}}{2}\,\mathrm{d}A \\
&= \int_0^1 \int_0^{4x} \frac{\sqrt{16x^2+5}}{2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{\sqrt{16x^2+5}}{2}y\right]_0^{4x}\,\mathrm{d}x \\
&= \int_0^1 2x\sqrt{16x^2+5}\,\mathrm{d}x.
\end{aligned}$$

Using u-sub, let $u = 16x^2+5$, then $\mathrm{d}u = 32x\,\mathrm{d}x \Rightarrow \mathrm{d}u/32 = x\,\mathrm{d}x$. The bounds change from $x = 0 \Rightarrow u(0) = 5$ and $x = 1 \Rightarrow u(1) = 21$. Therefore, we have

$$\begin{aligned}
\int_0^1 2x\sqrt{16x^2+5}\,\mathrm{d}x &= \int_5^{21} \frac{2\sqrt{u}}{32}\,\mathrm{d}u \\
&= \frac{1}{16}\left[\frac{2}{3}u^{3/2}\right]_5^{21} \\
&= \frac{1}{24}\left[21^{3/2} - 5^{3/2}\right].
\end{aligned}$$

□

Solution to 6. For Stokes' theorem, you have to state the following:

Given a closed boundary ∂_S of a surface S, which is piecewise smooth and positively oriented, we apply Stokes' theorem. Given that the closed boundary ∂_S has an orientation induced on it, we use the right hand rule to get the orientation of the surface S to keep the orientation consistent.

Then, you find the curl, the bounds, and the surface element. Remember,

$$\boxed{\oint_{\partial_S} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot \mathrm{d}\mathbf{S} = \iint_S (\nabla \times \mathbf{F})\cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{\|\mathbf{r}_x \times \mathbf{r}_y\|}\|\mathbf{r}_x \times \mathbf{r}_y\|\,\mathrm{d}A = \iint_S (\nabla \times \mathbf{F})\cdot \mathbf{r}_x \times \mathbf{r}_y\,\mathrm{d}A}$$

□

Solution to 7. For the Divergence Theorem, you have to state the following:

Given a closed and piecewise smooth surface S that bounds a solid region E in space, we apply the Divergence Theorem. The outward orientation of S is induced naturally. Remember,

$$\boxed{\oiint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F})\,\mathrm{d}V}$$

□