

Abstract Linear Algebra: Homework 4

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Xiaojing Chen–Murphy

Hashem A. Damrah

UO ID: 952102243

Problem 1. Let $T : \mathbb{R}^{1 \times 3} \rightarrow \mathbb{R}^{2 \times 3}$ be a linear transformation defined by

$$T(x, y, z) = (x + z, 2x - z).$$

If $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$, $\mathcal{B}' = \{\gamma_1, \gamma_2\}$, where

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0), \quad \gamma_1 = (0, 1), \quad \text{and} \quad \gamma_2 = (1, 0).$$

Find the matrix ${}_{\mathcal{B}' \leftarrow \mathcal{B}} T$.

Solution. Evaluating T at α_1, α_2 , and α_3 we get

$$T(\alpha_1) = T(1, 0, -1) = (0, 3) = 0\gamma_1 + 3\gamma_2$$

$$T(\alpha_2) = T(1, 1, 1) = (2, 1) = 1\gamma_1 + 2\gamma_2$$

$$T(\alpha_3) = T(1, 0, 0) = (1, 2) = 2\gamma_1 + 1\gamma_2.$$

Therefore, the matrix ${}_{\mathcal{B}' \leftarrow \mathcal{B}} T$ is

$${}_{\mathcal{B}' \leftarrow \mathcal{B}} T = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

□

Problem 2. Let D be the differentiation operator on $\mathbb{P}^3(\mathbb{R})$, i.e.

$$D(g(x)) = g'(x) \text{ for } g(x) \in \mathbb{P}^3(\mathbb{R}).$$

(Note: D is a linear transformation on $\mathbb{P}^3(\mathbb{R})$)

(i) Let $\mathcal{B} = \{1, x, x^2, x^3\}$ be the standard basis for $\mathbb{P}^3(\mathbb{R})$. Find the matrix $[D]_{\mathcal{B}}$.

(ii) Let $\mathcal{B}' = \{x^3, x^2, x, 1\}$ be the basis for $\mathbb{P}^3(\mathbb{R})$. Find the matrix $[D]_{\mathcal{B}'}$.

Solution. Just as in problem 1, we evaluate D at the elements of \mathcal{B} to get

$$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$

$$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$

$$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$

$$D(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3.$$

From that, we get

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad [D]_{\mathcal{B}'} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}.$$

□

Problem 3. Let T be a linear transformation on the vector space $V = \mathbb{R}^{2 \times 2}$ defined by

$$T(A) = 2A + A^T.$$

Let $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Find the matrix representation $[T]_{\mathcal{B}}$.

Solution. We evaluate T at the elements of \mathcal{B} to get

$$\begin{aligned} T(E_{11}) &= 2E_{11} + E_{11}^T = 3E_{11} + 0E_{12} + 0E_{21} + 0E_{22} \\ T(E_{12}) &= 2E_{12} + E_{12}^T = 0E_{11} + 2E_{12} + E_{21} + 0E_{22} \end{aligned}$$

Therefore, the matrix representation $[T]_{\mathcal{B}}$ is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \quad \square$$

Problem 4. Let V be a two-dimensional vector space over \mathbb{F} , and let \mathcal{B} be an ordered basis for V . If T is a linear transformation on V and $[T]_{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $T^2 - (a + d)T + (ad - bc)I = 0$.

Solution. The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation. Given an $A \in \mathbb{R}^{2 \times 2}$ and its characteristic polynomial is $P_A(\lambda) = \det(\lambda I - A)$, then substituting A for λ in $P_A(\lambda)$ results in the zero matrix.

Since $[T]_{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic polynomial is

$$P_T(\lambda) = \det(\lambda I - [T]_{\mathcal{B}}) = \det \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a + d)\lambda + (ad - bc).$$

By the Cayley-Hamilton theorem, the matrix $[T]_{\mathcal{B}}$ satisfies its characteristic polynomial, giving us

$$[T]_{\mathcal{B}}^2 - (a + d)[T]_{\mathcal{B}} + (ad - bc)I = 0.$$

Since $[T]_{\mathcal{B}}$ is a matrix representation of the linear transformation T , the equation above is equivalent to $T^2 - (a + d)T + (ad - bc)I = 0$. \square

Problem 5. Suppose that T is a linear transformation on a two-dimensional vector space such that T is neither the zero nor the identity linear transformation. Prove that if $T^2 = T$, there is an ordered basis \mathcal{B} for V such that $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(Hint: Construct a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ such that $T(\mathbf{v}_1) = \mathbf{v}_1$ and $T(\mathbf{v}_2) = \mathbf{0}$.)

Solution. Since $T^2 = T$, this means T is idempotent. The eigenvalues of an idempotent operator must satisfy the equation

$$\lambda^2 = \lambda.$$

Therefore, the eigenvalues of T are $\lambda = 0$ and $\lambda = 1$. Since 1 is an eigenvalue for T , there exists a non-zero vector $\mathbf{v}_1 \neq \mathbf{0}$ such that $T(\mathbf{v}_1) = \mathbf{v}_1$. Since T has two eigenvalues, there must exist another nonzero vector \mathbf{v}_2 such that $T(\mathbf{v}_2) = \mathbf{0}$. This, $\mathbf{v}_1 \in \text{Range}(T)$ and $\mathbf{v}_2 \in \text{Ker}(T)$.

Now, we must show that \mathbf{v}_1 and \mathbf{v}_2 are linearly independent. Suppose that \mathbf{v}_1 and \mathbf{v}_2 are linearly dependent, then there exist scalars α and β such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{0}$. Applying T to both sides of the equation gives

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = T(\mathbf{0}) = \mathbf{0}.$$

By the linearity of T , we have

$$\alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) = \alpha\mathbf{v}_1 + \beta\mathbf{0} = \alpha\mathbf{v}_1 = \mathbf{0}.$$

Since $\mathbf{v}_1 \neq \mathbf{0}$, this implies that $\alpha = 0$. But this means that $\mathbf{v}_2 = \mathbf{0}$, which is a contradiction. Therefore, \mathbf{v}_1 and \mathbf{v}_2 are linearly independent.

Since \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, they form a basis for V . Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis for V . In the basis, we write

$$T(\mathbf{v}_1) = 1\mathbf{v}_1 + 0\mathbf{v}_2 \quad \text{and} \quad T(\mathbf{v}_2) = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Therefore, the matrix representation of T in the basis \mathcal{B} is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad \square$$

Problem 6. Let V be an n -dimensional vector space over \mathbb{R} . Let T be a linear transformation on V . If

$$T^n = 0, \text{ and } T^{n-1} \neq 0, \text{ prove that there exists a basis } \mathcal{B} \text{ such that } [T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

(Hint: Construct a set of the form $\{x, T(x), T^2(x), \dots, T^{n-1}(x)\}$ and show that this set is a basis of V .)

Solution. Since $T^{n-1} \neq 0$, there exists some nonzero vector $\mathbf{x} \in V$ such that $T^{n-1}(\mathbf{x}) \neq 0$. Define the set of vectors

$$S = \{\mathbf{x}, T(\mathbf{x}), T^2(\mathbf{x}), \dots, T^{n-1}(\mathbf{x})\}.$$

We will show that S is a basis for V .

Suppose there exist scalars $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$ such that

$$c_0\mathbf{x} + c_1T(\mathbf{x}) + c_2T^2(\mathbf{x}) + \cdots + c_{n-1}T^{n-1}(\mathbf{x}) = \mathbf{0}.$$

Applying T^{n-1} to both sides of the equation gives

$$c_0T^{n-1}(\mathbf{x}) + c_1T^n(\mathbf{x}) + c_2T^{n+1}(\mathbf{x}) + \cdots + c_{n-1}T^{2n-2}(\mathbf{x}) = \mathbf{0}.$$

Since $T^{n-1}(\mathbf{x}) \neq 0$, it follows that $c_0 = 0$. Applying T^{n-2} to the original equation gives

$$c_1T^{n-2}(\mathbf{x}) = 0.$$

Since $T^{n-2}(\mathbf{x}) \neq 0$, it follows that $c_1 = 0$. Continuing this process until we reach c_{n-1} , we find that $c_0 = c_1 = \cdots = c_{n-1} = 0$. Therefore, S is linearly independent.

Since S contains n linearly independent vectors and V is an n -dimensional vector space, S is a basis for V .

Let $\mathcal{B} = \{\mathbf{x}, T(\mathbf{x}), T^2(\mathbf{x}), \dots, T^{n-1}(\mathbf{x})\}$ be the basis for V . Since $T(\mathbf{x}) = T(\mathbf{x}) \in \mathcal{B}$, it can be represented as $\mathbf{e}_2 = (0, 1, 0, \dots, 0)^T$. Similarly, $T^2(\mathbf{x}) = T^2(\mathbf{x}) \in \mathcal{B}$ can be represented as $\mathbf{e}_3 = (0, 0, 1, 0, \dots, 0)^T$.

Continuing this process, we find that $T^{n-1}(\mathbf{x}) = T^{n-1}(\mathbf{x}) \in \mathcal{B}$ can be represented as $\mathbf{e}_n = (0, 0, 0, \dots, 1)^T$. Therefore, the matrix representation of T in the basis \mathcal{B} is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \quad \square$$

Problem 7. Suppose $A \in \mathbb{R}^{n \times n}$ is an invertible matrix. Prove that AB and BA are similar matrices for any $B \in \mathbb{R}^{n \times n}$.

Solution. Since A is invertible, let $P = A$. We can then compute

$$A^{-1}(AB)A = (A^{-1}A)BA = BA.$$

Thus, we can express BA as

$$BA = A^{-1}(AB)A.$$

Therefore, AB and BA are similar matrices. \square

Problem 8. Let $A, B \in \mathbb{R}^{n \times n}$. Prove the following statements.

(i) If A and B are similar, then $\text{Tr}(A) = \text{Tr}(B)$.

(ii) $AB - BA = I$ is impossible.

Hint: You may use the results from Homework 2 Problem 1.

Solution to (i). By definition, two matrices are similar if there exists an invertible matrix P such that $B = P^{-1}AP$. Using the cyclic property of the trace, we have

$$\text{Tr}(B) = \text{Tr}(P^{-1}AP) = \text{Tr}(A).$$

Therefore, if A and B are similar, then $\text{Tr}(A) = \text{Tr}(B)$. \square

Solution to (ii). Assume, for contradiction, that there exists $A, B \in \mathbb{R}^{n \times n}$ such that

$$AB - BA = I.$$

Using the linearity of the trace

$$\text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA) = \text{Tr}(I).$$

From Homework 2 Problem 1, we know that $\text{Tr}(AB) = \text{Tr}(BA)$, so

$$\text{Tr}(AB) - \text{Tr}(AB) = 0 \neq n = \text{Tr}(I).$$

This is a contradiction, so the equation $AB - BA = I$ is impossible. \square

Problem 9. True or False. (No explanation needed.)

In the following statements (i) - (iii): Let V and W be finite-dimensional vector spaces. Let $T, U : V \rightarrow W$ be linear transformations. Let β and γ be ordered basis for V and W , respectively.

(i) Let $[T]_{\beta}^{\gamma} = [U]_{\beta}^{\gamma}$ implies $T = U$.

- (ii) If $\dim(V) = n$ and $\dim(W) = m$, then $[T]_{\beta}^{\gamma} \in \mathbb{R}^{n \times m}$.
- (iii) $[T(\mathbf{v})]_{\gamma} = [T]_{\beta}^{\gamma}[\mathbf{v}]_{\beta}$ for all $\mathbf{v} \in V$.
- (iv) Let $A \in \mathbb{R}^{n \times n}$. If $A^2 = I$, then $A = I$ or $A = -I$.
- (v) Let $A \in \mathbb{R}^{m \times n}$. Suppose $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined by $L_A(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$. Then $[L_A]_{\beta} = A$, where β is the standard basis for \mathbb{R}^n .

Solution to (i). It's true. If $[T]_{\beta}^{\gamma} = [U]_{\beta}^{\gamma}$, then T and U have the same matrix representation with respect to the bases β and γ . Since a linear transformation is uniquely determined by its action on a basis, having the same matrix implies that T and U act identically on all basis vectors, and hence they must be the same transformation, i.e., $T = U$. \square

Solution to (ii). It's false. The matrix representation $[T]_{\beta}^{\gamma}$ has dimensions $m \times n$, not $n \times m$, because T maps from an n -dimensional space to an m -dimensional space. The correct statement would be $[T]_{\beta}^{\gamma} \in \mathbb{R}^{m \times n}$. \square

Solution to (iii). It's true. This is a fundamental property of matrix representations of linear transformations. Given a vector $\mathbf{v} \in V$, its image under T has coordinates $[T(\mathbf{v})]_{\gamma}$, which are obtained by multiplying the matrix representation $[T]_{\beta}^{\gamma}$ by the coordinate vector $[\mathbf{v}]_{\beta}$. \square

Solution to (iv). It's false. The equation $A^2 = I$ means A is an involutory matrix, but this does not imply that A must be either I or $-I$. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

satisfies $A^2 = I$ but is neither I nor $-I$. \square

Solution to (v). It's true. By definition, the standard matrix representation of the linear map L_A induced by A is exactly A , since the standard basis vectors get mapped directly according to the columns of A . \square