1. Use the following conclusion to solve the given problems.

Let $A \in \mathbb{R}^{n\times m}$ *and* $B \in \mathbb{R}^{m\times n}$ *with* $n \geq m$*. Then* $\det(\lambda I_n - AB) = \lambda^{n-m}\det(\lambda I_m - BA)$.

1). Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}^T\mathbf{x} = 1$. Find the eigenvalues for $I_n - 2\mathbf{x}\mathbf{x}^T$.

2). Let $\mathbf{x} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mathbb{R}^n$, and $\mathbf{y} = \begin{pmatrix} b_1 \\ \cdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$. Find the eigenvalues for $I_n - \mathbf{x}\mathbf{y}^T$.

2. Prove that an upper triangular matrix with zeros in all the diagonal entries is nilpotent. (Note: A matrix A is nilpotent if and only if there exists a positive integer k such that A^k=0.)

3. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Prove that $A^{-1} = g(A)$, for some polynomial $g(x)$ with $\deg(g(x)) = n - 1$.

4. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Define

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2 \qquad (*).$$

Prove that $(*)$ defines an inner product on $\mathbb{R}^2$.

5. Let $A \in \mathbb{C}^{n\times n}$ and assume A is Hermitian positive-definite. Prove that $(\mathbf{x}, \mathbf{y}) = \mathbf{y}^*A\mathbf{x}$ defines an inner product on $\mathbb{C}^n$.

6. If V is a vector space over $\mathbb{R}$, verify the following polarization identity for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$(\mathbf{x}, \mathbf{y}) = \frac{1}{4}||\mathbf{x} + \mathbf{y}||^2 - \frac{1}{4}||\mathbf{x} - \mathbf{y}||^2.$$

7. Let V be an inner product space. Prove the following triangular inequality for any $\mathbf{x}, \mathbf{y} \in V$:

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||.$$

8. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be orthonormal vectors in $\mathbb{R}^n$. Show that $A\mathbf{x}_1, \cdots, A\mathbf{x}_n$ are also orthonormal if and only if $A \in \mathbb{R}^{n\times n}$ is orthogonal.

9. True or False. (No explanation needed.)

1). An inner product is a scalar-valued function on the set of ordered pairs of vectors.

2). An inner product is linear in both components.

3). If $(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{v}$ in an inner product space, then $\mathbf{w} = \mathbf{0}$.

4). A set of orthonormal vectors must be linearly independent.

5). A set of orthogonal vectors must be linearly independent.

6). A matrix in $\mathbb{R}^{n\times n}$ is orthogonal if and only if its column vectors are orthogonal.