Definition.

(i) **Regular Curve:** A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0$, for all $t \in I$.

(ii) **Regular Surface:** A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a nbd of $V \subset \mathbb{R}^3$ and a map $\mathbf{x} : U \subset \mathbb{R}^2 \to V \cap S$ such that, (1) $\mathbf{x}$ is differentiable, (2) $\mathbf{x}$ is a homeomorphism, and for each $q \in U$, $\mathrm{d}\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

How to Show a Surface is Regular: A subset $S \subset \mathbb{R}^3$ is a regular surface if for every $p \in S$, there exists an open set $U \subset \mathbb{R}^2$ and a map $\mathbf{x} : U \to \mathbb{R}^3$ such that:

- $\mathbf{x}$ is differentiable,
- $\mathbf{x}$ is a homeomorphism onto its image,
- $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent $\Rightarrow \mathbf{x}_u \times \mathbf{x}_v \neq 0$.

Common strategies:

- **Explicit parametrization:** Define $\mathbf{x}(u, v)$ and verify regularity by checking $\mathbf{x}_u \times \mathbf{x}_v \neq 0$.
- **Graph of a function:** $S = \{(x, y, f(x, y))\}$. Use parametrization $\mathbf{x}(x, y) = (x, y, f(x, y))$.
- **Level set:** $S = \{(x, y, z) : f(x, y, z) = 0\}$ with $\nabla f \neq 0 \Rightarrow$ use Implicit Function Theorem.

(iii) **Differentiable function:** Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset $V \subset S$. Then, f is differentiable at $p \in V$ if, for some $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, with $p \in \mathbf{x}(U) \subset V$, $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$.

(iv) **Differentiable Map Between S_1 and S_2:** A continuous map $\Phi : V_1 \subset S_1 \to S_2$ is differentiable at $p \in V_1$ if, $\mathbf{x}_1 : U_1 \subset \mathbb{R}^2 \to S_1$, $\mathbf{x}_2 : U_2 \subset \mathbb{R}^2 \to S_2$, with $p \in \mathbf{x}_1(U)$ and $\Phi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map $\mathbf{x}_2^{-1} \circ \Phi \circ \mathbf{x}_1 : U_1 \to U_2$ is differentiable at $q = \mathbf{x}_1^{-1}(p)$.

(v) **First Fundamental Form:** For a surface parametrized by $\mathbf{x}(u, v)$, the first fundamental form $I_p(\mathbf{w}) = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$, where E, F, and G are given by

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad \text{and} \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

(vi) **Area of a Surface**: The area of a regular surface S and a surface of revolution given by

$$A = \iint_D \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \iint_D \|\mathbf{x}_u \times \mathbf{x}_v\|\,\mathrm{d}u\,\mathrm{d}v \quad \text{and} \quad A = 2\pi \int_0^\ell \rho(s)\,\mathrm{d}s,$$

where ℓ is the length of the curve, and $\rho(s)$ is the distance from the axis of rotation to the curve.

(vii) **Orientation of a Surface:** A surface S is *orientable* if it is possible to choose a continuous unit normal vector field $\mathbf{N}(p)$ on all of S. That is, there exists a continuous map $\mathbf{N} : S \to \mathbb{R}^3$ such that $\mathbf{N}(p)$ is perpendicular to the tangent plane at p and $\|\mathbf{N}(p)\| = 1$ for all $p \in S$.

How to Show a Surface is Orientable:

- If the surface has a global parametrization $\mathbf{x}(u, v)$ with $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ and $\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ is continuous on the domain, then S is orientable.
- If S is given as the graph of a differentiable function $z = f(x, y)$, define $\mathbf{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$.
- If S is defined implicitly as $f(x, y, z) = 0$ with $\nabla f \neq 0$ on S, then $\mathbf{N} = \frac{\nabla f}{\|\nabla f\|}$ gives a global orientation.

– If S cannot be covered by one consistent choice of normal vectors (e.g., Möbius strip), then it is non-orientable.

Theorem (Inverse Function Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in an open set $U \subset \mathbb{R}^n$. If $Df(a)$ is invertible at $a \in U$, then there exist open sets $V \ni a$ and $W \ni f(a)$ such that $f : V \to W$ is a diffeomorphism (i.e., f is bijective, differentiable, and f^{-1} is differentiable).*

Sketch of Proof. Let $A = Df(a)$. Define $g(x) = f(a) + A(x - a)$. Then g is a linear approximation of f. Let $\varphi(x) = f(x) - g(x)$ so that $\varphi(a) = 0$, and $D\varphi(a) = Df(a) - A = 0$. Hence, φ is small near a.

Define $T(x) = x - A^{-1}(f(x) - f(a))$. Show T is a contraction on a small ball $B_\delta(a)$ using the mean value theorem and continuity of Df near a. By Banach Fixed Point Theorem, T has a unique fixed point x such that $f(x) = y$ for all y near $f(a)$.

The map $x \mapsto f(x)$ is bijective near a, and f^{-1} is differentiable by differentiating the identity $f(f^{-1}(y)) = y$. □

Theorem (Implicit Function Theorem)**.** *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a C^1 function. Suppose $F(a, b) = 0$ for $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ and the $m \times m$ matrix $\frac{\partial F}{\partial y}(a, b)$ is invertible. Then, there exist open sets $U \ni a$ and $V \ni b$ and a C^1 function $g : U \to V$ such that*

$$F(x, g(x)) = 0 \quad \textit{for all } x \in U.$$

Moreover, $g(a) = b$, and the graph of g locally describes the level set $\{(x, y) : F(x, y) = 0\}$.

Sketch of Proof. Define $G(x, y) = F(x, y)$. Since F is C^1 and $\frac{\partial F}{\partial y}(a, b)$ is invertible, we apply the Inverse Function Theorem to the map $(x, y) \mapsto (x, F(x, y))$.

Locally, we can solve for y as a function of x:

$$F(x, y) = 0 \Rightarrow y = g(x)$$

where g is C^1 and satisfies $g(a) = b$. The existence and differentiability of g follows from the fact that we can locally invert the y-coordinates while keeping x fixed. This gives a local parametrization of the level set $\{F(x, y) = 0\}$ by $x \mapsto (x, g(x))$. □

Quick References

Normal Vectors (parametrized surfaces) (graph $z = f(x, y)$) (curve)

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}, \quad \mathbf{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}, \quad \text{and} \quad \mathbf{N} = \frac{T'(t)}{\|T'(t)\|}.$$

Curvature and Torsion:

$$\begin{aligned} \kappa(t) &= \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \quad \text{and} \quad \tau(t) = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2} \\ \kappa(s) &= \|\alpha''(s)\| \quad \text{and} \quad \mathbf{B}'(s) = \tau(s)\mathbf{N}(s). \end{aligned}$$

Parametrization of a Möbius Strip and a Torus:

$$\begin{aligned} \mathbf{x}(u, v) &= \left\langle \left(2 - v \sin\left(\frac{u}{2}\right)\right) \sin(u), \left(2 - v \sin\left(\frac{u}{2}\right)\right) \cos(u), v \cos\left(\frac{u}{2}\right) \right\rangle \\ \mathbf{x}(u, v) &= \langle (R + r\cos(v))\cos(u), (R + r\cos(v))\sin(u), r\sin(v) \rangle. \end{aligned}$$

Given two normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ of two surfaces S_1 and S_2, we have

$$\begin{aligned} \mathbf{N}_1 \cdot \mathbf{N}_2 > 0 &\Rightarrow \text{same orientation} \\ \mathbf{N}_1 \cdot \mathbf{N}_2 < 0 &\Rightarrow \text{opposite orientation}. \end{aligned}$$