Fundamentals of Analysis I: Homework 5

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SECTION 2.4

Exercise 2.4.7 Let (a_n) be a bounded sequence.

(i) Prove that the sequence defined by $y_n = \sup(\{a_k \mid k \geq n\})$ converges.

(ii) The *limit superior* of (a_n), or $\lim_{n\to\infty} \sup(a_n)$, is defined by

$$\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} y_n,$$

where y_n is the sequence from part (i) of this exercise. Provide a reasonable definition for $\lim_{n\to\infty} \inf(a_n)$ and briefly explain why it always exists for any bounded sequence.

(iii) Prove that $\lim_{n\to\infty} \inf(a_n) \leq \lim_{n\to\infty} \sup(a_n)$ for every bounded sequence and give an example of a sequence for which the inequality is strict.

(iv) Show that $\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} \sup(a_n)$ if and only if $\lim_{n\to\infty} a_n$ exists. In this case, all three share the same value.

Solution to (i). Notice that as n increases, the set $\{a_k \mid k \geq n\}$ becomes smaller or stays the same. Thus,

$$\{a_k \mid k \geq n+1\} \subseteq \{a_k \mid k \geq n\}.$$

Because the supremum of a subset cannot exceed the supremum of the larger set containing it, we have

$$y_{n+1} = \sup\left(\{a_k \mid k \geq n+1\}\right) \leq \sup\left(\{a_k \mid k \geq n\}\right) = y_n.$$

Therefore, (y_n) is a decreasing sequence. Since (a_n) is bounded, then there exists an M such that $|a_n| \leq M$, for all $n \in \mathbb{N}$. This implies that $y_n \leq M$, for all $n \in \mathbb{N}$. Thus, (y_n) is a decreasing sequence bounded below by M. By the Monotone Convergence Theorem, (y_n) converges. □

Solution to (ii). The *limit inferior* of (a_n), or $\lim_{n\to\infty} \inf(a_n)$, is defined by

$$\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} x_n,$$

where $x_n = \inf(\{a_k \mid k \geq n\})$. The limit inferior always exists for any bounded sequence because (x_n) is an increasing sequence bounded above by M, where M is the upper bound of (a_n). □

Solution to (iii). Assume $\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} y_n$, where $y_n = \sup\left(\{a_k \mid k \geq n\}\right)$ and $\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} z_n$, where $z_n = \inf\left(\{a_k \mid k \geq n\}\right)$. Since y_n is the least upper bound for the set $\{a_k \mid k \geq n\}$ and z_n is the greatest lower bound for the set $\{a_k \mid k \geq n\}$, then

$$z_n \leq y_n \quad \text{for all } n \in \mathbb{N}.$$

Taking the limit of both sides, we get

$$\lim_{n\to\infty} z_n \leq \lim_{n\to\infty} y_n.$$

Thus, we get

$$\lim_{n\to\infty} \inf(a_n) \leq \lim_{n\to\infty} \sup(a_n).$$

Example. Consider the sequence $(a_n) = (-1)^n + {}^1\!/\!{}_n$ as $n \to \infty$.

(i) For an even n, $a_n = 1 + {}^1\!/\!{}_n$.

(ii) For an odd n, $a_n = -1 + {}^1\!/\!_n$.

For $\lim_{n\to\infty} \sup(a_n)$, for large n the supremum of the terms $\{a_k \mid k \geq n\}$ will be close to 1. For $\lim_{n\to\infty} \inf(a_n)$, for large n the infimum of the terms $\{a_k \mid k \geq n\}$ will be close to -1. Thus, $\lim_{n\to\infty} \inf(a_n) < \lim_{n\to\infty} \sup(a_n)$. $\square$

Solution to (iv). Assume $\lim_{n\to\infty} a_n = a$ exists. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$. This implies that for sufficiently large n, all terms in the set $\{a_k \mid k \geq n\}$ are within ε of a, so both $y_n = \sup(\{a_k \mid k \geq n\})$ and $z_n = \inf(\{a_k \mid k \geq n\})$ are within ε of a. Thus, $y_n \to a$ and $z_n \to a$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup(a_n)$.

Assume $\lim_{n\to\infty} \inf(a_n) = a = \lim_{n\to\infty} \sup(a_n)$. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|y_n - a| < \varepsilon$ and $|z_n - a| < \varepsilon$. Since $z_n \leq a \leq y_n$, then for all $n \geq N$, it follows that a_n is squeezed within ε of a. By Squeeze Theorem, $\lim_{n\to\infty} a_n = a$.

Therefore, $\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup(a_n)$ if and only if $\lim_{n\to\infty} a_n$ exists. $\square$

SECTION 2.5

Exercise 2.5.1 Give an example of each of the following, or argue that such a request is impossible.

(i) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(ii) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(iii) A sequence that contains subsequences converging to every point in the infinite set $\{1, {}^1\!/\!_2, {}^1\!/\!_3, {}^1\!/\!_4, \dots\}$.

(iv) A sequence that contains subsequences converging to every point in the infinite set $\{1, {}^1\!/\!_2, {}^1\!/\!_3, {}^1\!/\!_4, \dots\}$, and no subsequences converging to points outside of this set.

Solution to (i). Impossible, as by the Bolzano-Weierstrass theorem, a convergent subsequence of that subsequence exists, and that sub-subsequence is also a subsequence of the original sequence. $\square$

Solution to (ii). Consider the sequence $(2+{}^1\!/\!_n) \to 2$. The subsequences $(1+{}^1\!/\!_n) \to 1$ and $({}^1\!/\!_n) \to 0$ converge to 1 and 0 respectfully and the original sequence does not contain 0 or 1. $\square$

Solution to (iii). We can construct (a_n) by defining it as follows

(i) For each $k \in \mathbb{N}$, repeat the terms 1, $\frac{1}{2}$, $\frac{1}{3}$, $\dots$, $\frac{1}{k}$ in that order.

(ii) For example, start with 1, $\frac{1}{2}$, 1, $\frac{1}{3}$, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\dots$ and continue so that every $\frac{1}{k}$ appears infinitely many times.

Each point $\frac{1}{k}$ in the set $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ has an associated subsequence of (a_n) consisting entirely of terms equal to $\frac{1}{k}$, which converges to $\frac{1}{k}$ itself. For example,

(i) The subsequence (a_{n_j}) with $a_{n_j} = 1$ for all j converges to 1.

(ii) The subsequence $(a_{n_j}) = \frac{1}{2}$ for all j converges to $\frac{1}{2}$.

(iii) Similarly, there are subsequences converging to $\frac{1}{3}$, $\frac{1}{4}$, and so on. $\square$

Solution to (iv). Impossible, the sequence must converge to zero which is not in the set.

Let $\varepsilon > 0$. Choose N large enough that ${}^1\!/\!_n < {}^\varepsilon\!/\!_2$ for all $n \geq N$. We can find a subsequence $(b_k) \to {}^1\!/\!_n$, meaning $|b_k - {}^1\!/\!_n| < {}^\varepsilon\!/\!_2$ for some $k \in \mathbb{N}$. Using the Triangle Inequality, we get $|b_k - 0| \leq |b_n - {}^1\!/\!_n| + |{}^1\!/\!_n + 0| < {}^\varepsilon\!/\!_2 + {}^\varepsilon\!/\!_2 = \varepsilon$. $\square$

Exercise 2.5.3

(i) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(ii) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associate. Why doesn't our proof in (i) apply to this example?

Solution to (i). Let s_n be the original grouping of the terms and converge to L. Let s'_n be a subsequence of s_n that has a different grouping of addition. By Theorem 2.5.2, since s_n, the original sequence, converges, then s'_n, the subsequence, also converges to L. □

Solution to (ii). The example discussed at the end of Section 2.1 where infinite addition was shown not to be associative was the series $1-1+1-1+\cdots$. The proof in (i) does not apply to this example because the series does not converge. The series $1 - 1 + 1 - 1 + \cdots$ does not converge because the sequence of partial sums oscillates between 0 and 1. □

SECTION 2.6

Exercise 2.6.2 Give an example of each of the following, or argue that such a request is impossible.

(i) A Cauchy sequence that is not monotone.

(ii) A Cauchy sequence with an unbounded subsequence.

(iii) A divergent monotone sequence with a Cauchy subsequence.

(iv) An unbounded sequence containing a subsequence that is Cauchy.

Solution to (i). By Theorem 2.6.2, a converging sequence is also Cauchy. Therefore, the following sequence $x_n = \frac{(-1)^n}{n}$ is Cauchy. □

Solution to (ii). Impossible, since all Cauchy sequences converge, meaning they are bounded. □

Solution to (iii). Impossible, if a subsequence was Cauchy, it would converge implying it would be bounded. But this would also imply that the original sequence would be bounded since the subsequence is a monotone sequence. That would imply that the original sequence is monotone and bounded, meaning it would converge. □

Solution to (iv). The sequence $a_n = (2, 1/2, 3, 1/3, \dots)$ has the subsequence $(1/2, 1/3, \dots)$ which is Cauchy. □

Exercise 2.6.3 If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(i) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(ii) Do the same for the product $(x_n y_n)$.

Proof. Let $\varepsilon > 0$. Since (x_n) converges, then $(\exists N_1 \in \mathbb{N})(\forall n \geq N_1)[|x_n - x| < \varepsilon/2]$. Same thing for (y_n), $(\exists N_2 \in \mathbb{N})(\forall n \geq N_2)[|y_n - y| < \varepsilon/2]$. Choose $N = \max(\{N_1, N_2\})$. Then, for all $n \geq N$, we have $|(x_n + y_n) - (x - y)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore, $(x_n + y_n)$ converges, meaning it's a Cauchy sequence. $\square$

Proof. Let $\varepsilon > 0$. Since (x_n) converges, then $(\exists N_1 \in \mathbb{N})(\forall n \geq N_1)[|x_n - x| < \varepsilon/2M_1]$. Same thing for (y_n), $(\exists N_2 \in \mathbb{N})(\forall n \geq N_2)[|y_n - y| < \varepsilon/2M_2]$. Choose $N = \max(\{N_1, N_2\})$. Bound $|x_n| \leq M_1$ and $|y_n| \leq M_2$, for all $n \in \mathbb{N}$. Then, for all $n \geq N$, we have

$$\begin{aligned}
|(x_n y_n) - (xy)| &= |(x_n y_n) - (y_n x) + (y_n x) - (xy)| \\
&\leq |x_n y_n - y_n x| + |y_n x - xy| \\
&\leq M_1 \cdot |x_n - x| + M_2 \cdot |y_n - x| \\
&< M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} = \varepsilon.
\end{aligned}$$

Therefore, $(x_n y_n)$ converges, meaning it's a Cauchy sequence. $\square$