Funds of Anal I: Homework 2

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Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

Exercise 1.3.1

(i) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(ii) Now, state and prove a version of Lemma 1.3.8 for greatest lower bound.

Solution 1.3.1

(i) **Definition 1.3.2B.** A real number t is the *infimum* or *greatest lower bound* of a set S if

  (a) t is a lower bound of S, and

  (b) if t' is any lower bound of S, then $t \leq t'$.

(ii) **Lemma 1.3.8B.** *Assume $s \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $s = \inf(A)$ if and only if for all $\varepsilon > 0$, there exists an element $a \in A$ such that $s + \varepsilon > a$.*

Proof: Assume $s = \inf(A)$ and consider $s + \varepsilon$ for some $\varepsilon > 0$. Then, $s + \varepsilon$ cannot be a lower bound on A because (ii) implies all lower bounds b must be such that $s \leq b$. Therefore, there must exist an element $a \in A$ such that $s + \varepsilon > a$.

Conversely, for all $\varepsilon > 0$, there exists an $a \in A$ such that $s + \varepsilon > a$. Then, $s + \varepsilon$ is not a lower bound for all ε, which is the same as saying every lower bound b must have $b \leq s$ (ii). $\square$

Exercise 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup(A)$.

Solution 1.3.7

Proof: If a is an upper bound for the set $A \subseteq \mathbb{R}$ and is an element of A, then, by definition, $a = \max(A) = \sup(A)$. $\square$

Exercise 1.3.9

(i) If $\sup(A) < \sup(B)$, show that there exists an element $b \in B$ that is an upper bound for A.

(ii) Give an example to show that this is not always the case if we only assume $\sup(A) \leq \sup(B)$.

Solution 1.3.9

(i) **Proof:** We'll prove this case by case.

Case: 1 If $\sup(B) = \max(B)$, then we can choose $b = \sup(B)$ and $b \in B$.

Case: 2 Since $\sup(B) > \sup(A)$, then, by the Theorem 1.4.3 [Density of $\mathbb{Q}$ in $\mathbb{R}$], there exists some c in between $\sup(A)$ and $\sup(B)$. Let $c = {}^{\sup(A) + \sup(B)}/_{2}$. Then, $c \in B$, since $\sup(B) > c$. Therefore, $\sup(B) > c > \sup(A)$ and $c \in B$. □

(ii) Let $A = (-\infty, 1]$ and $B = (-\infty, 1)$, then $\sup(A) = 1$ and $\sup(B) = 1$. However, there is no element $b \in B$ that is an upper bound for A.

Exercise 1.3.11

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(i) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup(A) \leq \sup(B)$.

(ii) If $\sup(A) < \inf(B)$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(iii) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup(A) < \inf(B)$.

Solution 1.3.11

(i) True: If $A \subseteq B$, this means that $\forall x \in A \Rightarrow x \in B$. This means that $\sup(A) \in B$, but not always the case that $\sup(B) \in A$. Therefore, $\sup(A) \leq \sup(B)$.

(ii) True: Since $\sup(A) < \sup(B)$, then, by the Theorem 1.4.3 [Density of $\mathbb{Q}$ in $\mathbb{R}$], there exists some c in between $\sup(A)$ and $\sup(B)$. Let $c = {}^{\sup(A)\,+\,\sup(B)}/_2$. Then, $c \in \mathbb{R}$ and $a < c < b$ for all $a \in A$ and $b \in B$.

(iii) False: Let $A = (-\infty, 1]$ and $B = [1, \infty)$, then $1 < 1$.

Exercise 1.4.1

Recall that **I** stands for the set of irrational numbers.

(i) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(ii) Show that if $a \in \mathbb{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(iii) Part 1 can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is **I** closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution 1.4.1

(i) **Proof:** The rational numbers a and b can be expressed as $a = {}^{p}/_{q}$ and $b = {}^{r}/_{s}$, where $p, q, r, s \in \mathbb{Z}$ and $q, s \neq 0$. Then, $a + b = {}^{p}/_{q}{}^{r}/_{s} = \frac{ps+rq}{qs}$, where $ps + rq \in \mathbb{Z}$ and $qs \in \mathbb{Z}$. Therefore, $a + b \in \mathbb{Q}$. Then, $ab = {}^{p}/_{q} \cdot {}^{r}/_{s} = {}^{pr}/_{qs}$, where $pr \in \mathbb{Z}$ and $qs \in \mathbb{Z}$. Therefore, $ab \in \mathbb{Q}$. $\square$

(ii) **Proof:** Suppose $a + t \in \mathbb{Q}$, then, by 1, $(a + t) - a \in \mathbb{Q}$, contradicting the initial assumption that $t \in \mathbf{I}$. $\square$

(iii) No, **I** is not closed under addition and multiplication. For example, $\sqrt{2} - \sqrt{2} = 0$ and $\sqrt{2} \cdot \sqrt{2} = 2$.

Exercise 1.4.3

Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution 1.4.3

Proof: Suppose $x \in \bigcap\limits_{n=1}^{\infty}(0, 1/n)$, then we have $0 < x < 1/n$ for all $n \in \mathbb{N}$. However, this is impossible by the Archimedean Property. Therefore, $\bigcap\limits_{n=1}^{\infty}(0, 1/n) = \emptyset$. $\square$

Exercise 1.4.4

Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup(T) = b$.

Solution 1.4.4

Proof: The intersection of $\mathbb{Q}$ and $[a, b]$ is the interval $[a, b]$. Since b is an upper bound for T, then $b = \sup(T)$. $\square$