Introduction to Toplogy I: Homework 4

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Exercise 1.12. Let
$$W = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, 0\right\},$$
with the metric induced from the usual one on $\mathbb{R}$. Let (X, d_X) be another metric space. Given a sequence $p_1, p_2, p_3, \cdots \in X$ and a point $\ell \in X$, prove that $p_n \to \ell$ if and only if the map $f : W \to X$ defined by
$$\begin{cases} f(1/n) = p_n, \\ f(0) = \ell \end{cases}.$$

Solution. Assume $p_n \to \ell$. Each point $1/n$ is isolated in W, meaning that there exists $r > 0$ such that $B_W(1/n, r) = \{1/n\}$. Hence for any $\varepsilon > 0$ choose $\delta = r$. Then whenever $\mathrm{d}_W(x, 1/n) < \delta$ we have $x = 1/n$ and therefore $\mathrm{d}_X(f(x), f(1/n)) = 0 < \varepsilon$. Thus f is continuous at each $1/n$. Let $\varepsilon > 0$. Since $p_n \to \ell$, there exists N such that for all $n \geq N$ we have $\mathrm{d}_X(p_n, \ell) < \varepsilon$. In W the open interval $(-1/(N+1), 1/(N+1)) \cap W$ is an open neighborhood of 0 containing 0 and all points $1/n$ with $n \geq N+1$. Choose $\delta = 1/(N+1)$. Then if $x \in W$ and $\mathrm{d}_W(x, 0) < \delta$, necessarily $x = 0$ or $x = 1/n$ with $n \geq N+1$. In either case $\mathrm{d}_X(f(x), f(0)) = \mathrm{d}_X(p_n, \ell) < \varepsilon$ (when $x = 0$ the distance is 0). Thus f is continuous at 0. Combining the two parts, f is continuous on all of W.

Conversely, assume f is continuous. Let $\varepsilon > 0$. By continuity of f at 0 there exists $\delta > 0$ such that whenever $x \in W$ and $\mathrm{d}_W(x, 0) < \delta$ we have $\mathrm{d}_X(f(x), f(0)) < \varepsilon$. Choose N with $1/N < \delta$. Then for every $n \geq N$ we have $\mathrm{d}_W(1/n, 0) = 1/n < \delta$, so $\mathrm{d}_X(p_n, \ell) = \mathrm{d}_X(f(1/n), f(0)) < \varepsilon$. Hence $p_n \to \ell$.

Therefore $p_n \to \ell$ if and only if f is continuous. □

Exercise 3.4. Let $S \subset [0, 1]$, and let $A \subset C([0, 1])$ be the set of continuous functions that vanish on S as in Example 3.4. Prove that A is closed in the sup metric.

Hint: You could prove this directly from the definitions, or you could use Proposition 2.10 and Proposition 1.10 together with Example 1.8(c), which is stated for evaluation at $x = 0$ but which we can see is equally valid for evaluation at any $x \in [0, 1]$.

Solution. Let $f_n \in A$ and $f_n \to f$ in the sup norm. Fix $s \in S$. For every n we have $f_n(s) = 0$. Hence
$$|f(s)| = \lim_{n\to\infty} |f(s) - f_n(s)| \leq \lim_{n\to\infty} \mathrm{d}_\infty(f, f_n) = 0,$$
so $f(s) = 0$. As $s \in S$ was arbitrary, f vanishes on S, i.e. $f \in A$. Therefore A is closed. □

Exercise 4.1. Prove that the sequence of piecewise-linear functions $f_1, f_2, f_3, \cdots \in C([0, 1])$ introduced at the beginning of the section is Cauchy in the L^1 metric.

Solution. Let f_n be the piecewise-linear function with $f_n(0) = 0$, $f_n(1/n) = 1$, and $f_n(1) = 1$. Let $g \equiv 1$. For each n,
$$\mathrm{d}_1(f_n, g) = \int_0^1 |f_n(x) - 1| \mathrm{d}x = \frac{1}{2n}.$$
Hence for $m, n \geq$ ‘1,
$$\mathrm{d}_1(f_m, f_n) \leq \mathrm{d}_1(f_m, g) + \mathrm{d}_1(g, f_n) = \frac{1}{2m} + \frac{1}{2n} \leq \frac{1}{\min\{m, n\}}.$$
Given $\varepsilon > 0$ choose $N > 1/\varepsilon$. Then for $m, n \geq N$,
$$\mathrm{d}_1(f_m, f_n) \leq \frac{1}{\min\{m, n\}} \leq \frac{1}{N} < \varepsilon.$$
Thus (f_n) is Cauchy in the L^1 metric. □

Exercise 4.3. Let $p_1, p_2, p_3, \cdots$ be a Cauchy sequence in a metric space (X, d). Prove that the sequence is bounded, meaning that there is a point $q \in X$ and a radius $R > 0$ such that $p_n \in B_R(q)$. (In fact, for any $q \in X$ you can find such a radius R, and in particular for $X = \mathbb{R}^n$ you can take $q = 0$.)

Hint: Start by applying the definition of Cauchy with $\varepsilon = 1$ to get an N such that if $m, n \geq N$ then $\mathrm{d}(p_m, p_n) < 1$, and in particular $\mathrm{d}(p_N, p_n) < 1$.

Solution. Since (p_n) is Cauchy, pick $\varepsilon = 1$. Then there exists N such that for all $m, n \geq N$, $\mathrm{d}(p_m, p_n) < 1$. Fix $q := p_N$ and define

$$R := \max\left\{1, \max_{1 \leq k \leq N-1} \mathrm{d}(p_N, p_k)\right\}.$$

If $n \geq N$ then $\mathrm{d}(p_n, q) = \mathrm{d}(p_n, p_N) < 1 \leq R$. If $n < N$ then by definition $\mathrm{d}(p_n, q) = \mathrm{d}(p_n, p_N) \leq R$. Hence for every n we have $\mathrm{d}(p_n, q) \leq R$, so $p_n \in B_R(q)$. Thus (p_n) is bounded. $\square$