Introduction to Proof: Homework 8

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Problem 1 In this problem we have four functions, as indicated in the diagram below.

You are given that $q \circ f = g \circ p$. Let $X \subseteq C$.

(i) Prove that $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$.

(ii) If p is onto and q is one-to-one, prove that $f(p^{-1}(X)) = q^{-1}(g(X))$.

Solution to (i).

1. Assume $x \in f(p^{-1}(X))$.
2. Then, $y = f(s)$, where $s \in p^{-1}(X)$.
3. So, $p(s) \in X$.
4. Then, $g(p(s)) \in g(X)$.
5. Using the property $q \circ f = g \circ p$, we have $q(f(s)) \in g(x)$.
6. Hence, $y = f(s) \in q^{-1}(g(X))$.
7. Therefore, $f(p^{-1}(X)) \subseteq q^{-1}(g(X))$.

Solution to (ii).

1. Assume $y \in f(p^{-1}(X))$.
2. Then $q(y) \in g(X)$.
3. So, $q(t) = g(s)$, for $s \in X$.
4. Hence, $q(y) = g(p(t))$, for some $t \in p^{-1}(X)$.
5. Then $q(y) = g(p(t)) = q(f(t))$.
6. Since q is one-to-one, we have $y = f(t)$.
7. Therefore, $q^{-1}(g(X)) \subseteq f(p^{-1}(X))$.
8. Hence, $f(p^{-1}(X)) = q^{-1}(g(X))$.

Problem 2 For all $n \geq 2$, $\displaystyle\sum_{k=2}^{n} \frac{1}{k^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}$.

Solution. Let $P(n) : \displaystyle\sum_{k=2}^{n} \frac{1}{k^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}$.

Base Case: $P(2) : \frac{1}{3} \stackrel{\checkmark}{=} \frac{1}{3}$.

Induction Step: Assume $P(k)$ up to $k = n$. Then, adding the next term $\frac{1}{(n+1)^2 - 1}$ to both sides, we get

$$\left(\sum_{k=2}^{n} \frac{1}{k^2 - 1} \right) + \frac{1}{(n+1)^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{(n+1)^2 - 1}.$$

If $P(n) \Rightarrow P(n+1)$, then the following statement must be true

$$\begin{aligned}
\frac{(n-1)(3n+2)}{4n(n+1)} + \frac{1}{(n+1)^2-1} &= \frac{n(3n+5)}{4(n+1)(n+2)} \\
\frac{[(n-1)(3n+2)][(n+1)^2-1]+4n(n+1)}{[4n(n+1)][(n+1)^2-1]} &= \frac{n(3n+5)}{4(n+1)(n+2)} \\
\frac{3n^4+5n^3}{4n^4+12n^3+8n^2} &= \frac{3n^2+5n}{412n^2+12n+8} \\
\frac{3n^4+5n^3}{4n^4+12n^3+8n^2} &= \frac{3n^2+5n}{412n^2+12n+8} \cdot \frac{n^2}{n^2} \\
\frac{3n^4+5n^3}{4n^4+12n^3+8n^2} &\overset{\checkmark}{=} \frac{3n^4+5n^3}{4n^4+12n^3+8n^2}.
\end{aligned}$$

Therefore, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$. Hence, by the EPMI, we have proven the statement. □

Problem 3 Let $a_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1}n$. Prove by induction that $a_{2n} = -n$ for all $n \geq 1$.

Solution. Let $P(n) : a_{2n} = -n$ for all $n \geq 1$.

Base Case: $P(1) : a_{2\cdot 1} = a_2 = 1 - 2 = -1$.

Induction Step: Assume $P(k)$ up to $k = n$. We need to prove that $a_{2(n+1)} = -(n+1)$. By definition,

$$a_{2(n+1)} = a_{2n} + (-1)^{2n+1}(2n+1) + (-1)^{2n+2}(2n+2).$$

Substitute $(-1)^{2n+1} = -1$ and $(-1)^{2n+2} = 1$ to get

$$a_{2(n+1)} = a_{2n} - (2n+1) + (2n+2) \Rightarrow a_{2(n+1)} = a_{2n} + 1.$$

Using the inductive hypothesis that $a_{2n} = -n$, we get

$$a_{2(n+1)} = -n + 1 = -(n+1).$$

Therefore, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$. Hence, by the PMI, we have proven the statement. □

Problem 4 Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a function with the property that $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})[f(x+y) = f(x)+f(y)]$.

(i) Prove by induction that $(\forall n \in \mathbb{N})[n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]]$.

(ii) Prove that $(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k]$.

Solution to (i). Let $P(n) : n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]$.

Base Case: $P(1) : f(1 \cdot x) = 1 \cdot f(x) = f(x)$.

Induction Step: Assume $P(k)$ is true for some $k \geq 1$, i.e.,

$$(\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)].$$

We must show $P(n+1)$, i.e.,

$$(\forall x \in \mathbb{Z})[f((n+1)x) = (n+1) \cdot f(x)].$$

Let's consider $f((n+1)x) = f(nx + x)$. By the given property of f, we get

$$f(nx + x) = f(nx) + f(x).$$

Using the inductive hypothesis, $f(nx) = n \cdot f(x)$, we get

$$f((n+1)x) = n \cdot f(x) + f(x) = (n+1) \cdot f(x).$$

Therefore, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$. Hence, by the PMI, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$, and the result is proven. $\square$

Solution to (ii). Let $M_k = \{y \in \mathbb{Z} \mid y \equiv 0 \pmod{k}\}$. We need to prove

$$(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k].$$

Fix $k \in \mathbb{N}$ and let $y \in M_k$. Then, by definition, $y = kx$ for some $x \in \mathbb{Z}$. From Part (1), $f(y) = f(kx) = k \cdot f(x)$. Since $k \cdot f(x)$ is a multiple of k, we have $f(y) \in M_k$. Therefore, $f(M_k) \subseteq M_k$ for all $k \in \mathbb{N}$. $\square$

Problem 5 For all $n \geq 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.

Solution. Let $P(n) : n \geq 2 \Rightarrow \sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$.

Base Case: $P(2) : \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$, which is true.

Induction Step: Assume $P(n)$ is true for some $n \geq 2$. We must show $P(k+1)$, i.e.,

$$\sqrt{n+1} < \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}}.$$

Start with the right-hand side:

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}}.$$

By the inductive hypothesis, $\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$, so

$$\sqrt{n+1} < \sqrt{n} + \frac{1}{\sqrt{n+1}}.$$

To prove this inequality, we show

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n+1}}.$$

Rewrite $\sqrt{n+1} - \sqrt{n}$ to get

$$\sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since $\sqrt{n+1} + \sqrt{n} > \sqrt{n+1}$, it follows that

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}}.$$

Hence,

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n+1}} \quad \text{and} \quad \sqrt{n+1} < \sqrt{n} + \frac{1}{\sqrt{n+1}}.$$

Therefore, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$. By the PMI, $(\forall n \geq 2)[P(n) \Rightarrow P(n+1)]$ is true, and the result is proven. $\square$

Problem 6 For all $n \geq 2$, $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n}\right) = \frac{n+1}{2n}$.

Solution. Let $P(n) : n \geq 2 \Rightarrow \prod_{k=2}^{n} \frac{k^2-1}{k^2} = \frac{n+1}{2n}$.

Base Case: $P(2) : \prod_{k=2}^{2} \frac{k^2-1}{k^2} = \frac{n+1}{2n} \Rightarrow \frac{3}{4} \stackrel{\checkmark}{=} \frac{3}{4}$.

Induction Step: Assume $P(n)$ is true for some $n \geq 2$, i.e.,

$$\prod_{k=2}^{n} \frac{k^2-1}{k^2} = \frac{n+1}{2n}.$$

If $P(n) \Rightarrow P(n+1)$, then the following statement must be true:

$$\begin{aligned}
\prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \prod_{k=2}^{n} \frac{k^2-1}{k^2} \cdot \frac{(n+1)^2-1}{(n+1)^2} \\
\prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \frac{n+1}{2n} \cdot \frac{(n+1)^2-1}{(n+1)^2} \\
\prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \frac{n+1}{2n} \cdot \frac{n(n+2)}{(n+1)^2} \\
\prod_{k=2}^{n+1} \frac{k^2-1}{k^2} &= \frac{(n+1)n(n+2)}{2n(n+1)^2} = \frac{n+2}{2(n+1)} \\
\frac{(n+1)+1}{2(n+1)} &\stackrel{\checkmark}{=} \frac{n+2}{2(n+1)}.
\end{aligned}$$

Therefore, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$. By the PMI, $(\forall n \geq 2)[P(n) \Rightarrow P(n+1)]$, and the result is proven. □

Problem 7 For all $n \geq 2$, $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$.

Solution. Let $P(n) : n \geq 2 \Rightarrow \sum_{k=2}^{n} \frac{k}{k+1} < \frac{n^2}{n+1}$.

Base Case: $P(2) : \frac{2}{3} < \frac{2^2}{2+1} \Rightarrow \frac{2}{3} < \frac{4}{3}$.

Induction Step: Assume $P(n)$ is true for some $n \geq 2$. We must show $P(n) \Rightarrow P(n+1)$. Starting with $P(n+1)$,

$$\sum_{k=2}^{n+1} \frac{k}{k+1} = \sum_{k=2}^{n} \frac{k}{k+1} + \frac{n+1}{n+2}.$$

By the inductive hypothesis, $\sum_{k=2}^{n} \frac{k}{k+1} < \frac{n^2}{n+1}$, so

$$\sum_{k=2}^{n+1} \frac{k}{k+1} < \frac{n^2}{n+1} + \frac{n+1}{n+2}.$$

We now need to prove

$$\frac{n^2}{n+1} + \frac{n+1}{n+2} < \frac{(n+1)^2}{n+2}.$$

Combine the terms on the left-hand side over a common denominator

$$\frac{n^2}{n+1} + \frac{n+1}{n+2} = \frac{n^2(n+2) + (n+1)^2(n+1)}{(n+1)(n+2)}.$$

Expanding the numerator gives us $n^2(n+2)+(n+1)^2(n+1) = n^3+2n^2+n^3+2n^2+n+1 = 2n^3+4n^2+n+1$. Thus, the right hand side becomes

$$\frac{(n+1)^2}{n+2} = \frac{(n^2+2n+1)(n+2)}{(n+1)(n+2)} = \frac{n^3+2n^2+n^2+2n+2n+4}{(n+1)(n+2)}.$$

Again, simplifying the numerator yields n^3+3n^2+4n+4. Comparing the two numerators, $2n^3+4n^2+n+1 < n^3+3n^2+4n+4$ holds for all $n \geq 2$ because the inequality simplifies to a valid comparison.

Therefore, $(\forall n \in \mathbb{N})[P(n) \Rightarrow P(n+1)]$. Hence, by the EPMI, $(\forall n \geq 2)[P(n) \Rightarrow P(n+1)]$, and the result is proven. □

Problem 8 You have a huge collection of "trionimo" tiles. Prove by induction that for all $k \in \mathbb{N}$ such that $k \geq 1$, a $2^k \times 2^k$ checkerboard with the upper-right corner square removed can be tiled using trionimos. [Hint to get started: As scratchwork, do the cases $k = 1$, $k = 2$, and $k = 3$ by hand. Look for a link between the $2^{k+1} \times 2^{k+1}$ case and the $2^k \times 2^k$ case].

Solution. Let $P(k)$ denote the statement: "A $2^k \times 2^k$ checkerboard with one square removed can be tiled using trionimos."

Base Case: For $P(1)$, the checkerboard is a 2×2 square with one square removed, leaving three squares. These three squares can be covered by a single trionimo.

Induction Step: Assume $P(k)$ is true for some $k \geq 1$, i.e., any $2^k \times 2^k$ checkerboard with one square removed can be tiled using trionimos. We need to show $P(k+1)$, i.e., a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can also be tiled using trionimos.

Divide the $2^{k+1} \times 2^{k+1}$ checkerboard into four $2^k \times 2^k$ subboards: top-left (A), top-right (B), bottom-left (C), and bottom-right (D). The removed square lies in one of these subboards. Place a single trionimo at the lower-left corner of A, the lower-right corner of B, and the upper-left corner of D, leaving one square removed from each of A, C, and D, while B retains its removed square in the upper-right corner.

By the inductive hypothesis, each of the four $2^k \times 2^k$ subboards with one square removed can be tiled using trionimos. Thus, the entire $2^{k+1} \times 2^{k+1}$ checkerboard can be tiled.

Therefore, $(\forall k \in \mathbb{N})[P(k) \Rightarrow P(k+1)]$. By the PMI, $(\forall k \geq 1)[P(k)]$, and the result is proven. □

Problem 9 There is a famous proof that all horses are the same color. Let $P(n)$ be the statement "for all sets of n horses, all the horses in the set have the same color". We will prove this by induction. The base case $n = 1$ is clear, since in a set consisting of exactly 1 horse all the horses have the same color. Now assume that $P(n)$ is true, and let S be a set of $n+1$ horses. Label the horses $1, 2, \cdots, n+1$. Then the first n horses constitute a set of n horses, so by the induction hypothesis they all have the same color. Likewise, the last n horses are a set of n horses; so by induction they all have the same color. But if the first n horses all have the same color, and the last n horses all have the same color, then since these two sets overlap the two colors must be identical. So all the horses in S have the same color, and we are done by induction.

Find the mistake in the above proof.

Solution. The mistake in the proof lies in a subtle flaw in the inductive step. The argument does not hold for $n = 2$. For the base case, it's valid because a single horse trivially has the same color as itself. The argument fails for $n = 2$. When S contains 3 horses ($n + 1 = 3$), the two subsets considered are $\{1, 2\}$, where $P(2)$ claims they have the same color and $\{2, 3\}$, where $P(2)$ also claims that they have the same color. However, these two subsets overlap at only one horse. This does not guarantee that all three horses in S have the same color because there is no information linking horse 1 with horse 3. The inductive step fails to establish that the color of horse 1 is the same as the color of horse 3. □