SOLUTIONS TO HOMEWORK 3

Warning: Little proofreading has been done.

1. Section 2.2

Exercise 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ *verconges* to $x \in \mathbb{R}$ if there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ it is true that $n \geq \mathbb{N}$ implies $|x_n - x| < \varepsilon$.

Give an example of a vercongent sequence. Can you give an example of a vergonent sequence that is divergent? What exactly is being described in this strange definition?

Solution. Solving the last part provides answers to the other parts, so we start there.

We claim that $(x_n)_{n\in\mathbb{N}}$ verconges to x if and only if $(x_n)_{n\in\mathbb{N}}$ is bounded, regardless of what x is.

First, suppose $(x_n)_{n\in\mathbb{N}}$ is bounded. Then there is $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Set $\varepsilon = M + |x| + 1$. Let $N \in \mathbb{N}$. Let $n \geq N$. Then

$$|x_n - x| \leq |x_n| + |x| \leq M + |x| < M + |x| + 1 = \varepsilon.$$

This shows that $(x_n)_{n\in\mathbb{N}}$ verconges to x.

Now assume that $(x_n)_{n\in\mathbb{N}}$ verconges to x. Let $\varepsilon > 0$ be as in the definition, so that for all $N \in \mathbb{N}$ it is true that $n \geq \mathbb{N}$ implies $|x_n - x| < \varepsilon$. In particular, we may take $N = 1$, so that for all $n \in \mathbb{N}$ we have $|x_n - x| < \varepsilon$. Set $M = \varepsilon + |x|$. Then for all $n \in \mathbb{N}$, we have

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < \varepsilon + |x| = M.$$

This shows that $(x_n)_{n\in\mathbb{N}}$ is bounded.

Now it is easy to give an example of a vercongent sequence, in fact one which does not converge: $x_n = (-1)^n$ for $n \in \mathbb{N}$ will do. □

Exercise 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limits.

(1) $\displaystyle\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(2) $\displaystyle\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$.

(3) $\displaystyle\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution. (1) Let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ such that

$$N \geq \frac{3}{25\varepsilon}. \qquad \text{(which shows} \quad \frac{1}{N} \leq \frac{25}{3}\varepsilon)$$

Let $n \in \mathbb{N}$ satisfy $n \geq N$. Then

$$\begin{aligned}\left|a_n - \frac{2}{5}\right| &= \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{-13}{2(2n+5)}\right| = \frac{3}{5(5n+4)} \\ &\leq \frac{3}{25n} \leq \frac{3}{25N} \leq \frac{3}{25} \times \frac{25}{3}\varepsilon = \varepsilon.\end{aligned}$$

This completes the proof. □

(2) Let $\varepsilon > 0$.
Choose $N \in \mathbb{N}$ such that $N > 2/\varepsilon$.
Let $n \in \mathbb{N}$ satisfy $n \geq N$. Then, using $n^3 + 3 \geq n^3$ at the first step, we have

$$|a_n - 0| = \frac{2n^2}{n^3 + 3} \leq \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

This completes the proof of (2). □

(3) Let $\varepsilon > 0$.
Choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon^3$.
Let $n \in \mathbb{N}$ satisfy $n \geq N$. Then, using $|\sin x| \leq 1$ in the first step,

$$|a_n - 0| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \frac{1}{\sqrt[3]{\varepsilon^{-3}}} = \frac{1}{\varepsilon^{-1}} = \varepsilon.$$

This completes the proof of (3). □

Exercise 2.2.3. Describe what we would have to demonstrate in order to disprove each of the following statements.

(1) At every college in the United States, there is a student who is at least seven feet tall.
(2) For all colleges in the United States, there exists a professor (at that college) who gives every student a grade of either A or B.
(3) There exists a college in the United States where every student is at least six feet tall.

Solution. (1) Find a suitable college in the United States, measure the heights of all the students at that college, and find that the heights are all less than seven feet.

(2) Find a suitable college in the United States, and for every professor at that college find some student who got from that professor a grade of neither A nor B.

(3) Go to every college in the United States, and find some student at that college who is less than six feet tall. □

Exercise 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(1) A sequence with an infinite number of ones that does not converge to one.
(2) A sequence with an infinite number of ones that converges to a limit not equal to one.
(3) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution. (1) The sequence $((-1)^n)$ contains infinitely many ones and it diverges.

(2) This is impossible. Suppose a_n is such a sequence and a_n converges to $a \neq 1$. Choose $ep = |1-a|/2 > 0$. Since there are infinitely many ones in the sequence, no matter how big is N, there exists an $a_{n_0} = 1$ with $n_0 > N$, for which

$$|a_{n_0} - a| = |1 - a| > |1 - a|/2 = \varepsilon.$$

This shows that a_n does not converge to a. Contradiction.

(3) Here is an example of such a sequence:

$$1, 0, \overbrace{1, 1}^{2}, 0, \overbrace{1, 1, 1}^{3}, 0, \overbrace{1, 1, 1, 1}^{4}, 0, \overbrace{1, 1, 1, 1, 1}^{5}, 0, ...$$

This sequence can also be defined symbolically as following: define a sequence of integers recursively by

$$n_1 = 0, \qquad n_{k+1} = n_k + k + 1, \quad k = 1, 2, 3, \ldots.$$

This defines $(n_2, n_3, n_4, \ldots ....) = (2, 5, 9, 14, \ldots)$. Then our sequence is defined by

$$a_{n_k} = 0, \quad k = 2, 3, ... \quad \text{and} \qquad a_m = 1, \quad \text{if } m \neq n_k \text{ and } m \in \mathbb{N}.$$

□

Exercise 2.2.6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Solution. Suppose $a \neq b$. Set $\varepsilon = \frac{1}{3}|a-b|$. Then $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, we have $|a_n - a| < \varepsilon$. Choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$, we have $|a_n - b| < \varepsilon$. Choose some $n \in \mathbb{N}$ with $n \geq \max(N_1, N_2)$. Then, by triangle inequality,

$$\begin{aligned} |a-b| &= |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| \\ &< \varepsilon + \varepsilon = 2\varepsilon = \tfrac{2}{3}|a-b| < |a-b|. \end{aligned}$$

This is a contradiction, so $a = b$. $\square$

The following proof does not use contradiction.

Alternate solution. We prove that for every $\varepsilon > 0$, we have $|a-b| < \varepsilon$. This will imply that $|a-b| = 0$ or $a = b$.

Let $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, we have $|a_n - a| < \frac{1}{2}\varepsilon$. Choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$, we have $|a_n - b| < \frac{1}{2}\varepsilon$. Choose some $n \in \mathbb{N}$ with $n \geq \max(N_1, N_2)$. Then, by triangle inequality,

$$|a-b| \leq |a - a_n| + |a_n - b| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof. $\square$

2. Section 2.3

Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.
(b) If $(x_n) \to x$, show that $\sqrt{x_n} \to x$.

Solution. (a) Let $\varepsilon > 0$. Since $(x_n) \to 0$, choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$, we have

$$|x_n| = |x_n - 0| < \varepsilon^2,$$

which implies, taking square root, that

$$\sqrt{x_n} = |\sqrt{x_n} - 0| < \varepsilon.$$

(b) The case $x = 0$ is in (a). We consider $x \neq 0$. Since $x_n \geq 0$, then $x > 0$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - x| < \sqrt{x}\varepsilon,$$

which shows, by $x_n - x = (\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})$, that

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\sqrt{x}\varepsilon}{\sqrt{x}} = \varepsilon.$$

This completes the proof. $\square$

Exercise 2.3.3: the Squeeze Theorem. Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Let $\varepsilon > 0$.

Choose $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, we have $|x_n - l| < \varepsilon$. Choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$, we have $|z_n - l| < \varepsilon$. Define $N = \max(N_1, N_2)$.

Let $n \in \mathbb{N}$ with $n \geq N$. Since $n \geq N_1$, we have

$$l - \varepsilon < x_n < l + \varepsilon.$$

Since $n \geq N_2$, we have

$$l - \varepsilon < z_n < l + \varepsilon.$$

Therefore

$$l - \varepsilon < x_n \leq y_n \leq z_n < l + \varepsilon.$$

Thus $|y_n - l| < \varepsilon$. $\square$