Taylor Polynomials

Recall, if $f(x)$ is differentiable, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ is Taylor series centered at a.

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$T_1(x) = f(a) + f'(a)(x-a)$ is simply the linearization.

For $f(x,y)$, what is n^{th} degree Taylor polynomial at (a,b)?

$$f(x,y) = a_{00} + a_{10}(x-a) + a_{01}(y-b) + a_{20}(x-a)^2 + a_{11}(x-a)(y-b)$$
$$+ a_{02}(y-b)^2 + a_{30}(x-a)^3 + a_{21}(x-a)^2(y-b)$$
$$+ a_{12}(x-a)(y-b)^2 + a_{03}(y-b)^3 + \cdots$$

$$f(a,b) = a_{00}$$

$$f_x = a_{10} + 2a_{20}(x-a) + a_{11}(y-b) + 3a_{30}(x-a)^2 + 2a_{21}(x-a)(y-b)$$
$$+ a_{12}(y-b)^2 + \cdots$$

$$f_x(a,b) = a_{10}$$

$$f_y = a_{01} + a_{11}(x-a) + 2a_{02}(y-b) + a_{21}(x-a)^2 + 2a_{12}(x-a)(y-b)$$
$$+ 3a_{03}(y-b)^2$$

$$f_y(a,b) = a_{01}$$

$f_{xx} = 2a_{20} + 2(3)a_{30}(x-a) + 2a_{21}(y-b) + \dots$

$f_{xx}(a,b) = 2a_{20} \implies a_{20} = \frac{1}{2} f_{xx}(a,b)$

$f_{xy} = a_{11} + 2a_{21}(x-a) + 2a_{12}(y-b) + \dots$

$f_{xy}(a,b) = a_{11}$

Under appropriate continuity assumptions, $f_{xy} = f_{yx}$

$f_{yy} = 2a_{02} + 2a_{12}(x-a) + 2(3)a_{03}(y-b) + \dots$

$f_{yy}(a,b) = 2a_{02} \implies a_{02} = \frac{1}{2} f_{yy}(a,b)$

This process can continue to find the general formula for a_{nm}.

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$+ \frac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)(y-b)^2$$

$$+ \dots$$

Note: $T_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is the tangent plane

In general, $T_n(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\partial^{(i+j)} f}{\partial x^i \partial y^j}(a,b) \frac{1}{i!\,j!}(x-a)^i (y-b)^j$

is the n^{th} degree Taylor polynomial of $f(x,y)$ at (a,b).

Ex: Find second degree Taylor polynomial of

$$f(x,y) = \frac{x-y}{3y+2x} \quad \text{at } (2,-1)$$

$f(2,-1) = 3$

$$f_x = \frac{5y}{(3y+2x)^2}$$

$f_x(2,-1) = -5$

$f_y(2,-1) = -10$

$$f_y = \frac{-5x}{(3y+2x)^2}$$

$$f_{xx} = \frac{-20y}{(3y+2x)^3}$$

$$f_{xy} = \frac{(3y+2x)^2(5) - 5y(2)(3y+2x)(3)}{(3y+2x)^4}$$

$$= \frac{5(3y+2x) - 30y}{(3y+2x)^3} = \frac{10x - 15y}{(3y+2x)^3}$$

$$f_{yy} = \frac{30x}{(3y+2x)^3}$$

$f_{xx}(2,-1) = 20$

$f_{xy}(2,-1) = 35$

$f_{yy}(2,-1) = 60$

$$T_2(x,y) = 3 - 5(x-2) - 10(y+1) + 10(x-2)^2$$

$$+ 35(x-2)(y+1) + 30(y+1)^2$$

$\approx f(x,y)$ for (x,y) near $(2,-1)$

Recall, $f(2.01, -1.01) = 3.0\overline{5}$

$L(2.01, -1.01) = T_1(2.01, -1.01) = 3.05$

$$T_2(2.01, -1.01) = 3 - 5(0.01) - 10(-0.01) + 10(0.01)^2 + 35(0.01)(-0.01) + 30(-0.01)^2$$

$$= 3.0505$$

The first degree Taylor polynomial (tangent plane) agrees with f in first three decimal places

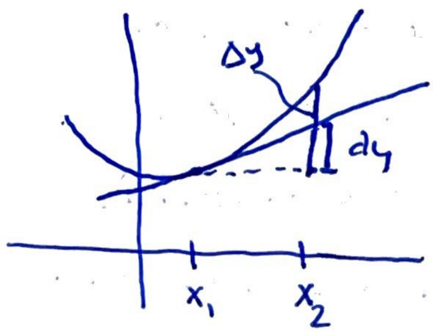
The second degree Taylor polynomial agrees with f in first five decimal places

Differentials

Recall, suppose $y = f(x)$ is differentiable at $x = x_1$

The tangent line to $y = f(x)$ at $(x_1, f(x_1))$ is

$$y = f(x_1) + f'(x_1)(x - x_1)$$



Over $[x_1, x_2]$,

$dx = \Delta x = x_2 - x_1$

$\Delta y = f(x_2) - f(x_1)$; Change in function.

$dy = f'(x)\,dx$

At $x = x_2$,

$dy = f'(x_2)(x_2 - x_1)$; Change in tangent line

If x_2 is close to x_1, then

$\Delta y \approx dy$ Change in function over $[x_1, x_2]$ can be approximated by change in tangent line (linearization) over $[x_1, x_2]$.

Now suppose $z = f(x,y)$, we have independent differentials dx and dy and define $dz = f_x\,dx + f_y\,dy$

If $P(a, b, f(a,b))$ is a point on the surface and

$dx = x - a$ and $dy = y - b$, then

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

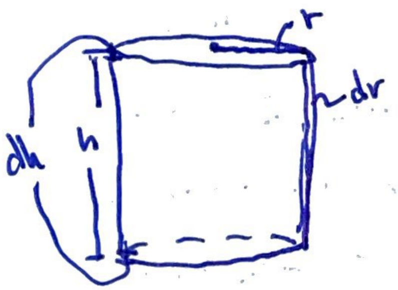
Recall, tangent plane is $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
$= f(a,b) + dz$

$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ Change in surface

$dz = z - f(a,b)$ Change in tangent plane.

If Δx and Δy are small, $\Delta z \approx dz$

Ex: Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.



Volume $V = \pi r^2 h$

Estimate ΔV from $(1.95, 9.8)$ to $(2, 10)$.

$r = 2$ cm $dr = -0.05$ cm

$h = 10$ cm $dh = -0.2$ cm

$$\Delta V = V(1.95, 9.8) - V(2, 10)$$

$$\approx dV = V_r\, dr + V_h\, dh$$

$$= (2\pi r h)\, dr + (\pi r^2)\, dh$$

$$= 2\pi(2)(10)(-0.05) + (\pi(4))(-0.2)$$

$$= -2.8\pi$$

Volume is approximately, 2.8π cm^3