Introduction to Abstract Algebra I: Homework 7

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Exercise 8.39. Show that if $\varphi : G \to G'$ and $\gamma : G' \to G''$ are group homomorphisms, then $\gamma \circ \varphi : G \to G''$ is also a group homomorphism.

Solution. Suppose we have three groups, $(G, \oplus)$, $(G', \otimes)$, and $(G'', \circ)$. Let $\varphi : G \to G'$ and $\gamma : G' \to G''$ be group homomorphisms. We want to show that the composition $\gamma \circ \varphi : G \to G''$ is also a group homomorphism. Take any two elements $a, b \in G$. Then, we have

$$(\gamma \circ \varphi)(a \oplus b) = \gamma(\varphi(a \oplus b)).$$

Since φ is a group homomorphism, we know that

$$\gamma(\varphi(a \oplus b)) = \gamma(\varphi(a) \otimes \varphi(b)).$$

Now, since γ is also a group homomorphism, we have

$$\gamma(\varphi(a) \otimes \varphi(b)) = \gamma(\varphi(a)) \circ \gamma(\varphi(b)).$$

Therefore, we can conclude that

$$(\gamma \circ \varphi)(a \oplus b) = (\gamma \circ \varphi)(a) \circ (\gamma \circ \varphi)(b).$$

This shows that $\gamma \circ \varphi$ preserves the group operation, and hence it is a group homomorphism. □

Exercise 8.41. Prove the following about S_n if $n \geq 3$.

(i) Every permutation in S_n can be written as a product of at most $n - 1$ transpositions.

Solution to (i). Let $\sigma \in S_n$ and write σ as a product of disjoint cycles (including 1-cycles for fixed points):

$$\sigma = \gamma_1 \gamma_2 \cdots \gamma_r,$$

where γ_i has length $m_i \geq 1$ and $\sum_{i=1}^{r} m_i = n$.

A k-cycle can be written as a product of $k - 1$ transpositions; for distinct elements $a_1, \ldots, a_k$,

$$(a_1, a_2, \ldots, a_k) = (a_1, a_k)(a_1, a_{k-1}) \cdots (a_1, a_2),$$

which uses exactly $k - 1$ transpositions (and a 1-cycle uses 0).

Applying this to each γ_i we express σ as a product of

$$\sum_{i=1}^{r} (m_i - 1) = \left(\sum_{i=1}^{r} m_i \right) - r = n - r,$$

transpositions. Since $r \geq 1$, we have $n - r \leq n - 1$, so σ is a product of at most $n - 1$ transpositions. □

Exercise 8.43. Show that for every subgroup H of S_n for $n \geq 2$, either all the permutations in H are even or exactly half of them are even.

Solution. Let H be a subgroup of S_n for $n \geq 2$. Take the $\operatorname{sgn} : S_n \to \{1, -1\}$ group homomorphism, and restrict it to H, giving us $\operatorname{sgn}|_H : H \to \{1, -1\}$. The kernel of this homomorphism is given by

$$\ker(\operatorname{sgn}|_H) = \{\sigma \in H \mid \operatorname{sgn}(\sigma) = 1\} = H \cap A_n.$$

□

Exercise 9.6. Find the order of the given element of the direct product $(3, 10, 9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$.

Solution. The order of 3 in $\mathbb{Z}_4$ is 4, since $3 \times 4 \equiv 0 \pmod 4$. The order of 10 in $\mathbb{Z}_{12}$ is 6, since $10 \times 6 \equiv 0 \pmod{12}$. The order of 9 in $\mathbb{Z}_{15}$ is 5, since $9 \times 5 \equiv 0 \pmod{15}$.

To find the order of the element $(3, 10, 9)$ in the direct product $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$, we take the least common multiple (LCM) of the individual orders:

$$\operatorname{ord}(3, 10, 9) = \operatorname{lcm}(4, 6, 5) = 60.$$

Therefore, the order of the element $(3, 10, 9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60. □

Exercise 9.8. What is the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$ and $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$?

Solution. For $\mathbb{Z}_6 \times \mathbb{Z}_8$, the largest order among all the orders of all cyclic subgroups is given by

$$\max \operatorname{ord}(\mathbb{Z}_6 \times \mathbb{Z}_8) = \operatorname{lcm}(6, 8) = 24.$$

For $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$, the same thing applies, giving us

$$\max \operatorname{ord}(\mathbb{Z}_{12} \times \mathbb{Z}_{15}) = \operatorname{lcm}(12, 15) = 60.$$

Therefore, the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$ is 24, and for $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ it is 60. □

Exercise 9.18. Are the groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ isomorphic? Why or why not?

Solution. Breaking down each group into its primary components, we have

$$\begin{aligned} G = \mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} &\cong \mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\ H = \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} &\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5. \end{aligned}$$

The primary components of G and H are given as

$$\begin{gathered} G_{(3)} \cong \mathbb{Z}_3 \cong H_{(3)}, \quad G_{(5)} \cong \mathbb{Z}_5 \cong H_{(5)} \\ G_{(2)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3} \not\cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3} \cong H_{(2)}. \end{gathered}$$

Since the primary components corresponding to the prime 2 are not isomorphic, we conclude that the groups G and H are not isomorphic. □

Exercise 9.22. Find all abelian groups, up to isomorphism, of order 16. Find the invariant factors and find an isomorphic group of the form indicated in Theorem 9.14.

Solution. Factoring 16 into its prime power components, we have $16 = 2^4$. The abelian groups of order 16, up to isomorphism, are given by the following decompositions, 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$. Thus, the abelian groups of order 16 along with their invariant factors are

$$\begin{aligned} \mathbb{Z}_{16} &: (16) \\ \mathbb{Z}_8 \times \mathbb{Z}_2 &: (8, 2) \\ \mathbb{Z}_4 \times \mathbb{Z}_4 &: (4, 4) \\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 &: (4, 2, 2) \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 &: (2, 2, 2, 2). \end{aligned}$$

□

Exercise 9.26. How many abelian groups (up to isomorphism) are there of order 24, order 25, and of order (24)(25)?

Solution. For 24, we have the prime factorization $24 = 2^3 \times 3^1$. The number of abelian groups of order 2^3 is given by the number of partitions of 3, which are 3, $2+1$, and $1+1+1$. Thus, there are 3 abelian groups of order 2^3. For 3^1, there is only 1 abelian group. Therefore, the total number of abelian groups of order 24 is $3 \times 1 = 3$. They are

$$\mathbb{Z}_8 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

For 25, we have the prime factorization $25 = 5^2$. The number of abelian groups of order 5^2 is given by the number of partitions of 2, which are 2 and $1+1$. Thus, there are 2 abelian groups of order 25. They are

$$\mathbb{Z}_{25}, \quad \mathbb{Z}_5 \times \mathbb{Z}_5.$$

For $(24)(25) = 600$, we have the prime factorization $600 = 2^3 \times 3^1 \times 5^2$. From the previous calculations, we know there are 3 abelian groups of order 2^3, 1 abelian group of order 3^1, and 2 abelian groups of order 5^2. Therefore, the total number of abelian groups of order 600 is $3 \times 1 \times 2 = 6$. They are

$$\begin{aligned} \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}, &\quad \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5, \\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}, &\quad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}, &\quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5. \end{aligned}$$ □

Exercise 9.28. Use Exercise 27 to determine the number of abelian groups (up to isomorphism) of order $(10)^5$.

Solution. Exercise 27 states that if we have r abelian groups of order m and s abelian groups of order n, then there are rs abelian groups of order mn, provided that m and n are coprime. Notice that $10 = 2 \times 5$. Then, we have $10^5 = 2^5 \times 5^5$. The number of abelian groups of order 2^5 is given by the number of partitions of 5, which are 7 in total. Similarly, the number of abelian groups of order 5^5 is also given by the number of partitions of 5, which is again 7. Therefore, by Exercise 27, the total number of abelian groups of order $(10)^5$ is $7 \times 7 = 49$. □

Exercise 9.39. Let G be an abelian group. Show that the elements of finite order in G form a subgroup. This subgroup is called the *torsion subgroup* of G.

Solution. Let H be the set of all elements of finite order in the abelian group G. Notice that $e^1 = e$, so the identity element e of G has finite order and is in H.

Next, take any two elements $a, b \in H$. Suppose a has order m and b has order n. This means $a^m = e$ and $b^n = e$. Now, consider the element $a \cdot b$. Take the exponent mn and using the fact that G is abelian, we have

$$(a \cdot b)^{mn} = a^{mn} \cdot b^{mn} = (a^m)^n \cdot (b^n)^m = e \cdot e = e.$$

Thus, the order of $a \cdot b$ divides mn, which is finite. Therefore, $a \cdot b \in H$.

Finally, we need to show that if $a \in H$, then its inverse a^{-1} is also in H. If a has finite order m, then $a^m = e$. Taking inverses on both sides gives $(a^{-1})^m = e$. Thus, the order of a^{-1} is also finite, so $a^{-1} \in H$.

Since we have shown that the identity element is in H, that the product of any two elements in H is also in H, and that the inverse of any element in H is also in H, we conclude that H is a subgroup of G. This subgroup is called the torsion subgroup of G. □