Problem 1. Evaluate $\oint_{\partial_S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = \left\langle 2xy, z^3 + 3x^2, 3yz^2 - x \right\rangle$, ∂_S is the boundary of the triangle with vertices $P = (2, 0, 0)$, $Q = (0, 3, 0)$, and $R = (0, 0, 5)$ oriented from P to Q to R and back to P.

Solution. By the orientation induced on the surface, using the right hand rule, the orientation of the normal vector is pointing outwards. We can use Stokes' Theorem to convert the line integral into a surface integral,

$$\oint_{\partial_S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S},$$

where S is the surface bounded by $\partial_S = C$ and $\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$. We first compute the curl of $\mathbf{F}$

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & z^3 + 3x^2 & 3yz^2 - x \end{vmatrix} \\
&= \left\langle 3z^2 - 3z^2, 1, 6x - 2x \right\rangle \\
&= \langle 0, 1, 4x \rangle .
\end{aligned}$$

Finding the equation of the plane containing the triangle, we have

$$15x + 10y + 6z = 30 \Rightarrow z = 5 - \frac{5}{2}x - \frac{5}{3}y.$$

The normal vector to this plane is

$$\mathbf{n} = \left\langle \frac{5}{2}, \frac{5}{3}, 1 \right\rangle .$$

Computing the dot product, we have

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \langle 0, 1, 4x \rangle \cdot \left\langle \frac{5}{2}, \frac{5}{3}, 1 \right\rangle = \frac{5}{3} + 4x.$$

The projection onto the xy-plane is the triangle with vertices $P' = (2, 0)$, $Q' = (0, 3)$, and $R' = (0, 0)$. Thus, the boundaries are $0 \leq x \leq 2$ and $0 \leq y \leq -\frac{3}{2}x + 3$. Therefore, we have

$$\begin{aligned}
\oint_{\partial_S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} \\
&= \iint_D \left(\frac{5}{3} + 4x \right) \mathrm{d}A \\
&= \int_0^2 \int_0^{-3/2x+3} \left(\frac{5}{3} + 4x \right) \mathrm{d}y\, \mathrm{d}x \\
&= \int_0^2 \frac{5}{3}y + 4xy \bigg|_0^{-3/2x+3} \mathrm{d}x \\
&= \int_0^2 \frac{5}{3}\left(-\frac{3}{2}x + 3 \right) + 4x \left(-\frac{3}{2}x + 3 \right) \mathrm{d}x \\
&= \int_0^2 -6x^2 + \frac{19x}{2} + 5\, \mathrm{d}x \\
&= \left[-2x^3 + \frac{19x^2}{4} + 5x \right]_0^2 \\
&= -16 + 19 + 10 = 13.
\end{aligned}$$

□

Problem 2. Let S be the portion of the ellipsoid $4x^2 + y^2 + 16z^2 = 64$ above the xy-plane oriented upward. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$ where $\mathbf{F} = \left\langle xz, x^3 + 2y, e^{x^2 - y^2} \right\rangle$.

Solution. By the orientation induced on the surface, using the right hand rule, the orientation of the line integral is counterclockwise. We can use Stokes' Theorem to convert the surface integral into a line integral.

The boundary ∂_S occurs when $z = 0$, which gives the ellipse $4x^2 + y^2 = 64$. This also gives us a simpler form of $\mathbf{F}$,

$$\mathbf{F} = \left\langle 0, x^3 + 2y, e^{x^2 - y^2} \right\rangle.$$

Since

$$\frac{x^2}{16} + \frac{y^2}{64} = 1,$$

we can use the parametrization

$$\mathbf{r}(t) = \langle 4\cos(t), 8\sin(t), 0 \rangle,$$

where $0 \leq t \leq 2\pi$. Therefore, we have

$$\mathrm{d}\mathbf{r} = \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z \rangle = \langle -4\sin(t), 8\cos(t), 0 \rangle \, \mathrm{d}t.$$

Computing the dot product, we have

$$\begin{aligned}
\mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \left\langle 0, x^3 + 2y, e^{x^2 - y^2} \right\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, 0 \rangle \\
&= (x^3 + 2y) \, \mathrm{d}y \\
&= \big((4\cos(t))^3 + 2(8\sin(t)) \big) \cdot 8\cos(t) \, \mathrm{d}t \\
&= \big(64\cos^3(t) + 16\sin(t) \big) \cdot 8\cos(t) \, \mathrm{d}t \\
&= 512\cos^4(t) + 128\sin(t)\cos(t) \, \mathrm{d}t.
\end{aligned}$$

By Stokes' Theorem, we have

$$\begin{aligned}
\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \oint_{\partial_S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} 512\cos^4(t) + 128\sin(t)\cos(t) \, \mathrm{d}t \\
&= 512 \int_0^{2\pi} \cos^4(t) \, \mathrm{d}t + 128 \int_0^{2\pi} \sin(t)\cos(t) \, \mathrm{d}t \\
&= 512 \cdot \frac{3\pi}{4} + 128 \cdot 0 = 384\pi.
\end{aligned}$$

□

Problem 3. Verify Stokes' Theorem for the vector field $\mathbf{F} = \langle y, z, x \rangle$ and the hemisphere $y = \sqrt{1 - x^2 - z^2}$ orientated in the direction of the positive y-axis.
(That means, evaluate both $\oint_{\partial_S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ and $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$ showing that they are equal for the given field and surface.)

Solution. We're first going to compute the line integral over ∂_S. Given the orientation of the surface, using the right hand rule, the orientation of the line integral is clockwise in the xz-plane.

We're given that the surface is defined as $y = \sqrt{1 - x^2 - y^2}$. This means that $x^2 + y^2 \leq 1$. The boundary ∂_S occurs when $y = 0$, which gives us the circle $x^2 + z^2 = 1$. Therefore, using polar, we get the parameterization

$$\mathbf{r}(t) = \langle \sin(t), 0, \cos(t) \rangle,$$

where $r = 1$ and $0 \leq t \leq 2\pi$. We need to flip the parameter as we need to go clockwise, as polar goes counterclockwise. The differential is

$$\mathrm{d}\mathbf{r} = \mathbf{r}'(t)\,\mathrm{d}t = \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z \rangle\,\mathrm{d}t = \langle \cos(t), 0, -\sin(t) \rangle\,\mathrm{d}t.$$

Evaluating $\mathbf{F}(\mathbf{r}(t))$, we get

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0, \cos(t), \sin(t) \rangle.$$

Computing the dot product, we have

$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = \langle 0, \cos(t), \sin(t) \rangle \cdot \langle \cos(t), 0, -\sin(t) \rangle = -\sin^2(t).$$

Therefore, we have

$$\begin{aligned}
\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} -\sin^2(t)\,\mathrm{d}t \\
&= -\frac{1}{2} \cdot \int_0^{2\pi} 1 - \sin(2t)\,\mathrm{d}t \\
&= -\frac{1}{2} \cdot \left[t + \frac{1}{2}\cos(2t) \right]_0^{2\pi} \\
&= -\frac{1}{2} \cdot \left[\left(2\pi + \frac{1}{2} \right) - \left(0 - \frac{1}{2} \right) \right] \\
&= -\frac{2\pi}{2} = -\pi.
\end{aligned}$$

Now, we compute the double integral over S. Computing the curl of $\mathbf{F}$, we have

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} \\
&= \langle 0 - 1, -(1 - 0), 0 - 1 \rangle \\
&= \langle -1, -1, -1 \rangle.
\end{aligned}$$

Using spherical coordinates, we have

$$\mathbf{r}(\theta, \varphi) = \langle \sin(\varphi)\cos(\theta), \cos(\varphi), \sin(\varphi)\sin(\theta) \rangle,$$

where $\rho = 1$. We know that

$$\begin{aligned}
\mathbf{r}_\theta &= \langle -\sin(\varphi)\sin(\theta), 0, \sin(\varphi)\cos(\theta) \rangle \\
\mathbf{r}_\varphi &= \langle \cos(\varphi)\cos(\theta), -\sin(\varphi), \cos(\varphi)\sin(\theta) \rangle.
\end{aligned}$$

Since the orientation induced on the surface is in the direction of the positive y-axis, we have need to have the $\hat{\jmath}$-component of the normal positive. Therefore, the normal vector is

$$\begin{aligned}
\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_\varphi &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -\sin(\varphi)\sin(\theta) & 0 & \sin(\varphi)\cos(\theta) \\ \cos(\varphi)\cos(\theta) & -\sin(\varphi) & \cos(\varphi)\sin(\theta) \end{vmatrix} \\
&= \langle \sin^2(\varphi)\cos(\theta), \sin(\varphi)\cos(\varphi)\cos^2(\theta), \sin^2(\varphi)\sin(\theta) \rangle.
\end{aligned}$$

The dot product is

$$\begin{aligned}\operatorname{curl}(\mathbf{F})\cdot\mathbf{n} &= \langle -1,-1,-1\rangle\cdot\langle \sin^2(\varphi)\cos(\theta), \sin(\varphi)\cos(\varphi)\cos^2(\theta), \sin^2(\varphi)\sin(\theta)\rangle \\ &= -\left(\sin^2(\varphi)\cos(\theta)+\sin(\varphi)\cos(\varphi)\cos^2(\theta)+\sin^2(\varphi)\sin(\theta)\right).\end{aligned}$$

Therefore, the surface integral becomes

$$\begin{aligned}\iint_S \operatorname{curl}(\mathbf{F})\cdot \mathrm{d}\mathbf{S} &= \iint_D -\left(\sin^2(\varphi)\cos(\theta)+\sin(\varphi)\cos(\varphi)\cos^2(\theta)+\sin^2(\varphi)\sin(\theta)\right)\mathrm{d}A \\ &= -\left[\int_0^\pi\int_0^\pi \sin^2(\varphi)\cos(\theta)+\sin(\varphi)\cos(\varphi)\cos^2(\theta)+\sin^2(\varphi)\sin(\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\right] \\ &= -\int_0^\pi \cos(\theta)\,\mathrm{d}\theta\cdot\int_0^\pi \sin^2(\varphi)\,\mathrm{d}\varphi - \int_0^\pi \cos^2(\theta)\,\mathrm{d}\theta\cdot\int_0^\pi \sin(\varphi)\cos(\varphi)\,\mathrm{d}\varphi \\ &\quad -\int_0^\pi \sin(\theta)\,\mathrm{d}\theta\cdot\int_0^\pi \sin^2(\varphi)\,\mathrm{d}\varphi \\ &= 0-\frac{\pi}{2}\cdot 0 - 2\cdot\frac{\pi}{2} = -\pi.\end{aligned}$$

Therefore, we have

$$\oint_{\partial S}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F})\cdot\mathrm{d}\mathbf{S} = -\pi. \qquad \square$$

Problem 4. Evaluate $\oiint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ where $\mathbf{F}=\langle x^3+zy^2, 4x^2z+2yz, 4-z^2\rangle$ and S is the sphere $x^2+y^2+z^2=9$ orientated outward.

Solution. By the Divergence Theorem, we have

$$\oiint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_V \operatorname{div}(\mathbf{F})\,\mathrm{d}V,$$

where $\operatorname{div}(\mathbf{F})=\nabla\cdot\mathbf{F}$ and V is the volume of the sphere $x^2+y^2+z^2=9$. We first compute the divergence of $\mathbf{F}$

$$\operatorname{div}(\mathbf{F})=\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}\left(x^3+zy^2\right)+\frac{\partial}{\partial y}\left(4x^2z+2yz\right)+\frac{\partial}{\partial z}\left(4-z^2\right)=3x^2+2z-2z=3x^2.$$

Using spherical coordinates, we have

$$x=\rho\sin(\varphi)\cos(\theta),\quad y=\rho\sin(\varphi)\sin(\theta),\quad\text{and}\quad z=\rho\cos(\varphi).$$

Therefore, the divergence becomes

$$\operatorname{div}(\mathbf{F})=3x^2=3\rho^2\sin^2(\varphi)\cos^2(\theta).$$

The Jacobian determinant for this transformation is $\rho^2\sin(\varphi)$. The limits of integration are $0\le\rho\le 3$, $0\le\varphi\le\pi$, and $0\le\theta\le 2\pi$. Thus, we have

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V \\ &= \int_0^{2\pi}\int_0^\pi\int_0^3 3\rho^2\sin^2(\varphi)\cos^2(\theta)\cdot\rho^2\sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &= 3\cdot\int_0^{2\pi}\cos^2(\theta)\,\mathrm{d}\theta\cdot\int_0^\pi \sin^3(\varphi)\,\mathrm{d}\varphi\cdot\int_0^3\rho^4\,\mathrm{d}\rho \\ &= 3\cdot\left[\frac{1}{2}(\theta-\sin(\theta)\cos(\theta))\right]_0^{2\pi}\cdot\int_0^\pi(1-\cos^2(\varphi))\sin(\varphi)\,\mathrm{d}\varphi\cdot\left[\frac{\rho^5}{5}\right]_0^3\end{aligned}$$

$$
\begin{aligned}
&= 3 \cdot \frac{2\pi}{2} \cdot \left[\frac{1}{12}\left(\cos(3\varphi) - 9\cos(\varphi)\right)\right]_0^{\pi} \cdot \frac{243}{5} \\
&= 3\pi \cdot \frac{4}{3} \cdot \frac{243}{5} = \frac{972\pi}{5}.
\end{aligned}
$$

□

Problem 5. Verify the Divergence Theorem for the vector field $\mathbf{F} = \langle -x, y, z\rangle$ and the surface, S, is the boundary of the solid enclosed by the parabolic cylinder $y = 4 - x^2$ and the planes $y + 2z = 4$ and $z = 2$ with positive orientation.
(That means, evaluate both $\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ and $\iiint_V \operatorname{div}(\mathbf{F})\,\mathrm{d}V$ showing that they are equal for the given field and surface.)

Solution. First, we compute the double closed surface integral over S. Notice that

$$
\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{S_3} \mathbf{F} \cdot \mathrm{d}\mathbf{S},
$$

where S_1 is the parabolic cylinder, S_2 is the plane $y + 2z = 4$, and S_3 is the plane $z = 2$.

S_1: Parametrizing the surface, we get $\mathbf{r}(x, z) = \langle x, 4 - x^2, z\rangle$. We know that

$$
\begin{aligned}
\mathbf{r}_z &= \langle 0, 0, 1\rangle \\
\mathbf{r}_x &= \langle 1, -2x, 0\rangle.
\end{aligned}
$$

Therefore, the normal vector is

$$
\mathbf{n} = \mathbf{r}_z \times \mathbf{r}_x = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ 1 & -2x & 0 \end{vmatrix} = \langle 2x, 1, 0\rangle.
$$

Evaluating $\mathbf{F}(\mathbf{r}(x, z))$, we get

$$
\mathbf{F}(\mathbf{r}(x, z)) = \langle -x, 4 - x^2, z\rangle.
$$

Computing the dot product, we have

$$
\mathbf{F} \cdot \mathbf{n} = \langle -x, 4 - x^2, z\rangle \cdot \langle 2x, 1, 0\rangle = -2x^2 + 4 - x^2 = 4 - 3x^2.
$$

The intersection of the parabolic cylinder and the plane $y + 2z = 4$ is

$$
4 - 2z = 4 - x^2 \Rightarrow z = \frac{x^2}{2}.
$$

Projecting the surface onto the xz-plane, we get the bounds $-2 \leq x \leq 2$ and $x^2/2 \leq z \leq 2$. Thus, we have

$$
\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{D_1} 4 - 3x^2 \,\mathrm{d}A \\
&= \int_{-2}^{2} \int_{x^2/2}^{2} 4 - 3x^2 \,\mathrm{d}z\,\mathrm{d}x \\
&= \int_{-2}^{2} 4z - x^2 z \Big|_{x^2/2}^{2} \mathrm{d}x \\
&= \int_{-2}^{2} \frac{3x^4}{2} - 8x^2 + 8 \,\mathrm{d}x \\
&= \frac{3x^5}{10} - \frac{8x^3}{3} + 8x \Big|_{-2}^{2}
\end{aligned}
$$

$$= \frac{128}{15}.$$

S_2: Parametrizing the surface, we get $\mathbf{r}(x,y) = \left\langle x, y, 2 - \frac{4-y}{2} \right\rangle$. We know that

$$\begin{aligned} \mathbf{r}_y &= \left\langle 0, 1, -\frac{1}{2} \right\rangle \\ \mathbf{r}_x &= \langle 1, 0, 0 \rangle. \end{aligned}$$

Therefore, the normal vector is

$$\mathbf{n} = \mathbf{r}_y \times \mathbf{r}_x = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 1 & -\frac{1}{2} \\ 1 & 0 & 0 \end{vmatrix} = \left\langle 0, -\frac{1}{2}, -1 \right\rangle.$$

But we need the normal vector to be pointing outwards. Therefore, we have

$$\mathbf{n} = \left\langle 0, \frac{1}{2}, 1 \right\rangle.$$

Evaluating $\mathbf{F}(\mathbf{r}(x,y))$, we get

$$\mathbf{F}(\mathbf{r}(x,y)) = \left\langle -x, y, \frac{4-y}{2} \right\rangle.$$

Computing the dot product, we have

$$\mathbf{F} \cdot \mathbf{n} = \left\langle -x, y, \frac{4-y}{2} \right\rangle \cdot \left\langle 0, \frac{1}{2}, 1 \right\rangle = 2.$$

The bounds for this surface are $-2 \le x \le 2$ and $0 \le y \le 4 - x^2$. Thus, we have

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{D_2} 2 \, \mathrm{d}A \\ &= 2 \cdot \int_{-2}^{2} \int_{0}^{4-x^2} \mathrm{d}y \, \mathrm{d}x \\ &= 2 \cdot \int_{-2}^{2} 4 - x^2 \, \mathrm{d}x \\ &= 2 \cdot \left[4x - \frac{x^3}{3} \right]_{-2}^{2} \\ &= 2 \cdot \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] \\ &= \frac{64}{3}. \end{aligned}$$

S_3: Parametrizing the surface, we get $\mathbf{r}(x,z) = \langle x, y, 2 \rangle$. We know that

$$\begin{aligned} \mathbf{r}_x &= \langle 1, 0, 0 \rangle \\ \mathbf{r}_y &= \langle 0, 1, 0 \rangle. \end{aligned}$$

Therefore, the normal vector is

$$\mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle.$$

But we need the normal vector to be pointing outwards. Therefore, we have

$$\mathbf{n} = \langle 0, 0, -1 \rangle.$$

Evaluating $\mathbf{F}(\mathbf{r}(x, y))$, we get

$$\mathbf{F}(\mathbf{r}(x, y)) = \langle -x, y, 2 \rangle.$$

Computing the dot product, we have

$$\mathbf{F} \cdot \mathbf{n} = \langle -x, y, 2 \rangle \cdot \langle 0, 0, 1 \rangle = -2.$$

The bounds for this surface are $-2 \leq x \leq 2$ and $0 \leq y \leq 4 - x^2$. Thus, we have

$$\begin{aligned}
\iint_{S_3} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{D_3} -2 \, \mathrm{d}A \\
&= \int_{-2}^{2} \int_{0}^{4-x^2} -2 \, \mathrm{d}y \, \mathrm{d}x \\
&= -2 \cdot \int_{-2}^{2} 4 - x^2 \, \mathrm{d}x \\
&= -2 \cdot \left[4x - \frac{x^3}{3} \right]_{-2}^{2} \\
&= -2 \cdot \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] \\
&= -\frac{64}{3}.
\end{aligned}$$

Therefore, we have

$$\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \frac{128}{15} + \frac{64}{3} - \frac{64}{3} = \frac{128}{15}.$$

Now, we compute the triple integral over the volume. We first compute the divergence of $\mathbf{F}$

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = -1 + 1 + 1 = 1.$$

Therefore, we get

$$\iiint_V \operatorname{div}(\mathbf{F}) \, \mathrm{d}V = \iiint_V 1 \, \mathrm{d}V = \iiint_V \mathrm{d}V.$$

The volume limits are $-2 \leq x \leq 2$, since $y = 4 - x^2$ defines a parabola opening down from $y = 4$, $0 \leq y \leq 4 - x^2$, and $2 - y/2 \leq z \leq 2$. Thus, the integral becomes

$$\begin{aligned}
\iiint_V \mathrm{d}V &= \int_{-2}^{2} \int_{0}^{4-x^2} \int_{2-y/2}^{2} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-2}^{2} \int_{0}^{4-x^2} \frac{y}{2} \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{-2}^{2} \left. \frac{y^2}{4} \right|_{0}^{4-x^2} \mathrm{d}x \\
&= \int_{-2}^{2} \frac{(4 - x^2)^2}{4} \, \mathrm{d}x \\
&= \int_{-2}^{2} \frac{x^4}{4} - 2x^2 + 4 \, \mathrm{d}x
\end{aligned}$$

$$= \frac{x^5}{20} - \frac{2x^3}{3} + 4x\bigg|_{-2}^{2}$$
$$= \frac{128}{15}.$$

Therefore, we get

$$\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_V \operatorname{div}(\mathbf{F})\,\mathrm{d}V. \qquad \square$$