Math 307, Homework #9
Due Sunday, December 8

1. Prove that $\frac{d^n}{dx^n}(x^2e^x) = (x^2 + 2nx + n(n-1))e^x$, for all $n \in \mathbb{N}$.
2. Here are two facts that hold for all real numbers x and y:

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) && \text{(addition formula for sine)} \\ |x+y| &\le |x| + |y| && \text{(triangle inequality).} \end{aligned}$$

   Using these together with induction, prove that $(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})\big[\ |\sin(nx)| \le n|\sin(x)|\ \big]$.
3. Let f_n denote the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \ge 2$. Prove that $f_n < (\frac{7}{4})^n$ for all $n \in \mathbb{N}$.
4. Consider the sequence given by $a_n = 2a_{n-1} + 4a_{n-2}$ and initial conditions $a_0 = 0$, $a_1 = 3$. Prove that $3|a_n$ for all $n \ge 0$.
5. Imagine that you have an infinite supply of 6-cent and 11-cent stamps. Prove that for all $n \ge 50$, you can make a combination of your stamps that exactly totals n cents. Use 2nd principle of mathematical induction.
6. Define the "Tribonacci" sequence as follows: $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \ge 4$. Prove that for all natural numbers $n \ge 1$ one has $T_n < 2^n$.
7. Recall the Fibonacci sequence $f_0, f_1, f_2, \ldots$. The following is a faulty proof (using 2nd principle of mathematical induction) that all Fibonacci numbers are even:

   I. $f_0 = 0$, which is even.

   II. Suppose $n \in \mathbb{N}$ and f_k is even for all $0 \le k \le n$.

   Then $f_{n+1} = f_n + f_{n-1}$. By the induction hypothesis, both f_n and f_{n-1} are even. So f_{n+1} is the sum of two even numbers, hence even.

   III. By PSMI, f_n is even for all $n \in \mathbb{N}$.

   Goal: Find the mistake in the above proof.
8. In this question we will prove that you can lift any cow. Let $P(n)$ be the statement "You can lift the cow on day n of its life". We will use induction to prove $(\forall n \in \mathbb{N})(n \ge 1 \Rightarrow P(n))$.

   I. When the cow is born it is very small, and of course you can lift it. This proves $P(1)$.

   II. Suppose that you can lift the cow on day n. It grows very little by the next day, so you will still be able to lift it on day $n+1$.

   III. By induction, we have proven that you can lift the cow on every day of its life. Boy, you are strong.

   Explain what you think is wrong with this proof.
9. Prove that $(A \cap B) \times C = (A \times C) \cap (B \times C)$.
10. Suppose $A \subseteq X$ and $B \subseteq Y$. Prove that $(X \times Y) - [(A \times Y) \cup (X \times B)] = (X - A) \times (Y - B)$.