Functional Complex Variables I: Homework 2

Due on April 16, 2025 at 23:59

Weiyong He

Hashem A. Damrah
UO ID: 952102243

Exercise 1.8.1. Find the principal argument $\operatorname{Arg}(z)$ when

$$\text{(i) } z = \frac{i}{-2-2i}; \qquad \text{(ii) } z = \left(\sqrt{3}-i\right)^6.$$

Solution to (i). Simplifying the expression, we have

$$z = \frac{i}{-2-2i} \cdot \frac{-2+2i}{-2+2i} = \frac{i(-2+2i)}{8} = -\frac{1}{4} - \frac{1}{4}i.$$

The principal argument is

$$\operatorname{Arg}(z) = \tan^{-1}\left(\frac{-1/4}{-1/4}\right) = \tan^{-1}(1) = \frac{p}{4}.$$

Since z is in the third quadrant, we have

$$\operatorname{Arg}(z) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}.$$

□

Solution to (ii). Simplifying the expression using the Binomial Theorem, we have

$$z = \sum_{k=0}^{6} \binom{6}{k} (\sqrt{3})^{6-k}(-i)^k = \sum_{k=0}^{6} \binom{6}{k} (\sqrt{3})^{6-k}(-1)^k i^k.$$

Each term contributes either a real or imaginary value. At the end, all imaginary parts cancel out, and we are left with only the real part, giving us

$$z = \left(\sqrt{3}-i\right)^6 = -64.$$

Since the real number is negative, we have

$$\operatorname{Arg}(z) = \tan^{-1}\left(\frac{0}{-64}\right) = \pi.$$

□

Exercise 1.8.9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} \quad (z \neq 1),$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \quad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

Solution. We first establish the first identity. Let $S = 1 + z + z^2 + \cdots + z^n$. We compute $S - zS$ to get

$$S - zS = (1 + z + z^2 + \cdots + z^n) - (z + z^2 + z^3 + \cdots + z^{n+1}) = 1 - z^{n+1}.$$

Thus, provided $z \neq 1$, we have

$$S = \frac{1-z^{n+1}}{1-z}.$$

Now, we derive Lagrange's trigonometric identity. We write $z = e^{i\theta}$. So, $|z| = 1$ and $z^k = e^{ik\theta}$. By the geometric series formula,

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z},$$

since $z^k = e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$, taking the real part of both sides, we have

$$\sum_{k=0}^{n} \cos(k\theta) = \operatorname{Re}\left(\sum_{k=0}^{n} z^k\right) = \operatorname{Re}\left(\frac{1-z^{n+1}}{1-z}\right).$$

We know that $z^{n+1} = e^{i(n+1)\theta}$ and $z = e^{i\theta}$. Expanding $1 - e^{i\theta}$, we have

$$\begin{aligned} 1 - e^{i\theta} &= e^{i\theta/2}\left(e^{-i\theta/2} - e^{i\theta/2}\right) \\ &= e^{i\theta/2}\left(-2i\sin(\theta/2)\right). \end{aligned}$$

Expanding $1 - e^{i(n+1)\theta}$, we have

$$1 - e^{i(n+1)\theta} = -2i\sin\left(\frac{(n+1)\theta}{2}\right)e^{i(n+1)\theta/2}.$$

Thus, we have

$$\begin{aligned} \frac{1-z^{n+1}}{1-z} &= \frac{-2i\sin\left(\frac{(n+1)\theta}{2}\right)e^{i(n+1)\theta/2}}{e^{i\theta/2}\left(-2i\sin(\theta/2)\right)} \\ &= \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \cdot \frac{e^{i(n+1)\theta/2}}{e^{i\theta/2}} \\ &= \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \cdot e^{in\theta/2} \\ &= \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \cdot \left(\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta}{2}\right)\right). \end{aligned}$$

Taking the real part, we have

$$\operatorname{Re}\left(\frac{1-z^{n+1}}{1-z}\right) = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \cdot \cos\left(\frac{n\theta}{2}\right).$$

We can now use the identity $2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$, by letting

$$A = \frac{(n+1)\theta}{2} \quad \text{and} \quad B = \frac{n\theta}{2}.$$

This gives us

$$2\sin\left(\frac{(n+1)\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right) = \sin\left(\frac{(2n+1)\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right).$$

Therefore

$$\frac{\sin\left[(n+1)\theta/2\right]}{\sin\left[\theta/2\right]} \cdot \cos\left(\frac{n\theta}{2}\right) = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin\left(\theta/2\right)}.$$

Therefore, we have

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin\left(\theta/2\right)}.$$ ☐

Exercise 1.8.10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

$$\text{(i)}\ \cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta); \qquad \text{(ii)}\ \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

Solution to (i). De Moivre's formula states that

$$\cos(n\theta) + i\sin(n\theta) = \left(\cos(\theta) + i\sin(\theta)\right)^n,$$

we can expand the right-hand side, when $n = 3$, to get

$$(\cos(\theta) + i\sin(\theta))^3 = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta) \tag{1}$$

Separating the real part from equation 1, we have

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta). \qquad \square$$

Solution to (ii). Separating the imaginary part from equation 1, we have

$$\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta). \qquad \square$$

Exercise 1.10.3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principle root

$$\text{(i) } (-1)^{1/3}; \qquad \text{(ii) } z^5 = 8^{1/6}.$$

Solution to (i). We-writing -1 in polar form, we have

$$-1 = 1 \cdot \exp\left[i(-\pi + 2k\pi)\right].$$

Taking the cube root, we have

$$(-1)^{1/3} = \exp\left[i\left(-\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right].$$

The principal root is when $k = 0$, giving us

$$(-1)^{1/3} = \exp\left[-\frac{\pi}{3}i\right] = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

The other roots are when $k = 1$ and $k = 2$, giving us

$$\begin{aligned}
(-1)^{1/3} &= \exp\left[i\left(-\frac{\pi}{3} + \frac{2\pi}{3}\right)\right] = \exp\left[i\left(\frac{\pi}{3}\right)\right] &&= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) &&= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\
(-1)^{1/3} &= \exp\left[i\left(-\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] = \exp\left[i\left(\frac{5\pi}{3}\right)\right] &&= \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) &&= -1.
\end{aligned}$$

Thus, the three roots are

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \text{and} \quad -1. \qquad \square$$

Solution to (ii). We first simplify $8^{1/6} = (2^3)^{1/6} = 2^{1/2} = \sqrt{2}$. So we are solving the equation $z^5 = \sqrt{2}$. We write $\sqrt{2}$ in polar form

$$8^{1/6} = 8^{1/6}\exp\left(\frac{i\pi k}{3}\right),$$

for $k = 0, 1, 2, 3, 4$. The principal root is when $k = 0$, giving us

$$8^{1/6} = 8^{1/6}e^0 = \sqrt{2}.$$

The other roots are when $k = 1, 2, 3, 4$, giving us

$$\begin{aligned}
&\underline{k = 1:}\ \sqrt{2}\exp\left(\frac{i\pi}{3}\right) &&= \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) &&= \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) &&= \frac{1}{\sqrt{2}} + i\frac{\sqrt{3}}{\sqrt{2}} \\
&\underline{k = 2:}\ \sqrt{2}\exp\left(\frac{2i\pi}{3}\right) &&= \sqrt{2}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) &&= \sqrt{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) &&= -\frac{1}{\sqrt{2}} + i\frac{\sqrt{3}}{\sqrt{2}}
\end{aligned}$$

$$\underline{k=3:}\ \sqrt{2}\exp(i\pi) = \sqrt{2}\left(\cos(\pi) + i\sin(\pi)\right) = \sqrt{2}(-1+0i) = -\sqrt{2}$$
$$\underline{k=4:}\ \sqrt{2}\exp\left(\frac{4i\pi}{3}\right) = \sqrt{2}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = \sqrt{2}\left(-1 - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{\sqrt{2}} - i\frac{\sqrt{3}}{\sqrt{2}}$$
$$\underline{k=5:}\ \sqrt{2}\exp\left(\frac{5i\pi}{3}\right) = \sqrt{2}\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = \sqrt{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}} - i\frac{\sqrt{3}}{\sqrt{2}}.$$ □

Exercise 1.11.1. Sketch the following sets and determine which are domains:

(i) $A = |z-2+i| \leq 1$.

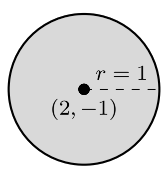
(ii) $A = |2z+3| > 4$.

(iii) $A = \operatorname{Im}(z) > 1$.

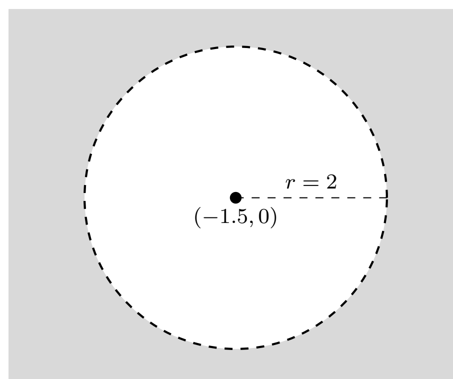
(iv) $A = \operatorname{Im}(z) = 1$.

(v) $A = 0 \leq \arg(z) \leq \pi/4$ $(z \neq 0)$.

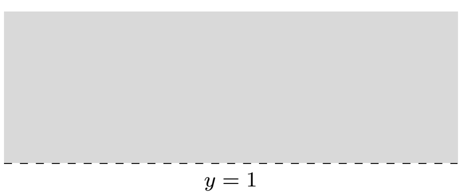
(vi) $A = |z-4| \geq |z|$.



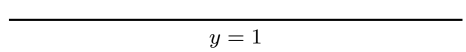
(a) $|z-2+i| \leq 1$



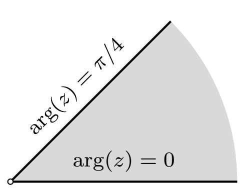
(b) $|2z+3| > 4$



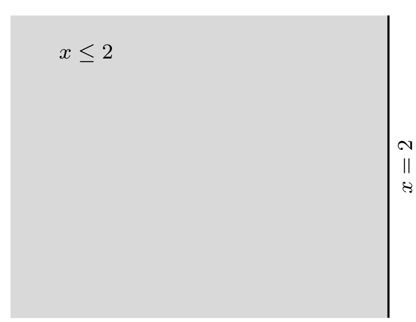
(c) $\operatorname{Im}(z) > 1$



(d) $\operatorname{Im}(z) = 1$



(e) $0 \leq \arg(z) \leq \pi/4$, $(z \neq 0)$



(f) $|z-4| \geq |z|$

Solution to (i). Let $z_0 \in \mathbb{C}$ such that $|z_0 - (2-i)| = 1$. Then, take $\varepsilon = 0.1$. Then, $V_\varepsilon(z_0)$ contains points with $|z-(2-i)| > 1$, i.e., outside the set. Since the graph is a closed disk, it's path connected, which implies connectedness. But, since the graph is closed, it isn't a domain. □

Solution to (ii). Let $z_0 \in A$. Then, $r := |z_0 + 3/2| > 2$. Let $\varepsilon := r - 2 > 0$. Then, for all $z \in V_\varepsilon(z_0)$, by the reverse triangle inequality, we have

$$\left|z + \frac{3}{2}\right| \geq \left|z_0 + \frac{3}{2}\right| - |z - z_0| > r - \varepsilon = 2.$$

So, $z \in A$. Therefore, $V_\varepsilon(z_0) \subset B$, meaning that A is open. Again, the graph is the exterior of an open disk, which is path connected. Therefore, A is a domain. $\square$

Solution to (iii). Let $z_0 \in x + iy \in A \Rightarrow y > 1$. Let $\varepsilon = y - 1 > 0$. Then, for any $z \in V_\varepsilon(z_0)$, we have

$$|\mathrm{Re}(z) - y| < \varepsilon \Rightarrow \mathrm{Re}(z) > y - \varepsilon = 1.$$

So, $\mathrm{Re}(z) > 1$, meaning that $z \in A$. Therefore, $V_\varepsilon(z_0) \subset A$, meaning that A is open. Again, the graph is the upper half-plane, which is path connected. Therefore, A is a domain. $\square$

Solution to (iv). Take any $z_0 = x + i \in A$. Any ε-neighborhood contains points where $\mathrm{Re}(z) \neq 1$, so it's not fully contained in A. Therefore, A is not open. The graph is a horizontal line, which is path connected. But, since the graph is closed, it isn't a domain. $\square$

Solution to (v). Take any point z on the boundary rays or very close to the origin. Any ε-neighborhood will contain points with argument less than 0 or more than $\pi/4$, or even $z = 0$. So no uniform ε-disk lies entirely in the set, meaning that A is not open. The graph is the union of two rays, which is path connected. But, since the graph is closed, it isn't a domain. $\square$

Solution to (vi). Let $z_0 = x + iy \in A$, with $x = 2$. Any neighborhood contains points with $\mathrm{Re}(z) > 2$, so the disk isn't contained in A. Therefore, A is not open. The graph is half the xy-plane, which is path connected. But, since the graph is closed, it isn't a domain. $\square$

Exercise 1.11.3. Which sets in Exercise 1.11.1 are bounded?

Solution. The only bounded set is the closed enclosed disk in part (i), as the maximum x-value is 3, the minimum x-value is 1, the maximum y-value is 1, and the minimum y-value is -2. All other sets are unbounded. $\square$

Exercise 2.20.1. Use results in Sec. 20 to find $f'(z)$ when

(i) $f(z) = 3z^2 - 2 + 4$.

(ii) $f(z) = \left(1 - 4z^2\right)^3$.

(iii) $f(z) = \dfrac{z - 1}{2z + 1}$ $(z \neq -1/2)$.

(iv) $f(z) = \dfrac{\left(1 + z^2\right)^4}{z^2}$ $(z \neq 0)$.

Solution to (i). We can use the power rule to find the derivative of $f(z) = 3z^2 - 2 + 4$. The derivative is given by

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z}(3z^2) + \frac{\mathrm{d}}{\mathrm{d}z}(-2) + \frac{\mathrm{d}}{\mathrm{d}z}(4) = 6z + 0 + 0 = 6z.$$ $\square$

Solution to (ii). Let $w = 1 - 4z^2$ and $W = w^3$. Then, we can use the chain rule to find the derivative of $f(z)$

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\mathrm{d}W}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}z} = 3w^2 \cdot (-8z) = -24z(1 - 4z^2)^2.$$ $\square$

Solution to (iii). We can use the quotient rule to find the derivative of $f(z) = \frac{z-1}{2z+1}$. The derivative is given by

$$f'(z) = \frac{(2z + 1)(1) - (z - 1)(2)}{(2z + 1)^2} = \frac{2z + 1 - 2z + 2}{(2z + 1)^2} = \frac{3}{(2z + 1)^2}.$$ $\square$

Solution to (iv). Let $w = 1 + z^2$ and $W = w^4$. Then, we can use the quotient rule to find the derivative of $f(z)$

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\mathrm{d}W}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}z} = 4w^3 \cdot (2z) = 8z(1+z^2)^3.$$

Now, we can use the quotient rule to find the derivative of $f(z) = \frac{(1+z^2)^4}{z^2}$, giving us

$$f'(z) = \frac{(z^2)(8z(1+z^2)^3) - ((1+z^2)^4)(2z)}{(z^2)^2} = \frac{8z^3(1+z^2)^3 - 2z(1+z^2)^4}{z^4} = \frac{2(3z^2-1)(1+z^2)^3}{z^3}. \qquad \square$$

Exercise 2.20.2. Using results in Sec. 20, show that

(i) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0),$$

of degree n ($n \geq 1$) is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

(ii) the coefficients in the polynomial $P(z)$ in part (i) can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \cdots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

Solution to (i). We apply the power rule

$$\frac{\mathrm{d}P}{\mathrm{d}z} = 0 + a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1}.$$

This derivative is a polynomial of degree $n-1$, and since polynomials are analytic, we have that $P(z)$ is differentiable everywhere. $\square$

Solution to (ii). We use the result from Section 20 about derivatives of power functions evaluated at 0. Specifically

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} z^m = \begin{cases} 0 & \text{if } m < k \\ \dfrac{m!}{(m-k)!} z^{m-k} & \text{if } m \geq k \end{cases}.$$

When we evaluate the k-th derivative of $P(z)$ at $z = 0$, only the term $z_k z^k$ contributes, since all higher powers vanish at $z = 0$, and lower powers differentiate to zero by the time we reach the k-th derivative. So,

$$P^{(k)}(z) = k!\, a_k + \text{terms with } z \text{ factors } \Rightarrow P^{(k)}(0) = k!\, a_k \Rightarrow a_k = \frac{P^{(k)}(0)}{k!}.$$

This is exactly the coefficient formula for a Taylor series centered at $z = 0$, and it confirms that any polynomial is its own Taylor series. $\square$

Exercise 2.23.1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(i) $f(z) = \overline{z}$.

(ii) $f(z) = z - \overline{z}$.

(iii) $f(z) = 2x + ixy^2$.

(iv) $f(z) = e^x e^{-iy}$.

Solution to (i). Let $z = x + iy$, then $\overline{z} = x - iy$. So,

$$f(z) = u(x, y) + iv(x, y) = x - iy \Rightarrow u(x, y) = x \quad \text{and} \quad v(x, y) = -y.$$

We compute the partial derivatives

$$\begin{aligned} u_x &= 1, \quad u_y = 0 \\ v_x &= 0, \quad v_y = -1. \end{aligned}$$

Now, we check the Cauchy-Riemann equations

$$u_x \neq v_y \quad \text{and} \quad u_y \neq -v_x.$$

Since the Cauchy-Riemann equations are not satisfied, we have that $f'(z)$ does not exist at any point. □

Solution to (ii). Again, let $z = x + iy$, then $\overline{z} = x - iy$. So,

$$f(z) = (x + iy) - (x - iy) = 2iy \Rightarrow u(x, y) = 0 \quad \text{and} \quad v(x, y) = 2y.$$

We compute the partial derivatives

$$\begin{aligned} u_x &= 0, \quad u_y = 0 \\ v_x &= 0, \quad v_y = 2. \end{aligned}$$

Now, we check the Cauchy-Riemann equations

$$u_x \neq v_y \quad \text{and} \quad u_x = v_y.$$

Since the Cauchy-Riemann equations are not satisfied, we have that $f'(z)$ does not exist at any point. □

Solution to (iii). Let $z = x + iy$, then $f(z) = 2x + ixy^2$. So,

$$f(z) = u(x, y) + iv(x, y) = 2x + ixy^2 \Rightarrow u(x, y) = 2x \quad \text{and} \quad v(x, y) = xy^2.$$

We compute the partial derivatives

$$\begin{aligned} u_x &= 2, \quad\ \ u_y = 0 \\ v_x &= y^2, \quad v_y = 2xy. \end{aligned}$$

The Cauchy-Riemann equations are only satisfied when $y = 0$ and $x = 1/y$. But, since they don't satisfy the Cauchy-Riemann equations at all points, we have that $f'(z)$ does not exist at any point. □

Solution to (iv). Let $z = x + iy$, then $f(z) = e^x e^{-iy}$. So,

$$f(z) = u(x, y) + iv(x, y) = e^x \cos(y) + ie^x \sin(y) \Rightarrow u(x, y) = e^x \cos(y) \quad \text{and} \quad v(x, y) = -e^x \sin(y).$$

We compute the partial derivatives

$$\begin{aligned} u_x &= e^x \cos(y), \quad\ \ u_y = -e^x \sin(y) \\ v_x &= -e^x \sin(y), \quad v_y = -e^x \cos(y). \end{aligned}$$

The equation $u_x = v_y$ is satisfied when $\cos(y) = 0$ but the second equation, $u_y = v_x$ is satisfied. But, since they don't satisfy the Cauchy-Riemann equations at all points, we have that $f'(z)$ does not exist at any point. □

Exercise 2.23.3. From results obtained in Secs. 21 and 22, determine where $f'(z)$ exists and find its value when

$$\text{(i) } f(z) = \frac{1}{z}; \qquad \text{(ii) } f(z) = x^2 + iy^2; \qquad \text{(iii) } f(z) = z\,\mathrm{Im}(z).$$

Solution to (i). We can write

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u(x,y) + iv(x,y).$$

Now, we can compute the partial derivatives

$$\begin{aligned} u_x &= \frac{x^2-y^2}{(x^2+y^2)^2}, \quad u_y = -\frac{2xy}{(x^2+y^2)^2} \\ v_x &= \frac{2xy}{(x^2+y^2)^2}, \quad v_y = \frac{x^2-y^2}{(x^2+y^2)^2}. \end{aligned}$$

Clearly, $u_x = v_y$ and $u_y = -v_x$. So, f is differentiable on $\mathbb{C}\setminus\{0\}$. Now, we just compute the trivial derivative to get

$$f'(z) = -\frac{1}{z^2}, \quad \text{for } z \neq 0. \qquad \square$$

Solution to (ii). We notice that $u(x,y) = x^2$ and $v(x,y) = y^2$. So, we can compute the partial derivatives

$$\begin{aligned} u_x &= 2x, \quad u_y = 0, \\ v_x &= 0, \quad\ \ v_y = 2y \end{aligned}$$

The Cauchy-Riemann equations hold only when $x = y$. Therefore, we get

$$f'(x+ix) = 2x + 0 = 2x. \qquad \square$$

Solution to (iii). We can write

$$\begin{aligned} f(z) = z\,\mathrm{Im}(z) = z\frac{z-\overline{z}}{2i} &= \frac{z^2 - z\overline{z}}{2i} \\ &= \frac{z^2 - (x^2+y^2)}{2i} \\ &= \frac{x^2 + 2ixy - y^2 - x^2 - y^2}{2i} \\ &= \frac{2ixy - 2y^2}{2i} \\ &= xy - \frac{y^2}{i} = xy + y^2 i \\ &= u(x,y) + iv(x,y). \end{aligned}$$

Now, we can compute the partial derivatives

$$\begin{aligned} u_x &= y, \quad u_y = x, \\ v_x &= 0, \quad v_y = 2y. \end{aligned}$$

Clearly, $u_x = v_y$ and $u_y = -v_x$ only when $x = 0 = y$. So, f is differentiable only at the origin. To compute the derivative at the origin, we use the limit definition:

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{z\,\mathrm{Im}(z)}{z} = \lim_{z\to 0} \mathrm{Im}(z) = 0. \qquad \square$$