

Fundamentals of Analysis II: Homework 6

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Yuan Xu 13:00

Hashem A. Damrah
UO ID: 952102243

Exercise 7.4.2.

(i) Let $g(x) = x^3$, and classify each of the following as positive, negative, or zero

$$(i) \int_0^{-1} g + \int_0^1 g \quad (ii) \int_1^0 g + \int_0^1 g \quad (iii) \int_1^{-2} g + \int_0^1 g.$$

(ii) Show that if $b \leq a \leq c$ and f is integrable on $[b, c]$, then it is still the case that $\int_a^b f = \int_a^c f + \int_c^b f$.

Solution to (i). Converting the integrals to their respective values, we have

$$\int_0^{-1} g + \int_0^1 g = - \int_{-1}^0 x^3 dx + \int_0^1 x^3 dx = -\left(-\frac{1}{4}\right) + \frac{1}{4} = \frac{1}{2}.$$

Therefore, the integral $\int_0^{-1} g + \int_0^1 g$ is positive.

Converting the integrals to their respective values, we have

$$\int_1^0 g + \int_0^1 g = - \int_0^1 x^3 dx + \int_0^1 x^3 dx = 0.$$

Therefore, the integral $\int_1^0 g + \int_0^1 g$ is zero.

Converting the integrals to their respective values, we have

$$\int_1^{-2} g + \int_0^1 g = - \int_{-2}^1 x^3 dx + \int_0^1 x^3 dx = -\left(-\frac{15}{4}\right) + \frac{1}{4} = 4.$$

Therefore, the integral $\int_1^{-2} g + \int_0^1 g$ is positive. \square

Solution to (ii). Assume $b \leq a \leq c$ and f is integrable on $[b, c]$. By the additivity of the integral, we have

$$\int_b^c f = \int_b^a f + \int_a^c f.$$

Rearranging the terms, we have

$$\begin{aligned} \int_b^a f &= \int_b^c f - \int_a^c f \\ \Rightarrow - \int_a^b f &= \int_b^c f - \int_a^c f \\ \Rightarrow \int_a^b f &= - \int_b^c f + \int_a^c f \\ \Rightarrow \int_a^b f &= \int_a^c f + \int_c^b f. \end{aligned}$$

\square

Exercise 7.4.3. Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

- (i) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.
- (ii) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.
- (iii) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

Solution to (i). True. Assume $|f|$ is integrable on $[a, b]$. By the integrability criterion, for any $\varepsilon > 0$, there is a partition P_ε such that

$$U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \varepsilon.$$

Since $-|f| \leq f \leq |f|$, we have

$$L(|f|, P_\varepsilon) \leq L(f, P_\varepsilon), \quad U(f, P_\varepsilon) \leq U(|f|, P_\varepsilon).$$

This implies that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \varepsilon.$$

By the integrability criterion, f is integrable on $[a, b]$. \square

Solution to (ii). False. Consider the function

$$g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then g is non-negative and positive on an infinite set of points. However, it is zero almost everywhere, so the integral $\int_a^b g = 0$. \square

Solution to (iii). True. Since g is continuous and $g(y_0) > 0$, there is an interval around y_0 where $g(x) > 0$. Since this is a nonzero interval and $g(x) \geq 0$, the integral over this interval is positive. Hence, $\int_a^b g > 0$. \square

Exercise 7.4.6. Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact:

(i) If f satisfies $|f(x)| \leq M$ on $[a, b]$, show

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

(ii) Prove that if f is integrable on $[a, b]$, then so is f^2 .

(iii) Now show that if f and g are integrable, then fg is integrable. (Consider $(f+g)^2$.)

Solution to (i). Using the difference of squares factorization

$$f(x)^2 - f(y)^2 = (f(x) - f(y))(f(x) + f(y)).$$

Taking absolute values,

$$|f(x)^2 - f(y)^2| = |f(x) - f(y)| \cdot |f(x) + f(y)|.$$

Since $|f(x)| \leq M$ and $|f(y)| \leq M$, we get

$$|f(x) + f(y)| \leq |f(x)| + |f(y)| \leq M + M = 2M.$$

Thus,

$$|f(x)^2 - f(y)^2| \leq 2M|f(x) - f(y)|. \quad \square$$

Solution to (ii). Since f is integrable, it satisfies the definition of integrability: for every $\varepsilon > 0$, there exists a partition P such that the difference between the upper sum and lower sum is less than ε . That is,

$$U(f, P) - L(f, P) < \varepsilon.$$

Now, we analyze f^2 . From Step 1, we have the inequality:

$$|f(x)^2 - f(y)^2| \leq 2M|f(x) - f(y)|.$$

This means that the function f^2 has variations that are controlled by f . Since f is integrable, the variation $|f(x) - f(y)|$ can be made arbitrarily small by refining the partition. Using this bound, we can show that $U(f^2, P) - L(f^2, P)$ also becomes arbitrarily small for sufficiently fine partitions. Therefore, f^2 is integrable. \square

Solution to (iii). Rearranging $(f + g)^2 = f^2 + 2fg + g^2$, we get

$$fg = \frac{(f + g)^2 - f^2 - g^2}{2}.$$

Since we have already established that if f is integrable, then f^2 is integrable, and since f and g are both given to be integrable, we conclude that f^2 , g^2 , and $(f + g)^2$ are all integrable.

Since the set of integrable functions is closed under addition and scalar multiplication, it follows that fg is also integrable. \square

Exercise 7.5.1.

- (i) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^x f$. Find a piecewise algebraic formula for $F(x)$ for all x . Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

- (ii) Repeat part (i) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Solution to (i). Define $f(x)$ as

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Evaluating $F(x)$, we get

$$F(x) = \int_{-1}^x -t dt.$$

We then have two cases to consider. If $x < 0$, then from -1 to x , we are in the negative region

$$F(x) = \int_{-1}^x -t dt = \frac{1}{2} - \frac{x^2}{2}.$$

If $x \geq 0$, then from -1 to x , we are in the positive region

$$F(x) = \int_{-1}^0 -t dt = \frac{1}{2} + \frac{x^2}{2}.$$

Therefore, the piecewise formula for $F(x)$ is

$$F(x) = \begin{cases} \frac{1}{2} - \frac{x^2}{2} & \text{if } x < 0 \\ \frac{1}{2} + \frac{x^2}{2} & \text{if } x \geq 0. \end{cases}$$

Each piece is a polynomial and polynomials are continuous, so we only need to check continuity at $x = 0$. Evaluate the left and right limits

$$\begin{aligned} \lim_{x \rightarrow 0^-} F(x) &= \lim_{x \rightarrow 0^-} \left(\frac{1}{2} - \frac{x^2}{2} \right) = \frac{1}{2} \\ \lim_{x \rightarrow 0^+} F(x) &= \lim_{x \rightarrow 0^+} \left(\frac{1}{2} + \frac{x^2}{2} \right) = \frac{1}{2} \\ F(0) &= \frac{1}{2}. \end{aligned}$$

Since the left and right limits are equal to the value of the function at $x = 0$, $F(x)$ is continuous at $x = 0$, making $F(x)$ continuous everywhere.

Again, since each piece is a polynomial, $F(x)$ is differentiable everywhere, except possibly at $x = 0$. Evaluate the left and right derivatives at $x = 0$,

$$F'(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}.$$

Evaluating the left and right derivatives at $x = 0$,

$$\lim_{x \rightarrow 0^-} F'(x) = \lim_{x \rightarrow 0^-} -x = 0$$

$$\lim_{x \rightarrow 0^+} F'(x) = \lim_{x \rightarrow 0^+} x = 0.$$

Since the left and right derivatives are equal at $x = 0$, $F(x)$ is differentiable at $x = 0$, making $F(x)$ differentiable everywhere.

Therefore, we have

$$F'(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Clearly, $F'(x) = f(x)$ for all x . □

Solution to (ii). Define $f(x)$ as

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Evaluating $F(x)$, we get

$$F(x) = \int_{-1}^x f dt.$$

We then have two cases to consider. If $x < 0$, then from -1 to x , we are in the region where $f(x) = 1$

$$F(x) = \int_{-1}^x 1 dt = x + 1.$$

If $x \geq 0$, then from -1 to x , we are in the region where $f(x) = 2$

$$F(x) = \int_{-1}^0 1 dt + \int_0^x 2 dt = 1 + 2x.$$

Therefore, the piecewise formula for $F(x)$ is

$$F(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 1 + 2x & \text{if } x \geq 0. \end{cases}$$

Each piece is a polynomial and polynomials are continuous everywhere, so we only need to check continuity at $x = 0$. Evaluate the left and right limits

$$\lim_{x \rightarrow 0^-} F(x) = \lim_{x \rightarrow 0^-} (x + 1) = 1$$

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow 0^+} (1 + 2x) = 1$$

$$F(0) = 1.$$

Since the left and right limits are equal to the value of the function at $x = 0$, $F(x)$ is continuous at $x = 0$, making $F(x)$ continuous everywhere.

Differentiating each piece, we get

$$F'(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Evaluating the left and right derivatives at $x = 0$,

$$\lim_{x \rightarrow 0^-} F'(x) = \lim_{x \rightarrow 0^-} 1 = 1$$

$$\lim_{x \rightarrow 0^+} F'(x) = \lim_{x \rightarrow 0^+} 2 = 2.$$

Since the left and right derivatives are not equal at $x = 0$, $F(x)$ is not differentiable at $x = 0$, making $F(x)$ differentiable everywhere except at $x = 0$.

Therefore, we have

$$F'(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Clearly, $F'(x) \neq f(x)$ for all $x \neq 0$. □

Exercise 7.5.2. Decide whether each statement is true or false, providing a short justification for each conclusion.

- (i) If $g = h'$ for some h on $[a, b]$, then g is continuous on $[a, b]$.
- (ii) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.
- (iii) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c .

Solution to (i). False. Consider the function $h : [-1, 1] \rightarrow \mathbb{R}$ defined as

$$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then, $g : [-1, 1] \rightarrow \mathbb{R}$ defined as

$$g(x) = \begin{cases} h'(x) = 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0, but $g = h'$. □

Solution to (ii). True. By the Fundamental Theorem of Calculus, if g is continuous on $[a, b]$, then g is the derivative of some function h on $[a, b]$. □

Solution to (iii). False. Consider the function

$$H(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then, $H(x) = 0$ and differentiable at 0, but h is not continuous at 0. □

Exercise 7.5.4. Show that if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Solution. Since f is continuous, by Theorem 7.5.1, part (ii), letting $F(x) = \int_a^x f = 0$ for all $x \in [a, b]$, we have $F'(x) = f(x) = 0$ for all $x \in [a, b]$. Therefore, $f(x) = 0$ everywhere on $[a, b]$. If f is not continuous, then this does not hold. □