Defn: Suppose f is a function of two variables, then the gradient of f denoted ∇f (stated grad f) is

$$\nabla f = \langle f_x, f_y \rangle.$$

Similarly, if $f = f(x,y,z)$, then

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

The directional derivative of $f(x,y,z)$ at (x_0, y_0, z_0) in the direction of the unit vector $\hat{u} = \langle a, b, c \rangle$ is

$$D_{\hat{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

$$= a f_x(x_0, y_0, z_0) + b f_y(x_0, y_0, z_0) + c f_z(x_0, y_0, z_0)$$

$$= \langle a, b, c \rangle \cdot \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle$$

$$= \hat{u} \cdot \nabla f(x_0, y_0, z_0)$$

Ex: Find the directional derivative of $f = x^2 \ln(xy - z)$ at $(2,4,7)$ toward $(3,2,6)$.

Direction: $\vec{u} = \langle 1, -2, -1 \rangle$

$$\hat{u} = \frac{1}{\sqrt{6}} \langle 1, -2, -1 \rangle$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \left\langle 2x \ln(xy - z) + \frac{x^2 y}{xy - z}, \frac{x^3}{xy - z}, \frac{-x^2}{xy - z} \right\rangle$$

$$\nabla f(2,4,7) = \langle 16, 8, -4 \rangle$$

$$D_{\hat{u}} f(2,4,7) = \hat{u} \cdot \nabla f(2,4,7)$$

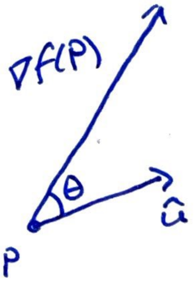
$$= \frac{1}{\sqrt{6}} \langle 1, -2, -1 \rangle \cdot \langle 16, 8, -4 \rangle$$

$$= \frac{1}{\sqrt{6}} (16 - 16 + 4)$$

$$= \frac{4}{\sqrt{6}}$$

Maximizing Directional Derivative

Let f be a differentiable function and $\hat{u}$ a unit vector. At a point P, ~~let~~ θ be angle between $\nabla f(P)$ and $\hat{u}$.



$$D_{\hat{u}} f(P) = \hat{u} \cdot \nabla f(P)$$
$$= |\hat{u}| |\nabla f(P)| \cos\theta$$
$$= |\nabla f(P)| \cos\theta$$

Since $-1 \le \cos\theta \le 1$, then for all directions, $\hat{u}$,

$$-|\nabla f(P)| \le D_{\hat{u}} f(P) \le |\nabla f(P)|$$

The directional ~~dire~~ derivative has a maximum value of $|\nabla f(P)|$ and occurs when $\cos\theta = 1$. Therefore $\theta = 0$ and $\nabla f(P)$ and $\hat{u}$ have the same direction.

The directional derivative has a minimum value of $-|\nabla f(P)|$ and it occurs when $\theta = \pi$ and $\hat{u}$ has opposite direction of $\nabla f(P)$.

In fact, for each c in $[-|\nabla f(P)|, |\nabla f(P)|]$, there exists a direction, $\hat{u}$, such that $D_{\hat{u}} f(P) = c$.

Ex: Find the maximum rate of change of $f(x,y) = x^2y + \ln x$

at $(1,2)$.

$\nabla f = \langle f_x, f_y \rangle = \langle 2xy + \frac{1}{x}, x^2 \rangle$

$\vec{\nabla} f(1,2) = \langle 5, 1 \rangle$

The max rate of change at $(1,2)$ is $|\nabla f(1,2)| = \sqrt{26}$

and it occurs in the direction of $\nabla f(P)$.

Tangent Plane

Let $F(x,y,z)$ be a function of three variables and then for each K in the range, $F=K$ is a level surface of F. ($F=K$ is an implicitly defined surface.) Let $P(x_0, y_0, z_0)$ be a point on the surface. Find the tangent plane to the surface at P.

Let C be any curve on the surface that passes through P.

Therefore C can be defined as $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and there exists t_0 such that $\vec{r}(t_0) = \vec{OP}$.

Since C lies on the surface, the components of $\vec{r}(t)$ satisfy $F = K$.

$$F(x(t), y(t), z(t)) = K$$

$$\frac{d}{dt}\left(F(x(t), y(t), z(t))\right) = \frac{d}{dt}(K)$$

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$$\nabla F \cdot \vec{r}\,'(t) = 0$$

In particular, at P,

$\nabla F(P) \cdot \vec{r}\,'(t_0) = 0$ ∴ $\nabla F(P)$ and $\vec{r}\,'(t_0)$ are orthogonal.

Since $\vec{r}\,'(t_0)$ is tangent to C at P, then $\nabla F(P)$ is orthogonal to C at P. This is true for all curves on the surface through P and therefore $\nabla F(P)$ is the normal vector of the tangent plane.