Multi-Variable Calculus I: Homework 2

Due on October 15, 2024 at 8:00 AM

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Problem 1

Determine if the points $A(3,1,2)$, $B(1,-1,1)$, $C(-2,4,3)$, and $D(-3,-5,0)$ are coplanar. That is, determine if the points lie in the same plane.

Solution 1

We first find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$.

$$\begin{aligned}\overrightarrow{AB} &= \langle 1-3, -1-1, 1-2\rangle &&= \langle -2,-2,-1\rangle \\ \overrightarrow{AC} &= \langle -2-3, 4-1, 3-2\rangle &&= \langle -5,3,1\rangle \\ \overrightarrow{AD} &= \langle -3-3, -5-1, 0-2\rangle &&= \langle -6,-6,-2\rangle.\end{aligned}$$

Then, we compute the scalar triple product $\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})$.

$$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 3 & 1 \\ -6 & -6 & -2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -5 & 1 \\ -6 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -5 & 3 \\ -6 & -6 \end{vmatrix}.$$

Therefore, we have $\overrightarrow{AC}\times\overrightarrow{AD} = \langle 0,-16,48\rangle$. Then, we have

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = \langle -2,-2,-1\rangle\cdot\langle 0,-16,48\rangle = (-2)(0)+(-2)(-16)+(-1)(48) = -16.$$

Since the scalar triple product is not zero, the points A, B, C, and D are not coplanar.

Problem 2

Suppose $\mathbf{a} \neq 0$.

1. True or False : If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$. Justify your answer.
2. True or False : If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$. Justify your answer.
3. True or False : If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$. Justify your answer.

Solution 2

1. False: The dot product $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ means that the projections of $\mathbf{b}$ and $\mathbf{c}$ in the direction of $\mathbf{a}$ are equal. However, the vectors $\mathbf{b}$ and $\mathbf{c}$ can be different.
2. False: The cross product $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ means that the vectors $\mathbf{b}$ and $\mathbf{c}$ have the same components perpendicular to $\mathbf{a}$. However, the vectors $\mathbf{b}$ and $\mathbf{c}$ can be different.
3. True: If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, that means that $\mathbf{b}$ and $\mathbf{c}$ have the same projection in the direction of $\mathbf{a}$. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, that means that $\mathbf{b}$ and $\mathbf{c}$ have the same components perpendicular to $\mathbf{a}$. Therefore, $\mathbf{b}$ and $\mathbf{c}$ must be the same.

Problem 3

① Find parametric equations and symmetric equations for the line through the point $P(5,3,1)$ and $Q(7,4,-3)$.

② At what point does the line intersect the xz-plane?

③ Where does the line intersect the plane $x - 3y - 2z = 8$?

Solution 3

① First, find $\overrightarrow{PQ}$

$$\overrightarrow{PQ} = \langle 7-5, 4-3, -3-1 \rangle = \langle 2, 1, -4 \rangle.$$

The parametric equations of a line with a point $P(x_0, y_0, z_0)$ and a direction vector $\langle a, b, c \rangle$ are

$$x = 5 + 2t$$
$$y = 3 + t$$
$$z = 1 - 4t,$$

and the symmetric equations are

$$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z-1}{-4}.$$

② To find the intersection point, set $y = 0$ in the parametric equations and solve for t, giving us $t = -3$. Substituting $t = -3$ into the parametric equations gives us the intersection point $(-1, 0, 13)$.

③ To find the intersection point, substitute the parametric equations for x, y, and z into the plane equation, giving us

$$(5 + 2t) - 3(3 + t) - 2(1 - 4t) = 8 \Rightarrow t = 2.$$

Substituting $t = 2$ into the parametric equations gives us the intersection point $(9, 5, -7)$.

Problem 4

Consider the points $P(2, 6, -1)$, $Q(-1, 8, -2)$, and $R(3, 7, 0)$.

① Find a vector orthogonal to the plane containing the points P, Q, and R.

② Find the area of the triangle with vertices P, Q, and R.

③ Find the equation of the plane containing the points P, Q, and R.

Solution 4

① First, find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$$\begin{aligned} \overrightarrow{PQ} &= \langle -1-2, 8-6, -2-(-1)\rangle = \langle -3, 2, -1\rangle \\ \overrightarrow{PR} &= \langle 3-2, 7-6, 0-(-1)\rangle \qquad = \langle 1, 1, 1\rangle. \end{aligned}$$

Now, find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}.$$

Therefore, we have $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 3, 2, -5\rangle$. Thus, the vector $\langle 3, 2, -5\rangle$ is orthogonal to the plane containing the points P, Q, and R.

② The area of the triangle with vertices P, Q, and R is given by

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{3^2 + 2^2 + (-5)^2} = \frac{1}{2}\sqrt{38}.$$

③ The general form of a plane equation is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

where $\langle a, b, c\rangle$ is a normal vector to the plane, and (x_0, y_0, z_0) is a point on the plane. We can use the point $P(2, 6, -1)$ and the normal vector $\langle 3, 2, -5\rangle$ to find the equation of the plane $3(x - 2) + 2(y - 6) - 5(z + 1) = 0$. Simplifying gives us the final equation

$$3x + 2y - 5z - 23 = 0.$$

Problem 5

Find the equation of the plane that contains the points $A(2,1,-1)$ and $B(0,-2,4)$ and is perpendicular to the plane $2x-y+3z=10$.

Solution 5

We first need to find the direction vector

$$\overrightarrow{AB} = \langle 0-2, -2-1, 4-(-1)\rangle = \langle -2,-3,5\rangle.$$

The normal vector to the plane $2x-y+3z=10$ is $\langle 2,-1,3\rangle$. Since the plane we are looking for is perpendicular to the given plane, the normal vector to the plane we are looking for is the cross product of the direction vector $\overrightarrow{AB}$ and the normal vector $\langle 2,-1,3\rangle$. Therefore, we have

$$\overrightarrow{AB} \times \langle 2,-1,3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & 5 \\ 2 & -1 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -3 & 5 \\ -1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 5 \\ 2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -3 \\ 2 & -1 \end{vmatrix}.$$

Therefore, we have $\overrightarrow{AB} \times \langle 2,-1,3\rangle = \langle -4,16,8\rangle$. We can use the point $A(2,1,-1)$ and the normal vector $\langle -4,16,8\rangle$ to find the equation of the plane $-4(x-2)+16(y-1)+8(z+1)=0$. Simplifying gives us the final equation

$$x-4y-2z=0.$$