Introduction to Toplogy I: Homework 2

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Exercise 1.5. In Example 1.8(a) we saw a sequence in $C([0,1])$ that converges in the L^1 metric but not in the sup metric. Prove that the reverse cannot happen: every sequence that converges in the sup metric converges in the L^1 metric.

Solution. Let (f_n) be a sequence in $C([0,1])$ and suppose

$$\mathrm{d}_\infty(f_n, f) := \sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0,$$

as $n \to \infty$, for some function f in $C([0,1])$. For each n and every $x \in [0,1]$ we have the pointwise inequality

$$|f_n(x) - f(x)| \leq \mathrm{d}_\infty(f_n, f).$$

Integrating both sides over $[0,1]$ yields

$$\int_0^1 |f_n(x) - f(x)| \mathrm{d}x \leq \int_0^1 \mathrm{d}_\infty(f_n, f)\, \mathrm{d}x = \mathrm{d}_\infty(f_n, f) \cdot 1.$$

Hence, with $\mathrm{d}_1(g,h) := \int_0^1 |g(x) - h(x)| \mathrm{d}x$, we get

$$\mathrm{d}_1(f_n, f) \leq \mathrm{d}_\infty(f_n, f) \to 0.$$

Therefore $\mathrm{d}_1(f_n, f) \to 0$, i.e. $f_n \to f$ in the L^1 metric on $C([0,1])$. □

Exercise 1.6. Let (X, d) be a metric space. Prove the reverse *triangle inequality*:

$$|\mathrm{d}(p,q) - \mathrm{d}(p,r)| \leq \mathrm{d}(q,r),$$

for all $p, q, r \in X$. Include an appropriate picture.

Solution. Let $p, q, r \in X$. By the triangle inequality we have

$$\mathrm{d}(p,q) \leq \mathrm{d}(p,r) + \mathrm{d}(r,q),$$

which can be rewritten as

$$\mathrm{d}(p,q) - \mathrm{d}(p,r) \leq \mathrm{d}(r,q) = \mathrm{d}(q,r).$$

Similarly, swapping q and r gives

$$\mathrm{d}(p,r) \leq \mathrm{d}(p,q) + \mathrm{d}(q,r) \Rightarrow \mathrm{d}(p,r) - \mathrm{d}(p,q) \leq \mathrm{d}(q,r),$$

or equivalently

$$-\mathrm{d}(q,r) \leq \mathrm{d}(p,q) - \mathrm{d}(p,r).$$

Combining these two inequalities yields the reverse triangle inequality:

$$|\mathrm{d}(p,q) - \mathrm{d}(p,r)| \leq \mathrm{d}(q,r).$$

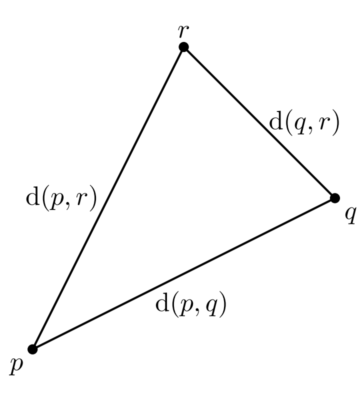

Figure 1: Visualization of the reverse triangle inequality in a metric space. □

Exercise 1.7. Let (X, d_X) and (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f : X \to Y$ be continuous at a point $p \in X$, and let $g : Y \to Z$ be continuous at $f(p)$. Prove that $g \circ f$ is continuous at p.

Solution. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces, let $f : X \to Y$ be continuous at $p \in X$, and let $g : Y \to Z$ be continuous at $f(p)$. We show $g \circ f$ is continuous at p.

Fix $\varepsilon > 0$. Since g is continuous at $f(p)$, there exists $\delta_2 > 0$ such that

$$\mathrm{d}_Y(y, f(p)) < \delta_2 \Rightarrow \mathrm{d}_Z(g(y), g(f(p))) < \varepsilon.$$

Since f is continuous at p, there exists $\delta_1 > 0$ such that

$$\mathrm{d}_X(x, p) < \delta_1 \Rightarrow \mathrm{d}_Y(f(x), f(p)) < \delta_2.$$

Now let $x \in X$ satisfy $\mathrm{d}_X(x, p) < \delta_1$. Then by the previous implication $\mathrm{d}_Y(f(x), f(p)) < \delta_2$, and therefore by the choice of δ_2 we have $\mathrm{d}_Z(g(f(x)), g(f(p))) < \varepsilon$. This shows that for every $\varepsilon > 0$, take $\delta = \delta_1$ to get

$$\mathrm{d}_X(x, p) < \delta \Rightarrow \mathrm{d}_Z(g(f(x)), g(f(p))) < \varepsilon.$$

Therefore, $g \circ f$ is continuous at p. □

Exercise 2.1. Sketch each subset of $\mathbb{R}^2$ and find its closure, interior, and boundary in the Euclidean metric:

(i) $A_1 = \{(x, y) \mid 0 < x \leq 1, 0 \leq y < 1\}$.

(ii) $A_2 = \{(x, y) \mid 0 < x \leq 1, y = 0\}$.

(iii) $A_3 = \{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$.

(iv) $A_4 = \{(x, y) \mid x \neq 0 \text{ or } y = 0\}$.

Solution to (i). The boundary of A_1 is the rectangle formed by the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The interior is the open rectangle bounded by those same lines, and the closure is the union of the interior and the boundary, that is, the closed rectangle $[0, 1] \times [0, 1]$.

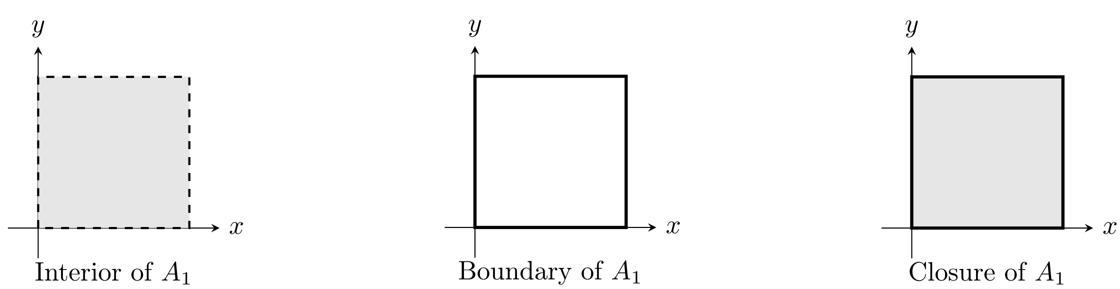

Figure 2: Visualization of the interior, boundary, and closure of $A_1 = [0, 1] \times [0, 1]$. □

Solution to (ii). The set $A_2 = \{(x, y) \mid 0 < x \leq 1,\ y = 0\}$ is a half-open line segment on the x-axis. Its interior is empty, since it has no open neighborhood contained in A_2. Its boundary is the closed segment $[0, 1] \times \{0\}$, and its closure is that same closed segment.

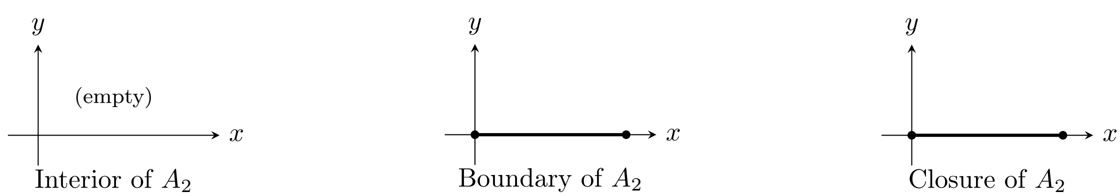

Figure 3: Visualization of the interior, boundary, and closure of $A_2 = \{(x, y) \mid 0 < x \leq 1,\ y = 0\}$. □

Solution to (iii). The set $A_3 = \{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ is dense in $\mathbb{R}^2$, since every open ball contains points with rational coordinates. However, its interior is empty because any open set in $\mathbb{R}^2$ also contains points whose x and y coordinates are both irrational. Therefore, its boundary and closure are both equal to $\mathbb{R}^2$.

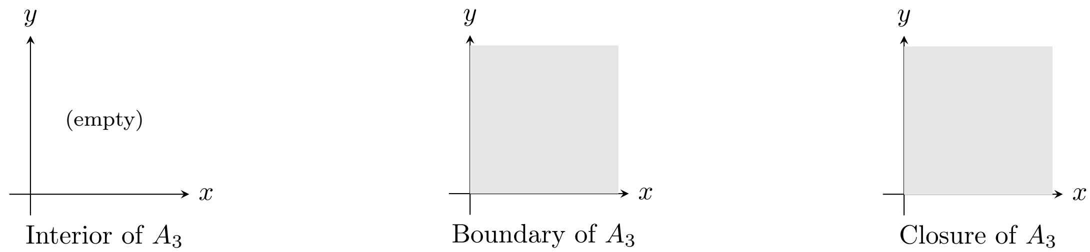

Figure 4: Visualization of the interior, boundary, and closure of $A_3 = \{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$. □

Solution to (iv). The set $A_4 = \{(x, y) \mid x \neq 0 \text{ or } y = 0\}$ is the entire plane except for the y-axis, with the x-axis added back in. Its interior is A_4 itself, since removing a one-dimensional subset from $\mathbb{R}^2$ does not affect openness. The boundary consists of the y-axis, and the closure is the whole plane $\mathbb{R}^2$.

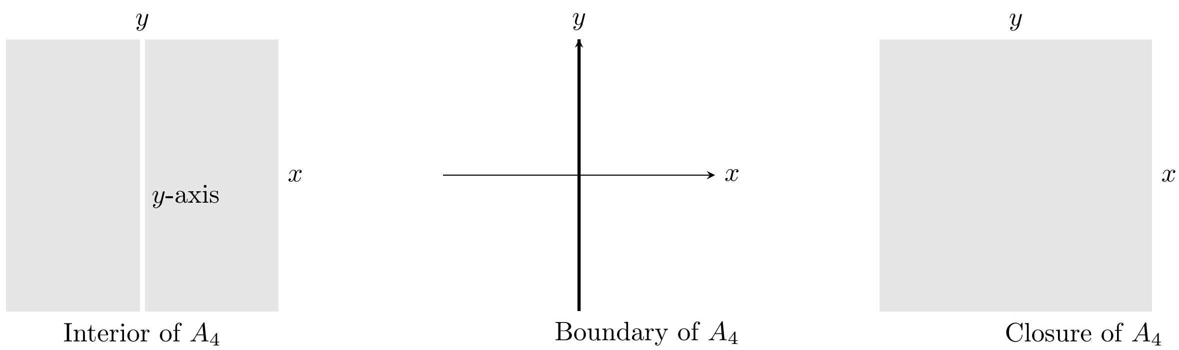

Figure 5: Visualization of the interior, boundary, and closure of $A_4 = \{(x, y) \mid x \neq 0 \text{ or } y = 0\}$. □

Exercise 2.3. Prove the analogue of Proposition 2.7 for closures, without appealing to Proposition 2.6.

(i) If $A \subset B$, then $\bar{A} \subset \bar{B}$.

(ii) $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

(iii) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

Give an example to show that the inclusion can be strict.

(iv) $\bar{\bar{A}} = \bar{A}$.

Solution to (i). Assume $A \subset B$. Let $p \in \bar{A}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects A. Since $A \subset B$, it follows that $B_r(p)$ also intersects B. Therefore, for every $r > 0$, the open ball $B_r(p)$ intersects B, which means $p \in \bar{B}$. Hence, $\bar{A} \subset \bar{B}$. □

Solution to (ii). Assume $p \in \bar{A} \cup \bar{B}$. Without loss of generality, assume $p \in \bar{A}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects A. Since $A \subset A \cup B$, it follows that $B_r(p)$ also intersects $A \cup B$. Therefore, for every $r > 0$, the open ball $B_r(p)$ intersects $A \cup B$, which means $p \in \overline{A \cup B}$. Hence, $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

Now, let $p \in \overline{A \cup B}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects $A \cup B$. This means that for each $r > 0$, there exists a point in either A or B that lies within the ball. If there are infinitely many such points in A, then p is a limit point of A and thus belongs to $\bar{A}$. Similarly, if there are infinitely many such points in B, then p is a limit point of B and thus belongs to $\bar{B}$. In either case, we have $p \in \bar{A} \cup \bar{B}$. Hence, $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

Combining both inclusions, we conclude that $\bar{A} \cup \bar{B} = \overline{A \cup B}$. □

Solution to (iii). Assume $p \in \overline{A \cap B}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects $A \cap B$. This means that for each $r > 0$, there exists a point in both A and B that lies within the ball. Therefore, for every $r > 0$, the open ball $B_r(p)$ intersects A and also intersects B. This implies that $p \in \bar{A}$ and $p \in \bar{B}$. Hence, $p \in \bar{A} \cap \bar{B}$, and we conclude that

$$\overline{A \cap B} \subset \bar{A} \cap \bar{B}.$$

To show that the inclusion can be strict, consider the metric space $(\mathbb{R}, \mathrm{d})$ with the usual metric. Let

$$A = (0, 1) \quad \text{and} \quad B = (1, 2).$$

Then,

$$A \cap B = (0, 1) \cap (1, 2) = \emptyset,$$

so

$$\overline{A \cap B} = \bar{\emptyset} = \emptyset.$$

However,

$$\bar{A} = [0, 1] \quad \text{and} \quad \bar{B} = [1, 2],$$

so

$$\bar{A} \cap \bar{B} = [0, 1] \cap [1, 2] = \{1\}.$$

Thus, $\overline{A \cap B} = \emptyset \subsetneq \{1\} = \bar{A} \cap \bar{B}$. □

Solution to (iv). Assume $p \in \bar{\bar{A}}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects $\bar{A}$. This means that for each $r > 0$, there exists a point in $\bar{A}$ that lies within the ball. Since $\bar{A}$ is the closure of A, it follows that for every $r > 0$, the open ball $B_r(p)$ also intersects A. Therefore, $p \in \bar{A}$. Hence, $\bar{\bar{A}} \subset \bar{A}$.

Now, let $p \in \bar{A}$. By definition of closure, for every $r > 0$, the open ball $B_r(p)$ intersects A. Since $A \subset \bar{A}$, it follows that for every $r > 0$, the open ball $B_r(p)$ also intersects $\bar{A}$. Therefore, $p \in \bar{\bar{A}}$. Hence, $\bar{A} \subset \bar{\bar{A}}$.

Combining both inclusions, we conclude that $\bar{\bar{A}} = \bar{A}$. □

Exercise 2.6. We saw in Example 2.8 that the inclusion $\operatorname{int} A \cup \operatorname{int} B \subset \operatorname{int}(A \cup B)$ can be strict. Prove however that if $\overline{A} \cap \overline{B} = \emptyset$ then $\operatorname{int} A \cup \operatorname{int} B = \operatorname{int}(A \cup B)$.

Solution. We always have

$$\operatorname{int} A \cup \operatorname{int} B \subset \operatorname{int}(A \cup B),$$

so it remains to prove the reverse inclusion under the assumption

$$\overline{A} \cap \overline{B} = \emptyset.$$

Let $x \in \operatorname{int}(A \cup B)$. Then there exists an open neighbourhood U of x with $U \subset A \cup B$. We will show that $x \in \operatorname{int} A$ or $x \in \operatorname{int} B$.

Suppose, for contradiction, that $x \notin \operatorname{int} A$ and $x \notin \operatorname{int} B$. Since $x \notin \operatorname{int} A$, every neighbourhood of x meets the complement $X \setminus A$. In particular U meets $X \setminus A$. But $U \subset A \cup B$, so any point of $U \cap (X \setminus A)$ must lie in B. Hence $U \cap B \neq \emptyset$, and because this holds for every neighbourhood of x we conclude $x \in \overline{B}$. By the same argument (swapping A and B) the assumption $x \notin \operatorname{int} B$ implies $x \in \overline{A}$. Thus

$$x \in \overline{A} \cap \overline{B},$$

contradicting the hypothesis that $\overline{A} \cap \overline{B} = \emptyset$.

Therefore at least one of $x \in \operatorname{int} A$ or $x \in \operatorname{int} B$ must hold, so

$$x \in \operatorname{int} A \cup \operatorname{int} B.$$

This proves $\operatorname{int}(A \cup B) \subset \operatorname{int} A \cup \operatorname{int} B$, and combined with the trivial inclusion we obtain

$$\operatorname{int}(A \cup B) = \operatorname{int} A \cup \operatorname{int} B.$$

□