Introduction to Proof: Homework 1

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Problem 1

In English we have lots of different ways of expressing conditional statements. For each of the following, rewrite it using the "If-Then" form in a way that preserves the meaning:

1. Passing the rest requires getting 70%.
2. My cat gets scared when the phone rings.
3. I get mad whenever you do that.
4. Being first in line guarantees getting a ticket.
5. You will fail the class unless you hand in your exam.

Solution 1

1. If I get 70% or more on the test, then I pass.
2. If the phone rings, then my cat gets scared.
3. If you do that, then I get mad.
4. If I'm first in line, then I'm guaranteed a ticket.
5. If you don't hand in your exam, then you will fail the class.

Problem 2

Recall the tautologies $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$, $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$, and $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$. Use these to write English language versions for the negation of each sentence below:

① If you wash the dishes, I will pay you 10.

② I will wash the car or I will mow the lawn.

③ I ate steak last night and I ate cereal this morning.

④ If I go to the movies, then I will see Star Wars or I will see Harry Potter.

⑤ If it is raining, then I will stay home or I will go to the movies.

⑥ If I am a monkey and you are a baboon then pigs can fly.

Solution 2

① If you won't wash the dishes, then I won't pay you $10.

② I will not wash the car and I will not mow the lawn.

③ I did not eat steak last night or I did not eat cereal this morning.

④ If I am not going to the movies, then I will not see Star Wars and I will not see Harry Potter.

⑤ If it is not raining, then I will not stay home and I will not go to the movies.

⑥ If I am not a monkey and you are not a baboon, then pigs don't fly.

Problem 3

Give the truth table for each of the following statements. Say which of the statements are tautologies.

1. $\neg[\neg P \wedge \neg Q]$.
2. $(P \vee Q) \Leftrightarrow (\neg P \Rightarrow Q)$.
3. $(Q \wedge \neg Q) \Rightarrow P$.
4. $Q \wedge (P \Rightarrow Q)$.
5. $(P \wedge Q) \Rightarrow (P \vee Q)$.
6. $(P \wedge Q) \Leftrightarrow (\neg Q \Rightarrow P)$.
7. $(P \vee Q) \Leftrightarrow (\neg Q \vee P)$.
8. $[P \Rightarrow (Q \Rightarrow R)] \Leftrightarrow [(P \wedge Q) \Rightarrow R]$.

Solution 3

P	Q	$P \vee Q$	$\neg P \Rightarrow Q$	$\Rightarrow$	$\Leftarrow$	$\Leftrightarrow$
True	True	True	True	True	True	True
True	False	True	True	True	True	True
False	True	True	True	True	True	True
False	False	False	True	False	True	True

$(P \vee Q) \Leftrightarrow (\neg P \Rightarrow Q)$: Tautology

P	Q	$P \Rightarrow Q$	$Q \wedge (P \Rightarrow Q)$
True	True	True	True
True	False	False	False
False	True	True	True
False	False	True	False

$Q \wedge (P \Rightarrow Q)$

P	Q	$P \wedge Q$	$\neg Q \Rightarrow P$	$\Rightarrow$	$\Leftarrow$	$\Leftrightarrow$
True	True	True	True	True	True	True
True	False	False	True	True	False	False
False	True	False	True	True	False	False
False	False	False	False	True	True	True

$[P \wedge Q] \Leftrightarrow [\neg Q \Rightarrow P]$

P	Q	$\neg P \wedge \neg Q$	$\neg[\neg P \wedge \neg Q]$
True	True	False	True
True	False	False	True
False	True	False	True
False	False	True	False

$\neg[\neg P \wedge \neg Q]$

P	Q	$P \wedge Q$	$P \vee Q$	$\Rightarrow$
True	True	True	True	True
True	False	False	True	True
False	True	False	True	True
False	False	False	False	True

$(P \wedge Q) \Rightarrow (P \vee Q)$: Tautology

P	Q	$P \vee Q$	$\neg Q \vee P$	$\Leftrightarrow$
True	True	True	True	True
True	False	True	True	True
False	True	True	False	False
False	False	False	True	True

$[P \vee Q] \Leftrightarrow [\neg Q \vee P]$

Problem 3 continued on next page . . .

P	Q	R	$P \Rightarrow [Q \Rightarrow R]$	$[P \wedge Q] \Rightarrow R$	$\Rightarrow$	$\Leftarrow$	$\Leftrightarrow$
True	True	True	True	True	True	True	True
False	True	True	True	True	True	True	True
False	False	True	True	True	True	True	True
False	False	False	True	True	True	True	True
True	False	False	True	True	True	True	True
True	True	False	False	False	True	True	True
False	True	False	True	True	True	True	True
True	False	True	True	True	True	True	True

$[P \Rightarrow (Q \Rightarrow R)] \Leftrightarrow [(P \wedge Q) \Rightarrow R]$: Tautology

Problem 4

Prove that $P, P \Leftrightarrow Q \vdash Q$ by filling in the blanks below. This rule of logic is called "Modus Ponens for the Biconditional (MPB)". Once you have proven it, you can use it in the other problems.

	Statement	Explanation
1.	P	??
2.	$P \Leftrightarrow Q$	??
3.	$P \Rightarrow Q$	??
4.	Q	??

Solution 4

	Statement	Explanation
1.	P	Hypothesis
2.	$P \Leftrightarrow Q$	Hypothesis
3.	$P \Rightarrow Q$	MPB, for 2, for 1
4.	Q	MP, for 3, for 1

Problem 5

Fill in the blanks to give a proof of $[P \wedge Q] \Rightarrow R$, $\neg R$, $P \vdash \neg Q$.

	Statement	Explanation
1.	$[P \wedge Q] \Rightarrow R$	??
2.	$\neg R$	??
3.	$\neg[P \wedge Q]$	??
4.	$\neg[P \wedge Q] \Leftrightarrow [\neg P \vee \neg Q]$	??
5.	??	MPB, for 4, for 3
6.	$[\neg P \vee \neg Q] \Rightarrow [P \Rightarrow \neg Q]$	??
7.	$P \Rightarrow \neg Q$	??
8.	??	Hypothesis
9.	??	??

Solution 5

	Statement	Explanation
1.	$[P \wedge Q] \Rightarrow R$	Hypothesis
2.	$\neg R$	Hypothesis
3.	$\neg[P \wedge Q]$	MT, for 1, for 2
4.	$\neg[P \wedge Q] \Leftrightarrow [\neg P \vee \neg Q]$	DeMorgan's Law
5.	$[\neg P \wedge \neg Q]$	MPB, for 4, for 3
6.	$[\neg P \vee \neg Q] \Rightarrow [P \Rightarrow \neg Q]$	Tautology
7.	$P \Rightarrow \neg Q$	MPB, for 6
8.	P	Hypothesis
9.	$P \Rightarrow \neg Q$	MP, for 7, for 8

Problem 6

Show that $\neg Q \Rightarrow \neg P, \neg Q \wedge R \vdash \neg P$.

Solution 6

	Statement	Explanation
1.	$\neg Q \Rightarrow \neg P$	Hypothesis
2.	$\neg Q \wedge R$	Hypothesis
3.	$\neg Q$	LCS, for 2
4.	$\neg P$	MP, for 1, for 3

Problem 7

Show that $Q \vee R, \neg R \vdash Q$.

Solution 7

	Statement	Explanation
1.	$Q \vee R$	Hypothesis
2.	$\neg R$	Hypothesis
3.	Q	DI, for 1, for 2

Problem 8

Show that $R \Rightarrow S$, $R \wedge Q \vdash R \wedge S$.

Solution 8

	Statement	Explanation
1.	$R \Rightarrow S$	Hypothesis
2.	$R \wedge Q$	Hypothesis
3.	R	LCS, for 2
4.	S	MP, for 1, for 3
5.	$R \wedge S$	CI, for 3, for 4

Problem 9

Show that $\neg[P \wedge R] \Rightarrow \neg S, \neg S \Rightarrow Q, \neg[P \wedge R] \vdash \neg S \wedge Q$.

Solution 9

	Statement	Explanation
1.	$\neg[P \wedge R] \Rightarrow \neg S$	Hypothesis
2.	$\neg S \Rightarrow Q$	Hypothesis
3.	$\neg[P \wedge R]$	Hypothesis
4.	$\neg S$	MP, for 1, for 3
5.	Q	MP, for 2, for 4
6.	$\neg S \wedge Q$	CI, for 4, for 5

$\square$

Problem 10

Show that $[P \lor R] \Rightarrow Q, P \vdash P \land Q$.

Solution 10

	Statement	Explanation
1.	$[P \lor R] \Rightarrow Q$	Hypothesis
2.	P	Hypothesis
3.	$P \Rightarrow (P \lor R)$	Tautology
4.	Q	MP, for 1, for 3
5.	$P \land Q$	CI, for 2, for 4

Problem 11

Show that $R \wedge \neg Q$, $P \Rightarrow Q \vdash R \wedge \neg P$.

Solution 11

	Statement	Explanation
1.	$R \wedge \neg Q$	Hypothesis
2.	$P \Rightarrow Q$	Hypothesis
3.	$\neg Q$	RCS, for 1

Problem 12

Show that Q, $[P \Rightarrow Q] \Rightarrow [R \wedge S] \vdash S$. (This can be done in 6 lines if you find the right tautology. Try working backwards).

Solution 12

	Statement	Explanation
1.	Q	Hypothesis
2.	$[P \Rightarrow Q] \Rightarrow [R \wedge S]$	Hypothesis
3.	$Q \Rightarrow [P \Rightarrow Q]$	Tautology
4.	$P \Rightarrow Q$	MP, for 3, for 1
5.	$R \wedge S$	MP, for 2, for 4
6.	S	LCS, for 5

Here's the truth table to show that $Q \Rightarrow [P \Rightarrow Q]$ tautology:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow [P \Rightarrow Q]$
True	True	True	True
True	False	False	True
False	True	True	True
False	False	True	True

Problem 13

Show that $Q \Rightarrow S$, $\neg S \vee R$, $\neg R \vdash \neg Q$.

Solution 13

	Statement	Explanation
1.	$Q \Rightarrow S$	Hypothesis
2.	$\neg S \vee R$	Hypothesis
3.	$\neg R$	Hypothesis
4.	$\neg S$	DI, for 2, for 3
5.	$\neg Q$	MP, for 1, for 4

Problem 14

There are five people considering going to the beach: Amanda, Bill, Chris, Debbie, and Eeyore. The following are true statements:

1. If Amanda goes to the beach, then Eeyore will not go.
2. If Debbie goes (to the beach), then Chris will go too.
3. If Debbie does not go, then Eeyore will go.
4. If Eeyore goes to the beach, then either Amanda will go or Bill will not go.

This is all too complicated for Chris, who gets fed up and decides not to go to the beach. Determine who goes to the beach, and who stays home.
Convert this into a problem in symbolic logic. I suggest you use "A" to stand for the proposition "Amanda goes to the beach". Give a symbolic logic proof, as we have been doing in the above exercises, where you use the rules of logic to deduce who does and does not go to the beach. You should have five hypotheses.

Solution 14

Let $A =$ Amanda goes to the beach, $B =$ Bill goes to the beach, $C =$ Chris goes to the beach, $D =$ Debbie goes to the beach, and $E =$ Eeyore goes to the beach. The statements can be written as:

	Statement	Explanation		Statement	Explanation
1.	$A \Rightarrow \neg E$	Hypothesis	6.	$\neg D$	MT, for 2, for 5
2.	$D \Rightarrow C$	Hypothesis	7.	E	MP, for 3, for 6
3.	$\neg D \Rightarrow E$	Hypothesis	8.	$A \lor \neg B$	MP, for 4, for 7
4.	$E \Rightarrow (A \lor \neg B)$	Hypothesis	9.	$\neg A$	MT, for 1, for 7
5.	$\neg C$	Hypothesis	10.	$\neg B$	DI, for 8, for 9

Therefore, if Chris does not go to the beach, then only Eeyore goes to the beach, while everyone else stays at home.

Problem 15

I have a penny, a dime, and a quarter. If you make a true statement, I will give you one of the coins. If you make a false statement, I will give you nothing. What statement can you make that will *guarantee* that you get the quarter?

Solution 15

Let P = "I have a penny", D = "I have a dime", Q = "I have a quarter", C = "You will give me a coin". The statement that guarantees that you get the quarter is $Q \vee \neg Q \vee \neg C$. If they don't give me a coin, then the statement is true; making them give me a quarter. If they do give me a coin, say a penny, then the statement is still true, since they aren't giving me a quarter.

Problem 16

Write out the operation tables for $(\mathbb{Z}_5, \cdot_5)$ and $(\mathbb{Z}_5, +_5)$.

Solution 16

$$
\begin{array}{c|cccc}
0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 \\
2 & 3 & 4 & 0 & 1 \\
3 & 4 & 0 & 1 & 2 \\
4 & 0 & 1 & 2 & 3
\end{array}
\qquad
\begin{array}{c|cccc}
0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 4 & 1 & 3 \\
3 & 3 & 1 & 4 & 2 \\
4 & 4 & 3 & 2 & 1
\end{array}
$$

$(\mathbb{Z}_5, +_5)$ $(\mathbb{Z}_5, \cdot_5)$

Problem 17

Solve the equation $(7 \cdot_{11} x) +_{11} 10 = 4$ in $\mathbb{Z}_{11}$. (As always in this course, you should explain your methods. Just writing down the answer with no explanation will not earn many points.)

Solution 17

Rewriting the equation gives us $7x + 10 = 4 \pmod{11}$, to reduce the tediousness of writing $\cdot_{11}$ and $+_{11}$ every time. Adding 10's additive inverse to both sides gives us $7x + 10 + 1 = 4 + 1 \pmod{11} \Rightarrow 7x = 5\text{mod}(11)$. Multiplying the equation by 7's multiplicative inverse gives us $8 \cdot 7x = 8 \cdot 5 \pmod{11} \Rightarrow x = 7$.
Checking to see if $x = 7$ is true gives us $7 \cdot 7 + 10 \pmod{11} = 5 + 10 \pmod{11} = 4 \pmod{11}$.

Problem 18

Make a table showing all the perfect squares in $\mathbb{Z}_{11}$ (I mean 0^2, 1^2, 2^2, and so on). Use this to help you find all solutions to the equation

$$x^2 + 4x + 10 = 0,$$

in $\mathbb{Z}_{11}$. (Hint: Complete the square.) I strongly suggest that you check your answers at the end, by plugging them back into the quadratic equation.

Solution 18

Here's the table showing all the perfect squares in $\mathbb{Z}_{11}$:

0	1	2	3	4	5	6	7	8	9	10
m^2	1	4	9	5	3	3	5	9	4	1

Adding 10's additive inverse to both sides gives us $x^2 + 4x = 1 \pmod{11}$. We can re-write $4x$ as $(2 \cdot 2)x$. We can then add a^2 to both sides of the equation to get

$$(x + 2)^2 = 5.$$

Using the table of perfect squares, we can see that $x + 2 = 4$ and $x + 2 = 7$ are the only solutions. Therefore, $x = 2$ and $x = 5$ are the solutions to the equation.

Checking to see if $x = 2$ and $x = 5$ are true gives us $2^2 + 2 \cdot 4 + 10 \pmod{11} \overset{\checkmark}{=} 0$ and $5^2 + 5 \cdot 4 + 10 \pmod{11} \overset{\checkmark}{=} 0$.