Functional Complex Variables I: Homework 4

Due on April 30, 2025 at 23:59

Weiyong He

Hashem A. Damrah
UO ID: 952102243

Exercise 2.23.6. Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}.$$

Verify that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$.

Solution. First, write $z = x + iy$, so $\overline{z} = x - iy$. Then compute $f(z)$ for $z \neq 0$

$$f(z) = \frac{(x - iy)^2}{x + iy} = \frac{x^2 - 2ixy - y^2}{x + iy}.$$

Multiply numerator and denominator by the conjugate of the denominator

$$f(z) = \frac{(x^2 - 2ixy - y^2)(x - iy)}{(x + iy)(x - iy)} = \frac{(x^2 - 2ixy - y^2)(x - iy)}{x^2 + y^2}.$$

Now expand the numerator

$$\begin{aligned} (x^2 - 2ixy - y^2)(x - iy) &= x(x^2 - 2ixy - y^2) - iy(x^2 - 2ixy - y^2) \\ &= x^3 - 2ix^2y - xy^2 - ix^2y + 2i^2xy^2 + iy^3 \\ &= x^3 - xy^2 - 3ix^2y - 2xy^2 + iy^3. \end{aligned}$$

(using $i^2 = -1$). So,

$$f(z) = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \cdot \frac{y^3 - 3x^2y}{x^2 + y^2}.$$

Define $u(x, y)$ and $v(x, y)$ by

$$u(x, y) = \begin{cases} x^3 - 3xy^2/x^2 + y^2 & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\text{and} \quad v(x, y) = \begin{cases} y^3 - 3x^2y/x^2 + y^2 & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

Now compute the partial derivatives at the origin

$$\begin{aligned} u_x(0, 0) &= \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} &&= 1 \\ u_y(0, 0) &= \lim_{h \to 0} \frac{u(0, h) - u(0, 0)}{h} = \lim_{h \to 0} \frac{-3 \cdot 0 \cdot h^2/h^2}{h} &&= 0 \\ v_x(0, 0) &= \lim_{h \to 0} \frac{v(h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{-3h^2 \cdot 0/h^2}{h} &&= 0 \\ v_y(0, 0) &= \lim_{h \to 0} \frac{v(0, h) - v(0, 0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} &&= 1. \end{aligned}$$

Therefore, at the origin

$$u_x = v_y = 1, \qquad u_y = -v_x = 0.$$

The Cauchy-Riemann equations are satisfied at $(0, 0)$. □

Exercise 2.23.8. Let a function $f(z) = u + iv$ be differentiable at a non-zero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form

Solution. Let $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 e^{i\theta_0}$. Since f is differentiable at z_0, the Cauchy-Riemann equations hold at that point, and the partial derivatives of u and v exist and are continuous near z_0.

Recall from Exercise 7 the relations between Cartesian and polar partials

$$u_x = \cos(\theta) \cdot u_r - \frac{\sin(\theta)}{r} \cdot u_\theta \quad \text{and} \quad v_x = \cos(\theta) \cdot v_r - \frac{\sin(\theta)}{r} \cdot v_\theta.$$

Similarly, for u_y and v_y

$$u_y = \sin(\theta) \cdot u_r + \frac{\cos(\theta)}{r} \cdot u_\theta \quad \text{and} \quad v_y = \sin(\theta) \cdot v_r + \frac{\cos(\theta)}{r} \cdot v_\theta.$$

By the Cauchy-Riemann equations,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Substitute the expressions above into these equations:

$$\begin{aligned} \cos(\theta) \cdot u_r - \frac{\sin(\theta)}{r} \cdot u_\theta &= \sin(\theta) \cdot v_r + \frac{\cos(\theta)}{r} \cdot v_\theta \\ \sin(\theta) \cdot u_r + \frac{\cos(\theta)}{r} \cdot u_\theta &= -\left(\cos(\theta) \cdot v_r - \frac{\sin(\theta)}{r} \cdot v_\theta\right). \end{aligned}$$

Now simplify the second equation:

$$\sin(\theta) \cdot u_r + \frac{\cos(\theta)}{r} \cdot u_\theta = -\cos(\theta) \cdot v_r + \frac{\sin(\theta)}{r} \cdot v_\theta.$$

So the Cauchy-Riemann equations in polar coordinates are:

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta.$$

Therefore, using the expressions from Exercise 7 and the polar form, we have derived the polar version of the Cauchy-Riemann equations at the point $z_0 = r_0 e^{i\theta_0}$. □

Exercise 2.25.7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughout D.

Solution. Since f is analytic on D, it satisfies the Cauchy-Riemann equations in D. Write

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$, and u and v are the real and imaginary parts of f, respectively. Given that $f(z) \in \mathbb{R}$ for all $z \in D$, we have $v(x, y) = 0$ for all $(x, y) \in D$.

Because f is analytic, the Cauchy-Riemann equations must hold:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Since $v(x, y) \equiv 0$, it follows that $v_x = 0 = v_y$ Substituting into the Cauchy-Riemann equations, we get $u_x = 0 = u_y$. Hence, all first partial derivatives of u vanish on D, so u is constant throughout D. Therefore,

$$f(z) = u(x, y) + iv(x, y) = \text{constant}.$$ □

Exercise 3.31.4. Show that

$$\text{(i)}\ \operatorname{Log}(1 + i)^2 = 2 \operatorname{Log}(1 + i); \qquad \text{(ii)}\ \operatorname{Log}(-1 + i)^2 \neq 2 \operatorname{Log}(-1 + i).$$

Solution to (i). We use the principal branch of the complex logarithm:

$$\operatorname{Log}(z) = \ln(|z|) + i \operatorname{Arg}(z) \qquad (-\pi < \operatorname{Arg}(z) \leq \pi).$$

Let $z = 1 + i$. Then

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{and} \quad \operatorname{Arg}(z) = \frac{\pi}{4}.$$

So,

$$\operatorname{Log}(1+i) = \ln(\sqrt{2}) + i\frac{\pi}{4}.$$

Then:

$$2\operatorname{Log}(1+i) = 2\ln(\sqrt{2}) + i \cdot 2 \cdot \frac{\pi}{4} = \ln(2) + i\frac{\pi}{2}.$$

Now consider $\operatorname{Log}((1+i)^2) = \operatorname{Log}(2i)$. Now, we can compute

$$|2i| = 2 \quad \text{and} \quad \operatorname{Arg}(2i) = \frac{\pi}{2}.$$

So

$$\operatorname{Log}((1+i)^2) = \ln(2) + i\frac{\pi}{2}.$$

Therefore, we get

$$\operatorname{Log}((1+i)^2) = 2\operatorname{Log}(1+i).$$

□

Solution to (ii). Let $z = -1 + i$. Then:

$$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \quad \text{and} \quad \operatorname{Arg}(z) = \frac{3\pi}{4},$$

since z is in the second quadrant So,

$$\operatorname{Log}(-1+i) = \ln(\sqrt{2}) + i\frac{3\pi}{4}.$$

Then:

$$2\operatorname{Log}(-1+i) = \ln(2) + i \cdot \frac{3\pi}{2}.$$

Now consider:

$$(-1+i)^2 = (-1)^2 + 2(-1)(i) + i^2 = 1 - 2i - 1 = -2i.$$

Then:

$$|-2i| = 2 \quad \text{and} \quad \operatorname{Arg}(-2i) = -\frac{\pi}{2}.$$

So:

$$\operatorname{Log}((-1+i)^2) = \operatorname{Log}(-2i) = \ln(2) + i(-\frac{\pi}{2}) = \ln(2) - i\frac{\pi}{2}.$$

Comparing:

$$\operatorname{Log}((-1+i)^2) = \ln(2) - i\frac{\pi}{2},$$

but

$$2\operatorname{Log}(-1+i) = \ln(2) + i\frac{3\pi}{2}.$$

These are not equal. Therefore,

$$\operatorname{Log}((-1+i)^2) \neq 2\operatorname{Log}(-1+i).$$

□

Exercise 4.38.2(iii). Evaluate the following integral

$$\int_0^\infty e^{-zt}\,\mathrm{d}t \qquad (\operatorname{Re}(z) > 0).$$

Solution. Computing the anti-derivative of e^{-zt}, we have

$$\int e^{-zt}\,\mathrm{d}t = -\frac{1}{z}e^{-zt} + C.$$

As $t \to \infty$, since $\mathrm{Re}(z) > 0$, we have $e^{-zt} \to 0$ exponentially. Therefore, we get

$$\int_0^\infty e^{-zt}\,\mathrm{d}t = 0 - \left(-\frac{1}{z}\cdot 1\right) = \frac{1}{z}.$$

□

Exercise 4.38.3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,\mathrm{d}\theta = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}.$$

Solution. Simplifying the integrand, we have

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,\mathrm{d}\theta = \int_0^{2\pi} e^{i(m-n)\theta}\,\mathrm{d}\theta.$$

Let $k = m - n$. Then, we have

$$\int_0^{2\pi} e^{ik\theta}\,\mathrm{d}\theta = \begin{cases} 2\pi & \text{if } k = 0 \\ 1/ik\cdot(e^{2\pi ik} - 1) = 0 & \text{if } k \neq 0 \end{cases}.$$

This is because $e^{2\pi ik} = 1$ for any integer k, so the numerator becomes 0 when $k \neq 0$. Therefore,

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,\mathrm{d}\theta = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}.$$

□

Exercise 4.38.5. Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \leq t \leq a$.

(i) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\,\mathrm{d}t = 2\int_0^a w(t)\,\mathrm{d}t.$$

(ii) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)\,\mathrm{d}t = 0.$$

Suggestion: In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of t, which is graphically evident.

Solution to (i). Suppose $w(t) = u(t) + iv(t)$ is even, i.e., $w(-t) = w(t)$. Then both $u(t)$ and $v(t)$ are even functions. Using the fact that the integral of an even real-valued function over a symmetric interval is twice the integral over $[0, a]$, we have

$$\int_{-a}^{a} w(t)\,\mathrm{d}t = \int_{-a}^{a} u(t)\,\mathrm{d}t + i\int_{-a}^{a} v(t)\,\mathrm{d}t = 2\int_0^a u(t)\,\mathrm{d}t + i\cdot 2\int_0^a v(t)\,\mathrm{d}t.$$

Thus,

$$\int_{-a}^{a} w(t)\,\mathrm{d}t = 2\int_0^a w(t)\,\mathrm{d}t.$$

□

Solution to (ii). Now suppose $w(t) = u(t) + iv(t)$ is odd, i.e., $w(-t) = -w(t)$. Then both $u(t)$ and $v(t)$ are odd functions. Since the integral of an odd real-valued function over a symmetric interval is zero, we have

$$\int_{-a}^{a} w(t)\,\mathrm{d}t = \int_{-a}^{a} u(t)\,\mathrm{d}t + i\int_{-a}^{a} v(t)\,\mathrm{d}t = 0 + i \cdot 0 = 0.$$

Therefore,

$$\int_{-a}^{a} w(t)\,\mathrm{d}t = 0.$$

□

Exercise 4.39.6. Let $y(x)$ be a real-valued function defined on the interval $0 \leq x \leq 1$ by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \leq 1 \\ 0 & \text{when } x = 0 \end{cases}.$$

(i) Show that the equation

$$z = x + iy(x) \qquad (0 \leq x \leq 1),$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \cdots)$ and $z = 0$.

(ii) Verify that the arc C in part (i) is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \leq \left|x^3 \sin\left(\frac{\pi}{x}\right)\right| \leq x^3,$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

Solution to (i). To find where the arc C intersects the real axis, we observe that this occurs when $\operatorname{Im}(z) = y(x) = 0$. Note that $y(x) = x^3\sin(\pi/x)$ for $x > 0$. Since $\sin(\pi/x) = 0$ when $\pi/x = n\pi$ (i.e., $x = 1/n$ for $n = 1, 2, \ldots$), it follows that $y(1/n) = 0$, and thus

$$z = \frac{1}{n} + i \cdot 0 = \frac{1}{n},$$

lies on the real axis. Also, since $y(0) = 0$, we have $z = 0 + i \cdot 0 = 0$ also lies on the real axis. Therefore, the arc intersects the real axis at the points $z = 1/n$ and $z = 0$. □

Solution to (ii). First, we show that $y(x)$ is continuous on $[0, 1]$. Since $|\sin(\pi/x)| \leq 1$, we have

$$|y(x)| = |x^3\sin(\pi/x)| \leq x^3 \longrightarrow 0 \quad \text{as } x \to 0.$$

Hence, $\lim_{x\to 0} y(x) = 0 = y(0)$, so $y(x)$ is continuous on $[0, 1]$. Computing the derivative, we have

$$y'(x) = \mathrm{d}vx\left(x^3\sin\left(\frac{\pi}{x}\right)\right) = 3x^2\sin\left(\frac{\pi}{x}\right) - \pi x\cos\left(\frac{\pi}{x}\right).$$

To analyze continuity at $x = 0$, observe that

$$|y'(x)| \leq 3x^2 + \pi x \to 0,$$

as $x \to 0$. So $\lim_{x\to 0} y'(x) = 0$. Also, for $x = 0$, we define

$$y'(0) := \lim_{x\to 0}\frac{y(x) - y(0)}{x - 0} = \lim_{x\to 0}\frac{x^3\sin(\pi/x)}{x} = \lim_{x\to 0} x^2\sin(\pi/x) = 0.$$

Thus, $y'(x)$ is continuous on $[0, 1]$, and $z(x) = x + iy(x)$ is continuously differentiable.

Therefore, the arc C is smooth. □