# Agricultural finance Lecture 6 Valuing investment under risk and uncertainty

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# Introduction

Frank Knight's distinction between risk and uncertainty:

- Risk known probabilities
- Uncertainty probabilities are not known

# Example



Choice between applying 90 pounds of nitrogen per acre or 110 pounds of nitrogen per acre. Regardless of the choice made by the farmer, one of two possible events (E) will occur (either event A or B). The probability The probability that event A will occur is given by $P(A)$ and the probability that event B will occur is given by $P(B)$. The combination of the farmer's actions an events results in four possible outcomes $O(a_1|A)$. with is the outcome of action $a_1$ given event $A$. Assume event A is the case where 30 inches of rain occurs while event B is the case where 35 inches of rain occurs.

# Payoffs



The actions are the choice of nitrogen to apply in each case.The payoffs to corn yield and profit per acre are given in the following table:

| Nitrogen per acre | Rainfall (inches per season) | |
|---|---|---|
| | 30 | 35 |
| | corn yield (bushels per acre) | |
| 90 | 41.71 | 46.46 |
| 110 | 44.30 | 49.35 |
| | Profit per acre | |
| 90 | 103.95 | 116.31 |
| 110 | 109.68 | 122.80 |

How does the producer decide between the two alternatives?

# Expected value of profit is one option



Assume $P[E = A] = 0.6$ and $P[E = B] = 0.40$ then

$$E[\pi|a_1] = P[E = a]116.31 + P[E = B]103.95 = 111.37$$

$$E[\pi|a_2] = P[E = A]122.80 + P[E = B]109.68 = 117.55$$

So because $E[\pi|a_2] > E[\pi|a_1]$ decision-maker should choose alternative $a_2$. But this alternative also has higher risks involved.

$$\sigma_2^2 = 41.31 \text{ and } \sigma_1^2 = 36.66$$

# Decision Tree

put tree diagram here

# Expected Utility

$$U(Y) = \frac{Y^{1-r}}{1-r}$$

Moss, calls this the power utility function. correct name is iso-elastic utility function of CARA

Differentiating

$$\frac{dU}{dY} = (1-r)\frac{Y^{-r}}{1-r} = Y^{-r}$$

The elasticity is

$$\frac{dU}{dY}\frac{Y}{U(Y)} = \frac{Y^{-r}Y}{\frac{Y^{1-r}}{1-r}} = 1-r$$

which can be see n to be constant

# Risk aversion



Arrow-Pratt measure of relative risk aversion

$$\rho = -Y\frac{U''(Y)}{U'(Y)}$$

Substituting first-derivative from previous slide we get:

$$= -rY\frac{Y^{-r-1}}{Y^{-r}}\} = rY^{-r}Y^{-r} = r$$

So this is constant and is the rate of relative risk aversion.

- risk averse $r > 0$
- risk neutral $r = 0$
- risk averse $r < 0$

# Graph of utilities for different degrees of risk aversion

# Milleron-Mityushin-Polterovich Theorem

*The demand curve slopes down if and only if the rate of relative risk aversion is less than or equal to four (4)*

# St Petersburg Paradox

Player bets on coin tosses. Pays fixed bet and wins reward $2^n$ on the $n-th$ toss.
After one toss expected payoff is

$$\frac{1}{2}2$$

with two tosses it is

$$\frac{1}{2}2+\frac{1}{4}4$$

and so on

$$\frac{1}{2}2+\frac{1}{4}4+\ldots+\frac{1}{2^n}2^n+\cdots=1+1+1+\cdots=\infty$$

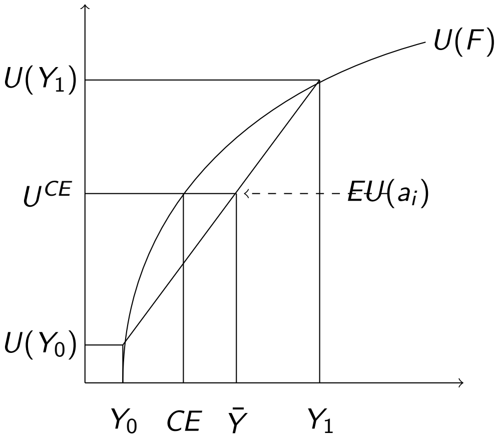
This motivated the introduction of expected utility theory so that the payoff remained bounded.

# Expected Utility theory

$$EU = P(A)U(Y) + P(B)U(Y)$$

$$P(A) + P(B) = 1$$

# Certainty Equivalents



# Certainty Equivalent-Algebra



The amount the decision maker is willing to pay to remain indifferent to a risky gamble.
Set expected utility equal to utility

$$EU = P(A)\frac{Y_2^{1-r}}{1-r} + P(B)\frac{Y_1^{1-r}}{1-r} = \frac{Y_{CE}^{1-r}}{1-r}$$

Then solve for $Y_{CE}$.
Risk premium is $\bar{Y} - Y^{CE}$.

# Solving

$$P(A)\frac{Y_2^{1-r}}{1-r} + P(B)\frac{Y_1^{1-r}}{1-r} = \frac{Y_{CE}^{1-r}}{1-r}$$

$$P(A)Y_2^{1-r} + P(B)Y_1^{1-r} = Y_{CE}^{1-r}$$

$$Y_{CE} = [P(A)Y_2^{1-r} + P(B)Y_1^{1-r}]^{\frac{1}{1-r}}$$

$$RP = P(A)Y_2 + P(B)Y_2 - Y_{CE}$$

A nice exercise now would be to look at how the risk premium varies with $r$ and with the probabilities.

# Numerical example



Using this data calculate the certainty equivalent and risk premium

| N | Rain | | | |
|---|---|---|---|---|
| | 30 | 35 | Mean | Standard deviation |
| | Profit | | | |
| 90 | 98056 | 113877 | 107548 | 7751 |
| 110 | 105390 | 122184 | 115467 | 8227 |

# Example

$$EU = P(A)U(Y) + P(B)U(Y)$$

Assume $P[E = A] = 0.6$ and $P[E = B] = 0.40$, $r = 0.5$

$$EU = 0.6U(Y) + 0.4U(Y)$$

$$= 0.6\frac{113877^{0.5}}{0.5} + 0.4\frac{98056^{0}.5}{0.5} = 655.459$$

$$Y_{CE} = 107,407$$

$$R_P = \bar{Y} - Y_{CE} = 142$$

risk premium is the maximum amount the producer is willing to pay to aoid the gamble (e.g. bu insurance)

# Alternative distributions

Above example had two outcomes (Bernoulli distribution) if we use a log-normal distribution and CRRA utility then the certainty equivalent utility is (lot's of handwaving)

$$U^{CE} = \frac{1}{1-r} \exp\left\{(1-r)(\mu + (1-r)\frac{\sigma^2}{2}\right\}$$

Assuming a normal distribution and a CARA (constant absolute risk aversion) (i.e. negative exponential utility) we would get:

$$U^{CE} = -\exp\left\{-\rho(\mu - \frac{\rho}{2}\sigma^2)\right\}$$

We will make use of this later of mean-variance analysis.

# Portfolio analysis



Example with four crops with 11 years of data

$$\min zV(\sum_{i=1}^{4} z_i r_r)$$

subject to $\sum_{i=1}^{4} z_i \bar{r}_i \geq \mu = 40000$

$$\sum_{i=1}^{4} z_i = 100$$

$$z_i \geq 0$$

# Variance and Mean



$$V(\sum_{i=1}^{4} z_i r_r) = \frac{1}{11}\sum_{i=1}^{11}\left(z_i\left[\sum_{j=1}^{4} r_{ij} - \bar{r}_i\right]\right)^2$$

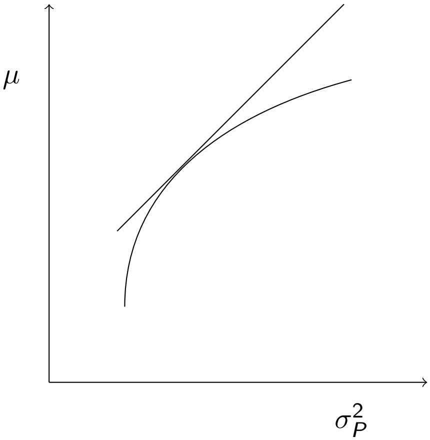
$$\bar{r}_i = \frac{1}{11}\sum_{j=1}^{11} r_{ij}$$

Then solve for $z$ using a non-linear (quadratic) programming package.

# Mean-variance (efficient) frontier

The mean variance approach involves maximizing a linearized utility

$$U(z) = \mu_P(z) - \frac{\rho}{2}\sigma_P^2(z)$$



# Mean-Variance Analysis

$$\sigma_P^2 = z^{'}\Omega z$$

where $\Omega$ is the variance covariance matrix.

# Capital Asset Pricing Model (CAPM)

- Asset pricing theory is what we turn to next
- Capital asset pricing model
- options pricing (Black-Scholes model
- Arbitrage pricing theory

# capital Asset Pricing Model

Utility based approaches have disadvantages, risk attitudes unknown, although risk preferences can be elicited experimentally and the genetics of risk preference is an emerging field of geno-econmic s (DR4L gene controls risk preference).
Alternative: Revealed preference approach.

# Sharpe 1964

$$U(E[w], \sigma_w)$$

$$R = \frac{w_t - w_1}{w_1}$$

or

$$w_t = Rw_1 + w_1$$

so we can substitute out $w_t$ and rewrite the tuility as

$$U = g(E[R], \sigma_R)$$

# Derivation

| Portfolio | Market | Asset $i$ | Riskless |
|---|---|---|---|
| Optimum | $w_m = 1$ | $w_i = 0$ | 0 |
| Candidate | $w_m = 1$ | $w_i = D$ | -D |

$$\sigma_P^2 = w_i^2\sigma_i^2 + w_m^2\sigma_m^2 + 2w_i w_m \sigma_{im}^2$$

$$\mu_P = R_f + w_m(E_m - E_i) + w_i(E_i - R_f)$$

# Derivation

$$\frac{\partial \sigma_P^2}{\partial w_i} = 2w_i\sigma_i^2 + 2w_m\sigma_{im}^2$$

At optimum this reduces to

$$2w_m\sigma_{im}^2$$

$$\frac{\partial \mu_P}{\partial w_i} = (E_i - R_f)$$

# Marginal Rates of Substitution



Now calculate the marginal rates of substitution for the mean and variance

$$\frac{\frac{\partial E_P}{\partial w_i}}{\frac{\partial \sigma_P^2}{\partial w_i}} = \frac{E_i - R_f}{2w_m\sigma_{im}^2} = \frac{E_j - R_f}{2w_m\sigma_{jm}^2} = \frac{E_m - R_f}{2w_m\sigma_m^2}$$

and rearrange to get:

$$E_i - R_f = (E_m - R_f)\frac{\sigma_{im}^2}{\sigma_m^2}$$

Then we set $\beta_{im} = \frac{\sigma_{im}^2}{\sigma_m^2} = \frac{Cov(R_i, R_m)}{Var[R_m]}$.

# CAPM



Derived from portfolio theorem by linearizing
In equilibrium:

$$E[R_i] = R_f + \beta_{im}(E[R_m] - R_f)$$

- $E[R_i]$ expected rate of return of the i-th asset
- $E[R_m]$ expected rate of return of the market portfolio
- $R_f$ risk-free rate of return
- $\beta_{im}$ market beta relative riskiness of stock.

# Estimation



We then estimate the following linear regression model to find $\beta$

$$R_{jt} + a_j + b_j R_{mt} + \epsilon_{jt}$$

This gives us the relative risk of each stock $\beta_j$

# Testing for market equilibrium

$$\hat{R}_j = \hat{\gamma}_0 + \hat{\gamma}_1 \hat{b}_j + u_j$$

In equilibrium $\hat{R}_j = \hat{b}_j$

# Options

- option to buy: call option
- option to sell: put option
- European vs American options (depends on boundary condition)
- Put call parity

# Options pricing

We will derive the Black-Scholes equation from the CAPM. This will require some intermediate steps first and a brief introduction to Ito's lemma.
The time increment of stock returns is

$$E(r_s dt] = E[\frac{dS_t}{dt}]$$

where

$$dS_t = rS_t dt + \sigma S_t dW_t$$

is an Ito stochastic differential equation (continuous sample path but not differentiable)

# Derivation



$$E[\frac{dS_t}{S_t}] = E[rdt] + E[\sigma dW_t]$$
$$= R_f dt + \beta(E[R_m] - R_f)dt$$

To get this substitute in $E[R_i] = R_f + \beta_{im}(E[R_m] - R_f)$ and note that $E[dW_t] = 0$

# Derivatives



An option is a derivative which means mathematically it is a function of the underlying stock $S$.
Denote the value of the derivative as $V_t(S_t, t)$. Then

$$E[r_v dt] = E[\frac{dV_t}{dt}] = R_f dt + \beta_V (E[R_m] - R_f) dt$$

# Ito's Lemma

Taylor expanding
$V(t, S_t)$ we get

$$dV = \frac{\partial V}{dt}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + ...$$

Now substitute in $dS_t = rS_t dt + \sigma S_t dW_t$

$$dV = \frac{\partial V}{dt}dt + \frac{\partial V}{\partial S}(rS_t dt + \sigma S_t dW_t) +$$
$$\frac{1}{2}\frac{\partial^2 V}{\partial S^2}(rS_t dt + \sigma S_t dW_t)^2 + ...$$

Now the key thing you need to know is that in the limit as $dt \to 0$ (i.e. things become continuous in time), then

$$dt^2 \text{ and } dtdW_t \to 0 \text{ and } dW_t^2 \to dt$$

so terms drop out and we get

$$dV = (\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + \sigma\frac{\partial V}{\partial S}dW_t$$

# Continuing the derivation...



Divide by $V_t$ and take expectations:

$$E\frac{dV}{V} = \frac{1}{V}(\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + E\sigma\frac{\partial V}{\partial S}dW_t$$

$$r_v dt = \frac{1}{V_t}(\frac{\partial V}{\partial t} + \ \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + \mu\frac{\partial V}{\partial S}\frac{1}{V_t}$$

The mean (drift) of $S$ is $rS_t$ so we now substitute this

$$r_v dt = \frac{1}{V_t}(\frac{\partial V}{\partial t} + \ \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + r_S S_t\frac{\partial V}{\partial S}\frac{1}{V_t}$$

$dt$ cancels.

# Relationship between $\beta_V$ and $\beta_S$



Take covariances between $r_v$ and $r_m$

$$Cov[r_v, r_m] = \frac{\partial V}{\partial S}\frac{S_t}{V_t}Cov[r_s, r_M]$$

which implies

$$\beta_V = \frac{\partial V}{\partial S}\frac{S_t}{V_t}\beta_S$$

Recall

$$E[\frac{dV_t}{V_t}] = R_f dt + \beta_V (E[R_m] - R_f)dt$$

Multiply this by $V_t$ to obtain:

$$E[dV_t] = R_f V_t dt + V_t \beta_V (E[R_m] - R_f)dt$$

then from the previous slide $\beta_V = \frac{\partial V}{\partial S}\frac{S_t}{V_t}\beta_S$ so

$$E[dV_t] = R_f V_t dt + V_t \frac{\partial V}{\partial S}\frac{S_t}{V_t}\beta_S (E[R_m] - R_f)dt$$

$$= R_f V_t dt + \frac{\partial V}{\partial S} S_t \beta_S (E[R_m] - R_f)dt$$

# Recall

$$dV = (\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW_t$$

Take expectations of this we get:

$$E[dV] = (\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt$$

using $\mu = E[\frac{dS}{S_t}] = R_f dt + \beta_S(E[R_M] - R_f)$

$$E[dV] = (\frac{\partial V}{\partial t} + (R_f S dt + \beta_S(E[R_M] - R_f))S dt)\frac{\partial V}{\partial S} +$$

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt$$

Finally set this equal to $= R_f V_t dt + \frac{\partial V}{\partial S} S_t \beta_S (E[R_m] - R_f) dt$
To get

$$(\frac{\partial V}{\partial t} + (R_f S + \beta_S (E[R_M] - R_f)) S) \frac{\partial V}{\partial S} +$$

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) = R_f V_t + \frac{\partial V}{\partial S} S_t \beta_S (E[R_m] - R_f)$$

which simplifies to

$$\frac{\partial V}{\partial t} + R_f S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - R_f V_t = 0$$

This is the Black-Scholes PDE (partial differential equation).

# Solution of Black-Scholes equation



- Classical approach in involves a change of variables and reparameterization
- Turn it into a heat equation with known solution
- Solution results in the Black-Scholes Formula

# The Black-Scholes Formula

For a call option.

$$C(t,S) = e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2)$$

where

- $$N(x) = \frac{1}{\sqrt{2\pi}} \in_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy$$
- $$d_1 = \frac{log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
- $$d_2 = d_1 - \sigma\sqrt{T-t}$$

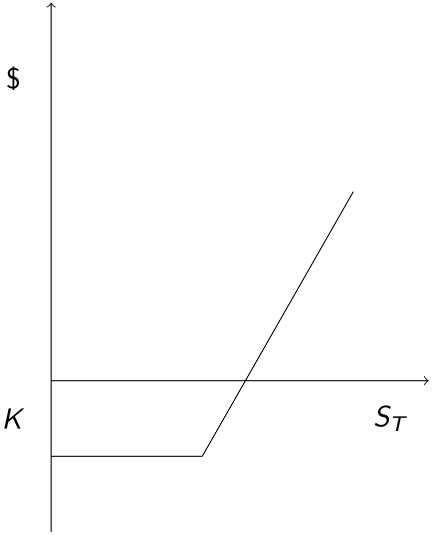
# Application: European call option



holder of option has the right to purchase (call) an asset at price K at date T. the payoff to the option is

$$\max[0, S_T - K] = [S_T - K]^+$$

# Diagram



# Limitations



- Assumes normal distribution, tail behavior may be important.
- Assumes continuous trading at least to an approximation (not always appropriate in some agricultural markets)
- Original price process needs to be appropriate for the observed asset
- Different price processes lead to different formulae for the call option.
- Illustrated is an approach to pricing rather than a single pricing formula.