Commodity Futures Markets Options

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Readings

- H. Geman, Ch.4. Agricultural Commodity Spot Markets, in: *Agricultural Finance*, John Wiley & Sons, 2015.

Put call parity

$$P(t) + S(t) = C(t) + ke^{r(T-t)}$$

- No taxes, no transaction costs: "frictionless markets"
- the underlying stock pays no dividend over the lifetime of the option
- interest rates r are constant. r is the continuously compounded interest rate
- Zero arbitrage. With zero initial wealth and zero risk at date 0 the final wealth will be zero at date T.

Position over interval (t, T)

Proof of put-call parity:

	t	T	
		$S(T) < k$	$S(T) > k$
buy the stock	$-S(t)$	$S(T)$	$S(T)$
buy the put	$-P(t)$	$k - S(T)$	0
sell the call	$+C(t)$	0	$-(S(T) - k)$
Sum	$-ke^{r(T-t)}$	k	k

Consequences of the Black-Scholes Formula

- Price formula has been established without any assumption on preferences and beliefs of market participants (separation assumption)
- From Put-call parity

$$P(t) = C(t) + ke^{r(T-t)} - S(t)$$

$$P(t) = -S(t)[1 - N(d_1)] + ke^{-r(T-t)}[1 - N(d_2)]$$

or

$$P(t) = ke^{r(T-t)}N(-d_2) - S(t)N(-d_2)$$

Portfolio

The value of the Portfolio (1 call option and n stocks):

$$V_P(t) = C_t + nS_t = -ke^{r(T-t)}N(d_2)$$

in the interval $(t, t+dt)$.
Evolution of the portfolio in this period is

$$\frac{dV(t)}{dt} = -rk^{r(T-t)}N(d_2)$$

$$dV = rk^{r(T-t)}N(d_2)dt$$

From the first equation

$$C_t = V_P(t) - nS(t) = V_P(t) + S_t\frac{\partial C}{\partial S}$$

Why?

Another approach to Black-Scholes

See pp. 68-69

$$V_P(t) = C(t) + nS(t)$$

then

$$dV_P(t) = dC(t) + ndS(t)$$

but

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and by Ito's lemma

$$dC(t) = \left[\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\right] dt + \left[\sigma S_t \frac{\partial C}{\partial S_t}\right] dW_t$$

Evolution of Portfolio Value

From

$$dV_P(t) = dC(t) + ndS(t)$$

substituting gives

$$dV_P(t) = \left[\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 + n\mu S_t\right] dt$$

$$+ \left[\sigma S_t \frac{\partial C}{\partial S_t} + n\sigma S_t\right] dW_t$$

Eliminating risk

Examine the last term

$$\left[\sigma S_t \frac{\partial C}{\partial S_t} + n\sigma S_t\right] dW_t$$

Set this $= 0$ and solve for n:

$$n = -\frac{\partial C}{\partial S_t}(t, S_t)$$

This tells you how many stocks to buy to eliminate risk.
But it also tells us why $C_t = V_P(t) - nS(t) = V_P(t) + S_t \frac{\partial C}{\partial S}$

Derive Black-Scholes from here

Setting $n = -\frac{\partial C}{\partial S_t}(t, S_t)$ we get

$$dV_P(t) = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}\right] dt$$

No arbitrage

Possibly even more fundamental than the LOP

$$dV_P(t) = rV_P(t)dt$$

and using

$$C(t) = V_P(t) + S_t \frac{\partial C}{\partial S}$$

Rearrange to get $V_P(t) = C(t) - S_t \frac{\partial C}{\partial S}$
Substituting we get

$$dV_P(t) = r\left(C(t) - S_t \frac{\partial C}{\partial S}\right) dt$$

Last step

$$dV_P(t) = r\left(C(t) - S_t \frac{\partial C}{\partial S}\right) dt$$

but

$$dV_P(t) = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}\right] dt$$

Equating these we get

$$\left[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}\right] dt = r\left(C(t) - S_t \frac{\partial C}{\partial S}\right) dt$$

or

$$\frac{\partial C}{\partial t} + S_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC(t) = 0$$

which is the Black-Scholes PDE (partial differential equation) with terminal value $C(T) = \max(0, S(T) - K)$. Then solve as previously shown via the Heat equation method or some other appropriate method

Delta Δ

From the solution of the Black-Scholes equation we had

$$C(t) = S(t)N(d_1) - ke^{r(T-t)}N(d_2)$$

where $N()$ are normally distributed probabilites evaluated at the values d_1 and d_2 respectively.
Now differentiate this with respect to $S(t)$ to get

$$\frac{\partial C}{\partial S} = N(d_1)$$

This expression $\frac{\partial C}{\partial S}$ is called Δ.
Because $N()$ is positive then $\frac{\partial C}{\partial S} > 0$

Leverage effect

$$\delta \approx \frac{\partial C}{\partial S}\delta S = N(d_1)\delta S$$

Divide both sides by $\frac{C}{S}$ and note $SN(d_1) > C$ results in

$$\frac{\delta C}{C} > \frac{\delta S}{S}$$

The return to a call option is greater than the return to the underlying asset.
Geman notes: Confident farmers buy call options to exploit the leverage effect but doubly expose themselves to risk by doing so.

Gamma Γ

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2} = N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} > 0, \forall S$$

$N'(d_1)$ is the standard normal density at d_1

Convex options

$$\Delta > 0$$

and

$$\Gamma > 0$$

- Imply $C(S)$ is convex. At ALL dates! Why?
- Clearly true at T from payoff graph

The End

Thanks for listening!