# Agricultural Marketing and Price Analysis
# Lecture 6
# Prices over time (commodity futures)

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# Spot prices - date 1

Recall

## Spot prices at date 1

$$\tilde{P}_1 = a - b(S_0 + \tilde{Q}_1 - S_1(\tilde{Q}_1))$$

$$P_1(Q_1) = \begin{cases} a - b(S_0 + Q_1) & \text{for} 0 \leq Q_1 \leq Q_1^* \\ a - b(S_0 + Q_1 - S^*(Q_1)) & \text{for } Q_1 > Q_1^* \end{cases}$$

# Spot prices - date 2

## Spot prices at date 2

$$\tilde{P}_2 = a - b(\tilde{Q}_2 + S_1(\tilde{Q}_1))$$

$$P_2(Q_1, \tilde{Q}_2) = \begin{cases} a - b(Q_2) & \text{for } 0 \leq Q_1 \leq Q_1^* \\ a - b(S^*(Q_1) + \tilde{Q}_2) & \text{for } Q_1 > Q_1^* \end{cases}$$

# Expected prices

$$E_0\left\{P_1(\tilde{Q}_1)\right\} = a - b(S_0 + \mu_1) + b\int_{Q_1^*}^{\infty} S^*(Q_1) df(Q_1) dQ_1$$

and

$$E_0(P_2(\tilde{Q}_1, \tilde{Q}_2)) = a - b\mu_2 - b\int_{Q_1^*}^{\infty} S^*(Q_1) f(Q_1) dQ_1$$

$$E_1\left\{P_2(\tilde{Q}_2|Q_1)\right\} = \begin{cases} a - b(\mu_2) & \text{for } 0 < Q_1 < Q_1^* \\ a - b(\mu_2 + S^*(Q_1)) & \text{for } Q_1 \geq Q_1^* \end{cases}$$

# Equilibrium

## Market efficiency

Efficient price discovery implies futures prices must equal he expected spot price (rational expectations assumption). Note that is this were not the case arbitrage possibilities would exist.

So

$$f_0^1 = E_0(P_1(\tilde{Q}_1)$$

$$f_0^2 = E_0(P_2(\tilde{Q}_1, \tilde{Q}_2))$$

$$f_1^2 = E_1(P_2(\tilde{Q}_2)|Q_1)$$

We need to evaluate the integral terms in thee expected price expressions

# Integral expression

To evaluate

$$\int_{Q_1^*}^{\infty} S^*(Q_1) f(Q_1) dQ_1$$

we need to know $f(Q_1)$. We will assume it follows a truncated normal distribution with $Q_1 \geq 0$.

# The Truncated Normal Distribution

First recall the normal density function:

$$P(X) = \frac{1}{\sigma\sqrt{2\pi}}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and the standard normal density:

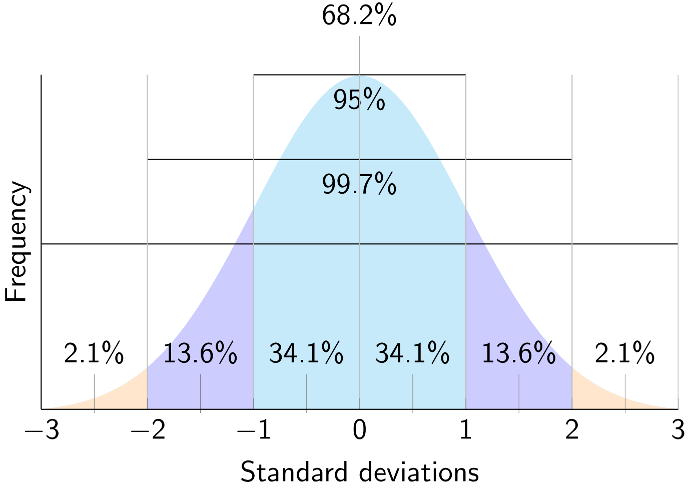
$$\phi(x) = \frac{1}{\sqrt{2\pi}}^{-\frac{(x-\mu)^2}{2}}$$

# Truncated Distributions are Conditional Distributions

$$P(x|a \leq x \leq b) = \frac{P(x)}{P(X \leq b) - P(X \leq a)}$$
$$= \frac{f(x)}{F(X \leq b) - F(X \leq a)}$$

# The Truncated Normal Distribution

Now we use the standard normal cumulative distribution function to “weight” the standard normal density by the area under under the density up to a truncation point.




Source: `http://johncanning.net/wp/?p=1202`

# Truncated Normal Density

The Normal distribution is given by

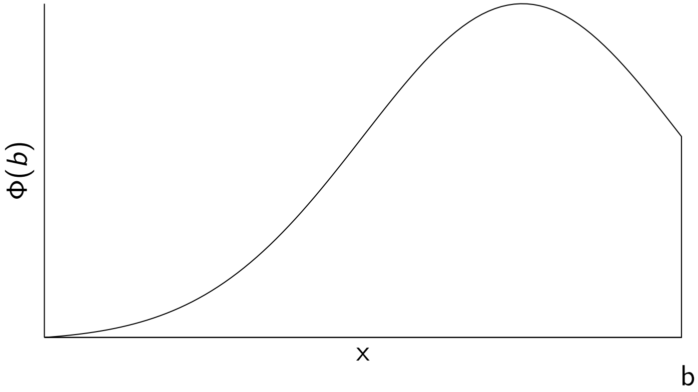
$$\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi$$

and

$$\lim_{x\to\infty} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = 1$$

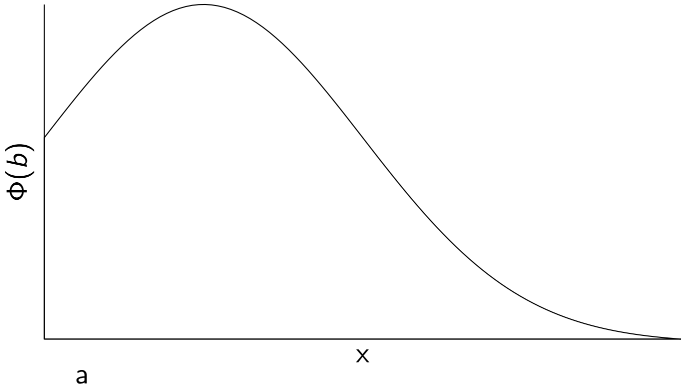
# Let us truncate this at another point say $b$

$$\Phi(b) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = const.$$



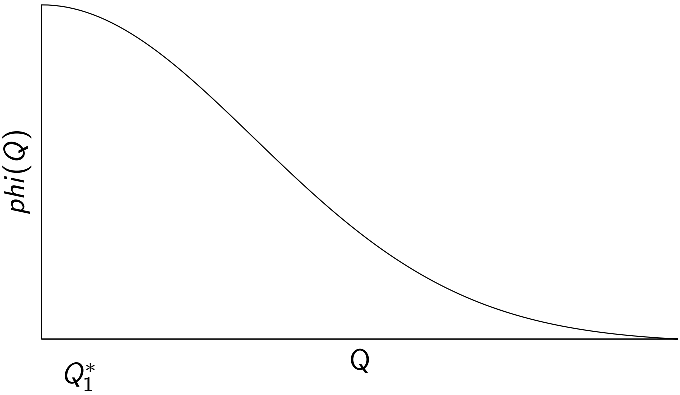
# Truncated lower tail

$$\Phi(b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^{\infty} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = const.$$



# Truncated Normal Distribution

$$\Phi(Q_1^*) = \frac{1}{\sigma\sqrt{2\pi}} \int_{Q_1^* \geq 0}^{\infty} e^{-\frac{(Q-\mu)^2}{2\sigma^2}} dQ = const.$$



# Truncated normal density for storage - First property

$$\frac{\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}$$

Note this is also known as a hazard function (ratio of density to inverse CDF)

# Second property

$$\int_a^{\infty} \phi(\frac{Q-\mu}{\sigma})dQ = \mu \left[1 - \Phi(\frac{a-\mu}{\sigma})\right] + \sigma\phi(\frac{a-\mu}{\sigma})$$

we will substitute the first property into the expected storage function and the second will be used to evaluate the integral.

# Next step

Recall

$$\int_{Q_1^*}^{\infty} S^*(Q_1)f(Q_1)dQ_1$$

First we will consider

$$\int_a^{\infty} Qf(Q)dQ$$

now substitute

$$f(Q) = \frac{\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}$$

# Next step

$$\int_a^\infty Q \frac{\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})} dQ$$

Note that $\Phi(\frac{a-\mu}{\sigma})$ is independent of $Q$. So

$$\frac{1}{1-\Phi(\frac{a-\mu}{\sigma})} \int_a^\infty Q\sigma^{-1}\phi(\frac{Q-\mu}{\sigma}) dQ$$

Now we use the second property

# Mean of a Truncated Random Variable with lower truncation a

$$\int_a^{\infty} Qf(Q)dQ = \frac{1}{1-\Phi(\frac{a-\mu}{\sigma})}\int_a^{\infty} Q\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})dQ$$

$$= \mu + \sigma\frac{\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}$$

Denoting the mean of the truncated random variable with $\mu_1$ and $\hat{\mu}_1$ is the scale parameter of $F(Q_1)$ then,

$$\hat{\mu}_1 + \sigma_1\frac{\phi(\frac{0-\hat{\mu}_1}{\sigma})}{1-\Phi(\frac{0-\hat{\mu}_1}{\sigma})} = \mu_1$$

This needs to be solved numerically for $\hat{\mu}_1$. Try using Sympy solve but with numerical values for the parameters.

Back to the task at hand evaluating the integral

$$\int_{Q_1^*}^{\infty} S^*(Q_1) f(Q_1) dQ_1$$

we previously derived $S^*(Q_1) = \frac{b(S_0+Q_1-\mu_2)-m}{2b}$ substituting we get

$$\int_{Q_1^*}^{\infty} \frac{b(S_0 + Q_1 - \mu_2) - m}{2b} f(Q_1) dQ_1$$

$$= \int_{Q_1^*}^{\infty} \frac{b(S_0 - \mu_2) - m}{2b} f(Q_1) dQ_1 + \int_{Q_1^*}^{\infty} \frac{(Q_1)}{2} f(Q_1) dQ_1$$

$$= \frac{b(S_0 - \mu_2) - m}{2b} \int_{Q_1^*}^{\infty} f(Q_1) dQ_1 + \frac{1}{2} \int_{Q_1^*}^{\infty} Q_1 f(Q_1) dQ_1$$

Next we evaluate the last two integrals

$$\int_{Q_1^*}^{\infty} f(Q_1)dQ_1 = \frac{1}{1-\Phi(Z_0)} \int_{Q_1^*}^{\infty} \sigma^{-1}\phi(\frac{Q_1-\hat{\mu}_1}{\sigma_1})dQ_1$$

$$= \frac{1-\Phi(Z_1)}{1-\Phi(Z_0)}$$

and

$$\int_{Q_1^*}^{\infty} Q_1 f(Q_1)dQ_1 = \frac{1}{1-\Phi(Z_0)} \int_{Q_1^*}^{\infty} Q_1\sigma^{-1}\phi(\frac{Q_1-\hat{\mu}_1}{\sigma_1})dQ_1$$

$$= \frac{\hat{\mu}_1\left[1-\Phi(Z_1)\right]+\sigma_1\phi(Z_1)}{1-\Phi(Z_0)}$$

# The Equilibrium Futures Prices - once again

$$f_0^1 = E_0(P_1(\tilde{Q}_1))$$
$$f_0^2 = E_0(P_2(\tilde{Q}_1, \tilde{Q}_2))$$
$$f_1^2 = E_1(P_2(\tilde{Q}_2)|Q_1)$$

$$E_0\left\{P_1(\tilde{Q}_1)\right\} = a - b(S_0 + \mu_1) + b\int_{Q_1^*}^{\infty} S^*(Q_1)df(Q_1)dQ_1$$

and

$$E_0(P_2(\tilde{Q}_1, \tilde{Q}_2)) = a - b\mu_2 - b\int_{Q_1^*}^{\infty} S^*(Q_1)f(Q_1)dQ_1$$

$$E_1\left\{P_2(\tilde{Q}_2|Q_1)\right\} = \begin{cases} a - b(\mu_2) & \text{for } 0 < Q_1 < Q_1^* \\ a - b(\mu_2 + S^*(Q_1)) & \text{for } Q_1 \geq Q_1^* \end{cases}$$

# Futures price in period zero

$$f_0^1 = E_0(P_1(\tilde{Q}_1)$$

$$= a - b(S_0 + \mu_1) + b \int_{Q_1^*}^{\infty} S^*(Q_1) df(Q_1) dQ_1$$

$$= a - b(S_0 + \mu_1) + \frac{b(S_0 - \mu_2) - m}{2} (\frac{1 - \Phi(Z_1)}{1 - \Phi(Z_0)})$$

$$+ \frac{b}{2} (\frac{\hat{\mu}_1 \left[1 - \Phi(Z_1)\right] + \sigma_1 \phi(Z_1)}{1 - \Phi(Z_0)})$$

# Summary

- The ability to price futures is is the basis for trading in futures markets. Traders will be set limits and base their trading strategy on sophisticated mathematical models of price processes to determine when arbitrage possibilities may occur. Increasingly automated trading is used in which human traders monitor a series of automated traders who trade based on underlying mathematical and computational models.
- We have considered a simple three-period trading model in discrete time. In reality trading firms employ a variety of methods including continuous-time models based on stochastic differential equations

# Summary

- These models draw on the finance literature more than the marketing literature and commities markets can be viewed as both a marketing and a financial instrument.
- A good and relatively accessible reference on the latter as it applies to Agricultural finance is Helyette Geman, Agricultural Finance.

# Basis spread explanations

What explains basis-price difference between spot and futures price) spreads observed in markets?

- Keynes theory of normal bacwardation
- The theory of convenience yield
- The theory of the transactions demand for inventoires
- Transportation bottlenecks and local supply and demand imbalances

# Convenience Yield

## Convenience yield

Firms hold additional stocks (above optimal) to lower marketing transaction costs

- purchase drives spot prices up
- sale drives futures prices down
- this leads to backwardation
- it also lowers the spread to where futures prices are less than the cost of carry (storage costs)
- convenience yield compensates for this by reducing marketing transaction costs

# Convenience yield model

Similar to two period storage model

Period 1: $n$ identical merchants each own $Q$ units of a commodity.

$$Q = nq$$

Storage:

$$S = ns$$

Period 2:

$q = 0$

# Equilibrium conditions

Equilibrium conditions (market demand = market supply):

$$X_1(P_1) = Q - S$$

$$X_2(P_2) = S$$

Assuming linear market demand that is the same in both periods:

$$X_1(P_1) = \frac{a}{b} - \frac{1}{b}P_1$$

$$X_2(P_2) = \frac{a}{b} - \frac{1}{b}P_2$$

# Period 2 behavior

Assume $n$ committed buyers (commitment here essentially means the seller and the buyer have a binding legal sales contract enforceable through the courts or through self-interest) and $k - n$ non-committed buyers. So in total $k$ buyers.

- zero transaction costs incurred if the merchant is able to fulfill demand of committed buyers
- Residual inventory sold to either non-committed buyer or other merchants
- If a merchants supply is insufficient to meet committed demand then the merchant must purchase from other merchants and resell to the committed buyer
- Purchasing from other merchants involves transaction costs proportion to size of trade (per unit transaction costs)

# Period 2 Transaction costs

Demands:

$$\frac{1X_2(P_2)}{0.5k(k+1}$$

$$\frac{2X_2(P_2)}{0.5k(k+1}$$

$$\frac{3X_2(P_2)}{0.5k(k+1}$$

so market demand is:

$$\sum_{i=1}^{k} \frac{iX_2(P_2}{0.5k(k+1} = X_2(P_2)$$

verify this!

# Random demand

Assume a merchant in period 1 will face random demand in period 2:

$$q^d(\theta, P_2) = \frac{\theta X_2(P_2)}{0.5k(k+1)}$$

where $\theta$ is a uniformly distributed random variable on support$[1, \ldots, k]$

# Lack of sufficient demand

- Representative merchant lacks inventory if $q^d(\theta, P_2) > s$
- $\theta > \theta^*(s)$ where $\theta^*(s) = \frac{0.5k9k+1s}{X_2(P_2)}$ merchant incurs a transaction cost due to insufficient inventory

Asume $\theta^*(s) < k$

# Expected shortfall

$$Z(s) = \frac{1}{k} \sum_{i=\theta^*(s)}^{k} \left( \frac{iX_2(P_2)}{0.5k(k+1)} - s \right)$$

Think cumulative distribution of $q^d(\theta, P_2) > s$ !!

# Transaction costs

expected period 2 transaction costs at date 0.

$$C(s) = \gamma Z(s)$$

Note $Z(s)$ is expected period 2 shortfall at date 1.

# Equilibrium price spread

Keep storing until the expected profit from storing equals zero, i.e.

$$P_2 = P_1 + m + C^{'}(s)$$

where $m$ is the unit carrying cost and $C^{'}(s)$ is the marginal change in the expected transaction cost.

# Equilibrium price spread

Keep storing until the expected profit from storing equals zero, i.e.

$$P_2 = P_1 + m + C^{'}(s)$$

where $m$ is the unit carrying cost and $C^{'}(s)$ is the marginal change in the expected transaction cost.
now substitute in the demands:

$$a - bS = a - b(Q - S) + m + C^{'}(s)$$

Reminder: We are after the optimal storage level $S$ and the optimal prices in each period.

# Differentiate $C(s)$

$$C(s) = \gamma Z(s)$$

so

$$C^{'}(s) = \gamma Z^{'}(s)$$

$$= -\gamma \left(1 - \frac{0.5(k+1)}{X_2(P_2)} s\right)$$

Proof: see footnote 6.

# Equilibrium

$$X_2(P_2) = S$$

identical merchants implies $s = \frac{S}{n}$
substituting gives

$$C^{'}(s) = -\gamma\left(1 - \frac{0.5(k+1)}{n}\right)$$

Note: $\frac{k+1}{n}$ is the ratio of buyers to sellers which lies between 1 and 2.

# Equilibrium storage

Recall:

$$a - bS = a - b(Q - S) + m + C^{'}(s)$$

so substituting

$$C^{'}(s) = -\gamma\left(1 - \frac{0.5(k+1)}{n}\right)$$

# Equilibrium storage

Recall:

$$a - bS = a - b(Q - S) + m + C^{'}(s)$$

so substituting

$$C^{'}(s) = -\gamma \left(1 - \frac{0.5(k+1)}{n}\right)$$

gives

$$a - bS = a - b(Q - S) + m - \gamma \left(1 - \frac{0.5(k+1)}{n}\right)$$

# Equilibrium storage

$$S^* = \frac{bQ - m + \gamma\left(1 - \frac{0.5(k+1)}{n}\right)}{2b}$$

This can be substituted into the spot prices in each period

$$P_1 = a - \frac{b}{2}\left(Q + \frac{m - \gamma\left(1 - \frac{0.5(k+1)}{n}\right)}{b}\right)$$

$$P_2 = a - \frac{b}{2}\left(Q - \frac{m - \gamma\left(1 - \frac{0.5(k+1)}{n}\right)}{b}\right)$$

# Price spread

Now let us look at the spread

$$P_2 - P_1 = m - (-\gamma C^{'})$$

because $C^{'} = -\gamma \left(1 - \frac{0.5(k+1)}{n}\right)$ which implies the price spread is less than the carrying cost. $(-\gamma C^{'})$ is a measure of convenience yield.

If you ignore convenience yield you will have the impression that LOP is violated. This is a warning to people doing empirical work (you need a convenience yield measure in your models).

# Summary

Main economic forces

- Commodity futures markets are complex and difficult to analyze
- arbitrage and intertemporal LOP
- convergence of spot and futures prices as maturity approaches
- production uncertainty
- stock-outs are important even in presence of uncertainty
- convenience yield
- we have not discussed hedging