

Agricultural finance Lecture 6

Valuing investment under risk and uncertainty

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Frank Knight's distinction between risk and uncertainty:

- ▶ Risk known probabilities
- ▶ Uncertainty probabilities are not known

Example

Choice between applying 90 pounds of nitrogen per acre or 110 pounds of nitrogen per acre. Regardless of the choice made by the farmer, one of two possible events (E) will occur (either event A or B). The probability that event A will occur is given by $P(A)$ and the probability that event B will occur is given by $P(B)$. The combination of the farmer's actions and events results in four possible outcomes $O(a_1|A)$, which is the outcome of action a_1 given event A. Assume event A is the case where 30 inches of rain occurs while event B is the case where 35 inches of rain occurs.

The actions are the choice of nitrogen to apply in each case. The payoffs to corn yield and profit per acre are given in the following table:

Nitrogen per acre	Rainfall (inches per season)	
	30	35
	corn yield (bushels per acre)	
90	41.71	46.46
110	44.30	49.35
	Profit per acre	
90	103.95	116.31
110	109.68	122.80

How does the producer decide between the two alternatives?

Expected value of profit is one option

Assume $P[E = A] = 0.6$ and $P[E = B] = 0.40$ then

$$E[\pi|a_1] = P[E = A]116.31 + P[E = B]103.95 = 111.37$$

$$E[\pi|a_2] = P[E = A]122.80 + P[E = B]109.68 = 117.55$$

So because $E[\pi|a_2] > E[\pi|a_1]$ decision-maker should choose alternative a_2 . But this alternative also has higher risks involved.

$$\sigma_2^2 = 41.31 \text{ and } \sigma_1^2 = 36.66$$

Decision Tree

put tree diagram here

Expected Utility

$$U(Y) = \frac{Y^{1-r}}{1-r}$$

Moss, calls this the power utility function. correct name is iso-elastic utility function of CARA

Differentiating

$$\frac{dU}{dY} = (1-r) \frac{Y^{-r}}{1-r} = Y^{-r}$$

The elasticity is

$$\frac{dU}{dY} \frac{Y}{U(Y)} = \frac{Y^{-r} Y}{\frac{Y^{1-r}}{1-r}} = 1-r$$

which can be seen to be constant

Arrow-Pratt measure of relative risk aversion

$$\rho = -Y \frac{U''(Y)}{U'(Y)}$$

Substituting first-derivative from previous slide we get:

$$= -rY \frac{Y^{-r-1}}{Y^{-r}} \} = rY^{-r} Y^{-r} = r$$

So this is constant and is the rate of relative risk aversion.

- ▶ risk averse $r > 0$
- ▶ risk neutral $r = 0$
- ▶ risk averse $r < 0$

Graph of utilities for different degrees of risk aversion

Milleron-Mityushin-Polterovich Theorem

The demand curve slopes down if and only if the rate of relative risk aversion is less than or equal to four (4)

St Petersburg Paradox

Player bets on coin tosses. Pays fixed bet and wins reward 2^n on the n – th toss.

After one toss expected payoff is

$$\frac{1}{2}2$$

with two tosses it is

$$\frac{1}{2}2 + \frac{1}{4}4$$

and so on

$$\frac{1}{2}2 + \frac{1}{4}4 + \dots + \frac{1}{2^n}2^n + \dots = 1 + 1 + 1 + \dots = \infty$$

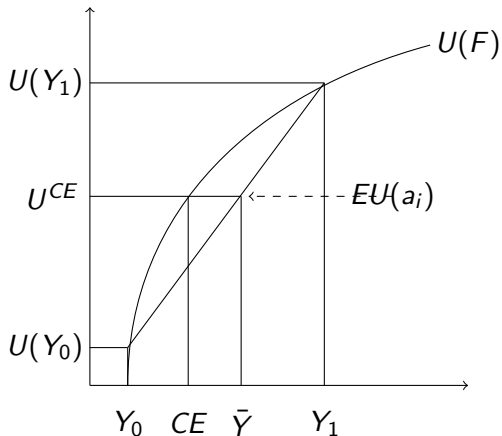
This motivated the introduction of expected utility theory so that the payoff remained bounded.

Expected Utility theory

$$EU = P(A)U(Y) + P(B)U(Y)$$

$$P(A) + P(B) = 1$$

Certainty Equivalents



Certainty Equivalent-Algebra

The amount the decision maker is willing to pay to remain indifferent to a risky gamble.

Set expected utility equal to utility

$$EU = P(A) \frac{Y_2^{1-r}}{1-r} + P(B) \frac{Y_1^{1-r}}{1-r} = \frac{Y_{CE}^{1-r}}{1-r}$$

Then solve for Y_{CE} .

Risk premium is $\bar{Y} - Y_{CE}$.

$$P(A) \frac{Y_2^{1-r}}{1-r} + P(B) \frac{Y_1^{1-r}}{1-r} = \frac{Y_{CE}^{1-r}}{1-r}$$

$$P(A)Y_2^{1-r} + P(B)Y_1^{1-r} = Y_{CE}^{1-r}$$

$$Y_{CE} = [P(A)Y_2^{1-r} + P(B)Y_1^{1-r}]^{\frac{1}{1-r}}$$

$$RP = P(A)Y_2 + P(B)Y_2 - Y_{CE}$$

A nice exercise now would be to look at how the risk premium varies with r and with the probabilities.

Numerical example

Using this data calculate the certainty equivalent and risk premium

N		Rain		
	30	35	Mean	Standard deviation
Profit				
90	98056	113877	107548	7751
110	105390	122184	115467	8227

Example

$$EU = P(A)U(Y) + P(B)U(Y)$$

Assume $P[E = A] = 0.6$ and $P[E = B] = 0.40$, $r = 0.5$

$$EU = 0.6U(Y) + 0.4U(Y)$$

$$= 0.6 \frac{113877^{0.5}}{0.5} + 0.4 \frac{98056^{0.5}}{0.5} = 655.459$$

$$Y_{CE} = 107,407$$

$$R_P = \bar{Y} - Y_{CE} = 142$$

risk premium is the maximum amount the producer is willing to pay to avoid the gamble (e.g. by insurance)

Above example had two outcomes (Bernoulli distribution) if we use a log-normal distribution and CRRA utility then the certainty equivalent utility is (lot's of handwaving)

$$U^{CE} = \frac{1}{1-r} \exp \left\{ (1-r) \left(\mu + (1-r) \frac{\sigma^2}{2} \right) \right\}$$

Assuming a normal distribution and a CARA (constant absolute risk aversion) (i.e. negative exponential utility) we would get:

$$U^{CE} = - \exp \left\{ -\rho \left(\mu - \frac{\rho}{2} \sigma^2 \right) \right\}$$

We will make use of this later of mean-variance analysis.

Example with four crops with 11 years of data

$$\min zV\left(\sum_{i=1}^4 z_i r_i\right)$$

subject to $\sum_{i=1}^4 z_i \bar{r}_i \geq \mu = 40000$

$$\sum_{i=1}^4 z_i = 100$$

$$z_i \geq 0$$

Variance and Mean

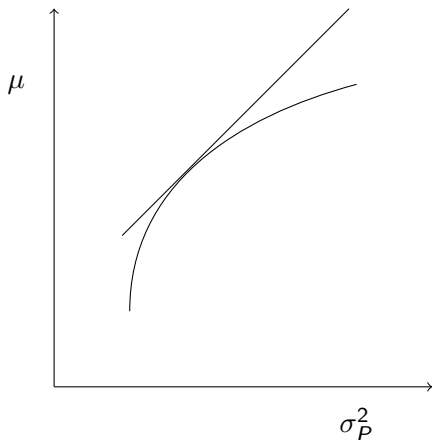
$$V\left(\sum_{i=1}^4 z_i r_i\right) = \frac{1}{11} \sum_{i=1}^{11} \left(z_i \left[\sum_{j=1}^4 r_{ij} - \bar{r}_i \right] \right)^2$$
$$\bar{r}_i = \frac{1}{11} \sum_{j=1}^{11} r_{ij}$$

Then solve for z using a non-linear (quadratic) programming package.

Mean-variance (efficient) frontier

The mean variance approach involves maximizing a linearized utility

$$U(z) = \mu_P(z) - \frac{\rho}{2}\sigma_P^2(z)$$



Mean-Variance Analysis

$$\sigma_P^2 = z' \Omega z$$

where Ω is the variance covariance matrix.

Capital Asset Pricing Model (CAPM)

- ▶ Asset pricing theory is what we turn to next
- ▶ Capital asset pricing model
- ▶ options pricing (Black-Scholes model)
- ▶ Arbitrage pricing theory

capital Asset Pricing Model

Utility based approaches have disadvantages, risk attitudes unknown, although risk preferences can be elicited experimentally and the genetics of risk preference is an emerging field of geno-economic s (DR4L gene controls risk preference).

Alternative: Revealed preference approach.

$$U(E[w], \sigma_w)$$

$$R = \frac{w_t - w_1}{w_1}$$

or

$$w_t = R w_1 + w_1$$

so we can substitute out w_t and rewrite the utility as

$$U = g(E[R], \sigma_R)$$

Portfolio	Market	Asset i	Riskless
Optimum	$w_m = 1$	$w_i = 0$	0
Candidate	$w_m = 1$	$w_i = D$	-D

$$\sigma_P^2 = w_i^2 \sigma_i^2 + w_m^2 \sigma_m^2 + 2w_i w_m \sigma_{im}^2$$

$$\mu_P = R_f + w_m(E_m - R_f) + w_i(E_i - R_f)$$

Derivation

$$\frac{\partial \sigma_P^2}{\partial w_i} = 2w_i\sigma_i^2 + 2w_m\sigma_{im}^2$$

At optimum this reduces to

$$2w_m\sigma_{im}^2$$

$$\frac{\partial \mu_P}{\partial w_i} = (E_i - R_f)$$

Marginal Rates of Substitution

Now calculate the marginal rates of substitution for the mean and variance

$$\frac{\frac{\partial E_P}{\partial w_i}}{\frac{\partial \sigma_P^2}{\partial w_i}} = \frac{E_i - R_f}{2w_m \sigma_{im}^2} = \frac{E_j - R_f}{2w_m \sigma_{jm}^2} = \frac{E_m - R_f}{2w_m \sigma_m^2}$$

and rearrange to get:

$$E_i - R_f = (E_m - R_f) \frac{\sigma_{im}^2}{\sigma_m^2}$$

Then we set $\beta_{im} = \frac{\sigma_{im}^2}{\sigma_m^2} = \frac{\text{Cov}(R_i, R_m)}{\text{Var}[R_m]}$.

Derived from portfolio theorem by linearizing
In equilibrium:

$$E[R_i] = R_f + \beta_{im}(E[R_m] - R_f)$$

- ▶ $E[R_i]$ expected rate of return of the i-th asset
- ▶ $E[R_m]$ expected rate of return of the market portfolio
- ▶ R_f risk-free rate of return
- ▶ β_{im} market beta relative riskiness of stock.

We then estimate the following linear regression model to find β

$$R_{jt} = a_j + b_j R_{mt} + \epsilon_{jt}$$

This gives us the relative risk of each stock β_j

Testing for market equilibrium

$$\hat{R}_j = \hat{\gamma}_0 + \hat{\gamma}_1 \hat{b}_j + u_j$$

In equilibrium $\hat{R}_j = \hat{b}_j$

- ▶ option to buy call option
- ▶ option to sell put option
- ▶ European vs American options (depends on boundary condition)
- ▶ Put call parity

We will derive the Black-Scholes equation from the CAPM.
This will require some intermediate steps first and a brief
introduction to Ito's lemma.

The time increment of stock returns is

$$E(r_s dt) = E\left[\frac{dS_t}{dt}\right]$$

where

$$dS_t = rS_t dt + \sigma S_t dW_t$$

is an Ito stochastic differential equation (continuous sample
path but not differentiable)

$$\begin{aligned}E\left[\frac{dS_t}{S_t}\right] &= E[rdt] + E[\sigma dW_t] \\&= R_f dt + \beta(E[R_m] - R_f)dt\end{aligned}$$

To get this substitute in $E[R_i] = R_f + \beta_{im}(E[R_m] - R_f)$ and note that $E[dW_t] = 0$

An option is a derivative which means mathematically it is a function of the underlying stock S .

Denote the value of the derivative as $V_t(S_t, t)$. Then

$$E[r_v dt] = E\left[\frac{dV_t}{dt}\right] = R_f dt + \beta_V(E[R_m] - R_f)dt$$

Ito's Lemma

Taylor expanding
 $V(t, S_t)$ we get

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \dots$$

Now substitute in $dS_t = rS_t dt + \sigma S_t dW_t$

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} (rS_t dt + \sigma S_t dW_t) + \\ &\quad \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (rS_t dt + \sigma S_t dW_t)^2 + \dots \end{aligned}$$

Now the key thing you need to know is that in the limit as $dt \rightarrow 0$
(i.e. things become continuous in time), then

$$dt^2 \text{ and } dt dW_t \rightarrow 0 \text{ and } dW_t^2 \rightarrow dt$$

so terms drop out and we get

$$dV = \left(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma \frac{\partial V}{\partial S} dW_t$$

Continuing the derivation...

Divide by V_t and take expectations:

$$E \frac{dV}{V} = \frac{1}{V} \left(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt + E \sigma \frac{\partial V}{\partial S} dW_t$$

$$r_v dt = \frac{1}{V_t} \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \mu \frac{\partial V}{\partial S} \frac{1}{V_t}$$

The mean (drift) of S is rS_t so we now substitute this

$$r_v dt = \frac{1}{V_t} \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt + rS_t \frac{\partial V}{\partial S} \frac{1}{V_t}$$

dt cancels.

Relationship between β_V and β_S

Take covariances between r_V and r_M

$$\text{Cov}[r_V, r_M] = \frac{\partial V}{\partial S} \frac{S_t}{V_t} \text{Cov}[r_S, r_M]$$

which implies

$$\beta_V = \frac{\partial V}{\partial S} \frac{S_t}{V_t} \beta_S$$