Commodities Futures Market - Lecture 6 Monte-Carlo Pricing

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May 15, 2017

Readings

Chapter 4 Section 4.8 Geman Agricultural Finance.

Monte-Carlo Method

Eckhardt, Roger (1987). "Stan Ulam, John von Neumann, and the Monte Carlo method" (PDF). Los Alamos Science, Special Issue (15): 131?137. `http://permalink.lanl.gov/object/tr?what=info:lanl-repo/lareport/LA-UR-88-9068`

> *"The first thoughts and attempts I made to practice [the Monte Carlo method] were suggested by a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield solitaire laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays.*

This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operation

The Law of Large Numbers

This is the basis for the Monte-Carlo Method.
Let $X_1, \ldots, X_n$ be a sequence of random variables with the same mean μ and variance σ^2 where σ^2 is finite.and pariwise uncorrelated, then the sequence of random variables U_n where $U_n = \frac{1}{n}[X_1 + \cdots + X_n]$ converges in probability to μ,.

$$Prob\left(|U_n - \mu| > \epsilon\right) \to 0 \text{ if } n \to \infty$$

You can read about this result in any basic statistics book for example OpenIntro Statistics.

Unkown distribution

In practice we don't know the distribution. However we can estimate the mean of X if we are able to make n indpendent random draws of X, $X_1, \ldots, X_n$.

Generalization

Metropolis (1956)

$$E(f(X)) =$$

the limit of the arithmetic average of the values of

$f(X)$ over n draws

First introduced into finance in the 1970's. Used earlier in industrial engineering and insurance.

Monte-Carlo Pricing for Plain Vanilla Options

$$\frac{dS}{S} = rdt + \sigma d\tilde{W}_t, \text{ where } (\tilde{W}_{t\geq 0} \text{ is Q-Brownian motion})$$

So this is the price dynamics under risk neutral Q Brownian motion. We have changed from one probability space to another. risk neutral measure is the set of probabilities that

$$C(t) = E_Q\left[\max\left(0, S(T) - k\right) e^{-r(T-t)} | F_t\right]$$

Interest rates are derministic then

$$C(T) = e^{-r(T-t)} E_Q\left[f(S(T)|F_t\right]$$

and

$$f(S(S(T)) = \max(0, S(T) - k)$$

We need only simulate $S(T)$.

Coefficient matching and solving stochastic differential equations

Given an SDE

$$dS = \mu S dt + \sigma S dW$$

and a function $S(t) = f(t, W(t))$(Note this is independent of S). Now take x equals a possible realization of $W(t)$ Ito's lemma gives:

$$dS = df = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2})dt + \frac{\partial f}{\partial x} dW$$

Now compare this with]

$$dS = \mu S dt + \sigma S dW$$

so

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} = \mu S$$

and

$$\frac{\partial f}{\partial t} = \sigma S = \sigma f$$

and $f(0,0) = S_0$

Solution

Solve these for $f()$

$$f(t, B) = e^{\sigma x + g(t)}$$

$$g'(t) = \mu - \frac{1}{2}\sigma^2$$

Integrate to get

$$g(t) = (\mu - \frac{1}{2}\sigma^2)t + C$$

more...

substitute back into $f(t, S)$ to get

$$f(t,S) = e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t + C}$$
$$= e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t + C}$$
$$S(t) = e^{C} e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t}$$

Now when $t = 0$ we get $S(0) = e^C$ so $C = \ln S(0)$ abnd replace x with $W(t)$.

$$S(t) = S_0 e^{\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t}$$

We can then use this to simulate the value of S and any functions of it including the call or put price option.

Solution

Risk neutral equation

$$S(T) = S(t)exp\left\{(r - \frac{\sigma^2}{2})(T - t + \sigma \hat{W}(T - t)\right\}$$

simulate $\hat{W}(T - t)$ which has the distribution $N(0, \sqrt{T - t})$.

Algorithm

$$S^{(1)}(T) = S(t) \exp\left\{(r - \frac{\sigma^2}{2})(T - t) + \sigma x_1\right\}$$

$$S^{(2)}(T) = S(t) \exp\left\{(r - \frac{\sigma^2}{2})(T - t) + \sigma x_2\right\}$$

etc.

$$b_1 = \max(0, S^{(1)(T) - k}$$

$$b_2 = \max(0, S^{(2)(T) - k}$$

etc.
Then calculate

$$C(t) \approx e^{-r(T-t)} \left[\frac{b_1 + b_2 + \ldots + b_N}{N}\right]$$

Random numbers

- Pseudo-random numbers
- Not purely random (cycling)
- Generate uniform random variates
- Transform these to other distribution
- Quasi-random numbers (don't cycle)
- These are based on number theoretic ideas (low discrepancy sequences)

R Packages

Read the randtoolbox Vignette.

- **randtoolbox** package for quasi-random numbers

Python modules

- **random** module
- For Halton sequences **ghalton**

Price options on a commodity

$$\frac{dS}{S} = (r - y)dt + \sigma d\hat{W}_t$$

Now

$$S^{(1)}(T) = S(t) \exp\left\{ (r - y - \frac{\sigma^2}{2})(T - t) + \sigma x_1 \right\}$$

$$S^{(2)}(T) = S(t) \exp\left\{ (r - y - \frac{\sigma^2}{2})(T - t) + \sigma x_2 \right\}$$

etc.

$$b_1 = \max(0, S^{(1)(T) - k}$$

$$b_2 = \max(0, S^{(2)(T) - k}$$

etc.

Call option price for a commodity spot

Then calculate

$$C(t) \approx e^{-(r-y)(T-t)}\left[\frac{b_1 + b_2 + \ldots + b_N}{N}\right]$$

To find the convenience yield solve

$$F(t, T) = S(t)e^{(r-y)(T-t)}$$

for y using observed data on $F(t, T), S(t)$.
And that's Monte-Carlo pricing!