Commodity Futures Markets Options

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Readings

- H. Geman, Ch.4. Agricultural Commodity Spot Markets, in: *Agricultural Finance*, John Wiley & Sons, 2015.

Options pricing

We will derive the Black-Scholes equation from the CAPM. This will require some intermediate steps first and a brief introduction to Ito's lemma.

The time increment of stock returns is

$$E(r_s dt] = E[\frac{dS_t}{dt}]$$

where

$$dS_t = rS_t dt + \sigma S_t dW_t$$

is an Ito stochastic differential equation (continuous sample path but not differentiable)

Derivation

$$E[\frac{dS_t}{S_t}] = E[rdt] + E[\sigma dW_t]$$
$$= R_f dt + \beta(E[R_m] - R_f)dt$$

To get this substitute in $E[R_i] = R_f + \beta_{im}(E[R_m] - R_f)$ and note that $E[dW_t] = 0$

Derivatives

An option is a derivative which means mathematically it is a function of the underlying stock S.
Denote the value of the derivative as $V_t(S_t, t)$. Then

$$E[r_v dt] = E[\frac{dV_t}{dt}] = R_f dt + \beta_V(E[R_m] - R_f)dt$$

Ito's Lemma

Taylor expanding $V(t, S_t)$ we get

$$dV = \frac{\partial V}{dt}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \ldots$$

Now substitute in $dS_t = rS_t dt + \sigma S_t dW_t$

$$dV = \frac{\partial V}{dt}dt + \frac{\partial V}{\partial S}(rS_t dt + \sigma S_t dW_t) +$$
$$\frac{1}{2}\frac{\partial^2 V}{\partial S^2}(rS_t dt + \sigma S_t dW_t)^2 + \ldots$$

Now the key thing you need to know is that in the limit as $dt \rightarrow 0$ (i.e. things become continuous in time), then

$$dt^2 \text{ and } dtdW_t \rightarrow 0 \text{ and } dW_t^2 \rightarrow dt$$

so terms drop out and we get

$$dV = (\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + \sigma\frac{\partial V}{\partial S}dW_t$$

Continuing the derivation...

Divide by V_t and take expectations:

$$E\frac{dV}{V} = \frac{1}{V}(\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + E\sigma\frac{\partial V}{\partial S}dW_t$$

$$r_v dt = \frac{1}{V_t}(\frac{\partial V}{\partial t} + \ \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + \mu\frac{\partial V}{\partial S}\frac{1}{V_t}$$

The mean (drift) of S is rS_t so we now substitute this

$$r_v dt = \frac{1}{V_t}(\frac{\partial V}{\partial t} + \ \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2})dt + r_S S_t\frac{\partial V}{\partial S}\frac{1}{V_t}$$

dt cancels.

Relationship between β_V and β_S

Take covariances between r_v and r_m

$$Cov[r_v, r_m] = \frac{\partial V}{\partial S}\frac{S_t}{V_t}Cov[r_s, r_M]$$

which implies

$$\beta_V = \frac{\partial V}{\partial S}\frac{S_t}{V_t}\beta_S$$

Recall

$$E[\frac{dV_t}{V_t}] = R_f dt + \beta_V(E[R_m] - R_f)dt$$

Multiply this by V_t to obtain:

$$E[dV_t] = R_f V_t dt + V_t \beta_V(E[R_m] - R_f)dt$$

then from the previous slide $\beta_V = \frac{\partial V}{\partial S}\frac{S_t}{V_t}\beta_S$ so

$$E[dV_t] = R_f V_t dt + V_t \frac{\partial V}{\partial S}\frac{S_t}{V_t}\beta_S(E[R_m] - R_f)dt$$

$$= R_f V_t dt + \frac{\partial V}{\partial S} S_t \beta_S(E[R_m] - R_f)dt$$

Recall

$$dV = (\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW_t$$

Take expectations of this we get:

$$E[dV] = (\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt$$

using $\mu = E[\frac{dS}{S_t}] = R_f dt + \beta_S(E[R_M] - R_f)$

$$E[dV] = (\frac{\partial V}{\partial t} + (R_f Sdt + \beta_S(E[R_M] - R_f))Sdt)\frac{\partial V}{\partial S} +$$

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt$$

Finally set this equal to $= R_f V_t dt + \frac{\partial V}{\partial S} S_t \beta_S (E[R_m] - R_f) dt$
To get

$$(\frac{\partial V}{\partial t} + (R_f S + \beta_S (E[R_M] - R_f)) S) \frac{\partial V}{\partial S} +$$

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) = R_f V_t + \frac{\partial V}{\partial S} S_t \beta_S (E[R_m] - R_f)$$

which simplifies to

$$\frac{\partial V}{\partial t} + R_f S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - R_f V_t = 0$$

This is the Black-Scholes PDE (partial differential equation).

Solution of Black-Scholes equation

- Classical approach in involves a change of variables and reparameterization
- Turn it into a heat equation with known solution
- Solution results in the Black-Scholes Formula
- We will do this in steps

Change of variables

$$S = Ke^x$$

$$t = T - \frac{\tau}{\frac{\sigma^2}{2}}$$

$$v(x,\tau) = \frac{1}{K}V(S,t) = \frac{1}{K}V(Ke^x, T - \frac{\tau}{\frac{\sigma^2}{2}})$$

Evaluate the derivatives

$$\frac{\partial V}{\partial t} + R_f S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - R_f V_t = 0$$

$$\frac{\partial V}{\partial t} = K \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} K \frac{\partial v}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = K \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}(\frac{\partial V}{\partial S}) = \frac{K}{S^2}(\frac{\partial^2 v}{\partial x^2}) - \frac{\partial v}{\partial x}$$

now substitute these into the Black-Scholes equation

$$v_\tau = v_{xx} + (\frac{r-\delta}{\frac{\sigma^2}{2}} - 1)v_x - \frac{r}{\frac{\sigma^2}{2}}v$$

Define
$\kappa = \frac{r-\delta}{\frac{\sigma^2}{2}}$ and $l = \frac{\delta}{\frac{\sigma^2}{2}}$ and substitute to get

$$v_\tau = v_{xx} + (\kappa - 1)v_x - (\kappa + l)v$$

PDE in constant coefficients

Further transformation

Set

$$\gamma = \frac{1}{2}(\kappa - 1)$$

and

$$\beta = \frac{1}{2}(\kappa + 1) = \gamma + 1$$

where

$$\beta^2 = \gamma^2 + \kappa$$

$$v(x,\tau) = e^{-\gamma x - (\beta^2 + l)\tau} u(x,\tau)$$

derivatives are

$$v_\tau = e^{-\gamma x - (\beta^2 + l)\tau} \left\{ -(\beta^2 + l)u + u_\tau \right\}$$

$$v_x = e^{-\gamma x - (\beta^2 + l)\tau} \left\{ -\gamma u + u_x \right\}$$

$$v_{xx} = e^{-\gamma x - (\beta^2 + l)\tau} \left\{ \gamma u^2 + 2\gamma u_x + u_{xx} \right\}$$

substituting gives

$$u_\tau = u_{xx} + (-2\gamma + \kappa - 1)u_x + \gamma(2\gamma - \kappa + 1)u$$

coefficients vanish when one substitutes γ back in.

The Heat Equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Separation of variables

Seek a solution of the form

$$u(x,\tau) = X(x)T(\tau)$$

substituting

$$X(x)T'(\tau) = X''(x)T(\tau)$$

Rearrange to get

$$\frac{T'(\tau)}{T(\tau)} = \frac{X''(x)}{X(x)}$$

Both sides constant so we get

$$\frac{T'(\tau)}{T(\tau)} = c$$

and

$$\frac{X''(x)}{X(x)} = c$$

This gives us two ordinary differential equations.

$$T'(\tau) = cT(\tau)$$

and

$$X''(x) = X(x)$$

The solution of which is

$$T(\tau) = T(0)e^{ct}$$
$$X(\tau) = d_1 e^{\sqrt{c}x} + d_2 e^{-\sqrt{c}x}$$

We then need to back transform the problem and employ boundary and initial conditions to pin-down constants.

The Black-Scholes Formula

For a call option.

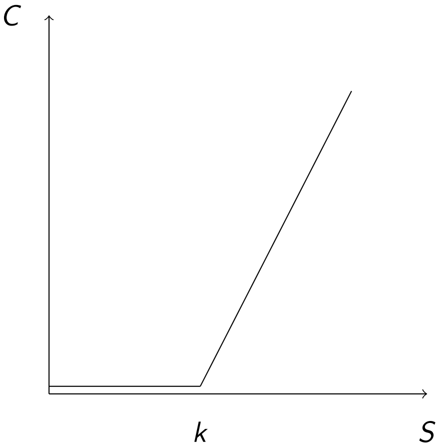
$$C(t,S) = e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2)$$

where

- $$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy$$
- $$d_1 = \frac{log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
- $$d_2 = d_1 - \sigma\sqrt{T-t}$$

Application: European call option

holder of option has the right to purchase (call) an asset at price K at date T. the payoff to the option is



$$C = \max(0, S - k)$$

Limitations

- Assumes normal distribution, tail behavior may be important.
- Assumes continuous trading at least to an approximation (not always appropriate in some agricultural markets)
- Original price process needs to be appropriate for the observed asset
- Different price processes lead to different formulae for the call option.
- Illustrated is an approach to pricing rather than a single pricing formula.

The End

Thanks for listening!