Agricultural finance Lecture 6 Binomial pricing

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December 15, 2016

Introduction

- mathematical tools underlying Black-Scholes are relativeley advanced
- Derivation from CAPM is relatively straightforward still requires some advanced mathematics (stochastic calculus)
- Therefore an even simpler approach has been developed: Binomial pricing which requires only elementary mathematics

Basic idea

Suppose current price of stock is $S = \$50$ and after one period the price will either increase to \$100 or fall to \$23. A call option with a strike price of $K = \$50$ is available. This expires at the end of the period. Money can be borrowed and lent at the prevailing interest rate of 25%. what is the value of the call?

Hedging

Consider the following hedge:

- Write 3 calls at C each
- Buy 2 shares at $50 each
- Borrow $40 at 25% to be paid back at the end of the period.

$$3C - 100 + 40 = 0$$

So $C = 20$

Arbitrage table

	Present date	Expiration date $S^* = \$25$	$S^* = \$100$
Write 3 calls	3C		-150
Buy 2 shares	-100	50	200
Borrow	40	-50	-50
Total			

Replication

An appropriately levered position in stock will replicate the future returns of a call

Binomial pricing

uS with probability q

S

dS with probability $1 - q$

Rate of return on Stock is either $u - 1$ or $d - 1$

Assumptions

- Interest r is constant
- Assume no taxes, transaction costs or margin requirements
- individuals are allowed to sell short
- $u > r > d$

Call option

Let C be the current value of the call, if the price increases it's value is C_u, if ithe stock price is uS. If the stock price is dS then the value of the call is C_d. We know that

$$C_u = \max[0, uS - K]$$

and

$$C_d = \max[0, dS - K]$$

So

$$C_u = \max[0, uS - K] \text{ with probability } q$$

$$C$$

$$C_d = \max[0, dS - K] \text{ with probability } 1 - q$$

Portfolio

Form a portfolio of A shares and B$ in riskless bonds:

$$AS + B \begin{cases} AuS + rB & \text{with probability } q \\ AdS + rB & \text{with probability } 1 - q \end{cases}$$

We can select A and B anyway we wish!!

Choose them so as

$$AuS + rB = C_u$$

$$AdS + rB = C_d$$

in other words choose a portfolio to replicate the payoffs of the call option
Solving we get

$$A = \frac{C_u - C_d}{(u-d)S}$$

$$B = \frac{uC_d - dC_u}{(u-d)r}$$

With no arbitrage the current value of the call C cannot be less than the current value of this hedging portfolio $AS + B$

No arbitrage

- Arbitrage would allow us to buy a call and sell the portfolio
- Suppose $AS + B < S - K$ then arbitraging would lead us to losing because anyone could buy a call from us and exercise it immediately

No arbitrage

$$C = AS + B$$

$$\frac{C_u - C_d}{(u-d)S} S + \frac{uC_d - dC_u}{(u-d)r}$$

$$= \frac{[(\frac{r-d}{u-d})C_u + (\frac{u-r}{u-d})C_d]}{r}$$

if $AS + B > S - K$ otherwise $C = S - K$

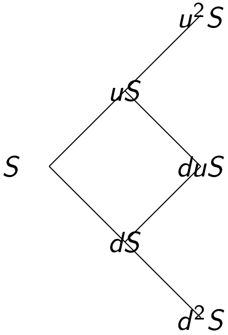
Define $p \equiv \frac{r-d}{u-d}$ and $1-p \equiv \frac{u-r}{u-d}$
So

$$C = \frac{[pC_u + (1-p)C_d]}{r}$$

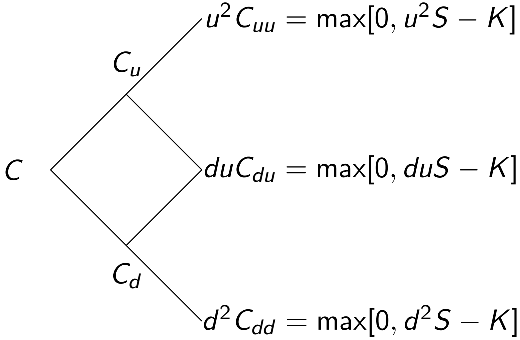
- Note that the value of the call does not depend on investors subjective beliefs about stock movements.
- Note that the value also doesn't depend on attitudes towards risk
- p is the equilibrium value of q if investors were risk neutral (risk neutral measure)

The value of a call is the expected discounted future value in a risk-neutral world.

Two-periods



Two-periods: Call value



Value of call with two periods remaining

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{r}$$

and

$$C_d = \frac{[pC_{du} + (1-p)C_{dd}]}{r}$$

Recursive solution

$$C = \frac{[pC_u + (1-p)C_d]}{r}$$

but

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{r}$$

and

$$C_d = \frac{[pC_{du} + (1-p)C_{dd}]}{r}$$

and

$$u^2 C_{uu} = \max[0, u^2 S - K]$$

$$duCdu = \max[0, duS - K]$$

$$d^2 C_{dd} = \max[0, d^2 S - K]$$

Recursive solution

$$C = \frac{[p^2 \max[0, u^2 S - K] + 2p(1-p)\max[0, duS - K]}{r^2} + \frac{(1-p)^2 \max[0, d^2 S - K]]}{r^2}$$

This should be beginning to look familiar.

N period Call

The value of an N period call option:

$$C = \frac{[\sum_{j=0}^{n} \binom{n}{j} p^j (1-p]^{n-j} \max[0, u^j d^{n-j} S - K]}{r^n}$$

Continuous limit

$h \equiv \frac{t}{n}$ is the elapsed time between stock price changes. Then as $h \rightarrow 0$ the N period call option formula will approach the Black-scholes formula.

Applications

- Option pricing (obviously)
- Real options theory

Real (investment) options

- NPV calculations for investment ignore questions concerning the timing of the investment
- NPV calculations are unable to effectively incorporate uncertainty when investments are irrversible
- Any investment problem can be treated as an option as such modified option pricing formulae can be used to evaluate investments
- The Real options approach is preferred to the net present value approach when investments are irrversible (timing matters) and in the presence of uncertainty.