Commodity Futures Markets Options

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Readings

- H. Geman, Ch.4. Agricultural Commodity Spot Markets, in: *Agricultural Finance*, John Wiley & Sons, 2015.

Put-call parity and Gamma Γ or are Put's convex?

$$S(t) + P(t) = C(t) + ke^{-r(T-t)}$$

Differentiating with respect to $S(t)$:

$$1 + \frac{\partial P}{\partial S(t)} = \frac{\partial C}{\partial S}$$

This is just

$$1 + \Delta_{Put} = \Delta_{call}$$

Rearranging $\delta_{Put} = -(1 - \Delta_{call}) = -(1 - N(d_1))$
Term in parentheses on the right lies between 0 and 1 so
Price of the put is a decreasing function of S

Γ_{Put}

$$\Gamma_{Put} = \frac{\partial \Delta_{Put}}{\partial S} = \frac{\partial \Delta_{call}}{\partial S} = \Gamma_{call}$$

So a Put option is also convex even though Δ is negative.

Theta θ

$$\theta_{call} = \frac{\partial C}{\partial t} = -\frac{S\sigma N'(d_1)}{2\sqrt{T-1}} - rke^{-r(t-t)}N(d_2)$$

You should be able to verify this yourself!! How?

Intrinsic value and time value

$$C(t) = S(t)N(d_1) - ke^{r(T-t)}N(d_2)$$

$$C(t) = \text{intrinsic value} + \text{Time value}$$

Vega (The streetfighter character not a Greek Letter)

$$Vega_{call} = \frac{\partial C}{\partial \sigma} = S\sqrt{T-t}N'(d_1)$$

Use put-call parity to show

$$Vega_{put} = \frac{\partial P}{\partial \sigma} = S\sqrt{T-t}N'(d_1)$$

Convexity and profit and loss

Consider a long call and Taylor expand profit and loss:

$$C(t+\delta t, S_t+\delta S_t)-C(t, S_t) = \frac{\partial C}{\partial t}\delta t+\frac{\partial C}{\partial S_t}\delta S_t+\frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}(\delta S_t)^2$$

Delta Hedging the Long Call

Now delta hedge this by adding $-\frac{\partial C}{\partial S} = -\Delta$ to this:

$$C^{cov}(t+\delta t, S_t+\delta S_t) - C^{cov}(t, S_t) = \frac{\partial C^{cov}}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 C^{cov}}{\partial S_t^2}(\delta S_t)^2$$

$$C^{cov}(t+\delta t, S_t+\delta S_t) - C^{cov}(t, S_t) = \frac{\partial C^{cov}}{\partial t}\delta t + \frac{1}{2}\Gamma_{option}(\delta S_t)^2$$

The call value is now covered by the Delta hedge so we denote it with C^{cov} rather than C. $\Gamma_{option} > 0$.

Stochastic volatility

These form a large class of models often estimated using using aRCH/GARCH econometric models and similar methods (discrete-time)

$$C^{cov}(t+\delta t, S_t+\delta S_t) - C^{cov}(t, S_t) = \frac{\partial C^{cov}}{\partial t}\delta t + \frac{1}{2}\Gamma_{option}(\delta S_t)^2 + vega_{call}\delta\sigma_t$$

Ownership pf physical assets can lead to profit changes due to stochastic volatility.

Options on commodity spot prices

Commodity price behavior is equivalent to that of a stock paying a continuous dividend equivalent to the convenience yield y (benefit from holding inventory).
Underlying rice dynamics is geometric Brownian motion:

$$\frac{dS}{S} = (\mu - y)dt + \sigma dW$$

Black-Scholes formula in this case is

$$C(t) = S(t)e^{-y(T-t)}N(d_1) - ke^{-r(T-t)}N(d_2)$$

$$\begin{cases} d_1 = & \frac{ln\left(\frac{S(t)e^{-y(T-t)}}{ke^{-r(T-t)}}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = & d_1 - \sigma\sqrt{T} \end{cases}$$

Comments

- Call options written on commoditties have a unique price and replicating portfolio (put-call parity)
- dynamic hedging holds (delta hedging is possible) as long as storage is possible and portfolio is rebalanced daily (continuously). Persihable items cannot be hedged in this way.
- Extension to other price processes possible (see next slide)
- Convenience yield is assumed constant. What if it isn't constant? How should we change the model to account for that?
- If interest rates aren't constant replace discount factor by a Bond price $B(t, T)$ with unit payoff.
- For options with maturities far in the future constant convenience yield cannot be assumed.

Other price process

Mean-reverting:

$$dS = (\mu - S)dt + \sigma dW$$

Ornstein-Uhlenbeck:

$$dS = a(\mu - S)dt + \sigma dW$$

Options on commodity futures (Black, 1976)

Fischer Black (1976) The pricing of commodity contracts, Journal of Financial Economics.

- Futures can be written on stocks or commodities.
- Maturity of future $T_f$must be greater than maturity T of the option (should be obvious)

$$C(t) = e^{-r(T-t)}\left[F^{T_f}(t)N(d_1) - kN(d_2)\right]$$

$$\begin{cases} d_1 = & \frac{ln\left(\frac{F^{T_f}(t)}{k}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = & d_1 - \sigma\sqrt{T} \end{cases}$$

Two observations

If $T = T_f$ then At maturity $S(T) = F(T, T)$ so

$$F(T, T) = S(t)e^{(r-y)(T-t)}$$

subsittuting this into the call pricing formula:

$$C(t) = e^{-r(T-t)}\left[S(t)e^{(r-y)(T-t)}N(d_1) - kN(d_2)\right]$$

which simplifies to

$$C(t) = \left[S(t)e^{-y(T-t)}N(d_1) - e^{-r(T-t)}kN(d_2)\right]$$

which is the Merton formula for value of a call option written on a spot price.

Pricing Put Options written on Futures Contracts

By Put call parity:

$$C9t) + ke^{-r(T-t)} = F(t, T_f)e^{-r(t-t)} + P(t)$$

Volatility smiles

Say you have data for options with different strikes. then

- Set the option price equal to the oberved market price of the option for each strike price.
- Invert the formula to obtain the volatility σ
- Plot σ against the different strike prices
- We will do this in a lab exercise.

The End

Thanks for listening!