Agricultural Marketing and Price Analysis Lecture 6 Prices over time (commodity futures)

Rodney Beard

November 7, 2016

Spot prices - date 1

Recall

Spot prices at date 1

$$\tilde{P}_1 = a - b(S_0 + \tilde{Q}_1 - S_1(\tilde{Q}_1))$$

$$P_1(Q_1) = \begin{cases} a - b(S_0 + Q_1) & \text{for} 0 \leq Q_1 \leq Q_1^* \\ a - b(S_0 + Q_1 - S^*(Q_1)) & \text{for } Q_1 > Q_1^* \end{cases}$$

Spot prices - date 2

Spot prices at date 2

$$\tilde{P}_2 = a - b(\tilde{Q}_2 + S_1(\tilde{Q}_1))$$

$$P_2(Q_1, \tilde{Q}_2) = \begin{cases} a - b(Q_2) & \text{for } 0 \leq Q_1 \leq Q_1^* \\ a - b(S^*(Q_1) + \tilde{Q}_2) & \text{for } Q_1 > Q_1^* \end{cases}$$

Expected prices

$$E_0\left\{P_1(\tilde{Q}_1)\right\} = a - b(S_0 + \mu_1) + b\int_{Q_1^*}^{\infty} S^*(Q_1)df(Q_1)dQ_1$$

and

$$E_0(P_2(\tilde{Q}_1, \tilde{Q}_2)) = a - b\mu_2 - b\int_{Q_1^*}^{\infty} S^*(Q_1)f(Q_1)dQ_1$$

$$E_1\left\{P_2(\tilde{Q}_2|Q_1)\right\} = \begin{cases} a - b(\mu_2) & \text{for } 0 < Q_1 < Q_1^* \\ a - b(\mu_2 + S^*(Q_1)) & \text{for } Q_1 \geq Q_1^* \end{cases}$$

Equilibrium

Market efficiency

Efficient price discovery implies futures prices must equal he expected spot price (rational expectations assumption). Note that is this were not the case arbitrage possibilities would exist.

So

$$f_0^1 = E_0(P_1(\tilde{Q}_1)$$
$$f_0^2 = E_0(P_2(\tilde{Q}_1, \tilde{Q}_2))$$
$$f_1^2 = E_1(P_2(\tilde{Q}_2)|Q_1)$$

We need to evaluate the integral terms in thee expected price expressions

Integral expression

To evaluate

$$\int_{Q_1^*}^{\infty} S^*(Q_1) f(Q_1) dQ_1$$

we need to know $f(Q_1)$. We will assume it follows a truncated normal distribution with $Q_1 \geq 0$.

The Truncated Normal Distribution

First recall the normal density function:

$$P(X) = \frac{1}{\sigma\sqrt{2\pi}}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and the standard normal density:

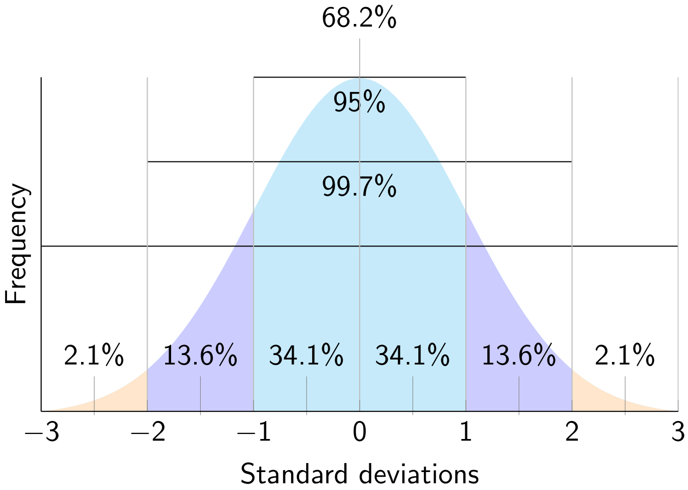
$$\phi(x) = \frac{1}{\sqrt{2\pi}}^{-\frac{(x-\mu)^2}{2}}$$

Truncated Distributions are Conditional Distributions

$$P(x|a \leq x \leq b) = \frac{P(x)}{P(X \leq b) - P(X \leq a)}$$
$$= \frac{f(x)}{F(X \leq b) - F(X \leq a)}$$

The Truncated Normal Distribution

Now we use the standard normal cumulative distribution function to "weight" the standard normal density by the area under under the density up to a truncation point.



Source: `http://johncanning.net/wp/?p=1202`

Truncated Normal Density

The Normal distribution is given by

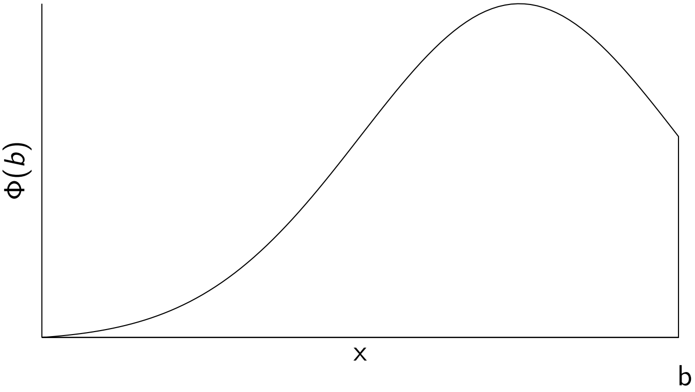
$$\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi$$

and

$$\lim_{x\to\infty} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = 1$$

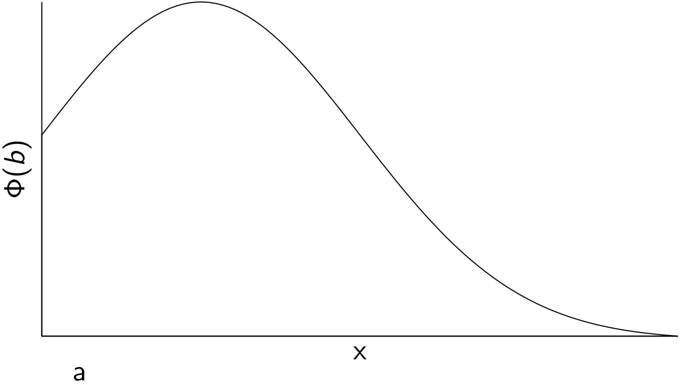
Let us truncate this at another point say b

$$\Phi(b) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = const.$$



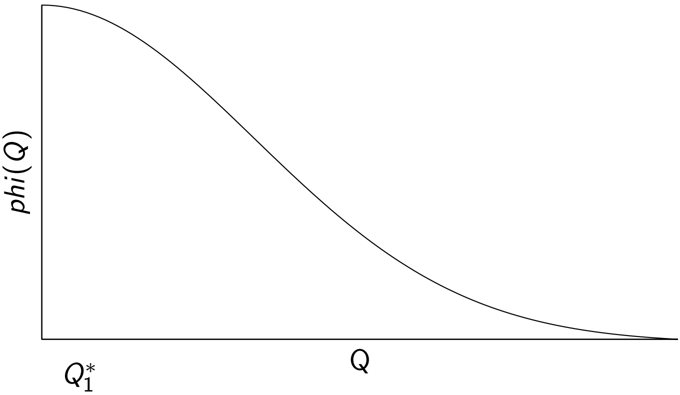
Truncated lower tail

$$\Phi(b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^{\infty} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = const.$$



Truncated Normal Distribution

$$\Phi(Q_1^*) = \frac{1}{\sigma\sqrt{2\pi}} \int_{Q_1^* \geq 0}^{\infty} e^{-\frac{(Q-\mu)^2}{2\sigma^2}} dQ = const.$$



Truncated normal density for storage - First property

$$\frac{\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}$$

Note this is also known as a hazard function (ratio of density to inverse CDF)

Second property

$$\int_a^{\infty} \phi(\frac{Q-\mu}{\sigma})dQ = \mu\left[1-\Phi(\frac{a-\mu}{\sigma})\right] + \sigma\phi(\frac{a-\mu}{\sigma})$$

we will substitute the first property into the expected storage function and the second will be used to evaluate the integral.

Next step

Recall

$$\int_{Q_1^*}^{\infty} S^*(Q_1) f(Q_1) dQ_1$$

First we will consider

$$\int_{a}^{\infty} Q f(Q) dQ$$

now substitute

$$f(Q) = \frac{\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}$$

Next step

$$\int_a^\infty Q \frac{\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})} dQ$$

Note that $\Phi(\frac{a-\mu}{\sigma})$ is independent of Q. So

$$\frac{1}{1-\Phi(\frac{a-\mu}{\sigma})} \int_a^\infty Q\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})dQ$$

Now we use the second property

Mean of a Truncated Random Variable with lower truncation a

$$\int_a^\infty Qf(Q)dQ = \frac{1}{1-\Phi(\frac{a-\mu}{\sigma})}\int_a^\infty Q\sigma^{-1}\phi(\frac{Q-\mu}{\sigma})dQ$$

$$= \mu + \sigma\frac{\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}$$

Denoting the mean of the truncated random variable with μ_1 and $\hat{\mu}_1$ is the scale parameter of $F(Q_1)$ then,

$$\hat{\mu}_1 + \sigma_1\frac{\phi(\frac{0-\hat{\mu}_1}{\sigma})}{1-\Phi(\frac{0-\hat{\mu}_1}{\sigma})} = \mu_1$$

This needs to be solved numerically for $\hat{\mu}_1$. Try using Sympy solve but with numerical values for the parameters.

Back to the task at hand evaluating the integral

$$\int_{Q_1^*}^{\infty} S^*(Q_1) f(Q_1) dQ_1$$

we previously derived $S^*(Q_1) = \frac{b(S_0+Q_1-\mu_2)-m}{2b}$ substituting we get

$$\int_{Q_1^*}^{\infty} \frac{b(S_0+Q_1-\mu_2)-m}{2b} f(Q_1) dQ_1$$

$$= \int_{Q_1^*}^{\infty} \frac{b(S_0-\mu_2)-m}{2b} f(Q_1) dQ_1 + \int_{Q_1^*}^{\infty} \frac{(Q_1)}{2} f(Q_1) dQ_1$$

$$= \frac{b(S_0-\mu_2)-m}{2b} \int_{Q_1^*}^{\infty} f(Q_1) dQ_1 + \frac{1}{2} \int_{Q_1^*}^{\infty} Q_1 f(Q_1) dQ_1$$

Next we evaluate the last two integrals

$$\int_{Q_1^*}^{\infty} f(Q_1)dQ_1 = \frac{1}{1-\Phi(Z_0)} \int_{Q_1^*}^{\infty} \sigma^{-1}\phi(\frac{Q_1-\hat{\mu}_1}{\sigma_1})dQ_1$$

$$= \frac{1-\Phi(Z_1)}{1-\Phi(Z_0)}$$

and

$$\int_{Q_1^*}^{\infty} Q_1 f(Q_1)dQ_1 = \frac{1}{1-\Phi(Z_0)} \int_{Q_1^*}^{\infty} Q_1\sigma^{-1}\phi(\frac{Q_1-\hat{\mu}_1}{\sigma_1})dQ_1$$

$$= \frac{\hat{\mu}_1\left[1-\Phi(Z_1)\right]+\sigma_1\phi(Z_1)}{1-\Phi(Z_0)}$$

The Equilibrium Futures Prices - once again

$$f_0^1 = E_0(P_1(\tilde{Q}_1)$$
$$f_0^2 = E_0(P_2(\tilde{Q}_1, \tilde{Q}_2))$$
$$f_1^2 = E_1(P_2(\tilde{Q}_2)|Q_1)$$

$$E_0\left\{P_1(\tilde{Q}_1)\right\} = a - b(S_0 + \mu_1) + b\int_{Q_1^*}^{\infty} S^*(Q_1)df(Q_1)dQ_1$$

and

$$E_0(P_2(\tilde{Q}_1, \tilde{Q}_2)) = a - b\mu_2 - b\int_{Q_1^*}^{\infty} S^*(Q_1)f(Q_1)dQ_1$$

$$E_1\left\{P_2(\tilde{Q}_2|Q_1)\right\} = \begin{cases} a - b(\mu_2) & \text{for } 0 < Q_1 < Q_1^* \\ a - b(\mu_2 + S^*(Q_1)) & \text{for } Q_1 \geq Q_1^* \end{cases}$$

Futures price in period zero

$$f_0^1 = E_0(P_1(\tilde{Q}_1)$$

$$= a - b(S_0 + \mu_1) + b\int_{Q_1^*}^{\infty} S^*(Q_1)df(Q_1)dQ_1$$

$$= a - b(S_0 + \mu_1) + \frac{b(S_0 - \mu_2) - m}{2}(\frac{1 - \Phi(Z_1)}{1 - \Phi(Z_0)})$$

$$+\frac{b}{2}(\frac{\hat{\mu}_1\left[1 - \Phi(Z_1)\right] + \sigma_1\phi(Z_1)}{1 - \Phi(Z_0)})$$

Summary

- The ability to price futures is is the basis for trading in futures markets. Traders will be set limits and base their trading strategy on sophisticated mathematical models of price processes to determine when arbitrage possibilities may occur. Increasingly automated trading is used in which human traders monitor a series of automated traders who trade based on underlying mathematical and computational models.
- We have considered a simple three-period trading model in discrete time. In reality trading firms employ a variety of methods including continuous-time models based on stochastic differential equations

Summary

- These models draw on the finance literature more than the marketing literature and commities markets can be viewed as both a marketing and a financial instrument.
- A good and relatively accessible reference on the latter as it applies to Agricultural finance is Helyette Geman, Agricultural Finance.