NOTES ON THE GORDON-SCHAEFER MODEL

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Thursday 5^{th} May, 2022

1. Introduction

The origin of fisheries economics lies in work by Warming in 1911 [9, 1]. Then in 1954 two important papers were published, one from an economics perspective by Gordon [5] and another published in the fisheries management literature by Schaefer [8]. Gordon formalized Warming's earlier work. These three papers and in particular the latter two make up the foundation of fisheries economics and in particular the main model that we use for fisheries policy work today. I should add there are other models, that relax many of the restrictive assumptions found here and extend this framework in order to explore other questions that cannot be addressed using the Gordon-Schaefer framework. So these notes are not the whole story just the first chapter.

2. Stock dynamics

A key question is how to represent growth? Things grow over time. If something $x(t)$ grows at a constant rate r then we can write the law by which it grows as

$$\frac{dx(t)}{dt} = r \tag{1}$$

This is a differential equation because it has a derivative on the left, it's not a differential equation because the derivative is on the left though, for example we can also write it like this

$$\frac{dx(t)}{dt} - r = 0 \tag{2}$$

Like most equations it consists of things we know and things we don't know, we want to solve the equation for the things we don't know. Let's say we know r and also t, can we solve for $x(t)$? This is different to solving a simple equation like $2x = 4$, because x is a scalar, whereas $x(t)$ is a function. $x(t)$ might be the stock of fish at time t. To be able to predict the stock at a given point in time, we need to know the stock at some initial point in time t_0. For convenience we wills et $t_0 = 0$.

The solution to $\frac{dx(t)}{dt} = r$ can be considered in steps firstly $x(t) = rr + c$ where c is an unknown constant. However, notice that when $t = 0$, then $x(0) = c$, so if we know the initial value , then we can find c In other words, if our problem is an initial value problem with known initial value, we can find $x(t) = rt + x(0)$. If we plot this equation against t we get a straight line.

We can do these steps in Python using SymPy

```
import math as math
from sympy import *

f, t, r = symbols('f t r ')

f = Function('f')
ode = Eq(diff(f(t), t), r)
ode
sol = dsolve(ode, f(t))  # solve the equation
sol

t = Symbol('t') #for a sinle symbol use capital S in Symbol()
x0 = Symbol('x0') #stock of fish at time 0 I've switched from

# using f(t) to x(t) with x(0) = x0
eq1 = Eq(sol.rhs.subs(t,0),x0)
eq1

C1, K, x0 = symbols('C1 K x0')
C2 = solve(eq1,C1)
C2

x = Symbol('x(t)')
eq3 = Eq(sol.rhs.subs(C1,C2),x)
equation = simplify(eq3.subs({Symbol('C1'): C2[0]}) )
equation

from sympy.plotting import plot
x0=0
r=0.01
plot(r*t + x0,(t, 0, 20),title='linear differential equation',show='TRUE')
```

☞**Click to run the code**[1]

Running this code will produce the following output. The code is available here.

An alternative assumption might be that the population grows proportionately to the current stock:

$$\frac{dx(t)}{dt} = rx(t), x(0) = x_0 \tag{3}$$

[1]You may need to save a copy of the notebook to run it.

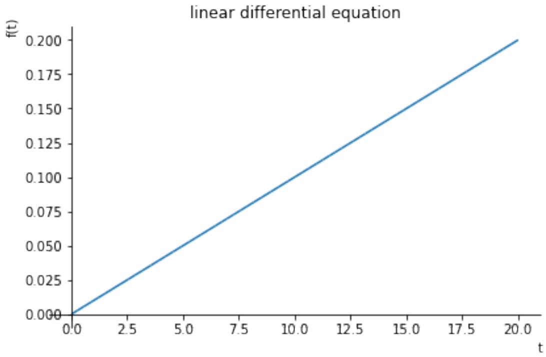

FIGURE 1. Linear differential equation

We solve this the same way, by rearranging to get $x(t)$ on one-side (This is called separation of variables):

$$\frac{1}{x(t)}dx = rdt \tag{4}$$

Integrating both sides

$$log(x(t) = rt + c \tag{5}$$
$$x(t) = e^{rt+c} \tag{6}$$
$$x(t) = x^{rt}e^{c} \tag{7}$$

Setting $t = 0$ we see that

$$x(0) = e^{c} \tag{8}$$

which can be substituted back in to (7) to obtain

$$x(t) = x(0)e^{rt} \tag{9}$$

This can be plotted against t. The following Python code derives the solution of the differential equation and plots it:

```
r,x0 = symbols('r, x0')
ode = Eq(diff(f(t), t), r*f(t))
ode

sol2 = dsolve(ode, f(t))
sol2

eq2 = Eq(sol2.rhs.subs(t,0),x0)
eq2

C1, K, x0 = symbols('C1 K x0')
```

```
C2 = solve(eq2,C1)
C2 #this is really C1

eq3 = Eq(sol2.rhs.subs(C1,C2),x)
equation = simplify(eq3.subs({Symbol('C1'): C2[0]}) )
equationx = Symbol('x(t)')

equation

x0=1
r=0.1
plot(x0*exp(r*t),(t, 0, 20),show='TRUE')
```

☞**Click to run the code**

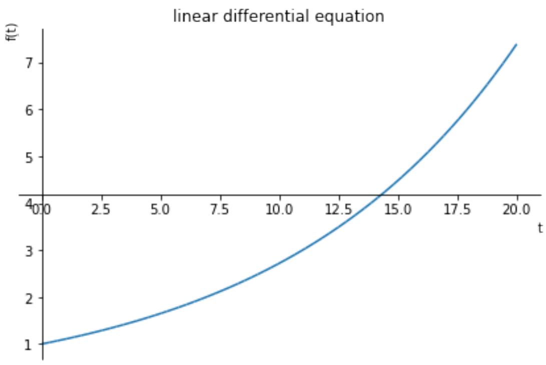

FIGURE 2. Linear differential equation $\frac{dx}{dt} = rx(t), x(0) = 1$

The exponential term e in the solution indicates that the stock would grow exponentially. The population (stock) of fish growing linearly or exponentially doesn't seem very realistic and comparing this to data would conform this. The model doesn't account for the fact that limited resources are available for the fish to consume, and that this will limit the growth of the stock. Therefore we wish to penalize the growth by a mortality term related to the density of the stock $\frac{X(t)}{K}$ given available resources K. ☞Discrete-time simulation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) \tag{10}$$

This can again be solved by using separation of variables (rearranging to the equation to get $x(t)$ on the left-hand side).

```
mport math as math
from sympy import *
t, r, K = symbols('t r K') #first define symbols
f = Function('f') #defien the symbol f as a function
```

```
ode = Eq(diff(f(t), t), r*f(t)*(1 - f(t)/K))
ode

sol = dsolve(ode, f(t))  # solve the equation
sol

t = Symbol('t')
x0 = Symbol('x0')
eq1 = Eq(sol.rhs.subs(t,0),x0)
eq1

C1 = symbols('C1')
C2 = solve(eq1,C1)
C2

x = Symbol('x(t)')
eq3 = Eq(sol.rhs.subs(C1,C2),x)
logequation = simplify(eq3.subs({Symbol('C1'): C2[0]}) )
logequation

x0 = 0.1
r = 0.3
K  = 1
plot(K*x0*exp(r*t)/(K +x0*exp(r*t) - x0),(t, 0, 20),show='TRUE')
```

☞**Click to run the code**

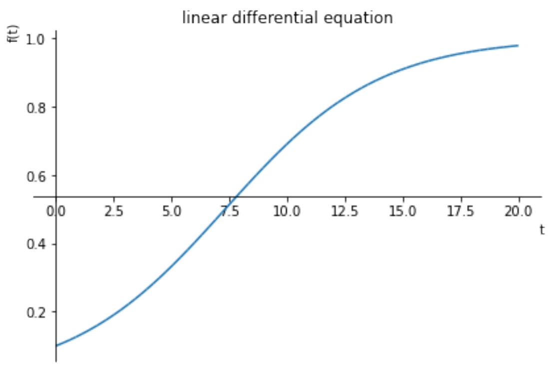

FIGURE 3. Non-linear differential equation $\frac{dx}{dt} = rx(t)(1 - \frac{x(t)}{K}), x(0) = 0.1, K = 1, r = 0.9$

So **??**Figure]fig3 depicts how the stock would grow if it weren't being fished. You should be able to verify by inspecting the graph that the stock will approach K in the limit, so in equilibrium (steady-state) the unfished biomass is$K$.

3. The Model

The Gordon-Schaefer fisheries model combines a biological population model with an economic model of rent in the fishery.

Typically one assumes that the biological model consists of a simple mass-balance relationship in which

$$\text{Stock change} = \text{Growth} - \text{Catch} \tag{11}$$

If growth exceed the catch, then the stock will grow, if the catch exceeds growth the stock will decline.

$$\text{Growth} > \text{Catch} \implies \text{stock increases} \tag{12}$$

In this chapter we will assume that the growth of the stock and catch balance exactly, so that the stock remains in equilibrium (steady-state).

$$\text{Growth} = \text{Catch} \implies \text{steady-state} \tag{13}$$

Assume that growth $F(x) = rx(t)(1 - \frac{x(t)}{K})$ and that catch $h(x) = qx(t)E(t)$ then in steady-state we have

$$F(x) = h(x) \implies rx(t)(1 - \frac{x(t)}{K}) = qx(t)E(t) \tag{14}$$

Cancelling the $x(t)$ on both sides we get

$$r(1 - \frac{x(t)}{K}) = qE(t) \tag{15}$$

and rearranging we obtain

$$x(t) = K(1 - \frac{q}{r}E) \tag{16}$$

This is the steady state level of the stock as a function of fishing effort. We can suppress the dependency on t because in steady-state time no longer influences the solution, the solution is "timeless". From this we can see that as fishing effort increases the stock goes down.

If we substitute this into the catch equation we get

$$h = qK(1 - \frac{q}{r}E)E \tag{17}$$

which is the sustainable yield-relationship. It is called the sustainable yield because it represents the catch or yield in steady-state.

We can plot this using the following Python code

```python
import math as math
from sympy import *
from sympy.plotting import plot
E = Symbol('E')
q = 0.01
K = 1
```

```
r = 2
plot(E*q*K*(1 - E*q/r),(E, 0, 200),title='sustainable yield',show='TRUE')
```

☞**Click to run the code**

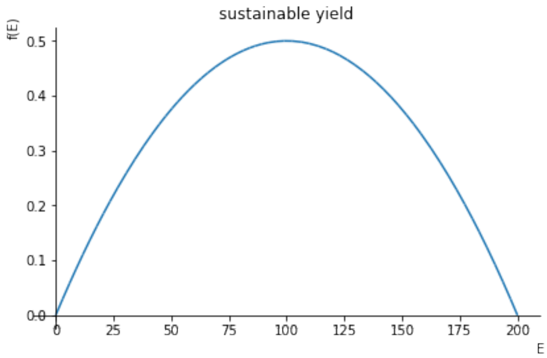

FIGURE 4. Maximum sustainable yield

This depicts the right hand side of (14). We can also plot the left-hand side.

To calculate the maximum sustainable yield we need to differentiate the sustainable yield function with respect to fishing effort

$$\frac{\partial h}{\partial E} = \frac{d}{dE}\left[qK(1-\frac{q}{r}E)E\right]$$
$$= qK(1-\frac{q}{r}E) - \frac{Kq^2}{r}E = 0$$
$$qK(1-2\frac{q}{r}E) = 0$$
$$E_{MSY} = \frac{r}{2q}$$

Substitution this back into the yield equation

$$h = qK(1-\frac{q}{r}E)E$$
$$= qK(1-\frac{q}{r}\frac{r}{2q})\frac{r}{2q}$$
$$\implies MSY = \frac{rK}{4}$$

The following calculates the maximum sustainable yield

```
import math as math
from sympy import *
r, X, K, q, E, p, c = symbols('r X K q E p c')

rent = p*q*X*E - c*E
```

```
eqn = r*X*(1- X/K) - q*X*E #balance: growth = catch

eqn

sol = solve(eqn, X)
sol #finds all solutions There are two.

print('X: ',sol[1])

simplify(rent.subs(X,sol[1]))

Y = q*X*E
Y = Y.subs(X,sol[1]) #normally this is a bad idea
Y #sustainable yield
Yss =Y
print('Y: ',Y)

focY = diff(Y,E)
print(focY)
Emsy = solve(diff(Y,E),E)
print('Emsy: ',Emsy[0])
```

☞**Click to run the code**

We can also examine the left-hand side of equation (14).

$$rX(1-\frac{X}{K})$$

The first-order conditions for a maximum of this are

$$r-2r\frac{X}{K}=0$$
$$X=\frac{K}{2}$$

In words we can interpret this as saying that the stock level corresponding to maximum sustainable yield is 50% of the unfished biomass in steady-state. This is potentially of relevance when thinking about target reference points although it should be notes that stock assessment models are based on a a different class of biological models to that studied here[2]. Substituting this back into the surplus production equation we see that it will exactly equal MSY. So that in steady-state the maximum of the surplus production function and the maximum of the sustainable yield curve are exactly equal.

```
import math as math
from sympy import *
from sympy.plotting import plot
```

[2]We will come back to examine this in the policy section.

```
X = Symbol('X')
K = 1
r = 2
plot(r*X*(1 - X/K),(X, 0, 200),title='sustainable␣yield',show='TRUE')
```

☞**Click to run the code**

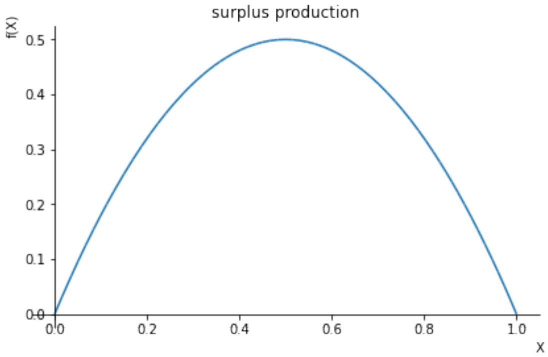

FIGURE 5. Surplus production

Noting that because these are equal in steady-state the maximum sustainable yield will be equal to the maximum of surplus production. If we divide by the total effort E we get the catch per unit effort (CPUE) $\frac{h}{E}$:

$$\text{CPUE} = qK(1 - \frac{q}{r}E) \tag{18}$$

It is worth considering the geometric relationships underpinning this.

Turning now to rent maximization, we wish to maximize total rent

$$\max_E pqX^*E - cE$$

where X^* is the steady state level of the stock.

4. Empirical estimation

To empirically estimate our model we can fit CPUE against fishing effort using linear regression

$$CPUE_i = \beta_0 + \beta_1 E_i + \epsilon_i, i = 1, \ldots, n \tag{19}$$

where is population regression model.

β_0, β_1 are the unknown coefficients we wish to estimate and ϵ_i is the error term.

This can be estimated using ordinary least squares (OLS) to obtain

$$b_0 = qK = CP\bar{U}E - b_1\bar{E} \tag{20}$$

and

$$b_1 = -\frac{q^2K}{r} = \frac{\sum_i^n (E_i - \bar{E})(CPUE_i - C\bar{PU}E)}{\sum_i^n (E_i - \bar{E})^2} \tag{21}$$

where terms with $\bar{.}$ are averages (arithmetic means).

Another way to think about this is that b_1 is the ratio of the covariance between effort and CPUE to the variance of fishing effort. $b_1 = \frac{Cov(E,CPUE)}{Var(E)}$. So we can see frrom this that covariance between effort and CPUE is related to the catchability and the variance of fishing effort is related to the growth rate of the stock. Fast growing stocks can be expected to have higher variance in fishing effort.

Note that b_0. and b_1 are estimates that one obtains from a regression analysis but q,K and r are the underlying "deep parameters" of the assumed biological model. b_0 and b_1 reflect these deep parameters.

Given data on catch and effort we can use the Python module Statsmodels to estimate the relationship between catch and effort. For a multiispecies fishery we may estimate the relationship between total catch and effort. However, it is also worth examining the relationship between the catch of specific species that make up the total catch and fishing effort. Sometime this is needed because specific fisheries management plans, e.g. tuna management plans specify a species specific catch limit and we wish to evaluate the whether the limit is consistent with economic efficiency.

5. Rent maximization

$$\max_E \Pi = ph(E) - cE \tag{22}$$

$$= pqK(1 - \frac{q}{r}E)E - cE \tag{23}$$

Differentiating this with respect to E we obtain

$$= .pqK - 2p\frac{q^2}{r}KE - c = 0 \tag{24}$$

$$E_{MEY} = r\frac{pqK - c}{pq^2K} \tag{25}$$

is the effort level corresponding to maximum economic yield.

Maximum economic yield is

$$h = Kr\frac{pqK - c}{pq^2K} - K\frac{q}{r}(r\frac{pqK - c}{pq^2K})^2 \tag{26}$$

We can plot this as follows

```
from sympy import *

r, X, K, q, E = symbols('r X K q E')

eqn = r*X*(1- X/K) - q*X*E #balance: growth = catch
```

```
eqn


sol = solve(eqn, X)
sol #finds all solutions There are two.

print('X: ',sol[1])

print('X: ',simplify(sol[1])) # provides the non-trivial solution sol[1] selects
# the first, try it!

p, q, c = symbols('p q c')
rent = p*q*X*E - c*E
simplify(rent.subs(X, sol[1]))

Y = q*X*E
Y = Y.subs(X, sol[1]) #normally this is a bad idea
Y #sustainable yield
Yss =Y
print('Y: ',Y)

revenue = p*Yss
print(revenue)
revenuess = revenue.subs({p:1000,K:1,q:0.01,r:1.2})
cost = 1*E
plot(revenuess,cost,(E,0,120),title='Rent maximization')
```

☞**Click to run the code**

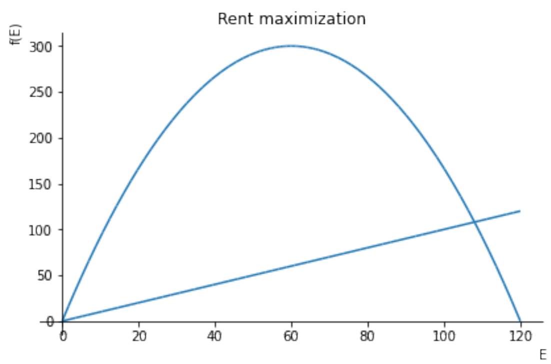

FIGURE 6. Rent maximization

The bionomic equilibrium is given by the point where total revenue equals total cost. This is the point that would be reached under free entry or no restrictions on fishing.

$$\Pi = pqK(1 - \frac{q}{r}E)E - cE = 0 \tag{27}$$

Solving for E we get

$$E_\infty = r\frac{pqK - c}{pq^2K} \tag{28}$$

Fishing at this level of activity completely dissipates rents and entry into the fishery would cease because no money is to be made[3].

6. Policy

The model presented here is designed to examine under what conditions overfishing occurs on the high seas. The idea is to compare bionomic equilibrium to the rent maximizing level of fishing which is the efficient point f rom an economic perspective. Because this is the benchmark model in fisheries economics it is often used and applied to other contexts besides the high seas. So for example in the Pacific the level of fishing effort corresponding to maximum economci yield is used to make recommendations concerning the number of fishing licenses to issue.

If effort is measured in days and we know the average number of days fished by fishing vessels then we can calculate the number of vessels to license by

$$\text{Vessels} = \text{days} \div \frac{\text{days}}{\text{vessel}} \tag{29}$$

over some licensing period, year, month etc.

The number of vessels licensed will then be the number that maximizes industry rents. Thought should be given as to whether this is in fact the desired objective. the same modelling approach can employed to explore alternative policy objectives other than rent maximization.

Note also that this model incorporates no discussion of licensing fees. No recommendation for how much to charge for fishing can be made based on this model. To do that a different model is needed that provides information on the demand for fishing. Given data on past license fees and requests for licenses in terms of fishing effort, it might be possible to estimate the demand for fishing licenses, typically there is insufficient variation in license fees over time to be able to effectively investigate this and and structural approaches to demand estimation need to be employed. This will be covered when we cover licensing in detail.

An alternative to trying to set an optimal license fee, would be to auction off available licenses. Simultaneously setting both the number of licenses and the fee is unlikely to be efficient as it it would require accurate information on the demand for licenses which is generally unavailable. Whether to choose the price (license fee) or the quantity (number of licenses) to regulate the fishery leads us into the whole debate on prices vs. quantities

[3]Entry into the fishery can be modelled using the dynamic open access model of the fishery which can be considered to be a short-run model rather than a long-run model. The model explored here is a long-run model.

which we will consider later. The final choice depends on characteristics of the fishery such as whether it is a search or schooling fishery.

7. Conclusion

In these notes I've tried to cover the basics of the Gordon-Schaefer fisheries model from start to finish. The notes contain Python code so you can run the derivations yourself and play with the figures. I also cover parameter estimation with data. The next set of notes will extend this model to cover issues with regulated fisheries and more sophisticated licensing models.

References

[1] Andersen, P. (1983). On Rent of Fishing Grounds: a translation of Jens Warming's 1911 Article, with an introduction. *History of Political Economy*, 15(3), 391-396.

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[3] Conrad, J., *Resource Economics*, Cambridge University Press; 2 edition (June 14, 2010).

[4] Fisher, A.C.. *Lecture notes on resource and environmental economics*, Springer 1st Edition, 2020.

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[9] Warming, J. 1911. Om Grundrente af Fiskgrunde.*Nationaløkonomisk Tidsskrift* 49:499–505.