Gambler's Ruin Problem Analysis

Author: *Alejandro Santiago Baca Eyssautier*

1. Introduction

The gambler's ruin problem is a classic problem in probability theory. It involves two players who repeatedly bet one unit of money per round. One of the players, denoted as I, starts with K units, and the other player, D, starts with $N-K$ units. The goal is to determine the probability that player I will lose all of their money (i.e., be ruined) based on the probability P of winning a round and the total units at stake.

This report simulates the gambler's ruin problem, plots the relationship between the ruin probability U_k and various parameters, and compares simulated results with theoretical expectations. The following steps are conducted:

1. Simulating multiple sample paths of the gambler's ruin problem.
2. Plotting U_k vs P for different values of K.
3. Plotting U_k vs K for different probabilities P.

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2. Concepts and Problem Formulation

1. **Problem Setup:**
   - **Players:**
     - I (player starting with K units)
     - D (player starting with $N-K$ units)
   - **Winning and Losing Probabilities:**
     - Player I wins with probability P.
     - Player D wins with probability $q = 1 - P$.
2. **Variables:**
   - X_n is the number of units player I has at round n.
   - $\tau = \min n \in \mathbb{N} : X_n = 0$ is the time of ruin.
   - $U_k = P(X_\tau = 0 \mid X_0 = k)$ is the probability that player I will be ruined starting with k units.
3. **Difference Equation:**

   The ruin probabilities satisfy the following difference equation:

$$U_{k+1} - U_k = \frac{q}{P}(U_{k-1} - U_k)$$

   With boundary conditions:

$$U_0 = 1, \quad U_N = 0$$

4. Theoretical Solution:

- If $P = q$, the ruin probability is:

$$U_k = 1 - \frac{K}{N}$$

- If $P \neq q$, the ruin probability is:

$$U_k = \frac{\left(\frac{q}{P}\right)^K - \left(\frac{q}{P}\right)^N}{1 - \left(\frac{q}{P}\right)^N}, \quad \text{for } P \neq q$$

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3. Simulations and Analysis

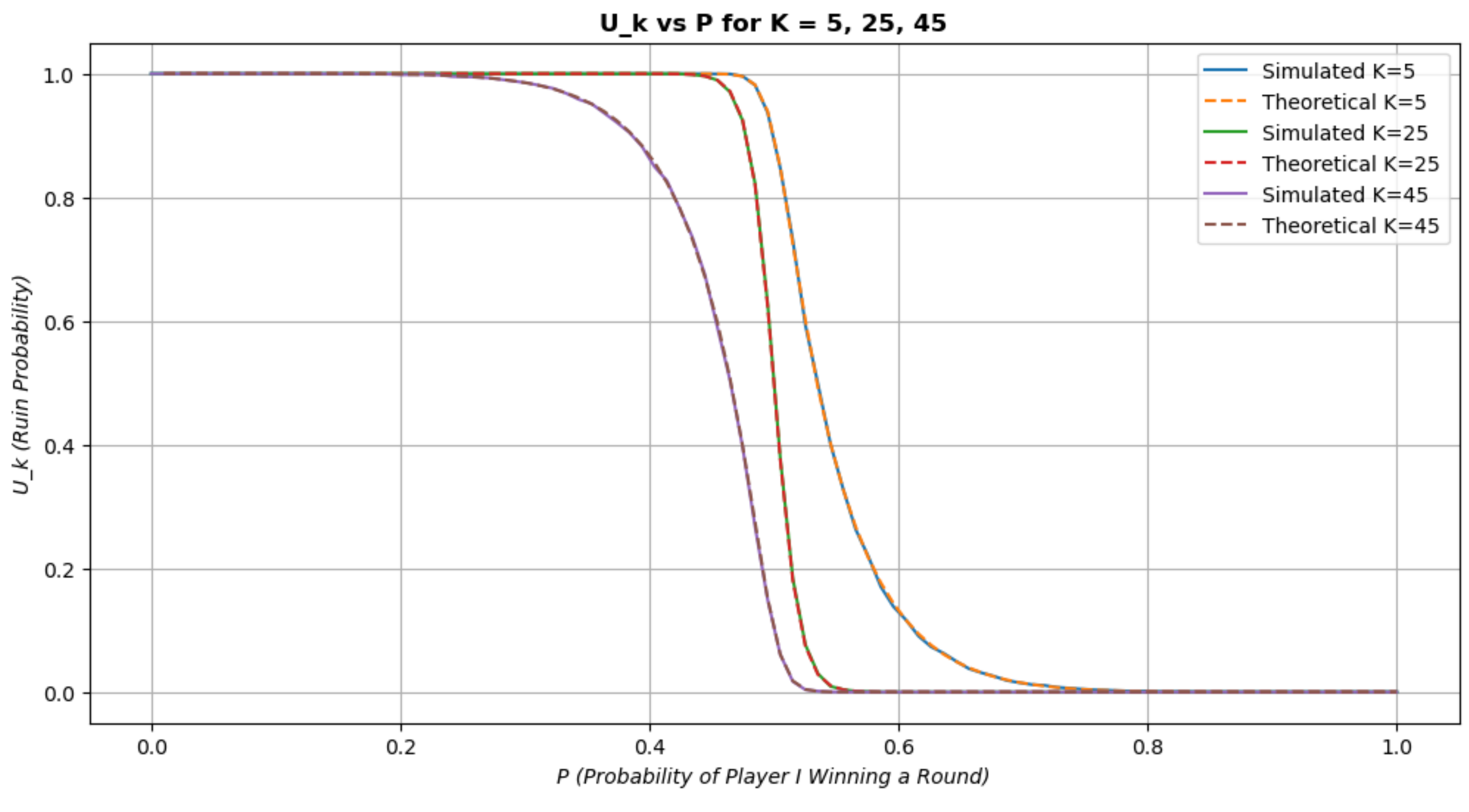
We simulate the gambler's ruin problem for $M = 10000$ sample paths. The simulation involves running multiple rounds where, at each step, player I wins or loses a unit with probability P and q, respectively. The process stops when player I either runs out of units or wins the total stake.

Then, we plot the ruin probability U_k as a function of the probability P for different values of K. The simulation results are compared with the theoretical predictions.

In [7]:
```
# Import necessary libraries
import numpy as np
import matplotlib.pyplot as plt
import warnings
warnings.filterwarnings("ignore")

# Parameters
N = 50  # Total units (sum of both players' units)
M = 10000  # Number of sample paths (simulations)
K_values = [5, 25, 45]  # Values of K to plot for Uk vs P
P_values = [0.25, 0.5, 0.75]  # Values of P to plot for Uk vs K
P_range = np.linspace(0.0, 1, 100) +  1e-10  # Range of P values to plot for U1 vs U2
```

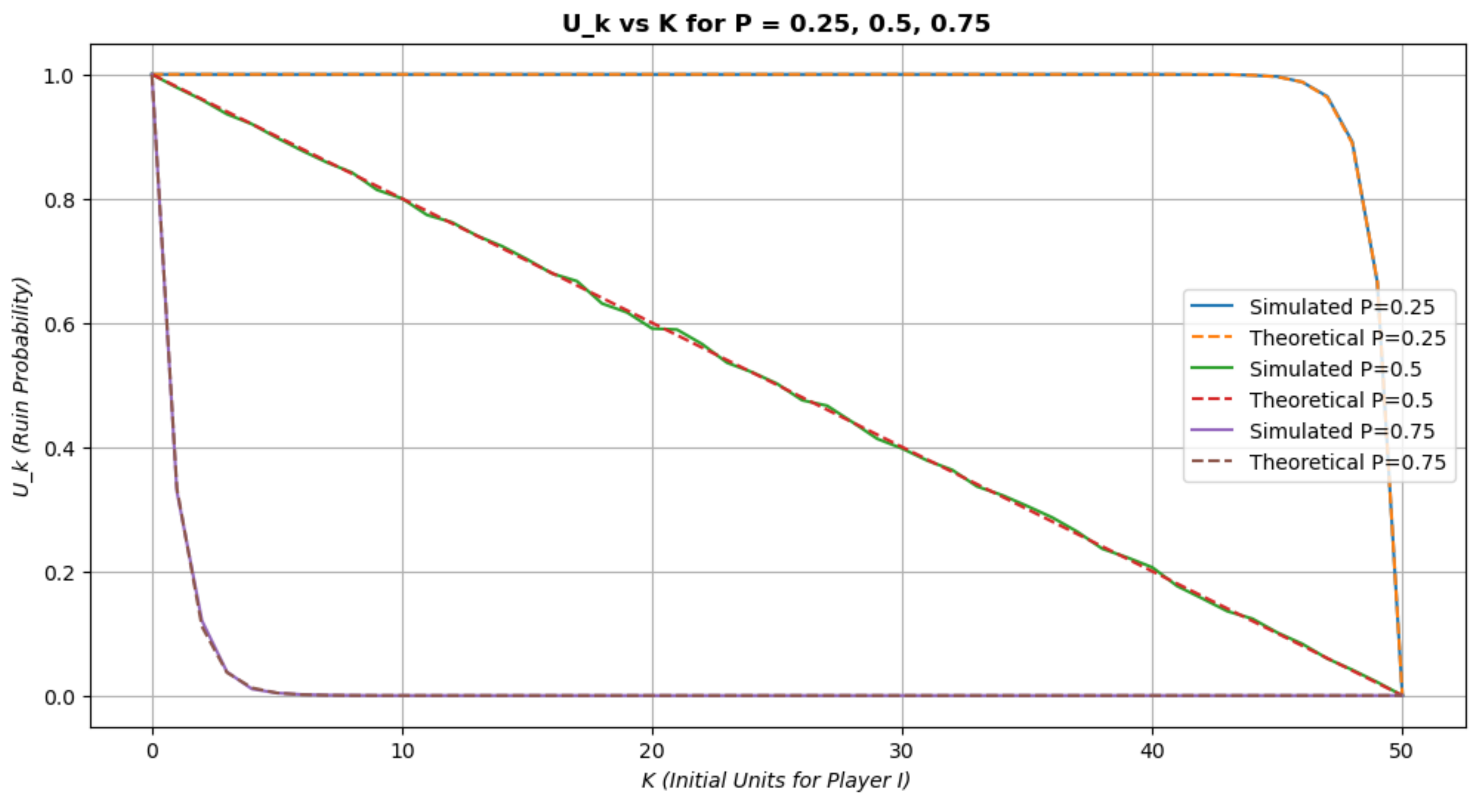
In [8]:
```
# Simulate and plot U_k vs P for different values of K
def simulate_ruin(K, N, P, M):
    """
    Simulate M paths of the gambler's ruin problem.

    Parameters:
    K - Initial units for player I
    N - Total units
    P - Probability that player I wins a round
    M - Number of sample paths (simulations)

    Returns:
    Probability that player I will be ruined (P(X_tau = 0 | X_0 = K))
    """
    wins = 0
    for _ in range(M):
        current_K = K
        while current_K > 0 and current_K < N:
            if np.random.rand() < P:
                current_K += 1  # Player I wins
            else:
                current_K -= 1  # Player D wins
        if current_K == 0:
            wins += 1  # Player I is ruined

    return wins / M


# Plot U_k vs P for different values of K
def theoretical_uk(K, N, P):
    """Theoretical probability of ruin for player I."""
    q = 1 - P
    if P == q:
        return 1 - (K / N)
    else:
        return ((q / P) ** K - (q / P) ** N) / (1 - (q / P) ** N)
```

In [9]:
```
# Plot U_k vs P for K = 5, 25, 45
plt.figure(figsize=(12, 6))
for K in K_values:
    U_k_sim = [simulate_ruin(K, N, P, M) for P in P_range]
    U_k_theory = [theoretical_uk(K, N, P) for P in P_range]
    plt.plot(P_range, U_k_sim, label=f'Simulated K={K}')
    plt.plot(P_range, U_k_theory, '--', label=f'Theoretical K={K}')

plt.title('U_k vs P for K = 5, 25, 45', fontweight='bold')
plt.xlabel('P (Probability of Player I Winning a Round)', fontstyle='italic')
plt.ylabel('U_k (Ruin Probability)', fontstyle='italic')
plt.legend()
plt.grid(True)
plt.show()
```



We next plot the ruin probability U_k as a function of K for three different values of P. Again, the simulation results are compared with the theoretical predictions.

In [13]:
```
# Plot U_k vs K for 3 different P's
plt.figure(figsize=(12, 6))
K_range = np.arange(0, N + 1)
for P in P_values:
    U_k_sim = [simulate_ruin(K, N, P, M) for K in K_range]
    U_k_theory = [theoretical_uk(K, N, P) for K in K_range]
    plt.plot(K_range, U_k_sim, label=f'Simulated P={P}')
    plt.plot(K_range, U_k_theory, '--', label=f'Theoretical P={P}')

plt.title('U_k vs K for P = 0.25, 0.5, 0.75', fontweight='bold')
plt.xlabel('K (Initial Units for Player I)', fontstyle='italic')
plt.ylabel('U_k (Ruin Probability)', fontstyle='italic')
plt.legend()
plt.grid(True)
plt.show()
```



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5. Results Analysis

1. Analysis of U_k vs P:

- For each K, we observe that as the probability P increases (i.e., player I has a higher chance of winning a round), the ruin probability decreases, as expected.
- For smaller K (e.g., $K = 5$), the ruin probability is higher even when P is large, indicating that starting with fewer units makes the player more vulnerable to ruin.
- The theoretical predictions and the simulated results are closely aligned, confirming the validity of the simulation.

2. Analysis of U_k vs K:

- As K increases, the probability of ruin decreases, as player I starts with more units and thus has a better chance of surviving until player D is ruined.
- For $P = 0.5$, the probability of ruin follows a more linear trend, as both players have equal chances of winning.
- For $P = 0.25$ and $P = 0.75$, the curves are steeper, reflecting the greater imbalance in winning probabilities.

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6. Conclusion

The gambler's ruin problem provides a fascinating glimpse into probabilistic decision-making and risk analysis. Through both simulation and theoretical analysis, we observe that:

- The ruin probability U_k is heavily influenced by both the starting units K and the probability P of winning each round.
- Theoretical solutions derived from the difference equation closely match the simulation results, verifying the accuracy of the simulation approach.
- This model could be extended by exploring other variations, such as adding a stopping condition or varying the bet size.

The simulation techniques presented here allow for a deeper understanding of the gambler's ruin problem and its applications in financial modeling, risk management, and decision theory.