1 Formulação do problema

Dados $f : [0,1] \to \mathbb{R}$ e constantes reais $\alpha > 0$ e $\beta \geq 0$, determine $u : [0,1] \to \mathbb{R}$ tal que

$$\begin{cases} -\alpha u_{xx}(x) + \beta u(x) = f(x), & x \in]0,1[, \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Exemplos de solução exata para o problema em (1):

- Ex. 1. Se $f(x) = -2\alpha + \beta x(x-1)$, então $u(x) = x(x-1)$.
- Ex. 2. Se $f(x) = (\alpha\pi^2 + \beta)\sin(\pi x)$, então $u(x) = \sin(\pi x)$.
- Ex. 3. Se $\alpha = 1$, $\beta = 0$ e $f(x) = 8$, então $u(x) = -4x(x-1)$.
- Ex. 4. Se $\alpha = \beta = 1$ e $f(x) = x$, então $u(x) = x + \frac{e^{-x} - e^{x}}{e - e^{-1}}$.

2 Problema aproximado - via método de Galerkin

Dados $f : [0,1] \to \mathbb{R}$ e constantes reais $\alpha > 0$ e $\beta \geq 0$, determine $u_h \in V_m = [\varphi_1, \varphi_2, \ldots, \varphi_m]$ tal que

$$\alpha \int_0^1 \frac{du_h}{dx}(x)\frac{dv_h}{dx}(x)dx + \beta \int_0^1 u_h(x)v_h(x)dx = \int_0^1 f(x)v_h(x)dx, \quad \forall v_h \in V_m. \tag{2}$$

2.1 Formulação matricial

Tomando $u_h(x) = \sum_{j=1}^{m} c_j\varphi_j(x)$ e $v_h = \varphi_i$, para $i = 1, 2, \ldots, m$, em (2), temos a forma matriz-vetor do problema aproximado, isto é, determinar um vetor $c \in \mathbb{R}^m$ tal que

$$Kc = F,$$

em que, para $i,\ j \in \{1, 2, \ldots, m\}$,

$$K_{i,j} = \alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_j}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_j(x)dx \quad \text{e} \quad F_i = \int_0^1 f(x)\varphi_i(x)dx.$$

3 Função base linear por partes

Dado $m \in \mathbb{N}$, considere a discretização uniforme do intervalo $[0,1]$, dada por $0 = x_0 < x_1 < \cdots < x_{m+1} = 1$, onde cada subintervalo tem comprimento $h = 1/(m+1)$.

Para cada $i \in \{1, \ldots, m\}$, defina

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & \forall x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h}, & \forall x \in [x_i, x_{i+1}], \\ 0, & \forall x \notin [x_{i-1}, x_{i+1}]. \end{cases} \qquad \text{e} \qquad \frac{d\varphi_i}{dx}(x) = \begin{cases} \dfrac{1}{h}, & \forall x \in]x_{i-1}, x_i[, \\ -\dfrac{1}{h}, & \forall x \in]x_i, x_{i+1}[, \\ 0, & \forall x \notin]x_{i-1}, x_{i+1}[. \end{cases}$$

Segue o gráfico de cada função φ_i:

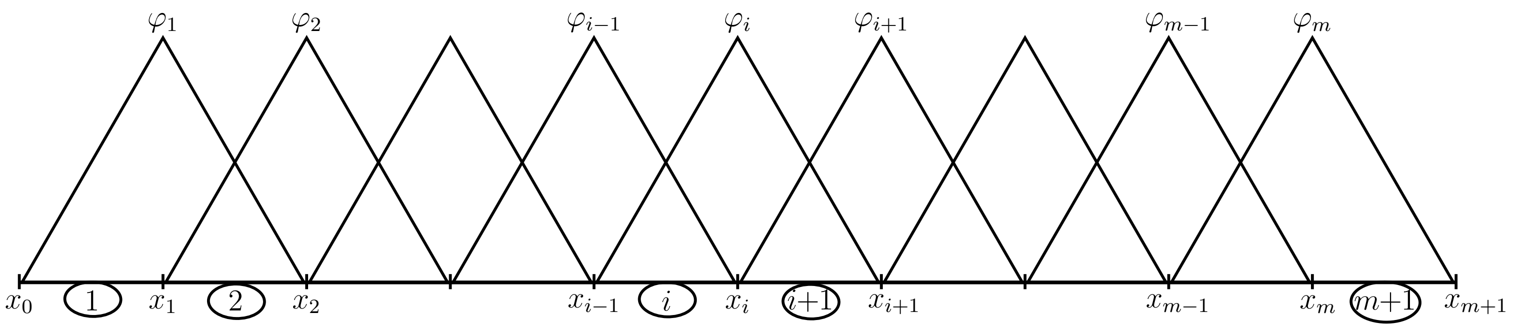


Figura 1: Função base linear por partes.

3.1 Cálculo da matriz K

Nesse caso específico de funções base lineares por parte, os únicos elementos possivelmente não nulos da matriz K são $K_{i,i}$, $K_{i,i+1}$ e $K_{i+1,i}$. Ademais, dado que K é uma matriz simétrica, basta calcular $K_{i,i}$ e $K_{i,i+1}$.

Para os elementos da diagonal, note que

$$
\begin{aligned}
K_{i,i} =&\alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_i}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_i(x)dx \\
=&\alpha\Big[\int_{x_{i-1}}^{x_i} \Big(\frac{1}{h}\Big)^2 dx + \int_{x_i}^{x_{i+1}} \Big(\frac{-1}{h}\Big)^2 dx\Big] + \beta\Big[\int_{x_{i-1}}^{x_i} \Big(\frac{x-x_{i-1}}{h}\Big)^2 dx + \int_{x_i}^{x_{i+1}} \Big(\frac{x_{i+1}-x}{h}\Big)^2 dx\Big] \\
=&\frac{\alpha}{h^2}\Big[\int_{x_{i-1}}^{x_i} dx + \int_{x_i}^{x_{i+1}} dx\Big] + \frac{\beta}{h^2}\Big[\int_{x_{i-1}}^{x_i} (x-x_{i-1})^2 dx + \int_{x_i}^{x_{i+1}} (x_{i+1}-x)^2 dx\Big] \\
=&\frac{2\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{(x-x_{i-1})^3}{3}\Big|_{x_{i-1}}^{x_i} - \frac{(x_{i+1}-x)^3}{3}\Big|_{x_i}^{x_{i+1}}\Big] \\
=&\frac{2\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{h^3}{3} + \frac{h^3}{3}\Big] \\
=&\frac{2\alpha}{h} + \frac{2\beta h}{3}.
\end{aligned}
$$

Quanto aos elementos $K_{i,i+1}$, temos

$$
\begin{aligned}
K_{i,i+1} =&\alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_{i+1}}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_{i+1}(x)dx \\
=&\alpha \int_{x_i}^{x_{i+1}} \frac{d\varphi_i}{dx}(x)\frac{d\varphi_{i+1}}{dx}(x)dx + \beta \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_{i+1}(x)dx \\
=&\alpha \int_{x_i}^{x_{i+1}} \Big(\frac{-1}{h}\Big)\Big(\frac{1}{h}\Big)dx + \beta \int_{x_i}^{x_{i+1}} \Big(\frac{x_{i+1}-x}{h}\Big)\Big(\frac{x-x_i}{h}\Big)dx \\
=&-\frac{\alpha}{h} + \frac{\beta}{h^2}\int_{x_i}^{x_{i+1}} (x_i+h-x)(x-x_i)dx \\
=&-\frac{\alpha}{h} + \frac{\beta}{h^2}\Big[\int_{x_i}^{x_{i+1}} h(x-x_i)dx - \int_{x_i}^{x_{i+1}} (x-x_i)^2 dx\Big] \\
=&-\frac{\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{h(x-x_i)^2}{2}\Big|_{x_i}^{x_{i+1}} - \frac{(x-x_i)^3}{3}\Big|_{x_i}^{x_{i+1}}\Big] \\
=&-\frac{\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{h^3}{2} - \frac{h^3}{3}\Big] \\
=&-\frac{\alpha}{h} + \frac{\beta h}{6}
\end{aligned}
$$

3.2 Cálculo do vetor força F

3.2.1 Caso em que $f(x) = 8$

$$
F_i = \int_0^1 f(x)\varphi_i(x)dx = 8\int_0^1 \varphi_i(x)dx = 8\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx = 8h.
$$

3.2.2 Caso em que $f(x) = x$

$$
\begin{aligned}
F_i = \int_0^1 f(x)\varphi_i(x)dx = &\int_{x_{i-1}}^{x_i} x\frac{x-x_{i-1}}{h}dx + \int_{x_i}^{x_{i+1}} x\frac{x_{i+1}-x}{h}dx \\
=&\frac{1}{h}\int_{x_{i-1}}^{x_i}(x-x_{i-1}+x_{i-1})(x-x_{i-1})dx - \frac{1}{h}\int_{x_i}^{x_{i+1}}(x-x_{i+1}+x_{i+1})(x-x_{i+1})dx \\
=&\frac{1}{h}\int_{x_{i-1}}^{x_i}\Big((x-x_{i-1})^2+x_{i-1}(x-x_{i-1})\Big)dx - \frac{1}{h}\int_{x_i}^{x_{i+1}}\Big((x-x_{i+1})^2+x_{i+1}(x-x_{i+1})\Big)dx \\
=&\frac{1}{h}\Big(\frac{(x-x_{i-1})^3}{3}+x_{i-1}\frac{(x-x_{i-1})^2}{2}\Big)\Big|_{x_{i-1}}^{x_i} - \frac{1}{h}\Big(\frac{(x-x_{i+1})^3}{3}+x_{i+1}\frac{(x-x_{i+1})^2}{2}\Big)\Big|_{x_i}^{x_{i+1}} \\
=&\frac{1}{h}\Big(\frac{h^3}{3}+x_{i-1}\frac{h^2}{2}\Big) - \frac{1}{h}\Big(\frac{h^3}{3}-x_{i+1}\frac{h^2}{2}\Big) \\
=&x_{i-1}\frac{h}{2}+x_{i+1}\frac{h}{2} \\
=&h\frac{x_{i-1}+x_{i+1}}{2} \\
=&hx_i.
\end{aligned}
$$

3.2.3 Caso geral via quadratura gaussiana

$$
\begin{aligned}
F_i = \int_0^1 f(x)\varphi_i(x)dx = &\int_{x_{i-1}}^{x_i} f(x)\frac{x-x_{i-1}}{h}dx + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1}-x}{h}dx \\
=&\int_{-1}^{1} f\big(x(\xi,i)\big)\frac{x(\xi,i)-x_{i-1}}{h}\frac{h}{2}d\xi + \int_{-1}^{1} f\big(x(\xi,i+1)\big)\frac{x_{i+1}-x(\xi,i+1)}{h}\frac{h}{2}d\xi \\
=&\frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i)\big)\frac{\xi+1}{2}d\xi + \frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i+1)\big)\frac{1-\xi}{2}d\xi \\
=&\frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i)\big)\phi_2(\xi)d\xi + \frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i+1)\big)\phi_1(\xi)d\xi \\
\approx&\frac{h}{2}\sum_{j=1}^{n_{pg}} W_j\Big[f\big(x(P_j,i)\big)\phi_2(P_j)+f\big(x(P_j,i+1)\big)\phi_1(P_j)\Big],
\end{aligned}
$$

em que $\{P_1, P_2, \ldots, P_{n_{pg}}\}$ e $\{W_1, W_2, \ldots, W_{n_{pg}}\}$ são os pontos e pesos de gauss.

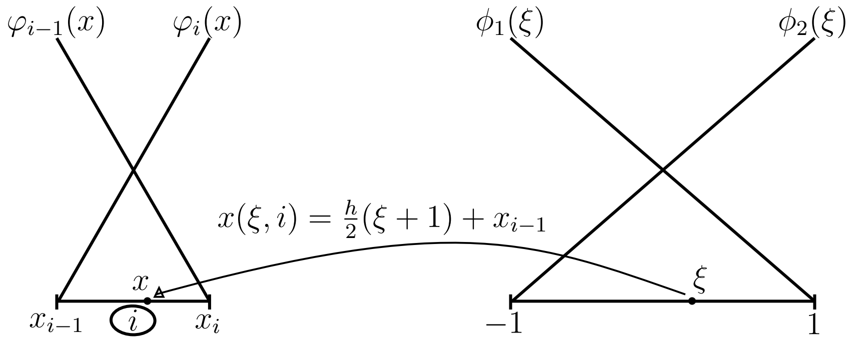


Figura 2: Mudança de variável. $\phi_1(\xi) = (1-\xi)/2$ e $\phi_2(\xi) = (1+\xi)/2$.

3.3 Cálculo do erro

O erro entre a solução exata e aproximada na norma do espaço $L^2(]0,1[)$ é dado por

$$\|u - u_h\| = \sqrt{\int_0^1 |u(x) - u_h(x)|^2 dx}.$$

Dessa forma, denotando $\mathcal{E}_h = \|u - u_h\|$, note que:

$$\begin{aligned}
\mathcal{E}_h^2 &= \int_0^1 |u(x) - u_h(x)|^2 dx \\
&= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} |u(x) - u_h(x)|^2 dx \\
&= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} |u(x) - \sum_{j=1}^{m} c_j \varphi_j(x)|^2 dx \\
&= \int_{x_0}^{x_1} |u(x) - c_1\varphi_1(x)|^2 dx + \sum_{i=2}^{m} \int_{x_{i-1}}^{x_i} |u(x) - c_{i-1}\varphi_{i-1}(x) - c_i\varphi_i(x)|^2 dx + \int_{x_m}^{x_{m+1}} |u(x) - c_m\varphi_m(x)|^2 dx \\
&= \frac{h}{2}\left[\int_{-1}^{1} |u\big(x(\xi,1)\big) - c_1\phi_2(\xi)|^2 d\xi + \sum_{i=2}^{m}\int_{-1}^{1} |u\big(x(\xi,i)\big) - c_{i-1}\phi_1(\xi) - c_i\phi_2(\xi)|^2 d\xi + \int_{-1}^{1} |u\big(x(\xi,m+1)\big) - c_m\phi_1(\xi)|^2 d\xi\right].
\end{aligned}$$

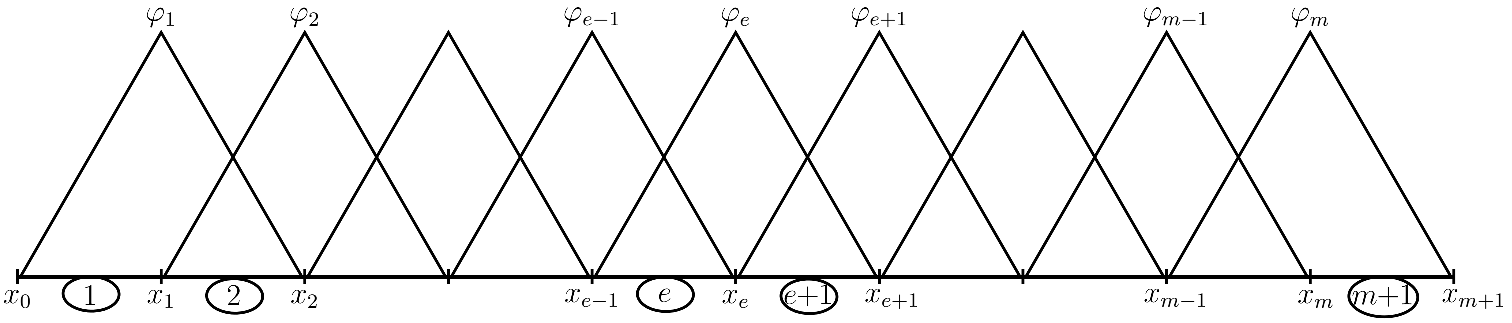
Daí, aplicando quadratura gaussiana, obtemos:

$$\begin{aligned}
\mathcal{E}_h^2 = \frac{h}{2}\Big[&\sum_{j=1}^{n_{pg}} W_j |u\big(x(P_j,1)\big) - c_1\phi_2(P_j)|^2 \\
&+ \sum_{i=2}^{m}\sum_{j=1}^{n_{pg}} W_j |u\big(x(P_j,i)\big) - c_{i-1}\phi_1(P_j) - c_i\phi_2(P_j)|^2 \\
&+ \sum_{j=1}^{n_{pg}} W_j |u\big(x(P_j,m+1)\big) - c_m\phi_1(P_j)|^2 \Big].
\end{aligned}$$

Podemos reescrever a expressão acima de forma mais compacta considerando $\bar{c} = [0; c; 0]$. Nesse caso, temos

$$\mathcal{E}_h^2 = \frac{h}{2} \sum_{i=1}^{m+1}\sum_{j=1}^{n_{pg}} W_j |u\big(x(P_j,i)\big) - \bar{c}_i\phi_1(P_j) - \bar{c}_{i+1}\phi_2(P_j)|^2$$

4 Matriz local e força local



Definimos a matriz global K e o vetor global F como

$$K_{i,j} = \alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_j}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_j(x)dx \quad \text{e} \quad F_i = \int_0^1 f(x)\varphi_i(x)dx,$$

com $i,\, j \in \{1, 2, \dots, m\}$. Com isso, dado que a integral sobre o domínio $[0, 1]$ pode ser escrita como a soma das integrais sobre cada subintervalo, temos que

$$K = \sum_{e=1}^{m+1} \mathcal{K}^e \quad \text{e} \quad F = \sum_{e=1}^{m+1} \mathcal{F}^e,$$

onde,

$$\mathcal{K}^e_{i,j} = \alpha \int_{x_{e-1}}^{x_e} \frac{d\varphi_i}{dx}(x)\frac{d\varphi_j}{dx}(x)dx + \beta \int_{x_{e-1}}^{x_e} \varphi_i(x)\varphi_j(x)dx \quad \text{e} \quad \mathcal{F}^e_i = \int_{x_{e-1}}^{x_e} f(x)\varphi_i(x)dx,$$

para $i,\, j \in \{1, 2, \dots, m\}$.

$\mathcal{K}^e$ e $\mathcal{F}^e$ representam a contribuição local do e-ésimo subintervalo para a matriz global K e para o vetor global F, respectivamente. Observe que, pela definição das funções φ,

$$\mathcal{K}^e_{i,j} = 0, \quad \text{se } i \notin \{e-1, e\} \text{ ou } j \notin \{e-1, e\}, \qquad \text{e} \qquad \mathcal{F}^e_i = 0, \quad \text{se } i \notin \{e-1, e\}.$$

Assim sendo, dado que a maioria das entradas de $\mathcal{K}^e$ e $\mathcal{F}^e$ é zero, é apropriado definir apenas a parte não necessariamente nula de $\mathcal{K}^e$ e $\mathcal{F}^e$. Faremos isso utilizando uma notação local para as funções da base. A relação entre a notação local e global está ilustrada na Figura 3.

$$K^e_{a,b} = \alpha \int_{x_1^e}^{x_2^e} \frac{d\varphi^e_a}{dx}(x)\frac{d\varphi^e_b}{dx}(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi^e_a(x)\varphi^e_b(x)dx \quad \text{e} \quad F^e_a = \int_{x_1^e}^{x_2^e} f(x)\varphi^e_a(x)dx, \quad \text{com} \quad a, b \in \{1, 2\}.$$

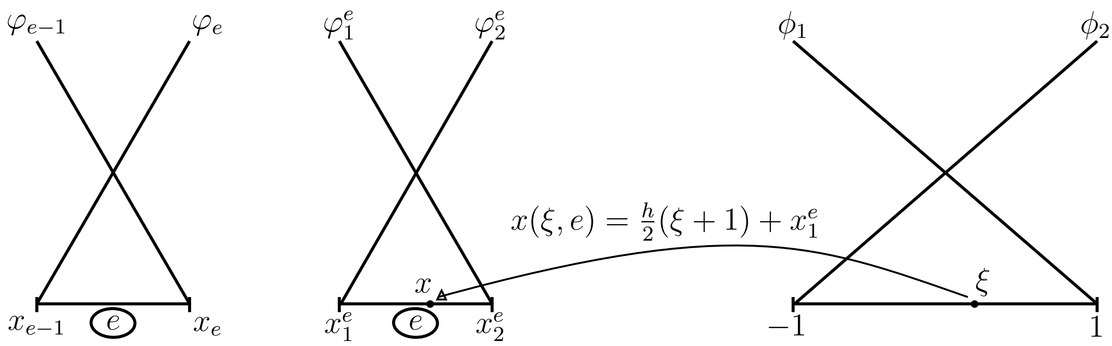


Figura 3: Elemento global, local e padrão referente a base linear.

Podemos construir a matriz global K a partir das matrizes locais K^e observando que:

$$K_{e,e} = K^e_{2,2} + K^{e+1}_{1,1}, \quad \text{para} \quad e \in \{1, 2, \dots, m\},$$

$$K_{e,e+1} = K^{e+1}_{1,2}, \quad \text{para} \quad e \in \{1, 2, \dots, m-1\},$$

$$K_{e+1,e} = K^{e+1}_{2,1}, \quad \text{para} \quad e \in \{1, 2, \dots, m-1\}.$$

Para construir o vetor global F a partir dos vetores locais F^e, basta notar que:

$$F_e = F^e_2 + F^{e+1}_1, \quad \text{para} \quad e \in \{1, 2, \dots, m\}.$$

4.1 Cálculo da matriz local K^e via quadratura gaussiana

$$
\begin{aligned}
K^e_{a,b} &= \alpha \int_{x_1^e}^{x_2^e} \frac{d\varphi_a^e}{dx}(x)\frac{d\varphi_b^e}{dx}(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x)\varphi_b^e(x)dx \\
&= \alpha \int_{-1}^{1} \frac{d\varphi_a^e}{dx}\big(x(\xi,e)\big)\frac{d\varphi_b^e}{dx}\big(x(\xi,e)\big)\frac{h}{2}d\xi + \beta \int_{-1}^{1} \varphi_a^e\big(x(\xi,e)\big)\varphi_b^e\big(x(\xi,e)\big)\frac{h}{2}d\xi \\
&= \alpha \int_{-1}^{1} \frac{d\phi_a}{d\xi}(\xi)\frac{2}{h}\frac{d\phi_b}{d\xi}(\xi)\frac{2}{h}\frac{h}{2}d\xi + \beta \int_{-1}^{1} \phi_a(\xi)\phi_b(\xi)\frac{h}{2}d\xi \\
&= \frac{2\alpha}{h} \int_{-1}^{1} \frac{d\phi_a}{d\xi}(\xi)\frac{d\phi_b}{d\xi}(\xi)d\xi + \frac{\beta h}{2} \int_{-1}^{1} \phi_a(\xi)\phi_b(\xi)d\xi \\
&= \frac{2\alpha}{h} \sum_{j=1}^{n_{pg}} W_j \frac{d\phi_a}{d\xi}(P_j)\frac{d\phi_b}{d\xi}(P_j) + \frac{\beta h}{2} \sum_{j=1}^{n_{pg}} W_j \phi_a(P_j)\phi_b(P_j).
\end{aligned}
$$

Observação: Temos que $\phi_a(\xi) = \varphi_a^e\big(x(\xi,e)\big)$. Derivando ambos os lados dessa igualdade em relação a ξ e aplicando a regra da cadeia no lado direito da igualdade, obtemos $\frac{d\phi_a}{d\xi}(\xi) = \frac{d\varphi_a^e}{dx}\big(x(\xi,e)\big)\frac{dx}{d\xi}(\xi,e)$. Daí, como $\frac{dx}{d\xi}(\xi,e) = h/2$, segue que $\frac{d\varphi_a^e}{dx}\big(x(\xi,e)\big) = \frac{d\phi_a}{d\xi}(\xi)\frac{2}{h}$.

4.2 Cálculo do vetor local F^e via quadratura gaussiana

$$
\begin{aligned}
F^e_a &= \int_{x_1^e}^{x_2^e} f(x)\varphi_a^e(x)dx \\
&= \int_{-1}^{1} f\big(x(\xi,e)\big)\varphi_a^e\big(x(\xi,e)\big)\frac{h}{2}d\xi \\
&= \frac{h}{2}\int_{-1}^{1} f\big(x(\xi,e)\big)\phi_a(\xi)d\xi \\
&\approx \frac{h}{2}\sum_{j=1}^{n_{pg}} W_j f\big(x(P_j,e)\big)\phi_a(P_j).
\end{aligned}
$$

4.3 Montagem da matriz global K a partir da matriz local K^e

Calcula uma matriz local K^e;
Inicializa a matriz global K como $zeros(m,m)$;
for $e = 2 : m$ **do**
    $K[e-1,e-1] += K^e[1,1]$
    $K[e,e-1] += K^e[2,1]$
    $K[e-1,e] += K^e[1,2]$
    $K[e,e] += K^e[2,2]$
end
for *e=1* **do**
    $K[e,e] += K^e[2,2]$
end
for *e=m+1* **do**
    $K[e-1,e-1] += K^e[1,1]$
end

4.4 Montagem do vetor global F a partir do vetor local F^e

```
Inicializa o vetor global F como zeros(m, 1);
for e = 2 : m do
    Calcula o vetor local F^e;
    F[e − 1]+ = F^e[1]
    F[e]+ = F^e[2]
end
for e = 1 do
    Calcula o vetor local F^e;
    F[e]+ = F^e[2]
end
for e = m + 1 do
    Calcula o vetor local F^e;
    F[e − 1]+ = F^e[1]
end
```