

# 1 Formulação do problema

Dados  $f : [0, 1] \rightarrow \mathbb{R}$  e constantes reais positivas  $\alpha$  e  $\beta$ , determine  $u : [0, 1] \rightarrow \mathbb{R}$  tal que

$$\begin{cases} -\alpha u_{xx}(x) + \beta u(x) = f(x), & x \in ]0, 1[, \\ u(0) = u(1) = 0. \end{cases} \quad (1)$$

Exemplos de solução exata para o problema em (1)

- Ex. 1. Se  $f(x) = -2\alpha + \beta x(x - 1)$ , então  $u(x) = x(x - 1)$ .
- Ex. 2. Se  $f(x) = (\alpha\pi^2 + \beta) \sin(\pi x)$ , então  $u(x) = \sin(\pi x)$ .
- Ex. 3. Se  $\alpha = 1$ ,  $\beta = 0$  e  $f(x) = 8$ , então  $u(x) = -4x(x - 1)$ .
- Ex. 4. Se  $\alpha = \beta = 1$  e  $f(x) = x$ , então  $u(x) = x + \frac{e^{-x} - e^x}{e - e^{-1}}$ .

## 2 Problema aproximado - via método de Galerkin

Dados  $f : [0, 1] \rightarrow \mathbb{R}$  e constantes reais positivas  $\alpha$  e  $\beta$ , determine  $u_h \in V_m = [\varphi_1, \varphi_2, \dots, \varphi_m]$  tal que

$$\alpha \int_0^1 \frac{du_h}{dx}(x) \frac{dv_h}{dx}(x) dx + \beta \int_0^1 u_h(x) v_h(x) dx = \int_0^1 f(x) v_h(x) dx, \quad \forall v_h \in V_m. \quad (2)$$

### 2.1 Formulação matricial

Tomando  $u_h(x) = \sum_{j=1}^m c_j \varphi_j(x)$  e  $v_h = \varphi_i$ , para  $i = 1, 2, \dots, m$ , em (2), temos a forma matriz-vetor do problema aproximado, isto é, determinar um vetor  $c \in \mathbb{R}^m$  tal que

$$Kc = F,$$

em que, para  $i, j \in \{1, 2, \dots, m\}$ ,

$$K_{i,j} = \alpha \int_0^1 \frac{d\varphi_i}{dx}(x) \frac{d\varphi_j}{dx}(x) dx + \beta \int_0^1 \varphi_i(x) \varphi_j(x) dx \quad \text{e} \quad F_i = \int_0^1 f(x) \varphi_i(x) dx.$$

## 3 Função base linear por partes

Dado  $m \in \mathbb{N}$ , considere  $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = 1$  uma partição uniforme de  $[0, 1]$ , ou seja, o diâmetro de cada elemento é dado por

$$h = 1/(m + 1).$$

Para cada  $i \in \{1, \dots, m\}$ , defina

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \forall x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & \forall x \in [x_i, x_{i+1}], \\ 0, & \forall x \notin [x_{i-1}, x_{i+1}]. \end{cases} \quad \text{e} \quad \frac{d\varphi_i}{dx}(x) = \begin{cases} \frac{1}{h}, & \forall x \in ]x_{i-1}, x_i[, \\ -\frac{1}{h}, & \forall x \in ]x_i, x_{i+1}[, \\ 0, & \forall x \notin ]x_{i-1}, x_{i+1}[. \end{cases}$$

Segue o gráfico de cada função  $\varphi_i$ :

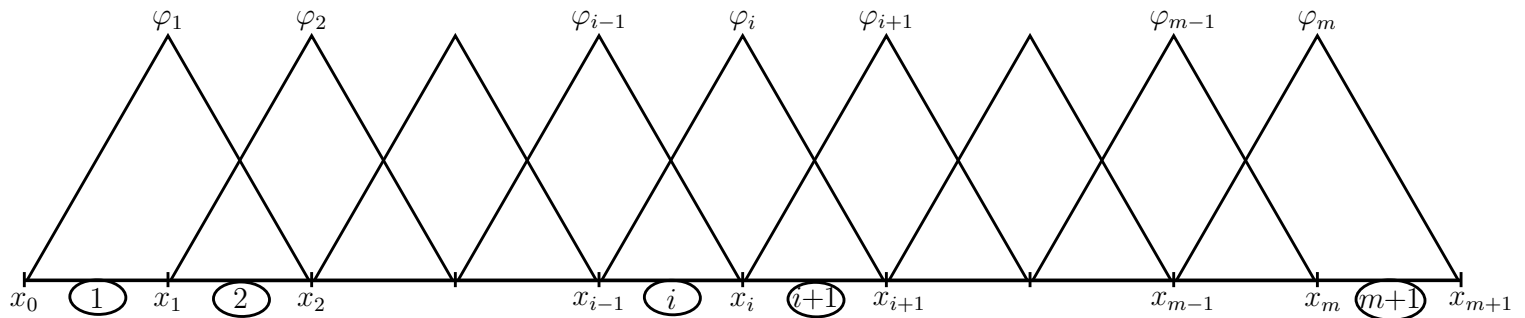


Figura 1: Função base linear por partes.

### 3.1 Cálculo da matriz $K$

Nesse caso específico de funções base lineares por parte, os únicos elementos possivelmente não nulos da matriz  $K$  são  $K_{i,i}$ ,  $K_{i,i+1}$  e  $K_{i-1,i}$ . Ademais, dado que  $K$  é uma matriz simétrica, basta calcular  $K_{i,i}$  e  $K_{i,i+1}$ .

Para os elementos da diagonal, note que

$$\begin{aligned} K_{i,i} &= \alpha \int_0^1 \frac{d\varphi_i}{dx}(x) \frac{d\varphi_i}{dx}(x) dx + \beta \int_0^1 \varphi_i(x) \varphi_i(x) dx \\ &= \alpha \left[ \int_{x_{i-1}}^{x_i} \left( \frac{1}{h} \right)^2 dx + \int_{x_i}^{x_{i+1}} \left( \frac{-1}{h} \right)^2 dx \right] + \beta \left[ \int_{x_{i-1}}^{x_i} \left( \frac{x - x_{i-1}}{h} \right)^2 dx + \int_{x_i}^{x_{i+1}} \left( \frac{x_{i+1} - x}{h} \right)^2 dx \right] \\ &= \frac{\alpha}{h^2} \left[ \int_{x_{i-1}}^{x_i} dx + \int_{x_i}^{x_{i+1}} dx \right] + \frac{\beta}{h^2} \left[ \int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 dx + \int_{x_i}^{x_{i+1}} (x_{i+1} - x)^2 dx \right] \\ &= \frac{2\alpha}{h} + \frac{\beta}{h^2} \left[ \frac{(x - x_{i-1})^3}{3} \Big|_{x_{i-1}}^{x_i} - \frac{(x_{i+1} - x)^3}{3} \Big|_{x_i}^{x_{i+1}} \right] \\ &= \frac{2\alpha}{h} + \frac{\beta}{h^2} \left[ \frac{h^3}{3} + \frac{h^3}{3} \right] \\ &= \frac{2\alpha}{h} + \frac{2\beta h}{3}. \end{aligned}$$

Quanto aos elementos  $K_{i,i+1}$ , temos

$$\begin{aligned} K_{i,i+1} &= \alpha \int_0^1 \frac{d\varphi_i}{dx}(x) \frac{d\varphi_{i+1}}{dx}(x) dx + \beta \int_0^1 \varphi_i(x) \varphi_{i+1}(x) dx \\ &= \alpha \int_{x_i}^{x_{i+1}} \frac{d\varphi_i}{dx}(x) \frac{d\varphi_{i+1}}{dx}(x) dx + \beta \int_{x_i}^{x_{i+1}} \varphi_i(x) \varphi_{i+1}(x) dx \\ &= \alpha \int_{x_i}^{x_{i+1}} \left( \frac{-1}{h} \right) \left( \frac{1}{h} \right) dx + \beta \int_{x_i}^{x_{i+1}} \left( \frac{x_{i+1} - x}{h} \right) \left( \frac{x - x_i}{h} \right) dx \\ &= -\frac{\alpha}{h} + \frac{\beta}{h^2} \int_{x_i}^{x_{i+1}} (x_i + h - x)(x - x_i) dx \\ &= -\frac{\alpha}{h} + \frac{\beta}{h^2} \left[ \int_{x_i}^{x_{i+1}} h(x - x_i) dx - \int_{x_i}^{x_{i+1}} (x - x_i)^2 dx \right] \\ &= -\frac{\alpha}{h} + \frac{\beta}{h^2} \left[ \frac{h(x - x_i)^2}{2} \Big|_{x_i}^{x_{i+1}} - \frac{(x - x_i)^3}{3} \Big|_{x_i}^{x_{i+1}} \right] \\ &= -\frac{\alpha}{h} + \frac{\beta}{h^2} \left[ \frac{h^3}{2} - \frac{h^3}{3} \right] \\ &= -\frac{\alpha}{h} + \frac{\beta h}{6} \end{aligned}$$

### 3.2 Cálculo do vetor força $F$

#### 3.2.1 Caso em que $f(x) = 8$

$$F_i = \int_0^1 f(x) \varphi_i(x) dx = 8 \int_0^1 \varphi_i(x) dx = 8 \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) dx = 8h.$$

### 3.2.2 Caso em que $f(x) = x$

$$\begin{aligned}
F_i &= \int_0^1 f(x) \varphi_i(x) dx = \int_{x_{i-1}}^{x_i} x \frac{x - x_{i-1}}{h} dx + \int_{x_i}^{x_{i+1}} x \frac{x_{i+1} - x}{h} dx \\
&= \frac{1}{h} \int_{x_{i-1}}^{x_i} (x - x_{i-1} + x_{i-1})(x - x_{i-1}) dx - \frac{1}{h} \int_{x_i}^{x_{i+1}} (x - x_{i+1} + x_{i+1})(x - x_{i+1}) dx \\
&= \frac{1}{h} \int_{x_{i-1}}^{x_i} \left( (x - x_{i-1})^2 + x_{i-1}(x - x_{i-1}) \right) dx - \frac{1}{h} \int_{x_i}^{x_{i+1}} \left( (x - x_{i+1})^2 + x_{i+1}(x - x_{i+1}) \right) dx \\
&= \frac{1}{h} \left( \frac{(x - x_{i-1})^3}{3} + x_{i-1} \frac{(x - x_{i-1})^2}{2} \right) \Big|_{x_{i-1}}^{x_i} - \frac{1}{h} \left( \frac{(x - x_{i+1})^3}{3} + x_{i+1} \frac{(x - x_{i+1})^2}{2} \right) \Big|_{x_i}^{x_{i+1}} \\
&= \frac{1}{h} \left( \frac{h^3}{3} + x_{i-1} \frac{h^2}{2} \right) - \frac{1}{h} \left( \frac{h^3}{3} - x_{i+1} \frac{h^2}{2} \right) \\
&= x_{i-1} \frac{h}{2} + x_{i+1} \frac{h}{2} \\
&= h \frac{x_{i-1} + x_{i+1}}{2} \\
&= hx_i.
\end{aligned}$$

### 3.2.3 Caso geral usando quadratura gaussiana

$$\begin{aligned}
F_i &= \int_0^1 f(x) \varphi_i(x) dx = \int_{x_{i-1}}^{x_i} f(x) \frac{x - x_{i-1}}{h} dx + \int_{x_i}^{x_{i+1}} f(x) \frac{x_{i+1} - x}{h} dx \\
&= \int_{-1}^1 f(x(\xi, i)) \frac{x(\xi, i) - x_{i-1}}{h} \frac{h}{2} d\xi + \int_{-1}^1 f(x(\xi, i+1)) \frac{x_{i+1} - x(\xi, i+1)}{h} \frac{h}{2} d\xi \\
&= \frac{h}{2} \int_{-1}^1 f(x(\xi, i)) \frac{\xi + 1}{2} d\xi + \frac{h}{2} \int_{-1}^1 f(x(\xi, i+1)) \frac{1 - \xi}{2} d\xi \\
&= \frac{h}{2} \int_{-1}^1 f(x(\xi, i)) \varphi_1(\xi) d\xi + \frac{h}{2} \int_{-1}^1 f(x(\xi, i+1)) \varphi_2(\xi) d\xi \\
&\approx \frac{h}{2} \sum_{j=1}^{N_{pg}} W_j \left[ f(x(P_j, i)) \varphi_1(P_j) + f(x(P_j, i+1)) \varphi_2(P_j) \right],
\end{aligned}$$

em que  $\{P_1, P_2, \dots, P_{N_{pg}}\}$  e  $\{W_1, W_2, \dots, W_{N_{pg}}\}$  são os pontos e pesos de gauss.

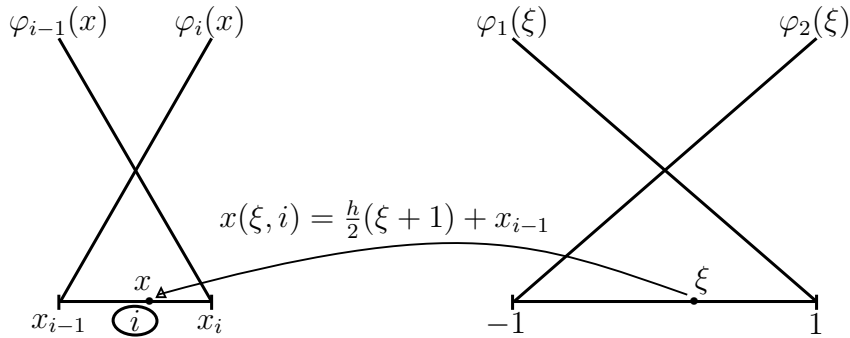


Figura 2: Mudança de variável.  $\varphi_1(\xi) = (\xi + 1)/2$  e  $\varphi_2(\xi) = (1 - \xi)/2$ .

### 3.3 Cálculo do erro

O erro entre a solução exata e aproximada na norma do espaço  $L^2(]0, 1[)$  é dado por

$$\|u - u_h\| = \sqrt{\int_0^1 |u(x) - u_h(x)|^2 dx}.$$

Dessa forma, denotando  $\mathcal{E}_h = \|u - u_h\|$ , note que:

$$\begin{aligned} \mathcal{E}_h^2 &= \int_0^1 |u(x) - u_h(x)|^2 dx \\ &= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} |u(x) - u_h(x)|^2 dx \\ &= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} |u(x) - \sum_{j=1}^m c_j \varphi_j(x)|^2 dx \\ &= \int_{x_0}^{x_1} |u(x) - c_1 \varphi_1(x)|^2 dx + \sum_{i=2}^m \int_{x_{i-1}}^{x_i} |u(x) - c_{i-1} \varphi_{i-1}(x) - c_i \varphi_i(x)|^2 dx + \int_{x_m}^{x_{m+1}} |u(x) - c_m \varphi_m(x)|^2 dx \\ &= \frac{h}{2} \left[ \int_{-1}^1 |u(x(\xi, 1)) - c_1 \varphi_2(\xi)|^2 d\xi + \sum_{i=2}^m \int_{-1}^1 |u(x(\xi, i)) - c_{i-1} \varphi_1(\xi) - c_i \varphi_2(\xi)|^2 d\xi + \int_{-1}^1 |u(x(\xi, m+1)) - c_m \varphi_1(\xi)|^2 d\xi \right] \end{aligned}$$

Daí, aplicando quadratura Gaussiana, obtemos:

$$\begin{aligned} \mathcal{E}_h^2 &= \frac{h}{2} \left[ \sum_{j=1}^{N_{pg}} W_j |u(x(P_j, 1)) - c_1 \varphi_2(P_j)|^2 \right. \\ &\quad + \sum_{i=2}^m \sum_{j=1}^{N_{pg}} W_j |u(x(P_j, i)) - c_{i-1} \varphi_1(P_j) - c_i \varphi_2(P_j)|^2 \\ &\quad \left. + \sum_{j=1}^{N_{pg}} W_j |u(x(P_j, m+1)) - c_m \varphi_1(P_j)|^2 \right]. \end{aligned}$$