1 Formulação do problema

Dados $f : [0, 1] \to \mathbb{R}$ e constantes reais positivas α e β, determine $u : [0, 1] \to \mathbb{R}$ tal que

$$\begin{cases} -\alpha u_{xx}(x) + \beta u(x) = f(x), & x \in]0, 1[, \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Exemplos de solução exata para o problema em (1)

- Ex. 1. Se $f(x) = -2\alpha + \beta x(x-1)$, então $u(x) = x(x-1)$.
- Ex. 2. Se $f(x) = (\alpha\pi^2 + \beta)\sin(\pi x)$, então $u(x) = \sin(\pi x)$.
- Ex. 3. Se $\alpha = 1$, $\beta = 0$ e $f(x) = 8$, então $u(x) = -4x(x-1)$.
- Ex. 4. Se $\alpha = \beta = 1$ e $f(x) = x$, então $u(x) = x + \frac{e^{-x}-e^{x}}{e-e^{-1}}$.

2 Problema aproximado - via método de Galerkin

Dados $f : [0, 1] \to \mathbb{R}$ e constantes reais positivas α e β, determine $u_h \in V_m = [\varphi_1, \varphi_2, \dots, \varphi_m]$ tal que

$$\alpha \int_0^1 \frac{du_h}{dx}(x)\frac{dv_h}{dx}(x)dx + \beta \int_0^1 u_h(x)v_h(x)dx = \int_0^1 f(x)v_h(x)dx, \quad \forall v_h \in V_m. \tag{2}$$

2.1 Formulação matricial

Tomando $u_h(x) = \sum_{j=1}^{m} c_j\varphi_j(x)$ e $v_h = \varphi_i$, para $i = 1, 2, \dots, m$, em (2), temos a forma matriz-vetor do problema aproximado, isto é, determinar um vetor $c \in \mathbb{R}^m$ tal que

$$Kc = F,$$

em que, para $i,\ j \in \{1, 2, \dots, m\}$,

$$K_{i,j} = \alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_j}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_j(x)dx \quad \text{e} \quad F_i = \int_0^1 f(x)\varphi_i(x)dx.$$

3 Função base linear por partes

Dado $m \in \mathbb{N}$, considere $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ uma partição uniforme de $[0, 1]$, ou seja, o diâmetro de cada elemento é dado por

$$h = 1/(m+1).$$

Para cada $i \in \{1, \dots, m\}$, defina

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & \forall x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h}, & \forall x \in [x_i, x_{i+1}], \\ 0, & \forall x \notin [x_{i-1}, x_{i+1}]. \end{cases} \qquad \text{e} \qquad \frac{d\varphi_i}{dx}(x) = \begin{cases} \dfrac{1}{h}, & \forall x \in]x_{i-1}, x_i[, \\ -\dfrac{1}{h}, & \forall x \in]x_i, x_{i+1}[, \\ 0, & \forall x \notin]x_{i-1}, x_{i+1}[. \end{cases}$$

Segue o gráfico de cada função φ_i:

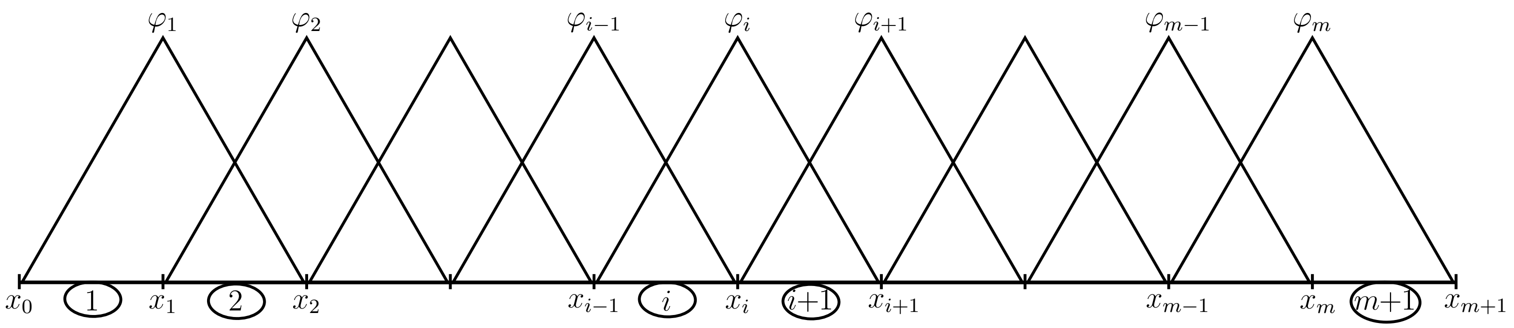

Figura 1: Função base linear por partes.

3.1 Cálculo da matriz K

Nesse caso específico de funções base lineares por parte, os únicos elementos possivelmente não nulos da matriz K são $K_{i,i}$, $K_{i,i+1}$ e $K_{i-1,i}$. Ademais, dado que K é uma matriz simétrica, basta calcular $K_{i,i}$ e $K_{i,i+1}$.

Para os elementos da diagonal, note que

$$\begin{aligned}
K_{i,i} =& \alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_i}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_i(x)dx \\
=& \alpha\Big[\int_{x_{i-1}}^{x_i} \Big(\frac{1}{h}\Big)^2 dx + \int_{x_i}^{x_{i+1}} \Big(\frac{-1}{h}\Big)^2 dx\Big] + \beta\Big[\int_{x_{i-1}}^{x_i} \Big(\frac{x-x_{i-1}}{h}\Big)^2 dx + \int_{x_i}^{x_{i+1}} \Big(\frac{x_{i+1}-x}{h}\Big)^2 dx\Big] \\
=& \frac{\alpha}{h^2}\Big[\int_{x_{i-1}}^{x_i} dx + \int_{x_i}^{x_{i+1}} dx\Big] + \frac{\beta}{h^2}\Big[\int_{x_{i-1}}^{x_i} (x-x_{i-1})^2 dx + \int_{x_i}^{x_{i+1}} (x_{i+1}-x)^2 dx\Big] \\
=& \frac{2\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{(x-x_{i-1})^3}{3}\Big|_{x_{i-1}}^{x_i} - \frac{(x_{i+1}-x)^3}{3}\Big|_{x_i}^{x_{i+1}}\Big] \\
=& \frac{2\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{h^3}{3} + \frac{h^3}{3}\Big] \\
=& \frac{2\alpha}{h} + \frac{2\beta h}{3}.
\end{aligned}$$

Quanto aos elementos $K_{i,i+1}$, temos

$$\begin{aligned}
K_{i,i+1} =& \alpha \int_0^1 \frac{d\varphi_i}{dx}(x)\frac{d\varphi_{i+1}}{dx}(x)dx + \beta \int_0^1 \varphi_i(x)\varphi_{i+1}(x)dx \\
=& \alpha \int_{x_i}^{x_{i+1}} \frac{d\varphi_i}{dx}(x)\frac{d\varphi_{i+1}}{dx}(x)dx + \beta \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_{i+1}(x)dx \\
=& \alpha \int_{x_i}^{x_{i+1}} \Big(\frac{-1}{h}\Big)\Big(\frac{1}{h}\Big)dx + \beta \int_{x_i}^{x_{i+1}} \Big(\frac{x_{i+1}-x}{h}\Big)\Big(\frac{x-x_i}{h}\Big)dx \\
=& -\frac{\alpha}{h} + \frac{\beta}{h^2}\int_{x_i}^{x_{i+1}} (x_i+h-x)(x-x_i)dx \\
=& -\frac{\alpha}{h} + \frac{\beta}{h^2}\Big[\int_{x_i}^{x_{i+1}} h(x-x_i)dx - \int_{x_i}^{x_{i+1}} (x-x_i)^2 dx\Big] \\
=& -\frac{\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{h(x-x_i)^2}{2}\Big|_{x_i}^{x_{i+1}} - \frac{(x-x_i)^3}{3}\Big|_{x_i}^{x_{i+1}}\Big] \\
=& -\frac{\alpha}{h} + \frac{\beta}{h^2}\Big[\frac{h^3}{2} - \frac{h^3}{3}\Big] \\
=& -\frac{\alpha}{h} + \frac{\beta h}{6}
\end{aligned}$$

3.2 Cálculo do vetor força F

3.2.1 Caso em que $f(x) = 8$

$$F_i = \int_0^1 f(x)\varphi_i(x)dx = 8\int_0^1 \varphi_i(x)dx = 8\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx = 8h.$$

3.2.2 Caso em que $f(x) = x$

$$
\begin{aligned}
F_i = \int_0^1 f(x)\varphi_i(x)dx &= \int_{x_{i-1}}^{x_i} x\frac{x-x_{i-1}}{h}dx + \int_{x_i}^{x_{i+1}} x\frac{x_{i+1}-x}{h}dx \\
&=\frac{1}{h}\int_{x_{i-1}}^{x_i}(x-x_{i-1}+x_{i-1})(x-x_{i-1})dx - \frac{1}{h}\int_{x_i}^{x_{i+1}}(x-x_{i+1}+x_{i+1})(x-x_{i+1})dx \\
&=\frac{1}{h}\int_{x_{i-1}}^{x_i}\Big((x-x_{i-1})^2 + x_{i-1}(x-x_{i-1})\Big)dx - \frac{1}{h}\int_{x_i}^{x_{i+1}}\Big((x-x_{i+1})^2 + x_{i+1}(x-x_{i+1})\Big)dx \\
&=\frac{1}{h}\Big(\frac{(x-x_{i-1})^3}{3} + x_{i-1}\frac{(x-x_{i-1})^2}{2}\Big)\Big|_{x_{i-1}}^{x_i} - \frac{1}{h}\Big(\frac{(x-x_{i+1})^3}{3} + x_{i+1}\frac{(x-x_{i+1})^2}{2}\Big)\Big|_{x_i}^{x_{i+1}} \\
&=\frac{1}{h}\Big(\frac{h^3}{3} + x_{i-1}\frac{h^2}{2}\Big) - \frac{1}{h}\Big(\frac{h^3}{3} - x_{i+1}\frac{h^2}{2}\Big) \\
&=x_{i-1}\frac{h}{2} + x_{i+1}\frac{h}{2} \\
&=h\frac{x_{i-1}+x_{i+1}}{2} \\
&=hx_i.
\end{aligned}
$$

3.2.3 Caso geral usando quadratura gaussiana

$$
\begin{aligned}
F_i = \int_0^1 f(x)\varphi_i(x)dx &= \int_{x_{i-1}}^{x_i} f(x)\frac{x-x_{i-1}}{h}dx + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1}-x}{h}dx \\
&= \int_{-1}^{1} f\big(x(\xi,i)\big)\frac{x(\xi,i)-x_{i-1}}{h}\frac{h}{2}d\xi + \int_{-1}^{1} f\big(x(\xi,i+1)\big)\frac{x_{i+1}-x(\xi,i+1)}{h}\frac{h}{2}d\xi \\
&=\frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i)\big)\frac{\xi+1}{2}d\xi + \frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i+1)\big)\frac{1-\xi}{2}d\xi \\
&=\frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i)\big)\varphi_2(\xi)d\xi + \frac{h}{2}\int_{-1}^{1} f\big(x(\xi,i+1)\big)\varphi_1(\xi)d\xi \\
&\approx\frac{h}{2}\sum_{j=1}^{N_{pg}} W_j\Big[f\big(x(P_j,i)\big)\varphi_2(P_j) + f\big(x(P_j,i+1)\big)\varphi_1(P_j)\Big],
\end{aligned}
$$

em que $\{P_1, P_2, \ldots, P_{N_{pg}}\}$ e $\{W_1, W_2, \ldots, W_{N_{pg}}\}$ são os pontos e pesos de gauss.

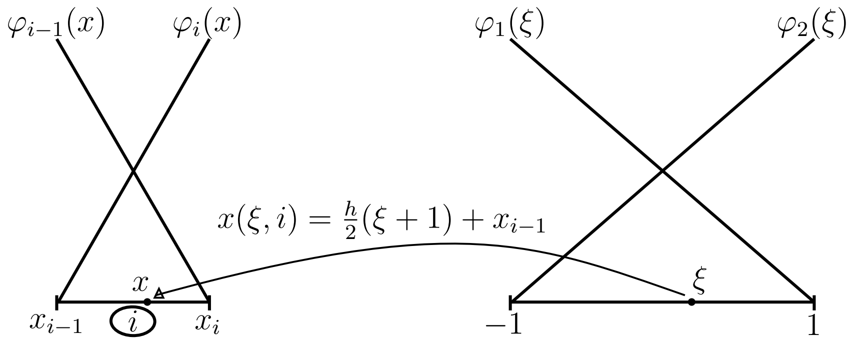

Figura 2: Mudança de variável. $\varphi_1(\xi) = (1-\xi)/2$ e $\varphi_2(\xi) = (1+\xi)/2$.

3.3 Cálculo do erro

O erro entre a solução exata e aproximada na norma do espaço $L^2(]0,1[)$ é dado por

$$\|u-u_h\| = \sqrt{\int_0^1 |u(x)-u_h(x)|^2 dx}.$$

Dessa forma, denotando $\mathcal{E}_h = \|u-u_h\|$, note que:

$$\begin{aligned}
\mathcal{E}_h^2 &= \int_0^1 |u(x)-u_h(x)|^2 dx \\
&= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} |u(x)-u_h(x)|^2 dx \\
&= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} |u(x)-\sum_{j=1}^{m} c_j\varphi_j(x)|^2 dx \\
&= \int_{x_0}^{x_1} |u(x)-c_1\varphi_1(x)|^2 dx + \sum_{i=2}^{m} \int_{x_{i-1}}^{x_i} |u(x)-c_{i-1}\varphi_{i-1}(x)-c_i\varphi_i(x)|^2 dx + \int_{x_m}^{x_{m+1}} |u(x)-c_m\varphi_m(x)|^2 dx \\
&= \frac{h}{2}\left[\int_{-1}^{1} |u\big(x(\xi,1)\big)-c_1\varphi_2(\xi)|^2 d\xi + \sum_{i=2}^{m}\int_{-1}^{1} |u(x(\xi,i))-c_{i-1}\varphi_1(\xi)-c_i\varphi_2(\xi)|^2 d\xi + \int_{-1}^{1} |u(x(\xi,m+1))-c_m\varphi_1(\xi)|^2 d\xi\right]
\end{aligned}$$

Daí, aplicando quadratura Gaussiana, obtemos:

$$\begin{aligned}
\mathcal{E}_h^2 =& \frac{h}{2}\Big[\sum_{j=1}^{N_{pg}} W_j |u\big(x(P_j,1)\big)-c_1\varphi_2(P_j)|^2 \\
&+ \sum_{i=2}^{m}\sum_{j=1}^{N_{pg}} W_j |u\big(x(P_j,i)\big)-c_{i-1}\varphi_1(P_j)-c_i\varphi_2(P_j)|^2 \\
&+ \sum_{j=1}^{N_{pg}} W_j |u\big(x(P_j,m+1)\big)-c_m\varphi_1(P_j)|^2\Big].
\end{aligned}$$