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Title

- Text

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Exercise 11: Numerical Integration IV

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Outline

1. Information
2. Goals
3. Theory/ Recap
4. Exercises

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Information

General

- Lecture material & problem sets available here
- Tutorial material available here

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Goals

Goals of Today

- Understand numerical integration for more dimensions
- Understand why conventional approaches might be problematic
- Know about probability basics
- Understand Monte Carlo quadrature

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Theory / Recap

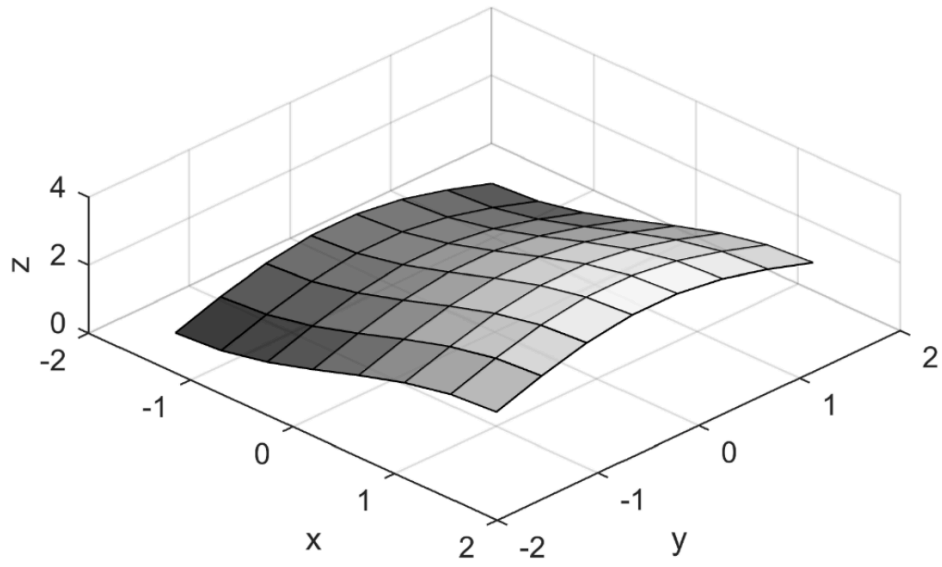
Multivariate numerical integration

- The exact integral:

$$I = \int_{\Omega} f(\mathbf{x}) d\mathbf{x}$$

- Approximation:

$$I \approx \sum w_i f(\mathbf{x}_i)$$

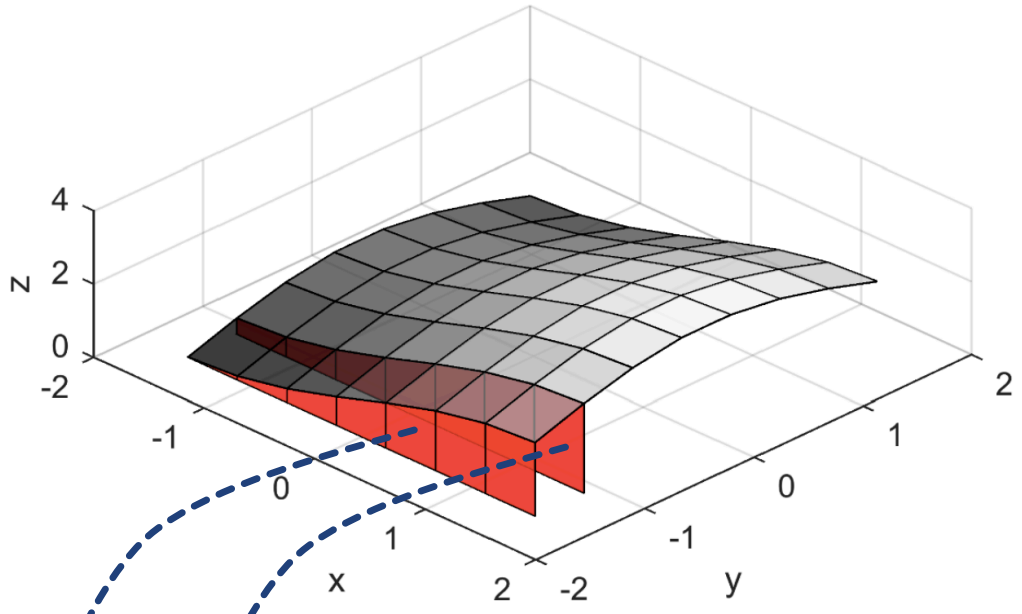


Example 1: Find weights

- Write down the weights for each $f_{i,j}$ resulting from the multivariate trapezoidal rule

- Griven:
  - 3x3 grid (matrix indexing)
  - $h = 1$ (all directions)

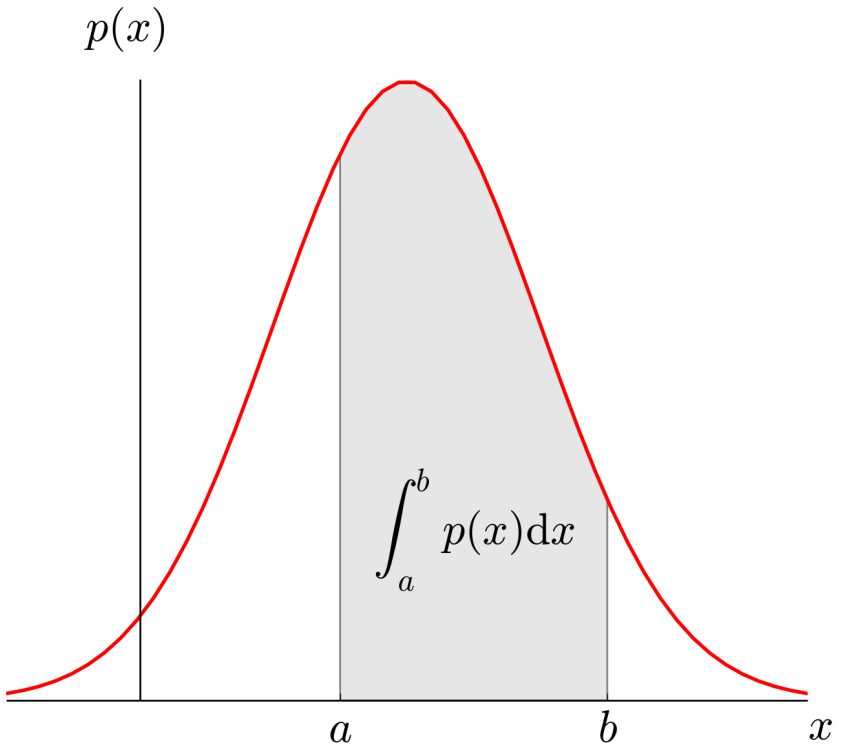
- How it‘s done:
  - Compute area of each slice (with tr. rule)
  - Apply tr. rule to the areas to get volume
  - Find the weights



$$\frac{h}{2}(A_1 + A_2) = V_1$$

Probability Review

- Discrete Random Variables
  - $p: \Omega \rightarrow \mathbb{R}$, where eg. Ω={0, 1, 2, …}
  - $\sum_{\Omega} p = 1, p \geq 0$
- Continuous Random Variables
  - $p: \Omega \rightarrow \mathbb{R}$, where eg. Ω=[0, 1]
  - $\int_{\Omega} p = 1, p \geq 0$



Marginalization, Conditional Probability, and Bayes Rule

- Marginalization:

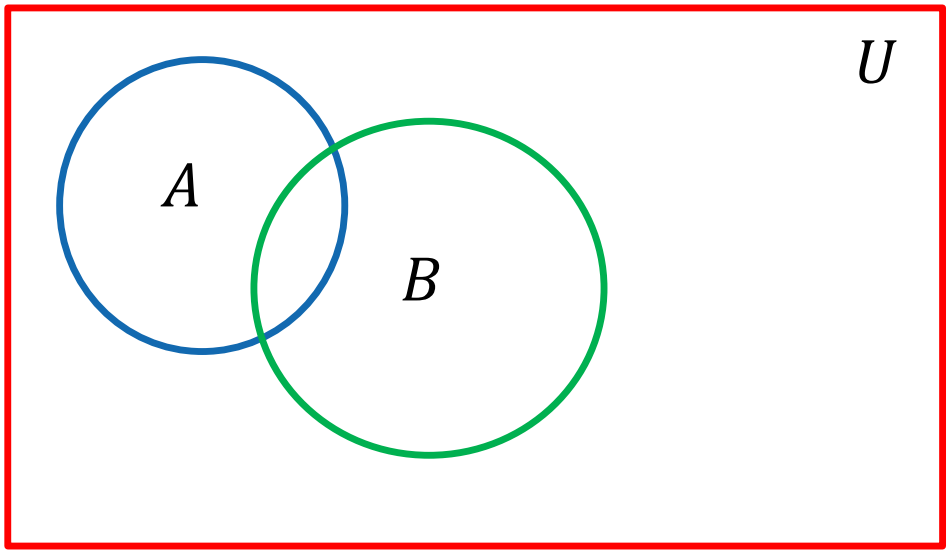
$$p(x) = \int_y p(x, y) dy$$

- Conditional Probability

$$p(x \mid y) = \frac{p(x, y)}{p(y)}$$

- Bayes Rule:

$$p(x \mid y) = \frac{p(y \mid x) p(x)}{p(y)}$$



Venn Diagram of Bayes Rule: Have a look at https://oscarbonilla.com/2009/05/visualizing-bayes-theorem/ for more info

Example 2: Probability Density

- PDF: $p(x, y) = c, x \in [-1, 1], y \in [-1, 1]$
- Find c such that it's a valid PDF
- Compute the probability that x is larger than 0, $P(x > 0)$

- Tipps:
  - $P(x > a) = \int_a^\infty p(x)dx$

Example 3: Bayes Spam Filtering

- Find the probability that an email containing the word „gold“ is spam, ie. $p(spam \mid \text{“}gold\text{“})$

- Given:
  - $p(\text{„}gold\text{“} \mid spam) = 0.6$
  - $p(\text{„}gold\text{“} \mid no\ spam) = 0.01$
  - $p(spam) = 0.2$

- Tipps:
  - $p(x \mid y) = \frac{p(y|x)p(x)}{p(y)}$

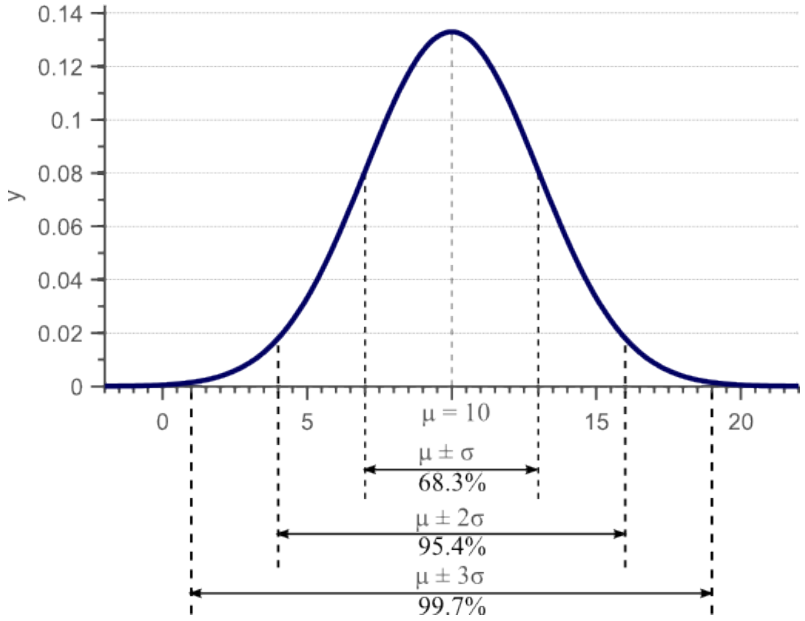
Expectation value and variance

- Expectation value:

$$\mathbb{E}[f(x)] = \int_{-\infty}^{\infty} f(x)p(x)dx$$

- Variance:

$$Var\big(f(x)\big) = \sigma^2 = \mathbb{E}[f^2(x)] - \mathbb{E}^2[f(x)]$$



Example 4: Expectation of a dice

- Compute the expectation value of a dice

- Tipps:
  - $\mathbb{E}[f(x)] = \int_{-\infty}^{\infty} f(x)p(x)dx$

Monte Carlo Quadrature

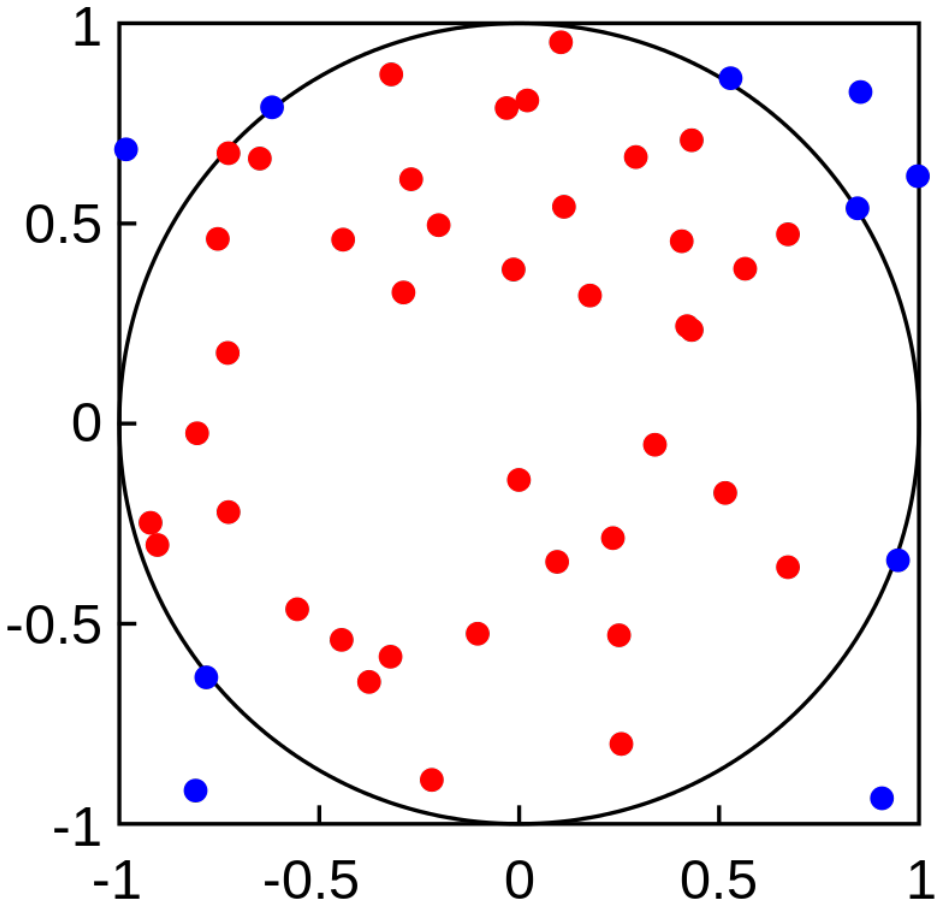
- Probability of hitting the circle:

$$p = \frac{A_{circle}}{A_{total}} \approx \frac{n_{inside}}{n_{inside} + n_{outside}}$$

- The area of the circle is therefore:

$$A_{circle} \approx \frac{n_{inside}}{n_{inside} + n_{outside}} \cdot A_{total}$$

- In general:
  1. Generate sample from known domain size, $x \sim p$
  2. Check if the sample is inside or outside
  3. Update the counters



Example 5: Monte Carlo for area

- We want to evaluate the integral

$$I = \int_{-1}^{1} x^2 dx$$

- Define $\varphi(x, y)$ st. $I = \int_{\mathbb{R}^2} \varphi(x, y) dx dy$
- Assume x, y are uniformly sampled from $[-1,1]\times[-1,1]$, compute $\mathbb{E}[\varphi(x, y)]$

- Tipps:
  - $\mathbb{E}[f(x)] = \int_{-\infty}^{\infty} f(x)p(x)dx$

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Exercises

Exercise 1

- Multivariate numerical integration

Exercise 2

- Monte Carlo Quadrature

Exercise 3

- Mini exercise for Bayes Rule

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Questions?