Title

- Text

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Exercise 9: Numerical Integration II

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Outline

1. Information
2. Goals
3. Theory / Recap (55')
4. Exercises (5')

Information

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General

- Slides by LAB are on the website (look at them before proceeding!)
- These slides: https://polybox.ethz.ch/index.php/s/9NFCvtriBbRBDnS

- **No office hour this week! (Thursday is off)**

Goals

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Goals of Today

- Understand Richardson extrapolation
- Understand how to use Richardson extrapolation
- Be aware that higher order approximations are not always better
- Understand how Richardson extrapolation can increase accuracy for numerical integration (Romberg integration)

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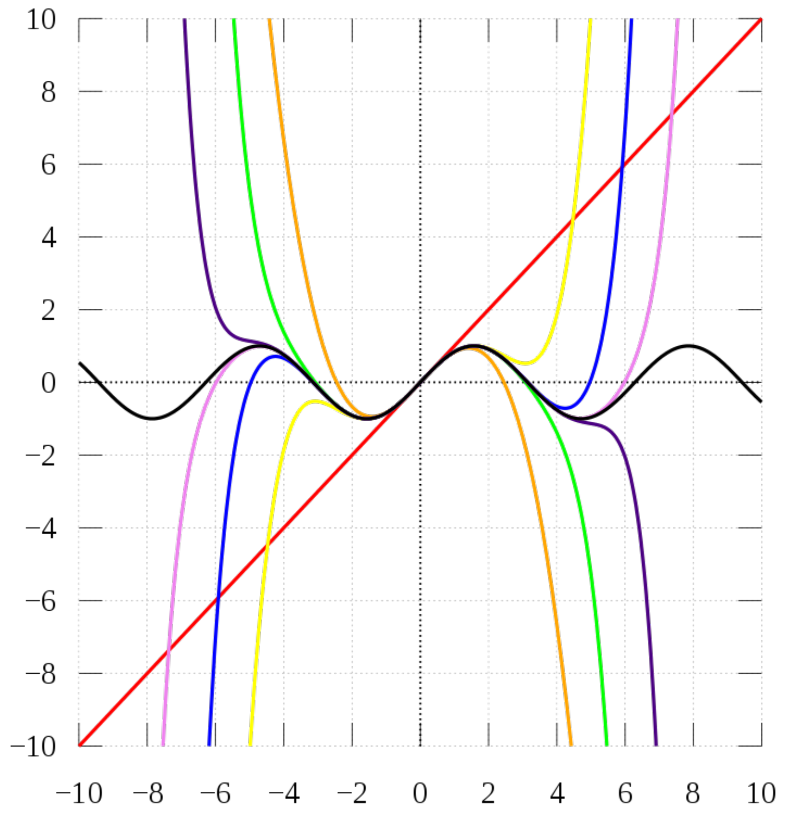
Theory / Recap

Taylor expansion

- A function $f(x)$ around the origin can be written as an infinite Series:
$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

- With...
$c_0 = f(0)$
$c_1 = f'(0)$
$c_2 = \frac{f''(0)}{2}$
etc.

- Note: Lower orders affect function the most



https://en.wikipedia.org/wiki/Taylor_series#/media/File:Sintay_SVG.svg

Exercise 1

- Write down the Taylor Series around 0 of

$$f(x) = \frac{1}{1-x}, \qquad x \in [-1, 1]$$

Richardson Extrapolation

- A function G depends on a stepsize h:

$$G(h)$$

- Perform a Taylor series expansion around 0:

$$G(h) = G(0) + c_1 h + c_2 h^2 + \cdots$$

- Decrease stepsize to $h/2$:

$$G(h/2) = \mathrm{G}(0) + \frac{\mathrm{c}_1}{2}\mathrm{h} + \frac{\mathrm{c}_2}{4} + \cdots$$

- Combine to have:

$$G_1(h) = G(0) + \mathcal{O}(h^2)$$

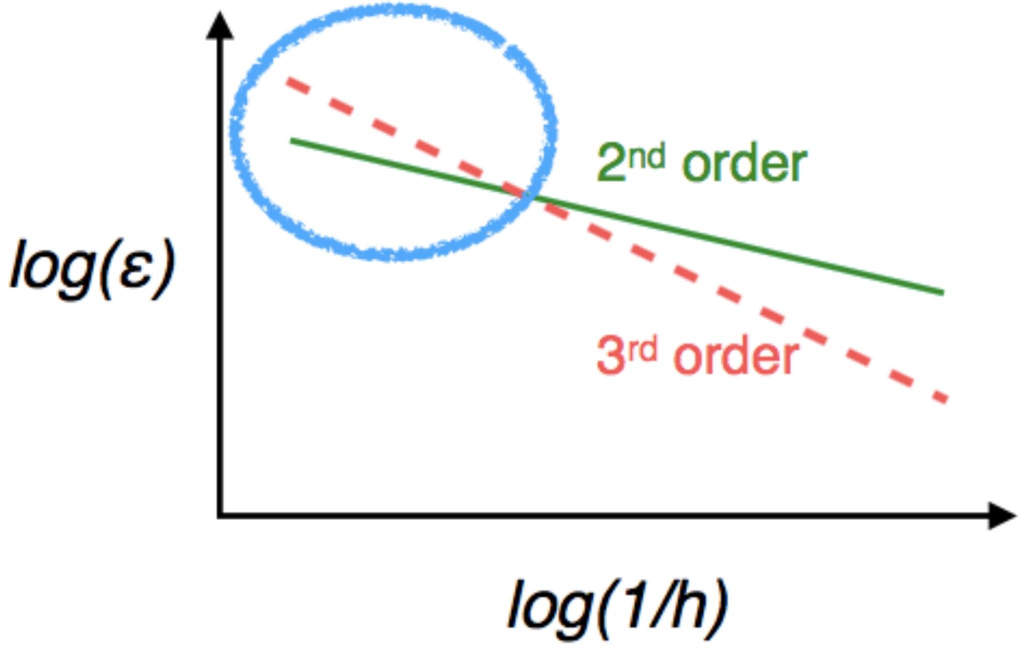
Iterative formulation

- In order to obtain $G_n(h)$ (order $n+1$) compute:

$$G_n(h) = \frac{1}{2^n - 1}\left(2^n G_{n-1}(h/2) - \mathrm{G_{n-1}(h)}\right) = \mathrm{G}(0) + \mathcal{O}(h^{n+1})$$

Higher order is not always better

- The order defines the **rate** at which the error drops
- However, the terms still depend on constants



Romberg Integration

- The Romberg Integration uses Trapezoidal rule & Romberg extrapolation:

$$I_k^n = \frac{1}{4^k - 1}\left(4^k I_{k-1}^{2n} - I_{k-1}^n\right)$$

- With…

n: Number of steps to divide interval

k: Iterations step, resp. order $k+1$

- Use composite trapezoidal rule as base case:

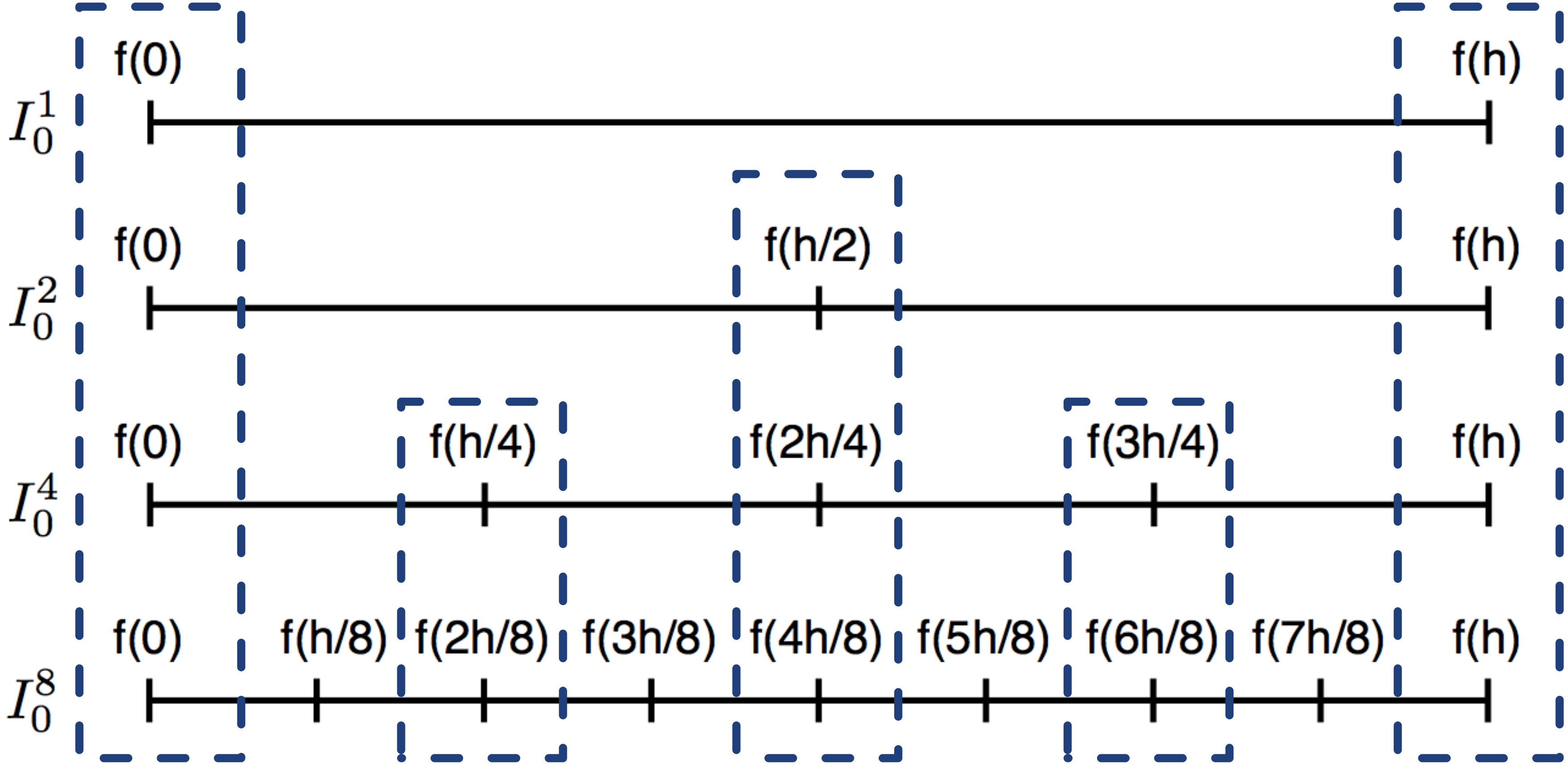
$$I_0^n = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f_i\right)$$

- With…

$h = (b-a)/n$

$f_i = f(a + i \cdot h)$

Recycle function evaluations (store in array!)



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Exercise 2

- Approximate

$$I = \int_{-1}^{1} x^2 dx$$

with

$$I_1^1$$

Formulas:

$I_k^n = \frac{1}{4^k - 1}\left(4^k I_{k-1}^{2n} - I_{k-1}^{n}\right)$

$I_0^n = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f_i\right)$

With $h = (b-a)/n$ and $f_i = f(a + i \cdot h)$

Exercises

Exercises

1. Improve Euler Forward approximation with Richardson Extrapolation
2. Write the pseudo code for Romberg integration
3. Implementation of Romberg integration

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Questions?