Title

- Text

Exercise 2: LSQ & Newton's Method

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Outline

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Information

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General

- Lecture material & problem sets available here
- Tutorial material available here

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Goals

Goals of Today

- Know about SVD
- Understand the meaning of the condition number
- Know how to compute pseudo-inverse
- Know how to use Newton's Method

Theory / Recap

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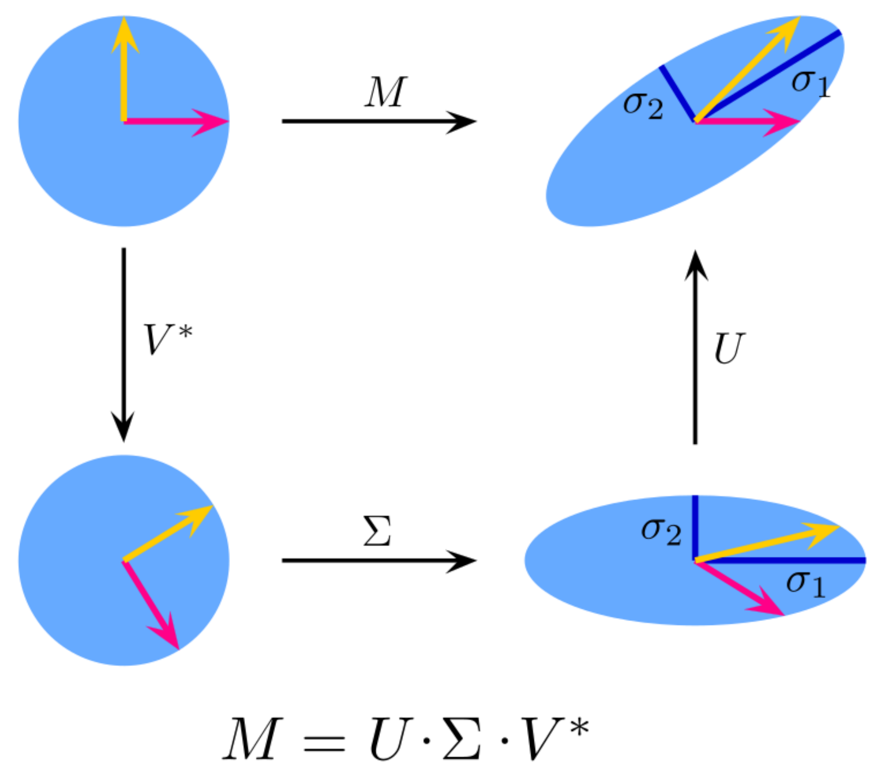
SVD

- $\mathrm{U, V}$: Orthogonal matrices
- Σ: non-squared diagonal Matrix
- Decomposition:
$$\mathrm{M = U\Sigma V^T}$$
- Inverse:
$$\mathrm{M^{-1} = V\Sigma^{-1}U^T}$$
- Note: $diag^{-1}(\sigma_i) = diag\left(\frac{1}{\sigma_i}\right)$



Conditional Number

- Assume a system $\mathrm{Ax} = \mathrm{b}$, condition number:

$$\kappa(\mathrm{A}) = \|\mathrm{A}\| \cdot \|\mathrm{A}^{-1}\| = \frac{\sigma_{max}(\mathrm{A})}{\sigma_{min}(\mathrm{A})}$$

- Describes impact of numerical error:

$$\|\mathrm{x}_{true} - \mathrm{x}_{num}\| \sim \kappa \cdot \|\mathrm{b}_{true} - \mathrm{Ax}_{num}\|$$

- Problem is well conditioned if κ is not too large

Example 1: Compute Conditional Number

- x-data: $\mathcal{X} = \left\{[1,\ 0,\ 0], [0,\ 2,\ 0], \left[0,\ 0, \frac{1}{2}\right]\right\}$
- Assume LSQ problem

- Compute the conditional number
- Is the problem well conditioned?

Pseudo Inverse

- Remember SVD:
$$M^{-1} = V\Sigma^{-1}U^T$$
- If $\sigma_i = 0 \iff M$ is singular
- If M is non-square the inv. doesn't exist
- Define the pseudo inverse:
  - Transpose Σ
  - Reciprocal of non-zero singular values
  - Zeros remain in Σ
  - $M^+ = V\Sigma^+U^T$

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{\sigma_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

Example 2: Compute the pseudo-inverse

- x-data: $\mathcal{X} = \left\{[1,\ 0,\ 0], \left[0,\ \frac{1}{2},\ 0\right]\right\}$
- Assume LSQ problem

- Compute the rank – is a pseudo-inverse required?
- Decompose the matrix as $\overline{\mathrm{X}} = \mathrm{U}\Sigma\mathrm{V}^{\mathrm{T}}$ (Tipp: U and V are identities – what dimensions?)
- Write down the pseudo inverse

Intervention: What have we done so far?

- We have some **data** $D = \{(\mathrm{x}_1, y_1), \dots, (\mathrm{x}_n, y_n)\}$
- We want to fit a **function with linear parameters** to the data, ie.: (*)
  - $y = w_0 \cdot x_1 + w_1 \cdot x_2$
  - $y = w_0 + w_1 \cdot x_1^2 + w_2 \cdot x_2^3$
- We decide on a **squared penalty** $\|\overline{\mathrm{X}}\mathrm{w} - \bar{\mathrm{y}}\|_2^2$
- Now we can optimize for w to **minimize the penalty**
- One way to do this is by **normal form**, if the **condition number** is large this is difficult since it has a “squared effect”
- Constructing the **pseudo-invers via SVD** weakens the effect of the large condition number

(*) Please note $\mathrm{x}_1^{(1)} = x_1$ etc.

Now something different: Newton's Method

- Newton's Method takes advantage of slope: Assume *somewhat* strict monotonic increase/ decrease towards root
- Derivation:
  - Approximate: $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$ (Taylor)
  - Discretize: $f\left(x^{(k)}\right) = f\left(x^{(k-1)}\right) + f'\left(x^{(k-1)}\right)\left(x^{(k)} - x^{(k-1)}\right)$
  - Set $f\left(x^{(k)}\right) = 0$: $\boxed{x^{(k)} = x^{(k-1)} - \frac{f\left(x^{(k-1)}\right)}{f'\left(x^{(k-1)}\right)}}$ (update rule)
- Iterate until $\left|x^{(k)} - x^{(k-1)}\right| < \epsilon$
- Link

Example 3: Using Newton's method

- Cube of volume 10 m^3 - What‘s the side length?

- Write down the function you are trying to find a root of
- Make an initial guess
- Iterate once using newton's method

- Use $x^0 = 0$ as an initial guess – what happens? Why?

Exercises

Q1

- Compute the condition number and find out what insights it reveals about a numerical problem
- Compute the pseudo inverse using SVD

Q2

- Pseudo-code of a recursive task
- Use newton’s method to compute the sqrt. (similar to example 3)
- Implement newton’s method

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Questions?