6 Numerical Integration pt. 2

PVK 2019: MAD

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Schedule

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    2. Taylor Expansion
    3. Richardson's Extrapolation
    4. Romberg Integration
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Theory

Setup

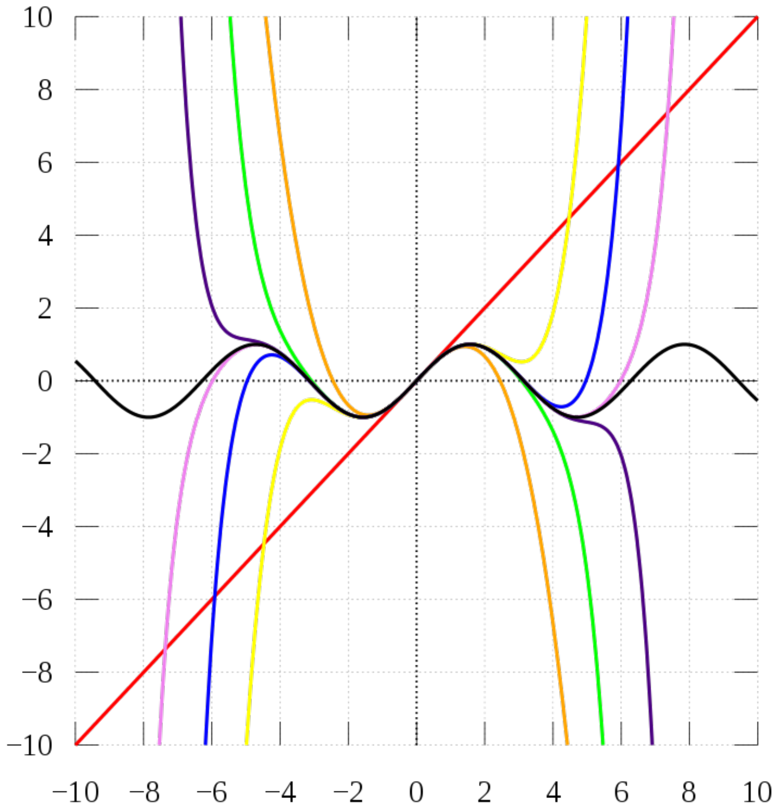
- Stepsize h
- Generic step-dependent function $G(h) = f(x; h)$

Taylor Expansion

- A function $f(x)$ around the origin can be written as an infinite Series:
$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

- With...
  $c_0 = f(0)$
  $c_1 = f'(0)$
  $c_2 = \frac{f''(0)}{2}$
  etc.

- Lower orders affect function the most



Richardson's Extrapolation

- A function G depends on a stepsize h:
$$G_0(h)$$
- Perform a Taylor series expansion around 0:
$$G_0(h) = G_0(0) + c_1 h + c_2 h^2 + \cdots$$
- Decrease stepsize to $h/2$:
$$G_0(h/2) = G_0(0) + \frac{c_1}{2} h + \frac{c_2}{4} h^2 + \cdots$$
- Combine to have:
$$G_1(h) = 2 \cdot G(h/2) - G(h) = G_0(0) + \mathcal{O}(h^2)$$

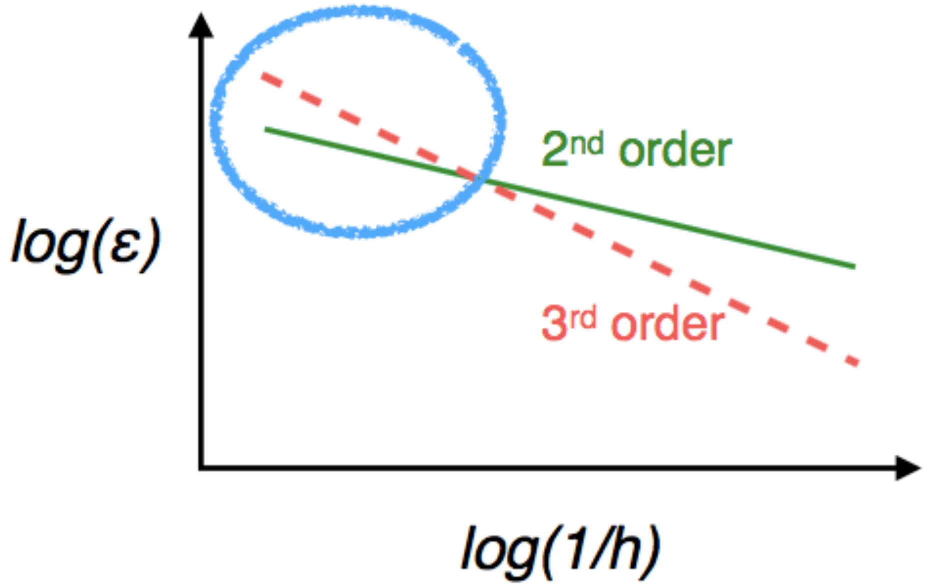
Richardson's Extrapolation cont.

- Iterative formulation:

$$G_n(h) = \frac{1}{2^n - 1}\left(2^n G_{n-1}(h/2) - G_{\mathrm{n-1}}(\mathrm{h})\right) = G(0) + \mathcal{O}(h^{n+1})$$

Richardson’s Extrapolation cont.

- The order defines the **rate** at which the error drops
- However, the terms still depend on constants



Romberg Integration

- The Romberg Integration uses Trapezoidal rule & Romberg extrapolation:

$$I_k^n = \frac{1}{4^k - 1}\left(4^k I_{k-1}^{2n} - I_{k-1}^n\right)$$

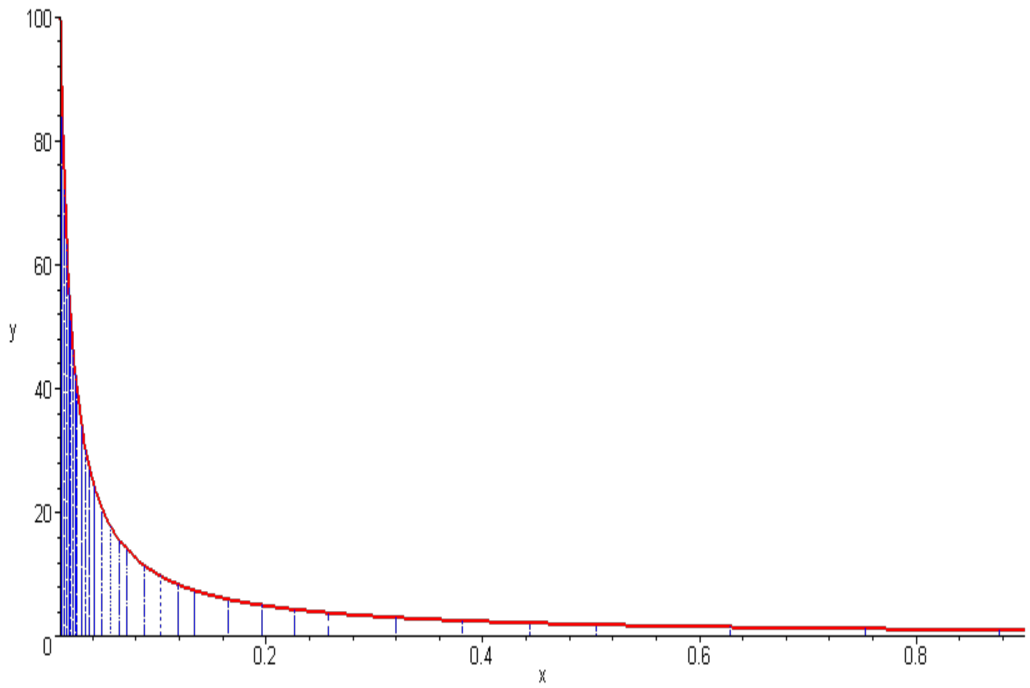
- With…
  - n: Number of steps to divide interval
  - k: Iterations step, resp. order $k + 1$
- Use composite trapezoidal rule as base case:

$$I_0^n = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f_i\right)$$

- With…
  - $h = (b - a)/n$
  - $f_i = f(a + i \cdot h)$

Adaptive Quadrature

- Use adaptive stepsize
- Decrease stepsize in specific region until desired accuracy is achieved



Adaptive Quadrature cont. Example

- Task:
  - Estimate the error $e(h/2)$ with Richardson's Extrapolation that a numerical scheme is generating at $h/2$

Adaptive Quadrature cont.

- Algorithm:
  1. Apply Simpsons rule on interval
  2. Split the interval into 2 intervals and apply Simpsons rule on both
  3. Estimate the error with Richardson’s
  4. If the error is smaller than threshold stop, else split again ...

Exercises

Exercise 1

a) Richardson's extrapolation method aims to improve the approximation by combining approximations of different step sizes: True / False.

b) In the context of Richardson's extrapolation the approximation of a function is expressed as:

$$G(h) = G(0) + c_1 h + c_2 h^2 + c_3 h^3 + \dots \tag{4}$$

$$G(h/2) = G(0) + \frac{1}{2} c_1 h + \frac{1}{4} c_2 h^2 + \dots \tag{5}$$

and a better approximation of $G(0)$ can be obtained from:

$$G_1(h) = 2G(h/2) - G(h) = G + c_2' h^2 + c_3' h^3 + \dots \tag{6}$$

Find an expression for c_3' in terms of c_3.

c) What is the leading error term for $G_1(h)$?

1. h
2. h^2
3. h^3
4. there is no error, $G_1(h)$ is the exact solution.

d) For the general approximating quantity $G_n(h)$, what is the leading error term?

1. h
2. h^n
3. h^{n+1}
4. h^{n-1}

Exercise 2

- Given:
  - $x = 0$
  - $h = 0.2$
  - $f(x) = x^2$

- Task:
  - Approximate the error $\epsilon\left(\frac{h}{2}\right)$ of the Euler Forward method and compare it to the true error:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Exercise 3

- Task:
  - Write pseudo code for Adaptive Quadrature using recursion, the rule of numerical integration is simpsons
  - The function itself takes a lower and and upper bound of an interval as an input (and is called with such in the recursion)

Homework

HW 1

- Write down the Taylor Series around 0 of

$$f(x) = \frac{1}{1-x}, \qquad x \in (-1, 1)$$

HW 2

- Derive Romberg Integration (rather difficult)
- Tip:
  - Construct and integral $I = \int_{-h}^{h} f(x)dx$ write $f(x)$ as a Taylor expansion around 0 and compute the integral
  - Set up the Trapezoidal rule as $I_T = h \cdot \big(f(-h) + f(h)\big)$ and express both terms $f(-h), f(h)$ as Taylor expansion around 0
  - Compare the two, all terms which are similar are accurate, as soon as the terms differ the error starts
  - At this point you should have shown that errors are odd 3, 5, …
  - This was for a single interval so the error is repeated N times, this will drop the leading term by the order of one (try to show this by expressing N through h)
  - Now your trapezoidal rule looks like this: $I_T = I + c_1 h^2 + c_2 h^4 + \cdots$. Apply Richardson's to this to eliminate the order 2 term

HW 3

a) Use the data provided in Fig. 6 to approximate $\int_1^5 y(x)dx$ using Romberg integration as accurately as possible. Keep your computations and final answer in fractions.

x	1	2	3	4	5
y	2	3	4	1	2

Figure 6: Data points used in Romberg integration.

HW 4

On the graphs shown in Fig. 8, qualitatively show how you expect the clustering of intervals to occur when using adaptive quadrature with the trapezoidal rule.

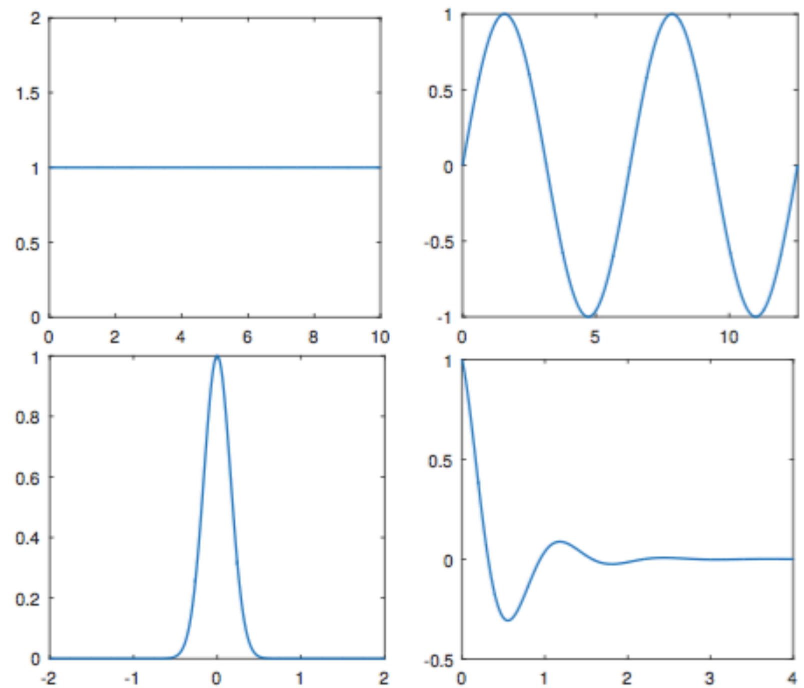


Figure 8: Draw clustering of intervals for adaptive quadrature with the trapezoidal rule.