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Exercise 10: Adaptive Quadrature & Gauss Quadrature

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Outline

1. Information
2. Goals
3. Theory (55’)
4. Exercises (5’)

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Information

General

- Slides by LAB are on the website (look at them before proceeding!)
- These slides: https://polybox.ethz.ch/index.php/s/9NFCvtriBbRBDnS

Goals

Goals of Today

- Understand the motivation behind adaptive quadrature
- Understand how adaptive quadrature (AQ) works
- Understand how recursion works and how it can be applied to AQ
- Understand derivation of Gauss Quadrature
- Understand how to use Gauss Quadrature with tables

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Theory

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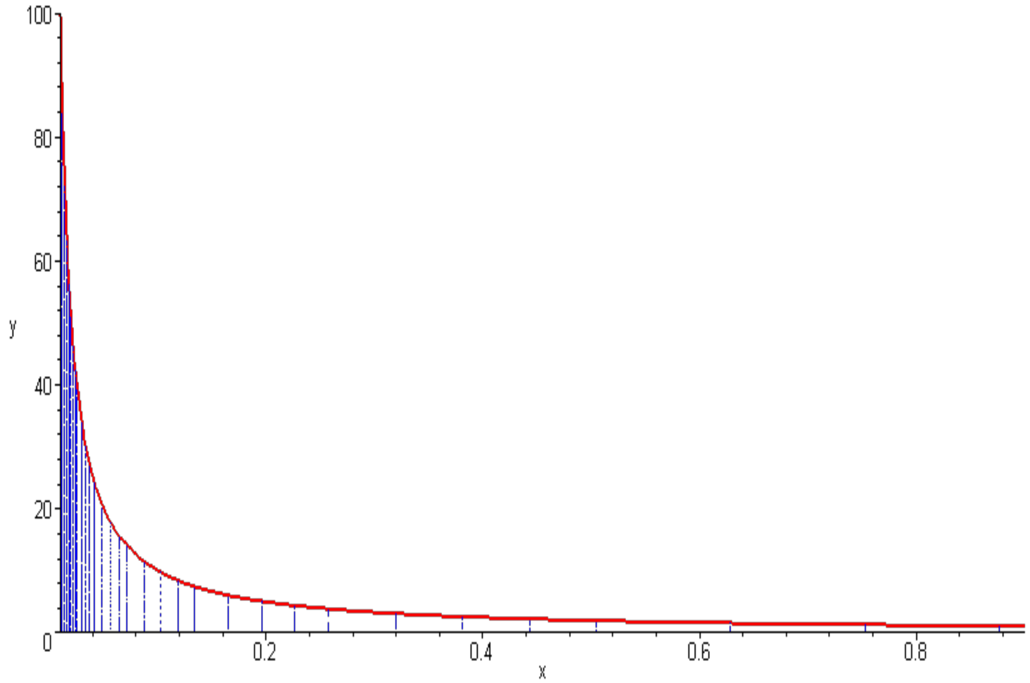
Adaptive Quadrature

- Use adaptive stepsize
- Decrease stepsize in specific region until desired accuracy is achieved



https://www.cs.uic.edu/~hogand/cs107/notes-numerical.html

Reminder: Error Estimate with Richardson's Extrapolation

- We can write a Taylor Series expansion of a numerical method:

$$G(h) = G(0) + c_1 h + c_2 h^2 + \cdots$$

- Hence the error generated by the stepsize $h/2$:

$$\epsilon(h/2) = |G(0) - G(h/2)| = \left| -\frac{c_1}{2} h - \frac{c_2}{4} h^2 + \cdots \right|$$

- On the other hand:

$$G(h/2) - G(h) = -\frac{c_1}{2}\, h - \frac{3c_2}{4}\, h^2 + \cdots$$

- Hence:

$$\epsilon(h/2) \approx |G(h/2) - G(h)|$$

Example 1

- Approximate the error $\epsilon\left(\frac{h}{2}\right)$ of the Euler Forward method and compare it to the true error:

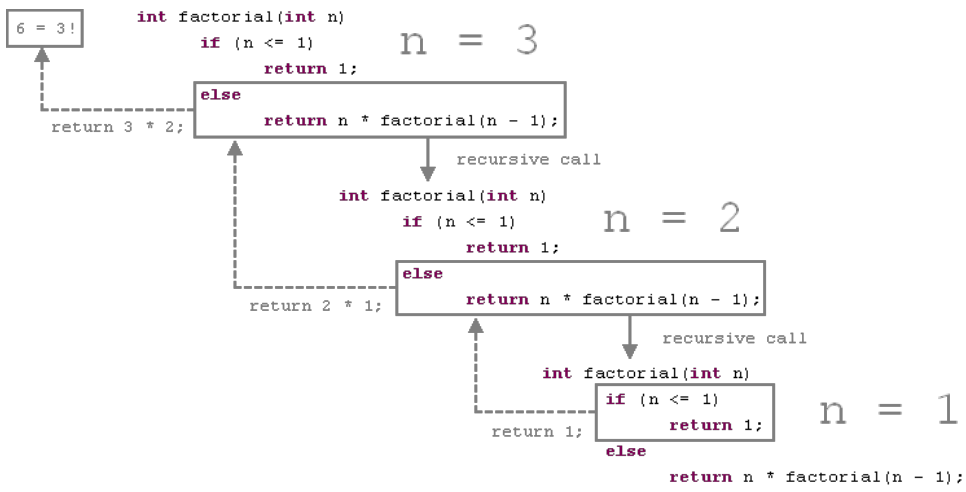
$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

- Given:
  - $x = 0$
  - $h = 0.2$
  - $f(x) = x^2$
- Remember:
  - $\epsilon(h/2) \approx |G(h/2) - G(h)|$

Reminder: Recursion

- The function calls itself

```
function f(x)
     if base case then
          Return base value
     else
          Return recursive value
     end if
end function
```



https://stackoverflow.com/questions/8183426/factorial-using-recursion-in-java

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Example 2

- Write pseudo-code for a function that finds the greatest common divisor, using Euclid's algorithm:

$$\mathrm{gcd}(x, y) = \begin{cases} x & if\ y = 0 \\ \mathrm{gcd}(y, x \bmod y) & if\ y > 0 \end{cases}$$

- Calculate the gcd of 8 and 12 with the algorithms, draw a diagram emphasizing the calls to the functions and the return values

Solution 2

```
function gcd(x, y)
     if y is 0 then
          Return x
     else
          Return gcd(y, x mod y)
     end if
end function
```

Adaptive Quadrature Algorithm

```
function ADAPTIVESIMPSON(a, b)
     Apply Simpson’s rule in [a, b]
     Split interval into [a, m] & [m, b]; with m = (a + b)/2
     Apply Simpson’s rule on [a, m] & [m, b]
     Estimate Error with Richardson’s extrapolation

     if error > ϵ then
          Return ADAPTIVESIMPSON(a, m) + ADAPTIVESIMPSON(m, b)
     else
          Return value of Simpson’s rule (accurate one)
     end if
end function
```

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Gauss Quadrature

- Decompose the function $f(x)$:

$$\int_a^b f(x)dx = \int_a^b \Phi(x)w(x)dx$$

- Approximate the integral by a sum over the function $\Phi(x)$ at specified supporting points x_i multiplied with weights α_i:

$$\int_a^b \Phi(x)\underbrace{w(x)}_{=1}dx \approx \sum_{i=1}^{n} \Phi(x_i)\alpha_i$$

Note

- Transform integral to be in the interval $[-1,\ 1]$ in order to use lookup-tables:

$$z = \frac{2x - (a + b)}{b - a}$$

- We then have:

$$\int_{-1}^{1} f(z)dz \approx \sum_{i=1}^{n} f(z_i)\alpha_i$$

- You can now look up z_i and α_i in tables
- Method is exact for polynomials of degree $2n - 1$

Example 3

- Find – by Gauss Quadrature - the exact integral value:

$$I = \int_0^2 x^3 dx$$

- Tipps:
  - Exact for order $2n-1$
  - $z = \frac{2x-(a+b)}{b-a}$

n=1	x_i	α_i
1	0	2
n=2	x_i	α_i
1	$-\sqrt{\frac{1}{3}} \approx -0{,}57735026919$	1
2	$\sqrt{\frac{1}{3}} \approx 0{,}57735026919$	1
n=3	x_i	α_i
1	$-\sqrt{\frac{3}{5}} \approx -0{,}774596669241$	$\frac{5}{9} \approx 0{,}555555555556$
2	0	$\frac{8}{9} \approx 0{,}888888888889$
3	$\sqrt{\frac{3}{5}} \approx 0{,}774596669241$	$\frac{5}{9} \approx 0{,}555555555556$
n=4	x_i	α_i
1	$-\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}} \approx -0{,}861136311594053$	$\frac{18-\sqrt{30}}{36} \approx 0{,}347854845137454$
2	$-\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}} \approx -0{,}339981043584856$	$\frac{18+\sqrt{30}}{36} \approx 0{,}652145154862546$
3	$\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}} \approx 0{,}339981043584856$	$\frac{18+\sqrt{30}}{36} \approx 0{,}652145154862546$
4	$\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}} \approx 0{,}861136311594053$	$\frac{18-\sqrt{30}}{36} \approx 0{,}347854845137454$

Exercises

Exercises (see lab slides for more information)

1. Integrate the “batman function” (see lab slides for equations) with AQ
2. Perform quadrature Trapezoidal Rule, Newton-Cotes, and Gauss Quadrature. Compare the methods. Think about **why** some work better than others and what conditions have to be met that some work better than others.

Questions?