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Exercise 7: Newton’s Method

MAD

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Outline

1. Information
2. Goals
3. Theory / Recap (30’)
4. Exercises (5’)

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Information

General

- Slides by LAB are on the website (look at them before proceeding!)
- These slides: https://polybox.ethz.ch/index.php/s/9NFCvtriBbRBDnS

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Goals

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Goals of Today

- Know what the root of a function is by **Intermediate Value Theorem**
- Briefly understand **Bisection Method**
- Understand derivation of **Newton's Method**
- Understand why Newton's Method might be better than Bisection
- Understand generalization of Newton's Method to systems

Theory / Recap

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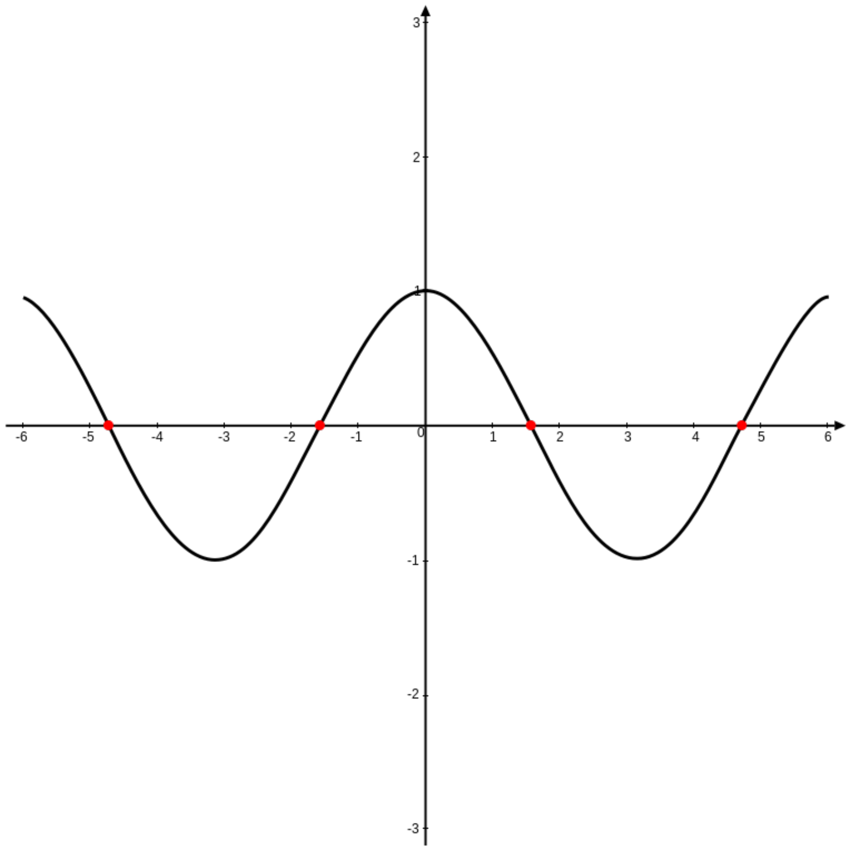
Root of a function

- $f(x^*) = 0$, x^* is a root of f
- **Intermediate Value Theorem**:
  - f is continuous
  - $\mathrm{sign}\, f(a) \neq \mathrm{sign}\, f(b)$
  - Then $x^* \in [a, b]$



https://en.wikipedia.org/wiki/Zero_of_a_function#/media/File:X-intercepts.svg

Bisection Method

- Bisection Method directly takes advantage of intermediate value theorem:

```
1. Initialize a, b
2. If sign f(a) == sign f(m)
      a ← m
    else
      b ← m
3. Repeat with m ← (a+b)/2
```

- Iterate until $|a - b| < \epsilon$
- Link

Newton's method

- Newton's Method takes advantage of slope: Assume *somewhat* strict monotonic increase/ decrease towards root
- Derivation:
  - Approximate: $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$ (Taylor)
  - Discretize: $f\left(x^{(k)}\right) = f\left(x^{(k-1)}\right) + f'\left(x^{(k-1)}\right)\left(x^{(k)} - x^{(k-1)}\right)$
  - Set $f\left(x^{(k)}\right) = 0$: $\boxed{x^{(k)} = x^{(k-1)} - \frac{f\left(x^{(k-1)}\right)}{f'\left(x^{(k-1)}\right)}}$ (update rule)
- Iterate until $\left|x^{(k)} - x^{(k-1)}\right| < \epsilon$
- Link
- Where does it fail?

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Issues with Newton’s Method

- Oscillation
- Stationary Point
- Bad convergence

Newton's Method for Systems

- $\mathbf{x}^* = [x_1^*, \ldots, x_N^*]^T$, $\mathbf{F}(\mathbf{x}^*) = [f_1(x_1^*), \ldots, f_N(x_N^*)]^T = \mathbf{0}$
- Update rule: $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - \boldsymbol{J}^{-1}\left(\mathbf{x}^{(k-1)}\right)\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$
- Define: $\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} = \boldsymbol{y}^{(k-1)}$
- Write update rule as: $\boldsymbol{J}\left(\mathbf{x}^{(k-1)}\right)\boldsymbol{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$

Example

System of Equations example

- $5x_1^2 - x_2^2 = 0; x_2 - 0.25(\sin x_1 + \cos x_2) = 0$
- Initial Guess: $x_1^{(0)} = 0;\ x_2^{(0)} = 0$
- Do 1 iterations…

Exercises

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Questions?