8 Numerical Integration pt. 3

PVK 2019: MAD

bacdavid@student.ethz.ch

gitlab.ethz.ch/bacdavid

Schedule

1. Theory
   1. Multivariate integration
   2. Monte Carlo integration
2. Exercises
3. Homework

Theory

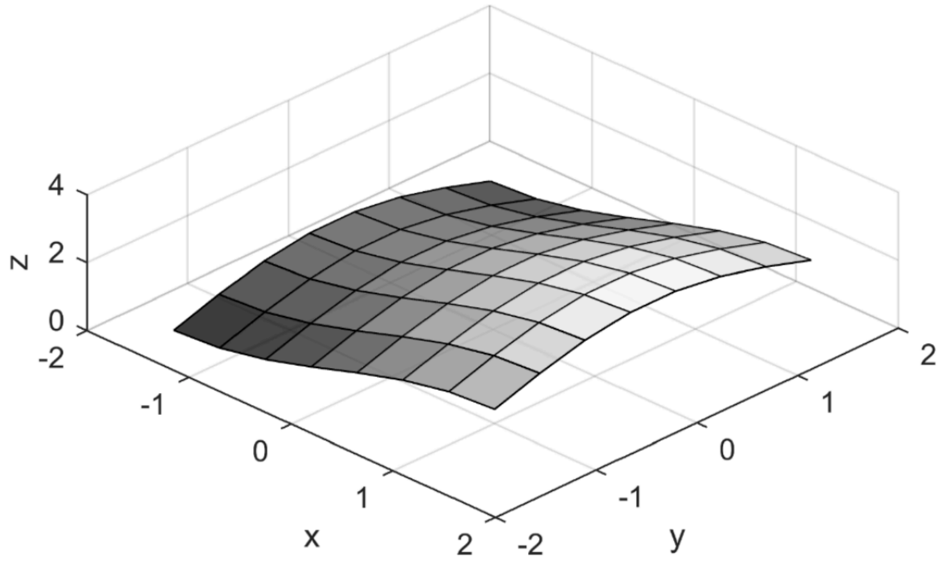
Setup

- Area A

Multivariate Integration

- The exact integral:

$$I = \int_{\Omega} f(\vec{x})d\vec{x}$$



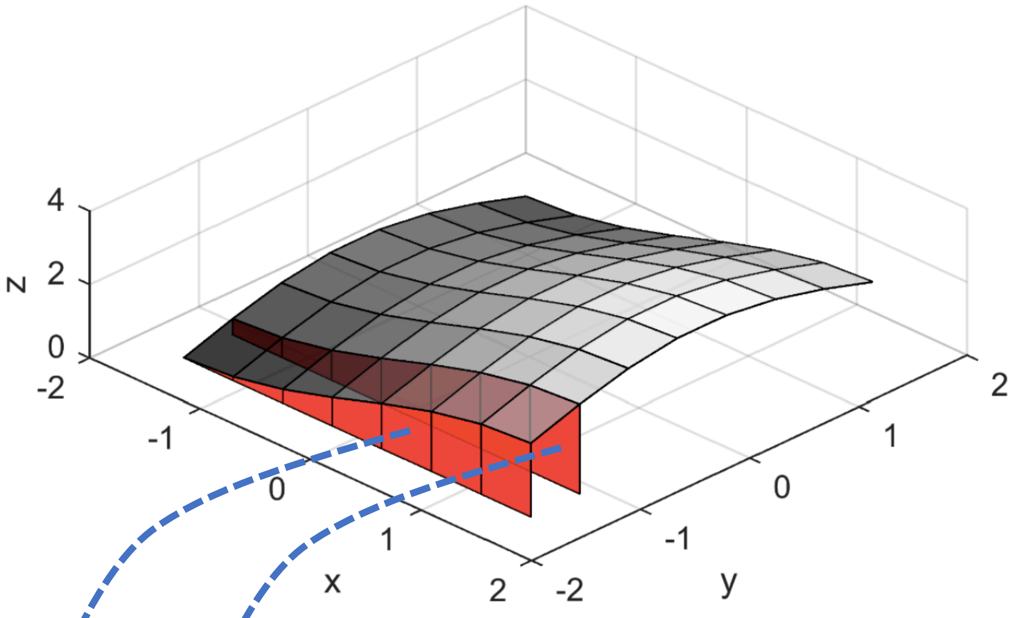
- Approximation:

$$I \approx \sum w_i f(\vec{x}_i)$$

Multivariate Integration cont. Example

- Griven:
  - 3x3 grid (matrix indexing)
  - $h = 1$ (all directions)

- Task:
  - Write down the weights for each $f_{i,j}$ resulting from the multivariate trapezoidal rule



$$\frac{h}{2}(A_1 + A_2) = V_1$$

Monte Carlo Quadrature

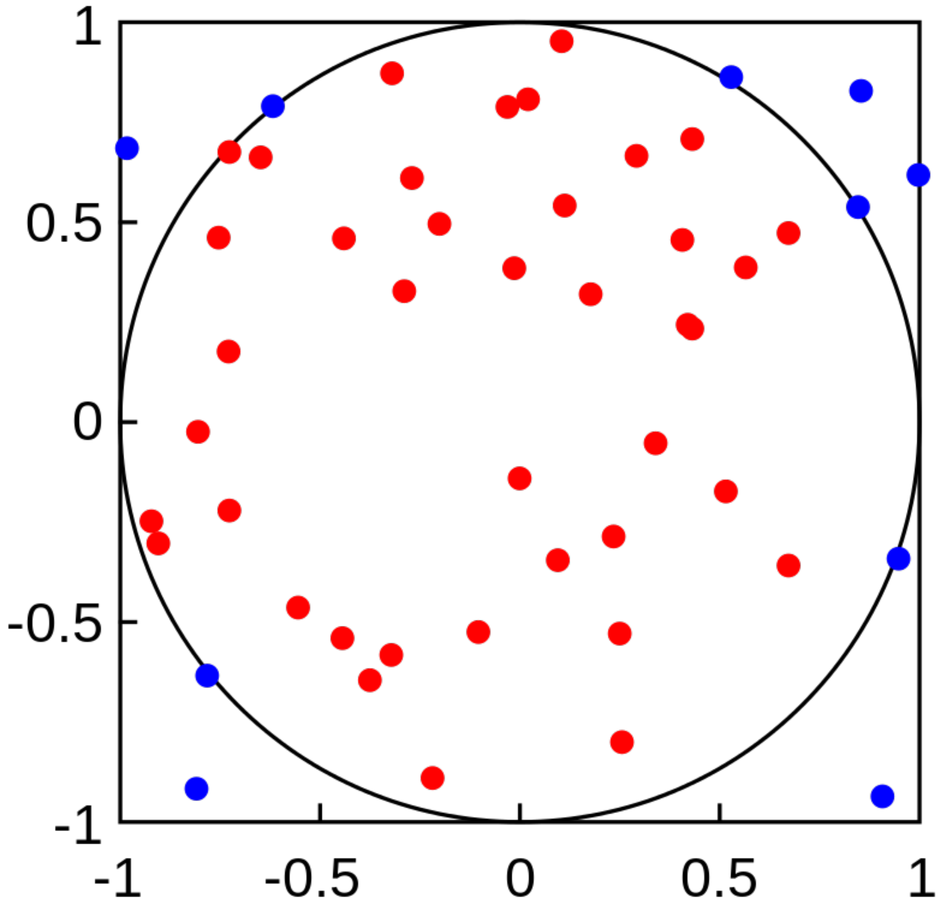
- Probability of hitting the circle:

$$p = \frac{A_{circle}}{A_{total}} \approx \frac{n_{inside}}{n_{inside} + n_{outside}}$$

- The area of the circle is therefore:

$$A_{circle} \approx \frac{n_{inside}}{n_{inside} + n_{outside}} \cdot A_{total}$$

- In general:
  1. Generate sample from known domain size, $x \sim p$
  2. Check if the sample is inside or outside
  3. Update the counters



Exercises

Exercise 1

Figure 9 shows the surface $z = 0.1x^2 + 0.1y^3 + 4$.

a) Find the exact volume enclosed between the surface and the xy plane, for $x \in [-3, 3]$ and $y \in [-3, 3]$. (Hint: you can make your computation easier by exploiting the fact that z is an odd function of y.)

b) Compute the volume using the midpoint (rectangle) rule, with 3 intervals along each of the x and y axes. What is the absolute error for your numerical approximation with respect to the exact value of the integral?

c) How is the error expected to change when you increase the number of function evaluations by a factor of 4?

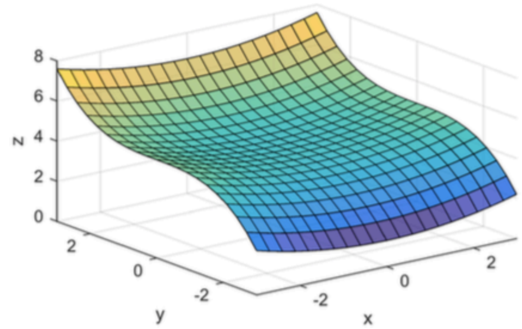


Figure 9

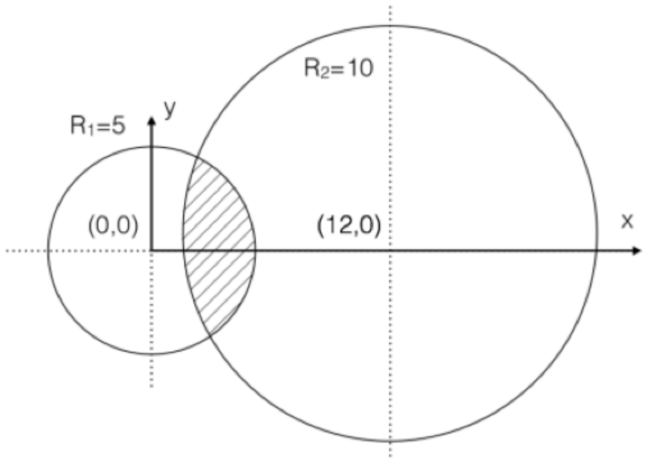
Exercise 2

a) Write a pseudocode to calculate the overlapping area of the two circles shown below using Monte Carlo Sampling. Assume that you have a function `random()`, which returns a uniformly distributed random number in the interval $[0, 1]$.

b) What would you have to change in your pseudocode if you wanted to estimate the error of the Monte Carlo sampling? Answer qualitatively, do not write any pseudocode.

c) How does the error of the method change if you use 10 times more samples?

d) How does the error of the method change if you use a two times larger sampling space?



Homework

HW 1

- Task:
  - We want to evaluate the integral

$$I = \int_{-1}^{1} x^2 dx$$

  - Define $\varphi(x, y)$ st. $I = \int_{\mathbb{R}^2} \varphi(x, y) dx dy$
  - Assume x, y are uniformly sampled from $[-1,1]\times[-1,1]$, compute $\mathbb{E}[\varphi(x, y)]$

HW 2

The battery in the circuit shown in Fig. 11 supplies a constant voltage of $V = 10$ volts. The current I in the circuit depends on the voltage, and the resistance R according to relation $V = IR$. The instantaneous power delivered by the battery is $P = I^2 R$. You are told that every time you switch on the circuit, R may randomly take on any positive value between 10Ω and 20Ω, following a given probability distribution $p(x)$. Furthermore, the value of R stays constant until you turn the circuit off.

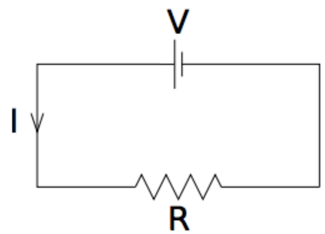


Figure 11

a) Write down the expression for P in terms of V and R.

b) Write down, in terms of V and $p(x)$, the expression for the expected (averaged over multiple switchings) power $\bar{P} = \mathbb{E}[P]$ delivered by the battery.

c) Approximate $\bar{P}$ using Monte Carlo method, given that 4 samples drawn from the given distribution $p(x)$ are as follows: $\{12, 17, 15, 16\}$.