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Title

- Text

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Exercise 8: Numerical Integration

MAD

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Outline

1. Information
2. Goals
3. Theory/ Recap
4. Exercises

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Information

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General

- Lecture material & problem sets available here
- Tutorial material available here

Goals

Goals of Today

- Understand why we use numerical integration
- Understand the basic approach for numerical integration
- Know how to integrate numerically with Rectangle, Trapezoidal, & Simpson’s rule
- Understand derivation of Newton-Cotes formula & how to use it

Theory / Recap

Integral

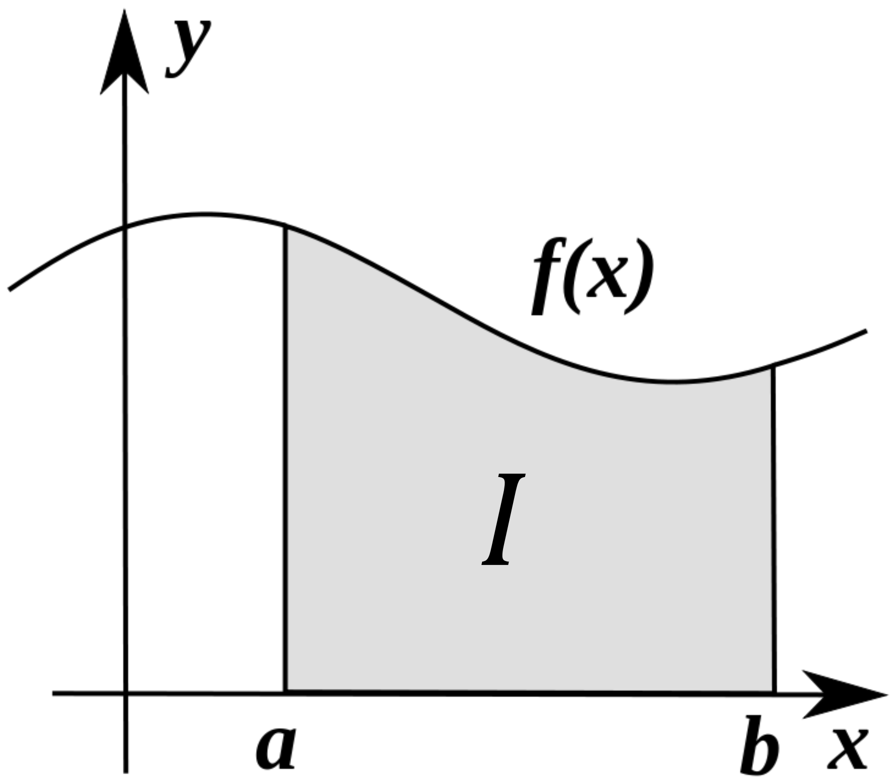
- Exact Integral:

$$I = \int_a^b f(x)dx$$

- …can be solved analytically or not

- Used for differential equations:

$$\dot{x} = f(x)$$



https://de.wikipedia.org/wiki/Integralrechnung#/media/File:Integral_as_region_under_curve.svg

Example 1: ODE

- Solve the ODE:

$$\dot{x} = t \cdot e^{t}, x(t_0) = x_0$$

- Tipp:

$$\int_a^b uv'dx = [uv]_a^b - \int_a^b u'vdx$$

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Numerical Integration

- Approximate:

$$I = \int_a^b f(x)dx = \sum_{i=0}^{N-1} I_i$$

- Approximate I_i: (table)
  - Rectangular Rule
  - Trapezoidal Rule
  - Simpson's Rule

Approximations for I_i

Rectangular Rule	$I_{Ri} = f\left(x_{i+1/2}\right) \cdot \Delta_i$
Trapezoidal Rule	$I_{Ti} = \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta_i$
Simpson's Rule	$I_{Si} = \frac{f(x_i)+4f\left(x_{i+1/2}\right)+f(x_{i+1})}{6} \cdot \Delta_i$

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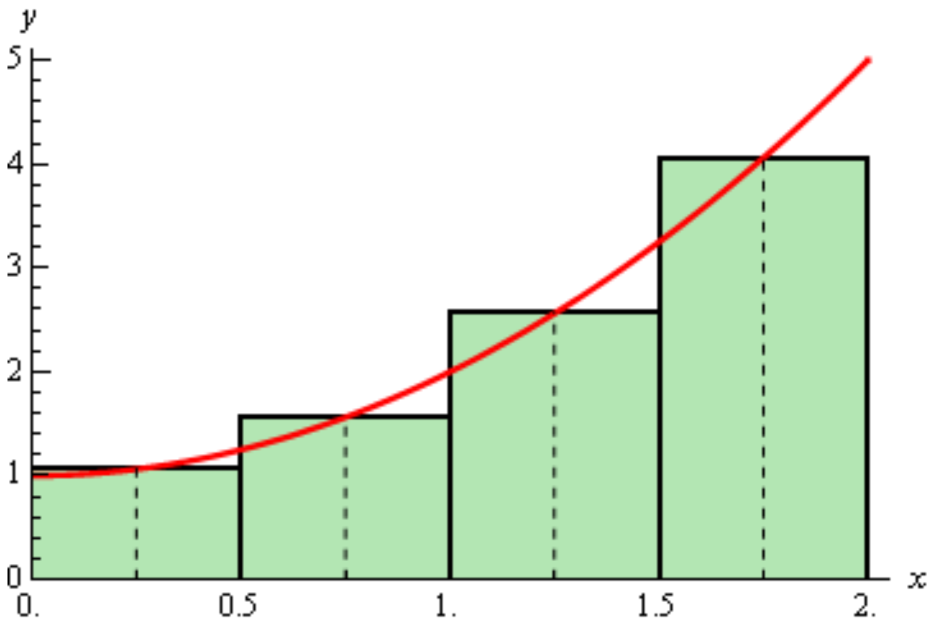
Rectangular Rule



http://tutorial.math.lamar.edu/Classes/CalcI/AreaProblem.aspx

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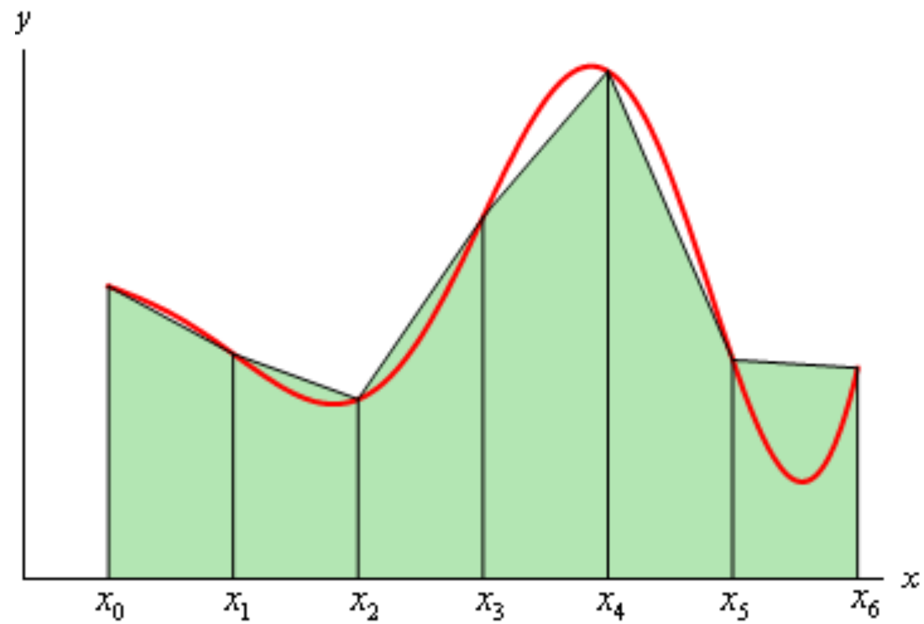
Trapezoidal Rule



http://tutorial.math.lamar.edu/Classes/CalcI/AreaProblem.aspx

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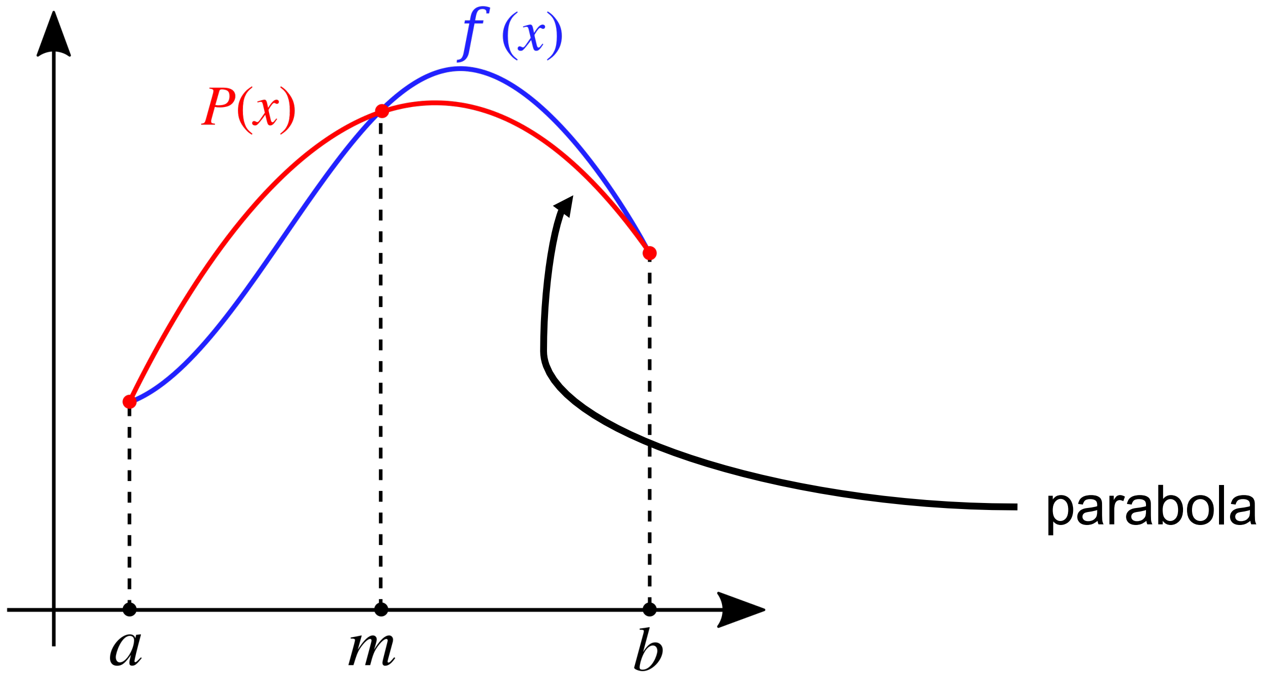
Simpson's Rule



https://de.wikipedia.org/wiki/Datei:Simpsons_method_illustration.svg

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Example 2: Numerical Integration

- Solve the integral

$$\int_{-1}^{1} x^2 dx$$

1. Exact
2. Rectangle Rule
3. Trapezoidal Rule
4. Simpson's Rule

Approximations for I_i

Rectangular Rule	$I_{Ri} = f\left(x_{i+1/2}\right) \cdot \Delta_i$
Trapezoidal Rule	$I_{Ti} = \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta_i$
Simpson's Rule	$I_{Si} = \frac{f(x_i)+4f\left(x_{i+1/2}\right)+f(x_{i+1})}{6} \cdot \Delta_i$

Newton-Cotes Formula

- Approximate:

$$I \approx (b-a) \cdot \sum_{k=0}^{n} C_k^n f(x_k)\,, \qquad with\ C_k^n = \frac{1}{b-a}\int_a^b l_k^n(x)dx \quad „weights“$$

- Properties of C_k^n:
  - $\sum_{k=0}^{n} C_k^n = 1$
  - $C_k^n = C_{n-k}^n$

- Note: Can be used for entire integral I or for smaller intervals I_i

Example 3: Derive an numerical integration rule

- use $n = 2$ and equally spaced x_0, x_1, x_2 to approximate I

$$I \approx (b-a) \cdot \sum_{k=0}^{n} C_k^n f(x_k),$$

$$with\ C_k^n = \frac{1}{b-a} \int_a^b l_k^n(x)dx \quad „weights“$$

- Tipp: $C_1^2 = \frac{2}{3}$

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Exercises

Exercise 1

- Recompute Simpson‘s Rule (without short-cuts)

Exercise 2

- Use numerical integration on a practical example
- Values are discretized (function unknown)

Exercise 3

- Extension of ex 2, but analytical function is given (exact integral can be computed)

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Questions?