3 Interpolation Extrapolation

p7

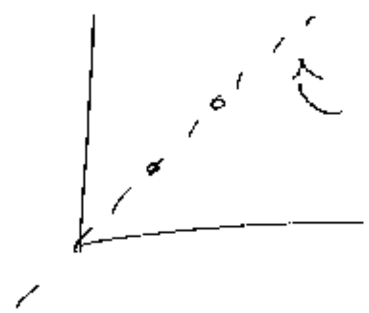
example

$$\text{degree} = N - 1$$
$$= \underline{1}$$

$\left(\text{from } \prod_{1 \le m \le N, \ne i} \cdots \right)$

guess function:

$\underline{f(x) = x}$



degree 1 => yes

p9

example

• 2 functions:

$$\begin{cases} S_1(x) = C_{11} + C_{12}x + C_{13}x^2 + C_{14}x^3 \\ S_2(x) = C_{21} + C_{22}x + C_{23}x^2 + C_{24}x^3 \end{cases}$$

• 8 unknown

• $S_1(x_1) = y_1$ $\quad S_1'(x_2) = S_2'(x_2)$
$S_1(x_2) = y_2$ $\quad S_1''(x_2) = S_2''(x_2)$
$S_2(x_2) = y_2$
$S_2(x_3) = y_3$

= 6 constr.
+ 2 ?

· Boundary Constraints: 1) Natural Splines $S_1''(x_1) = S_{N-1}''(x_N) = 0$
2) Parabolic Runout $S_1''(x_1) = S_1''(x_2)$
$S_{N-1}''(x_{N-1}) = S_{N-1}''(x_{N-2})$

P 13

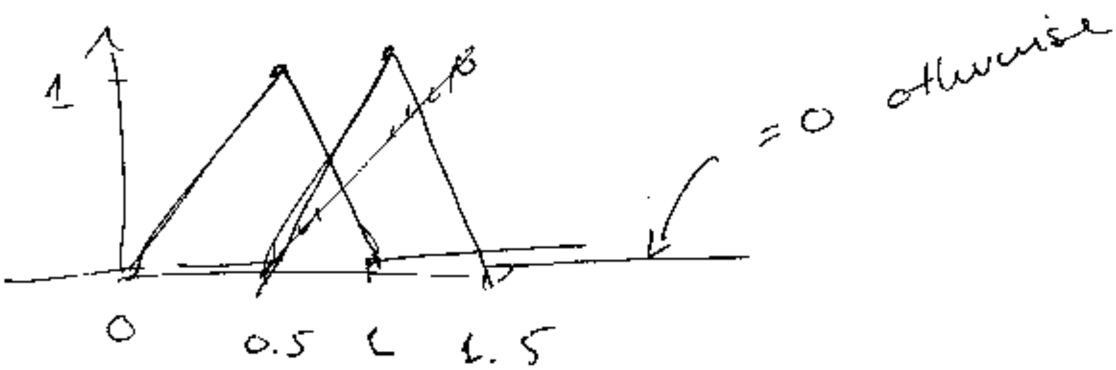
$$\left.\begin{array}{l} M = 1 \\ d = 1 \end{array}\right\} \text{size}(\vec{t}) = M + d + 1 = 3$$

$$\vec{t} = (0, 0.5, 1)$$

$$B_{1,0} = \begin{cases} 1 & 0 \leq x < 0.5 \\ 0 & \text{else} \end{cases}$$

$$B_{2,0} = \begin{cases} 1 & 0.5 \leq x < 1 \\ 0 & \text{else} \end{cases}$$

$$B_{1,1} = \begin{cases} \dfrac{x-0}{0.5-0} \cdot 1 + \cancel{\dfrac{1-x}{1-0.5} \cdot 0} & 0 \leq x < 0.5 \\ \cancel{\dfrac{x-0}{0.5-0} \cdot 0} + \dfrac{1-x}{1-0.5} \cdot 1 & 0.5 \leq x < 1 \end{cases}$$



P 27

- $L(x)$ is of order $N-1$
- If N increases the degree increases
- „Runge's phenomenon"
- Order is always $\underline{\underline{3}}$
- Cubic splines require an optimization process; Lagrange doesn't → computationally less expensive. Use Lagrange if N is small
- Go through all points

P 28

- Used for interpolation since we force boundary conditions making extrapolation ~~...~~ useless

P 29

$$c_{11} + c_{12}x_1 + c_{13}x_1^2 + c_{14}x_1^3 = y_1$$
$$c_{11} + c_{12}x_2 + c_{13}x_2^2 + c_{14}x_2^3 = y_2$$
$$c_{21} + c_{21}x_2 + c_{23}x_2^2 + c_{24}x_2^3 = y_2$$
$$c_{21} + c_{21}x_3 + c_{23}x_3^2 + c_{23}x_2^3 = y_3$$
$$c_{12} + 2c_{13}x_2 + 3c_{14}x_2^2 - c_{21} - 2c_{23}x_2 - 3c_{24}x_2^2 = 0$$
$$2c_{13} + 6 \cdot c_{14}x_2 - 2c_{23} - 6c_{24}x_2 = 0$$

... p 29

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & 0 & 0 & 0 & 0 \\ 1 & x_2 & x_2^2 & x_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 0 & 0 & 0 & 1 & x_3 & x_3^2 & x_3^3 \\ 0 & 1 & 2x_2 & 3x_2^2 & 0 & -1 & -2x_2 & -3x_2^2 \\ 0 & 0 & 2 & 6x_2 & 0 & 0 & -2 & -6x_2 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{24} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \\ 0 \\ 0 \end{bmatrix}$$

p30

$M = 3$

$\vec{t} = (0, 0.2, 0.4, 0.6, 0.8, 1)$

$d = \text{size}(\vec{t}) - M - 1 = \underline{\underline{2}}$

p31

- false : has to be in order
- true : $M \geq 1$, $\text{size}(\vec{t}) = 5$)

  $d = 5 - 1 - 1 = 3$

- false : $d - 0$

  $\rightarrow M = \text{size}(\vec{t}) - d - 1 = 4$

  $\rightarrow$ however, 2×2 knot vectors are repeated making the intervals have a "size of zero": so we end up with 2 basis functions

P 32

$\{1, \sin(x), \cos(x)\}$

$\tilde{V}_1 = 1 \quad ; \int_{-\pi}^{\pi} 1 \, dx = 2\pi$

$V_1 = \frac{1}{\sqrt{2\pi}}$

$\tilde{V}_2 = \sin(x) - \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin(x) \, dx}_{=0} \cdot \frac{1}{\sqrt{2\pi}} = \sin(x) \; ; \int_{-\pi}^{\pi} \sin^2(x) dx = \pi$

$V_2 = \frac{\sin(x)}{\sqrt{\pi}}$

$\tilde{V}_3 = \cos(x) - \text{const} \cdot \underbrace{\int_{-\pi}^{\pi} \cos(x) \, dx}_{=0} - \text{const} \cdot \underbrace{\int_{-\pi}^{\pi} \sin(x) \cos(x) \cdot \sin(x)}_{=0}$

$V_3 = \frac{\cos(x)}{\sqrt{\pi}}$

$\rightsquigarrow$ was orthogonal but not orthonormal.

P 34

a) 14 point $\to$ degree = 13

b) cubic splines

P 35

a) 5 point $\to$ degree 4 polynomial

$y_i = a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3 + a_4 x_i^4$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ & & \vdots & & \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & x_1 \\ \vdots & \\ 1 & x_5 \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

doesn't pass through all points!

p36

a) 2

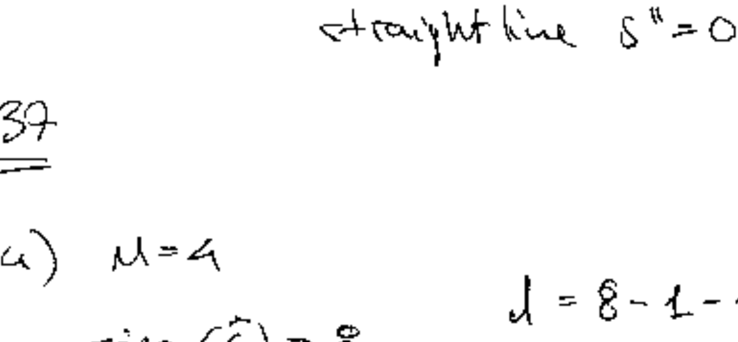
b) $S_1 = c_{11} + c_{12}x + c_{13}x^2 + c_{14}x^3$

$S_2 = c_{21} + c_{22}x + c_{23}x^2 + c_{24}x^3$

c) 8 constraints

d) Look at p29 for similar exercise

e)



p37

a) $M = 4$

size$(\vec{t}) = 8$

$d = 8 - 1 - 1 = 3$

b) „clamped" since they take value t_j where all others are zero & repeated knots p28

$$\langle x, y \rangle = \frac{1}{N} \sum x \cdot y$$

$$\tilde{\phi}_1 = \|\tilde{\phi}_1\| = \left(\tfrac{1}{2} (1+1) \right)^{0.5} = 1$$

$$\phi_1 = \underline{1}$$

$$\tilde{\phi}_2 = x - \left(\tfrac{1}{2} (-1+1) \right) \cdot 1 = x \;;\; \|\tilde{\phi}_2\| = 1$$

$$\phi_2 = x$$

$\Phi = \{1, x\}$ => was already orthonormal

$$\alpha_1 = \langle y_i, \phi_1(x_i) \rangle = \tfrac{1}{2} \sum y_i \cdot \phi_1(x_i) = \tfrac{1}{2}(0 \cdot 1 + 2 \cdot 1) = \underline{1}$$

$\underline{\underline{1:}}$ $y(x) = 1$

$$\alpha_2 = \tfrac{1}{2}(0 \cdot -1 + 2 \cdot 1) = 1$$

$\underline{\underline{2:}}$ $y(x) = \underline{x + 1}$ => no surprise (look at data)

p34

$$a_1 (1 - |x|) + a_2 \cdot (1 - |x - 3|) = y(x)$$

evaluate at the two data points:

$$a_1 - 2a_2 = 2$$
$$-2a_1 + a_2 = 10$$
$$\longrightarrow \quad a_1 = -\frac{22}{3}, \; a_2 = -\frac{14}{3}$$