6- Numerical Integration pt. 2

example

P11

$$G(h) = G(0) + c_1 h + c_2 h^2 + \dots$$

$$\varepsilon\left(\tfrac{h}{2}\right) = \left|G(0) - G\left(\tfrac{h}{2}\right)\right| = \left| \underline{-\frac{c_1}{2} h} - \frac{c_2}{4} h^2 + \dots \right|$$

exact value

same leading order

$$G\left(\tfrac{h}{2}\right) - G(h) = \underline{-\frac{c_1}{2} h} - \frac{3 c_2}{4} h^2 + \dots$$

$$\Longrightarrow \quad \varepsilon\left(\tfrac{h}{2}\right) \approx \left|G\left(\tfrac{h}{2}\right) - G(h)\right|$$

P14

a) true

b) $G_1(h) = 2G\left(\frac{h}{2}\right) - G(h) = \left(2G(0) + \cancel{c_1 h} + \frac{1}{2} c_2 h^2 + \frac{1}{4} c_3 h^3 + \dots\right)$

$$- \left(G(0) + \cancel{c_1 h} + c_2 h^2 + c_3 h^3 + \dots\right)$$

$$= \dots - \underline{\underline{\frac{3}{4} c_3 h^3}}$$

c) 2

d) 3

P 15

$\varepsilon(\frac{h}{2}) = |G(0) - G(\frac{h}{2})|$ „exact"

$\varepsilon(\frac{h}{2}) \approx |G(\frac{h}{2}) - G(h)|$ „approx."

$G(0) = f'(0) = \underline{0}$

$G(\frac{h}{2}) = G(0.1) = \frac{f(0.1)}{0.1} = \underline{0.1}$

$G(h) = G(0.2) = \frac{f(0.2)}{0.2} = 0.2$

$\varepsilon(\frac{h}{2}) = |0 - 0.1| = \underline{0.1}$

$\approx |0.1 - 0.2| = \underline{0.1}$

in this case the same => not always the case

P 16

```
fun Adaptive (a,b)
    Apply simpson's on [a,b]
    Split [a,m] & [m,b] ; m = (a+b)/2
    estimate error ε(h/2)

    if error > threshold
        return Adaptive (a,m) + Adaptive(m,b)

    else
        return value of simpson's rule

    end if
end fun
```

P 18

$$f(x) = \frac{1}{1-x} = c_0 + c_1 x + c_2 x^2 + \ldots$$

$$\underline{1} = (1-x)\left(c_0 + c_1 x + c_2 x^2 + \ldots\right)$$

compare LHS 2 RHS

$$= \underline{c_0} + \underline{c_1} x + c_2 x^2 + \ldots - \underline{c_0} x - c_1 x^2 - \ldots$$

$$1 = c_0 \longrightarrow c_0 = 1$$

$$0 = c_1 - c_0 \longrightarrow c_1 = c_0 = 1$$

$$\vdots$$

$$\longrightarrow c_i = 1$$

$$\Rightarrow \frac{1}{1-x} = 1 + x + x^2 + \ldots \quad \text{"geometric series"}$$

P 19 difficult

first show that the error term only has orders 2, 4, 6, ..

. Taylor expand $f(x)$ around 0:

$$I = \int_{-h}^{h} f(0) + f'(0)x + \ldots \, dx = \underline{2hf(0) + 0} + \frac{f''(0)}{6} h^3 + \ldots$$

Taylor exp.

$$I_{\text{Trapezoidal}} = h \cdot (f(-h) + f(h)) = h \cdot \Big(f(0) - f'(0)h + \frac{f''(0)}{2} h^2 - \ldots + f(0) + f'(0)h + \frac{f''(0)}{2} h^2 + \ldots\Big)$$

$$= \underline{2h f(0) + 0} + f''(0) h^3 + \ldots$$

- Compare the two terms: Terms are similar all the way up to order $\underline{3}$ for a single interval
- If we use multiple intervals the error is increased N times; where $N = \frac{b-a}{h}$
  → The order drops by one and becomes $\underline{\underline{2}}$

→ So now we can use Richardson's:

$$I_{Trapezoidal} = \underset{\text{true}}{I} + c_1 h^2 + c_2 h^4$$

only even terms; we showed that before

$$\frac{4 \cdot I_{Trapezoidal}\left(\frac{h}{2}\right) - I_{Trapezoidal}(h)}{3} = I - \frac{c_2}{4} h^4 - \ldots$$

dropped h^2 term

⇝ you can generalize this.

P 20

max number of intervals = 4

$n=1: \ h = \frac{4}{4} = 1$

$$I_0^4 = \frac{1}{2}\left(2+2+2\cdot 3+2\cdot 4+2\cdot 1\right)$$

$$= \underline{10}$$

$n=2: \ h = \frac{4}{2} = 2$

$$I_0^2 = 1\cdot\left(2+2+2\cdot 4\right) = \underline{14}$$

$n=1: \ h = \frac{4}{1} = 4$

$$I_0^1 = 2\cdot\left(2+2\right) = \underline{8}$$

then:

$$I_1^2 = \frac{1}{4^1-1}\cdot\left(4^1\cdot I_0^4 - I_0^2\right) = \cancel{26}\ \frac{26}{3}$$

$$I_1^1 = \frac{1}{4^1-1}\left(4^1\cdot I_0^2 - I_0^1\right) = \frac{48}{3}$$

$$I_2^1 = \frac{1}{4^2-1}\left(4^2\cdot I_1^2 - I_1^1\right) = \underline{\underline{\frac{1}{15}\cdot\left(16\cdot\frac{26}{3}-\frac{48}{3}\right)}}$$

P2f