Title

- Text

Exercise 12: Sampling

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Outline

1. Information
2. Goals
3. Theory/ Recap
4. Exercises

Information

General

- Lecture material & problem sets available here
- Tutorial material available here

Goals

Goals of Today

- Understand why we want to sample from a distribution
- Understand inverse CDF sampling
- Understand limitations of inverse CDF sampling
- Understand rejection sampling
- Understand Markov Chain Monte Carlo sampling

- **Difficult examples today! Don't be intimidated!**

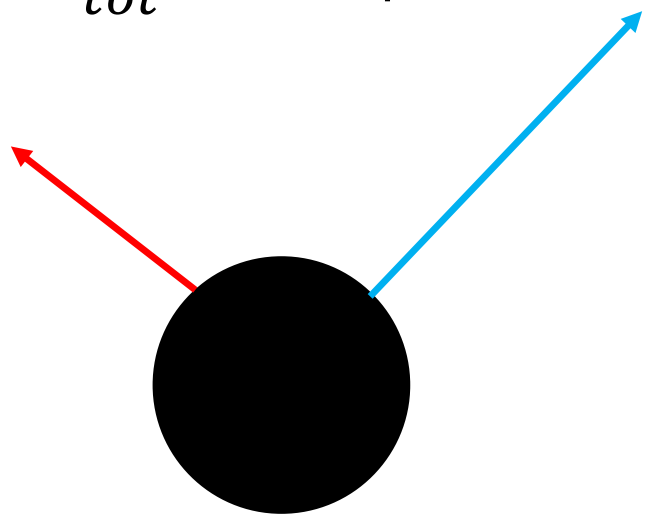
Theory / Recap

ETH zürich

Why do we sample?

- Simulate a probabilistic model
- Obtain statistical insights

$$F_{tot} = F^P + F^R$$



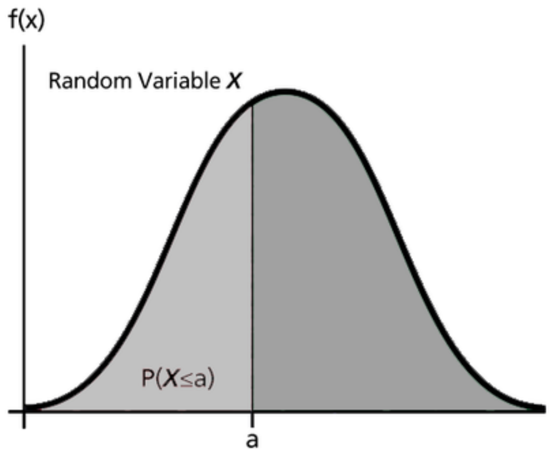
Inverse CDF sampling

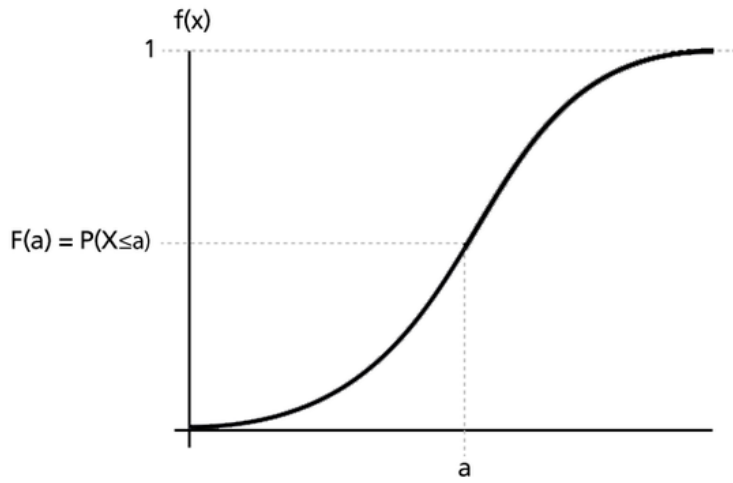
- Cumulative Distribution Function:

$$cdf(x) = \int_{-\infty}^{x} p(\bar{x})d\bar{x}$$

- Inverse CDF sampling:
  1. Sample $u \sim unif(0,1)$
  2. Generate sample $x = cdf^{-1}(u)$

- Difficult: Have to compute integral & inverse!





Example 1: Sampling from a uniform distribution

- Assume we want to sample from the uniform distribution $p(x) = \frac{1}{2}, x \in [-1, 1]$
- We can generate sample from $p(y) = 1, y \in [0, 1]$

- Tasks:
  - Set up the inverse sampling algorithm by:
    1. Compute the cdf of $p(x)$
    2. Compute the inverse of the cdf

  - Show that we are effectively sampling from $p(x)$ by using $p(x) = \frac{p(f(x))}{\left|\frac{f^{-1}(y)}{dy}\right|}$ **(ADVANCED)**

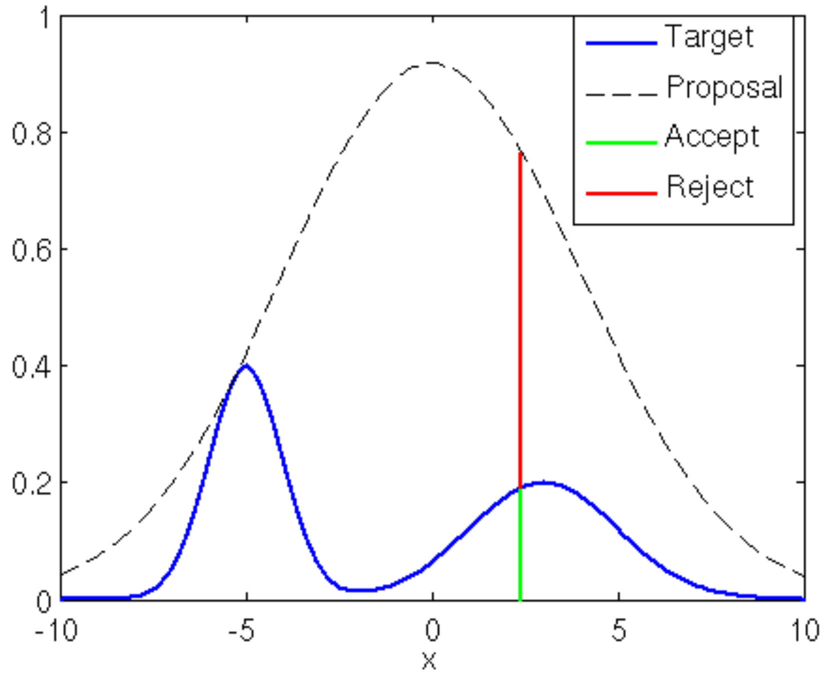
Rejection Sampling

- If we cannot compute $cdf^{-1}(x)$

- Define envelope “proposal”:
$$p(x) \leq L \cdot q(x)$$

- Rejection Sampling:
  1. Draw $y \sim q$ and $u \sim unif(0,1)$
  2. If $u \cdot L \cdot q(y) \leq p(y)$ keep the sample y, else discharge

- Note: We are not sampling from p directly, only evaluating!



Example 2: Rejection sampling with normal proposal

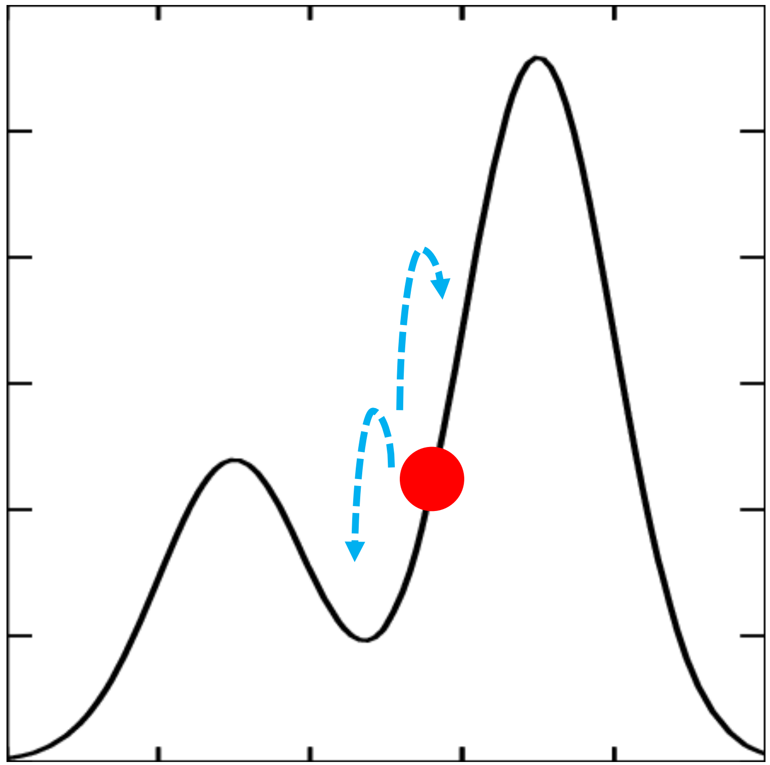
- We want to sample from a uniform distribution $p(x) = \frac{1}{2}, x \in [-1, 1]$
- But can only generate samples from $q(x) = 1/\sqrt{\pi} \cdot \exp -x^2$ (normal dist)

- Tasks:
  - Determine an optimal L such that $p(x) \leq L \cdot q(x)$, use $L_{optimal} = \left\| \frac{p(x)}{q(x)} \right\|_\infty$
  - You draw $y = 1.1$ and $u = 0.9$ – is the sample accepted or rejected?

- Tipps:
  - If $u \cdot L \cdot q(y) \leq p(y)$ keep the sample y, else discharge

Markov Chain Monte Carlo

- MCMC:
  1. Sample $y \sim \mathcal{N}(x, \sigma^2)$ and $u \sim unif(0,1)$
  2. If $u \cdot p(x) \leq p(y)$ set $x \leftarrow y$ else keep $x \leftarrow x$
- Analogy:
  - You hike in the mountains, you want to take more photos at the top and less at the bottom ("nice view")
  - If you move around the mountain according to MCMC and take a photo after every step you will end up with the desired photo collage
- Note: If $p(y) \geq p(x)$ we always move to y



Example 3: MCMC step

- Perform MCMC for two steps of MCMC
- The target distribution is $p(x) = 2x, x \in [0, 1]$
- You have access to the following values:
  - $x_1 = 0.5$
  - $s_1 = 0.1$ and $s_2 = -0.2$ both drawn from $s \sim \mathcal{N}(0, \sigma^2)$
  - $u_1 = 0.9$ and $u_2 = 0.3$ both drawn from $u \sim unif(0,1)$

- Tipps:
  - $y = \mu + s$ then $y \sim \mathcal{N}(\mu, \sigma^2)$ if $s \sim \mathcal{N}(0, \sigma^2)$
  - MCMC:
    1. Sample $y \sim \mathcal{N}(x, \sigma^2)$ and $u \sim unif(0,1)$
    2. If $u \cdot p(x) \leq p(y)$ set $x \leftarrow y$ else keep $x \leftarrow x$

Exercises

Q1

- Implement rejection sampling and MCMC for the Gibbs Distribution

Questions?