5 Numerical Integration pt. 1

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Schedule

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    1. Setup
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    5. Gauss Quadrature
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Theory

Setup

- Exact integral value I
- Exact subintegral I_i
- Approximate subintegrals I_{Ri}, I_{Ti}, I_{Si}
- Newton cotes weights C_k^n

Why numerical integration?

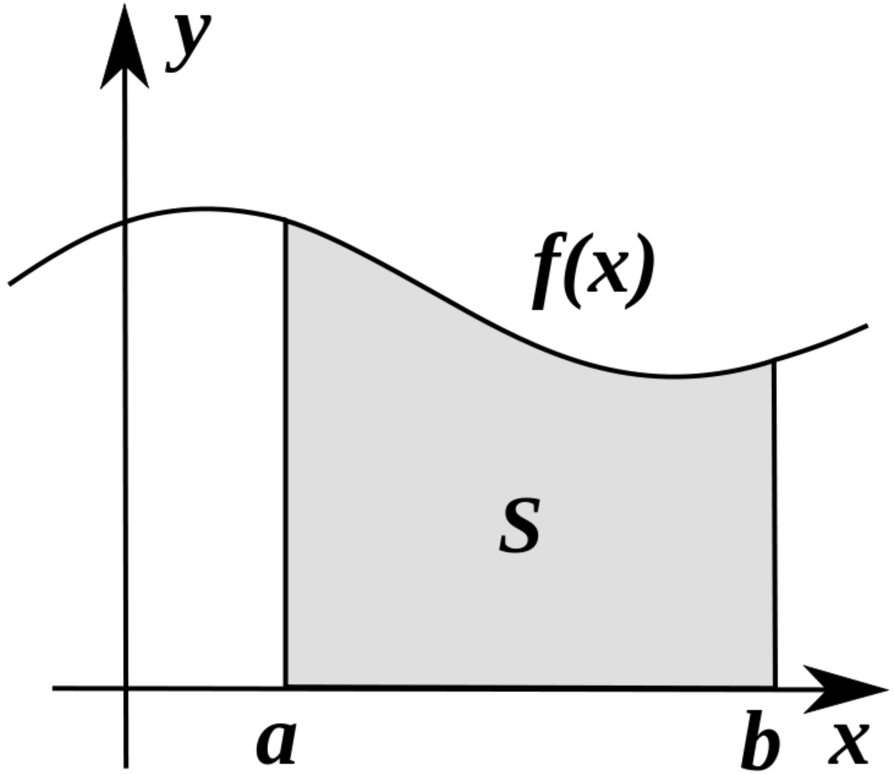
- Exact Integral:

$$I = \int_a^b f(x)dx$$

- …can be solved analytically or not

- Used for differential equations:

$$\dot{x} = f(x)$$



Numerical Integration

- Approximate:

$$I = \int_a^b f(x)dx = \sum_{i=0}^{N-1} I_i$$

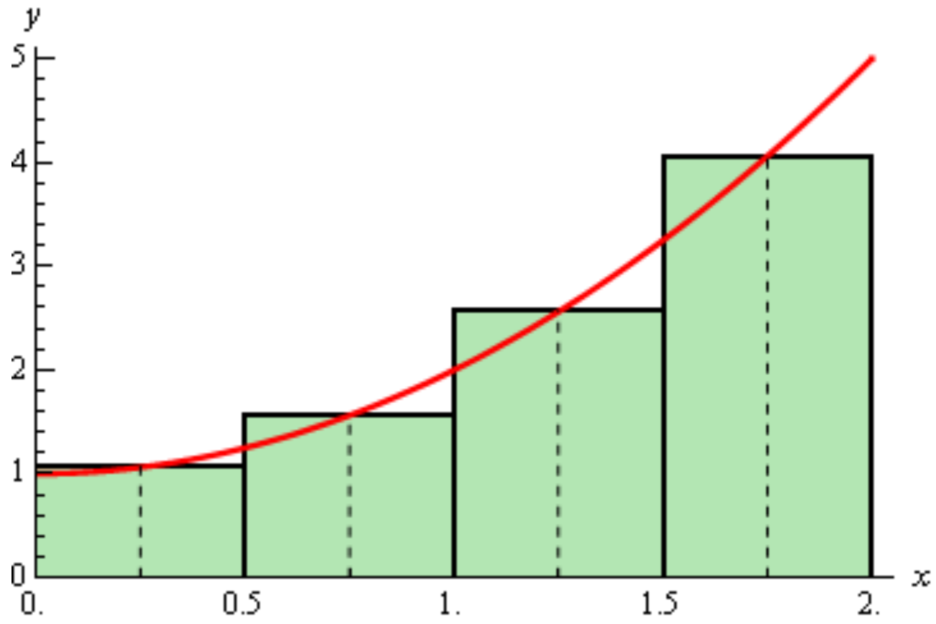
- Approximate I_i: (table)
  - Rectangular Rule
  - Trapezoidal Rule
  - Simpson's Rule

Approximations for I_i

Rectangular Rule	$I_{Ri} = f\left(x_{i+1/2}\right) \cdot \Delta_i$
Trapezoidal Rule	$I_{Ti} = \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta_i$
Simpson's Rule	$I_{Si} = \frac{f(x_i)+4f\left(x_{i+1/2}\right)+f(x_{i+1})}{6} \cdot \Delta_i$

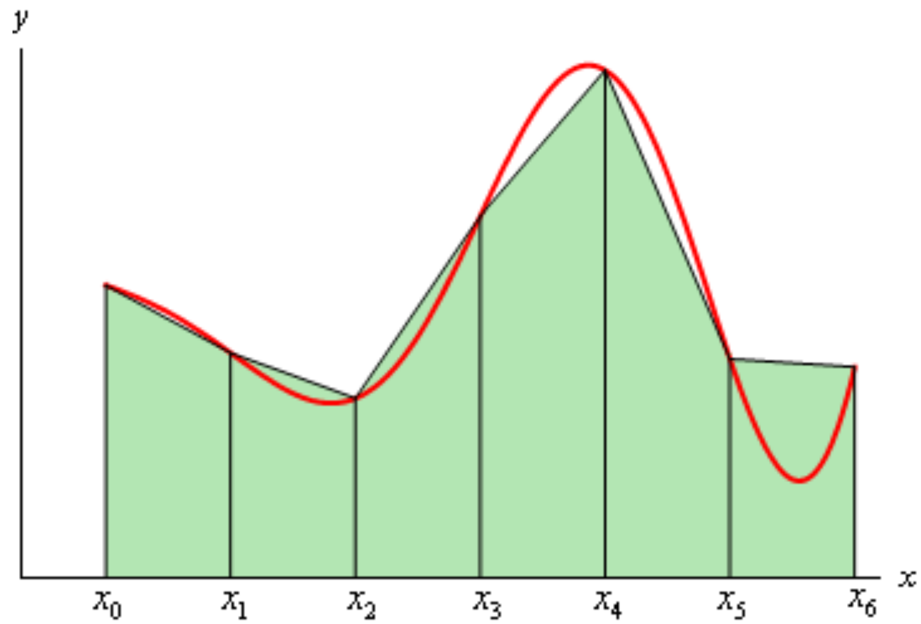
Numerical Integration cont.

- Rectangular Rule



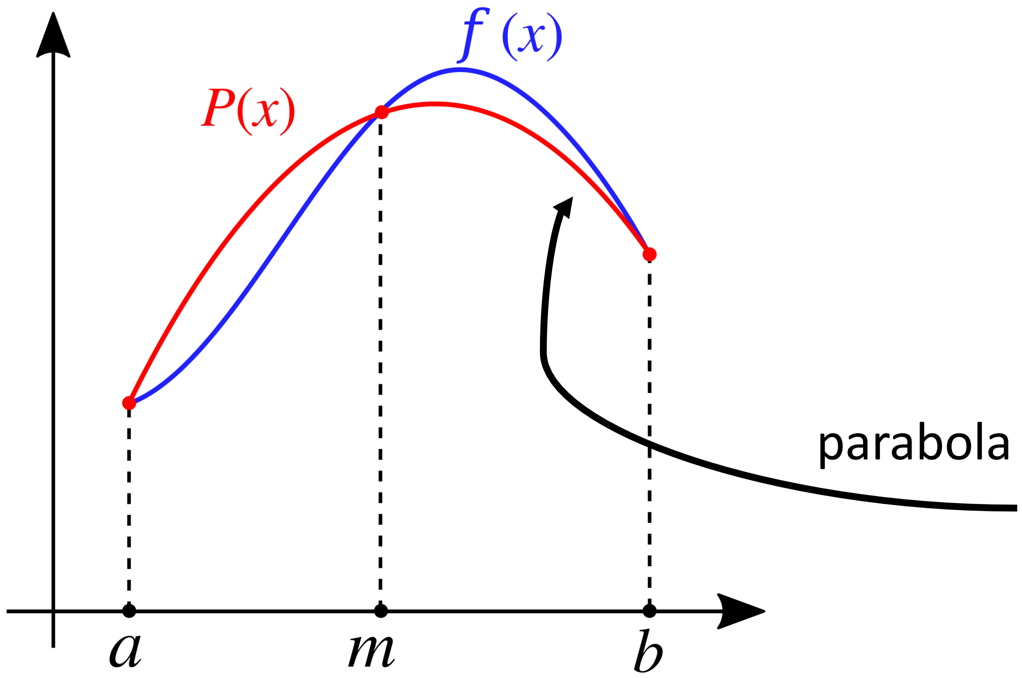
Numerical Integration cont.

- Trapezoidal Rule



Numerical Integration cont.

- Simpson's Rule



Numerical Integration cont. Example

- Task:
  - Solve the integral

$$\int_{-1}^{1} x^2 dx$$

1. Exact
2. Rectangle Rule
3. Trapezoidal Rule
4. Simpson's Rule

Approximations for I_i

Rectangular Rule	$I_{Ri} = f\left(x_{i+1/2}\right) \cdot \Delta_i$
Trapezoidal Rule	$I_{Ti} = \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta_i$
Simpson's Rule	$I_{Si} = \frac{f(x_i)+4f\left(x_{i+1/2}\right)+f(x_{i+1})}{6} \cdot \Delta_i$

Newton Cotes

- Approximate:

$$I \approx (b-a) \cdot \sum_{k=0}^{n} C_k^n f(x_k)\,, \qquad with\ C_k^n = \frac{1}{b-a}\int_a^b l_k^n(x)dx \quad „weights“$$

- Properties of C_k^n:
  - $\sum_{k=0}^{n} C_k^n = 1$
  - $C_k^n = C_{n-k}^n$

- Note: Can be used for entire integral I or for smaller intervals I_i

Newton Cotes cont. Example

- Task:
  - use $n = 2$ and equally spaced x_0, x_1, x_2 to approximate I

$$I \approx (b-a) \cdot \sum_{k=0}^{n} C_k^n f(x_k)\,,$$

$$with\ C_k^n = \frac{1}{b-a} \int_a^b l_k^n(x)dx \quad „weights“$$

- Tipp: $C_1^2 = \frac{2}{3}$

Gauss Quadrature

- What we have:

  n weights (α_i) + n abscissas (x_i) $= 2n$ parameters

- What we want: Integrate $P_{2n-1}(x)$ exactly (integral increases order by one):

$$\int_a^b f(x)dx \approx \int_a^b P_{2n-1}(x)dx = \sum_{i=1}^{n} \alpha_i \cdot f(x_i)$$

Gauss Quadrature cont.

- Transform integral to be in the interval $[-1,\ 1]$ in order to use lookup-tables:

$$z = \frac{2x - (a+b)}{b-a}$$

- We then have:

$$\int_{-1}^{1} f(z)dz \approx \sum_{i=1}^{n} \alpha_i \cdot f(z_i)$$

- You can now look up z_i and α_i in tables

Gauss Quadrature cont. Example

- Task:
  - Find – by Gauss Quadrature - the exact integral value:

$$I = \int_0^2 x^3 dx$$

- Tipps:
  - Exact for order $2n - 1$
  - $z = \frac{2x-(a+b)}{b-a}$

n=1	x_i	α_i
1	0	2
n=2	x_i	α_i
1	$-\sqrt{\frac{1}{3}} \approx -0{,}57735026919$	1
2	$\sqrt{\frac{1}{3}} \approx 0{,}57735026919$	1
n=3	x_i	α_i
1	$-\sqrt{\frac{3}{5}} \approx -0{,}774596669241$	$\frac{5}{9} \approx 0{,}555555555556$
2	0	$\frac{8}{9} \approx 0{,}888888888889$
3	$\sqrt{\frac{3}{5}} \approx 0{,}774596669241$	$\frac{5}{9} \approx 0{,}555555555556$
n=4	x_i	α_i
1	$-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} \approx -0{,}861136311594053$	$\frac{18-\sqrt{30}}{36} \approx 0{,}347854845137454$
2	$-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} \approx -0{,}339981043584856$	$\frac{18+\sqrt{30}}{36} \approx 0{,}652145154862546$
3	$\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} \approx 0{,}339981043584856$	$\frac{18+\sqrt{30}}{36} \approx 0{,}652145154862546$
4	$\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} \approx 0{,}861136311594053$	$\frac{18-\sqrt{30}}{36} \approx 0{,}347854845137454$

Exercises

none

Homework

HW 1

Consider this quadrature rule:

$$\int_{-1}^{1} f(x)\,dx \approx a\,(f(-1) + f(1)) + b\,(f'(-1) - f'(1))\,.$$

If $a = 1$ and $b = 0$ it is a trapezoidal rule and the formula is exact for $f(x) = 1$ and $f(x) = x$. Find a and b which make the formula exact also for $f(x) = x^2$.

HW 2

- Derive the 2 point Gauss Rule $I = \int_a^b f(x)dx \approx \alpha_1 f(x_1) + \alpha_2 f(x_2)$

- Tip:
  - Has to integrate a generic polynomial of degree $4 - 1 = 3$
  - Integrate the generic polynomial and compare coefficients
  - Solution in the script

HW 3

Lobatto quadrature is similar to Gaussian quadrature with the following differences:

- The integration points include the end points of the integration interval.
- It is accurate for polynomials up to degree $2n-3$, where n is the number of points

Let write the rule in the following way:

$$\int_{-1}^{1} f(x)\,dx \approx a\,(f(-1)+f(1)) + b\,(f(-\alpha)+f(\alpha))$$

It uses $n = 4$ points $(-1,\ 1,\ \alpha,\ -\alpha)$ and the formula should be exact for polynomials up to degree $2n-3=5$. Find α, a and b.