Exercise 8: Numerical Integration

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Outline

1. Information
2. Goals
3. Theory / Recap (55’)
4. Exercises (5’)

Information

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General

- Slides by LAB are on the website (look at them before proceeding!)
- These slides: https://polybox.ethz.ch/index.php/s/9NFCvtriBbRBDnS

- **No exercises next week (1. Mai), but lecture!**

Goals

Goals of Today

- Understand why we use numerical integration
- Understand the basic approach for numerical integration
- Know how to integrate numerically with Rectangle, Trapezoidal, & Simpson’s rule
- Understand derivation of Newton-Cotes formula & how to use it

Theory / Recap

Integral

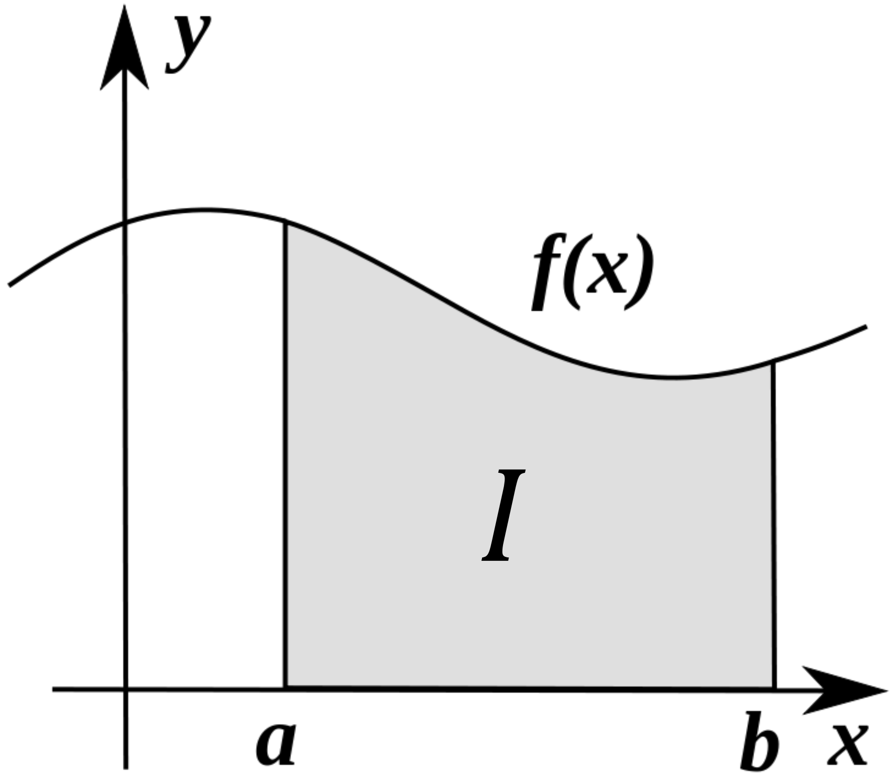
- Exact Integral:

$$I = \int_a^b f(x)dx$$

- …can be solved analytically or not

- Used for differential equations:

$$\dot{x} = f(x)$$



https://de.wikipedia.org/wiki/Integralrechnung#/media/File:Integral_as_region_under_curve.svg

Exercise 1

- Solve the ODE:

$$\dot{x} = t \cdot e^{t}$$

- Tipp:

$$\int_a^b uv'dx = [uv]_a^b - \int_a^b u'vdx$$

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Numerical Integration

- Approximate:

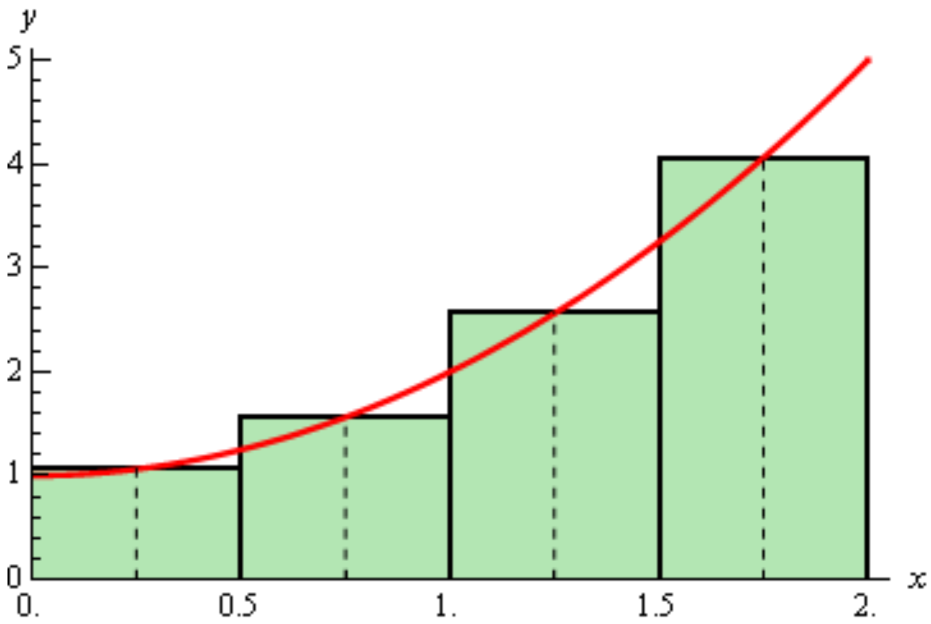
$$I = \int_a^b f(x)dx = \sum_{i=0}^{N-1} I_i$$

- Approximate I_i: (table)
  - Rectangular Rule
  - Trapezoidal Rule
  - Simpson's Rule

Approximations for I_i

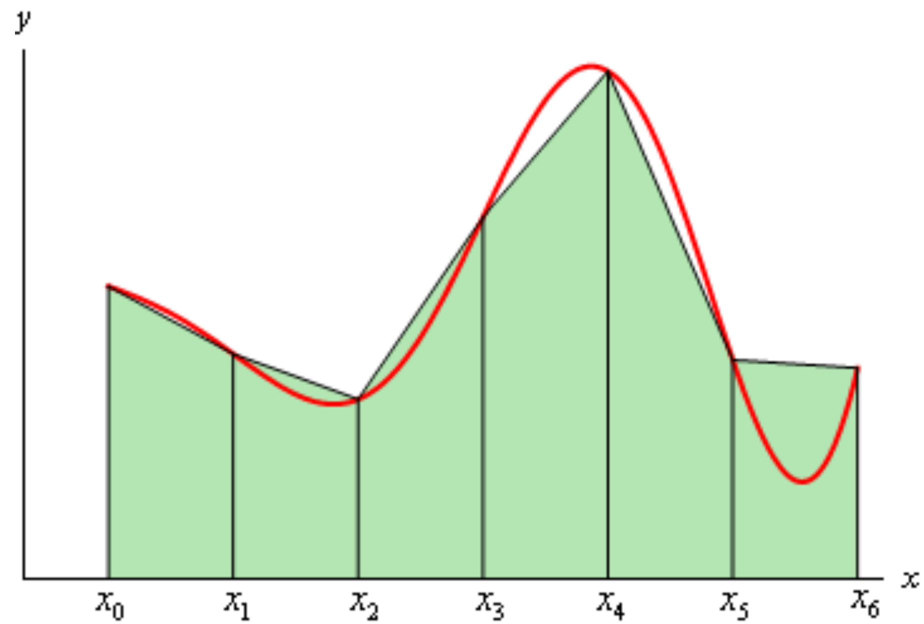
Rectangular Rule	$I_{Ri} = f\left(x_{i+1/2}\right) \cdot \Delta_i$
Trapezoidal Rule	$I_{Ti} = \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta_i$
Simpson's Rule	$I_{Si} = \frac{f(x_i)+4f\left(x_{i+1/2}\right)+f(x_{i+1})}{6} \cdot \Delta_i$

Rectangular Rule



http://tutorial.math.lamar.edu/Classes/CalcI/AreaProblem.aspx

Trapezoidal Rule



http://tutorial.math.lamar.edu/Classes/CalcI/AreaProblem.aspx

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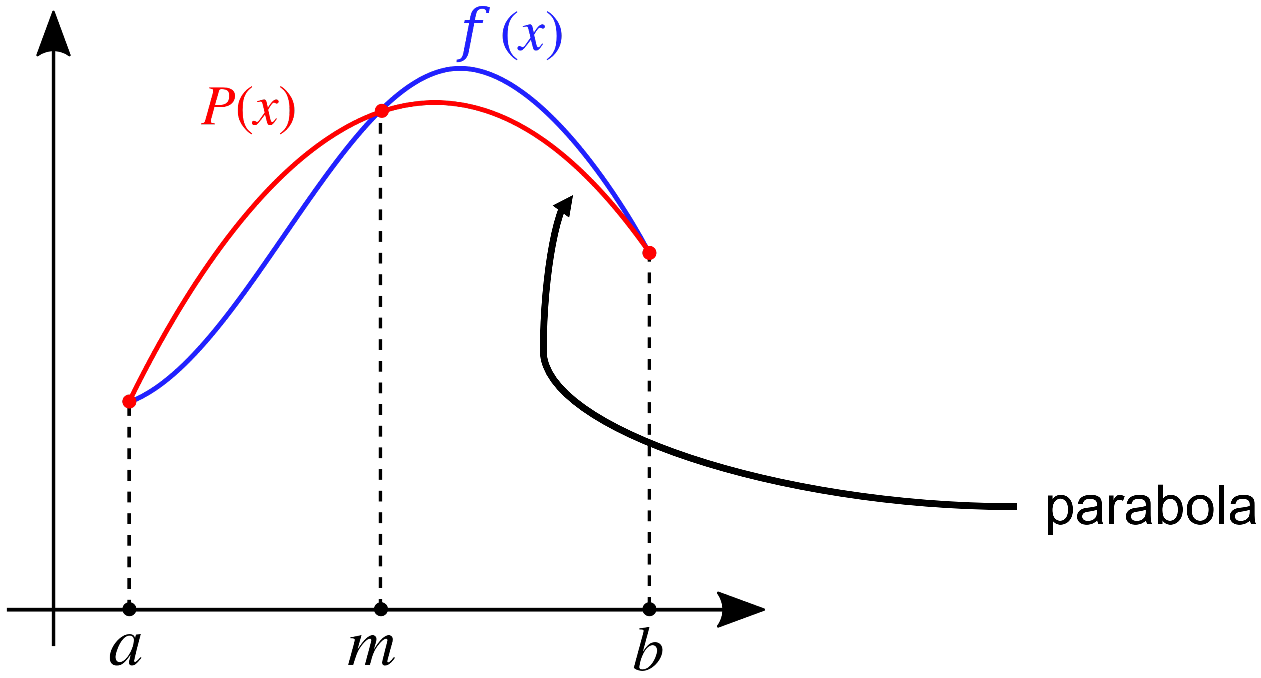
Simpson's Rule



https://de.wikipedia.org/wiki/Datei:Simpsons_method_illustration.svg

Exercise 2

- Solve the integral

$$\int_{-1}^{1} x^2 dx$$

1. Exact
2. Rectangle Rule
3. Trapezoidal Rule
4. Simpson's Rule

Approximations for I_i	
Rectangular Rule	$I_{Ri} = f\left(x_{i+1/2}\right) \cdot \Delta_i$
Trapezoidal Rule	$I_{Ti} = \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta_i$
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Newton-Cotes Formula

- Approximate:

$$I \approx (b-a) \cdot \sum_{k=0}^{n} C_k^n f(x_k)\,, \qquad with\; C_k^n = \frac{1}{b-a}\int_a^b l_k^n(x)dx \quad „weights“$$

- Properties of C_k^n:
  - $\sum_{k=0}^{n} C_k^n = 1$
  - $C_k^n = C_{n-k}^n$

- Note:
  - Can be used for entire integral I or for smaller intervals I_i
  - The “weights” C_k^n are used supporting points f_k

Exercise 3

- use $n = 2$ and equally spaced x_0, x_1, x_2 to approximate I

$$I \approx (b-a) \cdot \sum_{k=0}^{n} C_k^n f(x_k),$$

$$with\ C_k^n = \frac{1}{b-a} \int_a^b l_k^n(x) dx \quad „weights“$$

- Tipp: $C_1^2 = \frac{2}{3}$

Exercises

Exercises

1. Use Newton-Cotes to derive Simpson‘s Rule (no short-cuts)
2. Implement Simpson’s & Trapezoidal Integration (coding)

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Questions?