2 Non-Linear Systems

PVK 2019: MAD

bacdavid@student.ethz.ch

gitlab.ethz.ch/bacdavid

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Theory

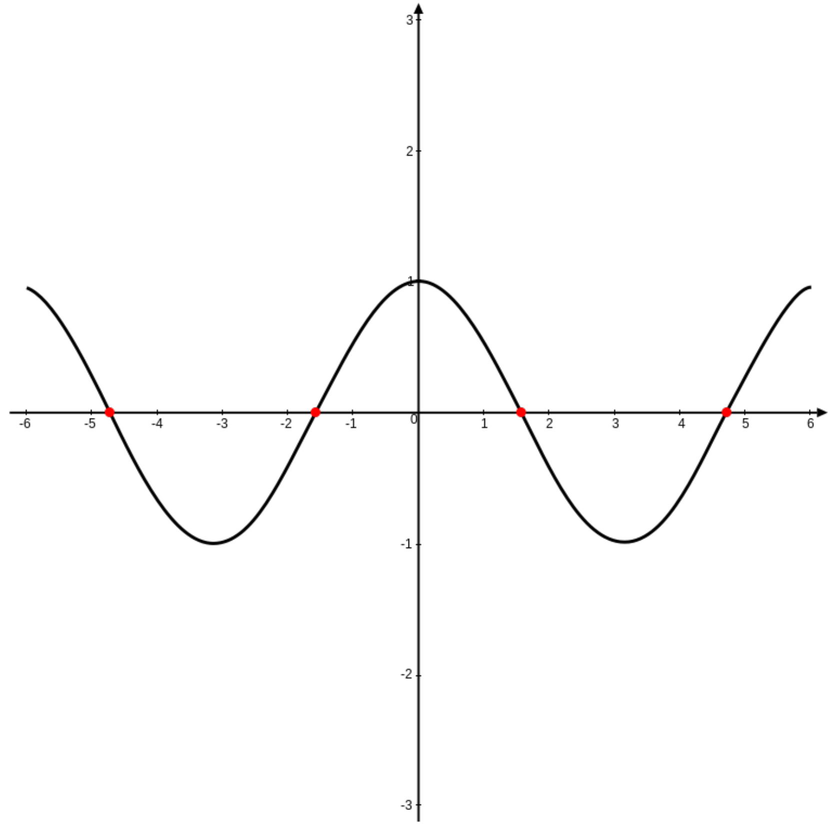
Setup

- Root of function in 1D x^*
- Root of function in nD $\vec{x}^* = [x_1^*, \ldots, x_n^*]^T$
- System of equations $\vec{F}(\vec{x})$
- Jacobian at $\vec{x}$ $J(\vec{x})$

Root of a function

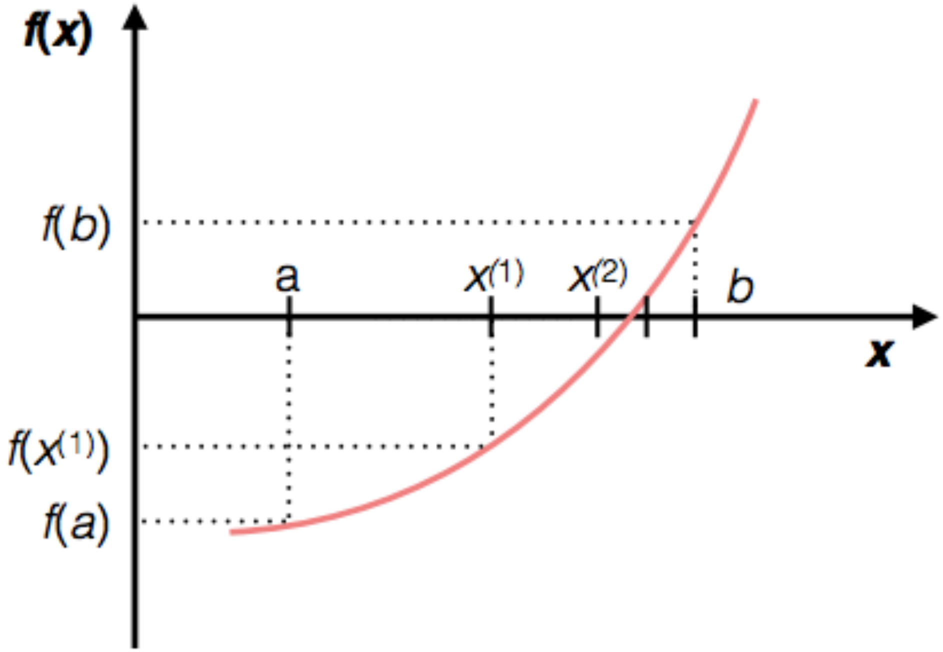
- $f(x^*) = 0$, x^* is a root of f

- **Intermediate Value Theorem**:
  - f is continuous
  - $\text{sign}\, f(a) \neq \text{sign}\, f(b)$
  - Then $x^* \in [a, b]$



Bisection Method

- Directly induced from intermediate value theorem
- Decrease interval size until sufficiently small, make sure that the bounds always have opposite sign



Newton’s Method in 1D

- Approximate a function by Taylor expansion:

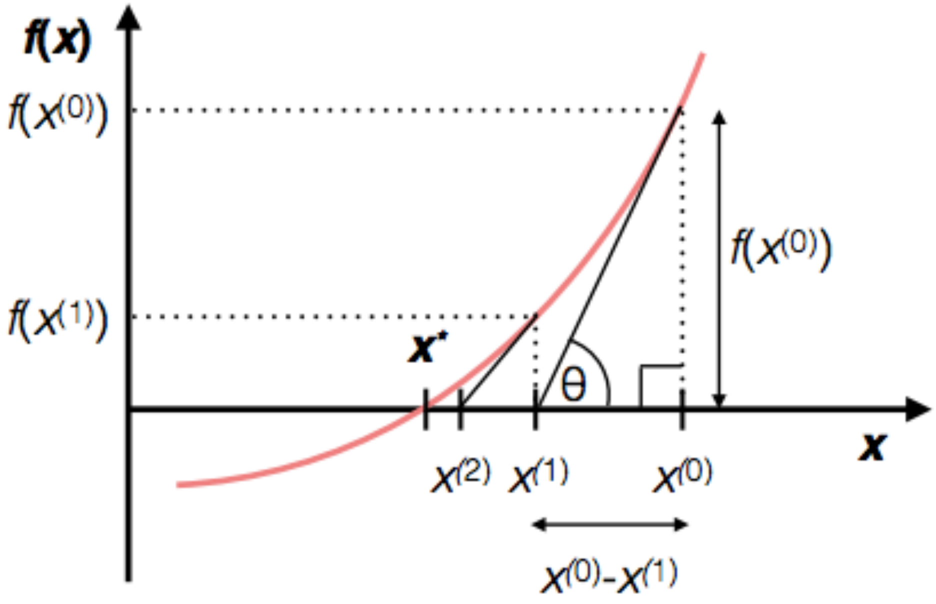
$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

- Discretize:

$$f\left(x^{(k)}\right) \\ = f\left(x^{(k-1)}\right) + f'\left(x^{(k-1)}\right)\left(x^{(k)} - x^{(k-1)}\right)$$

- Set $f\left(x^{(k)}\right) = 0$:

$$x^{(k)} = x^{(k-1)} - \frac{f\left(x^{(k-1)}\right)}{f'(x^{(k-1)})}$$



Newton’s Method in 1D cont.

- Resulting algorithm:

1. Start with initial guess x^0
2. Iterate $x^{(k)} = x^{(k-1)} - \frac{f\left(x^{(k-1)}\right)}{f'\left(x^{(k-1)}\right)}$
3. Stop when $\left|x^{(k)} - x^{(k-1)}\right| < \epsilon$

Newton’s Method in 1D cont. Example

- Given:
  - Cube of volume 10 m^3 - What‘s the side length?

- Task:
  - Write down the function you are trying to find a root of
  - Make an initial guess
  - Iterate once using Newton’s method
  - Use $x^0 = 0$ as an initial guess – what happens? Why?

Newton's Method in nD

- Replace the gradient with the Jacobian and the division by the inverse:

$$\vec{x}^{(k)} = \vec{x}^{(k-1)} - J^{-1}\left(\vec{x}^{(k-1)}\right)\vec{F}\left(\vec{x}^{(k-1)}\right)$$

- Define:

$$\vec{\boldsymbol{x}}^{(k)} - \vec{\mathbf{x}}^{(k-1)} = \vec{\boldsymbol{y}}^{(k-1)}$$

- Update rule:

$$J\left(\vec{x}^{(k-1)}\right)\vec{y}^{(k-1)} = -\vec{F}\left(\vec{x}^{(k-1)}\right)$$

Newton’s Method in nD cont.

- Resulting algorithm:

1. Start with initial guess $\vec{x}^0$
2. Iterate
    1. Solve for $\vec{y}^{k-1}$ in $J\left(\vec{x}^{(k-1)}\right)\vec{y}^{(k-1)} = -\vec{F}\left(\vec{x}^{(k-1)}\right)$
    2. Compute $\vec{x}^{(k)} = \vec{y}^{(k-1)} - \vec{x}^{(k-1)}$
3. Stop when $\left\|\vec{x}^{(k)} - \vec{x}^{(k-1)}\right\|_2 < \epsilon$

Newton's Method in nD cont. Example

- Given:
  - $x_1 + x_2 - x_1 x_2 + 2 = 0; x_1 \cdot \exp(-x_2) - 1 = 0$
  - Initial Guess: $x_1^{(0)} = 0;\ x_2^{(0)} = 0$

- Task:
  - Compute the Jacobian
  - Solve $J\left(\vec{x}^{(0)}\right) y^{(0)} = -\vec{F}\left(\vec{x}^{(0)}\right)$
  - Compute $\vec{x}^{(1)}$

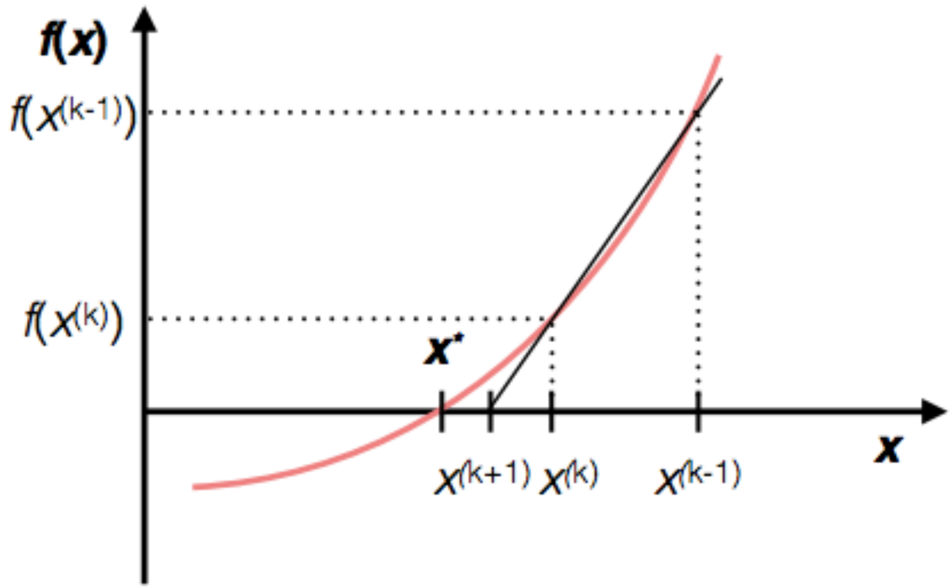
Secant Method

- Approximation of a gradient:

$$f'\left(x^{(k)}\right) \approx \frac{f\left(x^{(k)}\right) - f\left(x^{(k-1)}\right)}{x^{(k)} - x^{(k-1)}}$$

- Similar to Newton's Method but approximate gradient:

$$x^{(k-1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)\left(x^{(k)} - x^{(k-1)}\right)}{f\left(x^{(k)}\right) - f\left(x^{(k-1)}\right)}$$



Convergence Rate

- The error at time k:

$$E^{(k)} = x^{(k)} - x^*$$

- The convergence rate r:

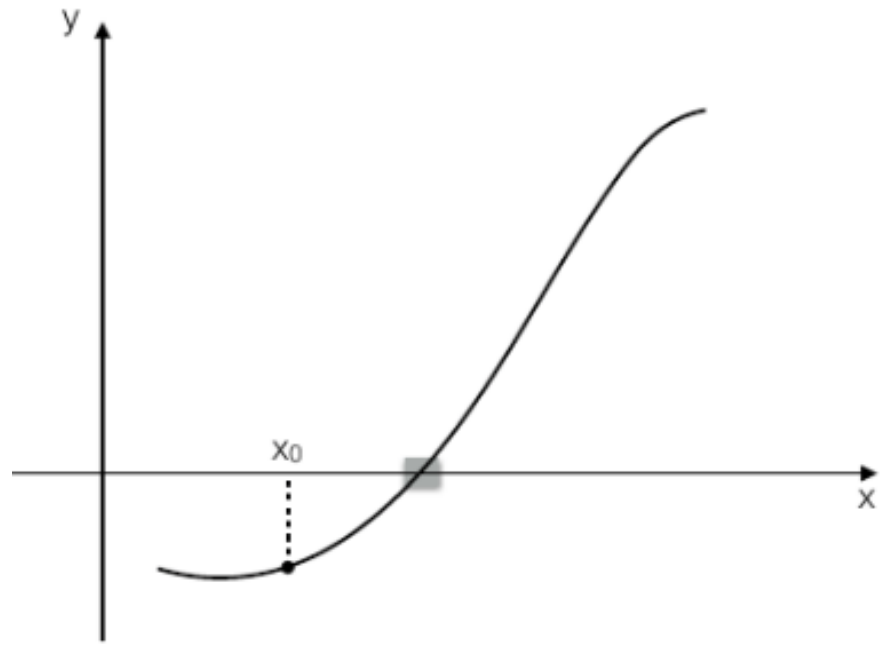
$$\lim_{k \to \infty} \frac{\left|E^{(k-1)}\right|}{\left|E^{(k)}\right|^r} = C$$

Exercises

Exercise 1

- Given:
  - Starting point x_0
  - The gray area is the area of termination

- Task:
  - Perform Newton's method graphically, starting at x_0
  - Is the given info enough to perform Secant method?



Exercise 2

- Task:
  - Solve $e^x - x - 1 = 0$ graphically
  - Solve $e^x = x + 2$ for the positive root using 3 Newton's Iterations
  - Justify your initial guess

Homework

HW 1

- Write pseudo code for bisection method

HW 2

- Find the minimum of x^2 with Newton's Method

HW 3

- Find the convergence rate of the Bisection method
- Solution in script

- Tip:
  - Convergence rate are based on worst case errors
  - Solution in skript