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Title

- Text

Exercise 3: Newton's Method for Systems

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Outline

1. Information
2. Goals
3. Theory/ Recap
4. Exercises

Information

General

- Lecture material & problem sets available here
- Tutorial material available here

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Goals

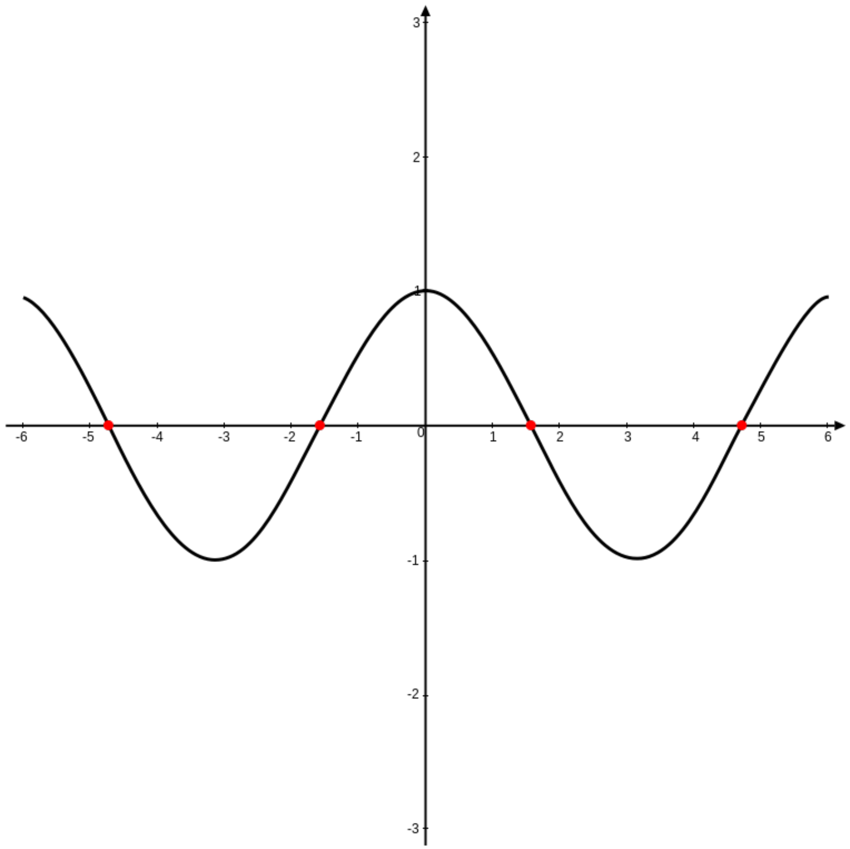
Goals of Today

- Understand Newton’s method (again)
- Understand issues related to Newton’s Method
- Understand how to generalize the method to a system of equations

Theory / Recap

Root of a function

- $f(x^*) = 0$, x^* is a root of f
- **Intermediate Value Theorem**:
  - f is continuous
  - $\text{sign}\, f(a) \neq \text{sign}\, f(b)$
  - Then $x^* \in [a, b]$



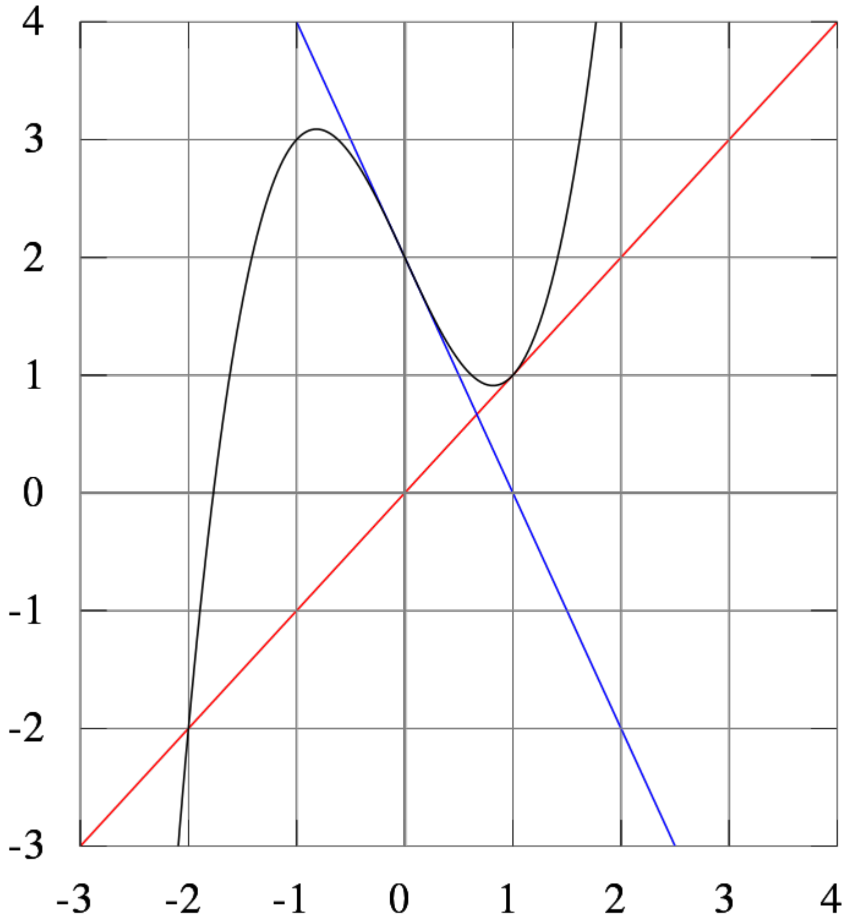
https://en.wikipedia.org/wiki/Zero_of_a_function#/media/File:X-intercepts.svg

Newton's Method

- Derivation:
  - Approximate: $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$ (Taylor)
  - Discretize: $f\left(x^{(k)}\right) = f\left(x^{(k-1)}\right) + f'\left(x^{(k-1)}\right)\left(x^{(k)} - x^{(k-1)}\right)$
  - Set $f\left(x^{(k)}\right) = 0$: $\boxed{x^{(k)} = x^{(k-1)} - \frac{f\left(x^{(k-1)}\right)}{f'\left(x^{(k-1)}\right)}}$ (update rule)
- Iterate until $\left|x^{(k)} - x^{(k-1)}\right| < \epsilon$
- Link

Issues

- Oscillation
- Stationary point



https://en.wikipedia.org/wiki/Newton%27s_method#/media/File:NewtonsMethodConvergenceFailure.svg

Newton's Method for Systems

- $\mathbf{x}^* = [x_1^*, \dots, x_N^*]^T$, $\mathbf{F}(\mathbf{x}^*) = [f_1(x_1^*), \dots, f_N(x_N^*)]^T = \mathbf{0}$
- Update rule: $\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - \boldsymbol{J}^{-1}\left(\mathbf{x}^{(k-1)}\right)\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$
- Define: $\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} = \boldsymbol{y}^{(k-1)}$
- Write update rule as: $\boldsymbol{J}\left(\mathbf{x}^{(k-1)}\right)\boldsymbol{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$

Example 1: System of Equations

- $x_1 + x_2 - x_1 x_2 + 2 = 0; x_1 \cdot \exp(-x_2) - 1 = 0$
- Initial Guess: $x_1^{(0)} = 0;\ x_2^{(0)} = 0$

- Compute the Jacobian
- Solve $\boldsymbol{J}\left(\mathbf{x}^{(k-1)}\right)\boldsymbol{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$
- Compute $\mathbf{x}^{(1)}$

Exercises

Q1

- Implement Newton's method for systems
- Solve a practical problem

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Questions?