3 Interpolation Extrapolation

PVK 2019: MAD

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Schedule

1. Theory
    1. Lagrange interpolation
    2. Cubic splines
    3. B-splines
    4. NURBS
    5. Inner product and Orthonormality
    6. Gram Schmidt
    7. Orthonormal Basis functions
    8. RBFs
2. Exercises
3. Homework

Theory

Setup

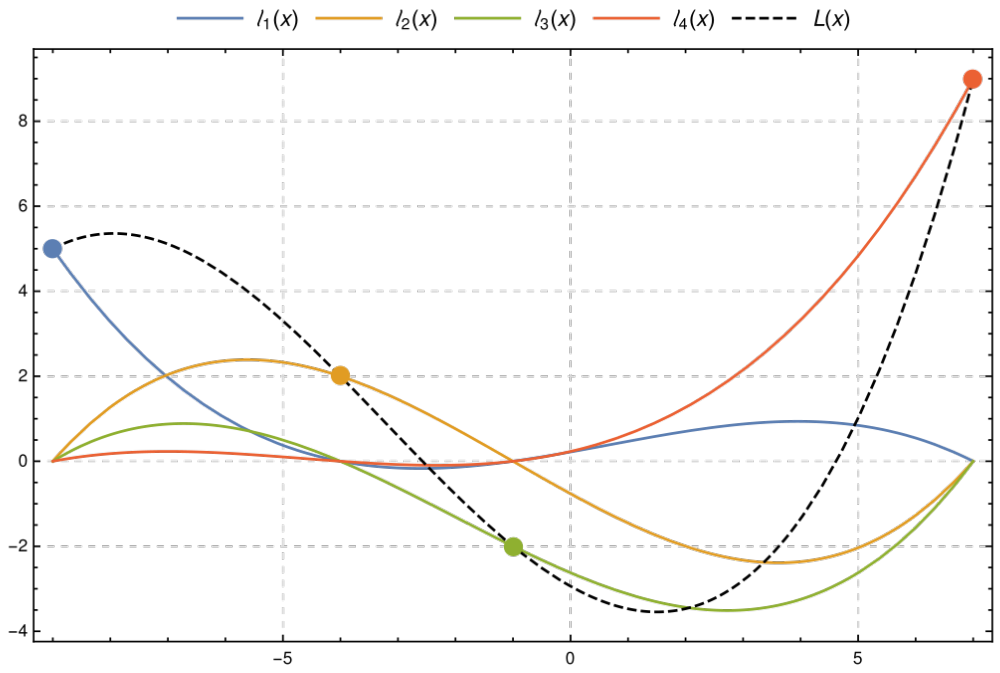
- Dataset $D = \{(x_1, y_1), \dots, (x_N, y_N)\}$
- Lagrange polynomial $l_i(x)$
- Lagrange interpolation $L(x)$
- Cubic function $S_j(x) = c_{j1} + c_{j2}x + c_{j3}x^2 + c_{j4}x^3$
- B-Spline $B_{i,d,\vec{t}}(x)$
- Knot vector $\vec{t} = (t_1, \dots, t_K)$

Lagrange interpolation

- Requirement for a function:
  - $l_i(x_i) = 1$
  - $l_i(x_m) = 0, m \neq i$

- Lagrange function:

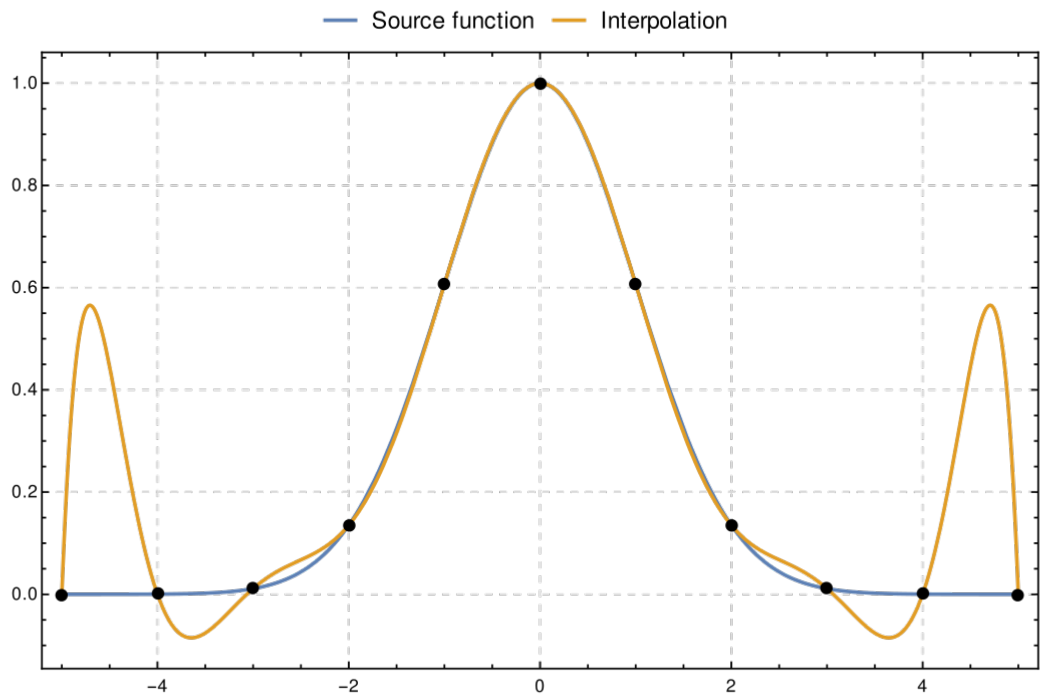
$$l_i(x) = \prod_{1 \le m \le N, \neq i} \frac{x - x_m}{x_i - x_m}$$



Lagrange interpolation cont.

- Construct a function that goes through every datapoint:

$$L(x) = \sum_{i=1}^{N} y_i \cdot l_i(x)$$



Lagrange interpolation cont. Example

- Given:
  - $D = \{(1, 1), (2, 2)\}$

- Task:
  - What degree does the Lagrange polynomial have?
  - Can you guess the function?
  - Write down the Lagrange Polynomial

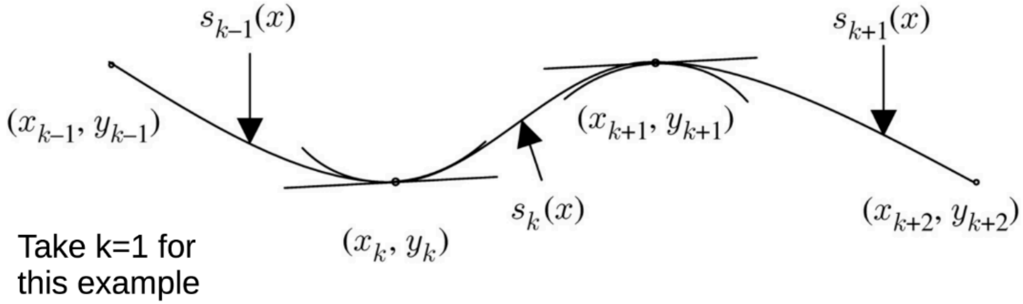
Cubic Splines

- Use a set of third order polynomials that go through all points:

$$S_j(x) = c_{j1} + c_{j2}x + c_{j3}x^2 + c_{j4}x^3,$$
$$x \in [x_j, x_{j+1}]$$

- Force them to be C^2 continuous:

$$S_j'' = S_{j+1}''$$



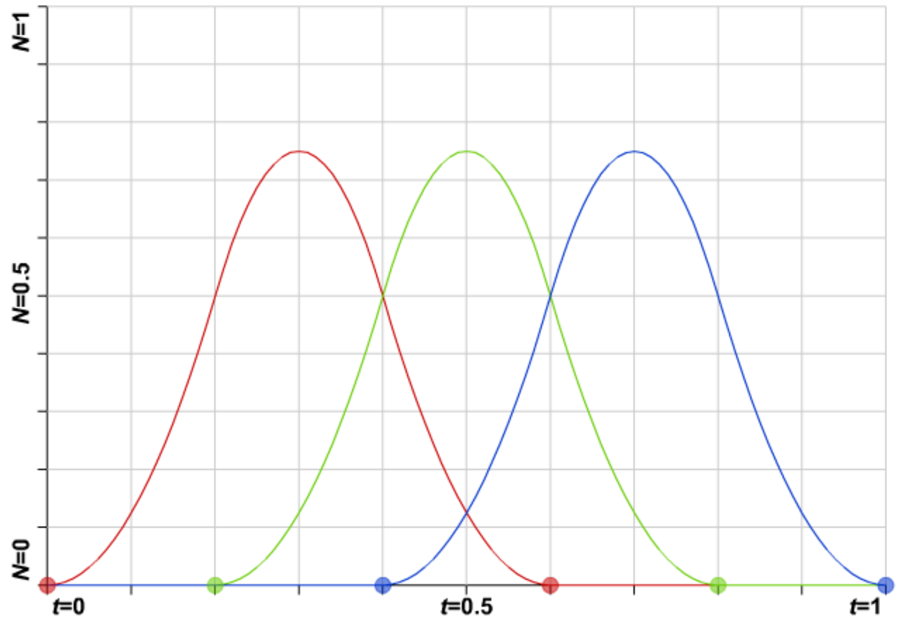
Cubic Splines cont. Example

- Given:
  - $D = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$

- Task:
  - How many cubic functions required?
  - How many unknowns?
  - Write down all constrains

B-Splines

- Basis function to define any other spline
- We are free to design them:
  - How many basis functions do we want? M
  - Of what degree should they be ? d
  - From where to where they should run? $\vec{t}$



B-Splines cont.

- Recursive definition:

$$B_{i,0,\vec{t}} = \begin{cases} 1 & if\ t_i \leq x < t_{i+1} \\ 0 & otherwise \end{cases}$$

$$B_{i,d,\vec{t}}(x) = \frac{x-t_i}{t_{i+d}-t_i} B_{i,d-1,\vec{t}}(x) + \frac{t_{i+d+1}-x}{t_{i+d+1}-t_{i+1}} B_{i+1,d-1,\vec{\mathrm{t}}}(x)$$

- With:

  d: degree of that B-spline
  i: iterator $1, \dots, M$
  $\vec{t}$: knot vector; in order; with $M + d + 1$ entries
  ex: $\vec{\mathrm{t}} = (0, 0.2, 0.4, 0.6, 0.8, 1)$

B-Splines cont.

- Only non-zero between $t_i \leq x \leq t_{i+d+1}$
- Continuous derivatives up to degree $d - 1$, if not overlapping knots
- If overlapping, derivative continuity drops by 1
- Use $0/0 = 0$ if knot values are repeated

B-Splines cont. Example

- $M = 1$
- $d = 1$
- $\vec{t}$ is equally spaced between 0 and 1

- Write down the function of the B-spline $B_{1,1,\vec{t}}(x)$
- Plot that spline
- Assume $M = 2$ and $\mathbf{t}$ is equally spaced between 0 and 2, plot that as well

NURBS

- Start of with data vectors $\vec{p}_i = (x_i, y_i)^{\mathbf{T}}, i = 1, \dots, N$
- Set $M = N$, decide on d, and space $\vec{t}$
- Construct NURBS:

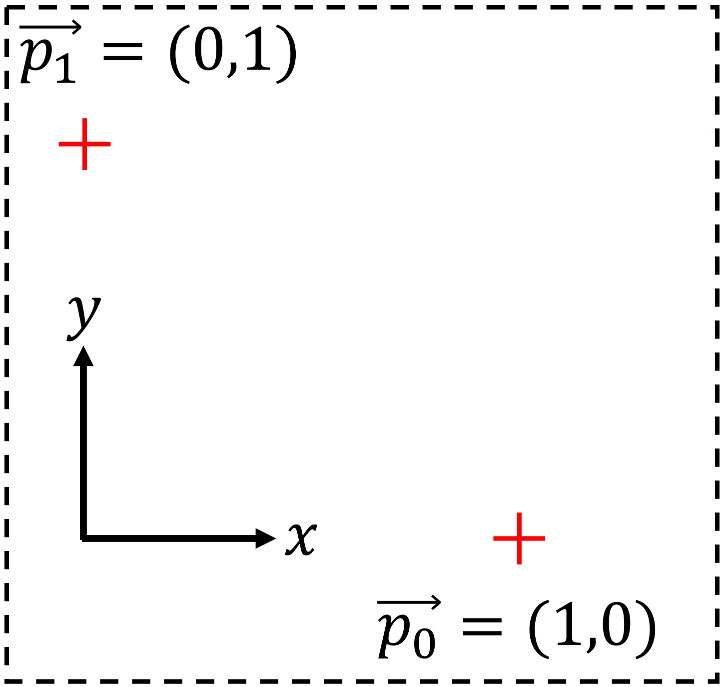
$$\vec{p}(s) = \sum_{i=1}^{N} R_{i,d,\mathbf{t}}(s)\vec{p}_i$$

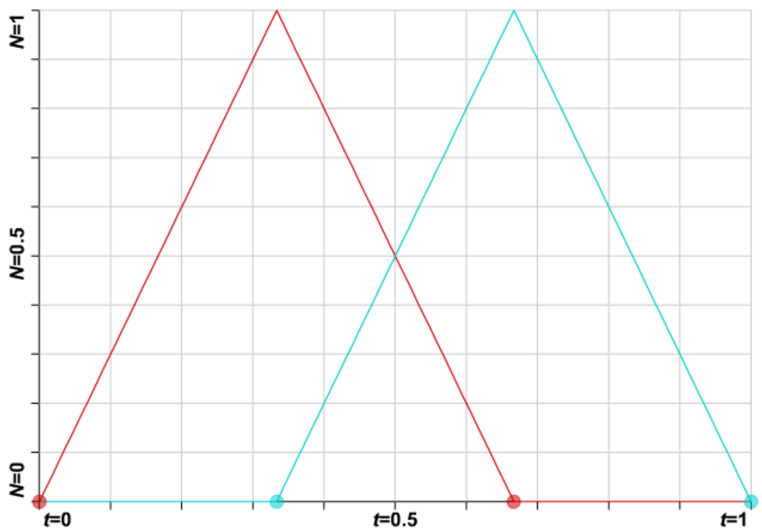
- With $R_{i,d,\mathbf{t}} = \frac{B_{i,d,\mathbf{t}}(s)w_i}{\sum_{j=1}^{N} B_{j,d,\mathbf{t}}(s)w_j}$ in order to weight data

NURBS cont.

- If non-weighted: $R_{i,d,\mathbf{t}}(s) = B_{i,d,\mathbf{t}}(s)$
- Due to the fact that $\sum_j B_{j,d,\mathbf{t}}\ (s) = 1$

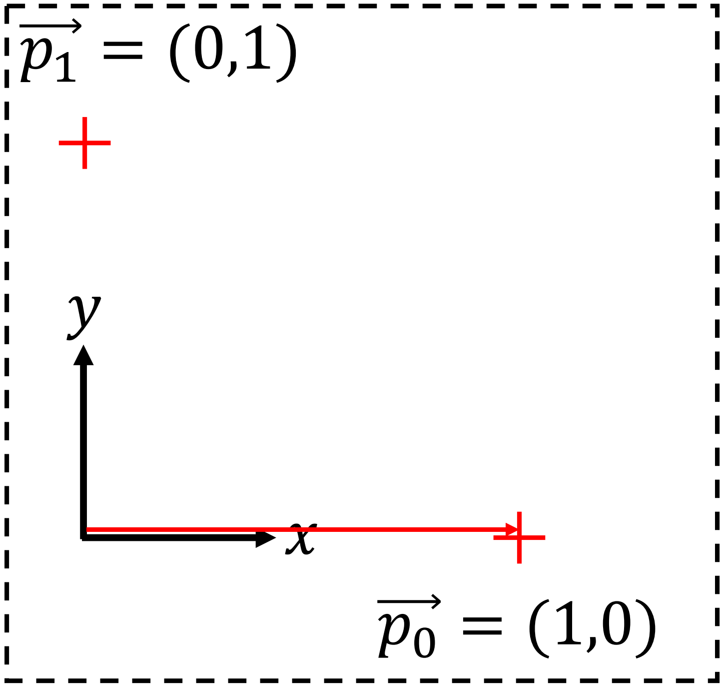
NURBS cont. Example

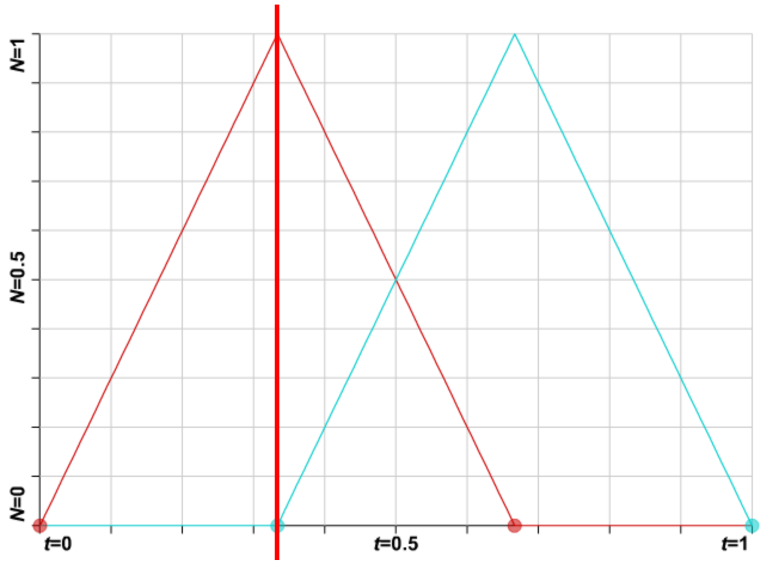




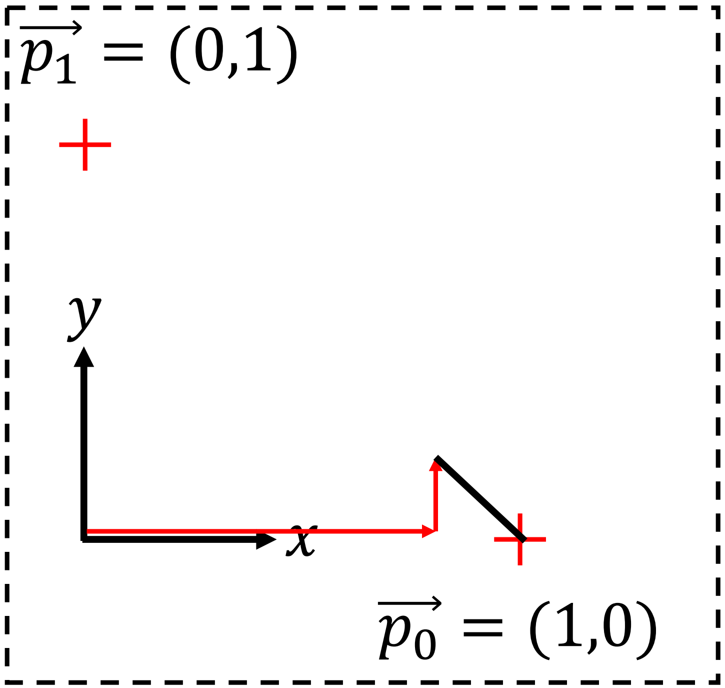
$$\vec{t} = (0, 0.33, 0.67, 1)$$

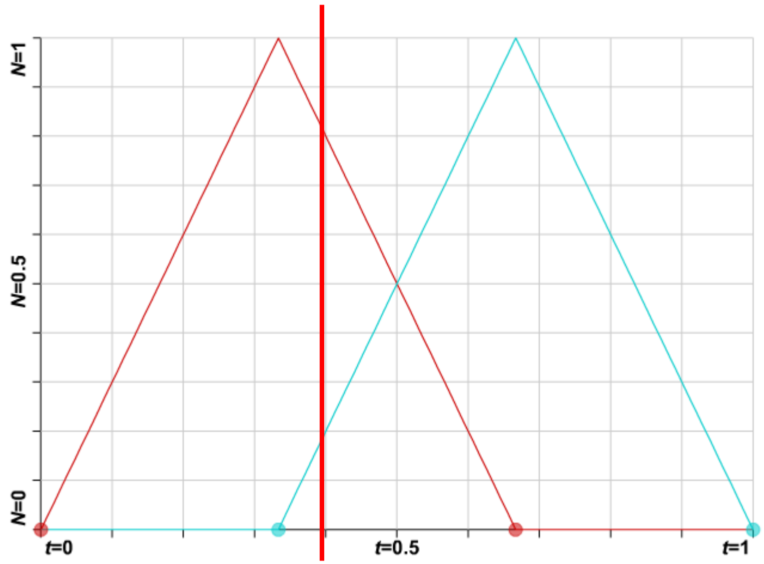
NURBS cont. Example



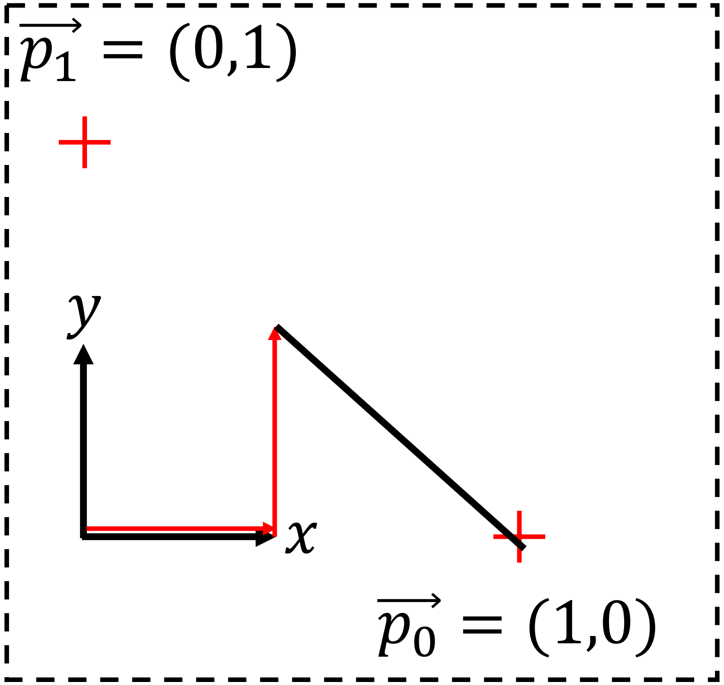


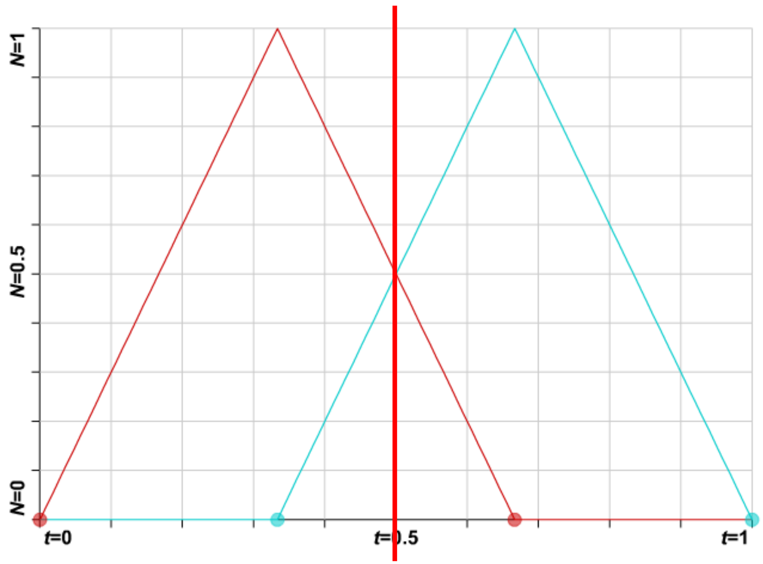
NURBS cont. Example



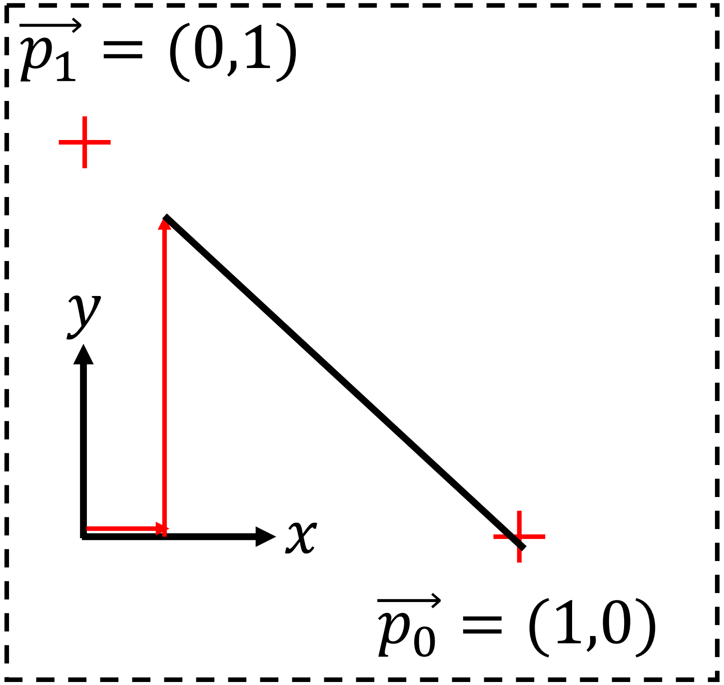


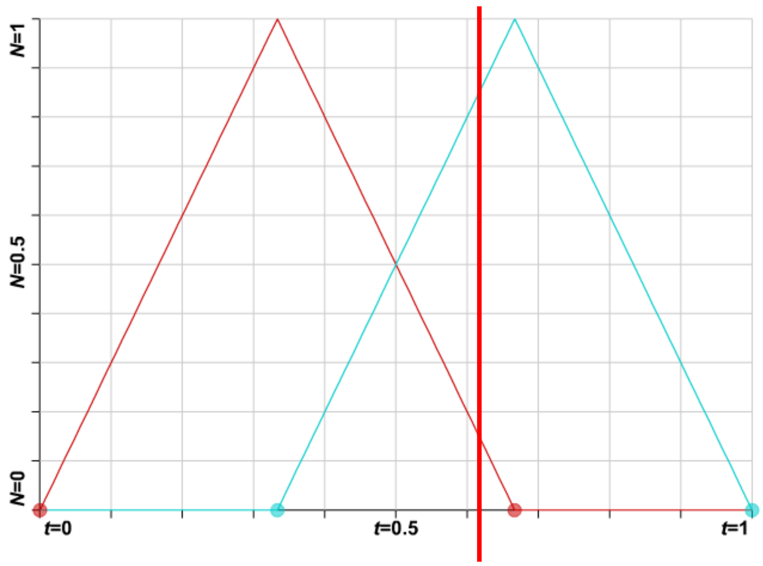
NURBS cont. Example



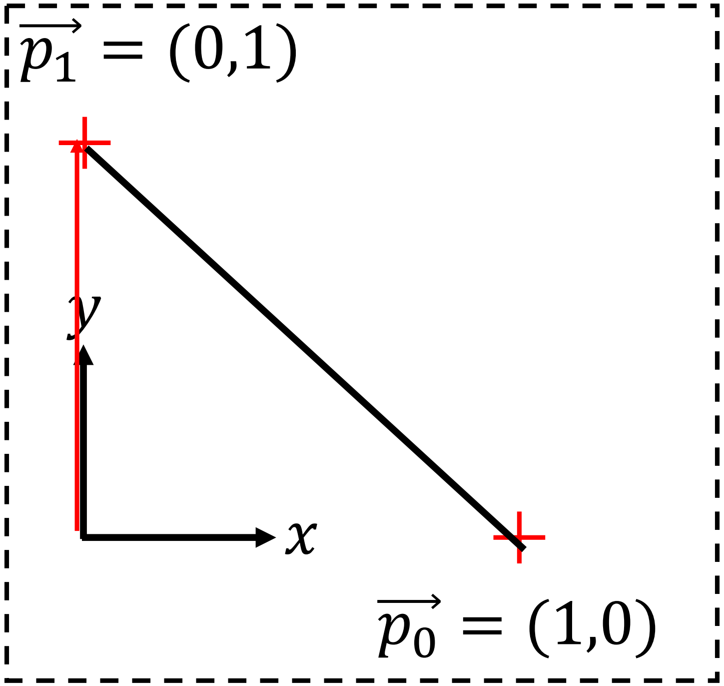


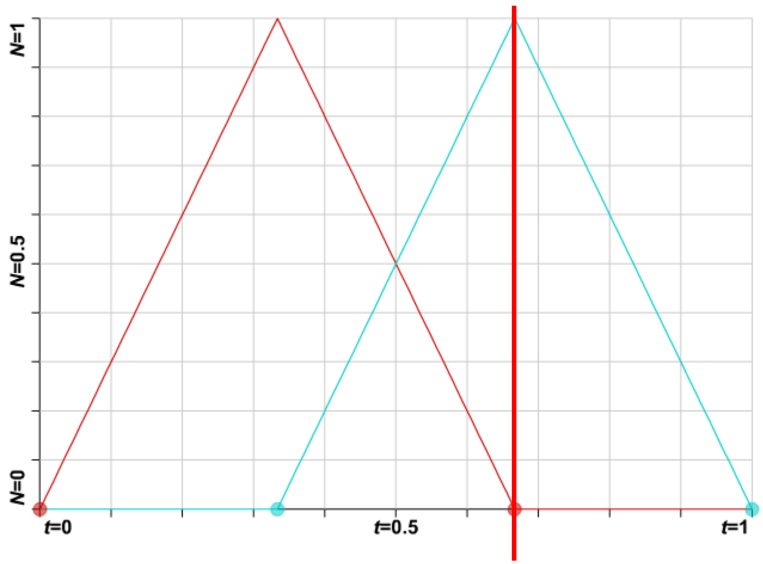
NURBS cont. Example





NURBS cont. Example

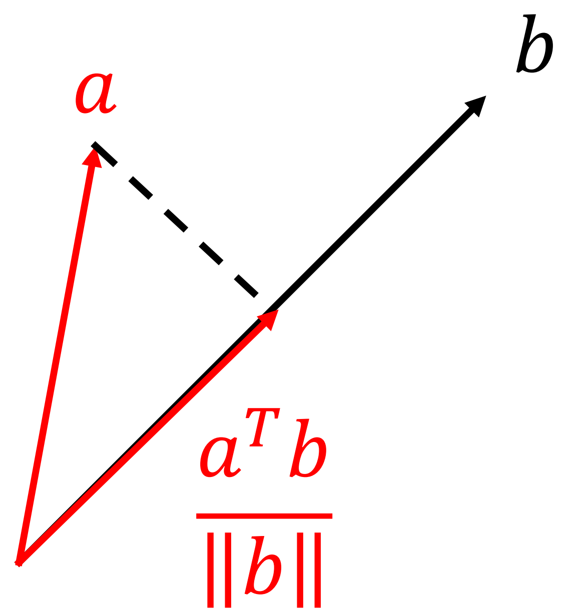




Inner product and Orthonormality

- Examples of inner product:
  - Real Numbers: $\langle x, y \rangle = x \cdot y$
  - Vectors: $\langle x, y \rangle = x^T y$
  - Random Vars.: $\langle x, y \rangle = \mathbb{E}_{p(x,y)}[xy]$
  - Functions: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$
- Orthonormality:
  - $\langle \phi_i, \phi_j \rangle = \delta_{ij}$
  - $\sqrt{\langle \phi_i, \phi_i \rangle} = 1$

Projection:



Gram Schmidt

Given a valid basis $\{w_1, \dots, w_k\}$, we want a orthonormal basis $\{v_1, \dots, v_k\}$

1. $\widetilde{v_1} = w_1$
2. $v_1 = \frac{\widetilde{v_1}}{\|v_1\|}$
3. $\widetilde{v_2} = w_2 - \langle v_1, w_2 \rangle v_1$
4. $v_2 = \frac{\widetilde{v_2}}{\|v_2\|}$
5. $\widetilde{v_3} = w_3 - \langle v_1, w_3 \rangle v_1 - \langle v_2, w_3 \rangle v_2$
6. $v_3 = \frac{\widetilde{v_3}}{\|v_3\|}$
7. …

Orthonormal basis functions

- A function as a linear combination:

$$y(x) = \sum_i \alpha_i \phi_i(x)$$

- As we have orthonormal basis (only dependent on ϕ_i - no recomputation when adding a basis function):

$$\alpha_i = \langle y, \phi_i \rangle$$

- We are dealing with data – use $\langle x, y \rangle = \mathbb{E}_{p(x,y)}[xy]$

$$\alpha_i = \langle y, \phi_i \rangle = \mathbb{E}_{p(y,\phi_i)}[y \cdot \phi_i] \approx \frac{1}{N} \sum_{j=1}^{N} y_j \cdot \phi_i(x_j)$$

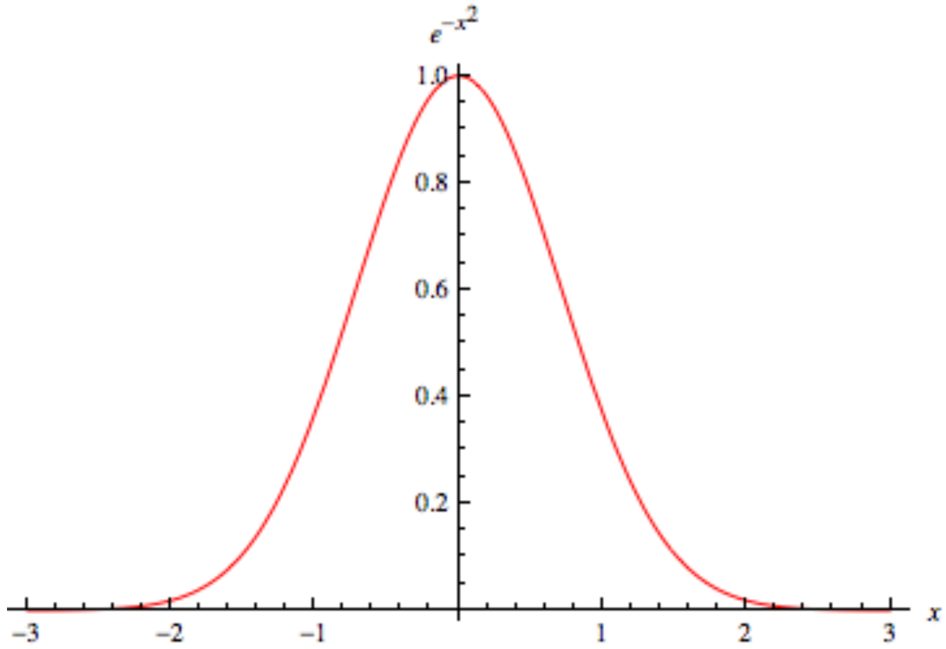
Radial Basis Functions (RBF)

- Functions that only depend on the distance from the origin (**symmetric**):

$$\phi(\mathrm{x}, \mathrm{c}) = \phi(\|\mathrm{x} - \mathrm{c}\|)$$

- Use for fitting as:

$$y(\mathrm{x}) = \sum_i \alpha_i \cdot \phi(\|\mathrm{x} - \mathrm{c_i}\|)$$



Exercises

Exercise 1

- Task:
  - Of what order is $L(x)$?
  - Do you see a problem?
  - How is this problem called?
  - What order is a cubic spline?
  - When would you use a cubic spline over Lagrange and vice-versa?
  - In what way are they similar?

Exercise 2

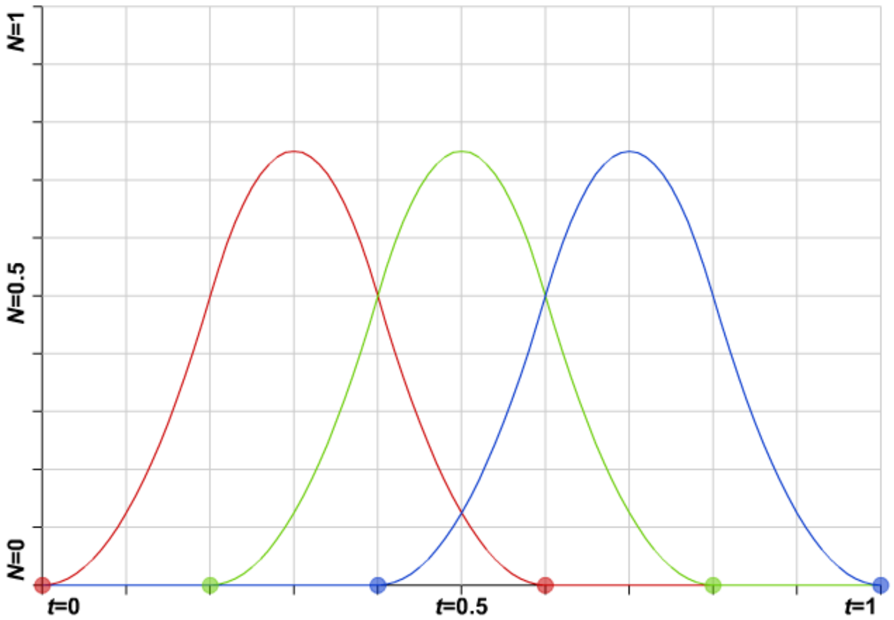
- Task:
  - Are cubic splines for interpolation or extrapolation? Why?

Exercise 3

- For the example on page 9, write out all the constraints and bring them to a format $A\vec{c} = b$, where $\vec{c}$ contains all parameters

Exercise 4

- Task:
  - What's $M, d, \vec{t}$?



Exercise 5

- Task (true or false):
  - Knot vector $(0, 0, 0.2, 0.7, 0.5, 1,1)$ is valid to build B-Splines basis functions.
  - The knot vector $(0, 0.25, 0.5, 0.75, 1)$ can be used to generate B-Splines of degree 3 at most.
  - The knot vector $(0, 0, 0.5, 1, 1)$ can generate at most 4 basis functions

Exercise 6

- Given the set $\{1, \sin(x), \cos(x)\}$ (Fourier Basis)
- Make the basis orthonormal wrt. $\int_{-\pi}^{\pi} f g dx$.
- Tipps:
  - $\int_{-\pi}^{\pi} \sin^2 x = \pi$
  - $\int_{-\pi}^{\pi} \cos^2 x = \pi$
  - $\int_{-\pi}^{\pi} \cos x = 0$
  - $\int_{-\pi}^{\pi} \sin x = 0$
  - $\int_{-\pi}^{\pi} \sin x \cos x = 0$

Homework

HW 1

a) What is the highest degree of Lagrange polynomial you can uniquely fit to the data points shown in Fig. 2?

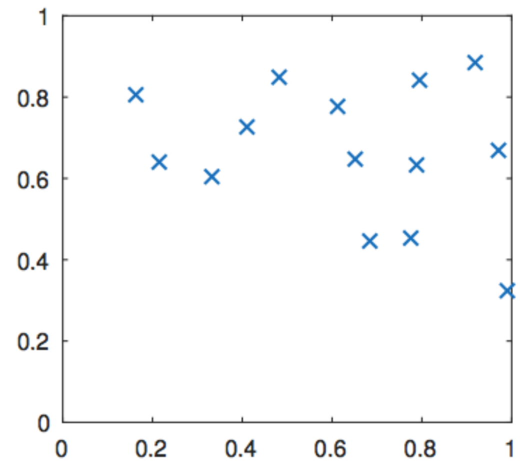


Figure 2

b) Can you think of a more appropriate interpolation method to fit the data points shown in Fig. 2? Do you think that the resulting lagrange interpolating polynomial would serve well for extrapolation near the endpoints of the data series? Explain your answer.

HW 2

You are given 5 data points (x_1, y_1), (x_2, y_2), $\cdots$, (x_5, y_5).

a) Write down an appropriate polynomial representation when constructing an interpolation curve using Lagrange polynomials. Write down the matrix system you would need to solve. Identify a problem with this approach.

b) Write down the matrix system you would need to solve when using linear least-squares to fit a straight line to the data and compare this to the results from the previous subquestion.

HW 3

Assume that you want to construct a natural cubic spline that passes through the points $(1,2)$, $(2,3)$, and $(3,5)$.

a) For the given data-points, how many segments would the resulting spline consist of?

b) Write down the general equation (do not evaluate coefficients yet!) for the cubic spline in each segment.

c) How many constraints are needed to construct this spline, and why?

d) Write down all the equations you need to enforce the constraints mentioned above.

e) Draw cubic splines to fit the data points shown in Fig. 4 with the following boundary conditions (make sure to explain your answers): (a) Clamped left, Natural right (b) Natural left, clamped right

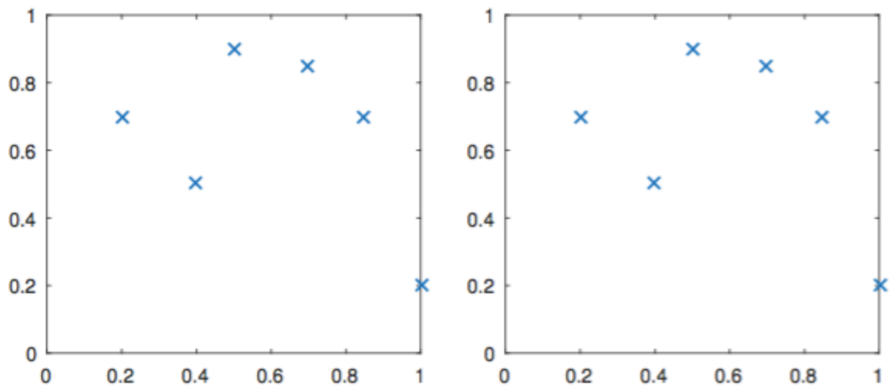
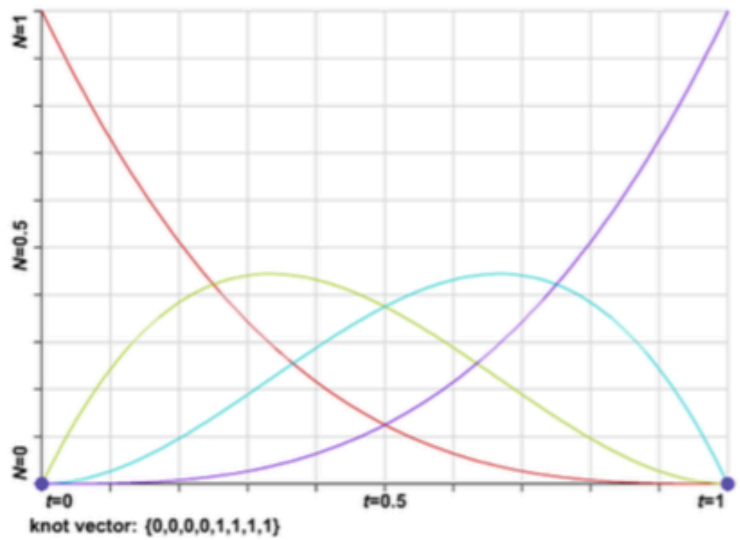


Figure 4: Figures with data points for cubic splines.

HW 4

You are given the set of B-splines shown in the following figure, with the corresponding knot vector $\{0, 0, 0, 0, 1, 1, 1, 1\}$.



a) What is the order of the B-splines? Explain your answer.

b) Are the B-splines "clamped" or "unclamped"? Explain your answer.

HW 5

- Given:
  - $D = \{(-1, 0), (1, 2)\}$
  - Basis functions $\{1, x\}$

- Task:
  - Orthonormalize the basis
  - Compute α_1 and write down the function
  - Now compute α_2 and write down the function containing α_1 and α_2

HW 6

- Given:
  - $D = \{(0, 2), (3, 10)\}$
  - Use $\phi(x) = 1 - |x - c_i|$ as basis function
  - Each RBF is centered at x_i

- Task:
  - Find α_1 and α_2 st. $y(x) = \sum_i \alpha_i \cdot (1 - |x - c_i|)$ goes through all data points exactly