Title

- Text

Exercise 6: Orthonormal Functions & Radial Basis Functions

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Outline

1. Information
2. Goals
3. Theory/ Recap
4. Exercises

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Information

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General

- Lecture material & problem sets available here
- Tutorial material available here

Goals

Goals of Today

- Know what an orthonormal basis is
- Know what the benefit of orthonormal functions is
- Know how to fit orthonormal functions to data

- Know what radial basis functions (RBF) are
- Know how to fit RBFs to data

Theory / Recap

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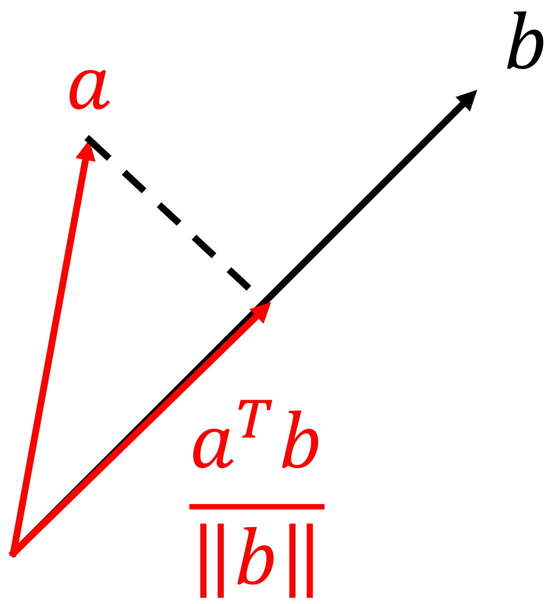
Inner Product

- Properties:
  - Conjugate symmetry:
    $$\langle x, y \rangle = \overline{\langle y, x \rangle}$$
  - Linear in first argument:
    $$\langle ax, y \rangle = a\langle x, y \rangle$$
  - Positive definite:
    $$\langle x, x \rangle > 0 \ \forall x \in V \backslash \{0\}$$

- Examples:
  - Real Numbers: $\langle x, y \rangle = x \cdot y$
  - Vectors: $\langle x, y \rangle = x^T y$
  - Random Vars.: $\langle x, y \rangle = \mathbb{E}_{p(x,y)}[xy]$
  - Functions: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx$

Projection:



Orthonormal Basis

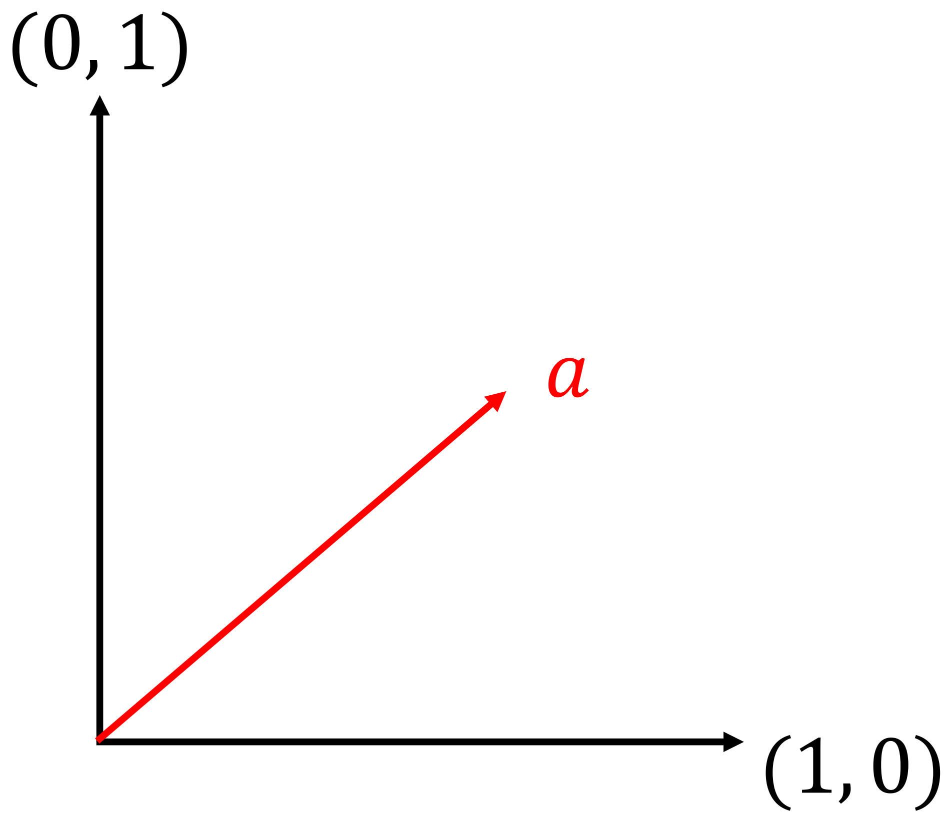
- Inner product of two basis vectors:

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}$$

- All basis vectors are of length one:

$$\sqrt{\langle \phi_i, \phi_i \rangle} = 1$$

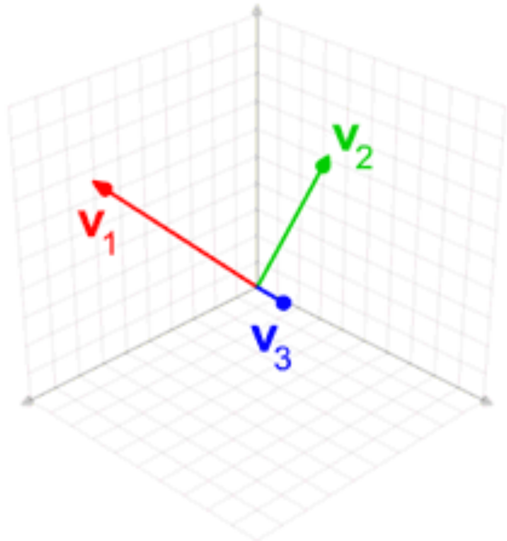
Note: Projecting a on to one of the basis vectors will yield its coordinate wrt to that basis vector.



Gram Schmidt Orthonormalization

Given a valid basis $\{w_1, \dots, w_k\}$, we want a orthonormal basis $\{v_1, \dots, v_k\}$

1. $\widetilde{v_1} = w_1$
2. $v_1 = \frac{\widetilde{v_1}}{\|v_1\|}$
3. $\widetilde{v_2} = w_2 - \langle v_1, w_2 \rangle v_1$
4. $v_2 = \frac{\widetilde{v_2}}{\|v_2\|}$
5. $\widetilde{v_3} = w_3 - \langle v_1, w_3 \rangle v_1 - \langle v_2, w_3 \rangle v_2$
6. $v_3 = \frac{\widetilde{v_3}}{\|v_3\|}$
7. ...



Example 1: Gram Schmidt for Fourier

- Given the set $\{1, \sin(x), \cos(x)\}$ (Fourier Basis)
- Make the basis orthonormal wrt. $\int_{-\pi}^{\pi} fg\,dx$.
- Tipps:
  - $\int_{-\pi}^{\pi} \sin^2 x = \pi$
  - $\int_{-\pi}^{\pi} \cos^2 x = \pi$
  - $\int_{-\pi}^{\pi} \cos x = 0$
  - $\int_{-\pi}^{\pi} \sin x = 0$
  - $\int_{-\pi}^{\pi} \sin x \cos x = 0$

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Data Fitting with Orthonormal Basis

- A function as a linear combination:

$$y(x) = \sum_i \alpha_i \phi_i(x)$$

- As we have orthonormal basis:

$$\alpha_i = \langle y, \phi_i \rangle$$

- We are dealing with data – use $\langle x, y \rangle = \mathbb{E}_{p(x,y)}[xy]$

$$\alpha_i = \langle y, \phi_i \rangle = \mathbb{E}_{p(y,\phi_i)}[y \cdot \phi_i] \approx \frac{1}{N} \sum_{j=1}^{N} y_j \cdot \phi_i(x_j)$$

Example 2: Fitting with orthonormal functions

- $D = \{(-1, 0), (1, 2)\}$
- Basis functions $\{1, x\}$

- Orthonormalize the basis
- Compute α_1 and write down the function
- Now compute α_2 and write down the function containing α_1 and α_2

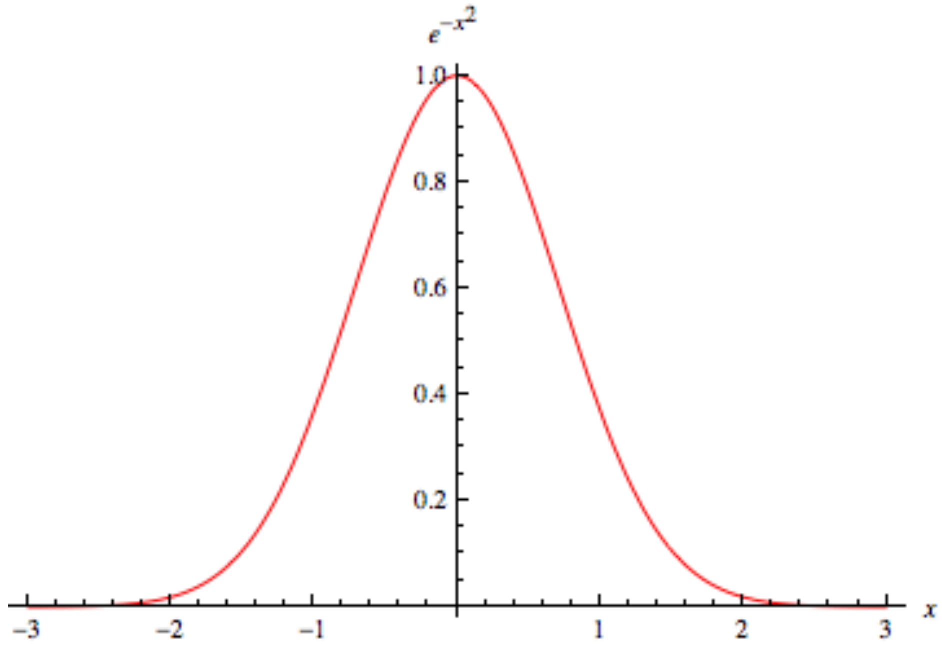
Radial Basis Functions

- Functions that only depend on the distance from the origin:

$$\phi(\mathrm{x}, \mathrm{c}) = \phi(\|\mathrm{x} - \mathrm{c}\|)$$

- Use for fitting as:

$$y(\mathrm{x}) = \sum_i \alpha_i \cdot \phi(\|\mathrm{x} - \mathrm{c}_\mathrm{i}\|)$$



Typical example: Gauss function

Example 3: RBF fitting

- $D = \{(0, 2), (3, 10)\}$
- Use $\phi(x) = 1 - |x - c_i|$ as basis function
- Each RBF is centered at x_i

- Find α_1 and α_2 st. $y(x) = \sum_i \alpha_i \cdot (1 - |x - c_i|)$ goes through all data points exactly

Exercises

Q1

- Perform Gram Schmidt
- See how adding an additional orthonormal function improves performance while we do not have to recompute the parameters

Q2

- Fit RBFs to some data points and see the resulting surface

Questions?