Title

- Text

Exercise 9: Numerical Integration II

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Outline

1. Information
2. Goals
3. Theory/ Recap
4. Exercises

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Information

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General

- Lecture material & problem sets available here
- Tutorial material available here

Goals

Goals of Today

- Understand Richardson extrapolation
- Understand how to use Richardson extrapolation
- Be aware that higher order approximations are not always better
- Understand how Richardson extrapolation can increase accuracy for numerical integration (Romberg integration)

Theory / Recap

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Taylor expansion

- A function $f(x)$ around the origin can be written as an infinite Series:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

- With...

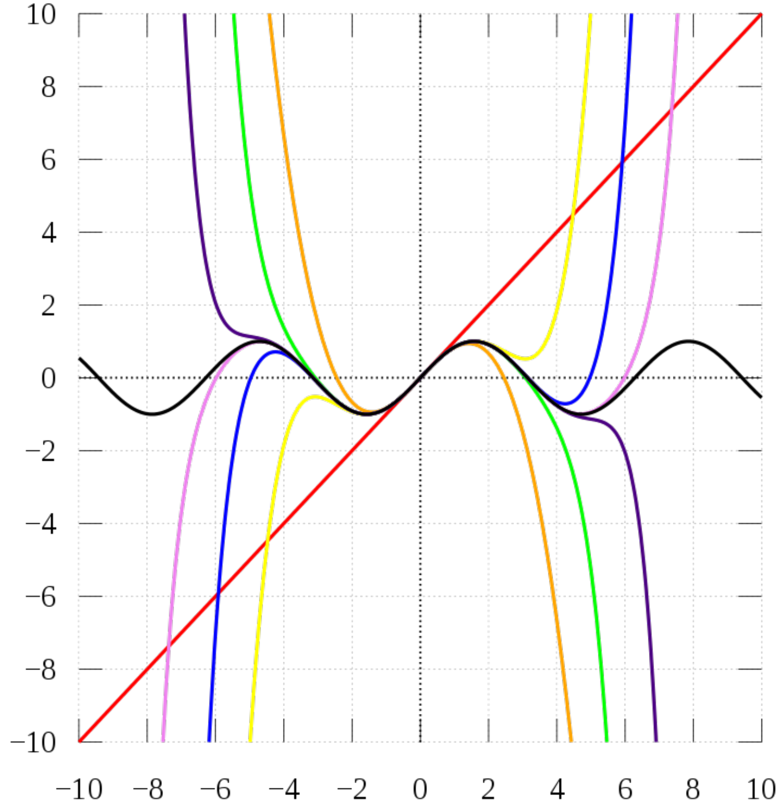
$c_0 = f(0)$

$c_1 = f'(0)$

$c_2 = \frac{f''(0)}{2}$

etc.

- Note: Lower orders affect function the most



https://en.wikipedia.org/wiki/Taylor_series#/media/File:Sintay_SVG.svg

Example 1: Taylor Expansion

- Write down the Taylor Series around 0 of

$$f(x) = \frac{1}{1-x}, \qquad x \in (-1, 1)$$

Richardson Extrapolation

- A function G depends on a stepsize h:

$$G_0(h)$$

- Perform a Taylor series expansion around 0:

$$G_0(h) = G_0(0) + c_1 h + c_2 h^2 + \cdots$$

- Decrease stepsize to $h/2$:

$$G_0(h/2) = G_0(0) + \frac{c_1}{2}\mathrm{h} + \frac{c_2}{4} + \cdots$$

- Combine to have:

$$G_1(h) = G_0(0) + \mathcal{O}(h^2)$$

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Iterative formulation

- In order to obtain $G_n(h)$ (order $n+1$) compute:

$$G_n(h) = \frac{1}{2^n - 1}\left(2^n G_{n-1}(h/2) - G_{\mathrm{n-1}}(\mathrm{h})\right) = G(0) + \mathcal{O}(h^{n+1})$$

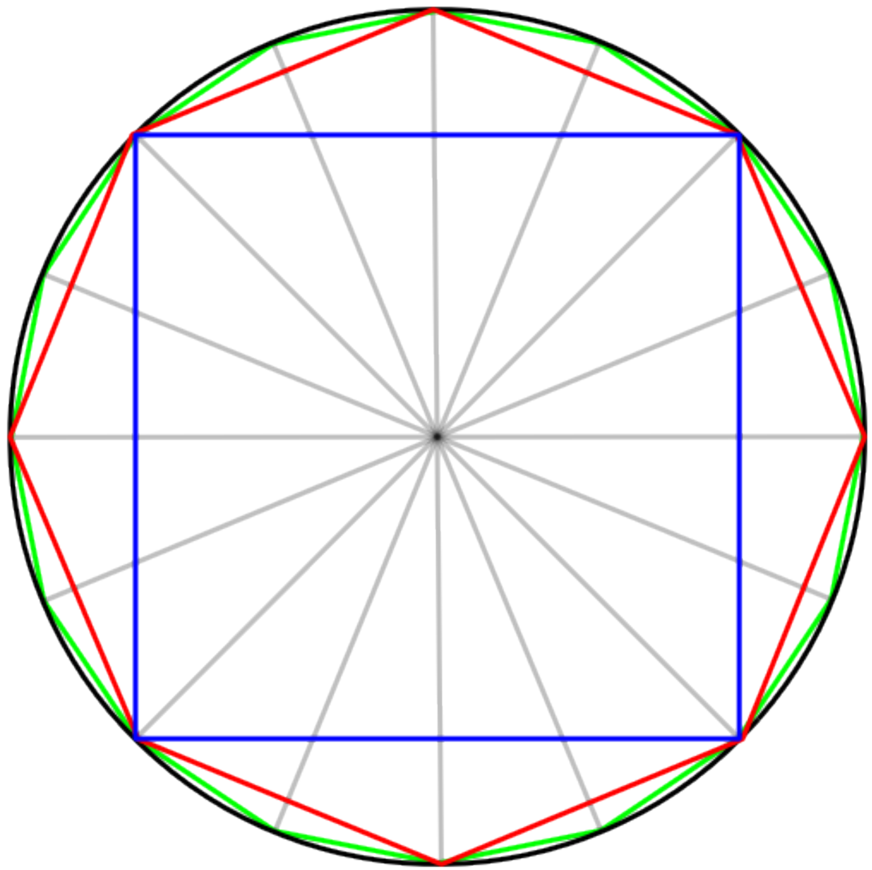
Example 2: Richardson Extrapolation for π

- Use Richardson Extrapolation to estimate π, please note that only even terms are in the Taylor series
- Use the circumference of a polygon to approximate:

$$\hat{\pi}_0^n = n \cdot \sin\frac{\pi}{n}$$

- Use $n = \{3, 6\}$ which corresponds to h_0 and $\frac{h_0}{2}$ correspondingly, to compute

$$\hat{\pi}_1^3 = \frac{4 \cdot \hat{\pi}_0^6 - \hat{\pi}_0^3}{3}$$



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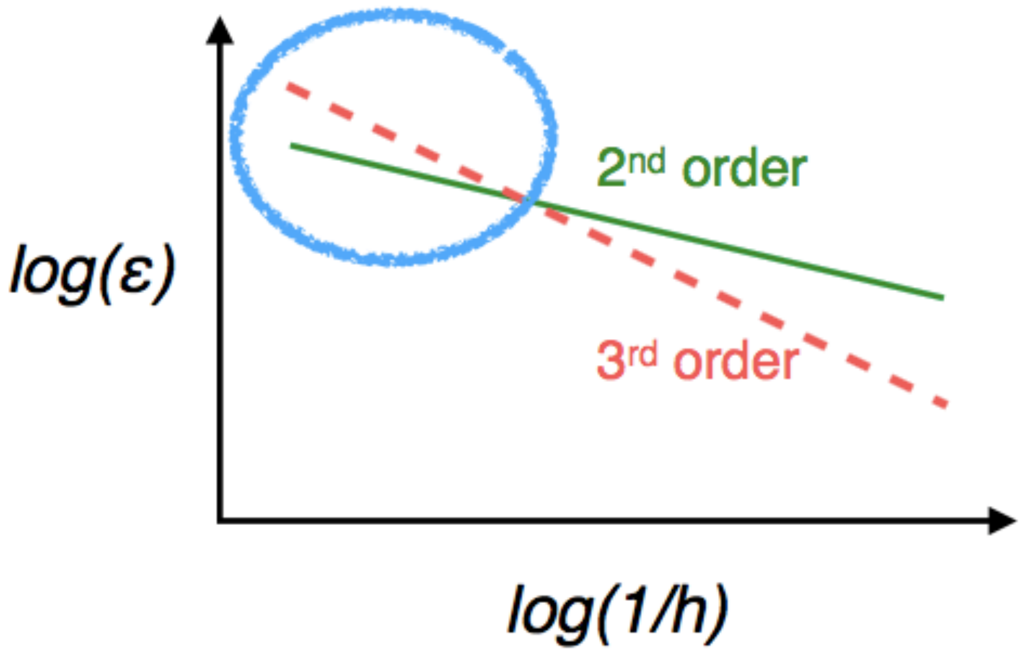
Higher order is not always better

- The order defines the **rate** at which the error drops
- However, the terms still depend on constants



Romberg Integration

- The Romberg Integration uses Trapezoidal rule & Romberg extrapolation:

$$I_k^n = \frac{1}{4^k - 1}\left(4^k I_{k-1}^{2n} - I_{k-1}^n\right)$$

- With…

  n: Number of steps to divide interval

  k: Iterations step, resp. order $k + 1$

- Use composite trapezoidal rule as base case:

$$I_0^n = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f_i\right)$$

- With…

  $h = (b - a)/n$

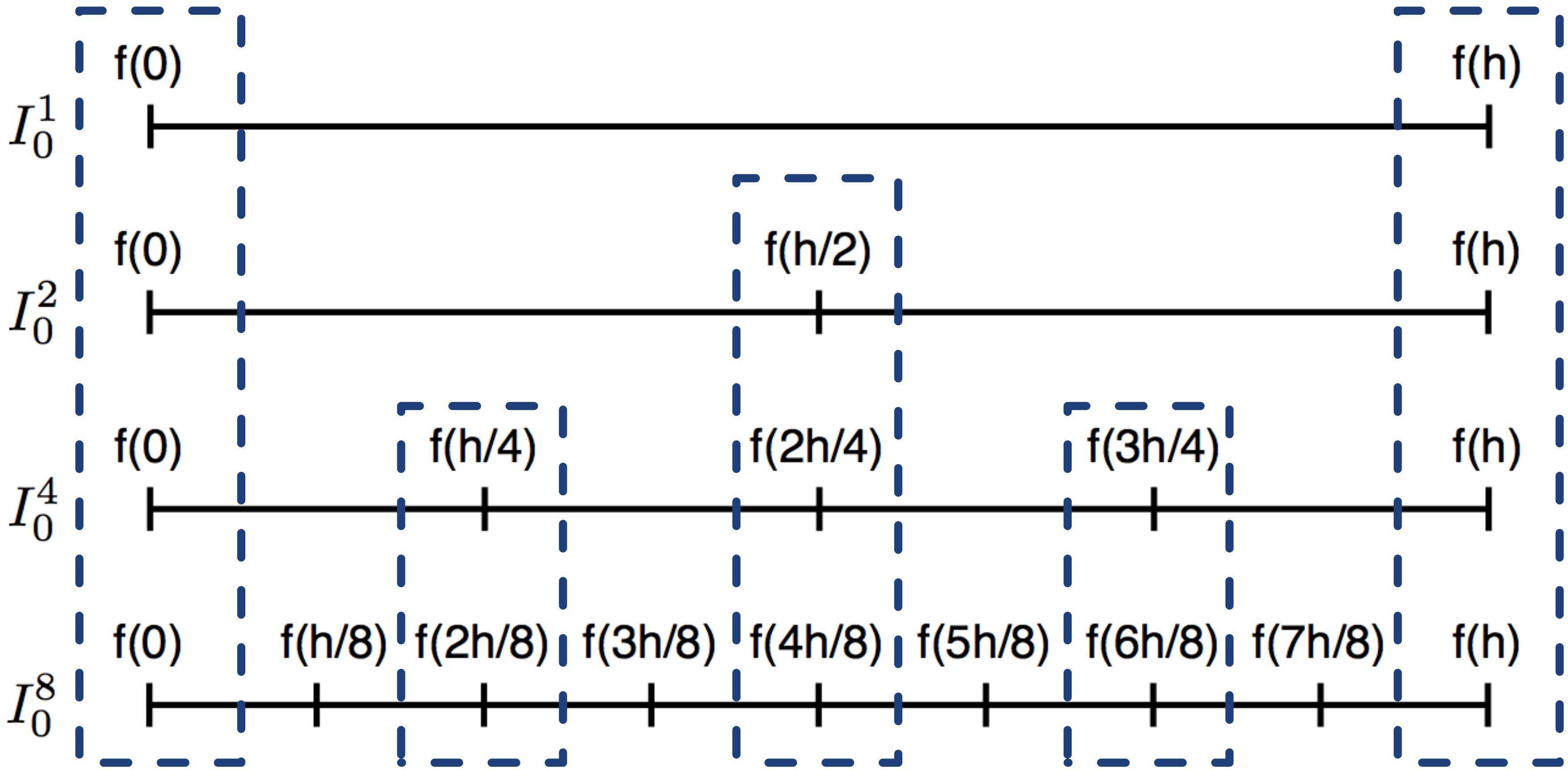
  $f_i = f(a + i \cdot h)$

Example ‘bonus’: Derive Romberg Integration
Solutions Provided

- Derive Romberg Integration by motivating all steps

- First, show that the error term holds terms of orders 2, 4, 6, … (Tipp: Taylor Exp)
  - $I = \int_{-h}^{h} f(0) + f'(0)x + \cdots dx \ = 2hf(0) + 0 + \cdots$
  - $I_{TR} = h\big(f(-h) + f(h)\big) = h\left(f(0) - f'(0)h + \frac{f''(0)}{2}h^2 - \cdots + f(0) + f'(0)h + \frac{f''(0)}{2}h^2\right) = 2hf(0) + 0 + f''(0)h^3 + \cdots$
  - Compare the two terms: The error is of order 3 for a single interval.
  - If multiple intervals are used the error is increased N times, where $N = \frac{b-a}{h}$. Hence the error term is now of order 2.
- Use Richardson to eliminate the terms one by one
  - $I_{TR} = I + c_1 \cdot h^2 + c_2 \cdot h^4 + \cdots$
  - To reduce one order by using half the step size: $\frac{4 \cdot I_{TR}\left(\frac{h}{2}\right) - I_{TR}(h)}{3} = I - \frac{c_2}{4} \cdot h^4 - \cdots$
  - One can generalize the above to any order

Recycle function evaluations (store in array!)



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Example 3: Romberg Integration

- Approximate

$$I = \int_{-1}^{1} x^2 dx$$

with

$$I_1^1$$

Formulas:

$I_k^n = \frac{1}{4^k - 1}\left(4^k I_{k-1}^{2n} - I_{k-1}^n\right)$

$I_0^n = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f_i\right)$

With $h = (b-a)/n$ and $f_i = f(a + i \cdot h)$

Exercises

Exercise 1

- Use Richardson Extrapolation for derivative approximation

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Exercise 2

- Write pseudo-code for Romberg Integration

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Exercise 3

- Implement Romberg Integration

Questions?