Proofs for the paper "Piecewise-Linear Motion Planning amidst Static, Moving, or Morphing Obstacles"

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1 Proof of Theorems 1 and 2

SDP$(r, s; \mathcal{D})$ is an obvious relaxation of MP$(s; \mathcal{D})$ and therefore $\rho_r \leq \rho^*$ for all r.

Theorem 1 (SDP$(r, s; \mathcal{D})$ is solvable) *The semidefinite program SDP$(r, s; \mathcal{D})$ has an optimal solution $\phi^r \in \mathbb{N}^{s(2n+1)}$.*

Proof 1 *Let $\phi \in \mathbb{N}^{s(2n+1)}$ be a feasible solution of SDP$(r, s; \mathcal{D})$. Recall that $g_m(t, \boldsymbol{x}) = R^2 - \|\boldsymbol{x}\|^2$. We first use the constraint $M_\phi(g_m(t, \boldsymbol{u}_i + t\boldsymbol{v}_i)) \succeq 0$ for all $t \in [\frac{(i-1)T}{s}, \frac{iT}{s}]$, to prove that $L_\phi(u_{ij}^{2r}) \leq \tau^{2r}$ and $L_\phi(v_{ij}^{2r}) \leq \tau^{2r}$ for all i, j, and some scalar τ. Let $i \in [s], j \in [n]$ be fixed, and write:*

$$u_{ij} + \frac{(i-1)T}{s} v_{ij} =: x\,; \quad u_{ij} + \frac{T}{s} v_{ij} =: y\,,$$

so that $u_{ij} = ax + by$ and $v_{ij} = a'x + b'y$ for some scalars a, a', b, b'. Observe that $M_\phi(g_m(t, \boldsymbol{u}_i + \frac{(i-1)T}{s}\boldsymbol{v}_i)) \succeq 0$ and $M_\phi(g_m(t, \boldsymbol{u}_i + \frac{iT}{s}\boldsymbol{v}_i)) \succeq 0$. This implies that $L_\phi(x^2 + y^2) \leq R^2$ and so $L_\phi(x^2) \leq R^2$ and $L_\phi(y^2) \leq R^2$. Next, again because of the localizing constraint with g_m, $L_\phi(x^2(R^2 - x^2 + y^2)) \geq 0$ so that $L_\phi(x^4) \leq R^4$; and similarly $L_\phi(y^4) \leq R^4$. Iterating the process yields

$$L_\phi(x^{2k})\,, L_\phi(y^{2k}) \leq R^{2k} \quad k = 0, \ldots, r\,. \tag{1}$$

Next, consider the bi-variate moment matrix H with rows and columns indexed by the monomials $(x^i y^j)_{i+j \leq r}$ and with entries

$$H((i,j),(i'j')) := L_\phi(x^{i+i'} y^{j+j'}), \quad \forall\,(i,j)\,,(i',j') \in \mathbb{N}^2_r\,.$$

*Using (1), $H \succeq 0$ and [**?**, Proposition 2.38], yields*

$$|L_\phi(x^i y^j)| \leq R^{i+j}\,, \quad \forall i, j\,:\,(i+j) \leq 2r. \tag{2}$$

Finally, recall that $u_{ij}=ax+by$ *and* $v_{ij}=a'x+b'y$*, and so*

$$L_\phi(u_{ij}^2)\;\leq\; a^2\,L_\phi(x^2)+\underbrace{2\,|ab\,L_\phi(x\,y)|}_{\leq 2|ab|R^2}+b^2\,L_\phi(y^2)\,.$$

Therefore $L_\phi(u_{ij}^2)\leq[(|a|+|b|)\,R]^2$*, and* $L_\phi(v_{ij}^2)\leq(|a'|+|b'|)R^2$*. Similarly one obtains* $L_\phi(u_{ij}^{2r})=L_\phi((ax+by)^{2r})\leq[(|a|+|b|)\,R]^{2r}=:V_{ij}^{2r}$ *and* $L_\phi(v_{ij}^{2r})=L_\phi((a'x+b'y)^{2r})\leq[(|a'|+|b'|)\,R]^{2r}=:W_{ij}^{2r}$*.*

As this holds for an arbitrary couple (i,j)*, letting* $\tau:=\max_{(i,j)}[V_{ij},W_{ij}]$*, one finally obtains* $L_\phi(u_{ij}^{2r}),L_\phi(v_{ij}^{2r})<\tau^{2r}$*, for all* $i\in[s]$*,* $j\in[n]$*.*

Similarly, in view of the constraint $L_\phi((\boldsymbol{u},\boldsymbol{v},z)^{\boldsymbol{\alpha}}h_i^z))=0$ *for all* $\boldsymbol{\alpha}\in\mathbb{N}_{2r-2}^{s(2n+1)}$*, it follows that* $L_\phi(z_i^2)=\sum_j L_\phi(v_{ij}^2)\leq n\tau^2$*, and iterating yields* $L_\phi(z_i^{2r})\leq n\tau^{2r}$*. By [?, Proposition 2.38], combining this with* $L_\phi(1)=1$ *and* $M_\phi\succeq 0$*, yields that* $|\phi_{\boldsymbol{\alpha}}|\leq n\tau^{2r}$ *for all* $\boldsymbol{\alpha}\in\mathbb{N}_{2r}^{s(2n+1)}$*. Hence the feasible set is bounded; as it is also closed it is compact and therefore SDP*$(r,s;\mathcal{D})$ *has an optimal solution, denoted* ϕ^r*. To see that the feasible set is closed, let* $(\phi_{\boldsymbol{\alpha}}^\ell)_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}$ *be such that* $\phi_{\boldsymbol{\alpha}}^\ell\to\phi_{\boldsymbol{\alpha}}$ *for all* $\boldsymbol{\alpha}\in\mathbb{N}_{2r}^{s(2n+1)}$*, as* $\ell\to\infty$*, and for instance consider the constraint* $M_{\phi^\ell}(g_k(t,\boldsymbol{u}_i+t\boldsymbol{v}_i))\succeq 0$ *for all* $t\in\left[\frac{(i-1)T}{s},\frac{iT}{s}\right]$*. Fix* $t\in\left[\frac{(i-1)T}{s},\frac{iT}{s}\right]$ *arbitrary, and let* $(\boldsymbol{u}_i,\boldsymbol{v}_i)\mapsto g_k(t,\boldsymbol{u}_i+t\boldsymbol{v}_i)=:g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i)\in\mathbb{R}[\boldsymbol{u},\boldsymbol{v}]$*. By construction, each entry of the localizing matrix* $M_{\phi^\ell}(g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i))$ *is linear in the variables* $(\phi_{\boldsymbol{\alpha}}^\ell)$*'s. Therefore,*

$$\lim_{\ell\to\infty}M_{\phi^\ell}(g_{kt}(\boldsymbol{u}_i,\boldsymbol{v}_i))\,(\succeq 0)\;=\;M_\phi(g_{kt}(\boldsymbol{u}_i,\boldsymbol{v}_i))\succeq 0.$$

Hence as this is true for arbitrary t*, it follows that*

$$M_\phi(g_k(t,\boldsymbol{u}_i+t\boldsymbol{v}_i))\succeq 0,\quad\forall t\in\left[\frac{(i-1)T}{s},\frac{iT}{s}\right].$$

A similar (and even simpler) reasoning applies to obtain $L_\phi((\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^\alpha h_i(\boldsymbol{u},\boldsymbol{v}))=0$ *and* $L_\phi((\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^\alpha h_i^z(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}))=0$*, for all* $\alpha\in\mathbb{N}_{2r-2}^{s(2n+1)}$*, as well as* $M_\phi(z)\succeq 0$*.*

Theorem 2 (Asymptotic convergence) *Let* $(\phi^r)_{r\in\mathbb{N}}$ *be a sequence of optimal solutions of SDP*$(r,s;\mathcal{D})$*, and complete each finite vector* ϕ^r *with zeros to make it an infinite sequence* $(\phi_{\boldsymbol{\alpha}}^r)_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}$*. Let* $(\phi_{\boldsymbol{\alpha}}^\star)_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}$ *be an arbitrary accumulation point of the sequence* $(\phi^r)_{r\in\mathbb{N}}$*. Then* $\rho_r\uparrow\rho^*$ *as* $r\to\infty$*, and:*

(a) $(\phi_{\boldsymbol{\alpha}}^\star)$ *is the moment vector of a measure* $d\phi^\star(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})$ *supported on* S*. For* $\phi^\star$*-almost all* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})$ *in* S*, the trajectory*

$$\boldsymbol{x}_i(t)\;:=\;\boldsymbol{u}_i+t\,\boldsymbol{v}_i\,,\quad t\in\left[\frac{(i-1)T}{s},\frac{iT}{s}\right]\,;\quad i\in[s],$$

is an optimal solution of MP$(s;\mathcal{D})$*.*

(b) If MP$(s;\mathcal{D})$ *has a unique optimal piece-wise linear trajectory with* s *pieces*

$$\boldsymbol{x}_i^*(t)\;:=\;\boldsymbol{u}_i^*+t\,\boldsymbol{v}_i^*\,,\;t\in\left[\frac{(i-1)T}{s},\frac{iT}{s}\right]\quad i\in[s],\tag{3}$$

then $\phi^\star = \delta_{(\boldsymbol{u}^*,\boldsymbol{v}^*,\boldsymbol{z}^*)}$ *is the unique accumulation point of* $(\phi^r)_{r\in\mathbb{N}}$ *and*

$$\lim_{r\to\infty} L_{\phi^r}(\boldsymbol{u}_i) \,=\, \boldsymbol{u}_i^* \qquad \lim_{r\to\infty} L_{\phi^r}(\boldsymbol{v}_i) \,=\, \boldsymbol{v}_i^*\,, \quad \forall i\in[s]. \tag{4}$$

In the proof below, for any infinite sequence $\{\phi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$ of pseudo-moments over some variable $\boldsymbol{y}$ and for any polynomial $g\in\mathbb{R}[\boldsymbol{y}]$, we use the notation $M_\phi^d(g(\boldsymbol{y}))$ to denote the localisation matrix $M_{\phi'}^d(g(\boldsymbol{y}))$, where $\phi'=\{\phi_{\boldsymbol{\alpha}} \mid |\boldsymbol{\alpha}|\le 2d\}$.

Proof 2 *Let* ϕ^r *be an arbitrary optimal solution of SDP*$(r,s;\mathcal{D})$ *which exists by Theorem 1. With* τ *as in the proof of Theorem 1, we obtain* $|\phi^r_{\boldsymbol{\alpha}}|\le n\tau^{2r}$ *for all* $\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}_{2r}$*. In fact and more precisely:*

$$|\phi^r_{\boldsymbol{\alpha}}| \,\le\, n\tau^{2j}\,,\quad \forall\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}_{2r}\,:\, 2j-1\,\le\,|\boldsymbol{\alpha}|\,\le 2j\,;\; j=1,\ldots,r.$$

Next complete the finite sequence $(\phi^r_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}_{2r}}$ *with zeros to make it an infinite sequence* $(\phi^r_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}$*. Define a new sequence* $\hat\phi^r=(\hat\phi^r_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}$ *with:*

$$\hat\phi^r_{\boldsymbol{\alpha}} \,:=\, \frac{\phi^r_{\boldsymbol{\alpha}}}{n\tau^{2j}}\,,\quad \forall\boldsymbol{\alpha}:\, 2j-1\le|\boldsymbol{\alpha}|\le 2j\,;\; j\in\mathbb{N},$$

so that $\sup_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}|\hat\phi^r_{\boldsymbol{\alpha}}|\,\le\,1$*. Hence* $\hat\phi^r$ *belongs to the unit ball* B_1 *of the Banach space* ℓ_∞ *of uniformly bounded infinite sequences, equipped with the sup-norm. By Banach-Alaoglu Theorem [**?**],* B_1 *is sequentially compact in the weak-*$\star$ *topology* $\sigma(\ell_\infty,\ell_1)$*. Therefore there exists a subsequence* $(r_\ell)_{\ell\in\mathbb{N}}$ *and an infinite sequence* $\hat\phi^\star=(\hat\phi^\star_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}\in B_1$*, such that*

$$\lim_{\ell\to\infty}\hat\phi^{r_\ell}_{\boldsymbol{\alpha}} \,=\, \hat\phi^\star_{\boldsymbol{\alpha}} \quad \forall\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}.$$

Equivalently, letting $\phi^\star_{\boldsymbol{\alpha}}:=n\tau^{2j}\,\hat\phi^\star_{\boldsymbol{\alpha}}$*, for all* $\boldsymbol{\alpha}$ *such that* $2j-1\le|\boldsymbol{\alpha}|\le 2j$*, and* $j\in\mathbb{N}$*, one obtains:*

$$\lim_{\ell\to\infty}\phi^{r_\ell}_{\boldsymbol{\alpha}} \,=\, \phi^\star_{\boldsymbol{\alpha}} \quad \forall\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}, \tag{5}$$

Fix $d\in\mathbb{N}$*, and denote* $M^d_{\phi^\star}$ *(resp.* $M^d_{\phi^\star}(z_i)$*) the moment matrix with moments up to order* $2d$ *(resp. the localizing matrix with respect to* z_i*) with moments up to order* $2d$ *(resp.* $2d-1$*). The convergence (5) yields* $M^d_{\phi^\star}\succeq 0$ *and* $M^d_{\phi^\star}(z_i)\succeq 0$*, for every* d*. In addition we have* $|\phi^\star_{\boldsymbol{\alpha}}|\,\le\,n\tau^{2r}\,<\,(n\tau)^{2r}$ *whenever* $|\boldsymbol{\alpha}|\,\le\,2r$*. Therefore by Proposition 2.38 in [**?**] it follows that* $|\phi^\star_{\boldsymbol{\alpha}}|\,\le\,(n\tau)^{|\boldsymbol{\alpha}|}$*, for all* $\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}$*. Then by Proposition 2.37 in [**?**]* $\phi^\star$ *has a representing measure (still denoted* $\phi^\star$*) on the (compact) box* $[-n\tau,n\tau]^{s(2n+1)}$*. In addition, by Theorem **??**,* $z_i\ge 0$ *for all* $i\in[s]$*, on* $\mathrm{supp}(\phi^\star)$*.*

Next, fix $\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}$ *arbitrary. Then by (5):*

$$\begin{aligned} 0 \;&=\; \lim_{\ell\to\infty} L_{\phi^{r_\ell}}((\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^{\boldsymbol{\alpha}}h_i(\boldsymbol{u},\boldsymbol{v}))\\ &=\; L_{\phi^\star}((\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^{\boldsymbol{\alpha}}h_i(\boldsymbol{u},\boldsymbol{v}))\;=\;\int(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^{\boldsymbol{\alpha}}h_i(\boldsymbol{u},\boldsymbol{v})\,d\phi^\star(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\end{aligned}$$

for all $0 \le i \le s$. *Similarly:*

$$\begin{aligned} 0 &= \lim_{\ell\to\infty} L_{\phi^{r_\ell}}((\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^{\boldsymbol{\alpha}} h_i^z(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})) \\ &= L_{\phi^\star}((\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^{\boldsymbol{\alpha}} h_i^z(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})) \;=\; \int (\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})^{\boldsymbol{\alpha}} h_i^z(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\, d\phi^\star(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\,, \end{aligned}$$

for all $i \in [s]$. *As this holds for arbitrary* $\boldsymbol{\alpha} \in \mathbb{N}^{s(2n+1)}$ *and the support of* $\phi^\star$ *is compact, one obtains* $h_i(\boldsymbol{u},\boldsymbol{v}) = 0$, *that is:*

$$u_{i,j} + \frac{iT}{s}\, v_{i,j} \;=\; u_{i+1,j} + \frac{iT}{s}\, v_{i+1,j}\,,$$

for all $i \in \{0,\dots,s\}$ *and all* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \in \mathrm{supp}(\phi^\star)$. *Similarly,* $h_i^z(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) = 0$ *for all* $i \in [s]$, *for* $\phi^\star$*-almost all* $(\boldsymbol{u},\boldsymbol{v}\,\boldsymbol{z})$. *That is,* $z_i^2 = (\frac{T}{s}\,\|\boldsymbol{v}_i\|)^2$. *Since* $z_i \ge 0$ *also holds on* $\mathrm{supp}(\phi^\star)$, *then equivalently* $z_i = \frac{T}{s}\|\boldsymbol{v}_i\|$ *for all* $i \in [s]$.

Next, let $t \in \left[\frac{(i-1)T}{s}, \frac{iT}{s}\right]$ *and* $k \in [m]$ *be fixed, arbitrary, and rewrite* $g_k(t, \boldsymbol{u}_i + t\boldsymbol{v}_i)$ *as* $g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i) \in \mathbb{R}[\boldsymbol{u}_i,\boldsymbol{v}_i]$. *Then with* $d \in \mathbb{N}$ *fixed arbitrary,*

$$\begin{aligned} 0 \;\preceq\; & \lim_{\ell\to\infty} M^d_{\phi^{r_\ell}}(g_k(\boldsymbol{u}_i + t\,\boldsymbol{v}_i)) \;=\; \lim_{\ell\to\infty} M^d_{\phi^{r_\ell}}(g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i)) \\ & = M^d_{\phi^\star}(g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i)) \;=\; M^d_{\phi^\star}(g_k(\boldsymbol{u}_i + t\,\boldsymbol{v}_i). \end{aligned}$$

Hence, as d *was arbitrary, one obtains* $M^d_{\phi^\star}(g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i)) \succeq 0$ *for all* d. *As the support of* $\phi^\star$ *is compact, by Theorem* **??**, $g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i) \ge 0$ *on the support of* $\phi^\star$. *That is, for all* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \in \mathrm{supp}(\phi^\star)$*:*

$$g_{k,t}(\boldsymbol{u}_i,\boldsymbol{v}_i) \quad (=\, g_k(t, \boldsymbol{u}_i + t\boldsymbol{v}_i)) \;\ge\; 0.$$

As this is true for all $t \in \left[\frac{(i-1)T}{s}, \frac{iT}{s}\right]$ *and* $k \in [m]$, *one obtains* $g_k(t, \boldsymbol{u}_i + t\,\boldsymbol{v}_i) \ge 0$ *for all* $t \in \left[\frac{(i-1)T}{s}, \frac{iT}{s}\right]$, *and for all* $i \in [s]$. *From what precedes, we conclude that for all* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \in \mathrm{supp}(\phi^\star)$, *the piecewise linear trajectory*

$$\boldsymbol{x}_i(t) \;:=\; \boldsymbol{u}_i + t\,\boldsymbol{v}_i \qquad t \in \left[\frac{(i-1)T}{s}, \frac{iT}{s}\right] \qquad i \in [s], \tag{6}$$

is feasible for $MP(s;\mathcal{D})$. *Finally, we also have*

$$\begin{aligned} \rho^* \ge \lim_{\ell\to\infty} \rho_{r_\ell} &= \lim_{\ell\to\infty} \sum_{i=1}^s L_{\phi^{r_\ell}}(z_i) \\ &= \int \sum_{i=1}^s \frac{T}{s}\, \|\boldsymbol{v}_i\|\, d\phi^\star(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \\ &= \int \underbrace{\sum_{i=1}^s \left(\int_{\frac{(i-1)T}{s}}^{\frac{iT}{s}} \|\dot{\boldsymbol{x}}_i(t)\|\, dt \right)}_{\ge \rho^*} d\phi^\star(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}) \\ &\ge \rho^* \end{aligned}$$

(with $\boldsymbol{x}_i(t)$ *as in (6)). Therefore for* $\phi^\star$*-almost all* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})$*, the piecewise linear trajectory* $\boldsymbol{x}(t)$ *(with* s *pieces) defined in (6) is optimal for problem MP*$(s;\mathcal{D})$*. This proves (a).*

(b) If the trajectory (3) is the unique optimal solution of MP$(s;\mathcal{D})$ *then necessarily* $\phi^\star = \delta_{(\boldsymbol{u}^*,\boldsymbol{v}^*,\boldsymbol{z}^*)}$ *(where* $z_i^* = \|\boldsymbol{v}_i^*\|$*,* $i \in [s]$*) because in (a) we have seen that for* $\phi^\star$*-almost all* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})$*, the piecewise linear trajectory* $\boldsymbol{x}(t)$ *in (6) is optimal, hence identical to* $\boldsymbol{x}^*$ *in (3). But then this proves that all accumulation points of* $(\phi^r)_{\boldsymbol{\alpha}\in\mathbb{N}^{s(2n+1)}}$ *are identical to the vector of moments of the Dirac measure* $\phi^\star := \delta_{(\boldsymbol{u}^*,\boldsymbol{v}^*,\boldsymbol{z}^*)}$*, and therefore (5) now becomes the global convergence (4).*