MATH103-21S2 Assignment 1

Differential equations and sequences

General instructions

Working together

You may do this assignment in groups of up to three people. Only *one* assignment per group is to be handed in, with *all* your names included. The assignment must be your own group's work but you may discuss ideas with other people.

Marking

This assignment is worth 9% of your final grade and will be marked out of 9. One mark is allocated for your introduction and conclusion, the clarity of your explanations, and for presentation.

There are two tasks to this assignment. Each will be marked out of 4 on the following basis:

- 4: Complete.
- 3: Good but some elements missing.
- 2: OK, got the main points but quite a bit missing.
- 1: Got some points but too many missing.

Presentation

Read the General Instructions in Learn on writing assignments for guidelines on presentation.

- *Include an introduction and a conclusion. A few sentences for each is enough. You may prefer to do this for each task rather than for the assignment as a whole.*
- *Include explanations of what you are doing at each stage of the assignment.* You will lose marks if you do not describe what is going on in properly constructed sentences.
- Graphs should have titles, and appropriately labelled axes. Include a legend if there is more than one curve on the graph.
- Handwritten work is fine. It is enough to write down the general solution of a differential equation and the particular solution determined by the initial conditions. *However, graphs must be done using Maple, Desmos, or some other suitable tool to ensure accuracy.*
- Your assignment should be *8 pages at the most.* (Here a 'page' means a single 'side' of A4.)

Deadline

The deadline is 5pm Thursday 16th September. Upload a picture/scan of your solutions in pdf format to the provided link on Learn (if you are making the submission for your group, include the full name(s) of your partner(s) in the submission comments).

Task One: Clinical pharmacokinetics

You are the principal pharmacologist for a drug company and are responsible for setting the optimal dosage for a course of medication for your company's new drug Delta-Defence. The course of medication will be one initial pill of a certain dose, followed by a sequence of identical pills of a possibly different dose. You need to specify the amount of drug in the first pill, the amount of drug in all subsequent pills, and the time between doses. The amount of drug should be reported in milligrams, and the time between doses in hours.

Your clinical trials indicate that, once the drug is consumed, the amount X of the drug remaining in a patient's bloodstream decreases at a rate proportional to X with decay constant $k = \frac{\ln 2}{8}$ per hour. You have also determined that the drug's minimum therapeutic concentration in blood is 10 micrograms per millilitre and its toxicity threshold is 40 micrograms per millilitre. In other words, for the drug to be effective, the drug's concentration in the patient's blood has to be maintained between 10 and 40 micrograms per millilitre.

You may assume that

- at the moment the patient consumes any pill, it instantly dissolves and entirely enters into their bloodstream;
- blood occupies 7 percent of a person's mass;
- the density of human blood is 1.06 kilograms per litre.

(a) Specify a course of medication for a 53 kg patient that **maximizes** the time between doses.

(b) Plot the drug concentration in the patient's blood over a meaningful time period from the moment the course of medication is commenced.

(c) Suppose the patient has a medical condition that changes the rate at which the drug is metabolized to $k = \ln 2$ per day. How would the course of medication change?

(d) What would you change about Delta-Defence if you wanted a healthy patient to be able to take just one pill a day?

Task Two: A continued fraction sequence

Construct the recursive sequences a_n and b_n as follows. We start with two 4-digit numbers by taking all the digits in the odd and even positions of your student ID respectively. Let a_1 be the smaller of the two numbers, and let a_2 be the larger of the two numbers. For each $n \geq 1$, set

- $a_{n+2} = a_n + a_{n+1}$;
- $b_n = \frac{a_{n+1}}{a_n}$

(a) Find and plot the first 10 terms of the sequence $\{b_n\}$.

(b) Prove that

$$b_{n+1} = 1 + \frac{1}{b_n}$$

and thus write down a formula for b_{10} that includes b_1.

(c) Prove by induction that

$$a_{n+1}a_{n-1} - a_n^2 = (-1)^n \left[a_1^2 + a_1 a_2 - a_2^2\right]$$

for all $n \geq 2$. [*Hint:* The condition $a_{n+2} = a_n + a_{n+1}$ means that $a_{n+2} - a_{n+1}$ may be rewritten as a_n.]

(d) Show that the sequences

$$\{b_{2k-1}\}_{k=1}^{\infty} = \{b_1, b_3, b_5, \ldots\}$$
$$\{b_{2k}\}_{k=1}^{\infty} = \{b_2, b_4, b_6, \ldots\}$$

are monotone. Determine whether each of them is increasing or decreasing. Note that your answer will depend on the digits of your student ID. You may find the previous identity helpful. [*Hint:* One approach is to take either the difference or ratio of successive terms and use the recurrence in the definition of a_n to create repeated terms.]

(e) Find constants c and d such that $c \leq b_n \leq d$ for all integers $n \geq 1$. [*Hint:* One approach is to make a guess and prove by induction.]

(f) In view of (d) and (e), it now follows from the monotone convergence theorem that the limits

$$b_{odd} = \lim_{k\to\infty} b_{2k-1}, \quad b_{even} = \lim_{k\to\infty} b_{2k}$$

exist. Note that we cannot yet conclude that the limits are the same. Show that

$$b_{odd} = 1 + \frac{1}{b_{even}}, \quad b_{even} = 1 + \frac{1}{b_{odd}}.$$

(g) Prove that

$$b_{odd} = b_{even} = \frac{1+\sqrt{5}}{2}.$$

Note that one cannot take limits in (b) for the entire sequence $\{b_n\}_{n=1}^{\infty}$ as it is not yet clear that the limits on either side of the formula in (b) exist.