Numerical Linear Algebra	March 2, 2025
Topic: Orthogonal Vectors and Matrices	
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1 Orthogonal Vectors and Matrices

Definition 1. *A hermitian conjugate or adjoint of an $m \times n$ matrix A is the $n \times m$ matrix A^* obtained by taking the complex conjugate of each entry and then taking the transpose.*

$$A^* = \overline{A}^T$$

Where

$$A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{mn}} \end{bmatrix}$$

If $A = A^*$, then A is said to be hermitian.

Definition 2. *An inner product is bilinear, meaning*

$$\begin{aligned} (x_1 + x_2)y &= x_1 y + x_2 y \\ x(y_1 + y_2) &= xy_1 + xy_2 \\ (\alpha x)(\beta y) &= \alpha\beta xy \end{aligned}$$

Theorem 1.1. *The vectors in an orthogonal set S are linearly independent.*

Proof. Suppose $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ for an orthogonal set $\{v_1, \ldots, v_k\}$. By bilinearity of the inner product,

$$0 = \langle \alpha_1 v_1 + \cdots + \alpha_k v_k, v_i \rangle = \alpha_1 \langle v_1, v_i \rangle + \cdots + \alpha_k \langle v_k, v_i \rangle.$$

Since $\langle v_j, v_i \rangle = 0$ for $j \neq i$, we get

$$\alpha_i \langle v_i, v_i \rangle = 0 \quad \Longrightarrow \quad \alpha_i = 0.$$

Hence, all α_i must be zero and the vectors are linearly independent. □

Key idea:

Inner products and Orthogonality can decompose arbitrary vectors into orthogonal components.

Theorem 1.2. *If $q_1, \dots, q_n$ are orthogonal, then where v is an arbitrary vector, and qv^T is a scalar. The vector $r = v - (q_1 v)q_1 - \cdots - (q_n v)q_n$ is orthogonal to $q_1, \dots, q_n$.*

Multiplying q_i by r gives

$$\begin{aligned} q_i r &= q_i v - (q_i q_1) v q_1 - \cdots - (q_i q_n) v q_n \\ &= q_i v - q_i v q_i q_i \\ &= 0 \end{aligned}$$

References

[1] Lloyd N. Trefethenós, David Baué, *Numerical Linear Algebra*, Northwestern University.