Numerical Linear Algebra March 3, 2025

Topic: Fundamentals

FAUST *Scribe: David Zhang*

1 Vandermonde Matrix

Definition 1. *A* ***Vandermonde matrix*** *is a matrix with the terms of a geometric progression in each row, i.e.:*

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}$$

If c is a column vector of coefficients, then the product of the Vandermonde matrix and c is a column vector of the values of the polynomial at the points $x_1, x_2, \ldots, x_m$.

Thus the product $(Ac)_i = p(x_i)$ *where* $p(x)$ *is the polynomial defined by the coefficients in c.*

2 *Orthogonal Vectors and Matrices*

Definition 2. *A hermitian conjugate or adjoint of an* $m \times n$ *matrix* A *is the* $n \times m$ *matrix* A^* *obtained by taking the complex conjugate of each entry and then taking the transpose.*

$$A^* = \overline{A}^T$$

Where

$$A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{mn}} \end{bmatrix}$$

If $A = A^*$, *then* A *is said to be hermitian.*

Definition 3. *An inner product is bilinear, meaning*

$$\begin{aligned} (x_1 + x_2)y &= x_1 y + x_2 y \\ x(y_1 + y_2) &= xy_1 + xy_2 \\ (\alpha x)(\beta y) &= \alpha\beta xy \end{aligned}$$

Theorem 2.1. *The vectors in an orthogonal set* S *are linearly independent.*

Proof. Suppose $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ for an orthogonal set $\{v_1, \ldots, v_k\}$. By bilinearity of the inner product,

$$0 = \langle \alpha_1 v_1 + \cdots + \alpha_k v_k, v_i \rangle = \alpha_1 \langle v_1, v_i \rangle + \cdots + \alpha_k \langle v_k, v_i \rangle.$$

Since $\langle v_j, v_i \rangle = 0$ for $j \neq i$, we get

$$\alpha_i \langle v_i, v_i \rangle = 0 \quad \Longrightarrow \quad \alpha_i = 0.$$

Hence, all α_i must be zero and the vectors are linearly independent. $\square$

Key idea:

Inner products and Orthogonality can decompose arbitrary vectors into orthogonal components.

Theorem 2.2. *If $q_1, \ldots, q_n$ are orthogonal, then where v is an arbitrary vector, and $q^T v$ is a scalar.*

The vector $r = v - (q_1 v) q_1 - \cdots - (q_n v) q_n$ is orthogonal to $q_1, \ldots, q_n$.

Multiplying q_i by r gives

$$\begin{aligned} q_i r &= q_i v - (q_i q_1) v q_1 - \cdots - (q_i q_n) v q_n \\ &= q_i v - q_i v q_i q_i \\ &= 0 \end{aligned}$$

3 Norms

Certain norms are more useful than other norms. These are the induced matrix norms, defined in terms of the behavior of a matrix as an operator between its normed domain and range spaces.

The induced norm $\|A\|_(m, n)$ of a matrix A is defined as the smallest number C such that

$$\|A\|_(m, n) \leq C \|x\|_n$$

Otherwise, this is the

$$\|A\|_(m, n) = \sup_{x \neq 0} \frac{\|Ax\|_m}{\|x\|_n} = \sup_{\|x\|_n = 1} \|Ax\|_m$$

Definition 4. *Cauchy-Schwarz and Holder's Inequality*

$$\langle x, y \rangle \leq \|x\|_2 \|y\|_2$$

$$\|xy\|_1 \leq \|x\|_p \|y\|_q$$

where p and q are conjugate exponents (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Other than $p = q$, this is possible for any $p > 1$ (e.g. $p = 3, q = \frac{3}{2}$).

4 Singular Value Decomposition

If you don't know what to do, take the SVD

$$A = U \Sigma V^*$$

The left singular values U represent the length of the semiaxises of an ellipsoid, where the right singular values V the preimages of the principal semiaxes.

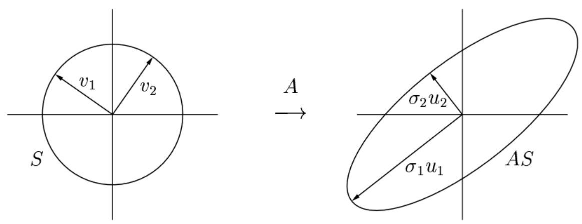

Figure 4.1. *SVD of a* 2×2 *matrix.*

Figure 1: SVD Circle Illustration

5 More on SVD

Definition 5. *Nondefective Matrix*

A matrix is nondefective if it has a full set of eigenvectors.

Theorem 5.1. *Matrix A can be made as the sum of r rank-one matrices*

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

The partial sums of the SVD are the best rank-r approximation to A. (the best energy)

References

[1] *Lloyd N. Trefethen, David Bau III,* Numerical Linear Algebra*, Northwestern University.*