CSCE 643	Feburary 6th, 2025

Topic 1 – The Configuration Space

Prof. Dylan Shell / Dr. Steven Lavelle *Scribe: David Zhang*

1 Overview

Definition 1. ***Configuration Space:*** *The state space for motion planning is a set of possible transformations that could be applied to the robot*

2 Basic Topological Concepts

2.1 Topological Spaces

Definition 2. ***An open set is*** *a interval with no boundary points. That must follow this rule (TLDR)*

Definition 3. ***A topological space*** *is a set $\mathbb{X}$ together with a collection of open sets O that satisfy the following properties:*

1. *The union of any number of open sets is an open set.*
2. *The intersection of a finite number of open sets is an open set*
3. *Both $\mathbb{X}$ and $\emptyset$ are open sets*

Definition 4. ***Special Points:*** *A point on the border of an open or closed set*

2.1.1 Special Point scenarios

Consider a point x that in the topological space $\mathbb{X}$, and a set U that is a subset of X. The following terms capture the position of point x relative to U.

1. If x is in U, then x is an interior point of U
2. If x is not in U, then x is an exterior point of U
3. If x is neither an interior point nor an exterior point of U, then x is a boundary point of U
4. If x is an interior or boundary point, it is a limit point (which the set of all limit points of U is called the closure of U)

Important Note: The closure is always a closed set, because it contains all the boundary points. That also means the open set contains none of the boundary points, making it open.

2.2 The Ball analogy (Example 4.1)

Think of an open set as a ball in $\mathbb{R}^n$ space, and the points that fill the ball are the interior points centered at some point x.

1. *All open sets can be represented by a countable union of open balls*
2. *Any function constructed from primitives that use the $<$ relation are open.*

2.3 Subspace Topology

The subspace topology is a topology that have all of its representative open sets be every subset to a larger topological space. $U \subseteq \mathbb{X}$

Definition 5. ***Hausdorff axiom:*** *For any distinct points x_1 and $x_2 \subseteq X$, there exists open sets O_1 and O_2 such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$*

In other words, points can be separated into distinct non-overlapping subsets.

Definition 6. ***Homeomorphism:*** *A function $f : \mathbb{X} \to \mathbb{Y}$ is a homeomorphism if it is bijective, and the function and its inverse are both continuous. Two topological spaces X and Y*

If two topological spaces X and Y are homeomorphic, then they are topologically equivalent, which means there exists a homeomorphism between them. This property is reflexive, symmetric, and transitive.

Definition 7. *A topological space is bounded if there exists a ball $B \subset \mathbb{R}^n$ if there exists a ball $B \subset \mathbb{R}^n$ such that $X \subset B$.*

Fact 8. *The mapping $x \to \frac{1}{x}$ makes two open sets $(0, 1)$ and $(1, \infty)$ homeomorphic.*

Definition 9. ***Topological Graph:*** *A graph for which every vertex corresponds to a point in X and every edge corresponds to a continuous, injective function from $\tau : [0, 1]$ to X.*

Definition 10. *Two graphs are **isomorphic** if there exists a bijective mapping, $f : V_1 \to V_2$ such that there is an edge between v_1 and v_2 if and only if there is an edge between $f(v_1)$ and $f(v_2)$ in the second graph.*

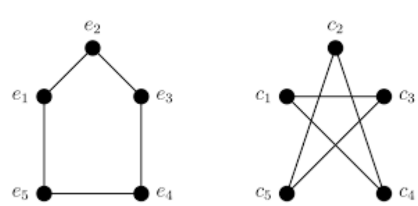

Figure 1: Isomorphic Graph Example

2.4 Manifolds

Definition 11. ***Manifold:*** *A topological space $M \subseteq \mathbb{R}^m$ is a manifold if for every point $x \in M$, there exists an open set $O \subset M$ such that O is homeomorphic to an open set in $\mathbb{R}^n$. $x \in O$ and O is homeomorphic to an open set in $\mathbb{R}^n$. n is the dimension of the manifold.*

In simple terms, it behaves at every point like our intuitive notions of a surface.

Corollary 12. *This naturally leads to $m \geq n$ leading to whitney's embedding theorem.*

3 What is a manifold?

A topological space that is "understood" is a manifold.

Observation 13. *S^1 is a circle.*

- *The circle can be stretched and deformed, into a square, so they are topologically equivalent*
- *As long as we don't cut or pinch the circle, they are equivalent (homeomorphic)*

S^2 is a sphere.

- *A sphere lives in 3-D Space*
- *S^2 is embedded in topological 3D space*

Note: A torus is not topologically equivalent to a sphere, because it has a hole in it.

Definition 14. ***Charts:*** *A chart is a cover of a topological space that gives coordinates to the space. They map a part of the manifold that to $\mathbb{R}^d$.*

Consider the earth as a manifold, the set of all locations on the earth. Of which we can problem sections of the earth as maps. And collections of these maps are called atlases. The maps are 2D images of the manifold to represent a 3D space. The altas may have portions of the map that overlap.

Definition 15. $\mathbb{S}^1 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$
It is a circle.

3.0.1 Identification in manifolds

Definition 16. ***Identification:*** *A general method of declaring that some points of a space are identical even though they originally were distinct.*

The figures showcase sections of a manifold and their identification. Arrows represent the mapping of the points. For example, for the cylinder, the top and the bottom are linked together mapping $(0, y)$ $(1, y)$.

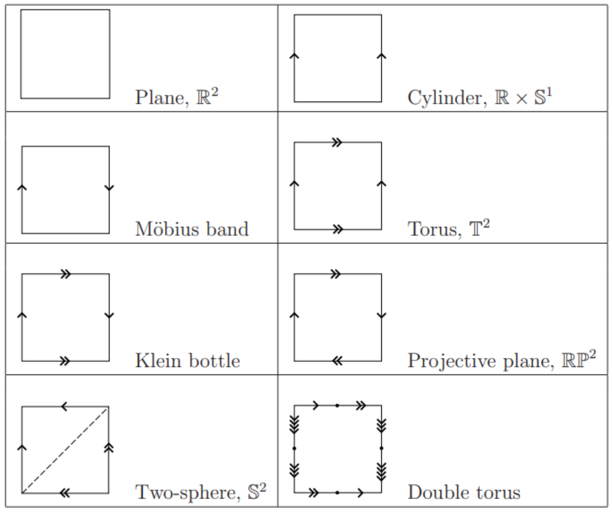

Figure 2: Identification Example

3.1 Paths and Connectivity

We essentially want to ***connect one part of space to another part of space****, and we can do this by utilizing a sequence of actions. Graphs are for discrete spaces, and topological spaces are for continuous spaces.*

Definition 17. ***Path:*** *A path is a continuous function from $\tau : [0, 1] \to X$. $\tau = x(s)$ where x is the path and s is the state index in the path.*

Definition 18. ***Connected Space:*** *A space is connected if there is a path or a union of two disjoint, nonempty, open sets.*

Definition 19. ***Homotopic Paths:*** *Two paths are homotopic if there exists a continuous function $h : [0, 1] \times [0, 1] \to X$ such that the paths deformed into the other. There cannot be a discontinuous jump between the paths.*

This also leads into algebraic topology, which is mainly characterized by groups

Definition 20. ***A group*** *is defined by the 4 properties:*

1. *Closure - multiplication / binary operations are closed within the group*
2. *Associativity - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*
3. *Identity - There exists an identity element e such that $a \cdot e = e \cdot a = a$*
4. *Inverse - For every element a, there exists an inverse a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$*

Bibliography. *Please give real bibliographical citations for the papers that we mention in class. See below for how to include a bibliography section. If you use BibTeX, integrate the .bbl file into your .tex source. You should reference papers like this: "The FKS dictionary originates in a paper* *by Fredman, Komlós and Szemerédi [1]." In general, the name of the authors should appear in text at most once (for the first citation); further citations look like: "Our proof follows that of [1]".*

Take a look at previous topics (TeX files are available) to see the details. A excellent source for bibliographical citations is DBLP. Just Google DBLP and an author's name.

References

[1] *M. Fredman, J. Komlós, E. Szemerédi,* Storing a Sparse Table with $O(1)$ Worst Case Access Time*, Journal of the ACM, 31(3):538-544, 1984.*