Multivariate Linear Regression

Univariate Linear Regression

In univariate linear regression, we had something like this:

Size in ft^2 (x)	**Price in \$100 (y)**
2104 ($x^{(1)}$)	460 ($y^{(1)}$)
1416 ($x^{(2)}$)	232 ($y^{(2)}$)
1534 ($x^{(3)}$)	315 ($y^{(3)}$)
852 ($x^{(4)}$)	178 ($y^{(4)}$)
...	...

And we calculate our hypothesis as:

$$h_\theta(x) = \theta_0 + \theta_1 x$$

Multivariate Linear Regression

We will have more features here, such as the dataset below:

Size (x_1)	**Bedrooms** (x_2)	**Floors** (x_3)	**House Age** (x_4)	**Price** (y)
2104	5	1	45	460
1416	3	2	40	232
1534	3	2	30	315
852	2	1	36	178
...	...	...	...	...

Here,

- n = Number of features
- $x^{(i)}$ = Input features of i$^{\text{th}}$ training example
- $x_j^{(i)}$ = Value of feature j in i$^{\text{th}}$ training example

For instance,

$$x^{(2)} = \begin{bmatrix} 1416 \\ 3 \\ 2 \\ 40 \end{bmatrix}$$

To get the values from the vectors,

$x_3^{(2)} = 2$

$x_2^{(3)} = 3$

$x_4^{(1)} = 45$

Our hypothesis will now be:

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3$$

Simplifying the Hypothesis

In multivariate linear regression, our hypothesis looks like

$$\begin{aligned} h_\theta(x) &= \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n \\ &= \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n \quad [Let\ x_0 = 1] \\ &= \theta^T x \end{aligned}$$

Here,

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}_{[(n+1)*1]} \qquad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}_{[(n+1)*1]}$$

Parameters = $\theta_0, \theta_1, \theta_2, \cdots, \theta_n = \theta$

Cost Function will be

$$J(\theta_0, \theta_1, \theta_2, \cdots, \theta_n) = \frac{1}{2m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$
$$=> J(\theta) = \frac{1}{2m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

Gradient Descent

repeat {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1, \theta_2, \cdots, \theta_n)$$

} [Simultaneously update θ_j for every $j = 0, 1, 2, \cdots, n$]

Or, we can say

repeat {

$$\theta_j := \theta_j - \alpha \cdot \frac{1}{m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) \cdot x_j^{(i)}$$

} [Simultaneously update θ_j for every $j = 0, 1, 2, \cdots, n$]

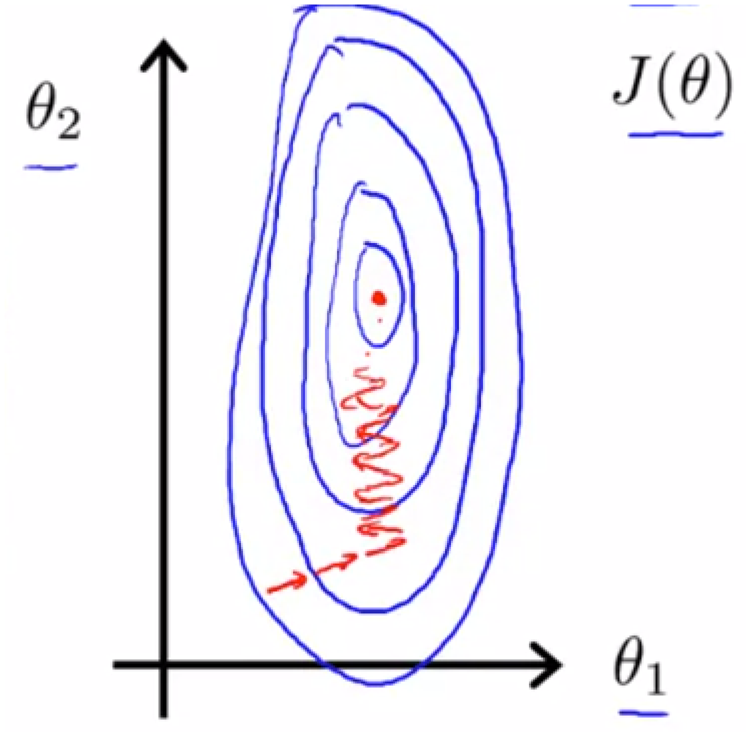
> When there are ‘n’ features, we define the cost function as
>
> $$J(\theta) = \frac{1}{2m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$
>
> For linear regression, which of the following are also equivalent and correct definitions of $J(\theta)$?
>
> √ $J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (\theta^T x^{(i)} - y^{(i)})^2$
>
> √ $J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} ((\sum_{j=0}^{n} \theta_j x_j^{(i)}) - y^{(i)})^2$
>
> √ $J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} ((\sum_{j=1}^{n} \theta_j x_j^{(i)}) - y^{(i)})^2$
>
> √ $J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} ((\sum_{j=0}^{n} \theta_j x_j^{(i)}) - (\sum_{j=0}^{n} y_j^{(i)}))^2$

Update Rules:

$$\begin{aligned}
\theta_0 &:= \theta_0 - \alpha \cdot \frac{1}{m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) \cdot x_0^{(i)} \\
\theta_1 &:= \theta_1 - \alpha \cdot \frac{1}{m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) \cdot x_1^{(i)} \\
\theta_2 &:= \theta_2 - \alpha \cdot \frac{1}{m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) \cdot x_2^{(i)} \\
&\vdots \\
\theta_n &:= \theta_n - \alpha \cdot \frac{1}{m} \cdot \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) \cdot x_n^{(i)}
\end{aligned}$$

Feature Scaling

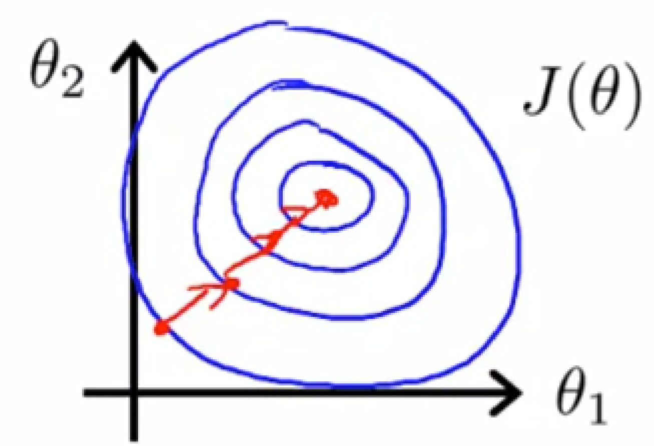
The objective of Feature Scaling is to make sure that the features are on a similar scale, so that gradient descent can converge more quickly.



For example,

- x_1 = Size range (0 to 2000 feet)
- x_2 = Number of bedroome (1-5)

For such a dataset, the plot of the cost function can be shaped eliptical. Gradient descent may oscillate back and forth and take a long time to reach the global minimum.



When we scale, gradient descent can find a much more direct path to the global minimum.

To scale x_1 and x_2 here,

$$x_1 = \frac{size}{2000} \qquad x_2 = \frac{bedrooms}{5}$$

Note: We want to get every feature into approzimately a $-1 \leq x_i \leq 1$ range

Some examples of feature scaling:

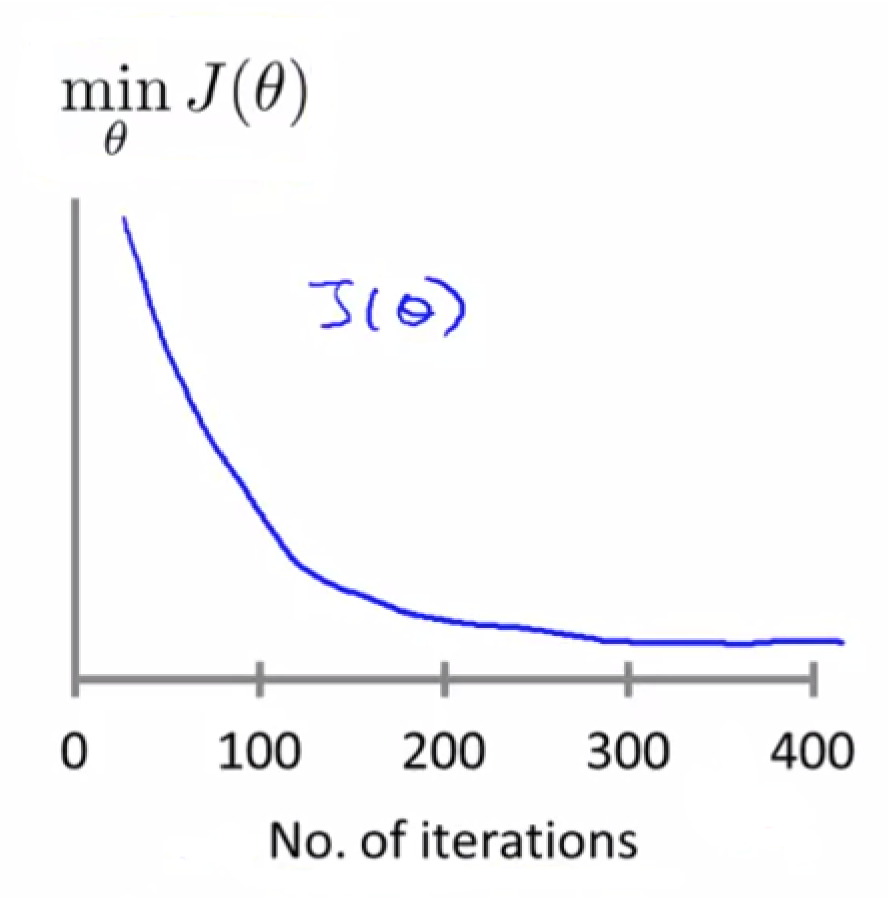
Properly Scaled	Poorly Scaled
$0 \leq x \leq 3$	$-100 \leq x \leq 100$
$-2 \leq x \leq 0.5$	$-0.0001 \leq x \leq 0.0001$

Mean Normalization

$$x_1 = \frac{size - 100}{2000} \qquad x_2 = \frac{bedroom - 2}{5}$$

$$x_i = \frac{old\, x_1 - \mu_i}{range\, of\, x_i}$$

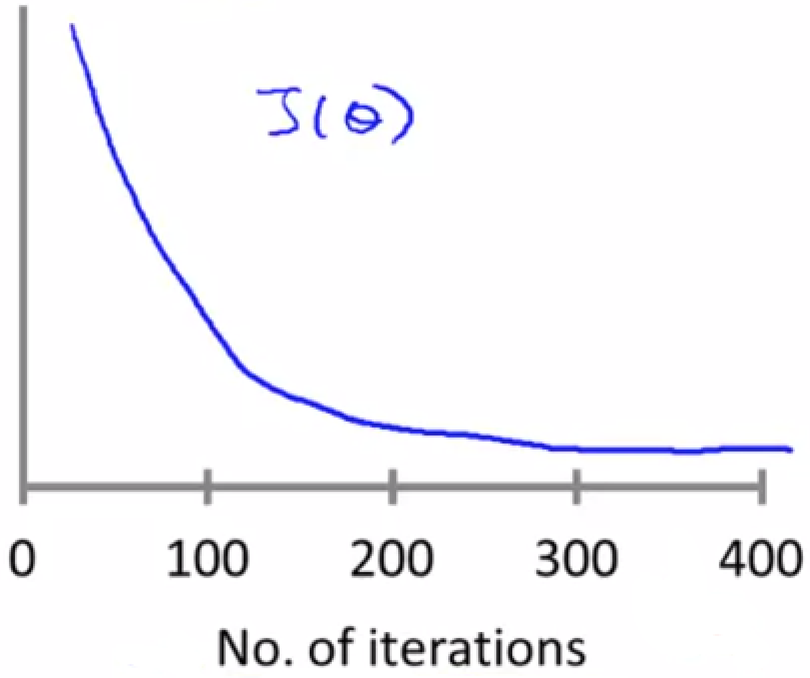
Debugging: Making sure Gradient Descent works correctly

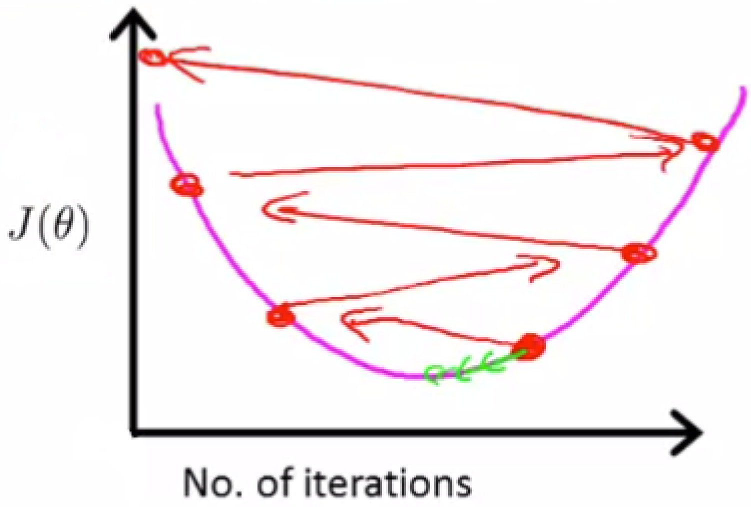


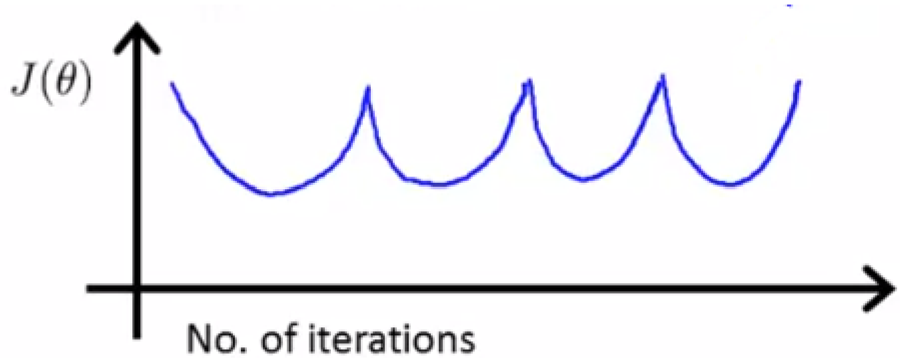
- The value of $J(\theta)$ should decrease after each iteration
- When the line flattens, it suggests that gradient descent has converged
- **Automatic Convergence Test:** Declare convergence if $J(\theta)$ decreases by less than 10^{-3} (or any other threshold) in one iteration

Examples of Gradient Descent not working:

$$\min_{\theta} J(\theta)$$

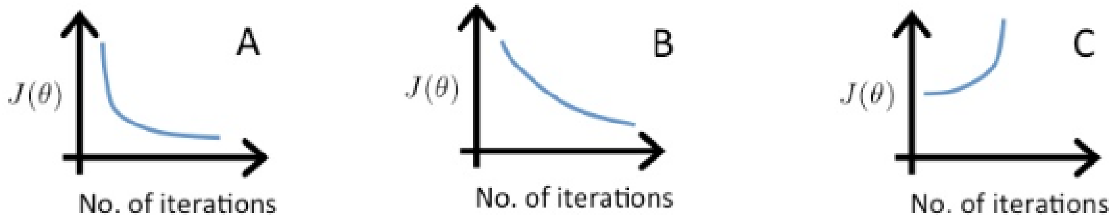






Use smaller values of the learning rate α to fix these, but if α is too small, gradient descent can be slow to converge

Examples of Learning Rates



A: $\alpha = 0.1$

B: $\alpha = 0.01$

C: $\alpha = 1$

Note: To choose α, start from small and try range of values multiplied by 3 each time. For instance,

$$\cdots \rightarrow 0.001 \rightarrow 0.003 \rightarrow 0.01 \rightarrow 0.03 \rightarrow 0.1 \rightarrow 0.3 \rightarrow 1 \rightarrow \cdots$$

Polynimial Regression

Suppose we have the housing prices problems. Given data of prices with the features “frontage” and “depth” of houses, we’ll have a hypothesis as:

$$h_\theta(x) = \theta_0 + \theta_1 \cdot frontage + \theta_2 \cdot depth$$

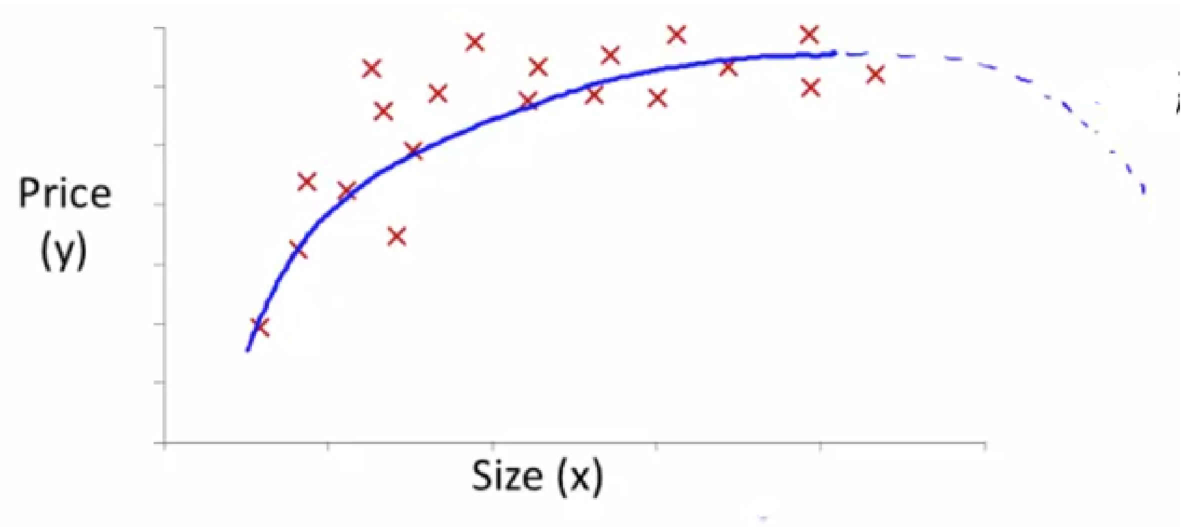
But we don’t necessarily need to define such hypothesis, instead we can determine which features really determines the prices, such as the “area” of the house. We can create a new feature:

$x = frontage * depth$

$h_\theta(x) = \theta_0 + \theta_1 x$

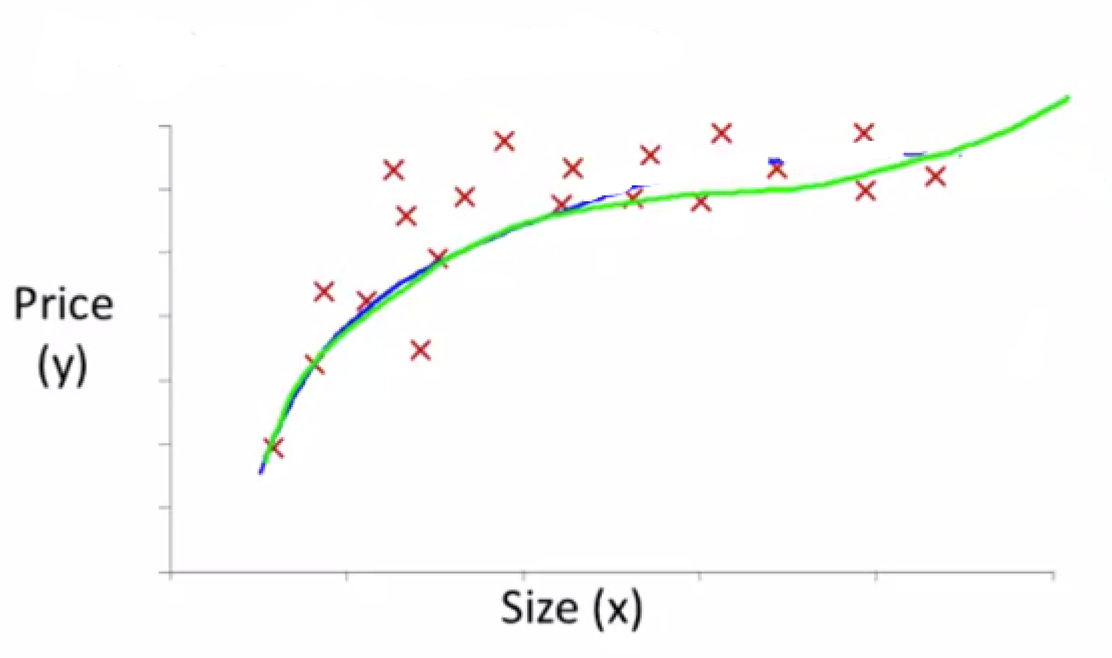
We can fit a **Quadratic Model** to the data and obtain a function like this:

$$h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2$$



However, it fits the data pretty well, but doesn’t make sense because quadratic functions go down, but housing functions never go down with increase in size. We can try to fit a **Cubic Function** instead and get a better fit:

$$h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3$$



$$\begin{aligned} h_\theta(x) &= \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\ &= \theta_0 + \theta_1 x + \theta_2 (size)^2 + \theta_3 (size)^3 \end{aligned}$$

We can also try the **Square Root Function** which flattens out a bit and doesn't come down

$$h_\theta(x) = \theta_0 + \theta_1 (size) + \theta_2 \sqrt{(size)}$$