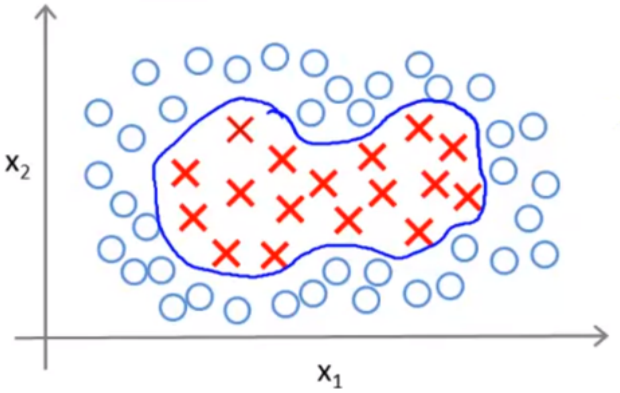
Kernels

Non-Linear Decision Boundary



One way to find the decision boundary in this case is to come up with polynomial features like, predict $y = 1$ if:

$$h_\theta(x) = \begin{cases} 1 & \text{if } \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1 x_2 + \theta_4 x_1^2 + \theta_5 x_2^2 + \dots \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We can simplify the notation here for $h_\theta(x) = 1$ as:

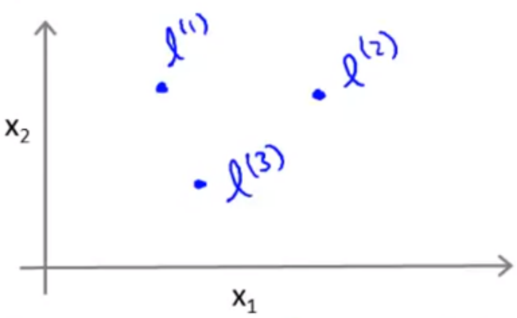
$$\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 + \dots$$

Where, $f_1 = x_1$, $f_2 = x_2$ and so on

The objective is to find a better choice of features than these higher order polynomials

Kernel

Let's pick three random points in the graph and call them **landmarks**.



Now the objective is:

> Given x, compute new features depending on proximity to the landmarks

Kernels (Gaussian Kernels)

$$
\begin{aligned}
f_1 &= \text{similarity}(x, l^{(1)}) = \exp\left(-\frac{||x-l^{(1)}||^2}{2\sigma^2}\right) \\
f_2 &= \text{similarity}(x, l^{(2)}) = \exp\left(-\frac{||x-l^{(2)}||^2}{2\sigma^2}\right) \\
&\vdots \\
&\text{Other Features}
\end{aligned}
$$

Where, $||x - l^{(1)}||$ is the Euclidean distance

Kernels and Similarity

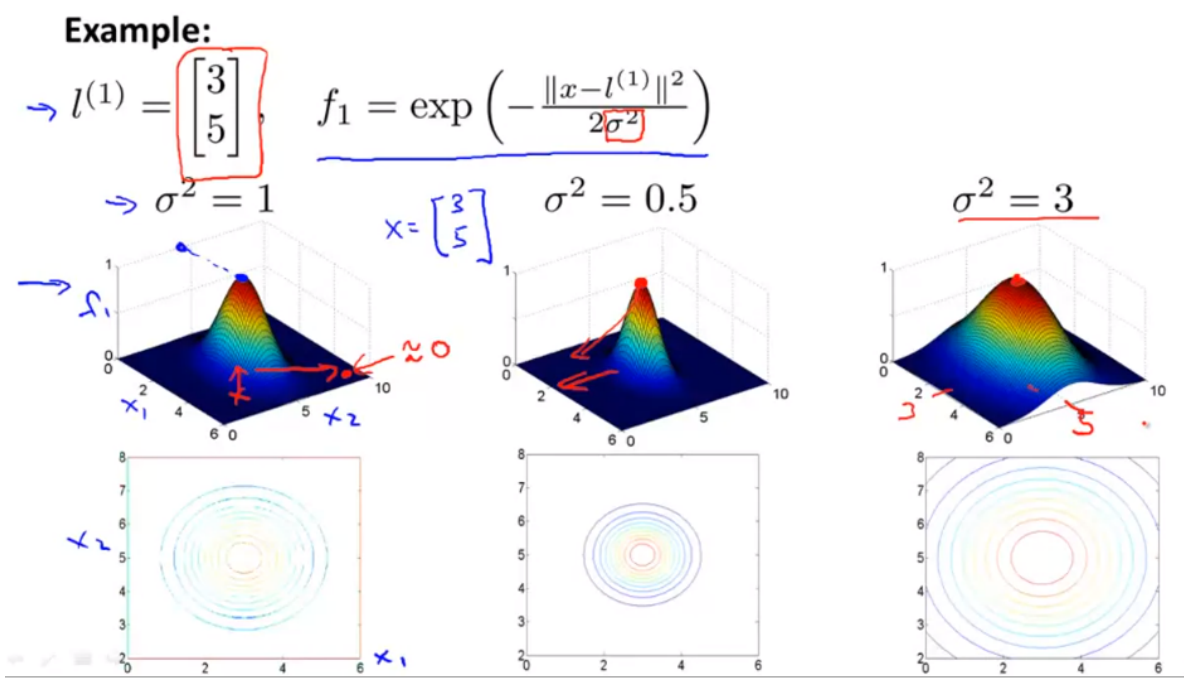
$$
\begin{aligned}
f_1 &= \text{similarity}(x, l^{(1)}) \\
&= \exp\left(-\frac{||x-l^{(1)}||^2}{2\sigma^2}\right) \\
&= \exp\left(-\frac{\sum_{j=1} n(x_j - l_j^{(1)})}{2\sigma^2}\right)
\end{aligned}
$$

If $x \approx l^{(1)}$:

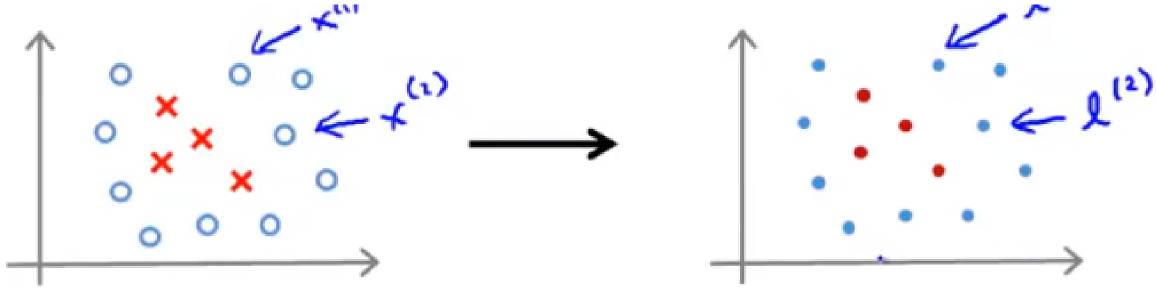
$$f_1 \approx \exp\left(-\frac{0^2}{2\sigma^2}\right) \approx 1$$

If x is far from, $l^{(1)}$:

$$f_1 = \exp\left(-\frac{(\text{Large Number})^2}{2\sigma^2}\right) \approx 0$$



Choosing the Landmarks



Given $(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})$

Choose $l^{(1)} = x^{(1)}, l^{(2)} = x^{(2)}, \dots, l^{(m)} = x^{(m)}$

Given example x:

- $f_1 =$ Similarity $(x, l^{(1)})$
- $f_2 =$ Similarity $(x, l^{(2)})$
- $\dots$

Then we will get a feature and the feature vector as,

$$f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \qquad f^{(i)} = \begin{bmatrix} f_0^{(i)} \\ f_1^{(i)} \\ f_2^{(i)} \\ \vdots \\ f_m^{(i)} \end{bmatrix}$$

Where, $f_0 = 1$ (Similar to an intercept term)

Now for training example $(x^{(i)}, y^{(i)})$:

$$X^{(i)} \rightarrow \begin{cases} f_1^{(i)} = \mathrm{sim}(x^{(i)}, l^{(1)}) \\ f_2^{(i)} = \mathrm{sim}(x^{(i)}, l^{(2)}) \\ \vdots \\ f_i^{(i)} = \mathrm{sim}(x^{(i)}, l^{(i)}) = \exp(-\frac{0}{2\sigma^2}) = 1 \\ \vdots \\ f_m^{(i)} = \mathrm{sim}(x^{(i)}, l^{(m)}) \end{cases}$$

SMV with Kernels

Hypothesis: Given x, compute features $f \in R^{m+1}$

Predict $y = 1$ if $\theta^T f \geq 0$

Training:

$$\min_\theta C \left[\sum_{i=1}^{m} y^{(i)} \mathrm{cost}_1(\theta^T f^{(i)}) + (1 - y^{(i)}) \mathrm{cost}_0(\theta^T f^{(i)}) \right] + \frac{\lambda}{2} \sum_{j=1}^{n} \theta_j^2$$

SVM Parameters:

- Large C: Lower bias, high variance

- Small C: Higher bias, low variance
- Large σ^2: Features f_i vary more smoothly (Higher bias, lower variance)
- Small σ^2: Features f_i vary less smoothly (Lower bias, higher variance)