Čech Cohomology

Spencer Slaton

April 2021

1 Introduction

This article serves as an overview of Čech cohomology and will cover its motivation, construction, and application. We assume only a basic knowledge of algebraic topology.

2 History

The Čech cohomology theory gets its name from Eduard Čech, a Czech mathematician active throughout the first half of the 20th century. He was the first to build on previous work and derive a homology theory from an open cover of any topological space (previous ideas required the space in question to satisfy additional hypothesis, namely compactness) [5]. Later, C. H. Dowker extended this idea to arbitrary open covers using refinements [1]. This realized the formation of the theory as it is known today. A fantastic overview is given by Eilenberg and Steenrod in their seminal work *Foundations of Algebraic Topology* [2]. Published in 1952, *Foundations* describes Čech cohomology in the language of nerves of an open cover. Today, we usually consider Čech cohomology of sheaves, which affords the theory a more general framework.

3 Construction

3.1 Coverings and Nerves

One of the reasons Čech cohomology is so interesting is because, in construction, it is closely related to the topology of the underlying space. Rather than cells or maps of simplices as one might consider in other homology theories, we consider open sets. Let X be a topological space and $\mathcal{U}$ with indexing set I be an open cover of X. The *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ is the collection of sets of I for which the $U_i \in \mathcal{U}$ indexed by the set has nontrivial intersection. Formally,

$$N(\mathcal{U}) = \{J : J \subseteq I \text{ and } \bigcap_{j \in J} U_j \neq \emptyset\}.$$

Then $N(\mathcal{U})$ is an abstract simplicial complex. This follows from the fact that removing one vertex from an n-simplex $K \subset N(\mathcal{U})$ necessarily gives an $(n-1)$-simplex $K^* \subset N(\mathcal{U})$ since $K^* \subset K$ and the sets indexed by K have nonempty intersection, so the sets indexed by K^* have nonempty intersection as well. So the boundary map on $N(\mathcal{U})$ is well-defined.

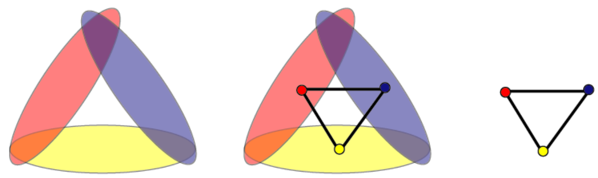

Figure 1: The construction of the nerve of three overlapping sets.

The abstract simplicial complex does not quite characterize X since $N(\mathcal{U})$ is subordinate to the open cover $\mathcal{U}$. To rectify this, is suffices to consider refinements of $\mathcal{U}$. By *refinements* we mean open covers $\mathcal{V}$ such that for all $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subset U$. We use the notation $\mathcal{U} < \mathcal{V}$ to denote that $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Let $\mathcal{U}$ be indexed by I as before, and $\mathcal{V}$ be a refinement of $\mathcal{U}$ indexed by J. Then we can define a map $p : J \to I$ that sends an element j to an element i where $V_j \subset U_i$. This map extends uniquely to a simplicial map $s_p : N(\mathcal{V}) \to N(\mathcal{U})$. Note that the choice of p is not necessarily unique, however, the resulting simplicial maps are "similar enough" in the sense that they are contiguous simplicial maps. The details of what this means and why this is true can be found in *Foundations*, but we will gloss over them here. What matters is we have sufficiently unique simplicial maps between our refinements.

For open covers $\mathcal{U} < \mathcal{V}$ there are homomorphisms

$$H_n(N(\mathcal{V}); G) \to H_n(N(\mathcal{U}); G) \text{ and } H^n(N(\mathcal{V}); G) \to H^n(N(\mathcal{U}); V)$$

between the usual homology and cohomology groups of the simplicial complexes $N(\mathcal{U})$ and $N(\mathcal{V})$ for any coefficient group G which are induced by s_p. The contiguous condition results in these homomorphisms being uniquely associated to the refinement [2].

We now step aside briefly to introduce the concept of a direct limit.

3.2 Direct Systems and Limits

A *direct system over I* is a partially ordered family of objects A_i indexed by I along with homomorphisms $f_{ij} : A_i \to A_j$ for each $i \leq j$. The homomorphisms satisfy the following two properties:

1. $f_{ii} = Id_{A_i}$ for all $i \in I$
2. $f_{ij} \circ f_{jk} = f_{ik}$ for all $i \leq j \leq k$.

On such a system we can define the *direct limit*, denoted $\varinjlim A_i$, by defining

$$\varinjlim A_i = \bigsqcup A_i \Big/ \sim$$

where $\sim$ is the equivalence relation $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some $k \in I$, i.e. if x_i and x_j eventually become equal along the direct system.

Categorically, the concept of direct limit is dual to the concept of inverse limit.

3.3 The Čech Groups

We can now define Čech homology and cohomology over a group G. Let H be the set of open covers of X. For fixed n, we claim the family $\{H^n(N(\mathcal{U}); G)\}_{\mathcal{U} \in G}$ of cohomology groups subordinate to all open covers of X forms a direct system with homomorphisms induced by s_p as described above. Call these homomorphisms $\pi^{\mathcal{U}}_{\mathcal{V}} : H^n(N(\mathcal{U}); G) \to H^n(N(\mathcal{V}); G)$. We know that the simplicial map s_p is the indentity map when $\mathcal{U} = \mathcal{V}$. So the induced map in cohomology is the identity as well. This is the first condition of being a direct system. Also if $\mathcal{U} < \mathcal{V} < \mathcal{W}$ we know that $\pi^{\mathcal{V}}_{\mathcal{W}} \circ \pi^{\mathcal{U}}_{\mathcal{V}}$ is corresponds to the composition of simplicial maps $N(\mathcal{U}) \to N(\mathcal{V}) \to N(\mathcal{W})$. Choosing this to be the simplicial map $N(\mathcal{U}) \to N(\mathcal{W})$ we see that this must generate the unique map $\pi^{\mathcal{U}}_{\mathcal{W}}$ so that $\pi^{\mathcal{V}}_{\mathcal{W}} \circ \pi^{\mathcal{U}}_{\mathcal{V}} = \pi^{\mathcal{U}}_{\mathcal{W}}$, the second condition of being a direct system.

Since $\{H^n(N(\mathcal{U}); G)\}_{\mathcal{U} \in G}$ is a direct system of groups, we define

$$\varinjlim \{H^n(N(\mathcal{U}); G)\}_{\mathcal{U} \in G} := H^n(X; G)$$

to be the nth Čech cohomology group of X. The definition of the nth Čech homology is defined similarly and is justified in the dual sense: instead of a direct system we get an inverse system, and taking its inverse limit rather than a direct limit we get

$$\varprojlim \{H_n(N(\mathcal{U}); G)\}_{\mathcal{U} \in G} := H_n(X; G).$$

4 General Remarks

Now that we have seen how the Čech homology and cohomology groups are constructed, let us make some observations and remarks. Firstly, if the space X is a single point, we can calculate the (co)homology groups directly. Let $\mathcal{U} = \{X\}$ be an open cover of X. Then $\mathcal{U}$ is a refinement of every open cover of X. It follows that both the direct and inverse systems contain only the object $H^n(N(\mathcal{U}); G)$ and ${}_n(N(\mathcal{U}); G)$, respectively. The direct and inverse limits are then just this singular object, and since $N(\mathcal{G})$ is just the one-vertex simplex, we conclude from simplicial (co)homology that

$$H^n(X; G) \cong H_n(X; G) \cong \begin{cases} G & n = 0 \\ 0 & n > 0. \end{cases}$$

We might hope that the Čech (co)homology theory works well with other common homology theories like simplicial, singular, and cellular homology, and to this end, this result for one-point spaces seems encouraging.

However, it turns out that for regular homology, the Čech formulation comes up short. The reason for this is purely algebraic. It turns out that inverse limits do not play well with the kinds of exact sequences one might normally derive for homology [2], while direct limits do not have this problem. It is for this reason that Čech cohomology, not Čech homology, is the title of this article, and the regular homology theory shall be ignored from here on.

Using the formulation of Čech cohomology involving sheaves, one can show that Čech cohomology and singular cohomology are equivalent when X is a CW-complex [3]. If X is more pathological, Čech cohomology may differ from singular cohomology. This provides another way to draw algebraic information from poorly behaved spaces.

5 Definition using Sheaves

Simply put, a *sheaf* is a map $\mathcal{F}$ that assigns to every open set U in a topological space a group $\mathcal{F}(U)$. There are several additional hypothesis in the definition that make this map behave well under restrictions and so forth, but we skip over them in favor of providing the relevant example: locally constant A-valued functions where A is an abelian group.

Let $\mathcal{A}$ be an abelian sheaf and $\mathcal{U}$ be an open cover of X indexed by I. An n-simplex is a collection $\sigma = (U_0, U_1, \ldots, U_n)$ of sets such that $\bigcap_{i=0}^n U_i$ is nonempty. We call this intersection the *support* of σ and denote it by $|\sigma|$. A *n-cochain* of $\mathcal{U}$ is a map f defined by

$$f(\sigma) = \mathcal{A}(|\sigma|).$$

Then the set of *n-cochains* forms an abelian group under addition denoted $C^n(\mathcal{U}; \mathcal{A})$. We make this into a cochain complex by defining the coboundary operator

$$\delta : C^n(\mathcal{U}; \mathcal{A}) \to C^{n+1}(\mathcal{U}; \mathcal{A})$$

by

$$\delta f(\sigma) = \sum_{i=0}^{n+1} (-1)^i r_{|\sigma|}^{|\sigma_i|} f(\sigma_i)$$

where $\sigma_i = (U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n)$ and $r_{|\sigma|}^{|\sigma_i|}$ is the sheaf restriction mapping. This mapping is part of the definition of a sheaf, and is a map $r_V^U : \mathcal{A}(U) \to \mathcal{A}(V)$. The cochain complex resulting from the definition of δ has a cohomology defined as the Čech cohomology subordinate to $\mathcal{U}$ with coefficients in $\mathcal{A}$. From here, the Čech cohomology of X is then defined in the same way as before, as the direct limit of this system with respect to refinements.

This definition in its entirety and in much the same notation can be found in reference [4].

6 Relationship with Transition Functions

As an application of Čech cohomology, there is a nice relationship between rank 1 complex vector bundles on a space X and elements of the first cohomology group $H^1(X, \mathcal{S})$. Here $\mathcal{S}$ denotes the sheaf of nonvanishing complex-valued functions on X, a group with respect to multiplication.

Let $E \to X$ be a complex line bundle. Recalling the definition of vector bundles, there exists an open cover $\mathcal{U} = \{U_i\}$ of X and transition functions

$$f_{ij} : U_i \cap U_j \to GL(1, \mathbf{C}) = \mathbf{C}^*.$$

One can show that $f_{ij} \cdot f_{jk} \cdot f_{ki} = 1$ on $U_i \cap U_j \cap U_k$ and $f_{ii} = 1$. This is precisely the *cocycle condition*, the condition that a Čech cochain is a cocycle. This defines a unique cocycle in the complex $C^1(\mathcal{U}, \mathcal{S})$, which defines a cohomology class in the direct limit $H^1(X, \mathcal{S})$. In this way we can construct a bijection between complex bundles on X and elements of $H^1(X, \mathcal{S})$ and classify all complex bundles of a given space [3][4].

With a little more work one can construct a long exact sequence in Čech cohomology from the short exact sequence of sheaves $0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^* \to 0$ and extract the map

$$\delta : H^1(X, \mathcal{S}) \to H^2(X, \mathbf{Z}).$$

The image in $H^2(X, \mathbf{Z})$ of the element in $H^1(X, \mathcal{S})$ corresponding to the transition functions is one definition of the first Chern class of $E \to X$ [4].

References

[1] Clifford Dowker. Mapping theorems for non-compact spaces. *American Journal of Mathematics*, 69:200–247, 1947.

[2] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology.* Princeton University Press, Princeton, 1952.

[3] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry.* John Wiley Sons, Ltd, 1994.

[4] Raymond Wells. *Differential Analysis on Complex Manifolds.* Springer, 2008.

[5] Eduard Čech. Théorie générale de l'homologie dans un espace quelconque. *Fundamenta Mathematicae*, 19:149–183, 1932.