Sufficient conditions for 2D Monotone Hermite Interpolant

October 22, 2014

1D Hermite Interpolation

Given the values of f & f_x at points x_0 & x_1 : $\{f_{(0)}, f_{x(0)}, f_{(1)}, f_{x(1)}\}$, a cubic Hermite interpolation at point x can be written as :

$$f^H(x) = f_{(0)}b_1(dx) + f_{(1)}b_1(1-dx) + h\big(f_{x(0)}b_2(dx) - f_{x(1)}b_2(1-dx)\big) \tag{1}$$

Where $dx = \frac{x-x_0}{x_1-x_0}$, & b_i are Hermite basis functions.
For such an interpolation to be monotonous, which is :

$$\min(f_{(0)}, f_{(1)}) \le f^H(x) \le \max(f_{(0)}, f_{(1)}) \quad \forall \ \{x : x_0 \le x \le x_1\} \tag{2}$$

we need the following conditions.

$$\left.\begin{array}{llll} 0 \le & f_{x(0)}, f_{x(1)} \le & \dfrac{3}{h}(f_{(1)} - f_{(0)}) & \text{if } (f_{(1)} - f_{(0)}) \ge 0 \\ 0 \ge & f_{x(0)}, f_{x(1)} \ge & \dfrac{3}{h}(f_{(1)} - f_{(0)}) & \text{if } (f_{(1)} - f_{(0)}) < 0 \end{array}\right\} \tag{3}$$

For such a set of conditions, let us define a shorthand notation as follows :

$$f_{x(0)}, f_{x(1)} << f_{(1)} - f_{(0)} \tag{4}$$

2D Hermite Interpolation

Similarly, for a given values of $\{f, f_x, f_y, f_{xy}\}$ at points $\{0, h\} \times \{0, h\}$: $\{f_{(0,0)}, ..., f_{xy(h,h)}\}$, a bi-cubic Hermite interpolation at point (x, y) can we written as : *(For convenience of notation h is taken to be 1, Yet, at some places 'h' will be seen. This is to have final results in general form).*

$$f^H_{(x,y)} = f^H_{(0,y)}b_1(x) + f^H_{(1,y)}b_1(1-x) + h\big(f^H_{x(0,y)}b_2(x) - f^H_{x(1,y)}b_2(1-x)\big) \tag{5}$$

where

$$\begin{aligned} f^H_{(0,y)} &= f_{(0,0)}b_1(y) + f_{(0,1)}b_1(1-y) + h\big(f_{y(0,0)}b_2(y) - f_{y(0,1)}b_2(1-y)\big) && (6)\\ f^H_{(1,y)} &= f_{(1,0)}b_1(y) + f_{(1,1)}b_1(1-y) + h\big(f_{y(1,0)}b_2(y) - f_{y(1,1)}b_2(1-y)\big) && (7)\\ f^H_{x(0,y)} &= f_{x(0,0)}b_1(y) + f_{x(0,1)}b_1(1-y) + h\big(f_{xy(0,0)}b_2(y) - f_{xy(0,1)}b_2(1-y)\big) && (8)\\ f^H_{x(1,y)} &= f_{x(1,0)}b_1(y) + f_{x(1,1)}b_1(1-y) + h\big(f_{xy(1,0)}b_2(y) - f_{xy(1,1)}b_2(1-y)\big) && (9) \end{aligned}$$

This way, we can interpret the 2D-cubic interpolation as a combination of 1D-cubic interpolations. Our target is to apply condition shown in eq. (3) in the eqs. (5) to (9).

- Firstly, we know from eq. (6) and (7) :

$$\begin{array}{rccl} \text{if } \ f_{y(0,0)}, f_{y(0,1)} & << & f_{(0,1)} - f_{(0,0)} & \\ \text{then } \ \min(f_{(0,1)}, f_{(0,0)}) & \le & f^H_{(0,y)} \ \le \ \max(f_{(0,1)}, f_{(0,0)}) & (10) \\ \text{if } \ f_{y(1,0)}, f_{y(1,1)} & << & f_{(1,1)} - f_{(1,0)} & \\ \text{then } \ \min(f_{(1,1)}, f_{(1,0)}) & \le & f^H_{(1,y)} \ \le \ \max(f_{(1,1)}, f_{(1,0)}) & (11) \end{array}$$

- **to-be-continued**.

Final Conditions

$$
\left.\begin{aligned}
f_{x(0,0)}, f_{x(1,0)} &<< & f_{(1,0)} - f_{(0,0)} \\
f_{x(0,1)}, f_{x(1,1)} &<< & f_{(1,1)} - f_{(0,1)} \\
f_{y(0,0)}, f_{y(0,1)} &<< & f_{(0,1)} - f_{(0,0)} \\
f_{y(1,0)}, f_{y(1,1)} &<< & f_{(1,1)} - f_{(1,0)} \\
f_{xy(0,0)}, f_{xy(0,1)} &<< & f_{x(0,1)} - f_{x(0,0)} \\
f_{xy(1,0)}, f_{xy(1,1)} &<< & f_{x(1,1)} - f_{x(1,0)} \\
f_{xy(0,0)}, f_{xy(1,0)} &<< & f_{y(1,0)} - f_{y(0,0)} \\
f_{xy(0,1)}, f_{xy(1,1)} &<< & f_{y(1,1)} - f_{y(0,1)} \\
f_{y(1,0)} - f_{y(0,0)}, f_{y(1,1)} - f_{y(0,1)} &<< & f_{(1,1)} + f_{(0,0)} - f_{(1,0)} - f_{(0,1)} \\
f_{x(0,1)} - f_{x(0,0)}, f_{x(1,1)} - f_{x(1,0)} &<< & f_{(1,1)} + f_{(0,0)} - f_{(1,0)} - f_{(0,1)}
\end{aligned}\right\} \tag{12}
$$