UNIT-2

Relations: Properties of Binary Relations, Relation Matrix and Digraph, Operations on Relations, Partition and Covering, Transitive Closure, Equivalence, Compatibility and Partial Ordering Relations, Hasse Diagrams.

RELATIONS

Relations:- Any set of ordered pairs defines a binary relation or simply a relation.

- Binary relation represents relationship between elements of two sets.
- If R is relation, a particular ordered pair say $(x, y) \in R$ can be written as xRy and can be read as "x is in relation R to y".

  Eg :

  1. x and y are two real numbers such that x > y
     $$R = \{(x, y) \mid x \text{ and y are real numbers and } x > y\}$$
  2. Represent the relation father to child.
     $$F = \{(x, y) \mid x \text{ is the father of } y\}$$
  3. S = {(1,2) (3,4) (5,6) (7,8) (john,b) (d,e)}

Domain of a relation:- Domain of the relation S is defined as the set of all first elements of the ordered pairs that belongs to S and is denoted by D or D(S).

$$\boxed{D(S) = \{x \mid (\exists y)\ (x, y) \in S\}}$$

Eg: S = {(1,2) (3,4) (a,t) (p,q)}
D(S) = {1,3,a,p}

Range of a relation:- Range of the relation S is defined as the set of all second elements of the ordered pairs that belongs to S and is denoted by R or R(S).

$$\boxed{R(S) = \{y \mid (\exists x)\ (x, y) \in S\}}$$

Eg: S = {(1,2) (3,4) (a,t) (p,q)}
R(S) = {2,4,t,q}

Operations on relations:- If R and S are two relations then $|R \cup S|$ defines a relation such that

- $x(R \cup S)y \Leftrightarrow$ xRy or xSy
- $x(R \cap S)y \Leftrightarrow$ xRy and xSy
- $x(R - S)y \Leftrightarrow$ xRy and $x \not{S} y$
- $x(\sim R)y \Leftrightarrow x \not{R} y$

Eg:- P = {(1,2) (2,4) (3,3)} Q = {(1,3) (2,4) (4,2)}
$P \cup Q, P \cap Q, D(P), D(Q), D(P \cup Q), R(P), R(Q), R(P \cup Q)$.

Show that i) D(P $\cup$ Q) = D(P) $\cup$ D(Q) ii) R(P $\cap$ Q) $\subseteq$ R(P) $\cap$ R(Q)

sol:-

$P \cup Q = \{(1,2)\,(1,3)\,(2,4)\,(3,3)\,(4,2)\}$	$D(P \cup Q) = \{1,2,3,4\}$
$P \cap Q = \{(2,4)\}$	$D(P \cap Q) = \{2\}$
$D(P) = \{1,2,3\}$	$R(P) = \{2,4,3\}$
$D(Q) = \{1,2,4\}$	$R(Q) = \{2,3,4\}$

$R(P \cup Q) = \{2,3,4\}$ | $R(P \cap Q) = \{4\}$

i) D(P $\cup$Q) = D(P) $\cup$D(Q)

LHS

$D(P \cup Q) = \{1,2,3,4\}$

RHS

$D(P) \cup D(Q)$ ={1,2,3}$\cup$ {1,2,4}

={1,2,3,4}

LHS = RHS

Hence proved.

ii) R(P $\cap$Q) $\subseteq$R(P) $\cap$R(Q)

LHS

R(P $\cap$Q) ={4}

RHS

R(P) $\cap$R(Q) = {2,3,4}

Clearly {4} $\subseteq${2,3,4} i.e R(P $\cap$Q) $\subseteq$R(P) $\cap$R(Q)

Hence Proved

Eg:- What is the range of relation $S = \{(x, x^2) \mid x \in N\}$ $T = \{(x, 2^x) \mid x \in N\}$ where N = {1,2,3,..}. Find $S \cup T$ and $S \cap T$.

sol:- $S = \{(x, x^2) \mid x \in N\}$ = { (1,1) (2,4) (3,9), ...}

$T = \{(x, 2^x) \mid x \in N\}$ = {(1,2) (2,4) (3,8),...}

$S \cup T$ = {(1,1) (1,2) (2,4) (3,8) (3,9),....}

$S \cap T$ = {(2,4)}

Properties of Binary relations in a set:-

1) Reflexive relation.
2) Symmetric relation.
3) Transitive relation.
4) Irreflexive relation.
5) Antisymmetric relation.

1) ***Reflexive relation***:- A binary relation R in a set X is said to be reflexive if for every $x \in X$, xRx that is for every $x \in X$, ordered pair $(x, x) \in R$.

   Eg1:- The relation ≤ is reflexive on the set of real numbers.

   Eg2:- The relation > is not reflexive on the set of real numbers

   Eg3:- The relation R on {1,2,3} given by

   R = {(1,1) (2,2) (2,3) (3,3)} which is reflexive

2) ***Symmetric relation***:- A relation R on set A is symmetric if whenever $(a,b) \in R$ then $(b,a) \in R$ i.e, $aRb \Rightarrow bRa$. This means if any one element is related to any other element, then second element is related to the first.

   Eg1:- The relation = on the set of real numbers is symmetric

   Eg2:- The relation ***"being a brother"*** is not symmetric on the set of all people but it is symmetric on the set of all males.

   Eg3:- The relation R on {1,2,3} given by

   R = {(1,1) (1,2) (1,3) (2,1) (2,2) (2,3) (3,1) (3,2) (3,3)} is symmetric.

3) ***Transitive relation***:- A relation R in a set X is said to be transitive, if xRy and yRz then xRz for every x,y and z. In other words, if $(x, y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$ for every $x, y, z \in R$.

Eg1:-The relations ≤, < and = are transitive on the set of real numbers.

Eg2:- The relation ***"is a mother"*** is not transitive.

Eg3: Consider A = {1,2,3}

$$R_1 = \{(1,1)(1,2)(2,3)(1,3)\}$$

$$R_2 = \{(1,1)(1,2)(2,2)(1,3)\}$$

$$R_3 = \{(1,1)(1,2)(1,3)(3,3)\}$$

R_1, R_2 and R_3 are transitive.

4) ***Irreflexive relation***:- A relation R in a set X is said to be irreflexive if for every $x \in X$ the ordered pair $(x, x) \notin R$

Eg:- Consider A = {1,2,3}

$R_1 = \{(1,2)(1,3)(2,1)(2,3)\}$ is Irreflexive.

5) ***Antisymmetric relation***:- A relation R in a set X is said to be antisymmetric if xRy and yRx then x = y for every $x, y \in X$.

Eg:- The relation equal to(=) on the set of real numbers is antisymmetric.

Eg:- S={1,2,……10} ,R={(x,y)/x+y=10}. What are the properties of relation R?

Sol:- Given S={1,2,3,….10}

R={(1,9),(2,8),(3,7),(4,6),(5,5),(6,4),(7,3),(8,2),(9,1)}

- (1,1),(2,2)….(9,9) $\notin$ R. So the given relation is not reflexive i.e irreflexive
- If (1,9) $\in$ R then (9,1) $\in$ R ,if (2,8) $\in$ R then (8,2) $\in$ R and so on .So the given relation is symmetric but not antisymmetric because if (1,9) $\in$ R then (9,1) $\in$ R but 1≠9]
- (1,9) $\in$ R and (9,1) $\in$ R, but (1,1) $\notin$ R. So the given relation is not transitive.

So the given relation is only irreflexive and symmetric

Relation matrix and Digraph:-

Relation matrix:- A relation R from a finite set A to finite set B can be represented by a matrix called the relation matrix of R.

Let $A = \{a_1, a_2, a_3, \ldots a_n\}$ $B = \{b_1, b_2, b_3, \ldots b_n\}$ be finite sets containing m and n elements respectively and R be the realtion from A to B. Then R can be represented m X n matrix.

$M_R = [r_{ij}]$ which is defined a follows:

$$r_{ij} \begin{cases} 1 & \text{if } a_i R b_i \\ 0 & \text{if } a_i \not R b_i \end{cases}$$

Note:- All the elements of M_R consists of 1's and 0's.

Eg1: A = {1,2,3,4} B = {b1,b2,b3} consider the relation R = {(1,b_2) (1,b_3) (3,b_2) (4,b_1) (4,b_3)} Determine the matrix.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Properties of a relation in a set:

1) If a relation is reflexive, then all the diagonal entries must be 1.

2) If a relation is symmetric, then the relation matrix is symmetric i.e, $r_{ij} = r_{ji}$ for every i and j.

3) If a relation is antisymmetric, then the relation matrix is symmetric i.e, $r_{ij}=1$ then $r_{ij}=0$ for $i \neq j$.

Eg2:- Let A={1,2,3,4}. Find the relation R, given relation matrix

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

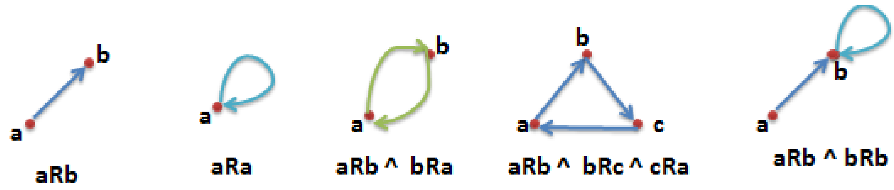
Sol:- R={(1,1),(1,3),(2,3),(3,1),(4,1),(4,2),(4,4)}

Graph of a relation:- A relation defined in a finite set can also be represented pictorially with the help of graph. The elements of A are represented by points or circles called **nodes.** These nodes are called as **vertices.**

If $(a_i, a_j) \in R$ then we connect the a_i and a_j by means of an arc and put an arrows on the arc in the direction from a_i to a_j. This is called an **edge.**

Note:-

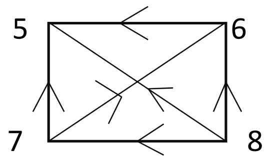
- If $a_i R a_j$ and $a_j R a_i$ then we draw two arcs between a_i and a_j with arrows pointing in both directions.
- If $a_i R a_i$ then we get an arc which starts from node a_i and return to a_i. This arc is called a loop.



Eg:-Given A={5,6,7,8}, R={(x,y)/x>y}.Give relation matrix and draw its graph.

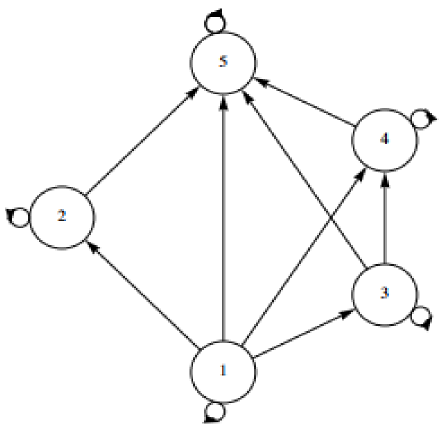
Sol: R={(8,5),(8,6),(8,7),(7,6),(7,5),(6,5)}

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



Eg:- Find the relation and corresponding relation matrix to the given graph on the set A={1,2,3,4,5}

Sol:



Sol: R={(1,1),(1,2),(1,3),(1,4),(1,5),(2,2),(2,5),(3,3),(3,4),(3,5),(4,4),(4,5),(5,5)}

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Equivalence Relation:-

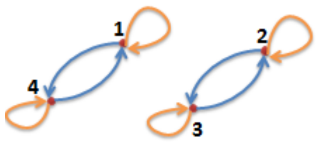
- A relation R on a set X is said to be equivalence relation if it is **reflexive, symmetric and transitive.**
- A relation R on a set X is said to be partial order relation if it is **reflexive, anti-symmetric and transitive.**

Eg:-

1) Equality of subsets of a universal set.
2) Equality of numbers on a set of real numbers.
3) Relation of propositions being equivalent in a collection of propositions.

Eg: Let X = { 1,2,3,4} R = {(1,1) (1,4) (4,1) (4,4) (2,2) (2,3) (3,2) (3,3)}. Prove that R is an equivalence relation.

$$M_r = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



Clearly R is reflexive, symmetric and transitive.

Note:- If R and S are transitive. Then $R \cup S$ need not be transitive.

Inverse relation:- A relation R from set A to set B has an inverse relation R^{-1} from B to A which is defined by $R^{-1} = \{(b,a) \mid (a,b \in R)\}$

Eg: Provide examples of inverse relation

A = {a,b,c} and B = {1,2} R= {(a,1) (a,2) (c,1)} is a relation from A to B.

The inverse relation of R is $R^{-1} = \{(1,a)(2,a)(1,c)\}$

Covering of a set:- Let S be a given set and A = {A_1,A_2,...A_n} where A_i = 1,2,..., m is the subset of S and $\bigcup_{i=1}^{m} A_i = S$. Then the set A is called a covering of S and the sets A_1,A_2,...A_m are said to cover S.

Eg:- S = {a,b,c} and consider the following subsets of S.

A_1 = {{a,b}, {b,c}} A_2 = {{a}, {a,c}} A_3 = {{a}, {b,c}} A_4 = {{a,b,c}} A_5 = {{a} ,{b}, {c}}
A_6 = {{a}, {a,b},{a,c}}

sol:-
A_1 = {{a,b} {b,c}} = {a,b,c} ✓
A_2 = {{a} {a,c}} = {a,c}
A_3 = {{a} {b,c}} = {a,b,c} ✓
A_4 = {{a,b,c}} = {a,b,c} ✓

$A_5 = \{\{a\} \{b\} \{c\}\} = \{a,b,c\}$ ✓
$A_6 = \{\{a\} \{a,b\}\{a,c\}\} = \{a,b,c\}$ ✓

Partition of a set:- A partition of a set S is collection of disjoint non-empty subsets of S that have S as their union.

Suppose $S = \{1,2,3,4,5,6\}$. The collection of sets $A_1 = \{1,2,3\}$ $A_2 = \{4,5\}$ $A_3 = \{6\}$ form a partition of S, since these sets are disjoint and their union is S.

- The sets $A_1, A_2, \ldots A_m$ are called blocks of partition.

Eg:- $S = \{a,b,c\}$ $A_1 = \{\{a,b\}, \{b,c\}\}$ $A_2 = \{\{a\}, \{b,c\}\}$ $A_3 = \{\{a,b,c\}\}$ $A_4 = \{\{a,\} \{b\}, \{c\}\}$ $A_5 = \{\{a\}, \{a,c\}\}$

Sol:- A_2, A_3, A_4 are partition of S and also they are covering
A_1 is a covering but it is not a partition of a set, since the set $\{\{a,b\}, \{b,c\}\}$ is not disjoint.
A_5 is not a partition, since the union of the subsets is not S.
A_3 has only one block and A_4 has three blocks.

- Every partition is also a covering. But the converse need not be true.

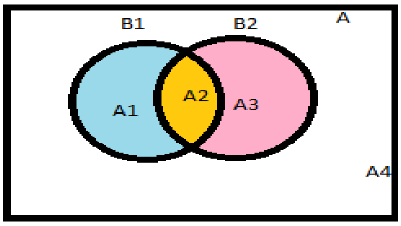
- **Minsets:-** Let B_1 and B_2 be subsets of a set A. let the sets A_1, A_2, A_3, A_4 can be described by B_1 and B_2 as follows:

$$A_1 = B_1 \cap B_2^c$$
$$A_2 = B_1 \cap B_2$$
$$A_3 = B_1^c \cap B_2$$
$$A_4 = B_1^c \cap B_2^c$$

Each of the sets A_1, A_2, A_3, A_4 is called a minset or minterm generated by B_1 and B_2.

Maxsets:- Let B_1 and B_2 be subsets of a set A. let the sets A_1, A_2, A_3, A_4 can be described by B_1 and B_2 as follows:

$$A_1 = B_1 \cup B_2^c$$
$$A_2 = B_1 \cup B_2$$
$$A_3 = B_1^c \cup B_2$$
$$A_4 = B_1^c \cup B_2^c$$



Each of the sets A_1, A_2, A_3, A_4 is called a maxset or maxterm generated by B_1 and B_2.

Eg:-Given $A=\{1,2,3,4,5,6\}$, $B_1=\{1,3,5\}$, $B_2=\{1,2,3\}$. Obtain minsets and maxsets of A generated by B_1 and B_2.

Sol: **Minsets:** B_1^c = U- B_1 =$\{1,2,3,4,5,6\} - \{1,3,5\} = \{2,4,6\}$
B_2^c = U- B_2 =$\{1,2,3,4,5,6\} - \{1,2,3\} = \{4,5,6\}$
$\therefore A_1=B_1 \cap B_2^c = \{1,3,5\} \cap \{4,5,6\}=\{5\}$
$\therefore A_2=B_1 \cap B_2 =\{1,3,5\} \cap \{1,2,3\} =\{1,3\}$
$\therefore A_3= B_1^c \cap B_2=\{2,4,6\} \cap \{1,2,3\} =\{2\}$
$\therefore A_4= B_1^c \cap B_2^c =\{2,4,6\} \cap \{4,5,6\} =\{4,6\}$

Maxsets : B_1^c = U- B_1 =$\{1,2,3,4,5,6\} - \{1,3,5\} = \{2,4,6\}$
B_2^c = U- B_2 =$\{1,2,3,4,5,6\} - \{1,2,3\} = \{4,5,6\}$
$\therefore A_1=B_1 \cup B_2^c = \{1,3,5\} \cup \{4,5,6\}=\{1,3,4,5,6\}$

$\therefore$ $A_2 = B_1 \cup B_2 = \{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}$

$\therefore$ $A_3 = B_1^c \cup B_2 = \{2,4,6\} \cup \{1,2,3\} = \{1,2,3,4,6\}$

$\therefore$ $A_4 = B_1^c \cup B_2^c = \{2,4,6\} \cup \{4,5,6\} = \{2,4,5,6\}$

Transitive Closure

Warshalls algorithm is used to find the transitive closure of a relation.

Consider a relation R defined on a set

$A = \{a_1, a_2, \ldots\ldots a_n\}$. If $\{x_1, x_2, \ldots\ldots x_n\}$ is a path in R, then any vertex other than x_1 and x_n is called *interior vertex* of the path. Also for $1 < k < n$, we define a Boolean matrix W_k as follows.

The (i, j)th element of W_k is 1 iff there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots\ldots a_n\}$. In other words, $W_n = M_R^{\infty}$.

If we define $W_0 = M_R$, then we can find a sequence $W_0, W_1, W_2, \ldots\ldots W_n$ whose first term is $W_0 = M_R$ and the last term is $W_n = M_R^{\infty}$.

Each matrix W_k can be computed from W_{k-1} using the following algorithm (**Warshall's algorithm**).

Step I. First transfer all 1's in W_{k-1} to W_k.

Step II. List the locations $p_1, p_2, \ldots$ in column k of W_{k-1}, where the entry is 1 and locations $a_1, a_2, \ldots\ldots$ in row k of w_{k-1}, where the entry is 1.

Step III. Put 1's at all the positions (p_i, q_j) of W_k (if they are not already there).

Example 2. *Using Warshall's algorithm, find the transitive closure of R defined on*

$A = \{1, 2, 3, 4\}$ and

$R = (1, 1), (1, 4), (2, 1), (2, 2), (3, 3), (4, 4).$

Sol. If M_R denotes the matrix representation of R, then (Take $W_0 = M_R$)

$$W_0 = M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } n = 4 \qquad \text{(As } M_R \text{ is a } 4 \times 4 \text{ matrix)}$$

We compute W_4 by using **Warshall's algorithm.**

For k = 1. In column 1 of W_0, 1's are at positions 1 and 2. Hence $p_1 = 1, p_2 = 2$

In row 1 of W_0, 1's are at positions 1 and 4.

Hence $q_1 = 1, q_2 = 4$. Therefore, to obtain W_1, we put 1 at the positions :

$\{(p_1, q_1), (p_1, q_2), (p_2, q_1), (p_2, q_2)\} = (1, 1), (1, 4), (2, 1), (2, 4)\}$. Thus

$$W_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For k = 2. In column 2 of W_1, 1's is at positions 2. Hence $p_1 = 2$.

In row 2 of W_1, 1's are at positions 1, 2 and 4.

Hence $q_1 = 1,\quad q_2 = 2,\quad q_3 = 4.$

Therefore, to obtain W_2, we put 1s at the positions:

$\{(p_1, q_1), (p_1, q_2), (p_1, q_3) = (2, 1), (2, 2), (2, 4)\}$. Thus (using W_1)

$$W_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For k = 3. In column 3 of W_2, '1' is at position 3.

Hence $p_1 = 3$

In row 3 of W_2, '1' is at position 3. Hence $q_1 = 3$

Thus, we put '1' at the position: $\{(p_1, q_1) = (3, 3)\}$. Thus (using W_2)

$$W_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For k = 4. In column 4 of W_3, 1's are at positions 1, 2 and 4. Hence $p_1 = 1, p_2 = 2, p_3 = 4$.
In row 4 of W_3, '1' is at position 4. Hence $q_1 = 4$

Therefore, we put 1's at the positions:

$\{(p_1, q_1), (p_2, q_1), (p_3, q_1) = (1, 4), (2, 4), (4, 4)\}$. Thus (using W_3).

$$W_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M_R^{\infty}$$

Hence, from the matrix M_R^{∞}, the transitive closure of R is given by

$$R^{\infty} = \{(1, 1), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Example 3. *Let A = {1, 2, 3, 4} and let R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the transitive closure of R using Warshall's algorithm.*

Sol. Let M_R denotes the matrix representation of R. Take $W_0 = M_R$, we have

$$W_0 = M_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } n = 4 \text{ (As } M_R \text{ is a } 4 \times 4 \text{ matrix)}$$

We compute W_4 by using warshall's algorithm.

For k = 1. In column 1 of W_0, '1' is at position 2. Hence $p_1 = 2$.

In row 1 of W_0, '1' is at position 2. Hence $q_1 = 2$. Therefore, to obtain W_1, we put '1' at the position: $\{(p_1, q_1) = (2, 2)\}$. Thus

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For k = 2. In column 2 of W_1, 1's are at positions 1, 2. Hence $p_1 = 1, p_2 = 2$.

In row 2 of W_1, 1's are at positions 1, 2 and 3. Hence $q_1 = 1, q_2 = 2, q_3 = 3$

Therefore, to obtain W_2, we put '1' at the positions :

$\{(p_1, q_1), (p_1, q_2), (p_1, q_3), (p_2, q_1), (p_2, q_2), (p_2, q_3) = (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$.

Thus (using W_1)

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For k = 3. In column 3 of W_2, 1's are at positions 1, 2. Hence $p_1 = 1, p_2 = 2$

In row 3 of W_2, '1' is at the position 4. Hence $q_1 = 4$

Therefore, to obtain W_3 we put 1's at the positions: $\{(p_1, q_1), (p_2, q_1) = (1, 4), (2, 4)\}$.

Thus (using W_2)

$$W_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For k = 4. In column 4 of W_3, 1's are at positions 1, 2, 3. Hence $p_1 = 1, p_2 = 2, p_3 = 3$

In row 4 of W_3, '1' is at no position, and no new 1's are added and hence $M_R^{\infty} = W_4 = W_3$.

Thus

$$W_4 = W_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_R^{\infty}$$

Thus, the transitive closure of R is given as

$R^{\infty} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$.

Compatibility relations:- A relation R on X is said to be a compatibility relation if it is both reflexive and symmetric.

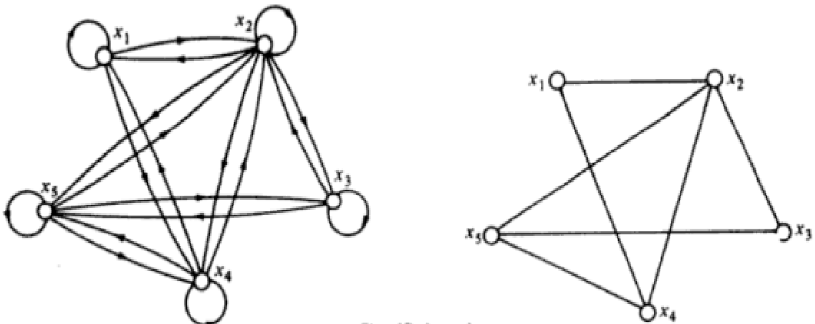
Eg: Let X = {ball, bed, dog, let, egg } and let the relation R be given by

$$R = \{(x, y) \mid x, y \in X \wedge (xRy) \text{ if x and y contain some common letter}\}$$

Compatibility relation is denoted by the symbol $\approx$. "A subset $A \subseteq X$ is called a maximal compatibility block if any element of A is compatible to every other element of A and no element of X – A is compatible to all elements of A.

Applications:-

- Compatibility relations are useful in solving certain minimisation problems of switching theory, particularly for incompletely specified minimisation problems.
  Let us denote x_1)ball x_2) bed x_3) dog x_4) let x_5) egg by respectively.



(a) Simplified graph (b)

Since ($\approx$) it is a compatibility relation it is not necessary to draw the loops at each element. Also it is not necessary to draw both aRb and bRa. Hence we can simplify fig - a to fig – b.

To find the maximal compatibility blocks. First draw a simplified graph of the compatibility relation and pick largest complete polygons. "largest complete polygons". we need a polygon in which any vertex is connected to every other vertex.

Eg: Triangle is always a complete polygon, but for a quadrilateral to be a complete polygon must have 2 diagonals present.

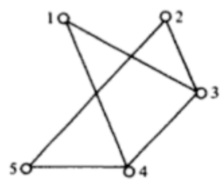
Note:- The relation matrix of a compatibility relation is symmetric and has its diagonal element unity. Therefore if is sufficient to give only the elements of the lower triangular part of the relation matrix.

$$\begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{ccccc} b & 1 & & & \\ c & 0 & 1 & & \\ d & 1 & 1 & 0 & \\ e & 0 & 1 & 1 & 1 \\ & a & b & c & d \end{array}$$

However compatibility relation defined a covering of the set {a,b,d} {b,c,e} {b,d,e} are compatibility blocks. The sets are not mutually disjoint sets and hence these sets are not a covering of X.

Eg:- Find the maximal compatibility blocks in the given graphs

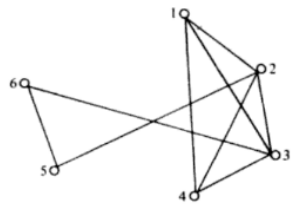
2	0			
3	1	1		
4	1	0	1	
5	0	1	0	1
	1	2	3	4



Hence largest complete polygons are {1,3,4}
The maximal compatibility block of the relation {1,3,4} {2,3} {4,5} {2,5}

Eg:- Find the maximal compatibility blocks in the given graphs

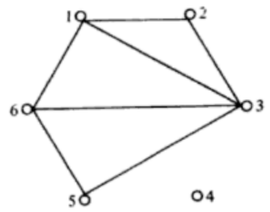
2	1				
3	1	1			
4	1	1	1		
5	0	1	0	0	
6	0	0	1	0	1
	1	2	3	4	5



Hence largest complete polygons are {1,2,3,4}
The maximal compatibility block of the relation {1,2,3,4} {2,5} {3,6} {5,6}

Eg:- Find the maximal compatibility blocks in the given graphs

2	1			
3	1	1		
5	0	0	1	
6	1	0	1	1
	1	2	3	5



Hence largest complete polygons are {1,2,3} {1,3,6} {3,5,6}
The maximal compatibility block of the relation {1,2,3} {1,3,6} {3,5,6} {4}

Partial Ordering:- A binary relation R on set S is called partial ordering or partial order if it is reflexive, anti-symmetric and transitive.

A set S together with a partial ordering R is called partially ordered set or **poset** and is denoted by **(S,R).**

Eg:- Show that relation '$\geq$' is a partial ordering on the set of integers.

sol:- Let Z be set of integers and the relation R = '$\geq$'

1) Since $a \geq a$ for every integers and the relation R = '$\geq$'
2) Let a and b be any two integers
   let aRb and bRa $\Rightarrow$ $a \geq b$ and $b \geq a$ $\Rightarrow$ $a = b$
   $\therefore$ The relation $\geq$ is anti – symmetric.
3) Let a ,b and c are integers
   let $a \geq b$ and $b \geq c$ $\Rightarrow$ $a \geq c$
   $\therefore$ The relation $\geq$ is transitive

$\therefore$ The relation $\geq$ is reflexive, anti-symmetric and transitive . Therefore $(Z, \geq)$ is a poset.

Eg:- Show that relation '/' is a partial ordering on the set of positive integers.

sol:- Let Z^+ be set of positive integers and the relation R = '/'

1) Since a/a for every integers and the relation R = /
2) Let a and b be any two integers
   let aRb and bRa $\Rightarrow$ a /b and b/a $\Rightarrow$ a = b
   $\therefore$ The relation / is anti – symmetric.
3) Let a ,b and c are integers
   let a /b and b/c $\Rightarrow$ a /c
   $\therefore$ The relation / is transitive

$\therefore$ The relation $\geq$ is reflexive, anti-symmetric and transitive . Therefore (Z^+, /) is a poset.

Hasse diagrams:- "A partial ordering $\leq$ on the set can be represented by means of a diagram known as **hasse diagram** or a **partially ordered set diagram** of **(P, $\leq$).**

To draw a hasse diagram, the following steps need to be followed:

1) Each element is represented by a small circle or dot.
2) The circle for $x \in P$ is drawn below the circle for $y \in P$ if x < y and a line is drawn between x and y if y covers x.
3) If x < y but y does not cover x, then x and y are not connected directly by a single line. But they are connected through one or more element of P.

Hence the set of ordered pairs in $\leq$ can be obtained from such a diagram.

Methodology to be followed to draw a Hasse diagram

- First cancel all the reflexive ordered pairs. For eg., (1,1),(2,2)….
- Then cancel all Transitive ordered pairs. For example if (1,3) and (3,5) belongs to R then cancel The pair(1,5)
- Now draw the hasse diagram such that lower magnitude element in the ordered pair should always lie below the element with larger magnitude.

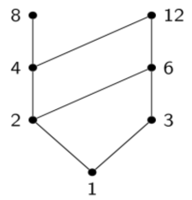
Eg: Draw the hasse diagram representing the partial ordering {(a,b) | a divides b} on {1,2,3,4,6,8,12}

Let P = {1,2,3,4,6,8,12}

(P, $\leq$) = {(1,1),(2,2),(3,3),(4,4),(6,6),(8,8),(12,12),(1,2), (1,3), (1,4) ,(1,6), (1,8), (1,12), (2,4), (2,6), (2,8) ,(2,12), (3,6), (3,12), (4,8), (4,12),(6,12)}

- First cancel all reflexive pairs
  $\therefore$ (P, $\leq$) = {(1,2), (1,3), (1,4) ,(1,6), (1,8), (1,12), (2,4), (2,6), (2,8) ,(2,12), (3,6), (3,12), (4,8), (4,12),(6,12)}
- Next cancel all transitive pairs
- $\therefore$ (P, $\leq$) = {(1,2), (1,3),(2,4), (2,6),(3,6), (4,8), (4,12),(6,12)}

$\therefore$ Hasse diagram of ({1,2,3,4,6,8,12},/) is

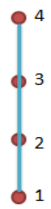


Eg: Draw the hasse diagram representing the partial ordering {(a,b) | a$\leq$ b} on {1,2,3,4}

(P, $\leq$) = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

- First cancel all reflexive pairs
- (P, $\leq$) = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}
- Next cancel all transitive pairs
  $\therefore$ (P, $\leq$) = {(1,2),(2,3),(3,4)}

$\therefore$ Hasse diagram is

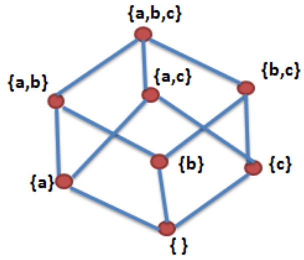


Eg: Draw the hasse diagram representing the partial ordering {A,B) | A$\subseteq$B} on {a,b,c}

Let P = {a,b,c}

(P, $\subseteq$) = $\{(\phi,\{a\})(\phi,\{b\})(\phi,\{c\})(\phi,\{a,b\})(\phi,\{a,c\})(\phi,\{b,c\})\ (\phi,\{a,b,c\})(\{a\},\{a,b\})$
$(\{a\},\{a,c\})\ \ (\{a\},\{a,b,c\})(\{b\},\{a,b\})(\{b\},\{b,c\})(\{b\},\{a,b,c\})(\{c\},\{a,c\})$
$(\{c\},\{b,c\})(\{c\},\{a,b,c\})$

Hasse diagram is



Least member :- Let (P, $\leq$) denote a partially ordered set, if there exists an element $y \in P$ such that $y \leq x\ \forall x \in P$ then y is called the least member in P relative to partial ordering $\leq$.

Great member :- Let (P, $\leq$) denote a partially ordered set, if there exists an element $y \in P$ such that $x \leq y\ \forall x \in P$ then y is called the great member in P relative to partial ordering $\leq$.

Note:-

- A maximal and minimal member need not be unique.
- Maximal and minimal elements are easily calculated from the hasse diagram. They are "top" and "bottom" elements in the diagram.

Eg:- Which elements of the poset ({2,4,5,10,12,20,25},/) are maximal and which are minimal?

Maximal elements are 12, 20 and 25
Minimal elements are 2 and 5