UNIT-2

Functions: Bijective Functions, Composition of Functions, Inverse Functions, Permutation Functions, Recursive Functions.

FUNCTIONS

Introduction: Function is defined as a particular class of relations. A relation or correspondence in each member of the first set is associated with one and only one member of the second set is called a function from the first set to the second.

Applications:

- Computer output can be considered as the input.
- A special class of functions is used in organizing files on computer.

Definition: Let X and Y be any two sets. A relation f from X to Y is called a function if for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$

$$X \xrightarrow{f} Y \text{ or } f : X \to Y$$

The definition of f^n requires that a relation must satisfy two additional conditions in order to quality as function. They are

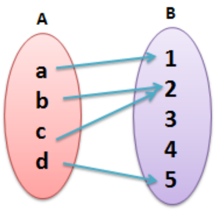
- For every $x \in X$ must be related to some the $y \in Y$ domain of f must be X and not merely a subset of X.
- Uniqueness $(x, y) \in f \wedge (x, z) \in f$
  $\Rightarrow y = z$

Notation:- $f : X \to Y$ function may also be called as mapping transformation, correspondence or operation.

Representation of a function:-

$A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$

$f : A \to B$ f(a) = 1 f(b) = 2 f(c) = 2 f(d) = 5



Note:-

- ✓ All the elements of A must be mapped to elements in B.
- ✓ And there is no compulsion of mapping of elements in B.

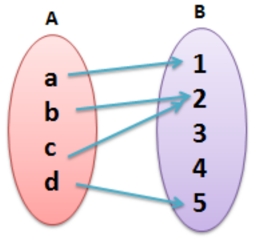
Domain & Co-domain:- If **f** is a function from A to B. Then A is called the ***domain*** of f denoted by "dom f", its members are the first co-ordinates of the ordered pairs belonging to f and set B is called ***co-domain***.

If $(x, y) \in f$, it is customary to write y = f(x)

y is called the image of x
x is called the pre-image of y.

The set consisting of all the images of the elements of A under the function f is called the range of f. It is denoted by f(A).

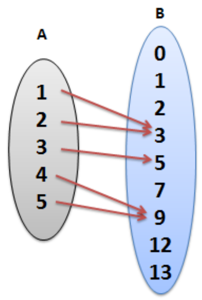
The range of a function $f: X \to Y$ is defined as $\{f(x) \in y \mid x \in X\}$



Domain = {a,b,c,d}
Co-domain = {1,2,3,4,5}
Range = {1,2,5}
image of a is 1
image of b is 2
preimage of 5 is d
preimage of 2 is b and c

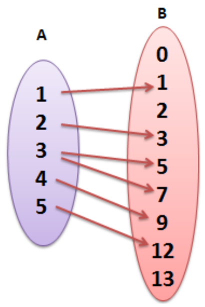
Eg: **Consider** $A = \{1,2,3,4,5\}$ $B = \{0,1,2,3,5,7,9,12,13\}$

i) f = {(1,3) (2,3) (3,5) (4,9) (5,9)}



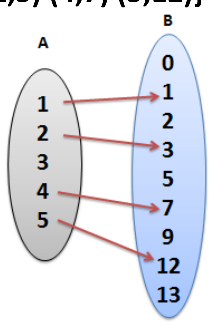
The components in B can repeat
Range = {3,5,9}

ii) f = {(1,1) (2,3) (3,5) (3,7) (4,9) (5,12)}



f is not a function from A to B, because different pairs (3,5) and (3,7) have same first component

iii) f = {(1,1) (2,3) (4,7) (5,12)}



f is not a function from A to B, because the elements in A has no image in B.

iv) f(x) = 1/x for x $\in$ R represents a function where R is the set of real numbers and f:R -> R

f(x) is not defined at x = 0

f(0)= 1/0 infinite

f(x) is not a function.

v) **f(x) = $\sqrt{x}$ for x$\in$R represents a function where R is the set of real numbers and f:R -> R**

f(x) is not a function. f(x) is not real for x < 0. Hence f: R-> R is not defined.

Equal functions:- Two functions $f : A \to B$ and $g : A \to B$ are said to be equal if and only if

i. $D_f = D_g$

ii. $Co - D_f = Co - D_g$

iii. $\forall x \in A, \; f(x) = g(x)$

If they are equal, they are denoted as f = g.

Eg: Let A = {1,2} B = {3,6} $f : A \to B$ defined f(x) = x^2+2 and $g : A \to B$ defined g(x) =3x.

if x = 1,

f(1) = 1^2+2 = 3

g(1) = 3(1) = 3

if x = 2,

f(2) = 2^2+2 = 6

g(2) = 3(2) = 6

Hence f = g.

Eg: If the function f is defined by f(x) = x^2+1 on the set {-2,-1,0,1,2}. Find the range of f.

f(-2) =$(-2)^2$+1 = 5

f(-1) =$(-1)^2$+1 = 2

f(0) =$(0)^2$+1 = 1

f(1) =$(1)^2$+1 = 2

f(2) =$(2)^2$+1 = 5

Range = {1,2,5}

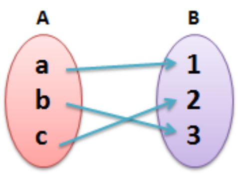
Bijective function:

A function f from A to B is said to be bijective function if f is both injective and surjective. i.e, both one to one and onto function.

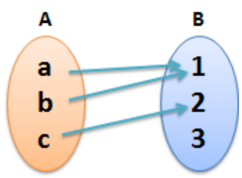
<u>One to one function:-</u> A mapping $f : X \to Y$ is called one to one or injective. if distinct elements of X are mapped with distinct elements of Y.For all elements of x and y in X such that

f(x) = f(y)

x = y



One to One



Not One to One

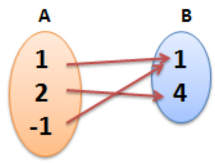
Eg: Determine f: Z -> Z given f(x) = x^2, x $\in$Z is a one to one function.

if x = 1, f(1) = 1^2 = 1

x = 2, f(2) = 2^2 = 4

x = -1, f(-1) = $(-1)^2$ = 1

It is not one to one because 1 and -1 has same image 1 which is against definition.

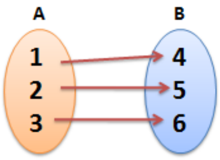


Not One to One

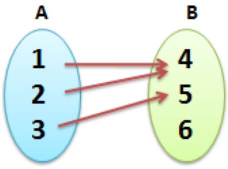
Onto function:- A mapping $f : X \rightarrow Y$ is called onto or surjective. if every element of Y is the image of some element in X, that is Y = R_f or f(x) = y.

The range of f is equal to entire co-domain Y. we say as f is a function of X onto Y.

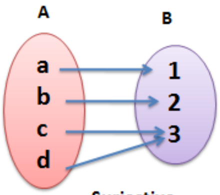
"f maps X onto Y or f is an onto function.



Surjective

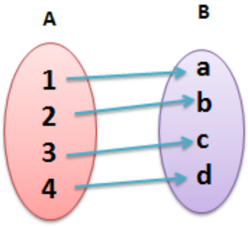


Not Surjective



Surjective

Eg: Let the function f : A to B where A = {1,2,3,4} and B = {a,b,c,d} f(1) = d, f(2) = b, f(3) = c and f(4) = a. Find whether it is bijective or not.



It is one to one function.
Range = Co-domain, every element of B is the image of some element in A.
Hence it is bijective function

Eg: Show that f: R->R defined by f(x) = 2x+1 for x$\in$R is a bijective map from R to R.

i) Proof of f being one-one

$$f(x) = f(y)$$
$$\Rightarrow 2x+1 = 2y+1$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

Thus f(x) = f(y) $\Rightarrow$ x = y
This implies f is one to one.

ii) Proof of f being one-one

$$f(x) = y$$
$$\Rightarrow 2x+1 = y$$
$$\Rightarrow 2x = y-1$$
$$\Rightarrow x = \frac{y-1}{2} \in R$$

Thus every element in the co-domain has pre-image in the domain.
Thus implies f is onto. Therefore f is bijective function.

Eg: Show that f: Z -> Z defined by f(x) = x+5 for x$\in$Z is a bijective map from Z to Z.

i) Proof of f being one-one

$$f(x) = f(y)$$
$$\Rightarrow x+5 = y+5$$
$$\Rightarrow x = y$$

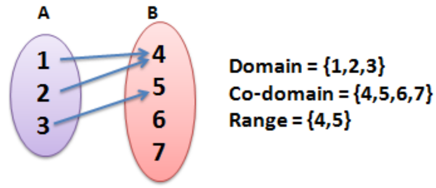
Thus f(x) = f(y) $\Rightarrow$ x = y
This implies f is one to one.

ii) Proof of f being one-one

$$f(x) = y$$
$$\Rightarrow x+5 = y$$
$$\Rightarrow x = y - 5 \in Z$$

Thus every element in the co-domain has pre-image in the domain.
Thus implies f is onto. Therefore f is bijective function.

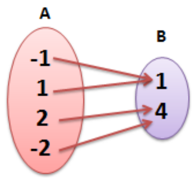
<u>Into function:</u> A function f from A to B is called into function. if and only if there exists atleast one element in B which is not image of any element in A. i.e, range of f is proper subset of co-domain of f.



<u>Types of functions:</u>

1) **Many to one:-** If two or more elements in the domain of f have same image element in the co-domain, then f is called many to one mapping i.e, $f : A \to B$ is many to one if it is not one to one.

   Eg: f(x) = x^2



2) **Identity function:-** Let X be any set and f be a function such that $f : X \to X$ is defined as f(x) = x for all $x \in X$. Then it is called identity function or identity transformation on X. It can be denoted by I or I_x.

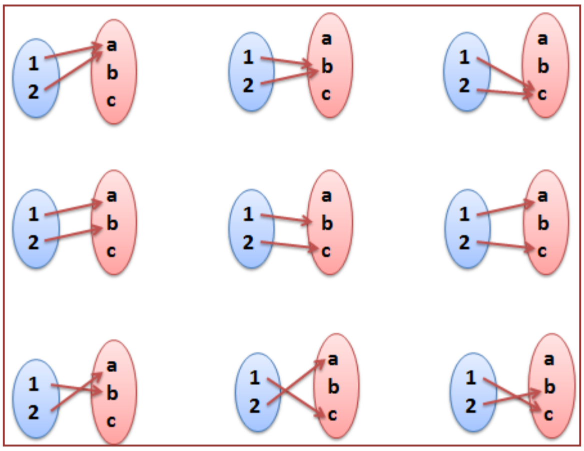
Identity function is both one to one and onto function

Note:- If A has m elements and B has n elements then the number of distinct functions from A to B is n^m.

Eg: Write all possible functions from X = {1,2} Y = {a,b,c}

m = 2 and n = 3

The number of distinct functions = 3^2 = 9



Eg: Which of the following are injections, surjections or bijections from R to R, R is the set of all real numbers.

i) f(x) = -2x

ii) g(x) = x^2-1

Sol:- ***f(x) = -2x***

a) Proof of f being one-one

$f(x) = f(y)$

$\Rightarrow -2x = -2y$

$\Rightarrow x = y$

Thus $f(x) = f(y) \Rightarrow x = y$

This implies f is one to one.

b) Proof of f being onto

$f(x) = y$

$\Rightarrow -2x = y$

$\Rightarrow x = -y/2 \in R$

$f(x) = f(-y/2)$

$= -2(-y/2)$

$= y$

Thus every element in the co-domain has pre-image in the domain.

Thus implies f is onto. Therefore f is bijective function.

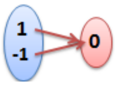
g(x) = x²-1

a) Proof of f being one-one

$g(x) = g(y)$

$\Rightarrow x^2-1= y^2-1$

$\Rightarrow x^2= y^2$



This implies g is not one to one.

b) Proof of f being onto

$g(x) = y$

$\Rightarrow x^2-1= y$

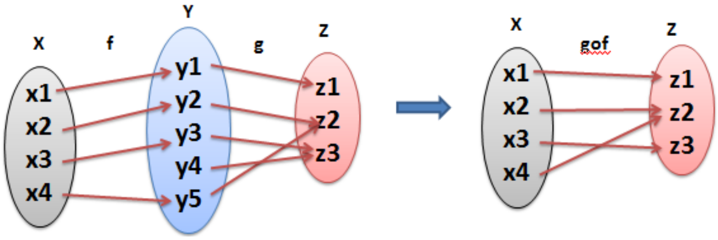
$\Rightarrow x = (y+1)^{1/2} \in R$

Thus implies g is not onto.

Therefore g is not bijective function.

Composition of functions:- Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The composition relation gof defined as gof = $\{(x,z) \mid (x \in X) \text{ and } (z \in Z) \text{ and } (y \in Y) \text{ and } y = f(x) \text{ and } z = g(y)\}$ is called composition of functions or relative product of functions f and g.

In other words let $f : X \to Y$ $g : Y \to Z$ be two functions. The composition of f and g written as gof is the function from X to Z defined as $gof(x) = g[f(x)] \quad \forall x \in X$

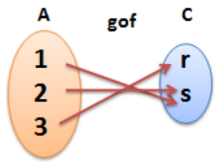


Eg: Let A = {1,2,3} B = {a,b} C = {r,s} $f : A \to B$ ***is defined as f(1) = a, f(2) = a, f(3) = b and*** $g : B \to C$ ***is defined as g(a) = s, g(b) = r find gof.***

Then $gof : A \to C$

$(gof)(1) = g(f(1)) = g(a) = s$

$(gof)(2) = g(f(2)) = g(a) = s$



(gof)(3) = g(f(3)) = g(b) = r

Eg: Let X = {1,2,3} f,g,h and s be functions from X to X given by f = {(1,2) (2,3) (3,1)} g = {(1,2) (2,1) (3,3)} h = {(1,1) (2,2) (3,3)} find fog,gof,sog,gos,sos,fos and fohog.

fog:

(fog)(1) = f(g(1)) = f(2) = 3
(fog)(2) = f(g(2)) = f(1) = 2
(fog)(3) = f(g(3)) = f(3) = 1
fog = {(1,3) (2,2) (3,1)}

gof:

(gof)(1) = g(f(1)) = g(2) = 1
(gof)(2) = g(f(2)) = g(3) = 3
(gof)(3) = g(f(3)) = g(1) = 2
gof = {(1,1) (2,3) (3,2)}

sog:

(sog)(1) = s(g(1)) = s(2) = 2
(sog)(2) = s(g(2)) = s(1) = 1
(sog)(3) = s(g(3)) = s(3) = 3
sog = {(1,2) (2,1) (3,3)}

gos:

(gos)(1) = g(s(1)) = g(1) = 2
(gos)(2) = g(s(2)) = g(2) = 1
(gos)(3) = g(s(3)) = g(3) = 3
gos = {(1,2) (2,1) (3,3)}

sos:

(sos)(1) = s(s(1)) = s(1) = 1
(sos)(2) = s(s(2)) = s(2) = 2
(sos)(3) = s(s(3)) = s(3) = 3
sos = {(1,1) (2,2) (3,3)}

fos:

(fos)(1) = f(s(1)) = f(1) = 2
(fos)(2) = f(s(2)) = f(2) = 3
(fos)(3) = f(s(3)) = f(3) = 1
fos = {(1,2) (2,3) (3,1)}

fohog:

(fohog)(1) = f[h(g(1))] = f[h(2)] = f(2) = 3
(fohog)(2) = f[h(g(2))] = f[h(1)] = f(1) = 2
(fohog)(3) = f[h(g(3))] = f[h(3)] = f(1) = 2
fohog = {(1,3) (2,2) (3,2)}

Eg: Let A = {1,2,3,4} and mapping $f : A \to A$ is defined as f = {(1,2) (2,3) (3,4) (4,1)} find the composite function f^2,f^3,f^4.

f^2 = fof

(fof)(1) = f(f(1)) = f(2) = 3
(fof)(2) = f(f(2)) = f(3) = 4
(fof)(3) = f(f(3)) = f(4) = 1
(fof)(4) = f(f(4)) = f(1) = 2
f^2 = {(1,3) (2,4) (3,1) (4,2)}

f^3 = fof^2

$(fof^2)(1) = f(f^2(1)) = f(3) = 4$
$(fof^2)(2) = f(f^2(2)) = f(4) = 1$
$(fof^2)(3) = f(f^2(3)) = f(1) = 2$
$(fof^2)(4) = f(f^2(4)) = f(2) = 3$
f^3 = {(1,4) (2,1) (3,2) (4,3)}

f^4 = fof^3

$(fof^3)(1) = f(f^3(1)) = f(4) = 1$
$(fof^3)(2) = f(f^3(2)) = f(1) = 2$
$(fof^3)(3) = f(f^3(3)) = f(2) = 3$
$(fof^3)(4) = f(f^3(4)) = f(3) = 4$
f^4 = {(1,1) (2,2) (3,3) (4,4)}

Eg: Let f(x) = x+2,g(x) = x-2, h(x) = 3x for x $\in$ R, Where R is the set of real numbers find fog, gof, fof, hof, hog, fohog.

fog

fog(x) = f[g(x)]
= f[x-2]
= (x-2)+2 = x

gof

gof(x) = g[f(x)]
= g[x+2]
= (x+2)-2 = x

fof

fof(x) = f[f(x)]
= f[x+2]
= (x+2)+2
= x+4

hof

hof(x) = h[f(x)]
= h[x+2]
= 3(x+2)
= 3x+6

hog

hog(x) = h[g(x)]
= h[x-2]
= 3(x-2) = 3x-6

fohog

fohog(x) = foh[g(x)]
= f[h(x-2)]
= f[3(x-2))]
= f(3x-6)
=3x-6+2 = 3x-4

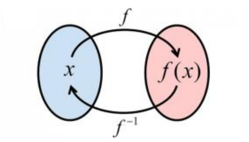
Inverse function: Let $f : A \rightarrow B$, $g : B \rightarrow A$ a map is called inverse of f if gof = I_A and fog = I_B i.e, g[f(x)] = x , $\forall x \in A$ and f[g(y)] = y , $\forall y \in B$.

Thus if f(x) = y then g(y) = g[f(x)] = x

The inverse of g of f is denoted by f^{-1}

$$f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$$

Note:- A function that has an inverse is said to be invertible. Not every function is invertible.

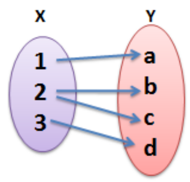


Eg: Let X = {a,b,c,d} and Y = {1,2,3,4} and let $f : X \rightarrow Y$ be defined by f ={(a,1) (b,2) (c,2) (d,3)} . Check whether f^{-1} is a function or not.

f ={(a,1) (b,2) (c,2) (d,3)}

f^{-1} ={(1,a) (2,b) (2,c) (3,d)}

The element 2 in Y is mapped to two elements in X, which is violating the rules. Hence f^{-1} is not a function.



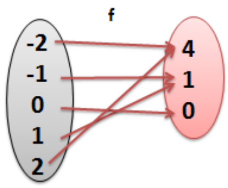
Eg: Let R is the set of real numbers and $f : R \rightarrow R$ be given by $f = \{(x, x^2) \mid x \in R\}$. Check whether f^{-1} is a function or not?

Let -2,-1,0,1,2

f(x) = x^2

x = -2, f(x) = 4
x = -1, f(x) = 1
x = 0, f(x) = 0
x = 1, f(x) = 1
x = 2, f(x) = 4

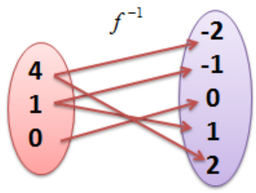
$\therefore f$ is a function



$f^{-1} = \{(x^2, x) \mid x \in R\}$

x = -2, $f^{-1}(-2) = 4$

...



The elements 4 and 1 are mapped to 2 elements.

$\therefore f^{-1}$ is not a function

Eg: Show that the function f(x) = x³ and g(x) = x^{1/3} for $x \in R$ are inverse to one another.

$f: R \to R$ is defined by f(x) = x^3

$g: R \to R$ is defined by g(x) = $x^{1/3}$

fog(x) = f[g(x)]

= f($x^{1/3}$)

= $(x^{1/3})^3$

= x = $I_x(x)$

gof(x) = g[f(x)]

= f(x^3)

= $(x^3)^{1/3}$

= x = $I_x(x)$

$\therefore g = f^{-1}$ and $f = g^{-1}$

i.e, f and g are inverse to one another.

Eg: Show that the mapping $f: R \to R$ be defined by f(x) = ax+b where a,b,x $\in R$. Define its inverse.

if x,y $\in R$

1) Proof of f being *one-one*

f(x) = f(y)

$\Rightarrow$ ax+b = ay+b

$\Rightarrow$ ax = ay

$\Rightarrow$ x = y

Thus f(x) = f(y) $\Rightarrow$ x = y

This implies f is one to one.

2) Proof of f being one-one

f(x) = y

$\Rightarrow$ ax+b = y

$\Rightarrow$ ax = y-b

$\Rightarrow x = \frac{y-b}{a}$ $\in$ R

Thus for x $\in R$, there exists $\frac{y-b}{a}$ $\in$ R such that

f($\frac{y-b}{a}$) = a($\frac{y-b}{a}$) + b

= y-b+b

= y

Thus every element in the co-domain has pre-image in the domain.

Thus implies f is onto.

Therefore f is bijective function.

$$\therefore f^{-1} = \frac{1}{a}(x-b)$$

Eg: If $f: R \to R$ such that f(x) = 2x+1 and $g: R \to R$ such that g(x) = x/3 then $(gof)^{-1} = f^{-1}og^{-1}$.

gof(x) = g(f(x))

$= g(2x+1) = \frac{2x+1}{3}$

Since f and g are one-one and onto. Therefore $(gof)^{-1}$ exists and defined as

$(gof)^{-1}$ (x)

f(x) = y

$\frac{2x+1}{3} = y$

$2x+1 = 3y$

$\Rightarrow 2x = 3y-1$

$\Rightarrow x = \frac{3y-1}{2}$

$\therefore (gof)^{-1}(x) = \frac{3x-1}{2}$

$f^{-1}og^{-1}$

$f^{-1}(x)$

f(x) = y

$2x+1 = y$

$\Rightarrow x = \frac{y-1}{2}$

$\therefore f^{-1}(x) = \frac{x-1}{2}$

$g^{-1}(x)$

g(x) = y

$\frac{x}{3} = y$

$\Rightarrow x = 3y$

$\therefore g^{-1}(x) = 3x$

$f^{-1}og^{-1}(x)$ = $f^{-1}(g^{-1}(x))$

$= f^{-1}(3x)$

$= \frac{3x-1}{2}$

$\therefore$ *$(gof)^{-1} = f^{-1}og^{-1}$*

Permutation function:-

A bijection mapping of a finite set A onto itself is called a "*permutation*". If A = {a_1, a_2,.... a_n} is a finite set and p is a bijection on A, we list the elements of A and the corresponding functional values $p(a_1)$, $p(a_2)$, ... $p(a_n)$ in the following form:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ p(a_1) & p(a_2) & \dots & p(a_n) \end{pmatrix}$$

If $P: A \to A$ is a bijective map, then the number of elements in the given set is called the **"degree"** of the permutation.

Eg: Let A = {1,2,3}. Write the permutation of this set $P: A \to A$ by f(1) = 2, f(2) = 1 and f(3) = 3

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Eg: Find all permutation of A = {1,2,3}.

Given A ={1,2,3}. Then $P: A \to A$ which is one-one and onto, is called a permutation.

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Product or composition of two permutation:-

Let f and g be two permuatations defined on A. Then f and g are bijections from A to A. Their composition fog and gof are bijections from A to A.

Eg: Let A = {1,2,3,4} and let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ *and* $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$. *Find fog and gof in permutation form.*

$$fog = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$$gof = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Note:-

- fog and gof are called the product of permutations f and g, which is also known as permutation multiplication.
- Let A = {1,2,3,....n}. Then permutation $\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}$ is called identity permutation of degree n

Eg: *If* $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Find fog and gof

$$fog = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$gof = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Inverse permutation:- Given a permutation f on the set A there exists a permutation f^{-1} on A is called inverse permutation such that $fof^{-1} = f^{-1}of = I$ = Identity permutation.

Eg: If $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ *then find* f^{-1} *and show that* $fof^{-1} = f^{-1}of = I$

$$f^{-1} = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$fof^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I$$

$$f^{-1}of = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \mathrm{I}$$

$$\therefore \; fof^{-1} = f^{-1}of = I$$

Cyclic Permutation:- A permutation that replaces n objects cyclically is called cyclic permutation or circular permutation . The length of a cycle is the number of elements permutated by cycle.

Eg: Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$ *be a permutation of degree 5. Find cycle permutation.*

f = (1 3 4 5) which of length 4, Here 2 is fixed.

Note:-

- Any permutation can be expressed as product of a finite no.of disjoint cycles.

$$\mathrm{f} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}$$
$$= (1\ 2\ 3\ 4\ 5)\ (8\ 9)$$

- A cycle of length 2 is called transposition.
- Every cycle is the product of transposition.

$$(1\ 2\ 3\ 4\ 5)\ (8\ 9)$$
$$= (1\ 2)(1\ 3)(1\ 4)(1\ 5)\ (8\ 9)$$

Inverse of a cyclic permutation:- To find the inverse of any cyclic permutation, we write its elements in reverse order.

Eg:- $(1\ 2\ 3\ 4\ 5)^{-1} = (5\ 4\ 3\ 2\ 1)$

Even and odd permutation:-

- ✓ A permutation is called ***even permutation*** if it can be expressed as the product of an even number of transpositions.
- ✓ A single cycle containing an odd number of elements is an *even permutation*.
- ✓ Eg: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$

$$= (1\ 2\ 4) \qquad \Rightarrow \text{cycle having odd number of elements}$$
$$= (1\ 2)(1\ 4) \qquad \Rightarrow \text{product of even number of transpositions}$$
$$\therefore \text{f is a even permutation}$$

- ✓ Eg: $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 8 & 6 & 1 & 4 & 3 \end{pmatrix}$

$$= (1\ 2\ 5\ 6)\ \ (3\ 7\ 4\ 8)$$
$$= (1\ 2)(1\ 5)(1\ 6)(3\ 7)(3\ 4)(3\ 8) \qquad \Rightarrow \text{product of even number of transpositions}$$
$$\therefore \text{f is a even permutation}$$

- ✓ A permutation is called ***odd permutation*** if it can be expressed as the product of an odd number of transpositions.
- ✓ A single cycle containing an even number of elements is an *odd permutation*.
- ✓ Eg: $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$

$$= (1\ 4\ 2\ 3) \qquad \Rightarrow \text{cycle having even number of elements}$$

$$=(1\ 4)(1\ 2)(1\ 3) \qquad \Rightarrow \text{product of odd number of transpositions}$$
$$\therefore \text{f is a odd permutation}$$

Note:-

- (odd elements) (odd elements) -- even permutation.
- (even elements) (even elements) -- even permutation.
- (odd elements) (even elements) -- odd permutation.
- Identity permutation -- even permutation.

Eg: Let the permutation of elements {1,2,3,4,5}

$$\alpha=\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}\quad \beta=\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}\quad \gamma=\begin{pmatrix}1&2&3&4&5\\5&4&3&1&2\end{pmatrix}\quad \delta=\begin{pmatrix}1&2&3&4&5\\3&2&1&5&4\end{pmatrix}$$

$find\ \alpha\beta,\ \beta\alpha, \alpha^2, \gamma\beta, \delta^{-1} and\ \alpha\beta\gamma.$ solve $\alpha x=\beta$

$$\alpha\beta=\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}o\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}$$
$$\beta\alpha=\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}o\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}$$
$$\alpha^2=\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}o\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\3&1&2&4&5\end{pmatrix}$$
$$\gamma\beta=\begin{pmatrix}1&2&3&4&5\\5&4&3&1&2\end{pmatrix}o\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\4&5&3&1&2\end{pmatrix}$$
$$\delta^{-1}=\begin{pmatrix}1&2&3&4&5\\3&2&1&5&4\end{pmatrix}^{-1}=\begin{pmatrix}3&2&1&5&4\\1&2&3&4&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\3&2&1&5&4\end{pmatrix}$$
$$\alpha\beta\gamma=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}o\begin{pmatrix}1&2&3&4&5\\5&4&3&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\4&3&5&1&2\end{pmatrix}$$
$$\alpha\, x=\beta$$
$$x=\alpha^{-1}\beta$$
$$\alpha^{-1}=\begin{pmatrix}2&3&1&4&5\\1&2&3&4&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\3&1&2&4&5\end{pmatrix}$$
$$x=\alpha^{-1}\beta=\begin{pmatrix}1&2&3&4&5\\3&1&2&4&5\end{pmatrix}o\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}$$
$$=\begin{pmatrix}1&2&3&4&5\\3&1&2&5&4\end{pmatrix}$$

Recursive function:-

Sometimes it is difficult to define an object explictly. However, it may be easy to define this object in terms of itself. This process is called recursion.

- Recursion is used to define sequence, functions and sets.

  Eg: The sequence 1,2,4,8,.... can be defined explicitly by the relation k(n) = 2^k for all integers $n \geq 0$, recursively defined as:

  a. k(0) = 1
  b. k(n+1) = 2 k(n) $\forall n \geq 0$

- To define a function with the set of non-negative integers as its domain, we consider the following 2 steps:
  - Step 1: Specify the value of the function at zero.
  - Step 2: Give a rule for finding its value at an integer from its values at smaller integers.

  This is called a recursive or inductive definition.

Definition: A function $f(x_1,x_2,\dots x_n)$ which maps every n-tuple in N^n to some element in N is called a total function i.e, $f: N^n \to N$

Eg: $f(x,y) = x+y$, $\forall x, y \in N$ $\Rightarrow$ total function.

If $f: D \to N$ where $D \subseteq N^n$ then f is called partial function.

Eg: $g(x,y) = x-y$, $\forall x, y \in N$ which satisfy $x \geq y \Rightarrow$ partial function.

Initial functions:- The following 3 functions namely

1. Zero function Z : $Z(x) = 0$, $\forall x$
2. Successor function S : $S(x) = x+1$, $\forall x$
3. Projection function $U_i^n = U_i^n(x_1, x_2, \dots x_n) = x_i$ for all n tuples $(x_1, x_2, \dots x_n)$, $1 \leq i \leq n$ are called initial functions.

- Projection function is also called generalised identity function.

  Eg: $U_1^1(x) = x$ for every $x \in N$ is the identity function.

$$U_1^2(x,y) = x$$
$$U_2^2(x,y) = y$$
$$U_2^3(4,3,8) = 3$$

Composition functions of more than one variable:-

Let $f_1(x,y)$ $f_2(x,y)$ and $g(x,y)$ be any three function. Then the composition of g with f_1 and f_2 is defined as function h(x,y).

$$\mathbf{h(x,y) = g(f_1(x,y), f_2(x,y))}$$

Eg: Let $f_1(x,y) = x+y$ $f_2(x,y) = x^2+y^2$ $g(x,y) = xy$

$h(x,y) = g(f_1(x,y), f_2(x,y))$
$= g((x+y), (x^2+y^2))$
$= (x+y)(x^2+y^2)$

Recursion: The operations are

- ✓ $f(x_1, x_2, \dots x_n, 0) = g(x_1, x_2, \dots x_n)$
- ✓ $f(x_1, x_2, \dots x_n, y+1) = h(x_1, x_2, \dots x_n, y, f(x_1, x_2, \dots x_n, y))$

  Eg: $f(x, y+1) = h(x, y, f(x,y))$

Primitive recursion function:-

A function f is called primitive recursive if and only if it can be obtained from the initial function by a finite number of operations of composition and recursion.

Eg: Show that function f(x,y) = x + y is a primitive recursion. Hence compute f(2,4).

Given that $f(x,y) = x + y$

$f(x,y+1) = x + y + 1$
$=(x + y) + 1$
$= f(x,y) + 1$

We define f(x,0) = x = $U_1^1(x)$

$\therefore$ f(x,y+1) = f(x,y) + 1

= S(f(x,y))

$= S(U_3^3(x, y, f(x, y)))$

We take $g(x) = U_1^1(x)$ and $h(x, y, z) = S(U_3^3(x, y, z))$

$\therefore f(x,0) = g(x)$, $\text{f(x, y+1)} = \text{h(x, y,z)}$

$\therefore f$ is obtained from intial functions U_1^1, U_3^3 and S by applyingcomposition and recursion once.

Hence f is primitive function.

$f(2,0) = 2$

$f(2,4) = S(f(2,3))$

$= S(S(f(2,2)))$

$= S(S(S(f(2,1))))$

$= S(S(S(S(f(2,0)))))$

$= S(S(S(S(2))))$

$= S(S(S(3)))$

$= S(S(4))$

$= S(5)$

$= 6$

Eg: Using recursion define multiplication functions * given by f(x,y) = x*y

Since multiplication of two natural numbers is simply repeated addition f has to be primitive recursive.

$f(x,0) = 0 * x = 0$

$f(x, y+1) = x*(y+1)$

$= (x*y)+1$

$= f(x, y)+1$

$= S(f(x, y), x)$

We can write $f(x,0) = 0 = Z(x)$ and

$f(x, y+1) = S(U_3^3(x, y, f(x, y)), U_1^3(x, y, f(x, y)))$

Eg: Show that the proper subtraction is primitive recurive and prove that 5 – 3 = 2.

We define the predecessor function P by P(0) = 0

$P(y+1) = y = U_1^2(y, P(y))$ which is a recursive function.Thus the subtraction function is

$f(x,0) = 0 = U_1^1(x)$

$f(x, y+1) = P(f(x, y)) = g(x, y, f(x, y))$

where $g(x, y, z) = P(z) = U_3^3(x, y, z)$

Hence proper subtraction function is a recursive function.

$f(5,0) = 5 + 0 = 5$

$f(5,3) = P(f(5,2))$

$$
\begin{aligned}
&= P(P(f(5,1))) \\
&= P(P(P(f(5,0)))) \\
&= P(P(P(5))) \\
&= P(P(4)) \\
&= P(3) \\
&= 2
\end{aligned}
$$

Ackermann's function:- The ackermann's function A(x,y) is defined by

I. $A(0, y) = y + 1$

II. $A(x+1,0) = A(x,1)$

III. $A(x+1, y+1) = A(x, A(x+1, y))$

We can compute the values of A(x,y) for fixed values of x and y by using above definition.

- ✓ A(x,y) is well defined and total function.
- ✓ In A(x,y) both x and y are inductive variables and there is no parameter. Also (x,y) is recursive but not primitive recursive.

Eg: $A(1,2)$

$$
\begin{aligned}
&= A(0+1,1+1) \\
&= A(0, A(1,1)) \\
&= A(0, A(0+1,0+1)) \\
&= A(0, A(0, A(1,0))) \\
&= A(0, A(0, A(0,1))) \\
&= A(0, A(0,2)) \\
&= A(0,3) \\
&= 4
\end{aligned}
$$

Eg: $A(2,2)$

$$
\begin{aligned}
&= A(1+1,1+1) \\
&= A(1, A(2,1)) \\
&= A(1, A(1+1,0+1)) \\
&= A(1, A(1, A(2,0))) \\
&= A(1, A(1, A(1+1,0))) \\
&= A(1, A(1, A(1,1))) \\
&= A(1, A(1, A(0+1,0+1))) \\
&= A(1, A(1, A(0, A(1, A(1,0))))) \\
&= A(1, A(1, A(0, A(1, A(0+1,0))))) \\
&= A(1, A(1, A(0, A(1, A(0,1))))) \\
&= A(1, A(1, A(0,2))) \\
&= A(1, A(1,3)) \\
&= A(1, A(0+1,2+1)) \\
&= A(1, A(0+1,2+1)) \\
&= A(1, A(0, A(0+1,1+1))) \\
&= A(1, A(0, A(0, A(1,1))))
\end{aligned}
$$

$$
\begin{aligned}
&= A(1, A(0, A(0, A(0+1,0+1)))) \\
&= A(1, A(0, A(0, A(0, A(1,0))))) \\
&= A(1, A(0, A(0, A(0, A(0+1,0))))) \\
&= A(1, A(0, A(0, A(0, A(0,1))))) \\
&= A(1, A(0, A(0, A(0,2)))) . \\
&= A(1, A(0, A(0,3))) \\
&= A(1, A(0,4)) \\
&= A(1,5) \\
&= A(0+1,4+1) \\
&= A(0, A(1,4)) \\
&= A(0, A(0+1,3+1)) \\
&= A(0, A(0, A(1,3))) \\
&= A(0, A(0, A(0+1,2+1))) \\
&= A(0, A(0, A(0, A(1,2)))) \\
&= A(0, A(0, A(0, A(0+1,1+1)))) \\
&= A(0, A(0, A(0, A(0, A(1,1)))))
\end{aligned}
$$

$$= A(0, A(0, A(0, A(0, A(0+1, 0+1)))))$$
$$= A(0, A(0, A(0, A(0, A(0, A(1,0))))))$$
$$= A(0, A(0, A(0, A(0, A(0, A(0,1))))))$$
$$= A(0, A(0, A(0, A(0, A(0,2)))))$$
$$= A(0, A(0, A(0, A(0,3))))$$
$$= A(0, A(0, A(0,4)))$$
$$= A(0, A(0,5))$$
$$= A(0,6)$$
$$= 7$$