UNIT-1

Mathematical Logic : Propositional Calculus: Statements and Notations, Connectives, Truth Tables, Tautologies, Equivalence of Formulas, Duality law, Tautological Implications, Normal Forms, Theory of Inference for Statement Calculus, Consistency of Premises, Indirect Method of Proof.
Predicate calculus: Predicative Logic, Statement Functions, Variables and Quantifiers, Free & Bound Variables, Inference theory for predicate calculus.

Introduction: ***Logic*** is the study of reasoning. One of the main aims of logic is to provide rules by which one can determine whether any particular argument or reasoning is valid. Any collection of rules needs a language in which these rules are stated. Natural languages are not always precise enough. So a formal language called "***Object language***" is used. A formal language is one in which the syntax is well defined. Inorder to avoid ambiguity, we use symbols in object languages. An additional reason to use symbols is that they are easy to write and manipulate. The study of object language requires the use of another language. For this purpose we can choose any of the natural languages. This natural language (preferably english) will be called as "***metalanguage***".

Propositional Calculus:

Statements and Notations:-

- Statements are the basic units of object language.
- Statements are also called as "*propositions*".
- Statements are of mainly 2 types
  - Primary statements
  - Compound statements

Primary statements:- A set of declarative sentences which cannot be further broken down into simpler sentences are called primary statements. These are also called as **atomic or primitive statements**. Each primary statement can have one and only one of the two possible values called "***truth values***". The truth values are "*true*" and "*false*" denoted by the symbols T and F respectively (sometimes with 1 or 0). Since we admit only 2 possible truth values, our logic is sometimes called a "***two-valued logic***".
Primitive statements are denoted by distinct symbols from the capital letters A,B,C......,P,Q,.....

> *"All declarative sentences to which it is possible to assign one and only one truth value are called statements".*

Examples:

1. India is a country.
2. 1 + 101 = 110.
3. Close the door.
4. This statement is false.
5. Delhi is the capital of Nepal.

Statements (1) and (5) have truth values *true* and *false* respectively. Statement (2) has truth value depending on the context. If both the numbers are treated as ***decimal*** the truth value is '*false*' but if they are considered as ***binary*** the truth value is '*true*'. So it has only one truth value depending on the context. Statement (3) is not a statement but an order. Statement (4) is not a statement as we can't properly assign a truth value to it.

So we can represent the statements as below:

P : India is a country

Q : 1 + 101 = 110

R : Delhi is the capital of Nepal.

Connectives:

These are the connecting words or expressions used to construct complicated statements from simpler statements. The statements constructed using simper statements (primary) are called ***compound statements***. Connectives in natural language are "and", "but", "or" etc.

Connectives in object language:

1) **Negation:** The negation of a statement is generally formed by introducing the word "not" at a proper place in the statement or by prefixing the statement with the phrase "It is not the case that". If "P" denotes a statement, then the negation of "P" is written as "$\neg$P" and read as "not P". If the truth value of "P" is "T", then the truth value of "$\neg P$" is "F" and vice-versa.

P	$\neg$P
T	F
F	T

Truth table for negation

Examples:

- P : London is a city.
  $\neg$P : London is not a city (or)
  $\neg$P : It is not the case that London is a city.

- P : I went to my class yesterday.
  $\neg$P : I did not go to my class yesterday.
  $\neg$P : I was absent from my class yesterday.
  $\neg$P : It is not the case that I went to my class yesterday.

Negation is a unary operation and the alternate symbols used are '~', bar ,NOT, (~p, $\overline{p}$, NOT p)

2) **Conjunction:** The conjunction of two statements P and Q is the statement $P \wedge Q$ which is read as "P and Q". The statement $P \wedge Q$ has the truth value T whenever both P and Q have truth value T, otherwise it has the truth value F.

P	Q	P $\wedge$ Q
T	T	T
T	F	F
F	T	F
F	F	F

Truth table

Examples:

- P : It is raining today.
  Q : There are 20 tables in this room.
  $P \wedge Q$: It is raining today and there are 20 tables in this room.
- Translate the following into symbolic statements : *Jack and Jill went up the hill*

  Step1:-

  Jack went up the hill.
  Jill went up the hill. (paraphrasing)

  Step2:-

  P : Jack went up the hill.
  Q : Jill went up the hill.

  Step3:-

  $P \wedge Q$

  Note:- $P \wedge Q$ and $Q \wedge P$ should have the same truth values.

- Roses are red and violets are blue.
  P : Roses are red.
  Q : Violets are blue.
  $P \wedge Q$
- He opened the book and started to read
  P: He opened the book
  Q: He started to read
  In this example $P \wedge Q$ is not the same as $Q \wedge P$. So conjunction should not be used.($Q \wedge P$ means he started to read and opened the book which is not meaningful)

3) Disjunction: The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q". The statement $P \vee Q$ has the truth value F only when both P and Q have truth value F, otherwise it has the truth value T.

P	Q	PVQ
T	T	T
T	F	T
F	T	T
F	F	F

Truth table

Examples:

- I shall watch the game on television or go to the game.

  Step1:-

  P : I shall watch the game on television.
  Q : I go to the game.

  Step2:-

  In this case "or" is used in the ***exclusive*** sense i.e, one or other possibility exists but not both. So "$\vee$" is not used here.

- There is something wrong with the bulb or with the wiring.
  Here the "or" was ***inclusive*** i.e, the intended meaning is clearly one or other or both. So "$\vee$" can be used here.

Step1:-

P : There is something wrong with the bulb.

Q : There is something wrong with the wiring.

Step2:-

$P \vee Q$

Statement Formulas and Truth tables:-

- ✓ The statements which do not contain any connectives are called ***atomic*** or *primary* or *simple* statements.
- ✓ The statements which contain one or more primary statements and some connectives are called ***molecular*** or *composite* or *compound* statements.
- ✓ Examples: Let P and Q are two statements, then some of the compound statements formed by using P and Q are
    - $\neg P$
    - $P \vee Q$
    - $(P \wedge Q) \vee (\neg P)$
    - $\neg P \vee (\neg Q)$ ...

Examples: Construct the truth table for the statement formulas

1) $((P \wedge Q) \vee \neg Q)$

P	**Q**	**$P \wedge Q$**	**$\neg Q$**	**$((P \wedge Q) V \neg Q)$**
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

2) $\neg (P \wedge Q) \vee (\neg R)$

P	**Q**	**R**	**$P \wedge Q$**	**$\neg(P \wedge Q)$**	**$\neg R$**	**$\neg(P \wedge Q) \vee \neg R$**
T	T	T	T	F	F	F
T	T	F	F	T	T	T
T	F	T	F	T	F	T
T	F	F	F	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	T	T

Conditional statements:- If P and Q are any two statements , then the statement $P \rightarrow Q$ which is read as "If P then Q" is called a conditional statement. The statement $P \rightarrow Q$ has a truth value F when Q has truth value F and P the truth value T, otherwise it has the truth value T.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement P is called the antecedent and Q the consequent in $P \to Q$

Examples:

- Express in english the statement $P \to Q$ where

  P : The sun is shining today.

  Q : 2 + 7 > 4

  Sol:- If the sun is shining today, then 2 + 7 > 4
- Write the following statement in symbolic form

  If either Jerry takes calculus or Ken takes sociology, then Larry will take English

  Sol:- P : Jerry takes calculus

  Q : Ken takes sociology.

  R : Larry takes English.

  $(P \vee Q) \to R$
- Write the following statement in symbolic form

  The crop will be destroyed if there is a flood.

  Sol:- A : The crop will be destroyed.

  B : There is a flood.

  $B \to A$

Example: Construct the truth table $(P \vee \neg Q) \to (P \wedge Q)$

P	**Q**	**¬Q**	**P∨ ¬Q**	**P∧Q**	**(P∨ ¬Q) → (P∧Q)**
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Biconditional statements: If P and Q are any two statements , then the statement $P \leftrightarrow Q$ which is read as “P If and only if Q” and abbreviated as “P iff Q” is called a biconditional statement. The statement $P \leftrightarrow Q$ has a truth value T whenever both P and Q have identical values. $P \leftrightarrow Q$ is also translated as “P is necessary and sufficient for Q”.

P	**Q**	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: Construct the truth table $(\neg P \wedge \neg Q) \leftrightarrow (P \wedge \neg Q)$

P	**Q**	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$P \wedge \neg Q$	$(\neg P \wedge \neg Q) \leftrightarrow (P \wedge \neg Q)$
T	T	F	F	F	F	T
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	T	F	F

Operator Precedence:-

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Logical NAND:- The logical NAND is an operation on two logical values, typically the values of two propositions, that produces a value of *false* if both of its operands are true. In other words, it produces a value of *true* if at least one of its operands is false.The truth table for P NAND Q (also written as P ↑ Q or P | Q) is as follows:

Logical NAND		
P	*Q*	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

The negation of a conjunction: $\neg(P \wedge Q)$, and the disjunction of negations: $(\neg P) \vee (\neg Q)$ can be tabulated as follows:

P	*Q*	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Logical NOR :The logical NOR is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if both of its operands are false. In other words, it produces a value of *false* if at least one of its operands is true. ↓ is also known as the Peirce arrow after its inventor, Charles Sanders Peirce, and is a Sole sufficient operator**.**

The truth table for **P NOR Q** (also written as **P ↓ Q** or **P ⊥ Q**) is as follows:

Logical NOR		
P	***Q***	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

The negation of a disjunction $\neg(P \vee Q)$, and the conjunction of negations $(\neg P) \wedge (\neg Q)$ can be tabulated as follows:

P	Q	$P\vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Well – Formed Formulas:-

A statement formula is an expression which is a string consisting of variables, parenthesis and connective symbols. A well-formed formula (wff) which is the recursive definition of a statement formula can be generated by the following rules:

1. A statement variable standing alone is a wff.
2. If A is a wff, then $\neg A$ is a wff.
3. If A and B are wff's, then $(A \wedge B), (A \vee B), (A \rightarrow B)$ and $(A \leftrightarrow B)$ are wffs.
4. A string of symbols containing the statement variables, connectives and parantheses is a wff, iff it can be obtained by finitely many applications of the rule 1, 2 and 3.

   Examples:

   $P \wedge Q$

   $(P \vee Q) \rightarrow R$

   $P \vee Q \rightarrow P$

   $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$

   $\neg \neg P$

   $P \neg \neg$ this is not a wff.

Tautologies :-

- ✓ A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called as a "*<u>universally valid formula"</u>* or a ***tautology*** or a *logical truth.*

  Examples:

  - A round circle.
  - A good- looking beautiful women.
  - A big giant.
- ✓ A statement formula which is false regardless of the truth values of the statements which replace the variables in it is called as a "***contradiction"***.
- ✓ A statement formula which is neither a tautology nor a contradiction is called a "**contingency".**

A straight forward method to determine whether a given formula is a tautology is to construct its truth table. But it becomes tedious when the number of distinct variables is large or when the formula is complicated.

The number of rows in a truth table is 2^n,where n is the number of distinct variables in the formula

Eg 1: Show that $P \vee \neg P$ is a tautology and $P \wedge \neg P$ is a contradiction.

P	¬P	P∨¬P	P∧¬P
T	F	T	F
F	T	T	F

Eg 2: Show that $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is a tautology.

P	Q	P→Q	¬Q	¬P	¬Q→¬P	(P→Q) ↔ (¬Q→¬P)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Equivalence of Formulas:- Let A and B be two statement formulas and let $P_1, P_2, \ldots P_n$ denote all the varibles occuring in both A and B. Consider an assignment of truth values to $P_1, P_2, \ldots P_n$ and the resulting truth values of A and B. If the truth value of A is equal to B for every one of the 2^n possible sets of truth values assigned to $P_1, P_2, \ldots P_n$, then A and B are said to be equivalent.

Examples:

- $\neg\neg P$ is equivalent to P.
- $P \vee P$ is equivalent to P.
- $(P \wedge \neg P) \vee Q$ is equivalent to Q.
- $P \vee \neg P$ is equivalent to $Q \vee \neg Q$.

Symbol for equivalence is $\Leftrightarrow$

Eg 1: Prove $\neg P \vee Q$ and $P \to Q$ are logically equivalent

P	Q	$\neg P$	$\neg P \vee Q$	$P \to Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Equivalent Formulas:

Equivalence	Name
$P \wedge T \Leftrightarrow P$ $P \vee F \Leftrightarrow P$	Identity Laws
$P \vee T \Leftrightarrow T$ $P \wedge F \Leftrightarrow F$	Domination Laws
$P \vee P \Leftrightarrow P$ $P \wedge P \Leftrightarrow P$	Idempotent Laws
$\neg(\neg P) \Leftrightarrow P$	Double Negation Law
$P \vee Q \Leftrightarrow Q \vee P$ $P \wedge Q \Leftrightarrow Q \wedge P$	Commutative Laws

Equivalence	Name
$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$ $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$	Associative Laws
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P$	Distributive Laws
$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	De Morgan Laws
$P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) \Leftrightarrow P$	Absorption Laws
$P \vee \neg P \Leftrightarrow T$ $P \wedge \neg P \Leftrightarrow F$	Negation Laws

Example: Show that $(P\wedge Q) \rightarrow (P\vee Q)$ is a tautology

$(P\wedge Q) \rightarrow (P\vee Q)$	$\Leftrightarrow$	$\neg(P\wedge Q) \vee (P\vee Q)$	
	$\Leftrightarrow$	$(\neg P \vee \neg Q) \vee (P\vee Q)$	(De Morgan's Law)
	$\Leftrightarrow$	$((\neg P \vee \neg Q \vee P) \vee Q$	(Associative Law)
	$\Leftrightarrow$	$((\neg P \vee P \vee \neg Q) \vee Q$	(Commutative Law)
	$\Leftrightarrow$	$(\neg P \vee P) \vee (\neg Q \vee Q)$	(Associative Law)
	$\Leftrightarrow$	$\mathbf{T} \vee \mathbf{T}$	(Negation Law)
	$\Leftrightarrow$	$\mathbf{T}$	

Hence proved.

Example: Show that $\neg(P \vee (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are equivalent.

$\neg(P \vee (\neg P \wedge Q))$	$\Leftrightarrow$	$\neg P \wedge \neg(\neg P \wedge Q)$	[De Morgan's Law]
	$\Leftrightarrow$	$\neg P \wedge (\neg(\neg P) \vee \neg Q)$	[De Morgan's Law]
	$\Leftrightarrow$	$\neg P \wedge (P \vee \neg Q)$	[Double negation law]
	$\Leftrightarrow$	$(\neg P \wedge P) \vee (\neg P \wedge \neg Q)$	[Distributive Law]
	$\Leftrightarrow$	$\mathbf{F} \vee (\neg P \wedge \neg Q)$	[Negation Law]
	$\Leftrightarrow$	$(\neg P \wedge \neg Q) \vee \mathbf{F}$	[Commutative Law]
	$\Leftrightarrow$	$(\neg P \wedge \neg Q)$	[Identity Law]

Equivalence Proved

Duality Law:-

Two formulas A and A^* are said to be duals to each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. These connectives $\wedge$ and $\vee$ are called duals of each other. If the formula A contains T or F, then A^* is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Eg :- Write the duals of

a) $(P\vee Q)\wedge R$ $\Leftrightarrow$ The dual is $(P\wedge Q)\vee R$

b) $(P\wedge Q)\vee T$ $\Leftrightarrow$ The dual is $(P\vee Q)\wedge F$

c) $\neg(P\vee Q)\wedge(P\vee\neg(Q\wedge\neg S))$ $\Leftrightarrow$ The dual is $\neg(P\wedge Q)\vee(P\wedge\neg(Q\vee\neg S))$

Tautological Implications:- The connectives $\wedge,\vee$ and $\leftrightarrow$ are symmetric i.e, $P\wedge Q \Leftrightarrow Q\wedge P$, $P\vee Q \Leftrightarrow Q\vee P$, $P\leftrightarrow Q \Leftrightarrow Q\leftrightarrow P$. But $P\rightarrow Q \neq Q\rightarrow P$.

For any statement formula $P\rightarrow Q$, the statement formula

- ✓ "$Q\rightarrow P$" is called converse,
- ✓ "$\neg P\rightarrow\neg Q$" is called inverse and
- ✓ "$\neg Q\rightarrow\neg P$" is called its contrapositive.

				Conditional	Converse	Inverse	Contrapositive
P	**Q**	$\neg P$	$\neg Q$	$P\rightarrow Q$	$Q\rightarrow P$	$\neg P\rightarrow\neg Q$	$\neg Q\rightarrow\neg P$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

A statement A is said to be ***tautologically imply*** a statement B iff $A \to B$ is a tautology. We shall denote this as $A \Rightarrow B$ and read as 'A implies B'.

Implications:

$$P \wedge Q \Rightarrow P \qquad (1)$$
$$P \wedge Q \Rightarrow Q \qquad (2)$$
$$P \Rightarrow P \vee Q \qquad (3)$$
$$\neg P \Rightarrow P \to Q \qquad (4)$$
$$Q \Rightarrow P \to Q \qquad (5)$$
$$\neg(P \to Q) \Rightarrow \neg Q \qquad (6)$$
$$P \wedge (P \to Q) \Rightarrow Q \qquad (7)$$
$$\neg P \wedge (P \vee Q) \Rightarrow Q \qquad (8)$$
$$(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R) \qquad (9)$$
$$(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R \qquad (10)$$
$$\neg(P \to Q) \Rightarrow P \qquad (11)$$
$$\neg Q \wedge (P \to Q) \Rightarrow \neg P \qquad (12)$$

Both implication and equivalence are transitive. To say that equivalence is transitive means if $A \Leftrightarrow B$ and $B \Leftrightarrow C$ then $A \Leftrightarrow C$. To say that implication is transitive means if $A \Rightarrow B$ and $B \Rightarrow C$ then $A \Rightarrow C$.

Eg:- *Show that implication is transitive.*

sol:- Let $A \Rightarrow B$ and $B \Rightarrow C$ i.e, $A \to B$ and $B \to C$ are tautologies. Hence ($A \to B$) $\wedge$ ($B \to C$) is also a tautology. But from (9) ($A \to B$) $\wedge$ ($B \to C$) $\Rightarrow (A \to C)$. Hence $A \to C$ is also a tautology. That means $A \Rightarrow C$.

Note:- $P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

$P \to Q \Leftrightarrow \neg P \vee Q$

Eg 1: Write an equivalent formula for $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$ *which does not contain the biconditional.*

sol:- $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$

$\Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q)) \vee ((R \to P) \wedge (P \to R))$

Eg 2: Write an equivalent formula for $P \wedge (Q \leftrightarrow R)$ which contains neither the biconditional nor the conditional.

sol:-

$$P \wedge (Q \leftrightarrow R)$$
$$\Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q))$$
$$\Leftrightarrow P \wedge ((\neg Q \vee R) \wedge (\neg R \vee Q))$$
$$\Leftrightarrow P \wedge (\neg Q \vee R) \wedge (\neg R \vee Q)$$

Normal Forms:

Let A (P_1,P_2,....P_n) be a statement formula where P_1P_2,....P_n are the atomic variables. If A has the truth value T for atleast one combination of truth values assigned to P_1,P_2,....P_n, the A is said to be satisfiable.

The problem of determining in a finite number of steps, whether a given statement formula is a tautology or a contradiction or atleast satisfiable is known as ***decision problem***.

Disjunctive Normal Forms (DNF's):-

We use the word "**product**" in place of **"conjunction**" and "**sum**" in place of "**disjunction**".

A product of the variables and their negations in a formula is called an ***elementary product***. Similarly, a sum of the variables and their negations is called an ***elementary sum.***

Let P and Q be any two atomic variables. Then P, $P \wedge \neg P \wedge Q$, $\neg Q \wedge \neg P \wedge P$, $P \wedge \neg P$ are some examples of elementary products. On the other hand, P, $\neg P \vee \neg Q$, $P \vee Q \vee \neg P$ etc are examples of elementary sums.

A formula which is equivalent to a given formula and which consists of **sum of elementary products** is called a **distinjunctive normal form** of the given formula.

Some examples:

- $(P \wedge Q \wedge \neg R \wedge S) \vee (\neg Q \wedge S) \vee (P \wedge S)$ is in disjunctive normal form.
- $(P \vee Q \vee \neg R \vee S) \wedge (\neg Q \vee S) \wedge \neg S$ is in conjunctive normal form.
- $(P \vee R) \wedge (Q \wedge (P \vee \neg Q))$ is not in a normal form.
- $\neg P \vee Q \vee R$ and $\neg P \wedge Q \wedge R$ are in both normal forms.

Eg 1: Obtain DNF for $(P \to Q) \to (\neg R \wedge Q)$

sol:-$(P \to Q) \to (\neg R \wedge Q)$

$(P \to Q) \to (\neg R \wedge Q)$	$\Leftrightarrow \neg (P \to Q) \vee (\neg R \wedge Q)$	$[P \to Q \Leftrightarrow \neg P \vee Q]$
	$\Leftrightarrow \neg (\neg P \vee Q) \vee (\neg R \wedge Q)$	$[P \to Q \Leftrightarrow \neg P \vee Q]$
	$\Leftrightarrow (P \wedge \neg Q) \vee (\neg R \wedge Q)$	[De Morgan's Law]

Eg 2: Obtain DNF for $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

sol:-

$P \to ((P \to Q) \wedge \neg (\neg Q \vee \neg P))$	
$\Leftrightarrow \neg P \vee [(P \to Q) \wedge \neg (\neg Q \vee \neg P)]$	$[P \to Q \Leftrightarrow \neg P \vee Q]$
$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P)]$	$[P \to Q \Leftrightarrow \neg P \vee Q]$
$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg(Q \wedge P))]$	[De Morgan's Law]
$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$	[Double Negation Law]
$\Leftrightarrow \neg P \vee [(\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))]$	[Distributive Law]
$\Leftrightarrow \neg P \vee [(\neg P \wedge P) \wedge Q)) \vee (Q \wedge Q) \wedge P)]$	[Associative Law]
$\Leftrightarrow \neg P \vee [(F \wedge Q) \vee (Q \wedge Q) \wedge P)]$	[Negation Law]
$\Leftrightarrow \neg P \vee [(F \wedge Q)) \vee (Q \wedge P)]$	[Idempotent Law]
$\Leftrightarrow \neg P \vee (F \vee (Q \wedge P))$	[Domination Law]
$\Leftrightarrow \neg P \vee (Q \wedge P)$	[Identity Law]

Conjunctive Normal Forms:- A formula which is equivalent to a given formula and which consists of **product of elementary sums** is called a **conjunctive normal** form of the given formula.

Eg 1: Obtain CNF for $\neg (\neg P \wedge (Q \vee \neg(R \wedge S)))$

sol:-

$\neg (\neg P \wedge (Q \vee \neg(R \wedge S)))$	$\Leftrightarrow \neg(\neg P) \vee \neg (Q \vee \neg (R \wedge S))$	[DeMorgan's Law]
	$\Leftrightarrow P \vee \neg (Q \vee \neg(R \wedge S))$	[Double Negation Law]
	$\Leftrightarrow P \vee (\neg Q \wedge \neg\neg (R \wedge S))$	[DeMorgan's Law]
	$\Leftrightarrow P \vee (\neg Q \wedge (R \wedge S))$	[Double Negation Law]
	$\Leftrightarrow (P \vee \neg Q) \wedge (P \vee R) \wedge (P \vee S)$	[Distributive Law]

Eg 2: Obtain CNF for $\neg((\neg P \to \neg Q) \wedge \neg R)$

sol:-

$$
\begin{aligned}
\neg((\neg P \to \neg Q) \wedge \neg R) &\Leftrightarrow \neg((\neg\neg P \vee \neg Q) \wedge \neg R) && [P \to Q \Leftrightarrow \neg P \vee Q] \\
&\Leftrightarrow \neg((P \vee \neg Q) \wedge \neg R) && \text{[Double negation Law]} \\
&\Leftrightarrow \neg(P \vee \neg Q) \vee \neg\neg R && \text{[DeMorgan's Law]} \\
&\Leftrightarrow \neg(P \vee \neg Q) \vee R && \text{[Double negation Law]} \\
&\Leftrightarrow (\neg P \wedge \neg\neg Q) \vee R && \text{[DeMorgan's Law]} \\
&\Leftrightarrow (\neg P \wedge Q) \vee R && \text{[Double negation Law]} \\
&\Leftrightarrow (\neg P \vee R) \wedge (Q \vee R) && \text{[Distributive Law]}
\end{aligned}
$$

Eg 3: Obtain CNF for $P \to (P \to Q) \wedge \neg(\neg Q \vee \neg P)$

sol:-

$$
\begin{aligned}
& P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)] \\
&\Leftrightarrow \neg P \vee [(P \to Q) \wedge \neg(\neg Q \vee \neg P)] && [P \to Q \Leftrightarrow \neg P \vee Q] \\
&\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg(Q \wedge P))] && [P \to Q \Leftrightarrow \neg P \vee Q] \\
&\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)] && \text{[Double negation Law]} \\
&\Leftrightarrow \neg P \vee [(\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P)] && \text{[Distributive Law]} \\
&\Leftrightarrow \neg P \vee [(\neg P \wedge Q \wedge P) \vee (Q \wedge P)] && \text{[Idempotent Law]} \\
&\Leftrightarrow \neg P \vee [(\neg P \wedge P \wedge Q) \vee (Q \wedge P)] && \text{[Commutative Law]} \\
&\Leftrightarrow \neg P \vee [(F \wedge Q) \vee (Q \wedge P)] && \text{[Negation Law]} \\
&\Leftrightarrow \neg P \vee (F \vee (Q \wedge P)) && \text{[Domination Law]} \\
&\Leftrightarrow \neg P \vee (F \vee Q) \wedge (F \vee P) && \text{[Distributive Law]} \\
&\Leftrightarrow \neg P \vee (Q \wedge P) && \text{[Identity Law]} \\
&\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee P) && \text{[Distributive Law]} \\
&\Leftrightarrow (\neg P \vee Q) \wedge T && \text{[Negation Law]} \\
&\Leftrightarrow (\neg P \vee Q) && \text{[Identity Law]}
\end{aligned}
$$

Principal Disjunctive Normal Forms:- Let P and Q be two statement variables. Let us construct all possible formulas which consists of conjunctions of P or its negations and conjunctions of Q or its negations. None of the formulas should contain both a variable and its negation.

- → For any two variables P and Q, $P \vee Q, P \vee \neg Q, \neg P \vee Q and \neg P \vee \neg Q$ are called ***minterms*** or **boolean conjunctions**.
- → For three variables P , Q and R, the min terms are $P \vee Q \vee R, P \vee Q \vee \neg R, P \vee \neg Q \vee R, \neg P \vee Q \vee R, \neg P \vee \neg Q \vee \neg R, \neg P \vee Q \vee \neg R,$ $P \vee \neg Q \vee \neg R and \neg P \vee \neg Q \vee \neg R$

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

From the truth table of minterms it is clear that no two minterms are equivalent. Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

For every truth value T in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of P and Q. The disjunction of these minterms will be equivalent to the given formula.

For a given formula, an equivalent formula consisting of ***"disjunctions of minterms only"*** is known as its **Principal Disjunctive Normal Form.** Such a form is also called the ***"sum of products cononical form".***

Constructing PDNF's using truth tables

Example:- The truth tables for $P \to Q, P \vee Q$ and $\neg(P \wedge Q)$ is given below. Obtain PDNF of these formulas.

P	Q	$P \to Q$	$P \vee Q$	$\neg(P \wedge Q)$
T	T	T	T	F
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

Sol:-

$$P \to Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$
$$P \vee Q \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$
$$\neg(P \wedge Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Example: Obtain PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

			A		B	C		
P	Q	R	$P \wedge Q$	$\neg P$	$\neg P \wedge R$	$Q \wedge R$	$A \vee B$	$A \vee B \vee C$
T	T	T	T	F	F	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	F	F	F	F	F	F
T	F	F	F	F	F	F	F	F
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	F	F	F
F	F	T	F	T	T	F	T	T
F	F	F	F	T	F	F	F	F

$$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$$

Constructing PDNF's without using truth tables:-

- *Step 1:* Replace conditionals and biconditionals by their equivalent formulas containing only $\vee, \wedge$ and $\neg$.
- *Step 2*: Negations are applied to the variables using De Morgan's laws.
- *Step 3:* Apply distributive laws.
- *Step 4:* Drop contradictory elementary products.
- *Step 5:* Minterms are obtained in disjunctions by introducing the missing factors.
- *Step 6:* Finally delete identical minterms.

Eg 1: Obtain PDNF of $P \to Q$

$$P \to Q \Leftrightarrow \neg P \vee Q$$

$$\Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T)$$
$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$
$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

Eg 2: Obtain PDNF of $P \vee Q$

$$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T)$$
$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$
$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

Principal Conjunctive Normal Forms:- For a given formula, an equivalent formula consisting of ***"conjunctions of maxterms only"*** is known as its **Principal Conjunctive Normal Form**. Such a form is also called the ***"products-of-sums cononical form".*** **Maxterms** are called "**boolean disjunctions**".

- → For any two variables P and Q, $P \vee Q, P \vee \neg Q, \neg P \vee Q and \neg P \vee \neg Q$ are called min terms.
- → For three variables P , Q and R, the min terms are $P \vee Q \vee R, P \vee Q \vee \neg R, P \vee \neg Q \vee R, \neg P \vee Q \vee R, \neg P \vee \neg Q \vee \neg R, \neg P \vee Q \vee \neg R,$ $P \vee \neg Q \vee \neg R and \neg P \vee \neg Q \vee \neg R$

Eg 1: Obtain PCNF of

I. $\neg(P \vee Q)$
II. $\neg(P \rightarrow Q)$
III. $\neg(P \leftrightarrow Q)$

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$P \rightarrow Q$	$\neg(P \rightarrow Q)$	$P \leftrightarrow Q$	$\neg(P \leftrightarrow Q)$
T	T	T	F	T	F	T	F
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	T	T	T	F	T	F

I. $\neg(P \vee Q) \Leftrightarrow (\neg P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)$
II. $\neg(P \rightarrow Q) \Leftrightarrow (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$
III. $\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \vee \neg Q) \wedge (P \vee Q)$

PCNF without using truth tables:-

Eg 1:- Obtain PCNF of $\neg(P \vee Q)$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$
$$\Leftrightarrow (\neg P \vee F) \wedge (\neg Q \vee F)$$
$$\Leftrightarrow [\neg P \vee (Q \wedge \neg Q)] \wedge [\neg Q \vee (P \wedge \neg P)]$$
$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$$
$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$$

Eg 2:- Obtain PCNF of $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P) \Leftrightarrow (P \vee R) \wedge ((Q \rightarrow P) \wedge (P \rightarrow Q))$

$$\Leftrightarrow (P \vee R) \wedge ((\neg Q \vee P) \wedge (\neg P \vee Q))$$
$$\Leftrightarrow (P \vee R \vee F) \wedge (\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)$$
$$\Leftrightarrow (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \wedge \neg R)) \wedge (\neg P \vee Q \vee (R \wedge \neg R))$$
$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$
$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

<u>Converting one P-normal form to other:-</u>

If we are given a PCNF, we can obtain its equivalent PDNF and viceversa.

Eg 1: Obtain PCNF of $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

Given one is a PDNF. Let it be S

$$S \Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$
$$\neg S \Leftrightarrow (P \wedge \neg Q) \qquad (\because \text{Remaining minterms of PDNF})$$
$$\neg(\neg S) \Leftrightarrow \neg(P \wedge \neg Q)$$
$$\Leftrightarrow \neg P \vee Q \qquad \text{which is the required PCNF.}$$

Ordering and Uniqueness of Normal Forms:- Given any 'n' statement variables, we first arrange them in fixed order.

1. If capital letters are used to denote variables, then they may be arranged in alphabetical order.
   Eg:- P, A, Q, S
   sol: A, P, Q, S
2. If subscripted letters are also used, then the following is an illustration of the order that may be used: A,B,....., x, A_1,B_1,.......Z_2,.....C_3.....
   Eg:- P_1,Q,R_3,S_1,T_2,Q_3
   sol:- Q,P_1,S_1,T_2,Q_3,R_3

Minterms:- If two variables P, Q are given then we can have 2^2=4 minterms (0 to 3).

0 – 0 0	$\neg P \wedge \neg Q$ - m_0	If there are 3 variables then we have 2^3 = 8 minterms (0 to 7)
1 – 0 1	$\neg P \wedge Q$ - m_1	
2 – 1 0	$P \wedge \neg Q$ - m_2	
3 – 1 1	$P \wedge Q$ - m_3	

0 – 0 0 0	5 – 1 0 1	$(\neg P \wedge \neg Q \wedge \neg R)$ -- m_0
1 – 0 0 1	6 – 1 1 0	..
2 – 0 1 0	7 – 1 1 1	..
3 – 0 1 1		$(P \wedge Q \wedge R)$ -- m_7
4 – 1 0 0		

Maxterms:- If two variables P, Q are given then we can have 2^2=4 maxterms (0 to 3).

0 – 0 0	3 – 1 1	$\neg P \vee Q$ - m_2
1 – 0 1	$P \vee Q$ - m_0	$\neg P \vee \neg Q$ - m_3
2 – 1 0	$P \vee \neg Q$ - m_1	

If there are 3 variables then we have $2^3 = 8$ minterms (0 to 7)

- Sum of products or sum of minterms can be designated by the notation Σ and product of sums or product of maxterms can be designated by the notation Π.

  Eg:- $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

  $\Rightarrow \Sigma(m_3, m_1, m_0)$

  $\Rightarrow \Sigma(m_0, m_1, m_3)$

  Eg:- $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$

  $\Rightarrow \Pi(M_2, M_3, M_1)$

  $\Rightarrow \Pi(M_1, M_2, M_3)$

The Theory of inference for statement calculus: The main function of logic is to provide rules of inference or principles of reasoning. The theory associated with such rules is known as "*Inference theory*" because it is concerned with the inference of a conclusion from certain premises. When a conclusion is derived from certain premises by using the accepted rules of reasoning, the process of derivation is called a *deduction or formal proof*.

In any argument, a conclusion is admitted to be true provided the premises are accepted as true and the reasoning used in deriving the conclusion from the premises follows certain accepted rules of logic inference. Such an argument is called a valid argument. Any conclusion that is arrived at by following these rules is called a *valid conclusion* and the argument is called a *valid argument.*

Validity using truth table: Let A and B be two formulas. We say that "B logically follows from A" or "B is a valid conclusion of the premise A" iff $A \rightarrow B$ is a tautology i.e, $A \Rightarrow B$.

Eg 1: Determine whether the conclusion C follows logically from the premises H_1 and H_2.

a) $H_1 : P \rightarrow Q,\ H_2 : P,\quad C : Q$
b) $H_1 : P \rightarrow Q,\ H_2 : \neg P,\quad C : Q$
c) $H_1 : P \rightarrow Q,\ H_2 : \neg(P \wedge Q),\quad C : Q$
d) $H_1 : \neg P,\ H_2 : P \leftrightarrow Q,\quad C : \neg(P \wedge Q)$
e) $H_1 : P \rightarrow Q,\ H_2 : Q,\quad C : P$

Sol:-

a)

P (H_2)	Q (C)	$P \rightarrow Q$ (H_1)
T	T	T
T	F	F
F	T	T
F	F	T

C is a valid conclusion of H_1 and H_2.

b)

P	$\neg$P (H_2)	Q (C)	$P \rightarrow Q$ (H_1)
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

C is not a valid conclusion from H_1 and H_2.
C is true only in 3rd row but not 4th.

c)

P	$\neg$P (C)	Q	$\neg(P \wedge Q)$ (H₂)	$P \rightarrow Q$ (H₁)
T	F	T	F	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	T

C is a valid conclusion of H_1 and H_2.

d)

P	$\neg$P (H_1)	Q	$P \leftrightarrow Q$ (H_2)	$\neg(P \wedge Q)$ (C)
T	F	T	T	F
T	F	F	F	T
F	T	T	F	T
F	T	F	T	T

C is a valid conclusion of H_1 and H_2.

e)

P (C)	Q (H_2)	$P \rightarrow Q$ (H_1)
T	T	T
T	F	F
F	T	T
F	F	T

C is not a valid conclusion of H_1 and H_2.

Rules of inference:-

There are two rules of inference, ‘P’ and ‘T’ through which we demonstrate that a particular formula is a valid consequence of a given set of premises.

Rule P : A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tatutologically implied by any one or more of the preceding formulas in the derivation.

Before we proceed, we list some important implications and equivalences that will be referred to frequently.

<u>Implications:-</u>

1) $P, P \rightarrow Q \Rightarrow Q$ (Modus ponens)
2) $P, Q \Rightarrow P \wedge Q$
3) $\neg P, P \vee Q \Rightarrow Q$
4) $\neg Q, P \rightarrow Q \Rightarrow \neg P$ (Modus tollens)
5) $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (Hypothetical Syllogism)
6) $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (Dilemma)

<u>Equivalences:-</u>

1) $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
2) $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$
3) $\neg(P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \neg Q$
4) $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \vee (Q \rightarrow P)$
5) $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Eg 1: Show that R is a valid inference from the premises $P \to Q, Q \to R$ and P.

sol: {1}	(1)	$P \to Q$	Rule P
{2}	(2)	P	Rule P
{1,2}	(3)	Q	Rule T, (1), (2), $P,\ P \to Q \Rightarrow Q$
{4}	(4)	$Q \to R$	Rule P
{1,2,4}	(5)	R	Rule T, (3), (4), $P,\ P \to Q \Rightarrow Q$

Eg 2: Show that $R \vee S$ *follows logically from the premises* $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$, *and* $(A \wedge \neg B) \to (R \vee S)$.

sol: {1}	(1)	$(C \vee D) \to \neg H$	Rule P
{2}	(2)	$\neg H \to (A \wedge \neg B)$	Rule P
{1,2}	(3)	$(C \vee D) \to (A \wedge \neg B)$	Rule T, (1), (2), $(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$
{4}	(4)	$(A \wedge \neg B) \to (R \vee S)$	Rule P
{1,2,4}	(5)	$(C \vee D) \to (R \vee S)$	Rule T, (3), (4), $(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$
{6}	(6)	$C \vee D$	Rule P
{1,2,4,6}	(7)	$R \vee S$	Rule T, (5), (6), $P, P \to Q \Rightarrow Q$

Eg 3: Show that $S \vee R$ *is a tautologically implied by* $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

sol: {1}	(1)	$P \vee Q$	Rule P
{1}	(2)	$\neg P \to Q$	Rule T, (1), $P \vee Q \Leftrightarrow \neg P \to Q$
{3}	(3)	$Q \to S$	Rule P
{1,3}	(4)	$\neg P \to S$	Rule T, (2), (3), $P \to Q\ \&\ Q \to R \Rightarrow P \to R$
{1,3}	(5)	$\neg S \to P$	Rule T, (4), $P \to Q \Leftrightarrow \neg Q \to \neg P$
{6}	(6)	$P \to R$	Rule P
{1,3,6}	(7)	$\neg S \to R$	Rule T, (5), (6), $P \to Q\ \&\ Q \to R \Rightarrow P \to R$
{1,3,6}	(8)	$S \vee R$	Rule T, (7), $P \to Q \Leftrightarrow \neg P \vee Q$

Eg 4: Show that $R \wedge (P \vee Q)$ *is a valid inference from the premises* $P \vee Q, Q \to R$ and $P \to M, \neg M$.

sol: {1}	(1)	$P \to M$	Rule P
{2}	(2)	$\neg M$	Rule P
{1,2}	(3)	$\neg P$	Rule T, (1), (2), $\neg Q, P \to Q \Rightarrow \neg P$
{4}	(4)	$P \vee Q$	Rule P
{1,2,4}	(5)	Q	Rule T, (3),(4), $\neg P, P \vee Q \Rightarrow Q$
{6}	(6)	$Q \to R$	Rule P
{1,2,4,6}	(7)	R	Rule T, (5), (6), $P, P \to Q \Rightarrow Q$
{1,2,4,6}	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7), $P, Q \Rightarrow P \wedge Q$

Eg 5: Show that $\neg Q,\ P \to Q \Rightarrow \neg P$.

sol: {1}	(1)	$P \to Q$	Rule P
{1}	(2)	$\neg Q \to \neg P$	Rule T, (1), $P \to Q \Leftrightarrow \neg Q \to \neg P$
{3}	(3)	$\neg Q$	Rule P
{1,3}	(4)	$\neg P$	Rule T, (2), (3), $P, P \to Q \Rightarrow Q$

Rule CP (or) Rule of conditional proof:- If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

- Rule CP is also called as deduction theorem and is generally used if the conclusion is of form $R \to S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Eg 1: Show that $R \to S$ can be derived from the premise $P \to (Q \to S), \neg R \vee P$, and Q.

sol: {1}	(1)	$\neg R \vee P$	Rule P
{2}	(2)	R	Rule P(Assumed premise)
{1,2}	(3)	P	Rule T, (1), (2), $\neg P, P \vee Q \Rightarrow Q$
{4}	(4)	$P \to (Q \to S)$	Rule P
{1,2,4}	(5)	$Q \to S$	Rule T, (3),(4), $P, P \to Q \Rightarrow Q$
{6}	(6)	Q	Rule P
{1,2,4,6}	(7)	S	Rule T, (5), (6), $P, P \to Q \Rightarrow Q$
{1,2,4,6}	(8)	$R \to S$	Rule CP

Consistency of premises and Indirect Method of Proof:-

Consistency of premises:- A set of formulas $H_1, H_2, \ldots H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \ldots H_m$.

A set of formulas $H_1, H_2, \ldots H_m$ is inconsistent if their conjunction implies a contradication i.e, $[H_1 \wedge H_2 \wedge \ldots H_m \Rightarrow R \wedge \neg R]$ where R is any formula.

Eg 1:- Show that the premises are $a \to (b \to c), d \to (b \wedge \neg c), a \wedge d$ *inconsistent*

Sol:- {1}	(1)	$A \wedge D$	Rule P
{1}	(2)	A	Rule T, (1) $P \wedge Q \Rightarrow P$
{1}	(3)	D	Rule T, (1) $P \wedge Q \Rightarrow Q$
{4}	(4)	$A \to (B \to C)$	Rule P
{1,4}	(5)	$B \to C$	Rule T, (2),(4) $P, P \to Q \Rightarrow Q$
{1,4}	(6)	$\neg B \vee C$	Rule T, (5) $P \to Q \Leftrightarrow \neg P \vee Q$
{7}	(7)	$D \to (B \wedge \neg C)$	Rule P
{7}	(8)	$\neg(B \wedge \neg C) \to \neg D$	Rule T, (7) $P \to Q \Leftrightarrow \neg Q \to \neg P$
{7}	(9)	$\neg B \vee C \to \neg D$	Rule T, (8) Demorgan's law
{1,4,7}	(10)	$\neg D$	Rule T, (6),(9) Modus ponens
{1,4,7}	(11)	$D \wedge \neg D$	Rule T, (3),(10) $P, Q \Rightarrow P \wedge Q$

Eg 2:- Show that the following premises are inconsistent

1. *If Jack misses many classes because of illness, then he fails high school*
2. *If Jack fails high school, then he is uneducated.*
3. *If Jack reads a lot of books, then he is not uneducated.*
4. *Jack misses many classes because of illness and reads a lot of books.*

Sol:- Let, P: jack misses many classes because of illness
Q: Jack fails high school.
R: Jack is uneducated
S: Jack reads a lot of books

Therefore the premises are:

$$P \to Q, Q \to R, S \to \neg R, P \wedge S$$

{1}	(1)	$P \to Q$	Rule P
{2}	(2)	$Q \to R$	Rule P

{1,2}	(3)	$P \to R$	Rule T, (1) (2) $P \to Q$ & $Q \to R \Rightarrow P \to R$
{4}	(4)	$S \to \neg R$	Rule P
{4}	(5)	$R \to \neg S$	Rule T, (4) $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1,2,4}	(6)	$P \to \neg S$	Rule T,(3),(5)

$(P \to Q) \wedge (Q \to R) \Leftrightarrow P \to R$

{1,2,4}	(7)	$\neg P \vee \neg S$	Rule T, (6) $P \to Q \Leftrightarrow \neg P \vee Q$
{1,2,4}	(8)	$\neg(P \wedge S)$	Rule T, (6) Demorgan's law
{9}	(9)	$P \wedge S$	Rule P
{1,2,4,9}	(10)	$(P \wedge S) \wedge \neg(P \wedge S)$	Rule T, (8),(9) $P, Q \Rightarrow P \wedge Q$

$(P \wedge S) \wedge \neg(P \wedge S) \Leftrightarrow F$

$\therefore$ The given premises are inconsistent

<u>Indirect method of Proof :</u>

To show that a conclusion 'C' follows logically from the premises $H_1, H_2, ----H_m$, we assume an additional premise $\neg C$.

If the new set of premises is inconsistent then they imply a contradiction. That shows $\neg C$ is false i.e C is true. Therefore C follows logically from the premises $H_1, H_2, ---- H_m$.

Eg 1:- Show that $\neg(P \wedge Q)$ *follows from* $\neg P \wedge \neg Q$.

Sol:-	{1}	(1)	$\neg\neg(P \wedge Q)$	Rule P (Assumed premise)
	{1}	(2)	$(P \wedge Q)$	Rule T, (1) $\neg(\neg P) \Rightarrow P$
	{1}	(3)	P	Rule T, (2), $P \wedge Q \Rightarrow P$
	{4}	(4)	$\neg P \wedge \neg Q$	Rule P
	{4}	(5)	$\neg P$	Rule T, (4) $P \wedge Q \Rightarrow P$
	{1,4}	(6)	$P \wedge \neg P$	Rule T,(3) (5) $P, Q \Leftrightarrow P \wedge Q$

$P \wedge \neg P \Leftrightarrow F$ i.e, our assumption is wrong.

$\therefore \neg(P \wedge Q)$ *follows from* $\neg P \wedge \neg Q$.

Eg 2:- Using indirect method, Show that $P \to Q, Q \to R, \neg(P \wedge R), P \vee R \Leftrightarrow R$

Sol:-	{1}	(1)	$\neg R$	Rule P (Assumed premise)
	{2}	(2)	$Q \to R$	Rule P
	{1,2}	(3)	$\neg Q$	Rule T,(1),(2), $P \to Q, \neg Q \Rightarrow \neg P$
	{4}	(4)	$P \to Q$	Rule P
	{1,2,4}	(5)	$\neg P$	Rule T,(3),(4) $P \to Q, \neg Q \Rightarrow \neg P$
	{6}	(6)	$P \vee R$	Rule P
	{6}	(7)	$\neg P \to R$	Rule T,(6) $\neg P \to Q \Leftrightarrow P \vee Q$
	{1,2,4,6}	(8)	R	Rule T,(5) (7) Modus ponens
	{1,2,4,6}	(9)	$R \wedge \neg R$	Rule T,(1) (8) $P, Q \Leftrightarrow P \wedge Q$

$R \wedge \neg R \Leftrightarrow F$ i.e, our assumption is wrong.

$\therefore$ The given conclusion follows from the premises.

PREDICATE CALCULUS

Predicate calculus:

By using propositional logic, it was not possible to express the fact that any atomic statements have some features in common. In order to investigate questions of this nature, we introduce the concept of a "predicate" in an atomic statement. The logic based on the analysis of predicates in any statement is called predicate logic.

Predicate: A part of declarative sentence describing the properties of a subject or relation among subjects is called predicate.

Ex: Ram is a bachelor -> predicate
Shyam is a bachelor

Radha
Shyam } Subject of the statement

" is a bachelor " -> refers to a property that the subject can have, is called the predicate.

- Predicate is denoted by capital letters.
- Subject is denoted by small letters.

From the example:

Radha is a girl
Seeta is a girl

" is a girl" is a predicate, let denote by "G".
" Radha " is denoted by 'r'.
" Seeta " is denoted by 's'.

$\therefore$ Radha is a girl $\Rightarrow$ G(r)
$\therefore$ Seeta is a girl $\Rightarrow$ G(s)

Types of predicates:

- 1 place predicate
- 2 place predicate
- 3 place predicate
- m place predicate

1 place predicate: If there is one name associated with a predicate then it is called a 1 place predicate.
Ex: Ravi is a teacher $\Rightarrow$T(s)

2 place predicate: If two names are associated with a predicate then it is called a 2 place predicate.
Ex: Ravi is senior than ramesh $\Rightarrow$ S(r,m)

3 place predicate: If three names are associated with a predicate then it is called a 3 place predicate.
Ex: Ravi sits between ram and ramesh $\Rightarrow$ S(r,a,h)
m place predicate: If 'm' names are associated with a predicate then it is called m place predicate.
Ex:Ram sits among ravi, raju, rahul.. $\Rightarrow$ S(a,b,c,d…………….)

Statement functions:

Consider the following statements:

Somu is mortal

India is mortal
A table is mortal

Let H be the predicate '*is mortal*', s be the name *somu*, i be the name '*india*' and t be the *table*. Then, H(s), H(i), and H(t) denote the above statements. If we write H(x) for "x is mortal", then H(s), H(i) and H(t) can be obtained form H(x) by replacing x by an appropriate name.

Note:

- → H(x) is not a statement, but when x is replaced by the name of a subject, it becomes a statement.
- → A simple statement function of one variable is defined as an expression consisting of a predicate symbol and an individual variable.
- → A statement function becomes a statement when the variable is replaced by the name of any subject.

Example 1: *Let P(x) denote the statement 'x > 3'. What are the truth values of P(4) and P(2)?*

- The statement P(4) can be obtained by setting x = 4 in the statement x > 3. Hence, P(4) is the statement '4 > 3', which is true.
- The statement P(2) can be obtained by setting x = 2 in the statement x > 3. Hence, P(2) is the statement '2 > 3', which is false.

Example 2: *Let Q(x) denote the statement 'x = y+3'. What are the truth values of Q(1,2) and Q(3,0) ?*

- The statement *Q(1,2)* can be obtained by setting x = 1 and y = 2 in the statement '*x = y+3*'. Hence, *Q(1,2)* is the statement '1 = 2 + 3', which is false.
- The statement *Q(3,0)* can be obtained by setting x = 3 and y = 0 in the statement '*x = y+3*'. Hence, *Q(3,0)* is the statement '3 = 0 + 3', which is true.

Quantifiers:

There are two types of quantifiers

1. Universal quantifier
2. Existential quantifier

1. Universal quantifier:

Consider the following statements

- ✓ All men are mortal
- ✓ Every apple is red

The statements can be written as

- ➢ For all x, if x is a man, then x is mortal
- ➢ For all x, if x is an apple, then x is red.

These statements can be symbolized as

$$(x)\quad (M(x) \to H(x));$$
$$(x)\quad (A(x) \to R(x));$$

$M(x)$: x is a man;	$A(x)$: x is an apple;
$H(x)$: x is mortal;	$R(x)$: x is red;

✦ We symbolize "for all x" by the symbol $(\forall x)$ or (x).It is called universal quantifier.

- The notation $(\forall x)P(x)$ denotes the universal quantification of $P(x)$.

2.Existential quantifier:

Consider the following statements.

- There exists a man
- Some men are clever
- Some real numbers are rational

The statements can be written as

- ✓ There exists an x such that x is a man
- ✓ There exists an x such that x is a man and x is clever
- ✓ There exists an x such that x is a real number and x is a rational number

These statements can be symbolized as

$(\exists x) \quad M(x)$;

$(\exists x) \quad (M(x) \wedge C(x))$;

$(\exists x) \quad (R_1(x)) \vee R_2(x))$;

$M(x)$: x is a man;

$C(x)$: x is clever;

$R_1(x)$: x is a real number;

$R_2(x)$: x is a rational number;

- We symbolize "there is at least one x such that" or "there exists an x such that" or " for some x" by the symbol $(\exists x)$.It is called universal quantifier.
- The notation $(\exists x)P(x)$ denotes the existential quantification of $P(x)$.

Example:- ***What is the truth value of the following quantifications?***

1. *$(\forall x)P(x)$, where $P(x)$ is the statement 'x < 2' and the universe discourse consists of all real numbers?*

   sol:- $P(x)$ *= x < 2*

   $P(x)$ is not true for every real number x, since P(3) is false.

   $\therefore (\forall x)P(x)$ is false.

2. *$(\forall x)P(x)$, where $P(x)$ is the statement '$x^2 < 10$' and the universe discourse consists of the positive integers not exceeding 4?*

   sol:- The statement $(\forall x)P(x)$ is $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, since the universe of discourse is {1,2,3,4}

   P(4) is the statement $4^2 < 10$, which is not true.

   $\therefore (\forall x)P(x)$ is false.

3. *$(\exists x)P(x)$ where $P(x)$ is the statement 'x > 3' and the universe discourse consists of all real numbers?*

   Sol:- $P(x)$ *= x > 3*

   when x =4 , 4 > 3 which is true

   $\therefore (\exists x)P(x)$ is true.

4. $(\exists x)P(x)$ *where* $P(x)$ *is the statement 'x = x+1' and the universe discourse is the set of all real numbers?*
Sol:- $P(x)$ *:* x = x + 1
Since $P(x)$ is false for every real number x, the existential quantification of $P(x)$ *is false.* $(\exists x)P(x)$ is false.

Example:

S(x): x is a student; I(x): x is intelligent; M(x): x likes music
Write the following statements in symbolic form.

1. All students are intelligent.
   For all x if x is a student then x is intelligent
   $(\forall x)$ [S(x) $\rightarrow$ I(x)]
2. Some intelligent students like music
   There exists an x such that x is a student and x is intelligent and x likes music
   (∃x)[S(x) $\wedge$ I(x) $\wedge$ M(x)]
3. Everyone who likes music is a stupid student
   For all x, if x likes music then x is a student and x is not intelligent(stupid)
   $(\forall x)$ [M(x) $\rightarrow$ S(x) $\wedge$ I′(x)]

Universe of discourse:-

The process of symbolizing a statement in predicate calculus, which is quite complicated, can be simplified by limiting the class of individuals or objects under consideration. This restricted class is called universe of discourse or simply the universe. If the discussion refers to human beings only, then the universe of discourse is the class of human beings.
Definition: The collection of values that a variable x can take is called x's universe of discourse.
Example: Symbolize the statement " All men are giants".
Solution : Using *G(x): x* is a giant.
M(x): x is a man.
The given statement can be symbolized as *(x) (M (x)* $\rightarrow$ *G(x)).* However, if we restrict the variable *x* to the universe which is the class of men, then the statement is *(x) G(x)*
The universe of discourse, if any, must be explicitly stated, because the truth value of a statement depends upon it.
For instance, consider the predicate *Q(x): x* is less than *5* and the statements *(x) Q(x)* and *(∃ x) Q(x).* If the universe of discourse is given by the set

1. {-1, 0, 1, 2, 4}
2. {3, –2, 7, 8, -2}
3. { 15, 20, 24} then *(x)Q(x)* is true for the universe of discourse *(1)* and false for *(2)* and *(3).* The statement *(∃ x)Q(x)* is true for both *(1)* and *(2),* but false for *(3).*

Free and Bound variables

Definition: – An expression like P (x) is said to have a free variable x (meaning, x is undefined). A quantifier (either ∀ or ∃) operates on an expression having one or more free variables, and binds one or more of those variables, to produce an expression having one or more bound variables.
Examples:
Eg1. ∃ x [x + y = z], x is bound but y and z are free variables.
Eg2: P (x,y) has 2 free variables, x and y.

Eg3: ∀x P (x , y) has 1 free variable y, and one bound variable x.
" P (x), where x=3 " is another way to bind x.
Given a formula containing a part of the form *(x)P(x)* or *(∃x) P(x),* such a part is called *x* bound part of that formula. Any occurrence of *x* in an *X*-bound part of a formula is called a bound occurrence of *x*, while any occurrence of *x* or any variable that is not a bound occurrence is called a free occurrence. The formula *P(x)* either in *(x) P (x)* or in *(∃x) P(x)* is described as the scope of the quantifier.

Theory of Inference for the Predicate Calculus:

Rule US: universal specification or instantiation

$$(x)A(x) \Rightarrow A(y)$$

From $(x)A(x)$, one can conclude $A(y)$

Rule ES: Existential specification

$$(\exists x)A(x) \Rightarrow A(y)$$

Rule EG*:* Existential generalization

$$A(x) \Rightarrow (\exists y)A(y)$$

From $A(x)$, one can conclude $(\exists y)A(y)$

Rule UG: universal generalization

$$A(x) \Rightarrow (y)A(y)$$

Ex1: Prove that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

{1}	(1)	$(\exists x)(P(x) \wedge Q(x)$	rule P
{1}	(2)	$(P(y) \wedge Q(y)$	rule ES
{1}	(3)	$P(y)$	rule T $P \wedge Q \Rightarrow P$
{1}	(4)	$(\exists x)(P(x)$	rule EG
{1}	(5)	$Q(y)$	rule T, $P \wedge Q \Rightarrow Q$
{1}	(6)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	rule T, P,Q$\Rightarrow P \wedge Q$

Ex 2: Show that *(x)(H(x) → M(x)) ∧ H(s) ⇔ M(s).*This problem is a symbolic translation of a well-known argument known as the **"Socrates argument**" which is given by:

All men are mortal. Socrates is a man. Therefore Socrates is a mortal.

If we denote *H (x): x* is a man, *M (x) : x* is a mortal, and s: Socrates, we can put the argument in the above form.

Solution:

{1}	(1)	*(x) (H(x) → M(x))*	***rule P***
{1}	(2)	*H(s) → M(s)*	***rule US****, (1)*
{3}	(3)	*H(s)*	***rule P***
{1,3}	(4)	*M(s)*	***rule T****, (2), (3)*

Note that in step 2 first we remove the universal quantifier.

Ex 3: Show that *(x)(P(x) →Q(x)) ∧ (x) (Q(x) →R(x)) ⇔ (x) (P(x) →R(x))*

Solution:

{1}	(1)	*(x)(P(x) → Q(x))*	***rule P***
{1}	(2)	*P(y) → Q(y)*	***rule US****, (l)*
{3}	(3)	*(x) (Q(x) → R(x))*	***rule P***

{3}	(4)	$Q(y) \rightarrow R(y)$	***rule* US,** (3)
{1,3}	(5)	$P(y) \rightarrow R(y)$	***rule T**, (2), (4)*
{1,3}	(6)	$(x)(P(x) \rightarrow R(x))$	***rule UG,** (5)*

Ex 4: Show that $(\exists x)M(x)$ follows logically from the premises $(x)(H(x) \rightarrow M(x))$ and $(\exists x)H(x)$

Solution:

{1}	(1)	$(\exists x)H(x)$	***rule P***
{1}	(2)	$H(y)$	***rule ES**, (1)*
{3}	(3)	$(x)(H(x) \rightarrow M(x))$	***rule P***
{3}	(4)	$H(y) \rightarrow M(y)$	***rule US, (**3)*
{1,3}	(5)	$M(y)$	***rule T**, (2), (4)*
{1,3}	(6)	$(\exists x)M(x)$	***rule EG**, (5)*