Example 4 : A population consists of five numbers 2, 3, 6, 8 and 11. Consider all possible samples of size two which can be drawn with replacement from this population. Find

(a) The mean of the population.

(b) The standard deviation of the population.

(c) The mean of the sampling distribution of means and

(d) The standard deviation of the sampling distribution of means (*i.e.*, the standard error of means).

[JNTU Nov 2004, April 2005 (Sets 3, 4), (A) Dec. 2009 (Set No. 4)]

Solution :

(a) ***Mean of the population is given by***

$$\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$$

(b) ***Variance of the population (σ^2) is given by***

$$\sigma^2 = \Sigma\frac{(x_i-\overline{x})^2}{n}$$

$$= \frac{(2-6)^2+(3-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5}$$

$$= \frac{16+9+0+4+25}{5} = 10.8$$

$\therefore \quad \sigma = \sqrt{10.8}$ *i.e.* $\sigma = 3.29$

(c) ***Sampling with replacement (Infinite population) :***

The total no. of samples with replacement is

$N^n = 5^2 = 25$ samples of size 2

Here N = population size and n = sample size listing all possible samples of size 2 from population 2, 3, 6, 8, 11 with replacement we get 25 samples

$$\begin{Bmatrix} (2,2) & (2,3) & (2,6) & (2,8) & (2,11) \\ (3,2) & (3,3) & (3,6) & (3,8) & (3,11) \\ (6,2) & (6,3) & (6,6) & (6,8) & (6,11) \\ (8,2) & (8,3) & (8,6) & (8,8) & (8,11) \\ (11,2) & (11,3) & (11,6) & (11,8) & (11,11) \end{Bmatrix}$$

Now compute the arithmetic mean for each of these 25 samples. The set of 25 means $\overline{x}$ of these 25 samples, gives rise to the distribution of means of the samples known as sampling distribution of means.

The samples means are

$$\begin{Bmatrix} 2 & 2.5 & 4 & 5 & 6.5 \\ 2.5 & 3 & 4.5 & 5.5 & 7 \\ 4 & 4.5 & 6 & 7.0 & 8.5 \\ 5 & 5.5 & 7 & 8 & 9.5 \\ 6.5 & 7 & 8.5 & 9.5 & 11 \end{Bmatrix} \quad \text{..... (I)}$$

and the mean of sampling distribution of means is the mean of these 25 means.

$$\mu_{\bar{x}} = \frac{\text{Sum of all sample means in (I)}}{25} = \frac{150}{25} = 6$$

Illustrating that $\mu_{\bar{x}} = \mu$.

(*d*) The variance $\sigma_{\bar{x}}^2$ of the sampling distribution of means is obtained by subtracting the mean 6 from each number in (I) and squaring the result, adding all 25 members thus obtained, and dividing by 25.

$$\sigma_{\bar{x}}^2 = \frac{(2-6)^2 + \ldots\ldots + (11-6)^2}{25} = \frac{135}{25} = 5.40 \quad \text{and thus} \quad \sigma_{\bar{x}} = \sqrt{5.40} = 2.32.$$

Clearly, for finite population involving sampling with replacement (or infinite population)

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \text{ or } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{3.29}{\sqrt{2}} = 2.32$$

Example 6 : A population consists of 5,10,14,18,13,24. Consider all possible samples of size two which can be drawn without replacement from the population. Find

(*a*) The mean of the population

(*b*) The standard deviation of the population

(*c*) The mean of the sampling distribution of means

(*d*) The standard deviation of sampling distribution of means.

[JNTU (H) Nov. 2010 (Set No.3)]

Solution : (*a*) The mean of the population μ is given by

$$\mu = \frac{\Sigma x}{n} = \frac{5+10+14+18+13+24}{6} = \frac{84}{6} = 14$$

(*b*) Variance of the population σ^2 is given by

$$\sigma^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$= \frac{1}{6}[(5-14)^2 + (10-14)^2 + (14-14)^2 + (18-14)^2 + (13-14)^2 + (24-14)^2]$$

$$= \frac{1}{6}[81+16+0+16+1+100] = \frac{214}{6} = 35.67$$

(*c*) All possible samples of size two (the no. of samples $= {}^6C_2 = 15$) and their means are shown in the following table.

Sample No.	Sample values	Total of Sample values	Sample mean
1	5,10	15	7.5
2	5,14	19	9.5
3	5,18	23	11.5
4	5,13	18	9
5	5,24	29	14.5
6	10,14	24	12
7	10,18	28	14
8	10,13	23	11.5
9	10,24	34	17
10	14,18	32	16
11	14,13	27	13.5
12	14,24	38	19
13	18,13	31	15.5
14	18,24	42	21
15	13,24	37	18.5
Total			210

$\therefore$ Mean of sample means $= \dfrac{210}{15} = 14$

i.e., The Mean of the sampling distribution of means is $\mu_{\bar{x}} = 14$

Illustrating that $\mu_{\bar{x}} = \mu$

(*d*) The variance of sampling dstribution of means

$$\sigma_{\bar{x}}^2 = \frac{1}{15}[(7.5-14)^2 + (9.5-14)^2 + (11.5-14)^2 + (9-14)^2$$

$$+ (14.5-14)^2 + ... + (21-14)^2 + (18.5-14)^2]$$

$$= \frac{1}{15}[42.25 + 20.25 + 6.25 + 25 + 0.25 + 4 + 0 + 6.25$$

$$+ 9 + 4 + 0.25 + 25 + 2.25 + 49 + 20.25]$$

$$= \frac{214}{15} = 14.2666$$

$\therefore$ Standard deviation of sampling distribution of means is $\sigma_{\bar{x}} = \sqrt{14.2666} = 3.78$

Example 9 : A population consists of six numbers 4,8,12,16,20,24. Consider all sam of size two which can be drawn without replacement from this popuation. Find

a) The population mean

b) The population standard deviation

c) The mean of the sampling distribution of means

d) The standard deviation of the sampling distribution of means

[JNTU (H) Nov. 2009 (Set No.

<u>**Solution :**</u> a) The mean of the population,

$$\mu = \frac{\Sigma x_i}{N} = \frac{4+8+12+16+20+24}{6} = \frac{84}{6} = 14$$

b) The variance of the population, $\sigma^2 = \frac{1}{N}\sum(x_i - \bar{x})^2$

$$= \frac{1}{6}[(4-14)^2 + (8-14)^2 + (12-14)^2 + (16-14)^2 + (20-14)^2 + (24-14)^2]$$

$$= \frac{1}{6}[100+36+4+4+36+100] = \frac{280}{6} = 46.67.$$

$\therefore$ The population standard deviation, $\sigma = \sqrt{46.67} = 6.83$

The number of samples that can be drawn from the population of size 6 witho replacement is ${}^6C_2$ *i.e.*, $\frac{6!}{2!4!}$ or 15.

The (15 samples are) sampling distribution is

$$\begin{Bmatrix} (4,8), (4,12), (4,16), (4,20), (4,24), (8,12), (8,16), (8,20), \\ (8,24), (12,16), (12,20), (12,24), (16,20), (16,24), (20,24). \end{Bmatrix}$$

The means are

$$\begin{Bmatrix} 6, 8, 10, 12, 14, \\ 10, 12, 14, 16, \\ 14, 16, 18, \\ 18, 20 \\ 22 \end{Bmatrix}$$

c) The mean of the sampling distribution of means is

$$\mu_{\bar{x}} = \frac{1}{15}[6+8+10+12+14+10+12+14+16+14+16+18+18+20+22]$$

$$= \frac{210}{15} = 14$$

d) The variance of the sampling distribution of means is

$$\sigma_{\bar{x}}^2 = \frac{1}{N}\sum (x_i - \bar{x})^2$$

$$= \frac{1}{15}[(6-14)^2 + (8-14)^2 + (10-14)^2 + (12-14)^2 + (14-14)^2 + (10-14)^2 +$$

$$(12-14)^2 + (14-14)^2 + (16-14)^2 + (14-14)^2 + (16-14)^2 + (18-14)^2 +$$

$$(18-14)^2 + (20-14)^2 + (22-14)^2]$$

$$= \frac{1}{15}[64+36+16+4+0+16+4+0+4+0+4+16+16+36+64]$$

$$= \frac{280}{15} = 18.67$$

Hence the standard deviation of the sampling distribution of means is

$$\sigma_{\bar{x}} = \sqrt{18.67} = 4.32\sqrt{2}.$$

Example 10 : Samples of size 2 are taken from the population 1,2,3,4,5,6 (*i*) with replacement and (*ii*) without replacement. Find

a) The mean of the population

b) Standard deviation of population

c) The mean of the sampling distribution of means

d) The standard deviation of the sampling distribution of means.

Verify that means of sampling distribution is equal to the mean of population and standard deviations of the means of sampling distribution are not equal to the standard deviation of the population. **[JNTU (H) Nov. 2009, (A) Nov. 2011 (Set No. 3)]**

Solution :

(*i*) (*a*) The mean of the population, $\mu = \dfrac{1+2+3+4+5+6}{6} = \dfrac{21}{6} = 3.5$

(*b*) The variance of the population is $\sigma^2 = \dfrac{1}{n}\sum (x_i - \mu)^2$

$$= \frac{1}{6}[(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2]$$

$$= \frac{1}{6}[6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25]$$

$$= \frac{17.50}{6} = 2.917$$

$\therefore$ The standard deviation of the population is, $\sigma = \sqrt{2.917} = 1.71$.

(*c*) Number of samples of size two with replacement is $N^n = 6^2 = 36$.

They are

$$\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$

$\therefore$ The number of samples, $n = 36$

Their corresponding means are

$$\left\{\begin{matrix} 1, & 1.5, & 2, & 2.5, & 3, & 3.5 \\ 1.5, & 2, & 2.5, & 3, & 3.5, & 4 \\ 2, & 2.5, & 3, & 3.5, & 4, & 4.5 \\ 2.5, & 3, & 3.5, & 4, & 4.5, & 5 \\ 3, & 3.5, & 4, & 4.5, & 5, & 5.5 \\ 3.5, & 4, & 4.5, & 5, & 5.5, & 6 \end{matrix}\right\}$$

The mean of the sampling distribution of means is

$$\mu_x = \frac{1}{36}[1+1.5+2+2.5+3+3.5+2.5+\ldots+6]$$

$$= \frac{126}{36} = 3.5\,.$$

(*d*) The standard deviation of the sampling distribution of means is

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{36}(x_i - \mu_{\bar{x}})^2}{n}$$

$$= \frac{1}{36}[(1-3.5)^2 + (1.5-3.5)^2 + (2-3.5)^2 + (2.5-3.5)^2 + (3-3.5)^2$$

$$+(3.5-3.5)^2 + (1.5-3.5)^2 + \ldots. + (6-3.5)^2]$$

$$= \frac{52.5}{36} = 1.46$$

$$\therefore\ \sigma_{\bar{x}} = \sqrt{1.46}$$

Example 16 : When a sample is taken from an infinite population, what happen to standard error of the mean if the sample size is decreased from 800 to 200.

[JNTU (A) Nov. 2010 (Set No.

Solution : The standard error of mean $= \dfrac{\sigma}{\sqrt{n}}$

Sample size $= n$. Let $n = n_1 = 800$.

Then $S.E_1 = \dfrac{\sigma}{\sqrt{800}} = \dfrac{\sigma}{20\sqrt{2}}$

When n_1 is reduced to 200

Let $n_2 = 200$. Then

$$S.E_2 = \frac{\sigma}{\sqrt{200}} = \frac{\sigma}{10\sqrt{2}}$$

$$\therefore S.E_2 = \frac{\sigma}{10\sqrt{2}} = 2\left[\frac{\sigma}{20\sqrt{2}}\right] = 2\,(S.E_1)$$

Hence if sample size is reduced from 800 to 200, S.E. of mean will be multiplied by 2

6b) $n = 300$

$\mu = 76$

$\sigma^2 = 256 \Rightarrow \sigma = 16.$

$\bar{X} \sim N(\mu, \sigma^2/\sqrt{n}).$

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

when,

$\bar{x}_1 = 75$

$$Z1 = \frac{\bar{x}_1 - \mu}{\sigma/\sqrt{n}} = \frac{75-76}{16./\sqrt{300}} = \frac{-1}{(16)(10)\sqrt{3}}$$

$Z1 = -0.361.$

when,

$\bar{x}_2 = 78$

$$Z2 = \frac{\bar{x}_2 - \mu}{\sigma/\sqrt{n}} = \frac{78-76}{16\sqrt{300}} = \frac{2}{160\sqrt{3}}$$

$Z_2 = 0.721$

$\therefore P(75 \leq \bar{x} \leq 78) = P(Z_1 \leq Z \leq Z_2)$

$= P(-0.361 \leq Z \leq 0.721)$

$= 0.1406 + 0.2642 = 0.4048$

Example 2 : What is the size of the smallest sample required to estimate an unkn[own] proportion to within a maximum error of 0.06 with atleast 95% confidence.

[JNTU Dec. 2004 S, April 2006, (K) Nov. 2009, (A) Nov. 2010 (Set No.

Solution : We are given

The maximum error, $E = 0.06$

Confidence limit = 95%

i.e. $(1-\alpha)\,100 = 95$

$$\Rightarrow \quad 1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \alpha/2 = 0.025$$

$$\therefore \quad z_{\alpha/2} = 1.96$$

Here P is not given. So we take $P = \frac{1}{2}$. Thus $Q = \frac{1}{2}$

Hence $n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (PQ)$

When P is unknown, sample size $n = \frac{1}{4}\left[\frac{z_{\alpha/2}}{E}\right]^2$

$$\therefore \quad n = \frac{1}{4}\left[\frac{1.96}{0.06}\right]^2 = 266.78 \approx 267$$

Example 6 : What is the maximum error one can expect to make with probability 0.90 when using the mean of a random sample of size $n = 64$ to estimate the mean of population with $\sigma^2 = 2.56$. [JNTU 2004S, (K) Nov. 2009, (H) May 2011 (Set No. 3)]

Solution : Here $n = 64$

The probability $= 0.90$

$\sigma^2 = 2.56 \Rightarrow \sigma = \sqrt{2.56} = 1.6$

Confidence limit $= 90\%$

$\therefore\ z_{\alpha/2} = 1.645$

Hence maximum error $E = z_{\alpha/2} \cdot \dfrac{\sigma}{\sqrt{n}} = 1.645 \times \dfrac{1.6}{\sqrt{64}} = 0.329.$

Example 15 : A random sample of size 100 is taken from a population with $\sigma = 5.1$. Given that the sample mean is $\bar{x} = 21.6$. Construct a 95% confidence interval for the population mean μ. **[JNTU 2001]**

Solution : Given $\bar{x}$ = sample mean = 21.6,

$z_{\alpha/2} = 1.96$, n = sample size = 100, $\sigma = 5.1$

$\therefore$ Confidence interval $= (\bar{x} - z_{\alpha/2} \cdot \sigma/\sqrt{n}\,,\ \bar{x} + z_{\alpha/2} \cdot \sigma/\sqrt{n})$

Now $\bar{x} - z_{\alpha/2} \cdot \dfrac{\sigma}{\sqrt{n}} = 21.6 - \dfrac{1.96 \times 5.1}{10} = 20.6$

and $\bar{x} + z_{\alpha/2} \cdot \dfrac{\sigma}{\sqrt{n}} = 22.6$

Hence (20.6, 22.6) is the confidence interval for the population mean μ.

Example 18 : Find 95% confidence limits for the mean of a normality distributed population from which the following sample was taken 15, 17, 10, 18, 16, 9, 7, 11, 13, 14.

[JNTU (H) Nov. 2009 (Set No. 4), (A) Nov. 2010 (Set No. 4), (K) May 2010S, May 2013 (Set No.4)]

<u>**Solution :**</u> We have $\bar{x} = \dfrac{15+17+10+18+16+9+7+11+13+14}{10} = 13$

$$s^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1}$$

$$= \frac{1}{9}[(15-13)^2 + (17-13)^2 + (10-13)^2 + (18-13)^2 + (16-13)^2$$

$$+ (9-13)^2 + (7-13)^2 + (11-13)^2 + (13-13)^2 + (14-13)^2] = \frac{40}{3}$$

Since $z_{\alpha/2} = 1.96$, we have

$$z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 1.96 \cdot \frac{\sqrt{40}}{\sqrt{10}.\sqrt{3}} = 2.26$$

$\therefore$ Confidence limits are $\bar{x} \pm z_{\alpha/2} \cdot \dfrac{s}{\sqrt{n}} = 13 \pm 2.26 = (10.74, 15.26)$

Example 19 : Determinne a 95% confidence interval for the mean of a normal distribution with variance 0.25, using a sample of $n = 100$ values with mean 212.3.

[JNTU(K) March. 2014 (Set No. 1)]

Solution : We have $n = 100$, $\bar{x} = 212.3$, S.D. $= \sigma = \sqrt{0.25}$ and $z_{\alpha/2} = 1.96$ (for 95%)

We know that 95% confidence interval is

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

Now $z_{\alpha/2} \cdot \dfrac{\sigma}{\sqrt{n}} = \dfrac{1.96 \times \sqrt{0.25}}{\sqrt{100}} = \dfrac{1.96 \times 0.5}{10} = \dfrac{0.98}{10} = 0.098$

$\therefore$ Confidence interval $= (212.3 - 0.098, 212.3 + 0.098)$

$= (212.202, 212.398)$.

$\therefore \mu_{\bar{x}}$ $\sqrt{n}$ $\sqrt{4}$

Example 23 : A random sample of 400 items is found to have mean 82 and S.D. of 18. Find the maximum error of estimation at 95% confidence interval. Find the confidence limits for the mean if $\bar{x} = 82$. **[JNTU Nov. 2008, (A) Nov. 2011, (H) May 2011 (Set No. 2)]**

Solution : Given standard deviation = σ = 18

Sample size = n = 400

$Z_{\alpha/2}$ for 95% confidence = 1.96 (from tables)

Sample mean = $\bar{x}$ = 82

$$\text{Maximum error, } E = Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} = \frac{1.96 \times 18}{20} = 1.764$$

The limits for the confidence are

$$\bar{x} - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$\therefore$ Confidence limits are 80.236 and 83.764

Example 25 : A sample of size 300 was taken whose variance is 225 and mean 54. Construct 95% confidence interval for the mean. **[JNTU (H) Nov. 2010 (Set No.1)]**

Solution : Since the sample size 300 is large (> 30), normal distribution is used as the sampling distrubution.

Here $n = 300$, $\bar{x}$ = sample mean = 54, $\sigma = \sqrt{225} = 15$

$$\therefore \text{ S.E. of } \bar{x} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{300}} = 0.866$$

95% confidence limits for the population mean are

$$\bar{x} \pm 1.96 \text{ (S.E. of } \bar{x}) = 54 \pm 1.96\,(0.866) = 54 \pm 1.697 = 55.697 \text{ and } 52.3$$

$\therefore$ The required confidence interval is (52.3, 55.7).