UNIT-5

Recurrence Relations : Generating Function of Sequences, Partial Fractions, Calculating Coefficient of Generating Functions, Recurrence Relations, Formulation as Recurrence Relations, Solving linear homogeneous recurrence Relations by substitution, generating functions and the Method of Characteristic Roots. Solving Inhomogeneous Recurrence Relations.

Recurrence relations

Definition: An equation that express a_n in terms of one or more of the previous terms of the sequence namely $a_0, a_1, a_2, \ldots a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-negative integer is called a **recurrence relation** for the sequence **$\{a_n\}$** or a **difference equation**.

If a_n terms of sequence satisfy the recurrence relation, then the sequence is called solution of the recurrence relation.

Eg-1: Let a_n be the sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3a_{n-2}$ for n = 2,3,… and $a_0 = 1$ and $a_1 = 2$. What are the values of a_2 and a_3.

Sol:- Given recurrence relation is $a_n = a_{n-1} + 3a_{n-2}$

$$a_2 = a_{2-1} + 3a_{2-2}$$
$$= a_1 + 3a_0 \quad = 2 + 3(1) \quad = 5$$
$$a_3 = a_{3-1} + 3a_{3-2}$$
$$= a_2 + 3a_1 \quad = 5 + 3(2) \quad = 11$$

Eg-2: Find the first five terms of the sequence if

- **I.** $a_n = a^2_{n-1}$, $a_1 = 2$
- **II.** $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1, a_1 = 2, a_2 = 0$
- **III.** $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1, a_1 = 1$

Sol:-

I. $a_n = a^2_{n-1}$, $a_1 = 2$

If n = 2, $a_2 = a^2_{2-1} = (a_1)^2 = 2^2 = 4$

If n = 3, $a_3 = a^2_{3-1} = (a_2)^2 = 4^2 = 16$

If n = 4, $a_4 = a^2_{4-1} = (a_3)^2 = 16^2 = 256$

If n = 5, $a_5 = a^2_{5-1} = (a_4)^2 = 256^2 = 65356$.

∴ The first five terms of the sequence are 2, 4, 16, 256, 65356.

II. $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1, a_1 = 2, a_2 = 0$

If n = 3,

$$a_3 = a_{3-1} + a_{3-3}$$
$$= a_2 + a_0 \quad = 0 + 1 = 1$$

If n = 4,

$$a_4 = a_{4-1} + a_{4-3}$$
$$= a_3 + a_1 \quad = 1 + 2 = 3$$

∴ The first five terms of the sequence are 1, 2, 0, 1, 3.

III. $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1, a_1 = 1$

If n = 2,

$$a_2 = 2a_{2-1} + 2^2 a_{2-2} = 2a_1 + 4a_0 = 2(1) + 4(1) = 2 + 4 = 6$$

If n = 3,

$a_3 = 3a_{3-1} + 3^2 a_{3-2} = 3a_2 + 9a_1 = 3(6) + 9(1) = 18+9=27$

If n = 4,

$a_4 = 4a_{4-1} + 4^2 a_{4-2} = 4a_3 + 16a_2 = 4(27) + 16(6)$

$= 108+96=204$

∴ The first five terms of the sequence are 1, 1, 6, 27, 204.

Eg-3: Determine whether the sequence $\{a_n\}$ is a solution of recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$

i. ***$a_n = -n+2$***

ii. ***$a_n = 3(-1)^n + 2^n - n + 2$***

Sol:-

i. ***$a_n = -n+2$***

Given $a_n = -n+2$

But question contains n-1 and n-2

$a_{n-1} = -(n-1) + 2$

$= -n+1+2$

$= -n+3$ ----- (1)

$a_{n-2} = -(n-2) + 2$

$= -n+2+2$

$= -n+4$ ----- (2)

Consider,

R.H.S $= a_{n-1} + 2a_{n-2} + 2n - 9$ (substitute 1 and 2 in R.H.S)

$= (-n+3) + 2(-n+4) + 2n - 9$

$= -n - 3 \cancel{-2n} + 8 + \cancel{2n} - 9$

$= -n+11-9$

$= -n+2$

$= a_n =$ L.H.S

∴ $a_n = -n+2$ is a solution of given recurrence relation.

ii. ***$a_n = 3(-1)^n + 2^n - n + 2$***

Given $a_n = 3(-1)^n + 2^n - n + 2$

But question contains n-1 and n-2

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$

$= 3(-1)^{n-1} + 2^{n-1} - n + 1 + 2$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3$ ----- (1)

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

$= 3(-1)^{n-2} + 2^{n-2} - n + 2 + 2$

$= 3(-1)^{n-2} + 2^{n-2} - n + 4$ ----- (2)

Consider,

R.H.S $= a_{n-1} + 2a_{n-2} + 2n - 9$ (substitute 1 and 2 in R.H.S)

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 2(3(-1)^{n-2} + 2^{n-2} - n + 4) + 2n - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 6(-1)^{n-2} + 2.2^{n-2} \cancel{-2n} + 8 + \cancel{2n} - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 6(-1)^{n-2} + 2.2^{n-2} + 8 - 9$

$= 6(-1)^{n-2} + 3(-1)^{n-1} + 2.2^{n-2} + 2^{n-1} - n + 2$

$$= \frac{6(-1)^n}{(-1)^2} + \frac{3(-1)^n}{(-1)^1} + 2.\frac{2^n}{2^2} + \frac{2^n}{2^1} - n + 2$$

$$= 6(-1)^n - 3(-1)^n + \frac{2^n}{2^1} + \frac{2^n}{2^1} - n + 2$$

$$= 3(-1)^n + 2.\frac{2^n}{2} - n + 2$$

$$= 3(-1)^n + 2^n - n + 2 \text{ = } a_n \text{ = L.H.S}$$

∴ $a_n = 3(-1)^n + 2^n - n + 2$ is a solution of given recurrence relation.

Eg-4:- Let $a_n=2^n+5(3^n)$ for n= 0, 1, 2....
i) Find a_0, a_1, a_2, a_3 and a_4
ii) Show that $a_2=5a_1-6a_0$, $a_3=5a_2-6a_1$ and $a_4=5a_3-6a_2$
iii) Show that $a_n=5a_{n-1}-6a_{n-2}$ for all n ≥2

Sol:- *Given a_n=2n+5(3n), n≥0*

i) $a_0 = 2^0+5.3^0 = 1+5(1) = 1+5 = 6$
$a_1 = 2^1+5(3^1) = 2+5(3) = 2+15 = 17$
$a_2 = 2^2+5(3^2) = 4+5(9) = 4+45 = 49$
$a_3 = 2^3+5(3^3) = 8+5(27) = 8+135 = 143$
$a_4 = 2^4+5(3^4) = 16+5(81) = 16+405 = 421$

ii) L.H.S = $5a_1 - 6a_0 = 5(17)-6(6) = 49 = a_2$ = R.H.S
L.H.S = $5a_2 - 6a_1 = 5(49) - 6(17) = 143 = a_3$ = R.H.S
L.H.S = $5a_3 - 6a_2 = 5(143)-6(49) = 421 = a_4$ = R.H.S
∴ Hence proved.

iii) L.H.S $= 5a_{n-1} - 6a_{n-2}$
$= 5\,[2^{n-1}+5(3^{n-1})] - 6[2^{n-2}+5(3^{n-2})]$
$= 2^{n-2}[-6+10] + 3^{n-2}[-30+75]$
$= 2^{n-2}x4 + 3^{n-2}x\ 45 \quad = 2^{n-2}x2^2 + 3^{n-2}x3^2x5$
$= 2^n+5(3^n) \quad = a_n \quad$ =R.H.S
∴ Hence proved.

Eg-5: A person deposits Rs.1000 in an account that yields 9% interest compounded yearly.

I. ***Set up recurrence relation for amount in account at the end of n years.***
II. ***Find an explicit formula for amount in account at end of n years.***
III. ***How much money will be in account after 100 years?***

Sol:-

I) Let S_n denote the amount in the account after n years.
But the amount in the account after n years = amount in the account after (n-1) years + interest for the n^{th} year. (Interest = 9/100 = 0.09)

$S_n = S_{n-1} + (0.09)\,S_{n-1}$
$= S_{n-1}(1 + 0.09)$
∴ $S_n = S_{n-1}(1.09)$

This is the required recurrence relation.

II) Explicit formula for Sn

$S_1 = S_{1-1}(1.09) = (1.09)\ S_0$

$S_2 = S_{2-1}(1.09) = (1.09)\ S_1 = (1.09)^2\ S_0$

$S_3 = S_{3-1}(1.09) = (1.09)\ S_2 = (1.09)^3\ S_0$

....

$S_n = S_{n-1}(1.09) = (1.09)\ S_{n-1} = (1.09)^n\ S_0$

$\therefore S_n = (1.09)^n\ S_0 = (1.09)^n \times 1000\ [S_0=1000]$ ---- (1)

Using mathematical induction we can prove the validity of equation (1)

When n = 0

$S_0 = (1.09)^0 \times 1000 = 1000$

$\therefore S_n$ is true for n = 0

We assume that $S_k = (1.09)\ k \times 1000$ is true ---- (2)

We need to prove that S_{k+1} is true i.e

$S_{k+1} = (1.09)\ k+1 \times 1000$ is true

From recurrence relation we have $S_{k+1} = S_k\ (1.09)$

$= (1.09)\ (1.09)\ k \times 1000$

$= (1.09)\ k+1 \times 1000$

$\therefore$ S_{k+1} is true.

Thus by the principle of mathematical induction S_n is true for all n.

$\therefore$ Explicit formula $S_n = (1.09)\ n \times 1000$

III) When n=100, we have

$S_{100} = (1.09)100 \times 1000$

= Rs. 1000 x (1.09)100

$\therefore$ Money in the account after 100 years = Rs. 1000 x (1.09)100

Eg-6: Suppose the number of bacteria in a colony triples every hour.

1) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

2) If 100 bacteria are used to begin a new colony, how many bacteria will be there in the colony in 10 hours.

Sol:- 1) Let a_n be the number of bacteria after n hours and a_{n-1} be the number of bacteria after n-1 hours.

$\therefore a_n = 3\ a_{n-1}$ is the required recurrence relation.

2) Explicit formula

Let $a_0 = 100$ then

$a_1 = 3\ a_0 = 3 \times 100$

$a_2 = 3\ a_1 = 3 \times 3 \times 100 = 3^2 \times 100$

$a_3 = 3\ a_2 = 3 \times 3 \times 3 \times 100 = 3^3 \times 100$

$a_n = 3\ a_{n-1} = 3^n \times 100$ -----(1)

By mathematical induction let's validate equation (1)

If n = 0 then

$a_0 = 3^0 \times 100$

$= 100$ which is true

$\therefore$ a_0 is true.

We assume that $a_k = 3^k \times 100$ is true

From (1) $a_{k+1} = 3a_k$

$= 3.3^k \times 100$

$= 3^{k+1} \times 100$

$\therefore a_{k+1}$ is true

Explicit formula is $a_n = 3^n \times 100$

If n = 10

$a_{10} = 3^{10} \times 100$

= 59,04,900

∴ No. of bacteria in the colony in 10 hours = 59,04,900

Linear homogeneous recurrence relations with constant coefficients:-

A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ where $c_1, c_2, \dots c_k$ are constants and $c_k \neq 0$ is called a linear homogeneous recurrence relation of **degree k** with constant coefficients.

Examples:

1. $a_n = 2a_{n-1}$ // homogeneous with constant coefficient.
2. $a_n - a_{n-1} = 3$ // Non- homogeneous
3. $a_n = 2a_{n-1} + a^2_{n-2}$ // It is not linear
4. $a_n = 2a_{n-1} + a_{n-2}$ // Linear homogeneous with constant coefficients.

Solving linear homogenous recurrence relations:-

A linear homogeneous recurrence relation can be solved by using 3 methods. They are:

1. Substitution (also called as Iteration).
2. Method of characteristic roots.
3. Generating functions.

1) Substitution:- In this method the recurrence relation for a_n is used repeatedly to solve for a general expression of a_n in terms of n.

Eg: Solve the recurrence relation $a_n = a_{n-1} + f(n)$, $n \geq 1$ by ***substitution.***

$a_1 = a_{1-1} + f(1) = a_0 + f(1)$

$a_2 = a_{2-1} + f(2) = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_{3-1} + f(3) = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$

$$a_n = a_{n-1} + f(n) = a_0 + \sum_{k=1}^{n} f(k)$$

More generally c is a constant

we can solve $a_n = C\, a_{n-1} + f(n)$

$a_1 = C\, a_0 + f(1)$

$a_2 = C\, a_1 + f(2)$

$= C(Ca_0 + f(1)) + f(2)$

$= C^2 a_0 + C\, f(1) + f(2)$

$a_3 = C\, a_2 + f(3)$

$= C(C^2 a_0 + C\, f(1) + f(2)) + f(3)$

$= C^3 a_0 + C^2 f(1) + C\, f(2) + f(3)$

$a_n = C\, a_{n-1} + f(n)$

$= C^n a_0 + C^{n-1} f(1) + C^{n-2} f(2) + \text{---} + C\, f(n-1) + f(n)$

$= C^n a_0 + \sum_{k=1}^{n} C^{n-k} f(k)$

2) The method of characteristic roots:- Consider the recurrence relation $a_n=c_1a^{n-1}+c_2a^{n-2}+....c_ka^{n-k}$ where $c_1,c_2....c_k$ are constants and $c_k \neq 0$.
The **characteristic equation** is given by $r^k-c_1r^{k-1}-c_2r^{k-2}....-\ c_{k-1}r-c_k=0$
The solutions to this equation are called **characteristic roots**.

Distinct Roots:- If the characteristic equation has distinct roots $r_1, r_2, r_3,....\ r_k$ then the formula is $a_n = c_1r_1^n + c_2r_2^n + c_3r_3^n + ...c_kr_k^n$ where $c_1,c_2,c_3,...c_k$ are constants.

Equal Roots:- If the characteristic equation has equal roots then the formula is **$a_n=c_1r_1^n+c_2nr_2^n$** where c_1,c_2 are constants.

Imaginary Roots:- If the characteristic equation has equal roots then the formula is $a_n=r^n(c_1 \cos n\Theta+c_2\sin n\Theta)$ where $r=\sqrt{(a^2+b^2)}$ & $\Theta=\tan^{-1}(b/a)$

S.No	Type of roots	Formula
1	Real and equal	$a_n= c_1r_1^n + c_2nr_2^n$
2	Real and distinct	$a_n=c_1r_1^n + c_2r_2^n$
3	Imaginary	$r^n(c_1 \cos n\Theta+c_2\sin n\Theta)$ where $r=\sqrt{(a^2+b^2)}$ & $\Theta=\tan^{-1}(b/a)$

Eg-1: Solve $a_n = a_{n-1} + 2a_{n-2}$, initial condition $a_0 = 0, a_1 = 1$.

Sol:- *Given recurrence relation is $a_n = a_{n-1} + 2a_{n-2}$*

$a_n - a_{n-1} - 2a_{n-2} = 0$ is second order linear homogeneous recurrence relation.

Characteristic equation is

$$r^2 - r - 2 = 0$$
$$r^2 - 2r + r - 2 = 0$$
$$r(r-2) + 1(r-2) = 0$$
$$(r-2)(r+1) = 0$$
$$r = 2,-1 \qquad \text{Distinct real roots}$$

The roots r_1 and r_2 are real and distinct, therefore solution is $a_n = c_1r_1^n + c_2r_2^n$ where c_1 and c_2 are constants.

∴ Solution is $a_n = c_1 2^n + c_2(-1)^n$.

Now the values of c1 and c2 must be calculated using the initial conditions.

If **$a_0 = 0$** $\Rightarrow n=0$

$$a_n = c_1 2^0 + c_2(-1)^0$$
$$0 = c_1 + c_2$$
$$c_1 = -c_2 \qquad \text{-----}\ (1)$$

If **$a_1 = 1$** $\Rightarrow n=1$

$$a_n = c_1 2^1 + c_2(-1)^1$$
$$1 = 2c_1 - c_2$$
$$2c_1 - c_2 = 1 \qquad \text{-----}\ (2)$$

By substituting (1) in (2)

$$2c_1 - c_2 = 1$$

$$2(-c_2)-c_2=1$$
$$-3c_2=1$$
$$c_2=-1/3 \qquad \text{----- (3)}$$

Solving (3) and (1)

$$c_1=-c_2$$
$$c_1=-(-1/3)$$
$$c_1=1/3$$

∴ Solution is $a_n = 1/3\, 2^n + (-1/3)(-1)^n$.

Eg-2: Solve $a_n = 8a_{n-1} - 16a_{n-2}$, initial conditions $a_0 = 16$, $a_1 = 80$.

Sol:- *Given recurrence relation is* $a_n = 8a_{n-1} - 16a_{n-2}$

$a_n - 8a_{n-1} + 16a_{n-2}= 0$ is second order linear homogeneous recurrence relation.

Characteristic equation is

$$r^2-8r+16=0$$
$$r^2-4r-4r+16=0$$
$$r(r-4)-4(r-4)=0$$
$$(r-4)(r-4)=0$$
$$r=4,4 \qquad \text{real roots and equal}$$

The roots r_1 and r_2 are real and equal, therefore solution is $a_n = c_1 r_1^n + c_2 n\, r_2^n$ where c_1 and c_2 are constants.

∴ Solution is $a_n = c_1 4^n + c_2 n\, 4^n$.

Now the values of c1 and c2 must be calculated using the initial conditions.

If $\mathbf{a_0 = 16} \Rightarrow n=0$

$$a_n = c_1 4^0 + c_2 0.\, 4^0$$
$$16 = c_1 + 0$$
$$c_1 = 16 \qquad \text{----- (1)}$$

If $\mathbf{a_1 = 80} \Rightarrow n=1$

$$a_n = c_1 4^1 + c_2 1.\, 4^1$$
$$80 = 4c_1 + 4c_2$$
$$c_1 + c_2 = 20 \qquad \text{----- (2)}$$

By substituting (1) in (2)

$$c_1 + c_2 = 20$$
$$16 + c_2 = 20$$
$$c_2 = 4.$$

∴ Solution is $a_n = 16.4^n + 4n.4^n$

$$= 4^n(16+4n)$$

Eg-3:- Solve the recurrence relation $a_n=2a_{n-1}-2a_{n-2}$, $a_0=1$, $a_1=2$.

Sol: Given recurrence relation is $a_n=2a_{n-1}-2a_{n-2}$

$a_n- 2a_{n-1}+2a_{n-2}=0$ is second order linear homogeneous recurrence relation.

Characteristic equation is $r^2-2r+2=0$

$a=1, b=-2, c=2$. Substitute these values in the formula $-b\pm\sqrt{b^2-4ac}/2a$

$=2\pm\sqrt{4-4(1)(2)}/2$ $=2\pm\sqrt{4-8}/2$ $=2\pm\sqrt{-4}/2$

$=2\pm\sqrt{4i^2}/2$ $=2\pm 2i/2$ $=2(1\pm i)/2$ $=i\pm 1$

Roots are imaginary.

∴Solution is $a_n=r^n\,(c_1 \cos n\Theta+c_2 \sin n\Theta)$ where $r=\sqrt{(a^2+b^2)}$ & $\Theta=\tan^{-1}(b/a)$

Here a=1 and b=1.

$r=\sqrt{1^2+1^2}$ $=\sqrt{1+1}$ $=\sqrt{2}$

$\Theta=\tan^{-1}(1/1)$ $=\tan^{-1}(1)$ $=45° =\pi/4$

∴ $i\pm 1 =\sqrt{2}\,(c_1 \cos n\pi/4+c_2 \sin n\pi/4)$

Now the values of c1 and c2 must be calculated using the initial conditions.

a_0=1 ⇒ n=0

$=\sqrt{2}^0(C_1 \cos 0\pi/4+C_2 \sin 0\pi/4)$

$=1(C_1 \cos 0+C_2 \sin 0)$ = C1=1---- (1)

a_1=2 ⇒ n=1

$\sqrt{2}^1(C_1 \cos\pi/4+C_2 \sin\pi/4)$

$\sqrt{2}(C_1 \cos\pi/4+C_2 \sin\pi/4)$

$=\sqrt{2}(C1\ l/\sqrt{2}+C_2\ l/\sqrt{2})$

$=\sqrt{2}(C_1 +C_2)/\sqrt{2}$ $= C_1 +C_2 = 2$ -----(2)

From (1) & (2) $C_1 +C_2 = 2$ $=1+ C_2 = 2$

$= C_2=2-1$ $= C_2=1$

∴ The required solution is $\sqrt{2}^n\,(\cos n\pi/4+\sin n\pi/4)$

Partial Fractions (Some Important Formulae)

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
Where x^2+bx+c cannot be factorised further	

Some Important formulae:

Series	*Expansion*
$(1+x)^{-1}$ (or) $1/(1+x)$	$1-x+x^2-x^3+x^4-.....$
$(1-x)^{-1}$ (or) $1/(1-x)$	$1+x+x^2+x^3+x^4+....$
$(1+x)^{-2}$ (or) $1/(1+x)^2$	$1-2x+3x^2-4x^3+....$
$(1-x)^{-2}$ (or) $1/(1-x)^2$	$1+2x+3x^2+4x^3+....$

Generating Functions: - Many counting problems are solved by using generating functions. The generating function of a sequence $a_0, a_1, a_2 \ldots a_n$ is given as

$$G(z) = a_0 + a_1 z^1 + a_2 z^2 + ... + a_n z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n$$ is called generating function of numeric function a.

Eg-1: Find the generating function for the sequence 2, 2, 2, 2, 2, 2

Sol:- $G(z) = 2+2z+2z^2+2z^3+2z^4+2z^5$

Eg-2: Find the generating function for the sequence 1,2,3,4… (Or) for the sequence {a_n}, a_n=n+1

Sol:- $G(z) = 1+2z+3z^2+4z^3+\ldots \Rightarrow 1/(1-z)^2 \Rightarrow (1-z)^{-2}$

Note:- *The generating function of a sequence* $a_0, a_1, a_2, \ldots a_n$ is denoted by $< a_0, a_1, a_2, \ldots a_n >$

Eg-3: Find the generating function of the sequence <5, 3,-4,-2, 0, 1>

Sol:- $G(z) = 5+3z-4z^2-2z^3+0z^4+1z^5 \Rightarrow G(z) = 5+3z-4z^2-2z^3+z^5$

Calculating Coefficient of Generating Functions:-

The coefficient of generating function of the form $(1+x+x^2+x^3+\ldots)^n$ *i.e* $\frac{1}{(1-x)^n}$ *is given by*

$$1+\binom{1+n-1}{1}x+\binom{2+n-1}{2}x^2+\ldots+\binom{r+n-1}{r}x^r$$

Eg-1: Find the coefficient of x^{16} in the expansion of $(x^2+x^3+x^4+....)^5$

Sol:- $(x^2+x^3+x^4+....)^5 = [x^2(1+x+x^2+x^3+\ldots)]^5$

$$\Rightarrow (x^2)^5(1+x+x^2+x^3+\ldots)^5 \quad \Rightarrow x^{10}.(1+x+x^2+x^3+\ldots)^5$$

$$\Rightarrow x^{10}.\frac{1}{(1-x)^5}$$

$$\frac{1}{(1-x)^5} \Rightarrow n=5$$

$$\Rightarrow 1+\binom{1+5-1}{1}x+\binom{2+5-1}{2}x^2+\ldots+\binom{r+5-1}{r}x^r$$

$$\therefore x^{10}.\frac{1}{(1-x)^5} = x^{10}\binom{r+5-1}{r}x^r \quad \Rightarrow x^{10+r}\binom{r+5-1}{r}x^r$$

But we want the coefficient of x^{16} $\Rightarrow x^{10+r} = x^{16} \Rightarrow 10+r = 16$

$$\Rightarrow r=16-10 \quad \Rightarrow r=6$$

$\therefore$ Coefficient of f x^{16} = $\binom{6+5-1}{6} \Rightarrow \binom{10}{6} \Rightarrow {}^{10}C_6$

$\Rightarrow {}^{10}C_4$ [$\because$ ${}^nc_r = {}^nc_{n-r}$] $\Rightarrow \frac{10\times9\times8\times7}{4\times3\times2\times1} \Rightarrow 210$

Eg-2: Find the coefficient of x^{20} in the expansion of $(x^3+x^4+x^5+....)^5$

Sol:- $(x^3+x^4+x^5+....)^5 = [x^3(1+x+x^2+x^3+...)]^5$

$\Rightarrow (x^3)^5(1+x+x^2+x^3+...)^5 \Rightarrow x^{15}.(1+x+x^2+x^3+...)^5$

$\Rightarrow x^{15}.\frac{1}{(1-x)^5}$

$\frac{1}{(1-x)^5} \Rightarrow n=5$

$\Rightarrow 1+\binom{1+5-1}{1}x+\binom{2+5-1}{2}x^2+...+\binom{r+5-1}{r}x^r$

$\therefore x^{15}.\frac{1}{(1-x)^5} = x^{15}\binom{r+5-1}{r}x^r \Rightarrow x^{15+r}\binom{r+5-1}{r}x^r$

But we want the coefficient of x^{20} $\Rightarrow x^{15+r} = x^{20} \Rightarrow 15+r = 20$

$\Rightarrow r=20-15 \Rightarrow r=5$

$\therefore$ Coefficient of f x^{20} = $\binom{5+5-1}{5} \Rightarrow \binom{9}{5} \Rightarrow {}^9C_5$

$\Rightarrow {}^9C_4$ [$\because$ ${}^nc_r = {}^nc_{n-r}$] $\Rightarrow \frac{9\times8\times7\times6}{4\times3\times2\times1} \Rightarrow 126$

3) Solution of recurrence relations using generating functions

Eg-1: Solve $a_n = 3a_{n-1}+2$, $n\geq1$ with initial condition $a_0 = 1$

Sol:- *Given $a_n = 3a_{n-1}+2$, $n\geq1$* ------ (1)

Let the generating function of the sequence $\{a_n\}$ be G(z) $=\sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq1}^{\infty} a_n z^n = 3\sum_{n\geq1}^{\infty} a_{n-1}z^n + 2\sum_{n\geq1}^{\infty} z^n$$

$$\Rightarrow \sum_{n\geq1}^{\infty} a_n z^n = 3z\sum_{n\geq1}^{\infty} a_{n-1}\frac{z^n}{z} + 2z\sum_{n\geq1}^{\infty}\frac{z^n}{z}$$

$$\Rightarrow G(z)-a_0 = 3z\sum_{n\geq1}^{\infty} a_{n-1}z^{n-1} + 2z\sum_{n\geq1}^{\infty} z^{n-1}$$

$$\Rightarrow G(z)-a_0 = 3z\,G(z)+2z\,(1+z+z^2+...)$$

$$\Rightarrow G(z)-1 = 3z\,G(z)+(\frac{2z}{1-z})$$

$$\Rightarrow G(z)-1 = 3z\,G(z)+2z(1-z)^{-1}$$

$$\Rightarrow G(z)-3z\,G(z) = 2z(1-z)^{-1}+1$$

$$\Rightarrow G(z)[1-3z] = \frac{2z}{1-z} + 1$$

$$\Rightarrow G(z) = \frac{2z + 1(1-z)}{(1-z)(1-3z)} = \frac{z+1}{(1-z)(1-3z)}$$

Let

$$\frac{1+z}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)}$$ [By using partial fractions]

Equating numerators on both sides ----- (2)

$\Rightarrow (1+z) = (1-3z)A + B(1-z)$

Put 1-z=0 $\Rightarrow$ z= 1 in B $(1-z)$ to calculate A in equation (2)

$\Rightarrow$ (1 +1) = (1 – 3) A $\Rightarrow 2 = -2A$

$\Rightarrow$ A = -2/2 $\Rightarrow A = -1$

Put (1-3z) =0 $\Rightarrow$ 1=3z $\Rightarrow$ z = 1/3 in $(1-3z)A$ to calculate B in equation (2)

$$(1+\frac{1}{3}) = B(1-\frac{1}{3})$$

$$\Rightarrow \frac{4}{3} = B(\frac{2}{3})$$

$$B = 2$$

$$\therefore G(z) = \frac{-1}{(1-z)} + \frac{2}{(1-3z)}$$

$\therefore a_n = 2(3)^n - 1(1)^n \quad \Rightarrow \; a_n = 2(3)^n - 1$

Eg-2: Solve $a_n = a_{n-1} + 2(n-1)$, initial condition $a_0 = 3$, n≥1

Sol:- *Given $a_n = a_{n-1} + 2(n-1)$, n≥1 ------ (1)*

Let the generating function of the sequence $\{a_n\}$ be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 1}^{\infty} a_n z^n = \sum_{n\geq 1}^{\infty} a_{n-1} z^n + 2\sum_{n\geq 1}^{\infty} n z^n - 2\sum_{n\geq 1}^{\infty} z^n$$

$$\Rightarrow \sum_{n\geq 1}^{\infty} a_n z^n = z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 2\sum_{n\geq 1}^{\infty} n.z^n - 2\sum_{n\geq 1}^{\infty} z^n$$

$$\Rightarrow G(z) - a_0 = z[G(z)] + 2[z + 2z^2 + 3z^3 + \ldots] - 2[z + z^2 + z^3 + \ldots]$$

$$\Rightarrow G(z) - a_0 = z[G(z)] + 2z[1 + 2z + 3z^2 + \ldots + (n+1)z^n] - 2[(1-z)^{-1} - 1]$$

$$\Rightarrow G(z) - zG(z) = 3 + 2z(1-z)^{-2} - 2(1-z)^{-1} + 2$$

$$\Rightarrow G(z)[1-z] = 3 + 2z(1-z)^{-2} - 2(1-z)^{-1} + 2$$

$$\Rightarrow G(z)[1-z] = 3 + \frac{2z}{(1-z)^2} - \frac{2}{(1-z)^1} + 2$$

$$\Rightarrow G(z) = \frac{5}{(1-z)} + \frac{2z}{(1-z)^3} - \frac{2}{(1-z)^2}$$

$$\therefore a_n = 5.1^n + 2n(n+1) - 2(n+1)$$

$$= 5.1^n + 2n^2 + 2n - 2n - 2$$

$$= 2n^2 + 3$$

Eg-3: Solve a_n -2 a_{n-1} – 3a_{n-2}=0, with a_0 = 3, a_1 = 1, n≥2

Sol:- *Given $a_n = 2a_{n-1} + 3a_{n-2}=0$, n≥2* ------ *(1)*

Let the generating function of the sequence {a_n} be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 2}^{\infty} a_n z^n = 2\sum_{n\geq 2}^{\infty} a_{n-1} z^n + 3\sum_{n\geq 2}^{\infty} a_{n-2} z^n z^n$$

$$\Rightarrow\ 2z\sum_{n\geq 2}^{\infty} a_{n-1}\frac{z^n}{z} + 3z^2\sum_{n\geq 2}^{\infty} a_{n-2} z^{n-2}$$

$$\Rightarrow\ G(z) - a_0 - a_1 z = 2z\,[\mathrm{G(z) - a_0}] + 3\mathrm{z}^2\, G(z)$$

$$\Rightarrow G(z) - a_0 - a_1 z = 2z\mathrm{G(z)} - 2\mathrm{z\ a_0} + 3\mathrm{z}^2\, G(z)$$

$$\Rightarrow G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -2\mathrm{z\ a_0} + a_0 + a_1 z$$

$$\Rightarrow G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -2\mathrm{z}\,(3) + 3 + (1)z$$

$$\Rightarrow G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -6\mathrm{z} + 3 + z$$

$$\Rightarrow G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -5\mathrm{z} + 3$$

$$\Rightarrow G(z)(1 - 2z - 3\mathrm{z}^2) = -5\mathrm{z} + 3$$

$$\Rightarrow G(z) = -5\mathrm{z} + 3/(1 - 2z - 3\mathrm{z}^2)$$

$$\Rightarrow G(z) = \frac{3 - 5\mathrm{z}}{(1 - 2z - 3z^2)} = \frac{3 - 5\mathrm{z}}{(1 - 3z)(1 + z)}$$

Let

$$\frac{3 - 5\mathrm{z}}{(1 - 3z)(1 + z)} = \frac{A}{(1 - 3z)} + \frac{B}{(1 + z)}$$

Equating numerators on both sides

⇒ 3-5z=A (1+z) +B (1-3z) ----- (2)

Put 1+z=0 ⇒ z=-1 in A (1+z) to calculate B in equation (2)

⇒ 3-5(-1) =B(1-3(-1)) ⇒ 3+5=B(1+3)

⇒ 8=B(4) ⇒ B=2

Equating coefficients of z on both sides in equation (2)

⇒ A-3B=-5 ⇒ A-3(2) =-5 ⇒ A-6—5

⇒ A=-5+6 ⇒ A=1

$$\therefore \mathrm{G(z)} = \frac{3 - 5\mathrm{z}}{(1 - 3z)(1 + z)} = \frac{1}{(1 - 3z)} + \frac{2}{(1 + z)}$$

$\therefore a_n = 1(3)^n + 2(-1)^n$

Eg-4: Solve a_n =4 a_{n-1} + 3n.2^n, with a_0 = 4, n≥1

Sol:- *Given a_n = 4 a_{n-1} + 3n.2^n, n≥1* ------ *(1)*

Let the generating function of the sequence {a_n} be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 1}^{\infty} a_n z^n = 4\sum_{n\geq 1}^{\infty} a_{n-1} z^n + 3\sum_{n\geq 1}^{\infty} n.2^n z^n$$

$$\Rightarrow \quad 4z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 3\sum_{n\geq 1}^{\infty} n.2^n z^n$$

$$\Rightarrow \quad 4z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 3\sum_{n\geq 1}^{\infty} n.(2z)^n$$

$$\Rightarrow G(z) - a_0 = 4zG(z) + 3(2z)\sum_{n\geq 1}^{\infty} n(2z)^{n-1}$$

$$\Rightarrow \quad G(z) - 4zG(z) = a_0 + 3(2z)\sum_{n\geq 1}^{\infty} n(2z)^{n-1}$$

$$\Rightarrow \quad G(z)(1-4z) = 4 + 6z[1 + 2.2z + 3.2z^2 \; + ..]$$

$$\Rightarrow \quad G(z)(1-4z) = 4 + 6z(1-2z)^{-2}$$

$$\Rightarrow \quad G(z) = \frac{4}{(1-4z)} + \frac{6\text{z}}{(1-4\text{z})\ (1-2z)^2}$$

Let $\dfrac{6\text{z}}{(1-4\text{z})\ (1-2z)^2} = \dfrac{A}{(1-4z)} + \dfrac{B}{(1-2z)} + \dfrac{C}{(1-2z)^2}$ [By using partial fractions]

Equating numerators on both sides

⇒ 6z = A(1-2z)^2+B(1-4z)(1-2z)+C(1-4z) ----- (2)

Put 1-4z=0 ⇒ -4z= -1 ⇒ z=1/4 in C $(1-4z)$ to calculate A in equation (2)

$\Rightarrow \; 6(1/4) = A(1-2(1/4))^2 + B(1-4(1/4))(1-2(1/4)) + C(1-4(1/4))$

⇒ 3/2= A(1-2/4)^2+B(1-4/4)*(1-2/4)+C(1-4/4)

⇒ 3/2 = A(1-1/2)^2+B(1-1)(1-1/2)+C(1-1)

⇒ 3/2 = A(1/2)^2+B(0)+C(0)

⇒ 3/2=A (1/4) ⇒ A= 3/2(4) ⇒ A= 3(2) ⇒ A=6

Put (1-2z)^2=0 ⇒ 1-2z =0 ⇒ 2z=1 ⇒ z=1/2 in A ((1-2z)^2 to calculate C in equation (2)

⇒ 6(1/2) = A (1-2(1/2))^2+B (1-4(1/2)) (1-2(1/2)) +C (1-4(1/2))

⇒ 3 = A (1-1)^2+B (1-2) (1-1) +C (1-2)

⇒ 3=A (0) +B (0) +C (-1) ⇒ 3= C (-1) ⇒ C=-3

Equating the coefficients of z on both sides in equation (2) to calculate B

⇒ 6z=-4zA-6zB-4zC ⇒ 6~~z~~ =~~z~~ (-4A-6B-4C) ⇒ 6=-4A-6B-4C

⇒ 6=-4(6)-6B-4(-3) ⇒ 6=-24+12-6B ⇒ 6= -12-6B

⇒ 6+12 = -6B ⇒ 18 = -6B ⇒ 6B = -18

⇒ B=-18/6 ⇒ B=-3

$$\therefore G(z) = \frac{4}{(1-4z)} + \frac{6}{(1-4z)} - \frac{3}{(1-2z)} - \frac{3}{(1-2z)^2}$$

$$\Rightarrow G(z) = \frac{10}{(1-4z)} - \frac{3}{(1-2z)} - \frac{3}{(1-2z)^2}$$

∴ a_n= 10(4)^n – 3(2)^n – 3(n+1)2^n

⇒ 10(4)^n – 3(2)^n – 3n (2)^n – 3(2)^n

⇒ 10(4)^n –2. 3(2)^n– 3n (2)^n

⇒ 10(4)^n –6(2)^n– 3n (2)^n

∴ a_n = ⇒ 10(4)^n -(2)^n(3n+6)

Solving Non-homogeneous or Inhomogeneous Recurrence Relation:- A linear inhomogeneous recurrence relation with constant coefficients of degree k is a recurrence relation of the form $a_n=C_1a^{n-1}+C_2a^{n-2}+....C_ka^{n-k}+G(n)$. G (n) is a function not identically zero. The general solution of this is given by $a_n = a_n^{(h)} + a_n^{(p)}$ where

$a_n^{(h)}$ = Associated linear homogeneous recurrence relation and

$a_n^{(p)}$ = particular solution.

S.No	G(n)	Particular Solution
1	x^n and root = x	$d.n.x^n$
2	x^n and root ≠ x	$d.x^n$
3	constant(eg:2,3,100 etc)	d or nd or n^2d
4	c_0+c_1n where c_0 & c_1 are constants	d_0+d_1n

Eg-1: Solve $a_n = 2a_{n-1} + 2^n$, initial condition $a_0 = 2$.

Sol:- *The given recurrence relation $a_n - 2a_{n-1} = 2^n$ is a first order Inhomogeneous recurrence relation.*

1) The associated homogeneous recurrence relation is $a_n - 2a_{n-1} = 0$

Characteristic equation is $r-2=0 \Rightarrow r=2$

∴ Homogeneous *solution is* $a_n^{(h)} = c_1 r_1^n$ *where c_1 is a constant.*

$\Rightarrow a_n^{(h)} = c_1 2^n$

2) Since R.H.S of the given is 2^n and 2 is a characteristic root, the particular solution be $a_n^{(p)}=d.n.2^n$

Substituting in the given relation we get

$dn2^n-2d(n-1)2^{n-1}=2^n \Rightarrow dn2^n- d(n-1)2^n =2^n$

$\Rightarrow 2^n[dn-d(n-1)] = 2^n \Rightarrow dn-d(n-1) =1$

$\Rightarrow dn-dn+d =1 \Rightarrow d=1$

$\therefore a_n^{(p)}=n.2^n$

∴*General solution is* $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1 2^n + n.2^n$

The value of c_1 must be calculated using initial condition.

If $\boldsymbol{a_0 = 2 \Rightarrow n=0}$

$a_n = c_1 2^n + n.2^n \Rightarrow 2 = c_1 2^0 + 0.2^n \Rightarrow c_1 = 2$

∴ *The required solution is* $a_n = 2.2^n + n.2^n$

Eg-2: Solve $a_n = 3a_{n-1} + 2n$, initial condition a0 = 1.

Sol: *The given recurrence relation $a_n - 3a_{n-1} = 2^n$ is a first order Inhomogeneous recurrence relation.*

1) The associated homogeneous recurrence relation is $a_n - 3a_{n-1} = 0$

Characteristic equation is $r-3=0 \Rightarrow r=3$

∴*Homogeneous solution is* $a_n^{(h)} = c_1 r_1^n$ *where c_1 is a constant.*

$\Rightarrow a_n^{(h)} = c_1 3^n$

2) Since R.H.S of the given equation is 2^n and 2 is not the characteristic root, the particular solution be $a_n^{(p)}=d2^n$

Substituting in the given relation we get

$d2^n-3d2^{n-1}=2^n \quad \Rightarrow \quad 2^n[d-3/2d] = 2^n$

$\Rightarrow d-3/2d=1 \quad \Rightarrow 2d-3d=2$

$\Rightarrow -d=2 \quad \Rightarrow d=-2$

$\therefore a_n^{(p)}=-2.2^n$

∴ General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n=c_1.3^n-2.2^n$

The value of c_1 must be calculated using initial condition.

If $\boldsymbol{a_0 = 1 \Rightarrow n=0}$

$\Rightarrow a_0=c_1.3^0-2.2^0 \quad \Rightarrow c_1-2=1 \quad \Rightarrow c_1=1+2 \quad \Rightarrow c_1=3$

∴ Required solution is $a_n=3.3^n-2.2^n$

Eg-3: Solve $a_n - 2a_{n-1} + a_{n-2} = 2$, initial condition $a_0 = 25$ and $a_1 = 16$.

Sol:- *The given recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 2$ is a second order Inhomogeneous recurrence relation.*

1) Associated homogeneous recurrence relation is $a_n - 2a_{n-1} + a_{n-2} = 0$

Characteristic equation is $r^2 - 2r + 1 = 0$

$\Rightarrow (r-1)^2 = 0 \quad \Rightarrow r = 1,1$

∴ Homogeneous solution is $a_n^{(h)} = c_1 1^n + c_2.n.1^n$

$= (c_1 + c_2.n).1^n$

2) Since R.H.S of the given(2) is a constant, the particular solution be $a_n = d$

Substituting in the given relation we get d-2d+d = 2

$\Rightarrow$ 0 = 2 which is impossible

So take nd

$\Rightarrow$ nd – 2 (n-1)d + (n-1)d = 2 $\Rightarrow$ nd - 2nd + 2d + nd – 2d = 2

$\Rightarrow \cancel{2nd} - \cancel{2nd} + \cancel{2d} - \cancel{2d} = 2$

$\Rightarrow$ 0 = 2 which is impossible

So take n^2 d

$\Rightarrow n^2 d - 2(n-1)^2 d + (n-2)^2 d = 2$

$\Rightarrow n^2 d - 2d(n^2 + 1 - 2n) + (n^2 + 4 - 4n) \ d = 2$

$\Rightarrow n^2 d - 2dn^2 - 2d + 4dn + dn^2 + 4d - 4nd = 2$

$\Rightarrow d(n^2 - 2n^2 - 2 + 4n + n^2 + 4 - 4n) = 2$

$\Rightarrow d(2) = 2$

$\Rightarrow d = 1$

$a_n^{(p)} = n^2.1$ is the particular solution.

∴ General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (c_1 + nc_2)1^n + n^2$

The values of c1 and c2 must be calculated using the initial conditions.

If $\boldsymbol{a_0 = 25} \Rightarrow n=0$

$a_n = (c_1 + nc_2)1^n + n^2$

$\Rightarrow a_0 = (c_1 + 0c_2)1^0 + 0^2$

$\Rightarrow 25 = c_1 + 0$

$\Rightarrow c_1 = 25$ -----(1)

If $a_1 = 16 \Rightarrow n=1$

$a_n = (c_1 + nc_2)1^n + n^2$

$\Rightarrow a_1 = (c_1 + 1c_2)1^1 + 1^2$

$\Rightarrow c_1 + c_2 + 1 = 16$

$\Rightarrow c_1 + c_2 = 15$ -----(2)

From (1) and (2)

$c_1 + c_2 = 15$

$\Rightarrow c_2 = 15 - 25$

$\Rightarrow c_2 = -10$

∴ Required solution is $a_n = (25 - 10n\ )1^n + n^2$

Eg-4: Solve $a_n - 7a_{n-1} + 10a_{n-2} = 6+8n$, initial conditions $a_0 = 1$ and $a_1 = 2$.

Sol:- *The given recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 6+8n$ is a second order Inhomogeneous recurrence relation*

1) Associated homogeneous recurrence relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Characteristic equation is $r^2-7r+10=0$

$\Rightarrow r^2-5r-2r+10 \Rightarrow r(r-5) -2(r-5)=0$

$\Rightarrow (r-5)(r-2) = 0 \Rightarrow r=5, r=2$

∴ Homogeneous Solution is $a_n^{(h)} = c_1 2^n + c_2 5^n$ where c_1 and c_2 are constants.

2) Let $a_n^{(p)} = d_0 + d_1 n$ be the particular solution as the R.H.S is of the form $c_0 + c_1 n$

$\Rightarrow (d_0+d_1n)-7(d_0+d_1(n-1)) +10(d_0+d_1(n-2)) = 6+8n$

$\Rightarrow (d_0+d_1n)-7(d_0+d_1n-d_1) +10(d_0+d_1n-2d_1) = 6+8n$

$\Rightarrow d_0+d_1n-7d_0-7d_1n+7d_1+10d_0+10d_1n-20d_1 = 6+8n$

$\Rightarrow (d_0-7d_0+10d_0) + (d_1n-7d_1n+7d_1+10d_1n-20d_1) = 6+8n$

$\Rightarrow (d_0-7d_0+10d_0) +d_1(n-7n+10n+7-20) = 6+8n$

$\Rightarrow 4d_0+ d_1(4n-13) = 6+8n$

$\Rightarrow (4d_0-13d_1) + d_1 4n = 6+8n$ ------ (1)

Equating coefficients on both sides of equation (1)

$(4d_0-13d_1) = 6$ ------ (2)

$d_1 4n = 8n$ ------ (3)

Calculating the value of d_1 from equation (3)

$d_1 4\cancel{n} = 8\cancel{n} \Rightarrow 4d_1=8 \Rightarrow d_1=8/4 \Rightarrow d_1=2$

Substituting the value of d_1 in equation (3)

$4d_0-13d_1=6 \Rightarrow 4d_0-13(2) = 6 \Rightarrow 4d_0 = 6+26$

$\Rightarrow 4d0 = 32 \Rightarrow d_0=32/4 \Rightarrow d_0=8$

∴$a_n^{(p)} = 8+2n$

General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$\Rightarrow c_1 2^n + c_2 5^n + 8 + 2n$

If $a_0 = 1 \Rightarrow n=0$

$c_1 2^0 + c_2 5^0 + 8 + 2(0) = 1 \Rightarrow c_1 + c_2 + 8 = 1 \Rightarrow c_1 + c_2 = 1-8 \Rightarrow c_1 + c_2 = -7$ ----- (4)

If $a_1 = 2 \Rightarrow n=1$

$c_1 2^1 + c_2 5^1 + 8 + 2(1) = 1 \Rightarrow c_1 2 + c_2 5 + 8 + 2 = 2 \Rightarrow 2c_1 + 5c_2 = 2-10 \Rightarrow 2c_1 + 5c_2 = -8$ ----- (5)

By solving (4) and (5) we will get the values of c_1 and c_2

$c_2 = 2,\ c1=-9$

∴ The required solution is $(-9)2^n+2.5^n+8+2n$

Eg-5: Solve $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$, initial conditions $a_0 = 1$ and $a_1 = 1$.

Sol:- *The given recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} - 3n - 2^n$ is a second order Inhomogeneous recurrence relation*

1) Associated homogeneous recurrence relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Characteristic equation is $r^2-4r+4=0$

$\Rightarrow (r-2)^2=0 \quad \Rightarrow \quad r=2, 2$

∴ Homogeneous solution is $a_n^{(h)} = c_1 2^n + c_2.n.2^n$ where c_1 and c_2 are constants.

2) R.H.S = $3n + 2^n$

∴ Particular solution $a_n^{(p)} = a_n^{(p1)} + a_n^{(p2)}$ where p1 for 3n and p2 for 2^n

$a_n^{(p1)} = d_0+d_1n$ [3n can be written as 3n+0 which is of the form c_0+c_1n]

$\Rightarrow (d_0+d_1n) - 4[d_0+d_1(n-1)] + 4[d_0+d_1(n-2)] = 3n$

$\Rightarrow (d_0+d_1n) - 4(d_0+d_1n-d_1) + 4(d_0+d_1n-2d_1) = 3n$

$\Rightarrow (d_0+d_1n) - (4d_0+4d_1n-4d_1) + (4d_0+4d_1n-8d_1) = 3n$

$\Rightarrow (d_0 \cancel{-4d_0} \cancel{+4d_0}) + d_1(n \cancel{-4n} +4 \cancel{+4n} -8) = 3n$

$\Rightarrow d_0 + d_1(n-4) = 3n \qquad \Rightarrow d_0 + d_1.n - 4d_1 = 3n$

$\Rightarrow (d_0 - 4d_1) + d_1.n = 3n$ ----- (1)

Equating coefficients of equation (1) on both sides

$d_1.n = 3n$ ----- (2)

$(d_0 - 4d_1) = 0$ ----- (3)

From equation (2) $d_1.\cancel{n} = 3\cancel{n} \qquad \Rightarrow d_1 = 3$

Substituting the value of d_1 in equation (3)

$\Rightarrow (d_0 - 4d_1) = 0 \qquad \Rightarrow (d_0 - 4(3)) = 0 \qquad \Rightarrow d_0 - 12 = 0 \quad \Rightarrow d_0 = 12$

∴ $a_n^{(p1)} = 12+3n$

$a_n^{(p2)} = d.n^2.2^n$[since R.H.S is 2^n and characteristic roots are (2,2)]

$\Rightarrow dn^2 2^n - 4d(n-1)^2 2^{n-1} + 4d(n-2)^2 2^{n-2} = 2^n$

$\Rightarrow dn^2 2^n - 4d(n^2+1-2n)2^{n-1} + 4d(n^2+4-4n)2^{n-2} = 2^n$

$\Rightarrow 2^n[dn^2 - 4/2d(n^2+1-2n) + 4/4d(n^2+4-4n)] = 2^n$

$\Rightarrow \cancel{2^n}[dn^2 - 2d(n^2+1-2n) + d(n^2+4-4n)] = \cancel{2^n}$

$\Rightarrow [dn^2 - 2d(n^2+1-2n) + d(n^2+4-4n)] = 1$

$\Rightarrow dn^2 - 2dn^2 - 2d + 4nd + dn^2 + 4d - 4nd = 1$

$\Rightarrow \cancel{2dn^2} - \cancel{2dn^2} \cancel{+4nd} \cancel{-4nd} - 2d + 4d = 1$

$\Rightarrow 2d = 1 \qquad \Rightarrow d = 1/2$

∴ $a_n^{(p2)} = ½ n^2.2^n \qquad = n^2.2^{n-1}$

∴ $a_n^{(p)} = a_n^{(p1)} + a_n^{(p2)} \quad \Rightarrow 12+3n+n^2.2^{n-1}$

General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)} \qquad \Rightarrow (c_1+c_2n)2^n + 12 + 3n + n^2.2^{n-1}$ ----- (4)

If $a_0 = 1 \Rightarrow$ n=0 [substitute the values in equation (4)]

$\Rightarrow (c_1+c_2.0)2^0 + 12 + 3(0) + 0^2.2^{0-1} = 1 \qquad \Rightarrow (c_1).1 + 12 = 1$

$\Rightarrow c_1 = 12-1 \qquad \Rightarrow c_1 = -11$

If $a_1 = 1 \Rightarrow$ n=1 [substitute the values of n, a_1 and c_1 in equation (4) to get the value of c_2]

$\Rightarrow (c_1+c_2.1)2^1 + 12 + 3(1) + 1^2.2^{1-1} = 1$

$\Rightarrow (c_1+c_2)2 + 12 + 3 + 1.2^0 = 1 \qquad \Rightarrow 2(c_1+c_2) + 15 + 1 = 1$

$\Rightarrow 2(c_1+c_2) + 16 = 1 \qquad \Rightarrow 2c_1 + 2c_2 = 1-16$

$\Rightarrow 2c_1 + 2c_2 = -15 \qquad \Rightarrow 2(-11) + 2c_2 = -15 \qquad \Rightarrow -22 + 2c_2 = -15$

$\Rightarrow 2\,c_2 = -15+22$ $\Rightarrow 2\,c_2 = 7$ $\Rightarrow c2= 7/2$

$\therefore$ The required solution is $(-11 + (7/2)\,n)\,2^{n+}12+3n+ n^2.2^{n-1}$