minimizing expected loss for hypothesis testing

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posterior probabilities & decision

- suppose you have two competing hypotheses:

  H_1 and H_2

- then
  - $P(H_1 \textit{ is true} \mid data)$ = posterior probability of H_1
  - $P(H_2 \textit{ is true} \mid data)$ = posterior probability of H_2
- potential decision criterion: choose the hypothesis with the higher posterior probability

reject H_1 if $P(H_1 \textit{ is true} \mid data) < P(H_2 \textit{ is true} \mid data)$

- alternative: consider a loss function

example: HIV testing with ELISA

H_1 : Patient doesn't have HIV

H_2 : Patient has HIV

- these are the only two possibilities.
- they are mutually exclusive hypotheses that cover the entire decision space.

loss functions & decisions

- $L(d)$: loss that occurs when decision d is made
- Bayesian testing procedure then minimizes the posterior expected loss
- possible decisions (actions):
  - d_1 : choose H_1 - decide that the patient doesn't have HIV
  - d_2 : choose H_2 - decide that the patient has HIV

making the right (or wrong) decision

$d = d_1$

- **right:** decide patient doesn't have HIV, and indeed they don't
$$L(d_1) = 0$$
- **wrong:** decide patient doesn't have HIV, but they do
$$L(d_1) = w_1$$

$d = d_2$

- **right:** decide patient has HIV, and indeed they do
$$L(d_2) = 0$$
- **wrong:** decide patient has HIV, but they don't
$$L(d_2) = w_2$$

loss depends on consequences

- consequences of making a wrong decision d_1 or d_2 are different
- wrong d_1:
  - decide that patient doesn't have HIV when in reality they do — false negative
  - potential consequences: no treatment and premature death!
- wrong d_2:
  - decide that the patient has HIV when in reality they don't — false positive
  - potential consequences: distress and unnecessary further investigation

back to example: HIV testing with ELISA

hypotheses

H_1: Patient does not have HIV

H_2: Patient has HIV

decision

d_1: Decide that patient does not have HIV

d_2: Decide that patient has HIV

losses

$$L(d_1) = \begin{cases} 0 & \text{if } d_1 \text{ is right} \\ w_1 = 1000 & \text{if } d_1 \text{ is wrong} \end{cases}$$

$$L(d_2) = \begin{cases} 0 & \text{if } d_2 \text{ is right} \\ w_2 = 10 & \text{if } d_2 \text{ is wrong} \end{cases}$$

posteriors

$P(H_1 \mid +) \approx 0.88$

$P(H_2 \mid +) \approx 0.12$

expected losses

$E[L(d_1)] = 0.88 \text{ x } 0 + 0.12 \text{ x } 1000 = 120$

$E[L(d_2)] = 0.88 \text{ x } 10 + 0.12 \text{ x } 0 = 8.8$

back to example: HIV testing with ELISA

hypotheses

H_1 : Patient does not have HIV

H_2 : Patient has HIV

decision

d_1 : Decide that patient does not have HIV

d_2 : Decide that patient has HIV

losses

$$L(d_1) = \begin{cases} 0 & \text{if } d_1 \text{ is right} \\ w_1 = \cancel{1000}\ 10 & \text{if } d_1 \text{ is wrong} \end{cases}$$

$$L(d_2) = \begin{cases} 0 & \text{if } d_2 \text{ is right} \\ w_2 = 10 & \text{if } d_2 \text{ is wrong} \end{cases}$$

posteriors

$P(H_1 \mid +) \approx 0.88$

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expected losses

$E[L(d_1)] = 0.88 \text{ x } 0 + 0.12 \text{ x } \cancel{1000}\ 10 = \cancel{120}\ 1.2$

$E[L(d_2)] = 0.88 \text{ x } 10 + 0.12 \text{ x } 0 = 8.8$

summary

- Bayesian methodologies allow for the integration of losses into the decision making framework easily
- in Bayesian testing we minimize the posterior expected loss