Predictive Inference with Integrals

In the lecture, we considered predicting the probability of Heads when there were two coins, each with different probabilities of landing Heads. The coin was drawn at random from a bag, so at first both coins were equally likely. But then the chosen coin was tossed twice, and we saw two Heads, so now the posterior probability of one of the coins, the one with the greater chance of Heads, is a little more larger than the other. The lecture showed how to use those posterior probabilities to calculate the probability that a third toss would be a Heads.

Before extending to the integral case, first consider the general discrete case. Suppose that there are K coins. The kth coin is known to have probability p_k of landing heads, for $k = 1, \ldots, K$. We draw a coin at random and toss it twice—suppose we see two Heads. We can now use the discrete form of Bayes' rule to calculate the posterior probability of each coin. If π_k is the prior probability for each coin, then the posterior probability of the first coin, given two Heads, is:

$$\begin{aligned}
\pi_1^* &= \frac{\mathbb{P}[\text{2 Heads} \mid \text{first coin}] \times \pi_1}{\sum_{k=1}^K \mathbb{P}[\text{2 Heads} \mid \text{coin } k] \times \pi_k} \\
&= \frac{\binom{2}{2} p_1^2 (1-p_1)^0 \times \pi_1}{\sum_{k=1}^K \binom{2}{2} p_k^2 (1-p_k)^0 \times \pi_k}.
\end{aligned}$$

This uses the binomial formula to calculate the probability of getting two Heads with each coin. Similar calculation can obtain the posterior probability π_k^* for each of the K coins.

To find the probability that the third toss is Heads, one uses conditional probability.

$$\begin{aligned}
\mathbb{P}[\text{toss 3 is Heads }] &= \sum_{k=1}^K \mathbb{P}[\text{ toss 3 is Heads and you have the } k\text{th coin}] \\
&= \sum_{k=1}^K \mathbb{P}[\text{toss 3 is Heads} \mid \text{coin } k] \times \mathbb{P}[\text{coin } k] \\
&= \sum_{k=1}^K p_k \times \pi_k^*. \qquad (1)
\end{aligned}$$

Now, for the integral. Suppose there are an uncountable number of coins, each with a different probability p of landing Heads. One can describe this situation as having a prior that is a continuous probability density function on the interval $[0, 1]$. For specificity, assume the prior is the Beta(1,1) distribution.

You observe two Heads, so from the lecture on Beta-Binomial conjugate families you know that your posterior belief about p, the chance of coming up Heads, is Beta(3, 1). Its probabilty density function is $\pi^*(p) = 3p^2$ for $0 \le p \le 1$.

Now look at equation (1). We replace the discrete summation by the continuous integral, the discrete posterior probabilities π_k^* by the continuous posterior density $\pi^*(p)$, and the discrete probabilities p_k by the continous probability p. Then the probability that the third toss is Heads is

$$\begin{aligned} \mathbb{P}[\text{toss 3 is Heads}] &= \int_0^1 p \times \pi^*(p)\, dp \\ &= \int_0^1 p \times 3p^2\, dp \\ &= 3/4. \end{aligned}$$