Duality in optimization, KKT and shadow prices

With applications to electricity markets

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These are some **basic notes** on **duality theory** and the **KKT conditions for optimality**, focusing on concepts rather than on detailed mathematical proofs. We end with a discussion on the meaning of **shadow prices**.

1. **What** is the dual problem?

Optimization problems may be viewed from either of two perspectives: the primal problem or the dual problem. If the primal is a minimization problem, then the dual is a maximization problem (and vice versa).

2. **Why is it useful** to formulate the dual problem?

Among other potential advantages:

- By **combining the primal and the dual** problems we can deduce conditions for checking if any point is a local optimum, the so called '**KKT conditions**'.
- The dual gives a **lower bound on the optimal value of the original problem** (we call the original optimization the 'primal problem'). This is very useful for numerical algorithms for optimization, as the dual can provide a cheap certificate for a lower bound of the global optimum. The proof is given later in this document.
- The dual might be **computationally easier to solve in some cases**. Furthermore, if the primal problem is convex, the solution of the dual is the same as the solution of the primal (because 'strong duality' holds in practically all relevant convex problems, explained later in this document).
- The dual gives **information about 'shadow prices'**: the value of a dual variable (defined later in this document) represents the change in the optimal value of the primal due to relaxing infinitesimally the constraint that the dual variable is associated with. This has an economic meaning, which we discuss later.
- The dual **gives information about feasibility of the primal**: if the dual is unbounded, the primal is infeasible, and vice versa. More details on this can be found at the beginning of section 5.2.2 in reference [1].

3. Formulating the dual

Let's consider a general optimization problem in standard form, with equality and inequality constraints. The primal problem is:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \\ & h_i(x) = 0 \end{array}$$

We discuss next how to solve this problem.

3.1 The Lagrangian function

To **help us solve the above problem**, we can make use of the Lagrangian function.

The basic idea in Lagrangian duality is to **take** the **constraints into account** by **augmenting** the **objective function** with a weighted sum of the constraint functions. The Lagrangian function is then:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Where 'λ' and 'ν' are called 'Lagrange multipliers' or 'dual variables'.

At this point it might be helpful to read the first section 'Summary and rationale' of the Wikipedia page on the method of Lagrange multipliers for solving optimization problems with just equality constraints. This gives an idea of why augmenting the objective function with the constraints is helpful for solving such problems.

But why is the Lagrangian function useful for constrained optimization? ('constrained' meaning that the objective function is subject to both equality and inequality constraints)

To begin with, the **value of the Lagrangian** is **always lower than** the **original objective function** for **any** given **feasible point**, when setting 'λ ≥ 0'. That is:

$\boxed{L(x, \lambda, \nu) \leq f_0(x)}$ for any feasible point 'x', i.e., meeting $f_i(x) \leq 0$ and $h_i(x) = 0$

Proof:

(from Prof. Spyros Chatzivasileiadis's course, slide 10)

Assume $\tilde{x}$ feasible point, i.e. $f_i(\tilde{x}) \leq 0,\ h_i(\tilde{x}) = 0,\ \lambda \geq 0.$ Then we have

This condition is important, as otherwise the product '$\lambda_i \cdot f_i(\tilde{x})$' would not be less than 0.

$$\sum_i \lambda_i f_i(\tilde{x}) + \sum_i \nu_i h_i(\tilde{x}) \leq 0$$

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_i \lambda_i f_i(\tilde{x}) + \sum_i \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$

So the **Lagrangian** is a **lower bound** for the objective function when evaluated in any feasible point.

3.2 The dual function

Given the proof above, by **minimizing** the **Lagrangian over 'x'** (which is an unconstrained optimization problem), we can get a **lower bound for** the **optimal value** of the original constrained optimization ('optimal value' meaning the value of the objective function at the optimal solution).

The infimum (global optimum) of the Lagrangian function is called the Lagrangian **dual function**:

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

The independent variables in the dual function 'g(λ, ν)' are just the dual variables 'λ' and 'ν', given that we have already minimized over the primal variable 'x'.

The **dual function** gives **lower bounds on the optimal value of** the **primal problem**. This can be easily seen by considering the result on the previous page:

$$g(\lambda, \nu) = \inf_x L(\tilde{x}, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

- This holds for every feasible point $\tilde{x}$, including the optimal point x^*.

Interestingly, the **dual function** is **always concave**, even if the primal problem is non-convex (see slides 8-9 and section 5.1.2 for the demonstration). You can find a **definition of convexity** later in this document.

3.2.1 Particular case: a linear program

Since it is the easiest problem to understand, let's consider the **particular case of** a **linear program**, i.e., an optimization problem where both the objective function and all constraints are linear:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0, \end{array}$$

(where the 'curved ≥' symbol simply means that all components of vector 'x' must be positive)

Its **corresponding Lagrangian** function is:

$$L(x, \lambda, \nu) = c^T x - \sum_{i=1}^{n} \lambda_i x_i + \nu^T (Ax - b) = (c + A^T \nu - \lambda)^T x - b^T \nu$$

(note that all vectors are defined as column vectors, hence the use of the 'transpose' operator; the minus sign for the λ term is because the inequality constraint is now '≥', not '≤' as in the previous page, so condition 'λ ≥ 0' still applies for the Lagrangian to be a lower bound of the objective function)

And its **corresponding dual function**:

$$g(\lambda, \nu) = \min_x L(x, \lambda, \nu) = -b^T\nu + \min_x (c + A^T\nu - \lambda)^T x$$

The optimization problem above has the following solution:

$$g(\lambda, \nu) = \begin{cases} -b^T\nu & \text{if } A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Since a lower bound of '−∞' is trivial, we are only interested in the upper solution. This result will be **relevant in the next section**.

3.3 The dual problem

Given that for each pair '(λ, ν)' the dual function gives us a lower bound for the optimal value of the primal problem, by **maximizing** the **dual function over** the **dual variables 'λ'** and **'ν'** we can **obtain the best lower bound** (i.e., the maximum lower bound possible).

This maximization is called the Lagrange **dual problem**:

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

There are some **constraints 'hidden'** or **'implicit' in** the **objective function** of the dual problem. We won't consider here the general case, as it is best to understand this in the **particular case of** a **linear program**.

Recall from the previous section on linear programming that the dual function is meaningless (i.e., unbounded) unless the upper condition in the following expression holds:

$$g(\lambda, \nu) = \begin{cases} -b^T\nu & \text{if } A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

So **the 'hidden' constraint is exactly that condition**:

$$A^T\nu - \lambda + c = 0$$

(equivalent conditions would appear in a more general, i.e., non-linear, primal problem; however, we won't discuss them here)

We can formulate an equivalent to the dual problem, explicitly including that condition:

$$\boxed{\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu - \lambda + c = 0 \\ & \lambda \succeq 0. \end{array}}$$

Note that typically it's **not necessary** to **write** the **dual problem 'by hand'**, as **optimization solvers give** the numerical **solution of** the **dual** too.

4. Weak and strong duality

As we have shown, the optimal value of the Lagrange dual problem, which we denote 'd*', is the best lower bound on 'p*' (optimal value of the primal) that can be obtained from the Lagrange dual function. We then have the simple but important inequality:

$$d^\star \leq p^\star$$

Which holds even if the original problem is not convex. This property is called '**weak duality**'.

We refer to the **difference 'p* – d*'** as the '**duality gap**' of the original problem**. If the duality gap is zero** for a given problem, then we say that '**strong duality**' holds.

Strong duality holds for convex problems[1], meaning that the **optimal value of the primal and dual problems is the same**. Given that a **linear program** is a **convex** problem, **strong duality holds** (the proof can be found in section 5.2.4 of [1] and in reference [2], but these are mathematically dense). On the other hand, **in general, strong duality does not hold for non-convex problems**.

5. KKT optimality conditions

Making use of what we have learned from duality, we are now ready to formulate optimality conditions for a constrained optimization problem (i.e., conditions that must be met by the optimal solution). These are very useful for actually solving an optimization problem, and are widely used by numerical solvers.

We will see next how to obtain the **Karush-Kuhn-Tucker (KKT)** conditions: these are **4 necessary** (but not sufficient) **conditions for optimality** of a general optimization problem (including non-convex, although with some caveats that are briefly discussed later). They are **first-order conditions** (they only include first derivatives).

- It might be helpful to first think of an **unconstrained optimization** problem, where we just need to minimize a given function: the **necessary condition** for **optimality** is to set the **derivative equal to zero**[2]. But this is not sufficient, higher order derivatives are necessary to check if a certain point where the derivative is zero is actually a minimum (see 'derivative test' for more info). Then, the **KKT** is the **equivalent necessary condition**, but **for constrained optimization** problems (accepting both equality and inequality constraints).

5.1 Deduction of the KKT

To obtain the KKT, we must **start assuming** that **strong duality holds** (later on we briefly comment on the use of KKT when strong duality does not hold, see section 5.3). This gives us the following

[1] Some regularity condition (e.g., Slater's condition) must be met in addition to the problem being convex, but most practical convex problems meet this condition, for which we omit further details.

[2] We assume a smooth function, so that we can compute the derivate; a similar approach using a 'subgradient' can be applied to non-smooth functions, but we won't discuss it here.

expressions, where 'x*' is an optimal solution to the primal problem (i.e., '$f_0(x^*) = p^*$', as used in section 4), and '(λ^*, ν^*)' is an optimal solution to the dual problem (i.e., '$g(\lambda^*, \nu^*) = d^*$'):

$$\begin{aligned} f_0(x^\star) &= g(\lambda^\star, \nu^\star) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \nu_i^\star h_i(x) \right) \\ &\le f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \nu_i^\star h_i(x^\star) \\ &\le f_0(x^\star). \end{aligned}$$

The 1st line states that the duality gap is zero (we are assuming strong duality). The 2nd line is simply the definition of the dual function (see section 3.2). To understand the inequality between the 2nd and 3rd lines, note that any minimization over 'x' (which is what the infimum is doing) will be lower or equal than the value of the same function evaluated in a point 'x*' that might not be the minimizer for this function ('x*' is the minimizer of the primal objective function '$f_0(x)$', but not necessarily of the Lagrangian function '$L(x,\lambda^*,\nu^*)$'). Finally, the 3rd line is the Lagrangian function, which we have already proven to be lower than the primal objective function for any feasible point (see section 3.1), which is the statement on the 4th line.

Note that the first term in the above expression is the same as the last term. So all the inequalities are actually equalities. By looking at the 3rd and 4th lines and setting an equality between them, we see that the following term must be zero:

$$\sum_{i=1}^m \lambda_i^\star f_i(x^\star) = 0$$

(remember that '$h_i(x^*) = 0$', which is a feasibility condition for the primal problem stated at the very beginning of section 3)

Since '$\lambda^* \geq 0$' and '$f_i(x^*) \leq 0$' (recall section 3.1), all values in the sum would be negative, so their total sum would never be zero, unless each of the products is zero:

$$\boxed{\lambda_i^\star f_i(x^\star) = 0,} \quad i = 1, \ldots, m.$$

This condition is known as **'complementary slackness'** (one of the four KKT conditions). It states that the product of an optimal dual variable and its corresponding inequality constraint evaluated at the optimal primal solution must be zero: **either** the **dual variable or** the **function in the constraint must be zero**.

- This shows an **interesting property**: an 'active' constraint (meaning that the 'less than or equal to' holds as an equality, '$f_i(x^*) = 0$') in the optimal solution implies that its corresponding dual variable is positive. On the other hand, an 'inactive' constraint (i.e., '$f_i(x^*) < 0$', which means that the constraint could be removed from the optimization without making any difference) implies that its corresponding dual variable is zero.

We can get two more optimality conditions from the deduction at the top of the previous page: note that for the inequality between the 3rd and 4th lines to hold, which we demonstrated in section 3.1, **primal feasibility** (meaning that '$f_i(x) \leq 0$' and '$h_i(x) = 0$') **and dual feasibility** (meaning that '$\lambda \geq 0$') **must hold**. Otherwise, the Lagrangian would not be a lower bound for the primal objective function (see section 3.1), and therefore our deduction in the previous page would not make sense. Then, '**primal feasibility**' and '**dual feasibility**' are the next two of the four KKT conditions:

$$\boxed{f_i(x^\star) \leq 0\ , \quad h_i(x^\star) = 0} \qquad\qquad \boxed{\lambda_i^\star \geq 0}$$

We have already obtained three of the four KKT conditions. The remaining one is called '**stationarity**', and to deduce it we must go back to the expression that we started with:

$$\begin{aligned} f_0(x^\star) &= g(\lambda^\star, \nu^\star) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^\star f_i(x) + \sum_{i=1}^{p} \nu_i^\star h_i(x) \right) \\ &= f_0(x^\star) + \sum_{i=1}^{m} \lambda_i^\star f_i(x^\star) + \sum_{i=1}^{p} \nu_i^\star h_i(x^\star) \\ &= f_0(x^\star). \end{aligned}$$

Where the inequalities have been replaced by equalities. Note that the 2nd and 3rd lines prove that the primal solution 'x^*' is in fact also a minimizer of the Lagrangian function '$L(x,\lambda^*,\nu^*)$' over 'x'. Therefore, the **gradient** of the **Lagrangian** function **must be zero** at the **optimal solution**, 'x^*' (this is also a condition for optimality in unconstrained optimization, recall the derivative test mentioned at the beginning of section 5):

$$\boxed{\nabla_x f_0(x^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla_x f_i(x^\star) + \sum_{i=1}^{p} \nu_i^\star \nabla_x h_i(x^\star) = 0}$$

Which is the fourth and last of the KKT conditions, the '**stationarity condition**'.

5.2 List of KKT conditions

To summarise the previous section, the **KKT** are **necessary conditions for optimality** **when strong duality holds**. They are however not sufficient, meaning that an optimal solution must meet the KKT conditions, but a point that meets the KKT conditions is not guaranteed to be an optimizer. A 'KKT point' might be a local or global minimum of the original optimization problem, a saddle point, or a local or global maximum.

The KKT are four conditions:

1) **Stationarity**: the gradient of the Lagrangian with respect to the primal variables must be zero.

$$\nabla_x f_0(x^\star) + \textstyle\sum_{i=1}^{m} \lambda_i^\star \nabla_x f_i(x^\star) + \sum_{i=1}^{p} \nu_i^\star \nabla_x h_i(x^\star) \quad = \quad 0$$

2) **Complementary slackness**: a dual variable is zero unless its associated inequality constraint is binding ('binding' meaning that the constraint is 'active' at the optimal solution).

$$\lambda_i^\star f_i(x^\star) = 0$$

3) **Primal feasibility**: the inequality and equality constraints in the primal problem must hold.

$$\begin{aligned} f_i(x^\star) &\leq 0 \\ h_i(x^\star) &= 0 \end{aligned}$$

4) **Dual feasibility**: all dual variables associated to inequality constraints must be non-negative (either zero or positive).

$$\lambda_i^\star \geq 0$$

Finally, a note on an important case when the KKT conditions are actually sufficient:

For a convex problem, the KKT are sufficient conditions for optimality [3]

5.3 Uses of the KKT

Given the above statement, KKT conditions can **transform** a **convex** optimization problem **into** a **set of algebraic equations and inequations** (which always include non-linear equations even if the optimization problem is linear, because complementary slackness implies the product of variables).

Therefore, the KKT play an important role in optimization. In a few special cases, it is possible to solve the KKT conditions (and therefore, the optimization problem) analytically. More generally, **many algorithms** for **convex optimization** are conceived as, or can be interpreted as, **methods for solving** the **KKT conditions**.

Finally, a quick note on solving optimization problems without formulating the dual problem. While we started the discussion here defining the dual problem, which led to the KKT conditions, solving a constrained optimization does not necessarily imply formulating its dual. That is, even without thinking of duality, there are methods to solve a constrained optimization problem simply by using the primal formulation. See for example the Simplex algorithm for linear programming.

5.4 KKT when strong duality does not hold

Although we have only deduced the KKT by assuming that strong duality holds, these conditions are **also useful** for **non-convex problems**. Non-convex problems have, in general, a non-zero duality gap: some non-convex problems might happen to meet strong duality but, in general, this property is not met.

[3] This is true for most practical convex problems, although it is in fact only true if the convex problem meets a regularity condition (e.g., Slater's), but this is typically met in practice. **For linear programs, the KKT are always sufficient** (the proof is given in section 5.5.3 of [1]).

The KKT are in fact also **necessary optimality conditions** for a **non-convex** smooth constrained optimization problem, if a mild 'constraint qualification' (i.e., regularity condition) is satisfied. We omit further details, but if you are interested in proofs, you can refer to, e.g., "Numerical Optimization" by Nocedal and Wright. For discrete optimization, which is non-smooth, the KKT simply cannot be directly computed (see the discussion in section 7 for more details).

Given that a 'KKT point' (i.e., a solution to the KKT conditions) might be a local or global minimum of the original optimization problem, a saddle point, or a local or global maximum, **further checks** are **needed** to **determine if** the **point is** in fact a **local optimizer**: second order optimality conditions can be used (or *n*-th order, if necessary). These conditions are conceptually similar to the derivative test for unconstrained optimization.

At this point, we could think of applying the following steps to solve a non-convex optimization problem (i.e., to obtain its global optimum):

1) Formulate the KKT conditions and solve them.
2) Apply further derivative tests to leave only the KKT points that are local minimizers.
3) Evaluate the objective function at all these local minimizers and take the point that gives the lowest objective value overall.

This way, we could obtain the global optimum to a non-convex problem. However, this procedure is intractable in practice for general non-convex problems: just Step 1 might be extremely difficult, as it implies solving a set of non-linear equations and inequations. Then, if Step 1 is solvable, Step 3 can imply function evaluations for an enormous number of points.

In conclusion, although the KKT gives us some valuable information about smooth non-convex problems, **finding** the **global optimum** of a **non-convex** problem is **extremely difficult** (even finding a local optimum is not typically done using Steps 1 and 2 above, but rather using 'interior point' methods as in the numerical solver Ipopt). Solving a non-convex optimization problem is in fact 'NP-hard', meaning in lay terms that it might be impossible to compute a solution in a reasonable amount of time.

6. On the concept of '**shadow prices**'

Dual values can be thought of as 'shadow prices', as they **measure** the **sensitivity of** the **optimal value** of the primal **to a change in the constraint**. The value of a dual variable represents the change in the optimal value of the primal due to relaxing infinitesimally the constraint that the dual variable is associated with (we omit the proof, refer to sections 5.4.4 and 5.6 of [1] for a thorough albeit dense explanation).

So we can **interpret the value** of the **dual variables** as '**prices**', if the optimization problem that they belong to is some economic minimization or maximization. The **primal** is typically interpreted as a '**resource allocation**' problem (e.g., how much electric power to produce from each of the generators in an electricity grid, in order to minimize overall cost), while the **dual** is a '**resource valuation**' problem (e.g., what is the value of electric power for the sellers and buyers involved). **Primal variables** are **quantities** while **dual variables** are **prices**.

- For **some mathematical details** on why the derivative of the objective function, evaluated at the optimal solution, and taken with respect to the 'slack' parameter of a constraint (i.e., the extra margin that we can give to that constraint) gives as a result the corresponding dual variable, see sections 1.2.3 and 1.3 of these notes from UC Berkeley.

- For **some intuition** on **practical applications** of shadow pricing, let's consider a very simple electricity market:

$$\begin{aligned} \underset{P_{g,i}}{\text{minimize}} \quad & 5 \cdot P_{g,1} + 10 \cdot P_{g,2} + 15 \cdot P_{g,3} \\ \text{subject to} \quad & \sum_{i=1}^{3} P_{g,i} = \text{Demand} \\ & 0 \text{ MW} \leq P_{g,i} \leq 20 \text{ MW}, \quad \forall i \end{aligned}$$

  Where we have 3 generators, each with a maximum capacity of 20 MW, and with marginal generation costs of 5 €/MWh, 10 €/MWh and 15 €/MWh (we consider that this is an hourly electricity market, so a generator operating at, say, 10 MW for an hour would produce an energy of 10 MWh).

  'Demand' is a parameter, i.e., a constant value in the optimization, not a decision variable. Suppose the value of this parameter 'Demand' is of 50 MWh.

  Now look at the 'Total Generation == Demand' constraint: if the fixed parameter 'Demand' decreases from 50 MWh to 49 MWh, the objective value would decrease. **By how much would it decrease?** By exactly the optimal value of the dual variable associated to the 'Generation == Demand' constraint. You can **check it yourself**, either solving this trivial optimization problem by hand (simply draw the staircase-like objective function and cut it by a vertical line of the 50 MWh of inelastic demand), or run this **Julia/JuMP code** (the JuMP library for the Julia programming language is a very useful tool for solving optimization problems, here are some instructions on how to get started with it).

- For **another example** related to **electricity**, but this time **pricing** a **service included in** an **inequality constraint**, consider **'Reserve'**: an electricity system needs to carry some volume of 'reserve', meaning some generators don't operate at full output to be able to rapidly increase power output if any other generation in the system fails, so that demand can still be supplied. The type of constraint that would be included in the above electricity market formulation is 'Reserve ≥ 10 MW' (the value of '10 MW' is arbitrary here). For details on the meaning of prices for reserve, see section 5.1 in this paper.

But **why should the price** of a commodity or service **be equal to the sensitivity of the optimal value?** The economic interpretation of the dual variables as prices is based on the **welfare theorems of microeconomics**. While these theorems are **mathematical idealizations** that almost never fully apply in real markets, if used carefully, they can be very **useful to guide** the **behaviour of** the **many selfish agents** operating in a liberalized economy. More on this in the next section.

6.1 Advantages and limitations of shadow prices

The **main advantage** of shadow prices is to **lead selfish agents to act in** a **socially optimal way**, under certain conditions: 1) **strong duality** must hold when formulating the market via an optimization problem; and 2) **perfect competition** must apply, meaning no market participant has the capability to distort the market outcome in their own benefit.

In a sense, this is Adam Smith's 'invisible hand of the market'. As we will see next, the two conditions listed above are rarely met in practice, which is why completely unregulated markets never lead to what's best for society.

Why are the two conditions above necessary?

1) If the **duality gap** is **larger than zero** (i.e., if strong duality does not hold), shadow prices don't correspond to an 'economic equilibrium', as the **optimal solution** of the **dual problem does not correspond to** the **optimal solution** of the **primal**. So some market participants would want to deviate from the solution given by the shadow prices, as they could be better off. You can refer to Chapter 6 of [3] for more details.

    - Therefore, **we typically seek convex markets**. Recall from section 4 in this document that strong duality holds practically always for convex problems, while it typically does not for non-convex problems.
    - **Physical models may be non-convex**, as is the case for the power flow equations in alternating current, which govern the behaviour of electricity grids, and therefore must be incorporated into formulations of electricity markets. More on this in section 6.3.2.

2) If **no market agent has** the capability to exert '**market power**' (i.e., we are in a perfectly competitive setting), the **solution of** the **equilibrium problem** and the **central optimization is the same**. We have introduced here some concepts such as 'equilibrium', which we discuss next in section 6.2, but the main message is that **shadow prices incentivize individuals to behave in** a **socially optimal way** in this setting.

    - For the **proof** of why the decisions of different selfish agents converge to the solution given by the shadow prices, if perfect competition holds, see Prof. Jalal Kazempour's slides.
    - **When does imperfect competition arise?** There can be imperfect competition if some market participants are sufficiently large or are strategically placed within a network, so they can influence prices by adapting their bidding strategies, increasing their own profit by doing so. We call these agents 'price-makers', while in perfect competition all agents are 'price-takers'.

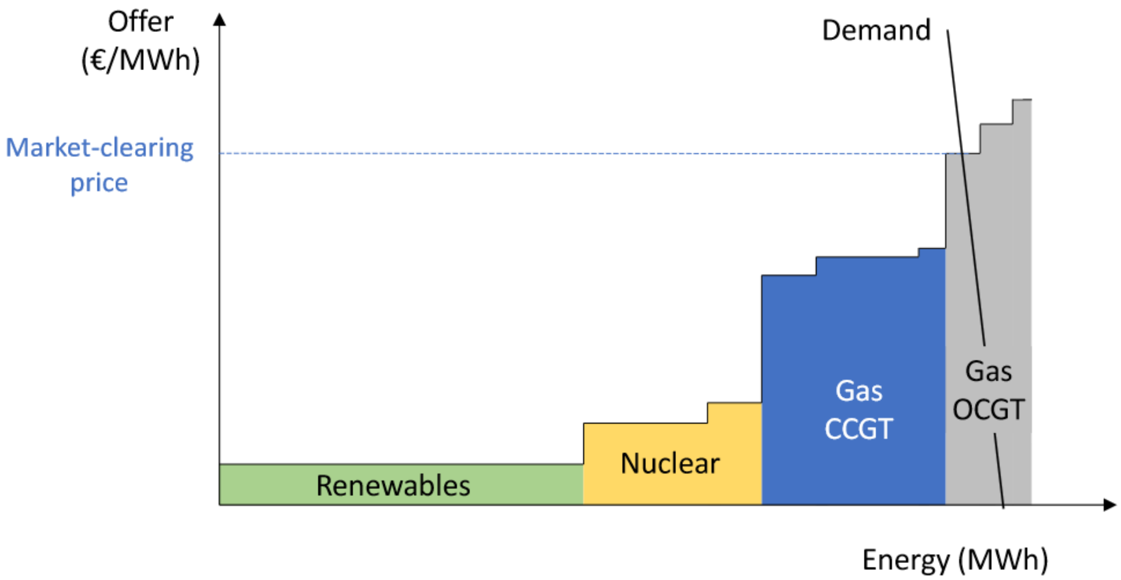
In summary, **non-convexities** and **market power** are important reasons why **shadow prices do not support** the **socially optimal solution**.

6.2 Economic equilibrium

To better understand the explanations above, it's important to define the concept of '**economic equilibrium**': it is a **market outcome** from which **participants do not have** an **incentive to unilaterally deviate.** Note the word 'unilaterally', since agreeing on strategic behaviour with other market participants would entail collusion, an illegal practice.

Consider a simple electricity market, as represented in the picture below**:**

- The **supply curve** is monotonically increasing: **producing more electricity increases** the **total cost**. Different generation technologies have different operating costs, based on their fuel and maintenance requirements.
- The **demand curve** is monotonically decreasing: **consumers** will **buy less electricity if** the **price increases**. We have chosen here a simple line for representing the so called 'inverse demand function', which is almost vertical since demand for electricity is quite 'inelastic' (meaning that we will consume roughly the same amount of electricity regardless of its price, since we need this commodity for our most basic needs).

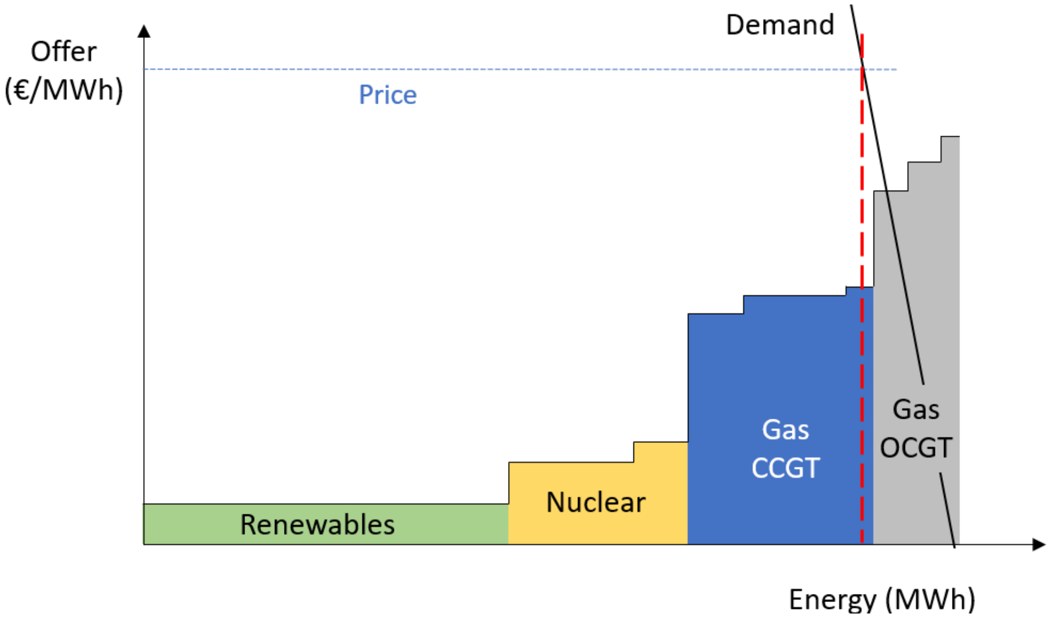


The equilibrium is reached at the **intersection of these two curves**. It is an equilibrium because the amount of electricity produced equals the amount of electricity purchased.

Neither producers nor consumers have an **incentive to deviate** from this solution:

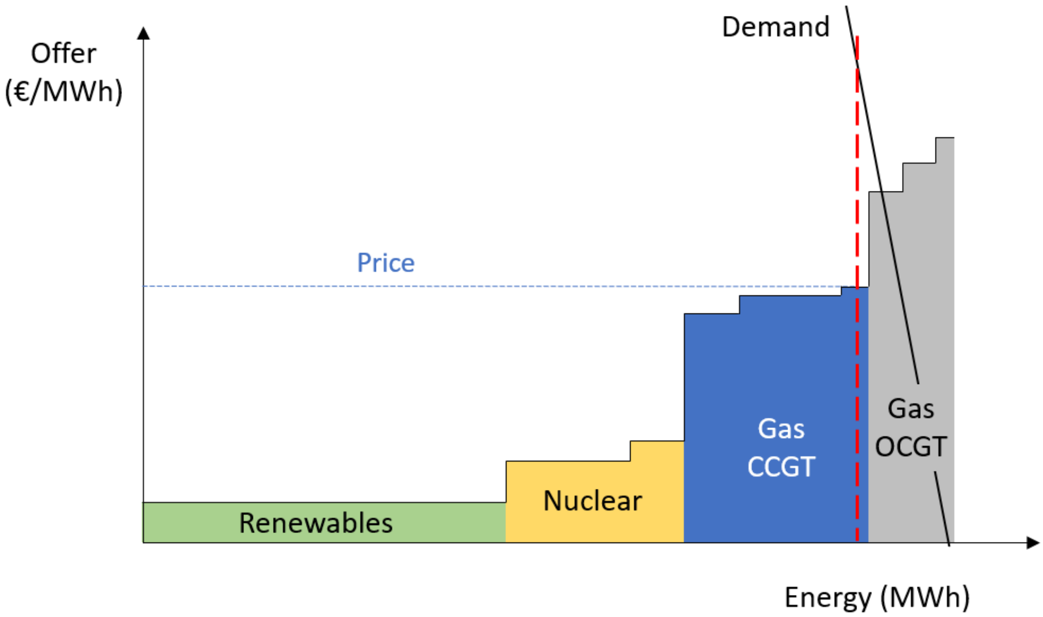
1) If **producers asked for** a **higher price**, they would sell less electricity, as consumers are not willing to purchase so much energy if the price is higher.
    - To better understand this in the picture, simply move upwards in the demand curve, starting from the equilibrium point: going upwards means paying a higher price, but it also means going to the left on the horizontal axis, that is, buying less energy.

    (see the next page)



- In this situation, producers would have an incentive to keep selling electricity, since they could still produce more electricity at a lower cost than the price they are being paid. But they would have to lower the price in order to sell more, as consumers need a lower price in order to increase the amount of electricity they purchase. This **takes producers back to** the **equilibrium point**.
- On the other hand, at the equilibrium point, producers cannot sell more electricity: given that their operating costs increase, no consumer is willing to pay the higher price that would be required to recover production costs for the extra MWh of electricity.

2) If **producers asked** instead **for** a **lower price**, some consumers would be willing to pay more than this, therefore creating an incentive for an increase in production. This would in turn lead to an increase in price, as the cost of the additional MWh's produced would be higher, since less efficient generation plants would be needed. This **takes producers** and **consumers** back again **to the equilibrium point**.



Another related concept is an '**equilibrium problem**': this is a generalization of an optimization problem, where **different agents solve their own optimization** to maximize their own profit (if they are suppliers) or utility (if they are consumers), while all of them are **subject to** some **linking conditions** (such as 'total generation must equal demand' in an electricity system).

- If you want to learn more about equilibrium problems, check out this paper by Profs. Antonio Conejo and Carlos Ruiz and the courses by Prof. Jalal Kazempour.

6.3 Further topics

6.3.1 Shadow prices in practice

In spite of their limitations, **shadow prices** are **widely used in practice**. For example, they are used to compute prices in **electricity markets**, although these markets certainly exhibit non-convexities which, as discussed in section 6.1, make shadow prices somewhat incoherent.

- We typically say that electricity markets use '**marginal pricing**', given that the price of electricity is set by the most expensive generator needed at any given time, the 'marginal' generator. **This is in fact the shadow price** of the 'generation must equal demand' constraint (see the simple optimization problem in page 10 and the first picture in section 6.2 to better understand this).
- **Non-convexities** are introduced by the power flow equations in alternating current, which we discuss in more detail in section 6.3.2. Other non-convexities are introduced by the on/off state of thermal generators, which is modelled through binary decision variables. The latter are discrete non-convexities, which make it impossible to compute shadow prices. Some mathematical tricks have been developed to overcome this problem, although none of them complete solves the issue. The meaning of **dual variables** in **discrete optimization** is discussed in section 7.

The main advantage of shadow pricing is that it **allows for decentralized decisions** (made by the different market participants) that **lead to** the **social optimum** solution, if used carefully (for example, when dealing with non-convexities, as mentioned above).

Given the limitations of shadow prices under non-convexities and imperfect competition (both of which are present in most electricity markets, certainly non-convexities are present in all of them), we may consider the **main alternative** to markets, which is **central planning**. However, an important challenge in central planning is how to **deal with uncertainty**:

- Let's take the case of the **electricity system** and consider which investments to make in new generators. We need to **make a guess on the future** growth of demand, the location of this demand (given that placing generation closer to demand reduces losses and avoids network congestion), and on which will be the most competitive generation technologies in terms of operating costs and technical characteristics.
    - It is **impossible to know** exactly how these variables will eventually turn out. For example, we could think that investing in 'carbon capture and storage' technologies along with thermal generation will be the best way to complement renewables by keeping some dispatchable generation for the future zero-emissions electricity grid.

This assumes that the cost of energy storage will remain prohibitive at large scale, and therefore dispatchable generation (e.g., gas fired power plants) is the most effective alternative. But, **what if the cost of storage drops dramatically** in the coming decade, as has happened with solar photovoltaic panels in recent years? The gas fired power plants with carbon capture and storage would then become '**stranded assets**', given that they have high running costs (they need fuel) while storage does not. But no one really knows if the cost of storage will indeed decrease sufficiently…

- If **private investors** are responsible for the **consequences** of **wrong decisions** (i.e., potential bankruptcy), they have a very clear **incentive to hedge** against the **uncertainty** (e.g., investing in different technologies which all show some promise). But there is also a clear incentive to not hedge excessively, as an overly conservative position also leads to significantly reduced benefits.
- The **errors made** by **central planners** on the other hand **might not** have clear **consequences** for the ones making the wrong call on an investment. In liberal democracies, governments change every few years, so it would be subsequent governments who have to deal with the consequences of decisions made by previous planners. Therefore, there is not such a big incentive to account for uncertainty appropriately.

So **two important problems** can be argued for **central planning**:

1) It is **difficult to make** the **planners accountable** for decisions that turn out wrong.
    - Note that private investors do get away with big mistakes sometimes, as with bailouts to banks during the 2008 global economic crisis. This is just an example of why markets need to be carefully regulated to work appropriately.
2) The mindset of central planners can **slow down innovation**, as they tend to be risk averse to avoid making big mistakes that catch attention. Innovation is a high risk, high gain approach, as many mistakes can be made, but the successes can lead to huge progress.
    - This does not apply to, for example, fundamental research. No private entity would invest in research in particle physics, although it may bring huge returns to society a few decades down the line (and it's simply extremely interesting to explore). This is just another example of why public support is critical for many activities.

As we have gotten into the realm of economics, this section is **much more speculative** than the rest of this document, which deals with rigorous mathematics. As such, the **claims** I have made above are based on my current understanding and vision of the field of economics, and they **can certainly be disputed**.

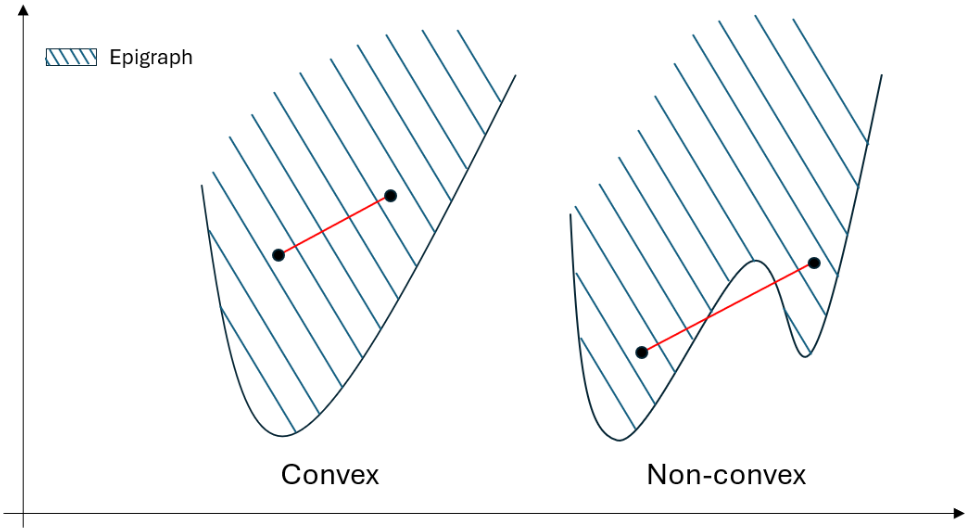
Finally, if you are interested in a first introduction to the topic of **prices** in **electricity markets**:

- Check **slides** 52 to 92 here, by Prof. Jalal Kazempour. With some mathematical detail, but using a simple and very clear example. A slightly more detailed version of those slides is available on **video** here.

6.3.2 Non-convexity and the AC OPF problem

Before jumping into the main topic for this section, it is useful to give an **intuitive definition of convexity**. For the formal definition of convexity, you can refer to section 1.1 of [1].

Consider two generic functions, as in the following picture:

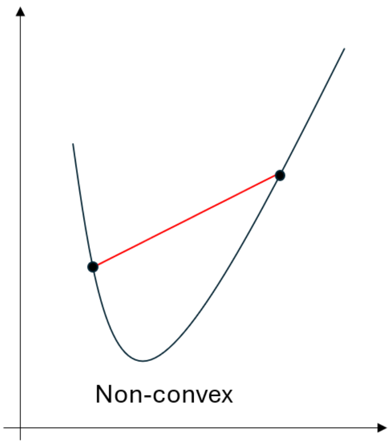


The intuitive **definition of convexity**, which is represented in the above picture, is the following:

A function is convex if the **line joining any two points** in its *epigraph* (the subspace above the function) **is contained within that epigraph**

For an optimization problem to be convex, **all constraints** and the **objective function must be convex**.

- Any **linear function** is **convex**.
- To check if a **quadratic function** is **convex**, one can write it in standard form (the one used here), and if the matrix included in that expression is '**positive semidefinite**,' then the function is convex.
- Any **equality constraint** which is **non-linear** introduces a **non-convexity**:



(note that the feasible space in the graph above is just the curve itself, since this corresponds to an equality constraint; therefore, the **line falls outside** the **feasible space**)

Now let's consider the meaning of **shadow prices** in **non-convex problems**. We only discuss here smooth non-convex functions, such as the ones in the pictures above. For a discussion on discrete optimization (all of which is non-convex), refer to section 7.

In particular, let's focus on the '*alternating current optimal power flow*' (**AC OPF**), a very important problem in electricity markets. The objective function of the AC OPF is to **minimize total generation costs**, so that generation equals demand plus losses, and complying with several technical limits (e.g., generators have a maximum capacity, transmission lines can carry a maximum current to avoid violating their 'thermal limit', etc.).

It is important to know that one **cannot choose** the **paths** along which **electric power flows**: these are governed by **Kirchhoff's laws**. We won't get into specifics here, but the **non-convexity** of the AC OPF problem can be clearly seen in the following **equality constraint**, which must be included in the optimization as it represents the flow of electric power across a circuit:

$$p_{ij} = \frac{1}{r_{ij}^2 + x_{ij}^2}\left[r_{ij}\left(v_i^2 - v_i v_j \cos(\delta_{ij})\right) + x_{ij}\left(v_i v_j \sin(\delta_{ij})\right)\right]$$

Which defines the flow of *active power* ('*p*') from node *i* to node *j* in a circuit. For a deduction of the power flow equations, you can refer to these notes by Dr Letif Mones.

In the above constraint, some decision variables such as 'δ_{ij}' (this is the 'phase angle difference of the voltage phasors at both ends of the line *i-j*', although this definition is somewhat irrelevant to our discussion) are inside trigonometric functions of sines and cosines. As these functions are **non-linear**, and they appear in an **equality constraint**, we can conclude that the constraint is non-convex.

Given that the AC OPF is non-convex, its **associated shadow prices <u>do not</u>** lead to the **socially optimal solution** (recall section 6.1). In theory, these prices could still be used as economic signals for generators and consumers, even though care would be needed due to the non-zero duality gap. In practice, prices for electricity are not directly computed from the AC OPF, as the primal formulation of this optimization problem is **intractable** for **large scale** electricity grids, and therefore the dispatch instructions for generators cannot be computed (recall the difficulty in solving non-convex problems discussed in section 5.4).

In order to solve the optimal dispatch for an electricity grid and compute prices from it, some **simplifications** are typically made to the AC OPF formulation, such as the so called (and confusingly so) '**DC OPF**'. This name may be confusing since this model has nothing to do with direct current (DC), but the name implies that we make an **approximation** of the **AC power flow equations** so that they resemble the equations that we would encounter in a DC electricity grid.

The simplifications made by DC OPF are typically **acceptable** in **transmission systems** (i.e., high voltage grids). However, **other approaches** to deal with the non-convexities of the AC OPF include **convex relaxations**, which we won't discuss here. For more details on AC OPF, DC OPF and relaxations of the former, refer to [3]. Note that optimizing electric power flows and computing meaningful prices is an **active area** of **research**.

Finally, a note on the feasible space of the AC OPF problem: it can be formed by several disconnected regions, as shown in Figures 3 and 4 of this paper.

6.3.3 Negative prices

Negative prices are a **curious phenomenon** that can be explained through duality theory.

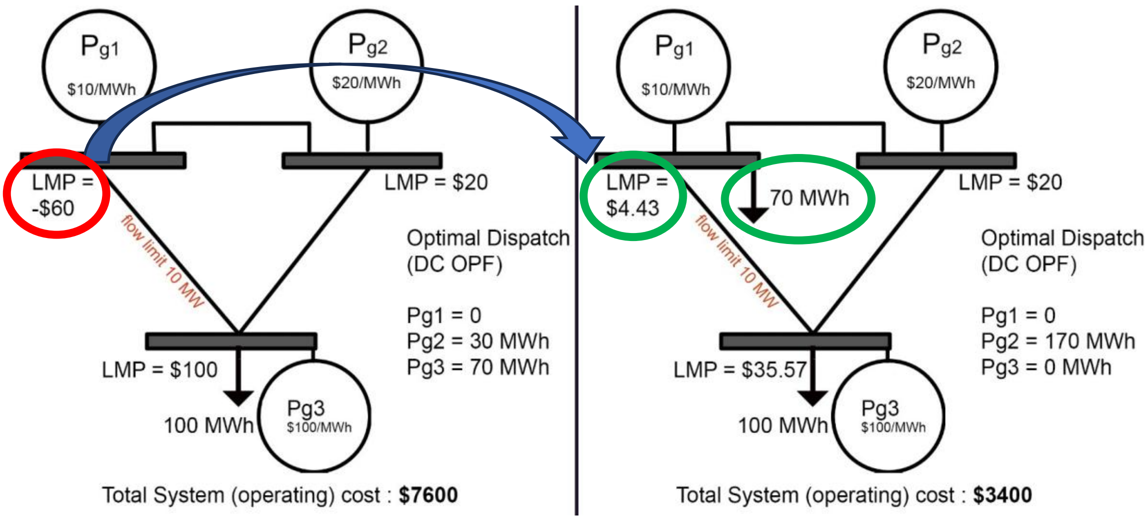
Let's first discuss the mathematics. **Dual variables** associated to **inequality constraints <u>must be non-negative</u>** for the Lagrangian function to be a lower bound of the primal objective function (refer to section 3.1 for a reminder). But dual variables associated to **equality constraints <u>are 'free'</u>**, that is, they may take both positive and negative values.

- If your recall the deduction in section 3.1, note that equality constraints must always be active, i.e., 'h(x) = 0', therefore 'v · h(x) = 0' regardless of 'v' being positive or negative.

Now let's consider a specific example of an electricity grid. First, it's useful to define the concept of '**Locational Marginal Price**' (LMP): this is the **price** of **electricity** at **each node** of the grid. It comes from the dual variable associated to the equality constraint of 'generation minus demand equals power exported' at each node, where the power exported follows Kirchhoff's laws.

- In **DC OPF**, the LMP can be different at each node of the grid if there is **network congestion** in any given line.
- In **AC OPF**, the LMP is always different in every node, due to **line losses** as determined by Joule's law (lines losses are neglected in DC OPF). Network congestion can of course also be present in AC OPF.

Take now this example of a DC OPF created by Prof. Kyri Baker, where we can see a negative price in node 1. **How can we interpret this result?**



- First note that **LMPs** arise from an equality constraint, therefore they **can be negative**.
- In this case, **network congestion** is causing the **negative LMP** in node 1, since we cannot use the cheapest generator P_{g1} without violating the flow limit of 10 MW in the line connecting nodes 1 and 3. Remember that line flows are determined by Kirchhoff's laws, so we cannot simply force power to flow instead through the lines connecting nodes 1 to 2, and then 2 to 3, to eventually reach the load of 100 MWh at node 3.

- However, **if we add some load in node 1**, for example of 70 MWh, we see that the **LMP becomes positive**, and more importantly, we see that the **total generation cost** (i.e., the optimal value of the primal problem) **has decreased!** We manage to go from a total cost of \$7,600 down to \$3,400. **How can the total cost decrease if we have increased demand?** The reason for this counterintuitive result is **Kirchhoff's laws**, as the increase in demand in node 1 allows now to use the cheapest generator P_{g1}, eventually decreasing overall costs.

You can find the Jupyter notebook **code** for the above example in Prof. Kyri Baker's GitHub page.

On **negative prices** in **real electricity markets**, these have become **quite common** in **Europe** since the beginning of the 2020's. Some of the **reasons why** we see negative prices, even though European markets do not solve an OPF (US electricity system operators do use the OPF for clearing the market) are:

- Negative prices can arise from the **inflexibility** of some **thermal plants** (nuclear is the best example for this, as it cannot turn on and off frequently): these plants may want to **remain synchronized** to the grid during **periods** of **high renewable output** (renewables can make prices drop to zero, as they have no fuel costs), and are even **willing to pay for doing so** (i.e., they submit a negative offer to the market for producing power).
    - Even though they incur fuel costs by remaining synchronized to the grid, and they have to pay the negative price on top of that, this might be a **sensible strategy** if the **start-up costs** of having to **re-synchronize again** in a few hours (after the period of excess renewable energy has ended) exceed the negative profit they incur during the hours with negative price.
- Another cause of negative prices are certain **financial instruments** such as contracts for differences (**CfD**) or power purchase agreements (**PPA**), which guarantee a certain price for some renewable generators, regardless of the price set at the wholesale electricity market. We call this guaranteed price the '**strike price**'.
    - If there is excess renewable generation, the power plants benefitting from these financial instruments would be willing to submit negative offers to the market, **up until** the **negative value** of **their strike price**: if their strike price is of 40 €/MWh, and the market clearing price turns out to be of -39 €/MWh, they still make a profit of 1 €/MWh (assuming zero fuel costs).

Finally, note that the **price** of **other services** such as **reserve**, discussed at the beginning of section 6, **cannot be negative**: unlike the price of energy, which comes from the power-balance equality constraint, the price of reserve comes from an **inequality constraint**, so the lowest possible price is zero.

6.3.4 Alternative market-clearing mechanisms

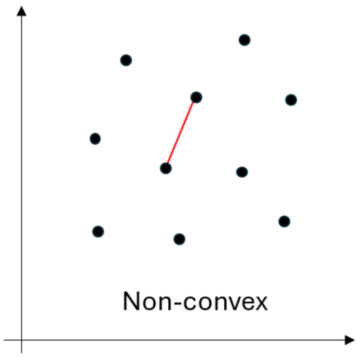
Given the limitations of shadow prices, different **market-clearing mechanisms** have been proposed. There are **4 desirable properties** that a market-clearing mechanism can meet, namely 1) market efficiency, 2) incentive compatibility, 3) cost recovery, and 4) revenue adequacy. **Marginal pricing** of electricity only meets properties 3 and 4, as shown here.

While **LMPs** (or some simplified version of them, as applied in Europe) are still the **most common practice** in electricity markets, some **alternative** methods for computing prices include '**pay-as-bid**' and '**Vickrey–Clarke–Groves**' (VCG). However, there is **always a trade-off**: no mechanism can meet the 4 desirable market properties, as proven in papers by Nobel prize winners Leonid Hurwicz and Roger Myerson.

7. KKT with integer variables

We discussed **smooth non-convex functions** in section 6.3.2. An important characteristic of these functions is that they are **differentiable** (even for non-smooth but **continuous functions** we can compute **subgradients**, which are practically as useful as gradients for optimization purposes). However, in **discrete optimization**, **neither gradients nor subgradients** can be computed.

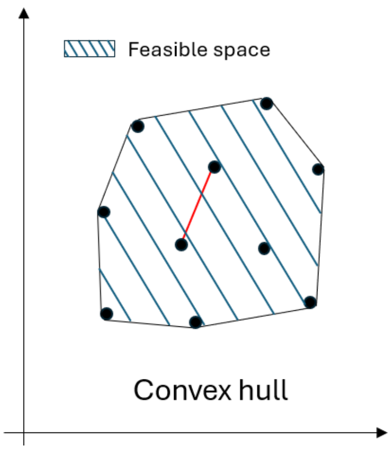
First, let's see why discrete optimization is non-convex. The **feasible space** in an optimization problem where **decision variables** must take **integer** values would look like a **cloud of points**:



The **line joining any two points** within the feasible space **lies outside** it, therefore the problem is non-convex, as we saw in section 6.3.2.

Even for optimization problems where we have **both continuous** and **integer variables** (we call these '**mixed-integer programs**') part of the feasible space would look like the above picture. The main message is that **integer variables** always introduce **non-convexities**.

From the picture above, we can draw an **intuitive definition** of the concept of '**convex hull**': the convex hull is obtained by thinking of a **rubber band** that we stretch and then let **compress around** the **cloud of points**:



The subspace within the convex hull is indeed convex, so this is a useful concept when dealing with discrete optimization. In fact, it is **sometimes used** to **compute shadow prices** in **electricity** markets, as it is one of the ways to deal with the on/off state of thermal generators modelled through **binary decision variables** in the so called '**Unit Commitment**' optimization problem. It is actually the method that has shown the best properties for this application, although at the expense of being **computationally expensive** (it might look simple in the graphic above, but that is just an illustrative example; it becomes much harder to do with millions of decision variables).

Moving on to the main topic of this section, **what do KKT conditions tell us about discrete optimization problems?** The answer is, **very little**.

- The main issue is that discrete problems are **not differentiable**, which is a **requirement for** computing the KKT **stationarity condition**.
- The **integrality constraints** (i.e., the requirement that decision variables are integer) imply that most **regularity conditions** are **not met**.
    - Recall from section 5.4 that meeting a regularity condition is **necessary** for **KKT solutions** to be **local optima**. We typically use Slater's condition, which in simple terms says that the feasible space must contain at least one 'strictly interior' point. Since the feasible space of purely integer programs is formed by isolated points, these are not strictly interior.

For more details on the two statements above, check out this book (be aware that it is mathematically dense) and this Stack Exchange post.

Note that we are focusing now on the **original optimization** problem with **integer variables**, not on its convex hull, which is in fact a relaxation of the original problem ('relaxation' meaning that the feasible space is larger than in the original problem). KKT conditions do apply to the convex hull, but the optimal solution of this relaxation will likely not coincide with the optimal solution of the original discrete problem. To better understand this, you can search for the concept of '**relaxation gap**', which is for example relevant in convex relaxations of the AC OPF problem (notably the 'Second-Order Cone' and the 'Semi-Definite' relaxations).

So **what can we do** for **computing prices** in **integer problems?** We have already mentioned the convex-hull approach, sometimes applied to the Unit Commitment problem in electricity markets. For simpler methods, you can read about the '**dispatchable**' and '**restricted**' approaches in this paper. This is still an active area of research.

References

[1] "Convex optimization", book by Stephen Boyd and Lieven Vandenberghe. 1st edition, 2004. https://web.stanford.edu/~boyd/cvxbook/

[2] Mosek modeling cookbook, https://docs.mosek.com/modeling-cookbook/index.html

[3] "Convex optimization of power systems", book by Josh Taylor. 1st edition, 2015. https://web.njit.edu/~jat94/cops.html

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