Technical Report
Distinctions and Similarities Between Log-Gaussian Cox Processes and Chi-squared Cox Processes

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1 Introduction

Cox processes (doubly stochastic Poisson processes) are a flexible class of models for clustered spatial point patterns. The *log-Gaussian Cox process* (LGCP) is by far the most widely studied and applied, owing to its tractability and natural connection to Gaussian random fields (GRFs).

An alternative—rarely used in practice—is the *chi-squared Cox process* (CSCP), in which the random intensity is given by the sum of squared Gaussian fields. Despite their similarities, the two processes induce different distributional and clustering properties. This document summarizes key properties of both models, and sketches possible avenues for publication that highlight the merits and potential uses of CSCPs.

2 Definitions

2.1 Log-Gaussian Cox Process (LGCP)

An LGCP is defined by

$$\Lambda(s) = \exp\{Z(s)\}, \qquad s \in D \subset \mathbb{R}^d,$$

where $Z(s)$ is a Gaussian random field with mean function $\mu(s)$ and covariance $C(s, s')$. Conditional on Λ, the point process X is Poisson with intensity function $\Lambda(s)$.

Key properties:

- $\mathbb{E}[\Lambda(s)] = \exp(\mu(s) + \frac{1}{2}\sigma^2)$.
- The pair correlation function is

  $$g(s, s') = \exp\{C(s, s')\},$$

  showing that clustering strength depends exponentially on covariance. In particular, $g(0) = \exp(\sigma^2)$ can be arbitrarily large.
- Widely used in practice: tractable through moment properties, simulation methods, and fitting algorithms (composite likelihood, Bayesian inference, INLA).

2.2 Chi-squared Cox Process (CSCP)

A CSCP is defined by

$$\Lambda(s) = \sum_{i=1}^{k} Z_i^2(s), \qquad s \in D,$$

where $\{Z_i(s)\}_{i=1}^k$ are independent Gaussian random fields with mean $\mu_i(s)$ and covariance functions $C_i(s, s')$.

Key properties:

- $\mathbb{E}[\Lambda(s)] = \sum_{i=1}^k \left(\mu_i(s)^2 + \sigma_i^2\right)$.
- $\mathrm{Var}[\Lambda(s)] = 2\sum_{i=1}^k \left(\sigma_i^4 + 2\mu_i^2\sigma_i^2\right)$.
- The pair correlation has the form

$$g(s, s') = 1 + \frac{\sum_{i=1}^k \left(\mathrm{Cov}(Z_i(s), Z_i(s'))\right)^2}{\left(\sum_{i=1}^k \sigma_i^2 + \mu_i^2\right)^2}.$$

  In the symmetric zero-mean case, $g(0) = 1 + 2/k$, so clustering strength at very short lags is bounded.

- Marginally, $\Lambda(s)$ follows a noncentral chi-squared distribution with k degrees of freedom and noncentrality parameters μ_i^2/σ_i^2. These marginals have lighter (exponential) tails compared to the lognormal marginals of an LGCP.

3 Similarities and Differences

Similarities

- Both are Cox processes driven by latent Gaussian random fields.
- Both induce clustering via spatial correlation in the latent fields.
- Both can represent multi-scale clustering when multiple latent fields with different correlation lengths are included.
- Both are second-order intensity reweighted stationary under suitable conditions, allowing use of summary statistics such as the pair correlation function.

Differences

- **Marginals:** LGCP intensities are lognormal with heavy tails; CSCP intensities are chi-squared/gamma-like with lighter exponential tails.
- **Pair correlation:** For LGCP, $g(s, s') = \exp\{C(s, s')\}$ and $g(0)$ can be arbitrarily large; for CSCP, $g(s, s')$ depends quadratically on covariance and $g(0)$ is bounded, yielding less explosive but potentially more interpretable clustering strength.
- **Role of k:** The degrees of freedom k in CSCP controls variability and relative clustering. For large k, the CSCP intensity approaches Gaussianity by the central limit theorem, reducing to near-homogeneous Poisson behavior.
- **Interpretability:** In LGCP, the log-scale linear predictor is directly interpretable. In CSCP, k may be viewed as the number of independent clustering mechanisms.

4 Covariance structure of the CSCP intensity field

Let the chi-squared Cox process (CSCP) be driven by k independent Gaussian random fields

$$Z_i(s) \sim \mathrm{GRF}\big(\mu_i,\, \sigma_i^2 \rho_i(\|s - s'\|)\big), \qquad i = 1, \dots, k,$$

where μ_i is the mean, σ_i^2 the marginal variance, and $\rho_i(h)$ the correlation function at spatial lag $h = \|s - s'\|$. The random intensity is

$$\Lambda(s) = \sum_{i=1}^{k} Z_i(s)^2.$$

4.1 Mean intensity

By independence across i,

$$\mathbb{E}\{\Lambda(s)\} = \sum_{i=1}^{k} \mathbb{E}\{Z_i(s)^2\} = \sum_{i=1}^{k} \big(\mu_i^2 + \sigma_i^2\big).$$

4.2 Covariance function (general case)

For any two locations s, s' with separation $h = \|s - s'\|$,

$$\mathrm{Cov}_\Lambda(h) = \mathrm{Cov}\big(\Lambda(s), \Lambda(s')\big) = \sum_{i=1}^{k} \mathrm{Cov}\big(Z_i(s)^2, Z_i(s')^2\big),$$

since the fields are independent across i.

Applying Isserlis' (Wick's) theorem for fourth moments of Gaussians,

$$\mathbb{E}[Z_i(s)^2 Z_i(s')^2] = \mathbb{E}[Z_i(s)^2]\,\mathbb{E}[Z_i(s')^2] + 2\,\big(\mathrm{Cov}(Z_i(s), Z_i(s'))\big)^2.$$

Thus

$$\mathrm{Cov}\big(Z_i(s)^2, Z_i(s')^2\big) = 2\big(\sigma_i^2 \rho_i(h)\big)^2 \;+\; 4\mu_i^2 \sigma_i^2 \rho_i(h).$$

Hence the full covariance is

$$\boxed{\mathrm{Cov}_\Lambda(h) = \sum_{i=1}^{k} \Big[\, 2\sigma_i^4 \rho_i(h)^2 \;+\; 4\mu_i^2 \sigma_i^2 \rho_i(h) \,\Big].}$$

4.3 Zero-mean special case

If $\mu_i = 0$ for all i, this reduces to

$$\boxed{\mathrm{Cov}_\Lambda(h) = 2\sum_{i=1}^{k} \sigma_i^4 \rho_i(h)^2.}$$

This case is often taken as the baseline in the literature (e.g. Møller and Waagepetersen, 2004 [1]).

4.4 Pair correlation function

The pair correlation function is defined as

$$g(h) \;=\; 1 + \frac{\mathrm{Cov}_\Lambda(h)}{\big(\mathbb{E}\{\Lambda(s)\}\big)^2}.$$

For the zero-mean, equal-variance case with exponential correlation $\rho_i(h) = \exp(-h/\phi_i)$,

$$g(h) - 1 \;=\; \sum_{i=1}^{k} b_i\, e^{-2h/\phi_i}, \qquad b_i = \frac{2\sigma_i^4}{\Big(\sum_{j=1}^{k} \sigma_j^2\Big)^2}.$$

4.5 Semilog transformation and scale identification

Writing $y(h) = g(h) - 1$, for two components we have

$$y(h) = b_S e^{-2h/\phi_S} + b_L e^{-2h/\phi_L}.$$

Taking logs,

$$\log y(h) = \log b_S - \tfrac{2h}{\phi_S} + \log\left(1 + \tfrac{b_L}{b_S} e^{-2h\left(\frac{1}{\phi_L} - \frac{1}{\phi_S}\right)}\right).$$

This is a log-sum-exp form: globally curved, but in limiting regimes it becomes linear:

$$\begin{aligned} \log y(h) &\approx \log b_S - \tfrac{2h}{\phi_S}, && \text{if small scale dominates,} \\ \log y(h) &\approx \log b_L - \tfrac{2h}{\phi_L}, && \text{if large scale dominates.} \end{aligned}$$

The crossover ("shoulder") occurs at

$$h^\star = \frac{\phi_S \phi_L}{2(\phi_L - \phi_S)} \log\left(\tfrac{b_S}{b_L}\right).$$

Thus, on a semilog plot of $g(h) - 1$, the CSCP naturally reveals its modular structure: straight-line regimes with slopes $-2/\phi_S$ and $-2/\phi_L$, with the transition at $h^\star$. See Figure 1.

Positioning relative to existing work

The model we describe as a *Chi-squared Cox process (CSCP)* is mathematically equivalent to the *permanental process* introduced by McCullagh and Møller (2006) [2], and previously foreshadowed in the boson process literature [3, 4]. Their work established the existence, joint densities, and moment properties of processes with intensity fields of the form $\Lambda(s) = \sum_{i=1}^{k} Z_i(s)^2$, with Z_i Gaussian random fields. In this sense, the theoretical foundation of CSCPs is already solidly in place.

What seems to be missing, however, is applied development. In contrast to log-Gaussian Cox processes (LGCPs), permanental/CSCPs have seen almost no uptake in spatial statistics practice. I think. In particular, several potentially valuable aspects have not been explored:

- **Diagnostics and interpretability:** For exponential correlations, $g(h) - 1$ decomposes as a sum of exponentials, which on a semilog plot reveals straight-line regimes with slopes $-2/\phi_i$. This provides a natural diagnostic and a modular interpretation of multiple scales of clustering — a feature absent in LGCPs, where scales combine inside the exponent.

- **Multi-scale modelling:** The modular structure suggests CSCPs may offer advantages in modelling both small-scale hotspots and large-scale background variation simultaneously. Identifying these scales empirically and contrasting them with LGCP fits has not yet been demonstrated.

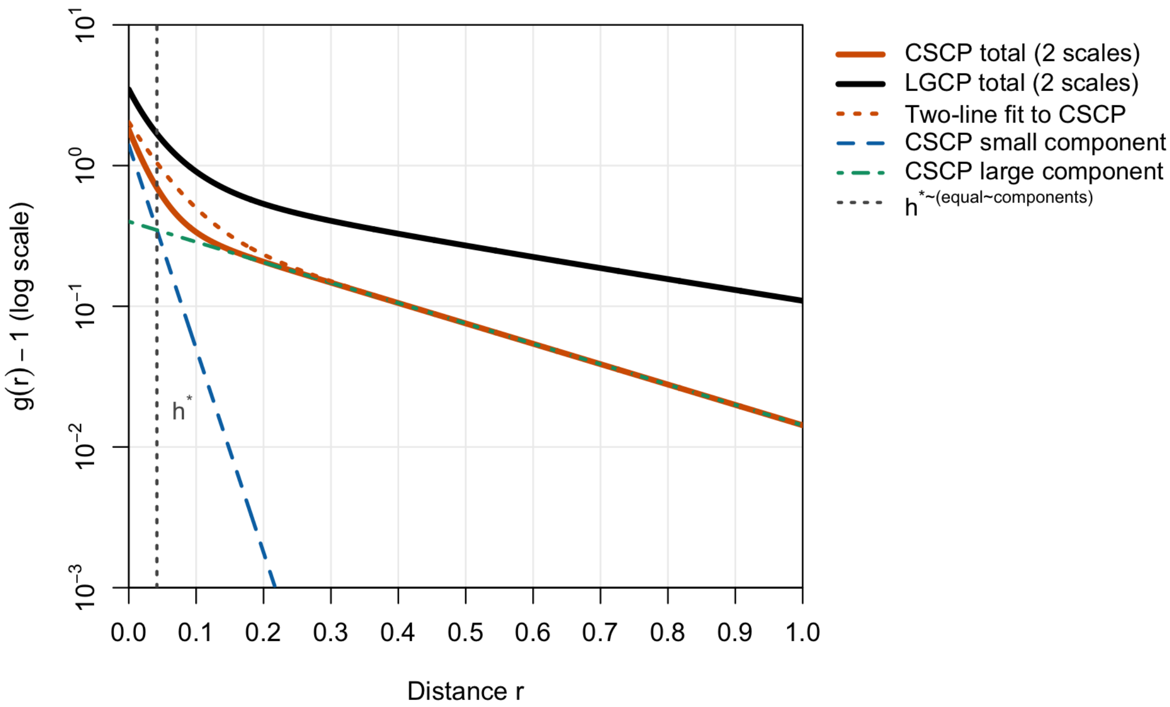

Figure 1: Semilogarithmic plot of $g(h)-1$ for a two-scale CSCP. Because $g(h)-1 = b_S e^{-2h/\phi_S} + b_L e^{-2h/\phi_L}$, the function is a sum of exponentials. On the semilog scale, this produces two approximately linear regimes: one at small lags with slope $-2/\phi_S$ (short-range clustering), and one at larger lags with slope $-2/\phi_L$ (long-range clustering). The vertical marker denotes the crossover distance $h^\star$, where the two component contributions are equal. This "shoulder" provides a natural diagnostic of the modular structure of CSCPs, in contrast to LGCPs where multiple scales are entangled in the exponent and the curve remains smoothly curved.

- **Estimation and heuristics:** Practical inference strategies (e.g. composite likelihood, minimum contrast, or heuristic initialisation from pair correlation diagnostics) remain underexplored.
- **Applied case studies:** To our knowledge, there are no published applications of CSCPs to real epidemiological or ecological point pattern datasets.

Accordingly, while the permanental/CSCP model itself is not new (some other relevant papers are [5, 6, 7, 8]), there is possibly scope for novel contributions in terms of applied methodology, diagnostics, and interpretability. Our work is positioned in this gap: to revisit CSCPs with emphasis on their practical merits and to provide the first systematic comparison with LGCPs in applied settings.

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