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Constructing the Equation of State for Compact Stars

Bachelor thesis

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1 Introduction

To this day, the nature of the equation of state [EOS] for neutron stars is highly debated. Previously discussed EOSs have emerged from various topics and models of physics. Each one has different parameters determining its impact on calculations.
For the past years, it has been common to derive an EOS from a theory and then use it to construct a mass-radius relation [MRR]. This method is obviously constrained by the fact that one cannot put an arbitrary number of theories into one EOS. That would result in a problem impossible to solve, even for computers. Therefore, the motivation is to minimize the number of parameters and constraints in the future.
The approach chosen in this thesis is inverting the above process. Instead of using an EOS to calculate masses and radii, those shall now take over the role as input parameters. For compact stars up to a certain mass, the EOS will be assumed to be known well enough. Above that mass a numerical reconstruction will take place, determining the EOS from a variety of MRRs.

2 Theoretical background

2.1 Compact Stars

At some point in a star's life, it is no longer capable of maintaining nuclear fusion in its core. Hydrogen burning first moves to a shell around the central region, further growing the previously produced helium core, up to the point where helium burning ignites. Depending on the star's mass, this process can either repeat until an iron core is formed or stop at some fusion product before that. In the latter case, i.e. for star masses up to $8\,\mathrm{M}_\odot$, the outer envelopes will escape the core and form a planetary nebula. What remains is a white dwarf, with an upper mass limit given by Chandrasekhar as about $1.4\,\mathrm{M}_\odot$. White dwarfs are the first known family of compact objects or -stars. Their interior structure depends on its progenitor star, hence their EOS is not universally defined ([1], p. 91). Their size is of order $10^3\,\mathrm{km}$, temperatures usually reside in the $10^4\,\mathrm{K}$ domain but can be one or two orders of magnitude higher ([1], p. 90).
If the fusion processes have produced an iron core, it will be compressed to densities exceeding multiple times the nuclear saturation density when the star turns into a supernova explosion. The type of remnant left behind is called neutron star, and the matter out of which it is formed is always of the same composition and state. Therefore, in contrast to white dwarfs, there is one unique EOS to describe all neutron stars.
Both kinds of compact stars are subject to fast neutrino and photon cooling. After at most a few million years temperatures will have dropped several orders of magnitude below $1\,\mathrm{MeV}$, so that on the nuclear scale the star can be treated as cold.

2.2 The Tolman-Oppenheimer-Volkoff equation

The equation used for determining the structure of compact stars will here be the Tolman-Oppenheimer-Volkoff [TOV] equation.
To make analytical and numerical calculations easier, the unit system is chosen to be $c = G = 1$, so that every unit is a power of length.
The derivation is based off the assumption that the star matter can be described as a perfect/ideal fluid. Further, the system shall not evolve in time, therefore staying spherically symmetric. In terms of the energy-momentum tensor we are left with:

$$T_{\mu\nu} = (\epsilon + P)\, u_\nu u_\mu - P g_{\mu\nu} \tag{1}$$

Spherical symmetry leads to a certain form of the metric, which then gives the stress-energy tensor components:

$$T_{\mu\nu} = diag(\epsilon e^{\nu(r)},\ Pe^{\lambda(r)},\ Pr^2,\ Pr^2 sin^2(\theta)) \tag{2}$$

Imposing hydrostatic equilibrium,

$$\nabla_\nu T^{\mu\nu} = 0 \tag{3}$$

and some calculation, one obtains the expression

$$content... \tag{4}$$

To determine what $\nu(r)$ looks like, and using the previously defined metric, one can calculate the non-zero components of the Ricci tensor:

$$\textit{content...} \tag{5}$$

The Einstein equations then yield more equations containing $\nu(r)$ and it's derivatives. After plugging some of the equations into one another, one obtains $\nu(r)$ and therefore, in conclusion, the full TOV equation:

$$\frac{dP}{dr} = \frac{(\epsilon + P)(m + 4\pi r^3 P)}{2mr - r^2} \tag{6}$$

Together with a second equation for the mass:

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \tag{7}$$

2.3 Numerical solution

2.3.1 Method

The TOV equations come in the form of a first-order ordinary differential equation system (ODE). In order to obtain sufficient precision, a 4th-order Runge-Kutta algorithm shall be used to solve said ODE. By looking at both equations, our ODE system is given in the form:

$$\dot{y}(t) = f(y(t), t) \tag{8}$$

where $\dot{y}(t)$ is a two component "vector":

$$\dot{y}(t) = \begin{pmatrix} dP/dr \\ dm/dr \end{pmatrix} \tag{9}$$

$f(y(t), t)$ then contains the right hand side of equations 6 and 7.
Numerically, solving the ODE is achieved by approximation of a step τ in the function variable t (in TOV, this is the radius r) with derivatives of the function that is the ODE solution (y).

$$y(t + \tau) = y(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{10}$$

with increments k_i:

$$\begin{aligned}
k_1 &= f(y(t), t) \cdot \tau \\
k_2 &= f(y(t) + k_1/2, t + \tau/2) \cdot \tau \\
k_3 &= f(y(t) + k_2/2, t + \tau/2) \cdot \tau \\
k_4 &= f(y(t) + k_3, t + \tau) \cdot \tau
\end{aligned}$$

All k_i are of course also two component "vectors".

2.3.2 Adaptive stepsize

In order to maintain a minimum level of precision, the program checks for the change in pressure resulting from one Runge-Kutta step. If said change is too large, the step size τ is varied by a factor and the Runge-Kutta step count is reduced by one. Upon repeating the step, the same check takes place, and if necessary another τ adjustion. If the check is successful, the program will accept the step result and proceed with the next one.

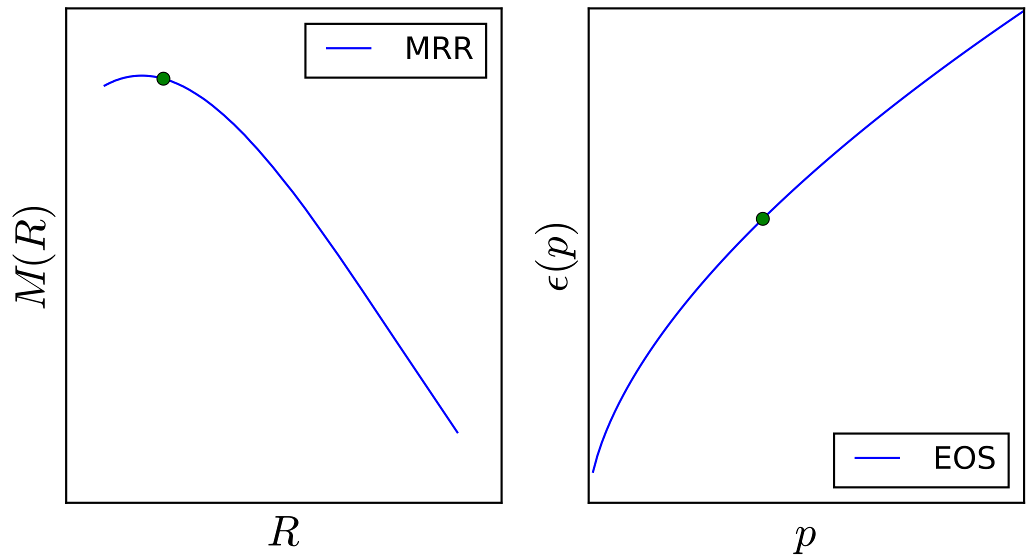

Figure 1: corresponding points in MRR and EOS. Point in EOS resembles values in the center of the star.

2.3.3 Implementation

In the actual program (written in C), the left hand side of our ODE system is implemented as a two-dimensional array of form y[N][x]. N will be two, for the whole program, as we will not add other equations to the system. The k_i are all one-dimensional arrays; that way x is only reflecting the step count. x will therefore be in range zero up to the number of iteration steps. We set an arbitrary maximum step amount to prevent segmentation faults.

3 Basic algorithm

The goal is to reconstruct the EOS with as few biasing and imposing of mathematical form as possible. Therefore, the only pre-determined input is an EOS corresponding to a given mass in the MRR (Fig. 1). From there, the EOS can be constructed without further constraint. Here, a straight line is added to the end of the EOS, that is, in the high density region (Fig. 2 b). The ODE solver starts at a point near the end of the given EOS and will calculate a mass with the (now small) piece of straight line and the EOS. If the mass does not fit the corresponding one in the MRR that was read in previously, the starting point on the line will be shifted to a higher pressure by a small step. At some very high pressure, a cut-off can be applied due to un-physical energy densities. If the starting point reaches the cut-off and the correct mass was not found, the slope of the line is varied. The latter also happens when the radius that corresponds to the calculated mass does not fit the one corresponding to the radius in the file. in Fig. 2 c) the slope and end point of the green line have been successfully varied and the next mass and radius are reconstructed using the variation of the red line. The green line is now a part of the given EOS. This process will be repeated until the end of the MRR file.

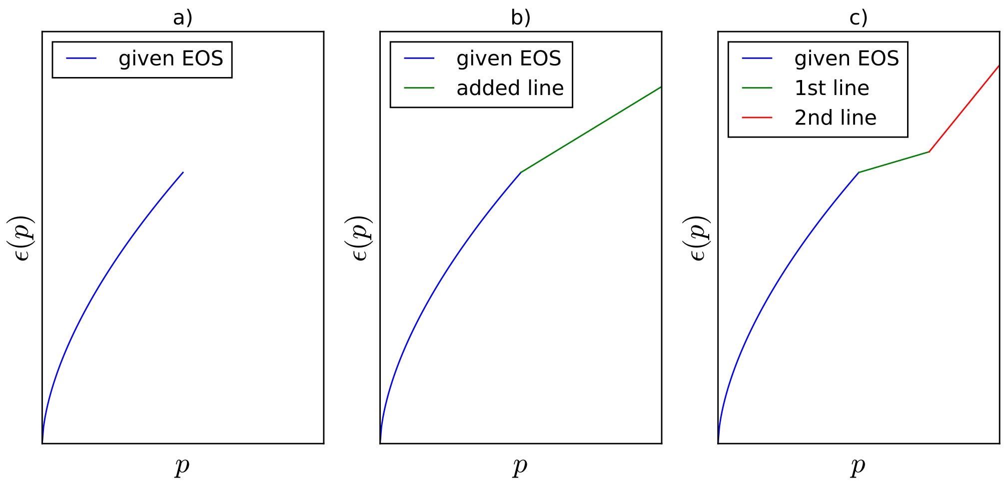

Figure 2: a) Given polytropic equation of state, b) added a constructed straight line, c) Successfully varied green line, red line for next mass

Messung der Fadenlänge	l (m)
1	
2	
3	
4	
5	
Mittelwert $\bar{l}$	
Standardabweichung σ_l	

References

[1] Glendenning, N. K.; Compact Stars: Nuclear Physics, Particle Physics, and General Relativity, Springer-Verlag New York, 2000, 1997