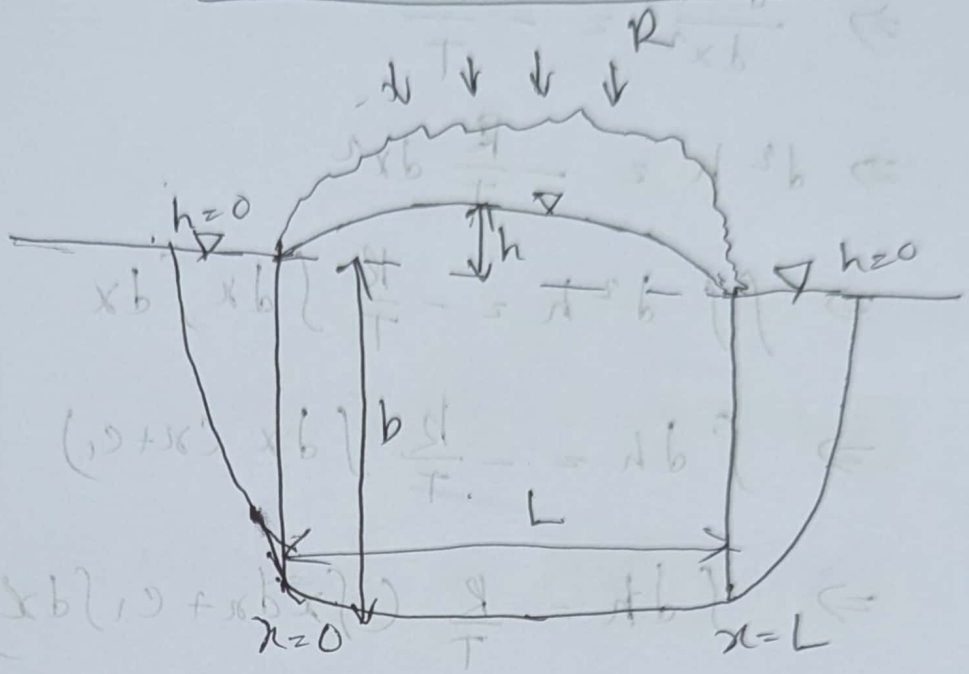
1-D Diffusion GW Flow



Here,

R = Recharge

h = hydraulic head

b = thickness of the aquifer

L = Length of island

Considering the problem have a steady state flow, we gets,

$$K \frac{d}{dx}\left((b+h)\frac{dh}{dx}\right) + R = 0$$

$$\Rightarrow Kb \frac{d^2h}{dx^2} + R = 0 \quad [\because b >> h]$$

$$\Rightarrow T \frac{d^2h}{dx^2} + R = 0 \quad [\because T = Kb]$$

$$\Rightarrow \frac{d^2h}{dx^2} = -\frac{k}{T}$$

$$\Rightarrow d^2h = -\frac{k}{T} dx^2$$

$$\Rightarrow \int\int d^2h = -\frac{k}{T} \int dx \int dx$$

$$\Rightarrow \int dh = -\frac{k}{T} \int dx\,(x + c_1)$$

$$\Rightarrow \int dh = -\frac{k}{T} \left(\int x\,dx + c_1 \int dx\right)$$

$$\Rightarrow h = -\frac{k}{T} \left(\frac{x^2}{2} + xc_1 + c_2\right)$$

The boundary condition is,

when, $x = 0$, $h = 0$ ..... (i)

when, $x = L$, $h = 0$ ..... (ii)

From (i) $\Rightarrow$

$$\therefore 0 = -\frac{k}{T} \left(\frac{0^2}{2} + (c_1 \times 0) + c_2\right)$$

$$\Rightarrow c_2 = 0$$

From (ii) $\Rightarrow$

~~when~~

$$0 = -\frac{k}{T} \left(\frac{L^2}{2} + Lc_1 + c_2\right)$$

$$\Rightarrow \quad L C_1 = -\frac{L^2}{2}$$

$$\Rightarrow \quad C_1 = -\frac{L}{2}$$

$$\therefore \quad h = \frac{-R}{T}\left(\frac{x^2}{2} - \frac{L}{2}x + 0\right)$$

$$\Rightarrow \quad h = \left(\frac{-Rx^2}{2T} + \frac{RLx}{2T}\right)$$

$$\Rightarrow \quad h = \frac{R}{2T}(Lx - x^2)$$

Numerical Solution

For 1-D steady state condition,

$$\frac{d^2h}{dx^2} = -\frac{R}{T}$$

Splitting the island hydraulic head into 10 parts along with two heads from river we get,

$h=0$ $h=0$

| h_0 | h_1 | h_2 | h_3 | h_4 | h_5 | h_6 | h_7 | h_8 | h_9 | h_{10} | h_{11} |

For h_1,

$$h_2 - 2h_1 + h_0 = -dx^2 \frac{R}{T}$$

For, $h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}$

$$h_1 - 2h_2 + h_3 = -dx^2 \frac{R}{T} \quad [\text{For } h_2]$$
$$h_2 - 2h_3 + h_4 = -dx^2 \frac{R}{T} \quad [\text{For } h_3]$$
$$h_3 - 2h_4 + h_5 = -dx^2 \frac{R}{T} \quad [\text{For } h_4]$$
$$h_4 - 2h_5 + h_6 = -dx^2 \frac{R}{T} \quad [\text{For } h_5]$$
$$h_5 - 2h_6 + h_7 = -dx^2 \frac{R}{T} \quad [\text{For } h_6]$$
$$h_6 - 2h_7 + h_8 = -dx^2 \frac{R}{T} \quad [\text{For } h_7]$$
$$h_7 - 2h_8 + h_9 = -dx^2 \frac{R}{T} \quad [\text{For } h_8]$$
$$h_8 - 2h_9 + h_{10} = -dx^2 \frac{R}{T} \quad [\text{For } h_9]$$
$$h_9 - 2h_{10} + h_{11} = -dx^2 \frac{R}{T} \quad [\text{For } h_{10}]$$

After organizing them for matrix visualization,

$$-2h_1 + h_2 = -dx^2 \frac{R}{T} - h_0$$

$$h_1 - 2h_2 + h_3 = -dx^2 \frac{R}{T}$$

$$h_2 - 2h_3 + h_4 = -dx^2 \frac{R}{T}$$

$$h_3 - 2h_4 + h_5 = -dx^2 \frac{R}{T}$$

$$h_4 - 2h_5 + h_6 = -dx^2 \frac{R}{T}$$

$$h_5 - 2h_6 + h_7 = -dx^2 \frac{R}{T}$$

$$h_6 - 2h_7 + h_8 = -dx^2 \frac{R}{T}$$

$$h_7 - 2h_8 + h_9 = -dx^2 \frac{R}{T}$$

$$h_8 - 2h_9 + h_{10} = -dx^2 \frac{R}{T}$$

$$h_9 - 2h_{10} + h_{11} = -dx^2 \frac{R}{T} - h_{11}$$

The matrix is —

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{bmatrix} = \begin{bmatrix} -dx^2 \frac{R}{T} - h_0 \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} \\ -dx^2 \frac{R}{T} - h_{11} \end{bmatrix}$$

$\therefore AX = B$ where, $X = A^{-1}B$

Given that,

Recharge, $R = 0.002$ m/d

Transmissivity, $T = 1500$ m²/d

Length, $L = 5000$ m

$\therefore dx = \frac{L}{10} = 500$ m

$dx^2 = 250000$.

$\therefore -dx^2 \frac{R}{T} = -0.3333$

From the matrix solution in excel we get,

$h_1 = 7.6667$ m

$h_2 = 3$ m

$h_3 = 4$ m

$h_4 = 4.6667$ m

$h_5 = 5$ m

$h_6 = 5$ m

$h_7 = 4.6667$ m

$h_8 = 4$ m

$h_9 = 3$ m

$h_{10} = 1.6667$ m

Parameters		
Recharge	0.002	m/d
Transmissivity	1500	m^2/d
Length	5000	m
dx	500	
dx^2	250000	

$dx^2(R/T)-h_0$	-0.333333
$dx^2(R/T)$	-0.333333
$dx^2(R/T)$	-0.333333
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$dx^2(R/T)$	-0.333333
$dx^2(R/T)$	-0.333333
$dx^2(R/T)-h_{11}$	-0.333333

Coefficient Matrix

-2	1	0	0	0	0	0	0	0	0		h1		-0.333333
1	-2	1	0	0	0	0	0	0	0		h2		-0.333333
0	1	-2	1	0	0	0	0	0	0		h3		-0.333333
0	0	1	-2	1	0	0	0	0	0		h4		-0.333333
0	0	0	1	-2	1	0	0	0	0	X	h5	=	-0.333333
0	0	0	0	1	-2	1	0	0	0		h6		-0.333333
0	0	0	0	0	1	-2	1	0	0		h7		-0.333333
0	0	0	0	0	0	1	-2	1	0		h8		-0.333333
0	0	0	0	0	0	0	1	-2	1		h9		-0.333333
0	0	0	0	0	0	0	0	1	-2		h10		-0.333333

Inverse Coefficient Matrix

-0.909090909	-0.818181818	-0.727273	-0.636364	-0.545455	-0.454545	-0.363636	-0.272727	-0.18181818	-0.090909		-0.333333		1.666667
-0.818181818	-1.636363636	-1.454545	-1.272727	-1.090909	-0.909091	-0.727273	-0.545455	-0.36363636	-0.181818		-0.333333		3
-0.727272727	-1.454545455	-2.181818	-1.909091	-1.636364	-1.363636	-1.090909	-0.818182	-0.54545455	-0.272727		-0.333333		4
-0.636363636	-1.272727273	-1.909091	-2.545455	-2.181818	-1.818182	-1.454545	-1.090909	-0.72727273	-0.363636		-0.333333		4.666667
-0.545454545	-1.090909091	-1.636364	-2.181818	-2.727273	-2.272727	-1.818182	-1.363636	-0.90909091	-0.454545	X	-0.333333	=	5
-0.454545455	-0.909090909	-1.363636	-1.818182	-2.272727	-2.727273	-2.181818	-1.636364	-1.09090909	-0.545455		-0.333333		5
-0.363636364	-0.727272727	-1.090909	-1.454545	-1.818182	-2.181818	-2.545455	-1.909091	-1.27272727	-0.636364		-0.333333		4.666667
-0.272727273	-0.545454545	-0.818182	-1.090909	-1.363636	-1.636364	-1.909091	-2.181818	-1.45454545	-0.727273		-0.333333		4
-0.181818182	-0.363636364	-0.545455	-0.727273	-0.909091	-1.090909	-1.272727	-1.454545	-1.63636364	-0.818182		-0.333333		3
-0.090909091	-0.181818182	-0.272727	-0.363636	-0.454545	-0.545455	-0.636364	-0.727273	-0.81818182	-0.909091		-0.333333		1.666667

Solution		
h1	1.666667	m
h2	3	m
h3	4	m
h4	4.666667	m
h5	5	m
h6	5	m
h7	4.666667	m
h8	4	m
h9	3	m
h10	1.666667	m