EE 514 Homework 2

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1. $[x, p]\,\Psi(x) = (xp - px)\,\Psi(x)$

$$= \left[x\left(-i\hbar \frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)x\right]\Psi(x)$$

$$= -i\hbar\left(x\frac{d\Psi(x)}{dx} - \frac{d}{dx}\left(x\,\Psi(x)\right)\right)$$

$$= -i\hbar\left(\cancel{x\frac{d\Psi(x)}{dx}} - \Psi - \cancel{x\frac{d\Psi(x)}{dx}}\right)$$

$$= i\hbar\,\Psi(x)$$

$$\Delta x \Delta p \geq \frac{1}{2}\left|\langle [x,p]\rangle\right|$$
$$\geq \frac{1}{2}\left|\langle i\hbar\rangle\right|$$
$$\geq \frac{1}{2} i\hbar$$

The result does not depend on $|\Psi\rangle$

2. a) $\frac{1}{2}\left|\langle[\sigma_x, \sigma_y]\rangle\right|$

$\frac{1}{2}\left|\langle \sigma_x\sigma_y - \sigma_y\sigma_x\rangle\right|$

$\cancel{\frac{1}{2}}\left|\langle \cancel{2} i\sigma_z\rangle\right| \quad |i| = 1$

$$= \left|\langle\sigma_z\rangle\right| \begin{cases} \text{if } |\Psi\rangle = |\uparrow\rangle, & 1 \\ \text{if } |\Psi\rangle = |\downarrow\rangle, & 1 \end{cases}$$

$$\Delta\sigma_x \Delta\sigma_y \geq 1$$

This tells us there will always be some uncertainty in our ability to measure spin in both the x and y directions on the same state.

$$\sigma_x\sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} \qquad \sigma_y\sigma_x = \begin{pmatrix} -i & 0 \\ 0 & i\end{pmatrix}$$

$$\sigma_x\sigma_y - \sigma_y\sigma_x = \begin{pmatrix} 2i & 0 \\ 0 & -2i\end{pmatrix} = 2i\,\sigma_z$$

$$\left|\langle\uparrow|\sigma_z|\uparrow\rangle\right| = \left|\mathrm{Tr}\left(\sigma_z|\uparrow\rangle\langle\uparrow|\right)\right|$$
$$= \left|\mathrm{Tr}\left(\sigma_z\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\right)\right|$$
$$= \left|\mathrm{Tr}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\right| = 1$$

$$\left|\langle\downarrow|\sigma_z|\downarrow\rangle\right| = \left|\mathrm{Tr}\left(\sigma_z|\downarrow\rangle\langle\downarrow|\right)\right|$$
$$= \left|\mathrm{Tr}\left(\sigma_z\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right)\right|$$
$$= \left|\mathrm{Tr}\begin{pmatrix}0 & 0\\0 & -1\end{pmatrix}\right| = 1$$

3. $\langle \sigma_x \rangle = \langle \uparrow | \sigma_x | \uparrow \rangle$

$= Tr(\sigma_x |\uparrow\rangle\langle\uparrow|)$

$= Tr\left(\sigma_x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$

$= Tr\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$

$= Tr\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$

$\Delta\sigma_x = \sqrt{\langle(\sigma_x - \langle\sigma_x\rangle)^2\rangle}$

$= \sqrt{\langle(\sigma_x - 0)^2\rangle}$

$= \sqrt{\langle(\sigma_x)^2\rangle}$

$= \sqrt{\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle}$

$= \sqrt{\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle}$

$= \sqrt{1} = 1$

4. a) $v_x \sigma_x$

$\det(\sigma_x - \lambda I) = 0$

$\det\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (-\lambda \cdot -\lambda) - (1 \cdot 1)$

$= \lambda^2 - 1$

$= (\lambda - 1)(\lambda + 1) = 0$

$\lambda = \pm 1$

$V_z \ \sigma_z$

$\det(\sigma_z - \lambda I) = 0$

$\det\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) = 0$

$\lambda = \pm 1$

$V_y \sigma_y$

$\det(\sigma_y - \lambda I) = 0$

$\det\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = (-\lambda \cdot -\lambda) - (-i \cdot i)$

$= \lambda^2 - 1$

$= (\lambda - 1)(\lambda + 1) = 0$

$\lambda = \pm 1$

$\vec{V} \cdot \vec{\sigma} = \begin{pmatrix} V_z & V_x - iV_y \\ V_x + iV_y & -V_z \end{pmatrix}$

$\det \begin{pmatrix} V_z - \lambda & V_x - iV_y \\ V_x + iV_y & -V_z - \lambda \end{pmatrix}$

$(V_z - \lambda)(-V_z - \lambda) - (V_x - iV_y)(V_x + iV_y)$

$-V_z^2 + \lambda^2 - V_x^2 - V_y^2 = 0$

$\lambda^2 - (\underbrace{V_x^2 + V_y^2 + V_z^2}_{\text{unit vector} = 1}) = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

b) $\vec{v}\cdot\vec{\sigma}\,|\pm v\rangle = (\pm 1)\,|\pm v\rangle$

$|\psi\rangle = \alpha|+v\rangle + \beta|-v\rangle$

$P_+|\psi\rangle = \alpha|+v\rangle$

$P_-|\psi\rangle = \beta|-v\rangle$

$P_\pm|\psi\rangle = \frac{1}{2}(I \pm \vec{v}\cdot\vec{\sigma})(\alpha|+v\rangle + \beta|-v\rangle)$

$= \frac{1}{2}(\alpha|+v\rangle + \beta|-v\rangle \pm \alpha(+1)|+v\rangle \pm \beta(-1)|-v\rangle)$

if $+$: $\frac{1}{2}(\alpha|+v\rangle + \cancel{\beta|-v\rangle} + \alpha|+v\rangle - \cancel{\beta|-v\rangle})$

$= \frac{1}{2}(2\alpha|+v\rangle) = \alpha|+v\rangle$

if $-$: $\frac{1}{2}(\cancel{\alpha|+v\rangle} + \beta|-v\rangle - \cancel{\alpha|+v\rangle} + \beta|-v\rangle)$

$= \frac{1}{2}(2\beta|-v\rangle) = \beta|-v\rangle$

5. a) $P_x = \langle\pm 1|\sigma_x|\pm 1\rangle$

$= Tr(\sigma_x|\pm 1\rangle\langle\pm 1|)$

$= Tr\left(\sigma_x\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)$ or $Tr\left(\sigma_x\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$

$= Tr\begin{pmatrix}0&0\\1&0\end{pmatrix} = 0$ or $Tr\begin{pmatrix}0&1\\0&0\end{pmatrix} = 0$

$P_y = \langle\pm 1|\sigma_y|\pm 1\rangle$

$= Tr(\sigma_y|\pm 1\rangle\langle\pm 1|)$

$= Tr\left(\sigma_y\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)$ or $Tr\left(\sigma_y\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$

$= Tr\begin{pmatrix}0&0\\i&0\end{pmatrix} = 0$ or $Tr\begin{pmatrix}0&-i\\0&0\end{pmatrix} = 0$

$$P_z = \langle \pm 1 | \sigma_z | \pm 1 \rangle$$
$$= Tr(\sigma_z | \pm 1 \rangle\langle \pm 1 |)$$
$$= Tr\left(\sigma_z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \quad \text{or} \quad Tr\left(\sigma_z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$= Tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \quad \text{or} \quad Tr\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = -1$$

$$P = (0, 0, \pm 1)$$

b)

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$P_x = Tr(\rho\sigma_x) = Tr\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = Tr\begin{pmatrix} \rho_{12} & \rho_{11} \\ \rho_{22} & \rho_{21} \end{pmatrix}$$
$$= \rho_{12} + \rho_{21}$$

$$P_y = Tr(\rho\sigma_y) = Tr\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) = Tr\begin{pmatrix} i\rho_{12} & -i\rho_{11} \\ i\rho_{22} & -i\rho_{21} \end{pmatrix}$$
$$= i(\rho_{12} - \rho_{21})$$
$$P_z = \rho_{11} - \rho_{22}$$

$$Tr(\rho) = \rho_{11} + \rho_{22} = 1$$
$$P_z = \rho_{11} - \rho_{22}$$
$$1 + P_z = 2\rho_{11}$$
$$\frac{1 + P_z}{2} = \rho_{11}$$
$$1 - P_z = \cancel{\rho_{11}} + \rho_{22} - (\cancel{\rho_{11}} - \rho_{22}) = 2\rho_{22}$$

$$1 - P_z = 2\rho_{22}$$

$$1 - P_z = \rho_{22}$$

$$P_x + iP_y = \rho_{12} + \rho_{21} + ii(\rho_{12} - \rho_{21})$$

$$= \rho_{12} + \rho_{21} + (-1)(\rho_{12} - \rho_{21})$$

$$= \rho_{12} + \rho_{21} - \rho_{12} + \rho_{21}$$

$$= 2\rho_{21}$$

$$\frac{P_x + iP_y}{2} = \rho_{21}$$

$$P_x - iP_y = \rho_{12} + \rho_{21} - ii(\rho_{12} - \rho_{21})$$

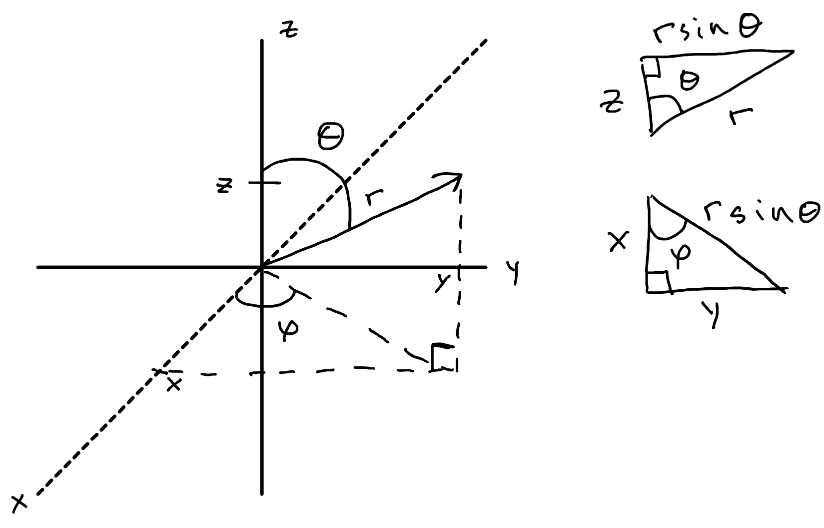
$$= \rho_{12} + \rho_{21} - (-1)(\rho_{12} - \rho_{21})$$

$$= \rho_{12} + \rho_{21} + \rho_{12} - \rho_{21}$$

$$= 2\rho_{12}$$

$$\frac{P_x - iP_y}{2} = \rho_{12}$$

c.



$$P_x = r\sin\theta\cos\varphi$$
$$P_y = r\sin\theta\sin\varphi$$
$$P_z = r\cos\theta$$

$$1 + P_z = 1 + r\cos\theta$$
$$P_x - iP_y = r\sin\theta\cos\varphi - ir\sin\theta\sin\varphi$$
$$= r\sin\theta(\cos\varphi - i\sin\varphi)$$
$$= r\sin\theta(e^{-i\varphi})$$

$$P_x + iP_y = r\sin\theta(\cos\varphi + i\sin\varphi)$$
$$= r\sin\theta(e^{i\varphi})$$

$$1 - P_z = 1 - r\cos\theta$$
$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & r\sin\theta e^{-i\varphi} \\ r\sin\theta e^{i\varphi} & 1 - r\cos\theta \end{pmatrix}$$

$$P = (0, 0, 1)$$

6. $$\mathrm{Tr}[(C - \langle C\rangle)^2 \rho]\,\mathrm{Tr}[\rho(D - \langle D\rangle)^2]$$
$$\left|\mathrm{Tr}[(C - \langle C\rangle)\rho^{1/2}\rho^{1/2}(D - \langle D\rangle)^2]\right|^2$$

$$\left|\mathrm{Tr}[\rho(D - \langle D\rangle)(C - \langle C\rangle)]\right|^2 = |Z|^2$$
$$[\mathrm{Re}(Z)]^2 + [\mathrm{Im}(Z]^2 \geq [\mathrm{Im}(Z)]^2$$
$$\left[\frac{1}{2i}(Z - Z^*)\right]^2 = \left[\frac{1}{2i}(\mathrm{Tr}(\rho DC) - \langle D\rangle\langle C\rangle - (\mathrm{Tr}(\rho CD) - \langle C\rangle\langle D\rangle))\right]^2$$
$$\left(\frac{1}{2i}\mathrm{Tr}[\rho(DC - CD)]\right)^2 = \left(\frac{1}{2i}\langle[D, C]\rangle\right)^2 = \frac{1}{2}|\langle[C, D]\rangle|$$

7. $E_1 = \alpha |1\rangle\langle 1| = \alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$

$E_2 = \alpha |-\rangle\langle -|$

$= \alpha \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \cdot \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)$

$= \frac{\alpha}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$

$= \frac{\alpha}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$

$= \frac{\alpha}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & \frac{\alpha}{2} \end{pmatrix}$

$E_3 = I - E_1 - E_2$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{2} & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & \frac{\alpha}{2} \end{pmatrix}$

$= \begin{pmatrix} 1 - \frac{\alpha}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 - \frac{3\alpha}{2} \end{pmatrix}$

$P_3 = \langle \psi | E_3 | \psi \rangle$

$\det(E_3 - \lambda I) = 0$

$\det\left(\begin{pmatrix} 1-\frac{\alpha}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1-\frac{3\alpha}{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$

$\det\begin{pmatrix} 1-\frac{\alpha}{2}-\lambda & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1-\frac{3\alpha}{2}-\lambda \end{pmatrix} = 0$

$(1-\frac{\alpha}{2}-\lambda)(1-\frac{3\alpha}{2}-\lambda) - (\frac{\alpha}{2})^2 = 0$

$1 - \frac{3\alpha}{2} - \lambda - \frac{\alpha}{2} + \frac{3\alpha^2}{4} + \frac{\alpha}{2}\lambda - \lambda + \frac{3\alpha}{2}\lambda + \lambda^2 - \frac{\alpha^2}{4} = 0$

$$1 - 2\alpha - 2\lambda + \frac{\alpha^2}{2} + 2\alpha\lambda + \lambda^2 = 0$$

$$\lambda^2 + (2\alpha - 2)\lambda + 1 - 2\alpha + \frac{\alpha^2}{2} = 0$$

$$\lambda = \frac{-\cancel{2}(\alpha - 1) \pm \sqrt{\cancel{4}(\alpha-1)^2 - \cancel{4}\left(1 - 2\alpha + \frac{\alpha^2}{2}\right)}}{\cancel{2}}$$

$$\lambda = 1 - \alpha \pm \sqrt{\alpha^2 - \cancel{2\alpha} + \cancel{1} - \cancel{1} + \cancel{2\alpha} - \frac{\alpha^2}{2}}$$

$$\lambda = 1 - \alpha \pm \sqrt{\frac{\alpha^2}{2}}$$

$$\lambda = 1 - \alpha \pm \frac{\alpha}{\sqrt{2}} \geq 0$$

$$1 \geq \alpha \pm \frac{\alpha}{\sqrt{2}}$$

$$1 \geq \alpha\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

$$1 \geq \alpha\left(\frac{\sqrt{2} \pm 1}{\sqrt{2}}\right)$$

$$\frac{\sqrt{2}}{\sqrt{2} \pm 1} \geq \alpha$$

$$\frac{\sqrt{2}}{\sqrt{2} + 1} \geq \alpha$$

8. $\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_N$

$$= \frac{1}{2}\left(I_1 + \vec{v}_1 \cdot \vec{\sigma}_1\right) \otimes \ldots \otimes \frac{1}{2}\left(I_N + \vec{v}_N \cdot \vec{\sigma}_N\right)$$

$$= \left(\frac{1}{2}\right)^N \Big[I_1 \otimes \ldots \otimes I_N + \vec{v}_1 \cdot \vec{\sigma}_1 \otimes I_2 \otimes \ldots \otimes I_N$$

$$+ I_1 \otimes \vec{v}_2 \cdot \vec{\sigma}_2 \otimes \ldots \otimes I_N$$

$$+ \vec{v}_1 \cdot \vec{\sigma}_1 \otimes \vec{v}_2 \cdot \vec{\sigma}_2 \otimes I_3 \otimes \ldots \otimes I_N$$

$$+ \vec{v}_1 \cdot \vec{\sigma}_1 \otimes I_2 \otimes \vec{v}_3 \cdot \vec{\sigma}_3 \otimes I_4 \otimes \ldots \otimes I_N$$

$$+ \ldots$$

$$\left(\frac{1}{2}\right)^N \sum_{i<j}^{N} v_i \sigma_i \otimes v_j \sigma_j \otimes I^{\otimes N}$$

9. a) $\mathrm{Tr}_B\, \rho_{AB} = \mathrm{Tr}_B \sum_{ij} \lambda_{ij} |a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j|$

$$= \sum_{ij} \lambda_{ij}\, \mathrm{Tr}_B \underbrace{\left(|a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j|\right)}$$

$$= \sum_{ij} \lambda_{ij} |a_i\rangle\langle a_i| \underbrace{\mathrm{Tr}_B |b_j\rangle\langle b_j|}_{\sum_k \langle v_k|b_j\rangle\langle b_j|v_k\rangle}$$

$$=$$

$$\sum_k \langle v_k|\left(\sum_{ij} \lambda_{ij} |a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j|\right)|v_k\rangle$$

$$= \sum_k \langle v_k|\rho_{AB}|v_k\rangle$$

b) No

c) $\sum_{\mu} \sum_{ij\eta\xi} \sum_{kl\alpha\beta} \lambda_\mu U_{ij\eta\xi} U^\dagger_{kl\alpha\beta} \cdot$

$$\cdot\, \mathrm{Tr}_B\left[|i\rangle\langle j|\rho_A|k\rangle\langle l| \otimes |\eta\rangle\langle\xi|\mu\rangle\langle\mu|\alpha\rangle\langle\beta|\right]$$

$$\sum_{\mu} \sum_{ij\eta\xi} \sum_{kl\alpha\beta} \lambda_\mu U_{ij\eta\xi} U^\dagger_{kl\alpha\beta} \cdot$$

$$\cdot\, |i\rangle\langle j|\rho_A|k\rangle\langle l|\, \mathrm{Tr}_B\left[|\eta\rangle\langle\xi|\mu\rangle\langle\mu\rangle\langle\alpha\rangle\langle\beta|\right]$$

$$\sum_{\mu} \sum_{ij\eta\xi} \sum_{kl\alpha\beta} \lambda_\mu U_{ij\eta\xi} U^+_{kl\alpha\beta} |i\rangle\langle j| \rho_A |k\rangle\langle l| \delta_{\eta\beta} \delta_{\xi\mu} \delta_{\mu\alpha}$$

$$\sum_{\mu\eta} \sum_{ij} \sum_{kl} \lambda_\mu U_{ij\eta\mu} U^+_{kl\mu\eta} |i\rangle\langle j| \rho_A |k\rangle\langle l|$$

$$\sum_{\mu\eta} \lambda_\mu \underbrace{\left[\sum_{ij} U_{ij\eta\mu} |i\rangle\langle j|\right]}_{\substack{\sum_{ij} \langle\eta|U_{ij}|\mu\rangle |i\rangle\langle j| \\ = \langle\eta|U|\mu\rangle}} \rho_A \underbrace{\left[\sum_{kl} U^+_{kl\mu\eta} |k\rangle\langle l|\right]}_{\substack{\sum_{kl} \langle\mu|U^+_{kl}|\eta\rangle |k\rangle\langle l| \\ = \langle\mu|U^+|\eta\rangle}}$$

$$\sum_{\mu\eta} \lambda_\mu \langle\eta|U|\mu\rangle \rho_A \langle\mu|U^+|\eta\rangle$$

10. $\sum_\alpha K_\alpha^+ K_\alpha = I$

$\rho' = \sum_\alpha K_\alpha \rho K_\alpha^+$

$Tr[\rho'] = 1 \quad Tr[\rho] = 1$

$Tr[\sum_\alpha K_\alpha \rho K_\alpha^+] = 1$

$Tr(\rho \sum_\alpha K_\alpha^+ K_\alpha) = 1$

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad K^+ = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}$$

$$K^+K = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha^*\beta + \gamma^*\delta \\ \alpha\beta^* + \gamma\delta^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

$$= \begin{pmatrix} A & z \\ z^* & B \end{pmatrix}$$

$$\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$$

$$\rho K^\dagger K = \begin{pmatrix} aA + bZ^* & aZ + bB \\ b^*A + (1-a)Z^* & b^*Z + (1-a)B \end{pmatrix}$$

$$Tr(\rho K^\dagger K) = aA + bZ^* + b^*Z + (1-a)B$$

let $a = \frac{1}{2}$, $b = 0$

$$Tr(\rho) = a + 1 - a = \frac{1}{2} + 1 - \frac{1}{2} = 1$$

$$Tr(\rho K^\dagger K) = \frac{1}{2}A + 0 \cdot Z^* + 0 \cdot Z + (1 - \frac{1}{2})B$$

$$= \frac{1}{2}A + \frac{1}{2}B = 1$$

let $A = 1$, $B = 1$

$$\frac{1}{2}(1) + \frac{1}{2}(1) = 1$$

$$K^\dagger K = \begin{pmatrix} 1 & Z \\ Z^* & 1 \end{pmatrix}$$

let $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{2}}$, $\gamma = \frac{i}{\sqrt{2}}$, $\delta = \frac{i}{\sqrt{2}}$

$$K^\dagger K = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha^*\beta + \gamma^*\delta \\ \alpha\beta^* + \gamma\delta^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq I$$