PHYS 550 Homework 5 - Neema Badihian

7.4.2. i)
$$\dot{\rho} = \gamma(Y\rho Y^{+} - \tfrac{1}{2}\{Y^{+}Y, \rho\})$$
$$= \gamma(Y\rho Y - \tfrac{1}{2}\{I, \rho\})$$
$$= \gamma(Y\rho Y - \rho)$$
$$= \gamma(\tfrac{1}{2}(I - v_x X + v_y I - v_z Z) - \tfrac{1}{2}(I + v_x X + v_y I + v_z Z))$$
$$= \gamma(-v_x X - v_z Z)$$
$$\tfrac{1}{2}\dot{v}_x = -\gamma v_x$$
$$\dot{v}_y = 0$$
$$\tfrac{1}{2}\dot{v}_z = -\gamma v_z$$
$$v_x(t) = v_x(0)e^{-2\gamma t}$$
$$v_y(t) = v_y(0)$$
$$v_z(t) = v_z(0)e^{-2\gamma t}$$
$$\rho(t) = \tfrac{1}{2}(I + v_x(0)e^{-2\gamma t}X + v_y(0)Y + v_z(0)e^{-2\gamma t}Z)$$

ii)
$$\rho(t) = \tfrac{1}{2}(I + v_x(0)e^{-2\gamma t}X + v_y(0)Y + v_z(0)e^{-2\gamma t}Z)$$
$$= |a|^2\rho(0) + |b|^2 Y\rho(0)Y$$
$$= (|a|^2 + |b|^2)\tfrac{1}{2}(I + v_y(0)Y) + (|a|^2 - |b|^2)\tfrac{1}{2}(v_x(0)X + v_z(0)Z)$$
$$K_0 = \sqrt{\frac{1+e^{-2\gamma t}}{2}}\, I$$
$$K_1 = \sqrt{\frac{1-e^{-2\gamma t}}{2}}\, Y$$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = \frac{1+e^{-2\gamma t}}{2} I + \frac{1-e^{-2\gamma t}}{2} Y^\dagger Y$$

$$= \frac{1}{2}(1+e^{-2\gamma t}+1-e^{-2\gamma t}) I$$

$$= I$$

7.4.3. i) $\dot{\rho} = \frac{1}{2}\dot{\vec{v}}\cdot\vec{\sigma}$

$$= -i[H,\rho] + \gamma(Z\rho Z - \rho)$$

$$= -i[\vec{h}\cdot\vec{\sigma}, \tfrac{1}{2}(I+\vec{v}\cdot\vec{\sigma})] + \gamma(Z\tfrac{1}{2}(I+\vec{v}\cdot\vec{\sigma})Z - \tfrac{1}{2}(I+\vec{v}\cdot\vec{\sigma}))$$

$$\frac{1}{2}(\dot{v}_x X + \dot{v}_y Y + \dot{v}_z Z) = \begin{vmatrix} X & Y & Z \\ h_x & h_y & h_z \\ v_x & v_y & v_z \end{vmatrix} - \gamma(v_x X + v_y Y)$$

a) $\vec{h} = (0,0,h)$

$$\frac{1}{2}(\dot{v}_x X + \dot{v}_y Y + \dot{v}_z Z) = \begin{vmatrix} X & Y & Z \\ 0 & 0 & h \\ v_x & v_y & v_z \end{vmatrix} - \gamma(v_x X + v_y Y)$$

$$= -h(v_y X - v_x Y) - \gamma(v_x X + v_y Y)$$

$$= (-h v_y - \gamma v_x)X + (h v_x - \gamma v_y)Y$$

$$\dot{\vec{v}} = \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix}$$

$$= \begin{pmatrix} -2hv_y - 2\gamma v_x \\ 2hv_x - 2\gamma v_y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2\gamma & -2h & 0 \\ 2h & -2\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$= M\vec{v}$$

$$M \text{ eigenvalues} = \{0, -2ih - 2\gamma, 2ih - 2\gamma\}$$
$$M \text{ eigenvectors} = \{(0,0,1), (\tfrac{-i}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), (\tfrac{i}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0)\}$$

$$\vec{v}(t) = \exp(Mt)\,\vec{v}(0)$$

$$\vec{v}(t) = (e^{-2\gamma t}(v_x(0)\cos(2ht) - v_y(0)\sin(2ht)),$$
$$e^{-2\gamma t}(v_y(0)\cos(2ht) + v_x(0)\sin(2ht)),$$
$$v_z(0))$$

b) $\vec{h} = (h_x, h_y, 0)$

$$\tfrac{1}{2}(\dot{v}_x X + \dot{v}_y Y + \dot{v}_z Z) = \begin{vmatrix} X & Y & Z \\ h_x & h_y & 0 \\ v_x & v_y & v_z \end{vmatrix} - \gamma(v_x X + v_y Y)$$

$$= (v_z h_y - \gamma v_x)X + (-v_z h_x - \gamma v_y)Y + (h_x v_y - h_y v_x)Z$$

$$M = \begin{pmatrix} -2\gamma & 0 & 2h_y \\ 0 & -2\gamma & -2h_x \\ -2h_y & 2h_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2\gamma & 0 & 2h\sin(\phi) \\ 0 & -2\gamma & -2h\cos(\phi) \\ -2h\sin(\phi) & 2h\cos(\phi) & 0 \end{pmatrix}$$

$$M \text{ eigenvalues} = \{-2\gamma, -\gamma - \sqrt{\gamma^2 - 4h^2}, -\gamma + \sqrt{\gamma^2 - 4h^2}\}$$

let $\vec{h} = (h, 0, 0)$ & $|\gamma| > 2h$

$$v_x(t) = v_x(0)e^{-2\gamma t}$$

$$v_y(t) = e^{-\gamma t}\Big(v_y(0)\cosh(t\sqrt{\gamma^2 - 4h^2})$$
$$- (\gamma v_y(0) + 2h v_z(0))\frac{\sinh(t\sqrt{\gamma^2 - 4h^2})}{\sqrt{\gamma^2 - 4h^2}}\Big)$$

$$v_z(t) = e^{-\gamma t}\Big(v_z(0)\cosh(t\sqrt{\gamma^2 - 4h^2})$$
$$+ (\gamma v_z(0) + 2h v_y(0))\frac{\sinh(t\sqrt{\gamma^2 - 4h^2})}{\sqrt{\gamma^2 - 4h^2}}\Big)$$

let $|\gamma| = 2h$

$$\vec{v}(t) = (v_x(0)e^{-2\gamma t},$$

$e^{-\gamma t}(v_y(0) - \gamma t(v_y(0) + v_z(0))),$
$e^{-\gamma t}(v_z(0) + \gamma t(v_y(0) + v_z(0))))$

ii) $\vec{h} = (1,0,0)$

$$M = \begin{pmatrix} -2\gamma & 0 & 0 \\ 0 & -2\gamma & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

As t increases from 0 to $1/\gamma$ to $10/\gamma$, the point will start at the edge of the positive side of the x-axis and approach the origin.

The limit $t \to \infty$ will leave us with $\vec{v}(t) = (0,0,0)$

iii)

7.4.5. i) $\dot{\rho} = \gamma(\sigma_- \rho \sigma_-^+ - \frac{1}{2}\{\sigma_-^+ \sigma_-, \rho\})$

$= \gamma(|1\rangle\langle 0|\rho|0\rangle\langle 1| - \frac{1}{2}\{|0\rangle\langle 0|, \rho\})$

$= \gamma(|1\rangle\langle 0|(\frac{1}{2}(I + v_x X + v_y Y \pm v_z Z))|0\rangle\langle 1|$
$- \frac{1}{4}|0\rangle\langle 0|(I + v_x X + v_y Y + v_z Z) - \frac{1}{4}(I + v_x X + v_y Y + v_z Z)|0\rangle\langle 0|)$

$= \gamma(|1\rangle\langle 0|(\frac{1}{2}(v_x X + v_y Y + v_z Z))|0\rangle\langle 1|$
$- \frac{1}{2}(|0\rangle\langle 0|(\frac{1}{2}(v_x X + v_y Y + v_z Z)) + \frac{1}{2}(v_x X + v_y Y + v_z Z))|0\rangle\langle 0|))$

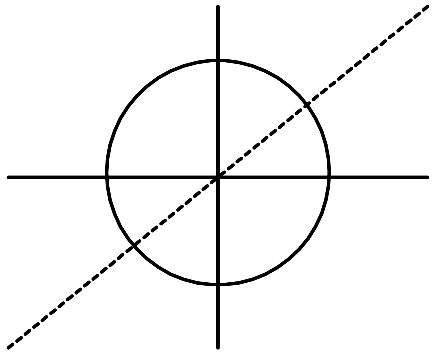
$= \gamma(-\frac{1}{4}v_x X - \frac{1}{4}v_y Y - \frac{1}{2}(1 + v_z)Z)$

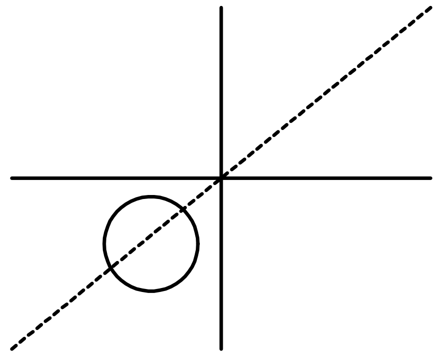
$\dot{\vec{v}} = (-\frac{\gamma}{2}v_x, -\frac{\gamma}{2}v_y, -\gamma(1 + v_z))$

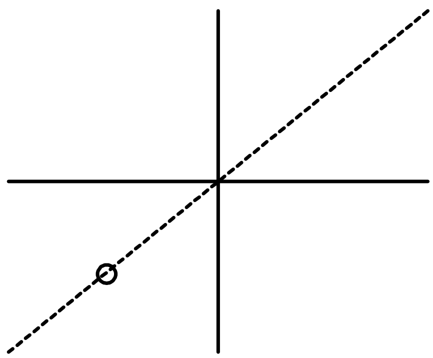
$\vec{v}(t) = (c_1' e^{-\gamma t/2}, c_2' e^{-\gamma t/2}, c_3' e^{-\gamma t} - 1)$

$\vec{v}(t) = (v_x(0)e^{-\gamma t/2}, v_y(0)e^{-\gamma t/2}, (1 + v_z(0))e^{-\gamma t} - 1)$

$t = 0$







$\rho = |1\rangle\langle 1|$

As $t \to \infty$, $\vec{v}(t) = (0, 0, -1)$

ii) $K_0 = \sqrt{1-p}\,|0\rangle\langle 0| + |1\rangle\langle 1|$
$K_1 = \sqrt{p}\,|1\rangle\langle 0|$

$\vec{v} = (\sqrt{1-p}\, v_x, \sqrt{1-p}\, v_y, (1-p)v_z - p)$

$K_0 = e^{-\gamma t/2}|0\rangle\langle 0| + |1\rangle\langle 1|$
$K_1 = \sqrt{1-e^{-\gamma t}}\,|1\rangle\langle 0|$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = e^{-\gamma t}|0\rangle\langle 0| + |1\rangle\langle 1| + (1-e^{-\gamma t})|0\rangle\langle 1|1\rangle\langle 0|$$
$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$
$$= I$$

7.4.7. i) <u>Amplitude</u>

$A_0 = |0\rangle\langle 0| + \sqrt{1-q}\,|1\rangle\langle 1|$
$A_1 = \sqrt{q}\,|0\rangle\langle 1|$

$$\sum_i A_i \rho A_i^\dagger = \begin{pmatrix} q + (1-q)a & b\sqrt{1-q} \\ b^*\sqrt{1-q} & (1-q)(1-a) \end{pmatrix}$$

Phase

$$P_0 = \sqrt{p}\, I$$
$$P_1 = \sqrt{1-p}\, Z$$

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

$$p\rho + (1-p) Z \rho Z = \begin{pmatrix} \rho_{00} & (2p-1)\rho_{01} \\ (2p-1)\rho_{10} & \rho_{11} \end{pmatrix}$$

Amplitude & Phase

$$\begin{pmatrix} q + (1-q) a & (2p-1) b \sqrt{1-q} \\ (2p-1) b^* \sqrt{1-q} & (1-q)(1-a) \end{pmatrix}$$

$$2p-1 = \frac{e^{-t/T_2}}{\sqrt{1-p}}$$

$$= e^{-t/T_2} e^{t/2T_1}$$

$$p = \frac{1}{2}\left(1 + e^{-\frac{t}{T_2} + \frac{t}{2T_1}}\right)$$

$$\frac{1}{T_\phi} = \frac{1}{T_2} - \frac{1}{2T_1}$$

$$p = \frac{1}{2}\left(1 + e^{-\frac{t}{T_\phi}}\right)$$

ii) $P_0 = |0\rangle\langle 0| + (2p-1)|1\rangle\langle 1|$
$P_1 = 2\sqrt{p(1-p)}\, |1\rangle\langle 1|$

$$K_0 = P_0 A_0$$
$$= |0\rangle\langle 0| + (2p-1)\sqrt{1-q}\, |1\rangle\langle 1|$$
$$= |0\rangle\langle 0| + e^{-t/T_\phi} e^{-t/2T_1} |1\rangle\langle 1|$$

$$K_1 = P_0 A_1$$
$$= \sqrt{q}\, |0\rangle\langle 1|$$
$$= \sqrt{1 - e^{-t/T_1}}\, |0\rangle\langle 1|$$

$$K_2 = P_1 A_0$$
$$= 2\sqrt{p(1-p)(1-q)}\, |1\rangle\langle 1|$$
$$= e^{-t/2T_1}\sqrt{1 - e^{-2t/T_\phi}}\, |1\rangle\langle 1|$$

iii) $K_\alpha = \sqrt{1/T_\alpha\, dt}\, L_\alpha$

$$e^{(-\alpha t)} = 1 - \alpha t + O(t^2)$$

$$K_1 = \sqrt{1-(1-t/T_1)}\,|0\rangle\langle 1|$$

$$K_2 = \sqrt{1-(1-2\,t/T_\phi)}\,|1\rangle\langle 1|$$

$$L_1 = 1/T_1\,|0\rangle\langle 1|$$
$$L_2 = 2/T_\phi\,|1\rangle\langle 1|$$

$$\mathcal{L} = \tfrac{1}{T_1}\left(|0\rangle\langle 1|\rho|1\rangle\langle 0| - \tfrac{1}{2}\{|1\rangle\langle 1|, \rho\}\right)$$
$$+\ 2/T_\phi\left(|1\rangle\langle 1|\rho|1\rangle\langle 1| - \tfrac{1}{2}\{|1\rangle\langle 1|, \rho\}\right)$$
$$= \tfrac{1}{T_1}\left(\sigma^-\rho\sigma^+ - \tfrac{1}{2}\{\sigma^+\sigma^-, \rho\}\right) + \tfrac{1}{2T_\phi}\left(Z\rho Z - \rho\right)$$

iv) $\mathcal{L} = \frac{1}{T_1}\left(\sigma^-\rho\sigma^+ - \frac{1}{2}\{\sigma^+\sigma^-, \rho\}\right)$

$$\tfrac{1}{T_\phi} = 0$$

$$\tfrac{1}{T_2} = \tfrac{1}{2T_1}$$

$$T_1 = 2T_1$$

v) $\frac{1}{T_2} - \frac{1}{2T_1} = \frac{1}{T_\phi}$

$$\tfrac{1}{T_\phi} \gtrsim 0$$

$$T_2 \leq 2T_1$$

7.4.8. i) $\mathcal{L}(I)(\rho) = \partial_t \rho = 0 \ \forall t$
$\mathcal{L}(I) = 0$

ii) 7.4.2. $\mathcal{L}(I) = \gamma(YIY - I)$
$= \gamma(Y^2 - I)$
$= 0$

unital

7.4.3. $\mathcal{L}(I) = -i[H, I] + \gamma(ZIZ - I)$
$= 0 + \gamma(Z^2 - I)$
$= 0$

unital

7.4.5. $\mathcal{L}(I) = \gamma(|1\rangle\langle 0| I |0\rangle\langle 1| - \frac{1}{2}\{|0\rangle\langle 0|, I\})$
$= \gamma(|1\rangle\langle 1| - |0\rangle\langle 0|)$
$= -\gamma Z$
$\neq 0$
not unital

7.4.7. $\mathcal{L}(I) = \frac{1}{T_1}(|0\rangle\langle 1| I |1\rangle\langle 0| - \frac{1}{2}\{|1\rangle\langle 1|, I\})$
$+ \frac{2}{T_\phi}(|1\rangle\langle 1| I |1\rangle\langle 1| - \frac{1}{2}\{|1\rangle\langle 1|, I\})$
$= \frac{1}{T_1} Z$

not unital

7.4.9. i) $\rho \mapsto \rho' = \begin{cases} \frac{1}{2}I & \text{w.p. } p \\ \rho & \text{w.p. } 1-p \end{cases}$

$\rho' = p\frac{I}{2} + (1-p)\rho$

$\rho \mapsto \rho' = (1 - \frac{3}{4}p)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)$

$K_0 = \sqrt{1 - \frac{3}{4}p}\, I$

$K_i = \sqrt{\frac{p}{4}}\, \sigma_i$ for $i = 1, 2, 3$

$\mathcal{L}(\rho) = \sum_i \gamma(\sigma_i \rho \sigma_i - \rho)$

$= \gamma(X\rho X + Y\rho Y + Z\rho Z - 3\rho)$

$2I - \rho = X\rho X + Y\rho Y + Z\rho Z$

$\mathcal{L}(\rho) = \gamma(2I - \rho - 3\rho)$

$= \gamma(2I - 4\rho)$

ii) $\mathcal{N}(\rho) = \begin{cases} \frac{1}{2}(Z\rho Z + \rho) \\ |0\rangle\langle 0| \\ I/2 \end{cases}$

<u>Phase</u>

$\mathcal{L}(\rho) = \gamma(Z\rho Z - \rho)$
$= \gamma(Z\rho Z + \rho - 2\rho)$
$= 2\gamma(\frac{1}{2}(Z\rho Z + \rho) - \rho)$

$\kappa = 2\gamma$
$\tilde{\Upsilon} = \rho$

$$\tilde{\kappa}(\mathcal{N}-\tilde{\mathcal{I}}) = 2\gamma\left(\tfrac{1}{2}(Z\rho Z+\rho)-\rho\right)$$

<u>Depolarizing</u>

$$\mathcal{L}(\rho) = \gamma(2I-4\rho)$$
$$= 4\gamma(I/2-\rho)$$
$$\kappa = 2\gamma$$
$$\tilde{\mathcal{I}} = \rho$$
$$\kappa(\mathcal{N}-\mathcal{I}) = 4\gamma(I/2-\rho)$$

8.2.1. No, the Lindbladian is not Hermitian.

$$\mathcal{L}^{+}(X) = \mathcal{L}(X^{+}) = \mathcal{L}(X)$$

<u>Counterexample</u>

$$\mathcal{L}(|0\rangle\langle 1|) = Z|0\rangle\langle 1|Z - |0\rangle\langle 1|$$
$$= -2|0\rangle\langle 1|$$
$$\mathcal{L}((|0\rangle\langle 1|)^{+}) = -2|1\rangle\langle 0|$$
$$\mathcal{L}(|0\rangle\langle 1|) \neq \mathcal{L}((|0\rangle\langle 1|)^{+})$$

8.5.3. i) $\dot{\vec{v}} = G_S \vec{v}$

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = \begin{matrix} -8v_x + v_y \\ -8v_y \\ -9v_z \end{matrix}$$

$$\dot{v}_x(t) = -8v_x(t) + v_y(0)e^{-8t}$$
$$v_y(t) = v_y(0)e^{-8t}$$
$$v_z(t) = v_z(0)e^{-9t}$$

$$v_x(t) = (\alpha+\beta t)e^{-8t}$$
$$(-8\alpha+\beta-8\beta t)e^{-8t} = (-8(\alpha+\beta t)+v_y(0))e^{-8t}$$
$$\beta = v_y(0)$$
$$v_x(0) = \alpha$$
$$v_x(t) = (v_x(0)+v_y(0)t)e^{-8t}$$

$$\vec{v}(t) = \left((v_x(0)+v_y(0)t)e^{-8t},\ v_y(0)e^{-8t},\ v_z(0)e^{-9t}\right)$$

ii) $\vec{v}(0) = (0,1,0)$

$$\vec{v}(t) = (te^{-8t}, e^{-8t}, e^{-9t})$$

$$\lim_{t\to\infty} \vec{v}(t) = \vec{0}$$

$$\lim_{t\to\infty} \rho(t) = I/2$$