PHYS 550 Final Project

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I. INTRODUCTION

In this paper, we analyze several quantum channels and their G matrices in order to understand the breakdown times and the effect of control fields on each respective channel. We begin in section II with the Bit-Flip and Bit-Phase-Flip channels. The differential equations are found for both channels in order to solve for the breakdown times. We then use our solutions to find the control field equations in section II C and analyze how the channels change. In sections III and IV we go through this same process for the Depolarizing and Amplitude Damping channels, only with the added work of finding the G matrices for both channels. Finally, we discuss our results in section V and the possibility of future research.

II. PROBLEM 8.5.6

As we can see, the structure of the $G^{(x)}$ and $G^{(y)}$ matrices are the following:

$$G^{(x)} = 2 \times \begin{pmatrix} 0 & -h_z & h_y \\ h_z & -\gamma & -h_x \\ -h_y & h_x & -\gamma \end{pmatrix} ; G^{(y)} = 2 \times \begin{pmatrix} -\gamma & -h_z & h_y \\ h_z & 0 & -h_x \\ -h_y & h_x & -\gamma \end{pmatrix} \tag{1}$$

The values for $\{\dot{v}_x, \dot{v}_y, \dot{v}_z\}$ for $G^{(x)}$ and $G^{(y)}$ are easily obtained from each row of the respective matrices using the following:

$$\dot{v}_j = a_{j0}v_x + a_{j1}v_y + a_{j2}v_z, \qquad j\epsilon\{0,1,2\} = \{x, y, z\} \tag{2}$$

A. Bit-Flip Channel ($G^{(x)}$ *case*) Breakdown Time

For $G^{(x)}$ we have:

$$\dot{v}_x = -2h_z v_y + 2h_y v_z \,, \tag{3a}$$
$$\dot{v}_y = 2h_z v_x - 2\gamma v_y - 2h_x v_z \,, \tag{3b}$$
$$\dot{v}_z = -2h_y v_x + 2h_x v_y - 2\gamma v_z \,. \tag{3c}$$

To solve for $\dot{v}_x$, we must first redefine $-2h_z v_y$ and $2h_y v_z$:

$$-2h_z v_y = -\frac{\dot{v}_y + 2\gamma v_y + 2h_x v_z}{v_x} v_y \,, \tag{4a}$$
$$2h_y v_z = \frac{-\dot{v}_z + 2h_x v_y - 2\gamma v_z}{v_x} v_z \tag{4b}$$

We can use these new definitions to redefine $\dot{v}_x$:

$$\dot{v}_x = -2h_z v_y + 2h_y v_z \tag{5a}$$
$$\dot{v}_x = -\frac{\dot{v}_y + 2\gamma v_y + 2h_x v_z}{v_x} v_y + \frac{-\dot{v}_z + 2h_x v_y - 2\gamma v_z}{v_x} v_z \tag{5b}$$
$$\dot{v}_x = \frac{-(\dot{v}_y v_y + \dot{v}_z v_z) - 2\gamma(v_y^2 + v_z^2)}{v_x} \tag{5c}$$
$$\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z = -2\gamma(v_y^2 + v_z^2) \tag{5d}$$

Operating under the assumption that $\dot{v}_x v_x + \dot{v}_y v_y = 0$, which is equivalent to the desired nature of the control fields, the equation above may be re-expressed:

$$\dot{v}_z v_z = -2\gamma(v_y^2 + v_z^2) \tag{6a}$$
$$\int_0^t v_z \frac{\partial v_z}{\partial t} \partial t = -2\gamma \int_0^t (v_y^2 + v_z^2) \partial t \tag{6b}$$
$$\frac{1}{2}(v_z^2(t) - v_z^2(0)) = -2\gamma \int_0^t (v_y^2 + v_z^2) \partial t \tag{6c}$$
$$v_z(t) = \sqrt{v_z^2(0) - 4\gamma \int_0^t (v_y^2 + v_z^2) \partial t} \tag{6d}$$

In order to determine an appropriate breakdown time t_b, the following assignment can be made:

$$v_z^2(0) = 4\gamma \int_0^{t_b} (v_y^2 + v_z^2) \partial t \tag{7}$$

As a result, Equation (8d) can be interpreted as:

$$v_z(t) = 2\sqrt{\gamma} \sqrt{\int_0^{t_b} (v_y^2 + v_z^2) \partial t - \int_0^t (v_y^2 + v_z^2) \partial t} \tag{8a}$$
$$v_z(t) = 2\sqrt{\gamma} \sqrt{\int_t^{t_b} (v_y^2 + v_z^2) \partial t} \tag{8b}$$
$$\frac{v_z^2(t)}{4\gamma} = \int_t^{t_b} (v_y^2 + v_z^2) \partial t \tag{8c}$$

If the special case is explored where, rather than treating $v_x^2 + v_y^2$ as a constant, both v_x^2 and v_y^2 are treated as constants separately, equation (6a) can be rewritten as:

$$\dot{v}_z v_z = -2\gamma(v_y^2 + v_z^2) \tag{9a}$$
$$= -2\gamma(C + v_z^2) \tag{9b}$$
$$\int_0^t \frac{v_z \frac{\partial v_z}{\partial t} \partial t}{C + v_z^2} = \int_0^t -2\gamma \partial t \tag{9c}$$
$$\frac{1}{2} ln|v_z^2(t) + C| - ln|v_z^2(0) + C| = -2\gamma t \tag{9d}$$
$$ln|v_z^2(t) + C| = ln|v_z^2(0) + C| - 4\gamma t \tag{9e}$$
$$v_z^2(t) + C = (v_z^2(0) + C)e^{-4\gamma t} \tag{9f}$$

To solve for the breakdown time t_b, $v_z(0)$ must first be defined. For the simplest solutions for t_b and $v_z(t)$, it may be defined the following way:

$$v_z(0) = \sqrt{(e^{4\gamma t_b} - C)} \tag{10a}$$
$$v_z^2(0) = e^{4\gamma t_b} - C \tag{10b}$$
$$v_z^2(t) + C = (e^{4\gamma t_b} - C + C)e^{-4\gamma t} \tag{10c}$$
$$v_z(t) = \sqrt{e^{4\gamma(t_b - t)} - C} \tag{10d}$$

$$\sqrt{(e^{4\gamma t_b} - C)} = v_z(0) \tag{11a}$$
$$e^{4\gamma t_b} = v_z^2(0) + C \tag{11b}$$
$$t_b = \frac{ln|v_z^2(0) + C|}{4\gamma} \tag{11c}$$

B. Bit-Phase-Flip Channel ($G^{(y)}$ *case*) Breakdown Time

For $G^{(y)}$ we have:

$$\dot{v}_x = -2\gamma v_x - 2h_z v_y + 2h_y v_z \,, \tag{12a}$$
$$\dot{v}_y = 2h_z v_x - 2h_x v_z \,, \tag{12b}$$
$$\dot{v}_z = -2h_y v_x + 2h_x v_y - 2\gamma v_z \,. \tag{12c}$$

To solve for $\dot{v}_y$, we must first redefine $2h_z v_x$ and $-2h_x v_z$:

$$2h_z v_x = -\frac{\dot{v}_x + 2\gamma v_x - 2h_y v_z}{v_y} v_x \,, \tag{13a}$$
$$-2h_x v_z = \frac{-\dot{v}_z - 2h_y v_x - 2\gamma v_z}{v_y} v_z \tag{13b}$$

We can use these new definitions to redefine $\dot{v}_y$:

$$\dot{v}_y = 2h_z v_x - 2h_x v_z \tag{14a}$$

$$\dot{v}_y = -\frac{\dot{v}_x + 2\gamma v_x - 2h_y v_z}{v_y} v_x + \frac{-\dot{v}_z - 2h_y v_x - 2\gamma v_z}{v_y} v_z \tag{14b}$$

$$\dot{v}_y = \frac{-(\dot{v}_x v_x + \dot{v}_z v_z) - 2\gamma(v_x^2 + v_z^2)}{v_y} \tag{14c}$$

$$\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z = -2\gamma(v_x^2 + v_z^2) \tag{14d}$$

Operating under the assumption that $\dot{v}_x v_x + \dot{v}_y v_y = 0$, which is equivalent to the desired nature of the control fields, the equation above may be re-expressed:

$$\dot{v}_z v_z = -2\gamma(v_x^2 + v_z^2) \tag{15a}$$

$$\int_0^t v_z \frac{\partial v_z}{\partial t} \partial t = -2\gamma \int_0^t (v_x^2 + v_z^2) \partial t \tag{15b}$$

$$\frac{1}{2}(v_z^2(t) - v_z^2(0)) = -2\gamma \int_0^t (v_x^2 + v_z^2) \partial t \tag{15c}$$

$$v_z(t) = \sqrt{v_z^2(0) - 4\gamma \int_0^t (v_x^2 + v_z^2) \partial t} \tag{15d}$$

In order to determine an appropriate breakdown time t_b, the following assignment can be made:

$$v_z^2(0) = 4\gamma \int_0^{t_b} (v_x^2 + v_z^2) \partial t \tag{16}$$

As a result, Equation (22d) can be interpreted as:

$$v_z(t) = 2\sqrt{\gamma} \sqrt{\int_0^{t_b} (v_x^2 + v_z^2) \partial t - \int_0^t (v_x^2 + v_z^2) \partial t} \tag{17a}$$

$$\frac{v_z^2(t)}{4\gamma} = \int_t^{t_b} (v_x^2 + v_z^2) \partial t \tag{17b}$$

If the special case is explored where, rather than treating $v_x^2 + v_y^2$ as a constant, both v_x^2 and v_y^2 are treated as constants separately, equation (18a) can be rewritten as:

$$\dot{v}_z v_z = -2\gamma(v_x^2 + v_z^2) \tag{18a}$$
$$= -2\gamma(C + v_z^2) \tag{18b}$$
$$\int_0^t \frac{v_z \frac{\partial v_z}{\partial t}\partial t}{C + v_z^2} = \int_0^t -2\gamma \partial t \tag{18c}$$
$$\frac{1}{2} ln|v_z^2(t) + C| - ln|v_z^2(0) + C| = -2\gamma t \tag{18d}$$
$$ln|v_z^2(t) + C| = ln|v_z^2(0) + C| - 4\gamma t \tag{18e}$$
$$v_z^2(t) + C = (v_z^2(0) + C)e^{-4\gamma t} \tag{18f}$$
$$\tag{18g}$$

To solve for the breakdown time t_b, $v_z(0)$ must first be defined. For the simplest solutions for t_b and $v_z(t)$, it may be defined the following way:

$$v_z(0) = \sqrt{(e^{4\gamma t_b} - C)} \tag{19a}$$
$$v_z^2(0) = e^{4\gamma t_b} - C \tag{19b}$$
$$v_z^2(t) + C = (e^{4\gamma t_b} - C + C)e^{-4\gamma t} \tag{19c}$$
$$v_z^2(t) + C = e^{4\gamma(t_b - t)} \tag{19d}$$
$$v_z(t) = \sqrt{e^{4\gamma(t_b - t)} - C} \tag{19e}$$

$$\sqrt{(e^{4\gamma t_b} - C)} = v_z(0) \tag{20a}$$
$$e^{4\gamma t_b} = v_z^2(0) + C \tag{20b}$$
$$t_b = \frac{ln|v_z^2(0) + C|}{4\gamma} \tag{20c}$$

C. Analysis of Bit-Flip and Bit-Phase-Flip Channels

As evidenced by Equations (10d) and (15d), $v_z(t)$ is a function of the difference between t_b and t; however, since it must remain real, the term under the radical must remain non-negative. For this to be true, since v_x^2 and v_y^2 were treated as constants, it follows that $t_b - t \geq \frac{ln|C|}{4\gamma}$ (where C is v_y^2 for the bit-flip channel and v_x^2 for the bit-phase-flip channel), meaning $t \leq t_b - \frac{ln|C|}{4\gamma}$. Once t passes this threshold, C can no longer stay constant, as that would lead to $v_z(t)$ becoming a complex value.

Furthermore, since v_x^2 and v_y^2 are treated as constants, $\dot{v_x}$ and $\dot{v_y}$ are both equal to 0, meaning the equations for the bit-flip channel can be rewritten as:

$$\dot{v}_y = 2h_z v_x - 2\gamma v_y - 2h_x v_z \tag{21a}$$
$$h_x(t) = \frac{h_z v_x(0) - \gamma v_y(0)}{v_z(t)} \tag{21b}$$
$$h_x(t) = \frac{h_z v_x(0) - \gamma v_y(0)}{\sqrt{e^{4\gamma(t_b - t)} - C}} \tag{21c}$$

$$\dot{v}_y = 2h_z v_x - 2\gamma v_y - 2h_x v_z \tag{22a}$$
$$h_z(t) = \frac{h_x v_z(t) + \gamma v_y(0)}{v_x(0)} \tag{22b}$$
$$h_z(t) = \frac{h_x \sqrt{e^{4\gamma(t_b - t)} - C} + \gamma v_y(0)}{v_x(0)} \tag{22c}$$

$$\dot{v}_y = 2h_y v_z - 2h_z v_y \tag{23a}$$
$$h_y(t) = \frac{h_z v_y(0)}{v_z(t)} \tag{23b}$$
$$= \frac{h_x \sqrt{e^{4\gamma(t_b - t)} - C} + \gamma v_y(0)}{v_x(0)\sqrt{e^{4\gamma(t_b - t)} - C}} v_y(0) \tag{23c}$$
$$= \frac{h_x v_y(0)}{v_x(0)} + \frac{\gamma v_y^2(0)}{v_x(0)\sqrt{e^{4\gamma(t_b - t)} - C}} \tag{23d}$$

and the equations for the bit-phase-flip channel can be rewritten as:

$$\dot{v}_x = -2\gamma v_x - 2h_z v_y + 2h_y v_z \tag{24a}$$
$$h_y(t) = \frac{h_z v_y(0) + \gamma v_x(0)}{v_z(t)} \tag{24b}$$
$$h_y(t) = \frac{h_z v_y(0) + \gamma v_x(0)}{\sqrt{e^{4\gamma(t_b - t)} - C}} \tag{24c}$$

$$\dot{v}_x = -2\gamma v_x - 2h_z v_y + 2h_y v_z \tag{25a}$$
$$h_z(t) = \frac{h_y v_z(t) - \gamma v_x(0)}{v_y(0)} \tag{25b}$$
$$h_z(t) = \frac{h_y \sqrt{e^{4\gamma(t_b - t)} - C} - \gamma v_x(0)}{v_y(0)} \tag{25c}$$

$$\dot{v}_y = 2h_z v_x - 2h_x v_z \tag{26a}$$

$$h_x(t) = \frac{h_z v_x(0)}{v_z(t)} \tag{26b}$$

$$= \frac{h_y\sqrt{e^{4\gamma(t_b-t)} - C} - \gamma v_x(0)}{v_y(0)\sqrt{e^{4\gamma(t_b-t)} - C}} v_x(0) \tag{26c}$$

$$= \frac{h_y v_x(0)}{v_y(0)} - \frac{\gamma v_x^2(0)}{v_y(0)\sqrt{e^{4\gamma(t_b-t)} - C}} \tag{26d}$$

In the case of the bit-flip channel, as $v_z(t)$ is in the denominator for $h_x(t)$, it stands to reason that as shown above, $t_b - t = \frac{ln|v_y^2|}{4\gamma}$ is the point where the control field $h_x(t)$ diverges, as the denominator being equal to 0 would lead to an infinite energy requirement to maintain constant coherence. A similar problem persists for $h_y(t)$; because the second term in this control field contains $v_z(t)$ in the denominator, it would diverge at $t_b - t = \frac{ln|v_y^2|}{4\gamma}$. It is easy to see how this applies directly to the negative branch of the equation, but since the positive branch is a function of two constants and the $h_x(t)$ control field, the same constraint still applies.

In the case of the bit-phase-flip channel, as $v_z(t)$ is in the denominator for $h_y(t)$, it stands to reason that as shown above, $t_b - t = \frac{ln|v_x^2|}{4\gamma}$ is the point where the control field $h_y(t)$ diverges, as the denominator being equal to 0 would lead to an infinite energy requirement to maintain constant coherence. A similar problem persists for $h_x(t)$; because the second term in this control field contains $v_z(t)$ in the denominator, it would diverge at $t_b - t = \frac{ln|v_x^2|}{4\gamma}$. It is easy to see how this applies directly to the negative branch of the equation, but since the positive branch is a function of two constants and the $h_y(t)$ control field, the same constraint still applies.

However, for both channels, since the $v_z(t)$ term is in the numerator for the $h_z(t)$ control field, the same constraint does not necessarily apply; since the denominator is treated as a constant, this control field does not run the same risk on its second branch. It is worth noting that one branch of $h_z(t)$ still is correlated to the one of the other control terms for each of the channels, meaning that the only truly unconstrained regime is the second branch of $h_z(t)$ in both cases.

From this, we can also make observations on the purity of the systems. Purity is defined as $P = \frac{1}{2}(1 + \|\vec{v}\|^2)$, which can be converted to $\frac{1}{2}(1 + v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}(1 + C + v_z^2) \Rightarrow v_z(t) = \sqrt{2P - 1 - C}$. In these uncontrolled system, as they are unital, the purity consistently decreases over time, as certain vectors for each channel tend towards 0. However, under the controlled system, the coherence remains constant while the condition is met, meaning that the only variable that impacts the decrease of the purity is $v_z(t)$. This remains the case until the divergence point is met, after which the coherence, and consequently the purity, continue to decrease, tending to the minimum purity for each channel, which is dependent on the initial states.

In an uncontrolled environment, the bit-flip channel maintains the x-axis on the Bloch sphere, whereas the minor axis on the y-z plane is constantly contracting. Similarly, the bit-phase-flip channel maintains one axis on the x-z plane on the Bloch sphere and the other on the y-axis.

Additionally, since these channels are unital, the center of the constantly evolving vector space rests at the origin of the Bloch sphere. However, within the setting defined above, the control fields act to try preserving $v_x(t)$ and $v_y(t)$, as the coherence is treated as a constant. To achieve this end, the control field treats the constant vector on the x-y plane as the major axis and $v_z(t)$ as the vector on the minor axis, thereby rotating the Bloch sphere to trade the contraction on the z-axis for a constant coherence. However, as this is a time-dependent process, there comes a divergence point where the energy required to maintain this constant rotation becomes too great, and as a result, the system collapses back to its uncontrolled state. As stated above, the divergence point is a function of the value of C, meaning the constant coherence value that the control fields attempt to preserve strongly dictates the amount of time the controls can operate before the system returns to an uncontrolled state.

III. PROBLEM 8.5.7

A. G Matrix Derivation

The depolarizing channel, which maintains a quantum state with probability $1-p$ and collapses to the maximally mixed state with probability p, is defined by the following transformation:

$$\mathcal{N} : \rho \to (1-p)\rho + p\pi \tag{27}$$

where π indicates the maximally mixed state, or $I/2$. With some simple algebra, this channel can be expanded to the following:

$$\mathcal{N} : \rho \to (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z) \tag{28}$$

Additionally, as determined via Exercise 7.4.9 in [1], the dissipation operator of the Lindbladian equation for the depolarizing channel is $\mathcal{L}_{\mathcal{D}} = \frac{2}{3}\gamma(I - 2\rho)$. Using this form of the dissipator term, the R matrix can be generated quite easily, as $R_{ij} = \langle F_i, \mathcal{L}_{\mathcal{D}}(F_j)\rangle = Tr[F_i \mathcal{L}_{\mathcal{D}}(F_j)]$. Furthermore, since the most general case of the quantum system is considered, no constraints on the Q matrix are provided, thereby operating on the same matrix used to solve the previous problem. Therefore, the following G matrix is created as the sum of the P and Q matrices operating on this system:

$$G = 2 \times \begin{pmatrix} -\frac{4}{3}\gamma & -h_z & h_y \\ h_z & -\frac{4}{3}\gamma & -h_x \\ -h_y & h_x & -\frac{4}{3}\gamma \end{pmatrix} \tag{29}$$

B. Breakdown Time Derivation

From this matrix, the differential equations that follow are:

$$\dot{v}_x = -\frac{8}{3}\gamma v_x - 2h_z v_y + 2h_y v_z\,, \tag{30a}$$

$$\dot{v}_y = 2h_z v_x - \frac{8}{3}\gamma v_y - 2h_x v_z\,, \tag{30b}$$

$$\dot{v}_z = -2h_y v_x + 2h_x v_y - \frac{8}{3}\gamma v_z\,. \tag{30c}$$

To solve for $\dot{v}_z$, we must first redefine $2h_xv_y$ and $2h_yv_x$ (the following process can be executed by solving for any one of the variables; $\dot{v}_z$ was an arbitrary choice):

$$2h_yv_x = \frac{\dot{v}_x + \frac{8}{3}\gamma v_x - 2h_zv_y}{v_z} v_x \,, \tag{31a}$$

$$2h_xv_y = \frac{-\dot{v}_y + 2h_zv_x - \frac{8}{3}\gamma v_y}{v_z} v_y \tag{31b}$$

Plugging these back into $\dot{v}_z$, we get the following equation:

$$\dot{v}_xv_x + \dot{v}_yv_y + \dot{v}_zv_z = -\frac{8}{3}\gamma(v_x^2 + v_y^2 + v_z^2)\,, \tag{32a}$$

$$\dot{v}_zv_z = -\frac{8}{3}\gamma(C + v_z^2)\,, \tag{32b}$$

$$\int_0^t \frac{v_z\frac{\partial v_z}{\partial t}\partial t}{C + v_z^2} = \int_0^t -\frac{8}{3}\gamma\partial t \tag{32c}$$

$$\frac{1}{2}(ln|v_z^2(t) + C| - ln|v_z^2(0) + C|) = -\frac{8}{3}\gamma t \tag{32d}$$

$$v_z^2(t) + C = (v_z^2(0) + C)e^{-\frac{16}{3}\gamma t} \tag{32e}$$

To solve for the breakdown time t_b, $v_z(0)$ must first be defined. For the simplest solutions for t_b and $v_z(t)$, it may be defined the following way:

$$v_z(0) = \sqrt{(e^{\frac{16}{3}\gamma t_b} - C)} \tag{33a}$$

$$v_z^2(0) = e^{\frac{16}{3}\gamma t_b} - C \tag{33b}$$

$$v_z^2(t) + C = (e^{\frac{16}{3}\gamma t_b} - C + C)e^{-\frac{16}{3}\gamma t} \tag{33c}$$

$$v_z(t) = \sqrt{e^{\frac{16}{3}\gamma(t_b - t)} - C} \tag{33d}$$

$$\sqrt{(e^{\frac{16}{3}\gamma t_b} - C)} = v_z(0) \tag{34a}$$

$$e^{\frac{16}{3}\gamma t_b} = v_z^2(0) + C \tag{34b}$$

$$t_b = \frac{3ln|v_z^2(0) + C|}{16\gamma} \tag{34c}$$

C. Analysis

As evidenced by Equation (23e), $v_z(t)$ is a function of the difference between t_b and t; however, since it must remain real, the term under the radical must remain non-negative. For this to be

true, since v_x^2 was treated as a constant, it follows that $t_b - t \geq \frac{ln|v_x^2|}{4\gamma}$, meaning $t \leq t_b - \frac{ln|v_x^2|}{4\gamma}$. Once t passes this threshold, v_x^2 can no longer stay constant, as that would lead to $v_z(t)$ becoming a complex value.
Furthermore, since v_x^2 and v_y^2 are treated as constants, $\dot{v_x}$ and $\dot{v_y}$ are both equal to 0, meaning those equations can be rewritten as:

$$\dot{v}_x = -\frac{8}{3}\gamma v_x - 2h_z v_y + 2h_y v_z \tag{35a}$$

$$h_y(t) = \frac{h_z v_y(0) + \frac{4}{3}\gamma v_x(0)}{v_z(t)} \tag{35b}$$

$$h_y(t) = \frac{h_z v_y(0) + \frac{4}{3}\gamma v_x(0)}{\sqrt{e^{\frac{16}{3}\gamma(t_b-t)} - C}} \tag{35c}$$

$$\dot{v}_x = -\frac{8}{3}\gamma v_x - 2h_z v_y + 2h_y v_z \tag{36a}$$

$$h_z(t) = \frac{h_y v_z(t) - \frac{4}{3}\gamma v_x(0)}{v_y(0)} \tag{36b}$$

$$h_z(t) = \frac{h_y\sqrt{e^{\frac{16}{3}\gamma(t_b-t)} - C} - \frac{4}{3}\gamma v_x(0)}{v_y(0)} \tag{36c}$$

$$\dot{v}_y = 2h_z v_x - \frac{8}{3}\gamma v_y - 2h_x v_z \tag{37a}$$

$$h_x(t) = \frac{h_z v_x(0) + \frac{4}{3}\gamma v_y(0)}{v_z(t)} \tag{37b}$$

$$h_x(t) = \frac{h_y\sqrt{e^{\frac{16}{3}\gamma(t_b-t)} - C}v_x(0) - \frac{4}{3}\gamma v_x^2(0) + \frac{4}{3}\gamma v_y^2(0)}{v_y(0)\sqrt{e^{\frac{16}{3}\gamma(t_b-t)} - C}} \tag{37c}$$

$$h_x(t) = \frac{h_y v_x(0)}{v_y(0)} - \frac{4\gamma v_x^2(0) + 4\gamma v_y^2(0)}{3v_y(0)\sqrt{e^{\frac{16}{3}\gamma(t_b-t)} - C}} \tag{37d}$$

As $v_z(t)$ is in the denominator for $h_y(t)$, it stands to reason that as shown above, $t_b - t = \frac{3ln|v_x^2+v_y^2|}{16\gamma}$ is the point where the control field $h_y(t)$ diverges, as the denominator being equal to 0 would lead to an infinite energy requirement to maintain constant coherence. A similar problem persists for $h_x(t)$; because the second term in this control field contains $v_z(t)$ in the denominator, it would diverge at $t_b - t = \frac{3ln|v_x^2+v_y^2|}{16\gamma}$. It is easy to see how this applies directly to the negative branch of the equation, but since the positive branch is a function of two constants and the $h_y(t)$ control field, the same constraint still applies.

However, since the $v_z(t)$ term is in the numerator for the $h_z(t)$ control field, the same constraint does not necessarily apply; since $v_y(0)$ is treated as a constant denominator, this control field does not run the same risk on its negative branch, which is simply a product of arbitrary constants. It is worth noting that the positive branch of $h_z(t)$ still is correlated to the $h_y(t)$ term, meaning that the only truly unconstrained regime is the negative branch of $h_z(t)$.

From this, we can also make observations on the purity of the system. Purity is defined as $P = \frac{1}{2}(1 + \|\vec{v}\|^2)$, which can be converted to $\frac{1}{2}(1 + v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}(1 + C + v_z^2) \Rightarrow v_z(t) = \sqrt{2P - 1 - C}$. In this uncontrolled system, as it is unital, the purity consistently decreases over time, as the vectors all tend to the origin, which inherently has a purity of 0. However, under the controlled system, the coherence remains constant while the condition is met, meaning that the only variable that impacts the decrease of the purity is $v_z(t)$. This remains the case until the divergence point is met, after which the coherence, and consequently the purity, continue to tend once again to 0.

In an uncontrolled environment, the depolarizing channel causes equal contraction along all three axes, slowly bringing the vector space of a qubit to the maximally mixed state. Additionally, since this channel is unital, the center of the constantly evolving vector space rests at the origin of the Bloch sphere. However, the definitions above create control fields that act to try preserving the length of the vector on the x-y plane, as the coherence is treated as a constant. To achieve this end, the control field treats the constant vector on the x-y plane as the major axis and $v_z(t)$ as the vector on the minor axis, thereby trading the contraction of $v_z(t)$ to slow down, or ideally halt, the decrease of the coherence. However, as this is a time-dependent process, there comes a divergence point where the energy required to maintain this constant transformation becomes too great, and as a result, the system collapses back to its uncontrolled state. As stated above, the divergence point is a function of the value of $v_x^2 + v_y^2$, meaning the constant coherence value that the control fields attempt to preserve strongly dictates the amount of time the controls can operate before the system returns to an uncontrolled state where all three axes continue to converge to the maximally mixed state.

IV. PROBLEM 8.5.8

A. G Matrix Derivation

The amplitude damping channel is defined by the following transformation:

$$|0\rangle \rightarrow |0\rangle \text{ with probability 1} \tag{38a}$$
$$|1\rangle \rightarrow |0\rangle \text{ with probability p} \tag{38b}$$

The Lindbladian equation of the amplitude damping channel can be described by:

$$\dot{\rho}(t) = \gamma(|0\rangle \langle 1| \rho |1\rangle \langle 0| - \frac{1}{2}\{|1\rangle \langle 1| , \rho\}) \tag{39a}$$

When we plug in the Pauli matrices for ρ, we get the following:

$$\mathcal{L}(X) = \gamma(|0\rangle\langle 1| X |1\rangle\langle 0| - \frac{1}{2}\{|1\rangle\langle 1|, X\}) = -\frac{\gamma}{2}X \tag{40a}$$
$$\mathcal{L}(Y) = \gamma(|0\rangle\langle 1| Y |1\rangle\langle 0| - \frac{1}{2}\{|1\rangle\langle 1|, Y\}) = -\frac{\gamma}{2}Y \tag{40b}$$
$$\mathcal{L}(Z) = \gamma(|0\rangle\langle 1| Z |1\rangle\langle 0| - \frac{1}{2}\{|1\rangle\langle 1|, Z\}) = -\gamma Z \tag{40c}$$

In order to find the G matrix for the amplitude damping matrix, we must first find the corresponding R matrix:

$$R_{ij} = Tr[F_i\mathcal{L}(F_j)] \tag{41}$$

Once we solve for each pair of values, we get the following matrix:

$$R = 2 \times \begin{pmatrix} -\gamma/2 & 0 & 0 \\ 0 & -\gamma/2 & 0 \\ 0 & 0 & -\gamma \end{pmatrix} \tag{42}$$

Finally, we add R and Q together ($G = R + Q$):

$$G = 2 \times \begin{pmatrix} -\gamma/2 & -h_z & h_y \\ h_z & -\gamma/2 & -h_x \\ -h_y & h_x & -\gamma \end{pmatrix} \tag{43}$$

B. Breakdown Time Derivation

To solve for the differential equations, we must first find $\vec{c}$ to solve for $\dot{\vec{v}} = G\vec{v} + \vec{c}$:

$$\mathcal{L}(I) = \gamma(|0\rangle\langle 1| I |1\rangle\langle 0| - \frac{1}{2}\{|1\rangle\langle 1|, I\}) \tag{44a}$$
$$\mathcal{L}(I) = \gamma Z \tag{44b}$$
$$c_j = \frac{1}{\sqrt{2}}Tr[\frac{1}{\sqrt{2}}\sigma_j\mathcal{L}(I)] = \frac{\gamma}{2}Tr[\sigma_j Z] \tag{44c}$$
$$\vec{c} = (0, 0, \gamma) \tag{44d}$$

The differential equations that follow are:

$$\dot{v}_x = -\gamma v_x - 2h_z v_y + 2h_y v_z \tag{45a}$$
$$\dot{v}_y = 2h_z v_x - \gamma v_y - 2h_x v_z \tag{45b}$$
$$\dot{v}_z = -2h_y v_x + 2h_x v_y - 2\gamma v_z + \gamma \tag{45c}$$

Now we need to redefine $-2h_y v_x$ and $2h_x v_y$ to solve for $\dot{v}_z$:

$$-2h_y v_x = \frac{-\dot{v}_x - \gamma v_x - 2h_z v_y}{v_z} v_x \tag{46a}$$
$$2h_x v_y = \frac{-\dot{v}_y + 2h_z v_x - \gamma v_y}{v_z} v_y \tag{46b}$$

$$\dot{v}_z = \frac{-\dot{v}_x - \gamma v_x - 2h_z v_y}{v_z} v_x + \frac{-\dot{v}_y + 2h_z v_x - \gamma v_y}{v_z} v_y - 2\gamma v_z + \gamma \tag{47a}$$

Assuming $\dot{v}_x v_x + \dot{v}_y v_y = 0$ and $v_x^2 + v_y^2$ is a constant, C, we get the following:

$$\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z = -\gamma v_x^2 - \gamma v_y^2 - 2\gamma v_z^2 + \gamma v_z \tag{48a}$$

$$\dot{v}_z v_z = -\gamma(C + 2v_z^2 - v_z) \tag{48b}$$

$$\dot{v}_z v_z = -2\gamma(\frac{C}{2} + v_z^2 - \frac{v_z}{2}) \tag{48c}$$

$$\int_0^t \frac{v_z \frac{\partial v_z}{\partial t} \partial t}{\frac{C}{2} + v_z^2 - \frac{v_z}{2}} = \int_0^t -2\gamma \partial t \tag{48d}$$

C. Analysis

The integral on the left side of (61e) is unsolvable, and as such, we are unable to find t_b or the control fields. If we were to treat γ, which is a function of the bath, as a time-based function, then we could define γ as $\frac{-0.125}{\frac{C}{2}+v_z^2-\frac{v_z}{2}}$. In this case, we would end up with:

$$\int_0^t \frac{v_z \frac{\partial v_z}{\partial t} \partial t}{\frac{C}{2} + v_z^2 - \frac{v_z}{2}} = \int_0^t \frac{\frac{1}{4}}{\frac{C}{2} + v_z^2 - \frac{v_z}{2}} \partial t \tag{49a}$$

$$\int_0^t \frac{v_z \frac{\partial v_z}{\partial t} \partial t}{\frac{C}{2} + v_z^2 - \frac{v_z}{2}} - \frac{\frac{1}{4}}{\frac{C}{2} + v_z^2 - \frac{v_z}{2}} \partial t = 0 \tag{49b}$$

$$\int_0^t \frac{v_z - \frac{1}{4}\frac{\partial v_z}{\partial t}}{\frac{C}{2} + v_z^2 - \frac{v_z}{2}} \partial t = 0 \tag{49c}$$

We may still write out the control fields without the solution for $v_z(t)$:

$$\dot{v}_x = -\gamma v_x - 2h_z v_y + 2h_y v_z \tag{50a}$$

$$h_y(t) = \frac{-h_z v_y(0) - \frac{1}{2}\gamma v_x(0)}{v_z(t)} \tag{50b}$$

$$\dot{v}_x = -\gamma v_x - 2h_z v_y + 2h_y v_z \tag{51a}$$

$$h_z(t) = \frac{h_y v_z(t) - \frac{1}{2}\gamma v_x(0)}{v_y(0)} \tag{51b}$$

$$\dot{v}_y = 2h_z v_x - \gamma v_y - 2h_x v_z \tag{52a}$$

$$h_x(t) = \frac{h_z v_x(0) - \frac{1}{2}\gamma v_y(0)}{v_z(t)} \tag{52b}$$

As for the geometric interpretation, let us begin by describing the amplitude damping channel before any manipulation. In an uncontrolled environment, the amplitude damping channel causes contraction along all three axes. Additionally, the center of the evolving vector space constantly tends in the negative direction of the z-axis, as it seeks to converge at $|0\rangle$. However, the definitions above create control fields that act to try preserving the length of the vector on the x-y plane, as the coherence is treated as a constant.

In a controlled environment, we see that $h_z(t)$ is the only control field without $v_z(t)$ in the denominator. We must make assumptions about $v_z(t)$ since we are unable to solve for the general case. If we assume that $v_z(t)$ has a divergence point with its breakdown time, then $h_x(t)$ and $h_y(t)$ will diverge at that point, as $v_z(t)$ deteriorates under the control field while $v_x(0)$ and $v_y(0)$ are preserved. Additionally, since the first term of $h_z(t)$ includes $h_y(t)$, only the second term is unconstrained if we assume we keep a constant γ. If our $v_z(t)$ results in our γ having a convergence point, then there is no unconstrained term in $h_z(t)$. As we are unable to solve for the breakdown time of $v_z(t)$, it might be possible that it does not have a breakdown time, though this is highly unlikely. If this is the case, the control field can theoretically be kept up indefinitely.

From this, we can also make observations on the purity of the system. Purity is defined as $P = \frac{1}{2}(1 + \|\vec{v}\|^2)$, which can be converted to $\frac{1}{2}(1 + v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}(1 + C + v_z^2) \Rightarrow v_z(t) = \sqrt{2P - 1 - C}$. In this uncontrolled system, the purity first decreases but then begins to increase over time, as the amplitude damping channel tends towards a pure quantum state rather than the maximally mixed state. As $v_z(t)$ gets closer to 0, the purity decreases, but once it passes $v_z(t) = 0$, the purity begins to increase again. In the case of no breakdown time, the coherence magnitude will never decrease. Because of this, $v_z(t)$ is the only variable that affects the purity of the system. If there is a breakdown time, then v_x and v_y would also affect purity post breakdown, but up until that point, $v_z(t)$ would be the only variable affecting the rate of change of purity. However, even as $v_x(t)$ and $v_y(t)$ affect the purity, the growth of the magnitude of $v_z(t)$ will inevitably bring the purity back to 1; therefore the control field actually works to reduce the purity compared to the uncontrolled environment rather than increase it as time increases, which is a rather curious observation requiring more research and investigation.

V. RESULTS AND DISCUSSION

From this paper, we observe that the breakdown time for each channel is directly influenced by the level of coherence that the control field attempts to maintain constantly; once the energy required to sustain the rotation becomes immensely high, the system will diverge and start becoming more decoherent. As coherence correlates to a vector on the x-y plane, the controls aim to stall the transformation on those axes, which in turn causes decay for $v_z(t)$ until the breakdown time. The results clearly show that a higher degree of coherence needing to be maintained in any channel leads to a lesser amount of time before the system requires too much energy to maintain such a transformation; however, there is still a lot of work to be done in this field. While the goal is to maintain a constant value of $v_x^2 + v_y^2$, the assumptions that are made to simplify the calculations maintain both v_x^2 and v_y^2 as separate constants, solving only a specific case of the problem at hand. As evidenced by the bit and bit-phase examples, solving for a specific t_b value gets immensely tricky in the general case, a problem also observed in [3].

Other interesting open questions mentioned by [3] include the expression of control fields in n-level systems instead of simply the two-level systems currently explored (both in terms of Hilbert space expansion and multi-qubit entanglement preservation, for which the Bloch sphere representation would no longer suffice) and the problem of discovering partial solutions given incomplete information about a channel or transformation.

Finally, another interesting point of further research would be to first explore a general solution for breakdown time for all unital channels, considering the similarities in the calculus for the bit, bit-phase, and depolarizing channels, and subsequently to explore the expansion of that solution to one that only considers the general case of constant coherence.

CONTRIBUTIONS

Neema Badihian: Created G matrices for depolarizing and amplitude damping channel and $\vec{c}$ vector for amplitude damping channel, worked on creating and solving differential equations for bit, bit-phase, and amplitude damping channels, analyzed control fields, geometric interpretation, and analytical unsolvability of breakdown time for amplitude damping channel, wrote introduction, created bibliography, assisted in all other contributions.

Saaketh Rayaprolu: Created G matrices for depolarizing and amplitude damping channel, worked on creating and solving differential equations for bit, bit-phase, depolarizing, and amplitude damping channels, analyzed breakdown times, control fields, and geometric interpretations for depolarizing, bit, amplitude damping, and bit-phase channels, wrote conclusion, assisted in all other contributions.

Effort: Equal parts for Neema Badihian and Saaketh Rayaprolu.

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