Matrix Multiplication & Transposition on Hypercube

Presented by Brett Duncan

Matrix Multiplication Applications

- Many problems can be solved using matrix multiplication or a variation of it. [1]
  - Finding the shortest distance between all pairs of vertices in a graph.
  - Transitive closure
  - Finding the radius, diameter, and centers of a graph.
  - Finding a breadth-first spanning tree
  - Topological sort

Parallel Matrix Multiplication on Hypercube

- Use $N = n^3 = 2^{3q}$ processors, where $n = 2^q$.
- Can visualize the processors as being arranged in an $n \times n \times n$ array, with processor P_r occupying position (i, j, k), where $r = in^2 + jn + k$ and $0 \leq i, j, k \leq n - 1$
  - The binary representation of r is
  - $r_{3q-1} r_{3q-2} \dots r_{2q} r_{2q-1} \dots r_q r_{q-1} \dots r_0$
  - The binary representations of i, j, and k are
  - $r_{3q-1} r_{3q-2} \dots r_{2q}, \quad r_{2q-1} r_{2q-2} \dots r_q, \quad r_{q-1} r_{q-2} \dots r_0$
  - Processors agreeing on one or two of the coordinates (i, j, k) form a hypercube.
  - Processors agreeing on one coordinate form a hypercube with n^2 processors.
  - Processors agreeing on two coordinates form a hypercube with n processors.

Parallel Matrix Multiplication on Hypercube

Step 1: The elements of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are distributed over the n^3 processors so that the processor in position (i, j, k) contains a_{ji} and b_{ik}. This is done as follows:

(1.1) Copies of data initially in $A(0, j, k)$ and $B(0, j, k)$, are sent to the processors in positions (i, j, k), where $1 \leq i \leq n-1$. As a result, $A(i, j, k) = a_{jk}$ and $B(i, j, k) = b_{jk}$, for $0 \leq i \leq n-1$.

(1.2) Copies of the data in $A(i, j, i)$ are sent to the processors in positions (i, j, k), where $0 \leq k \leq n-1$. As a result, $A(i, j, k) = a_{ji}$ for $0 \leq k \leq n-1$.

(1.3) Copies of the data in $B(i, i, k)$ are sent to the processors in positions (i, j, k), where $0 \leq j \leq n-1$. As a result, $B(i, j, k) = b_{ik}$ for $0 \leq j \leq n-1$.

Step 2: Each processor in position (i, j, k) computes the product

$$C(i, j, k) \leftarrow A(i, j, k) \times B(i, j, k).$$

Thus, $C(i, j, k) = a_{ji} \times b_{ik}$ for $0 \leq i, j, k \leq n-1$.

Step 3: The sum

$$C(0, j, k) \leftarrow \sum_{i=0}^{n-1} C(i, j, k)$$

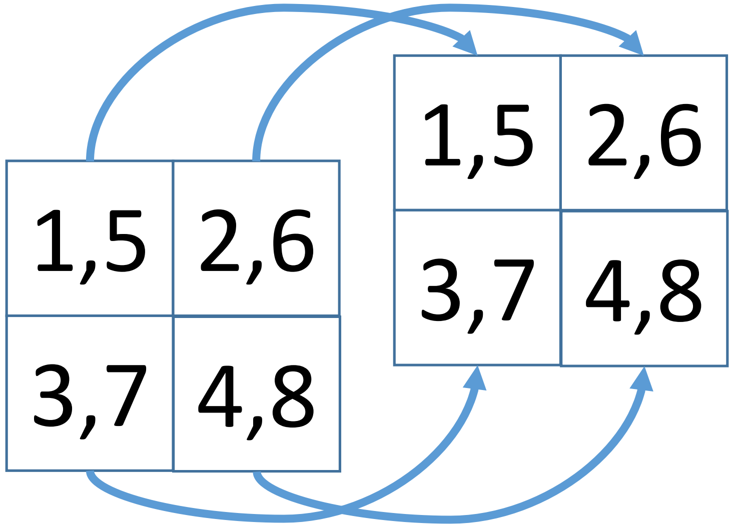
is computed for $0 \leq j, k \leq n-1$.

Parallel Matrix Multiplication on Hypercube (The idea)

Example: Multiplying 2×2 matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

- Step 1.1: Data in $A(0, j, k)$ and $B(0, j, k)$ are sent to processors in positions (i, j, k), where $1 \le i \le n - 1$.



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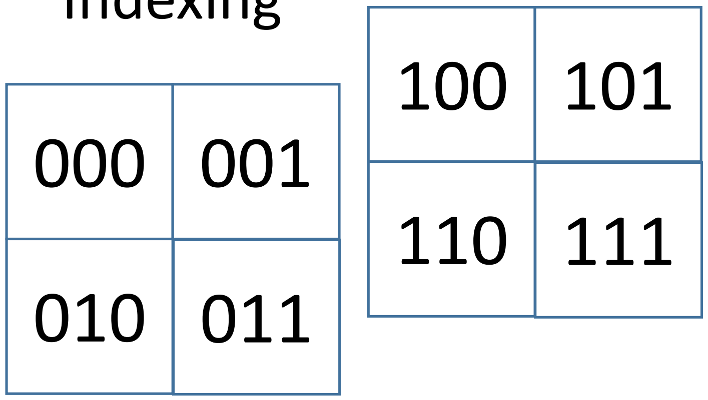
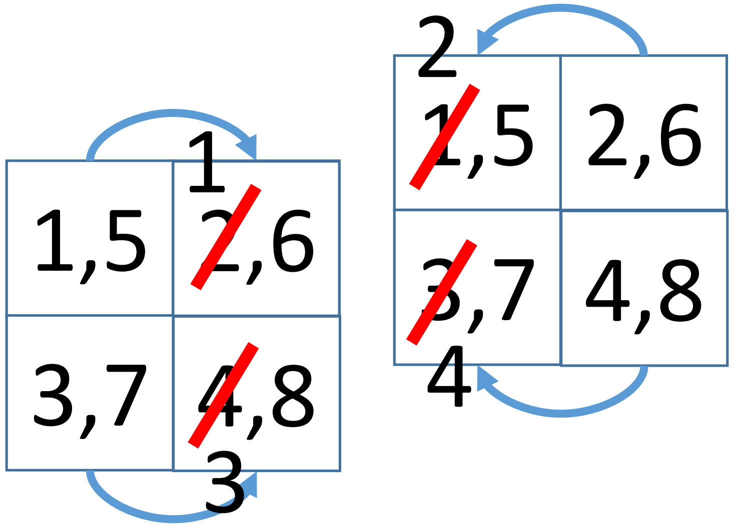


Parallel Matrix Multiplication on Hypercube (The idea)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

- Step 1.2: Copies of data in $A(i,j,i)$ are sent to the processors in positions (i,j,k), where $0 \leq k \leq n-1$.



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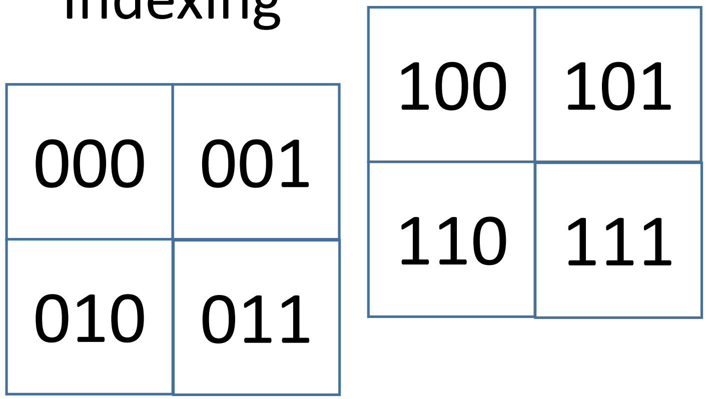
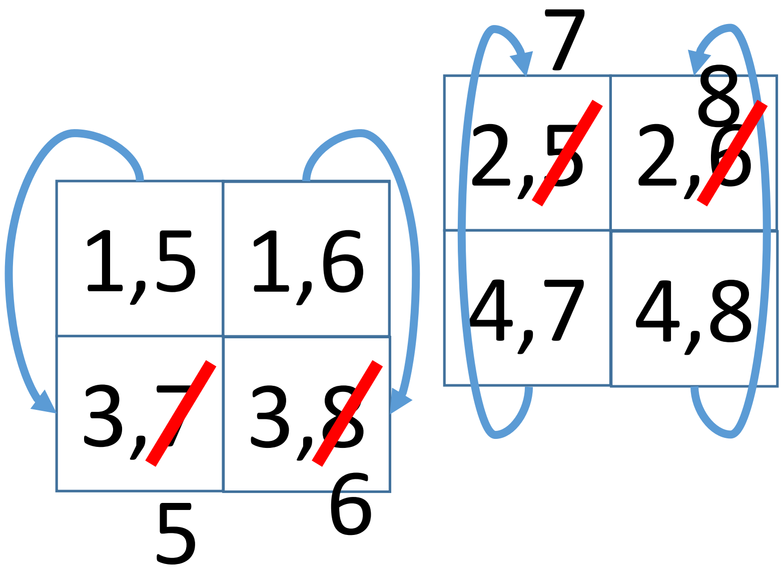


Parallel Matrix Multiplication on Hypercube (The idea)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

- Step 1.3: Copies of data in $B(i, i, k)$ are sent to the processors in positions (i, j, k), where $0 \leq j \leq n-1$.



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Parallel Matrix Multiplication on Hypercube (The idea)

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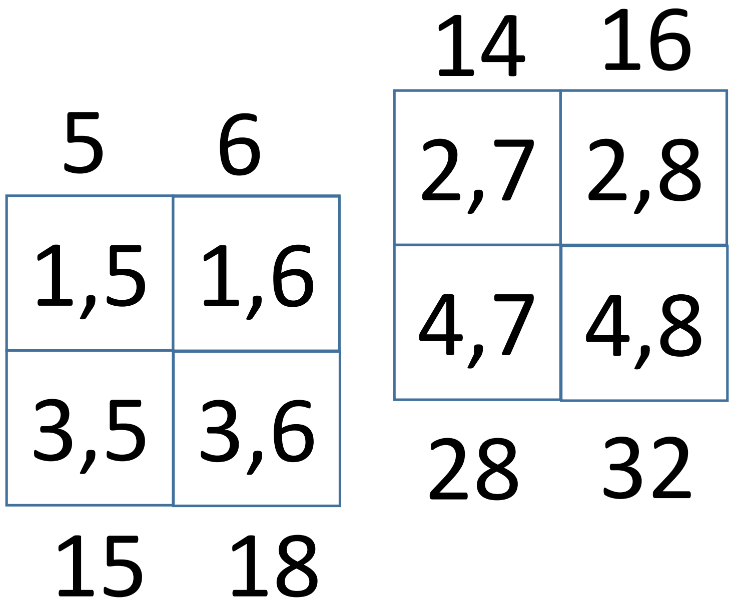
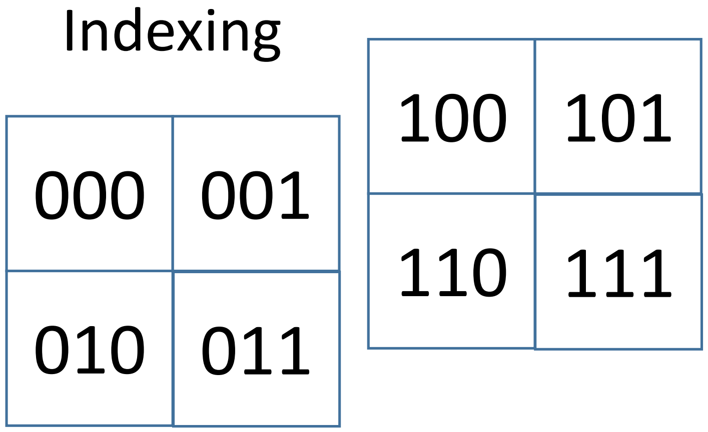
- Step 2: Each processor computes the product of their local A and B registers and stores it in their C register.





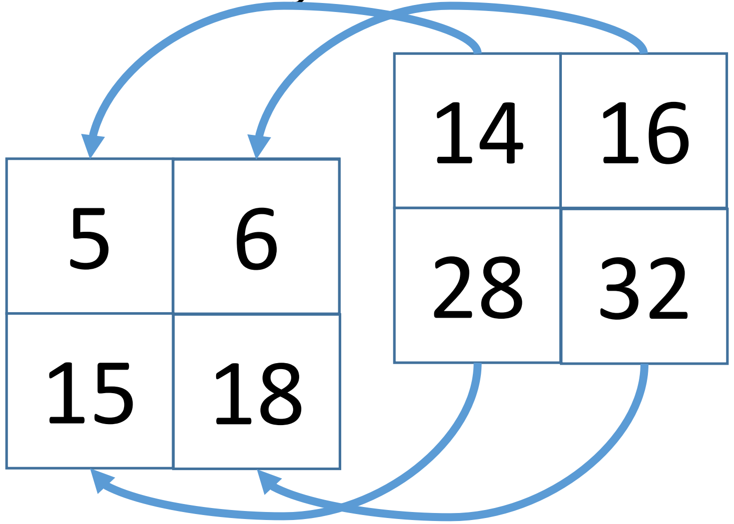
Parallel Matrix Multiplication on Hypercube (The idea)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

- Step 3: The sum

$$C(0,j,k) = \sum_{i=0}^{n-1} C(i,j,k)$$

is computed for $0 \le j,k \le n-1$.



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Parallel Matrix Multiplication on Hypercube (The idea)

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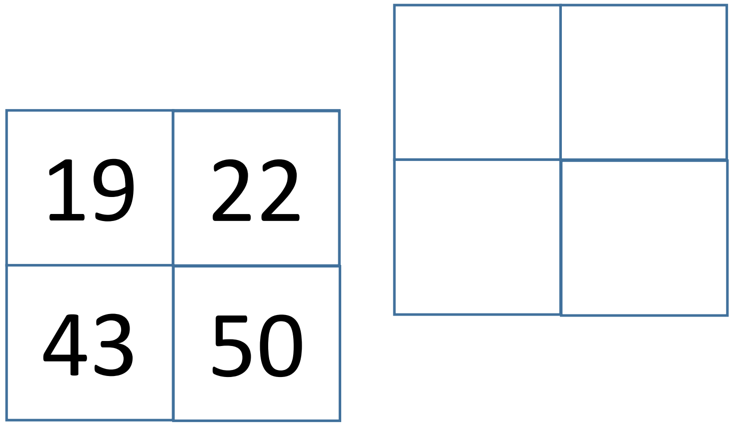

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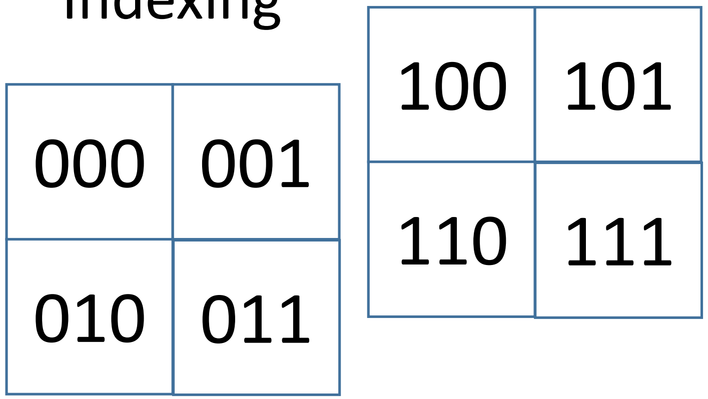


Parallel Matrix Multiplication on Hypercube

Algorithm HYPERCUBE MATRIX MULTIPLICATION (A, B, C)

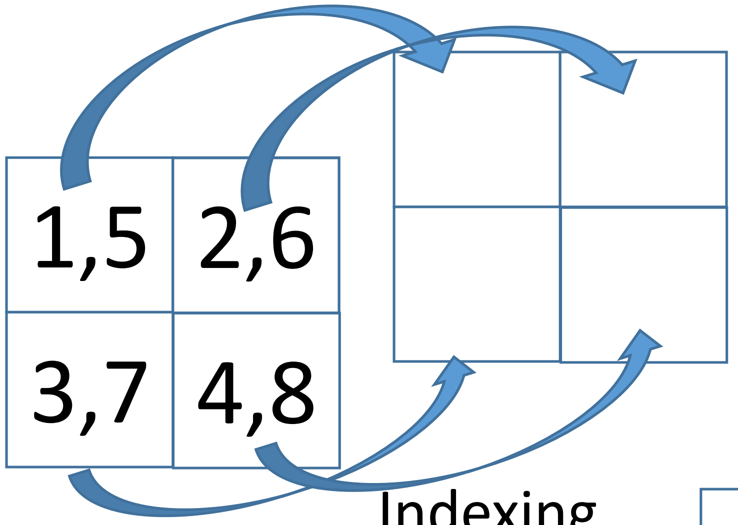
Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
    (i) $A_{r^{(m)}} \leftarrow A_r$
    (ii) $B_{r^{(m)}} \leftarrow B_r$
  end for
end for
(1.2) **for** $m = q - 1$ **downto** 0 **do**
  for all $r \in N(r_m = r_{2q+m})$ **do in parallel**
    $A_{r^{(m)}} \leftarrow A_r$
  end for
end for
(1.3) **for** $m = 2q - 1$ **downto** q **do**
  for all $r \in N(r_m = r_{q+m})$ **do in parallel**
    $B_{r^{(m)}} \leftarrow B_r$
  end for
end for

Step 2: **for** $r = 0$ **to** $N - 1$ **do in parallel**
  $C_r \leftarrow A_r \times B_r$
end for

Step 3: **for** $m = 2q$ **to** $3q - 1$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
    $C_r \leftarrow C_r + C_{r^{(m)}}$
  end for
end for. ■

Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$
$m = 2$



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  end for
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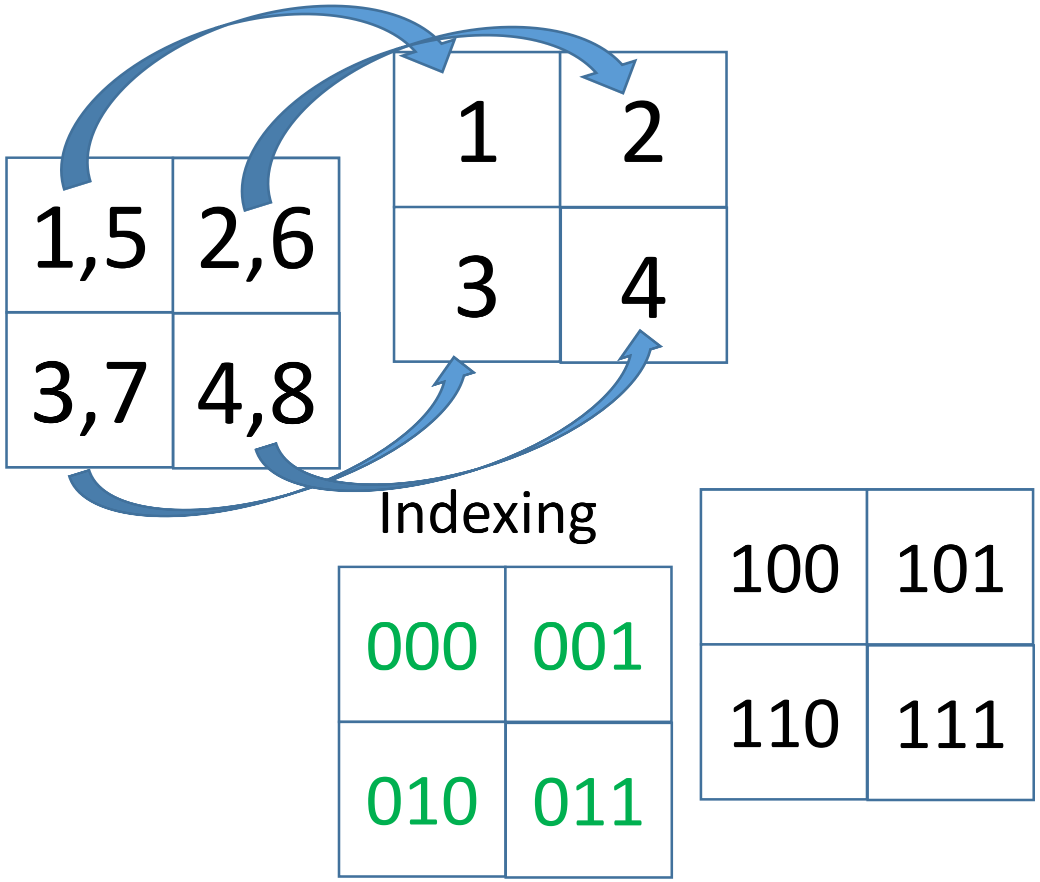

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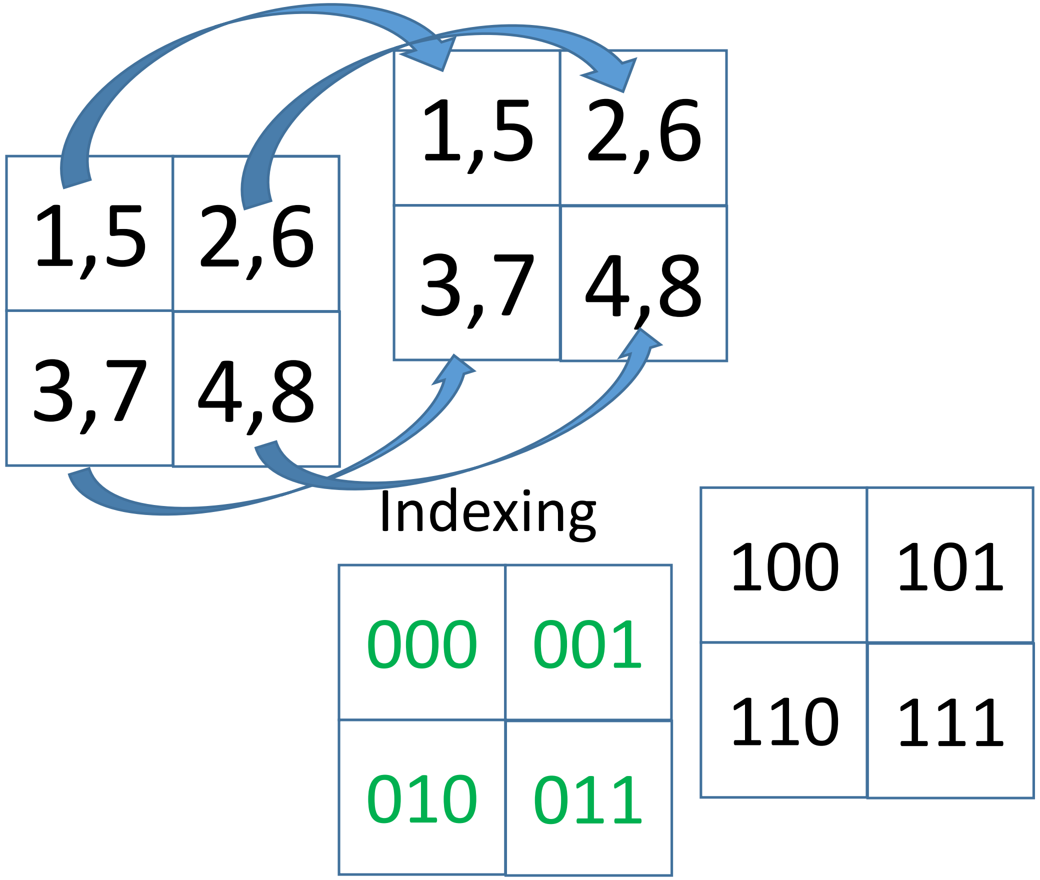

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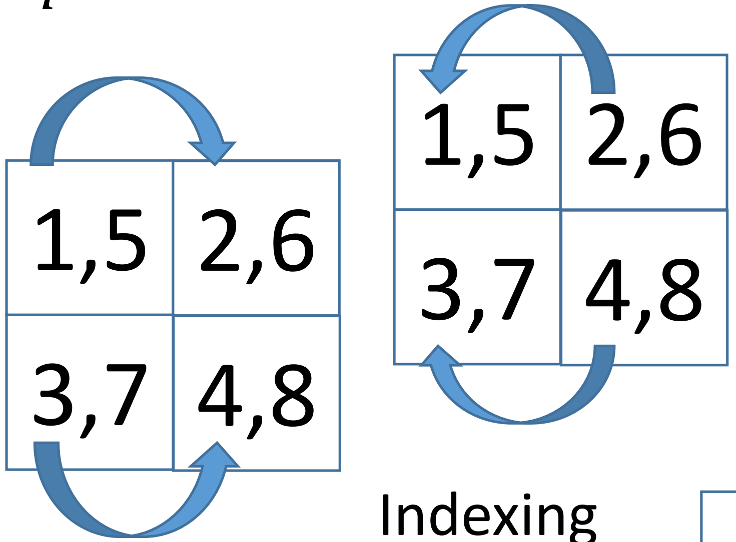
Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$
$m = 0$
$2q + m = 2$



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Use $N = n^3 = 2^{3q}$ processors.

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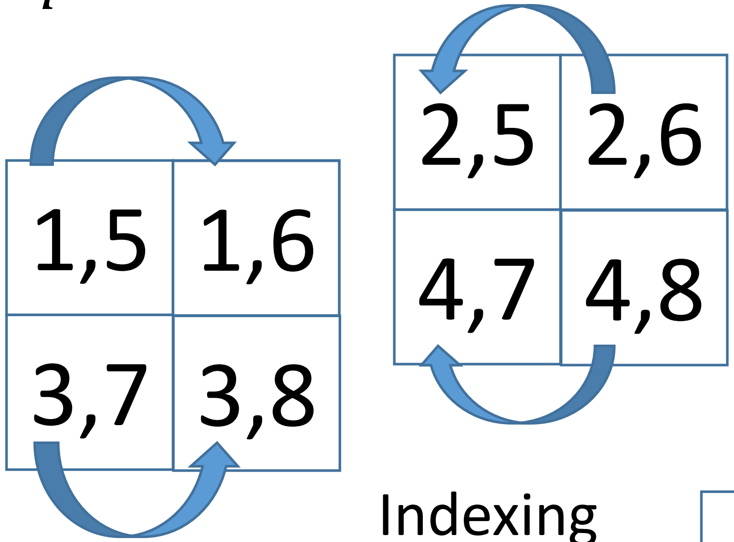
$m = 0$
$2q + m = 2$



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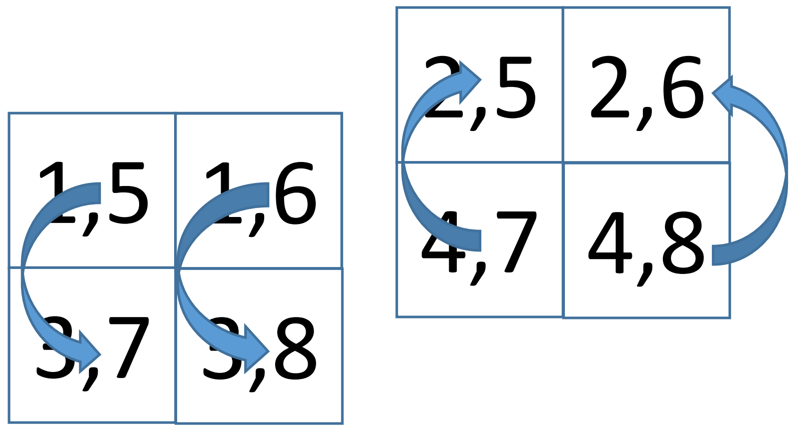
Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$

$m = 1$

$q + m = 2$



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Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$
$m = 1$
$q + m = 2$

1,5	1,6
3,5	3,6

2,7	2,8
4,7	4,8

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Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$
$m = 1$

1x5	1x6
3x5	3x6

2x7	2x8
4x7	4x8

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Parallel Matrix Multiplication on Hypercube

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end for. ■

Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$
$m = 1$

5	6
15	18

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Parallel Matrix Multiplication on Hypercube

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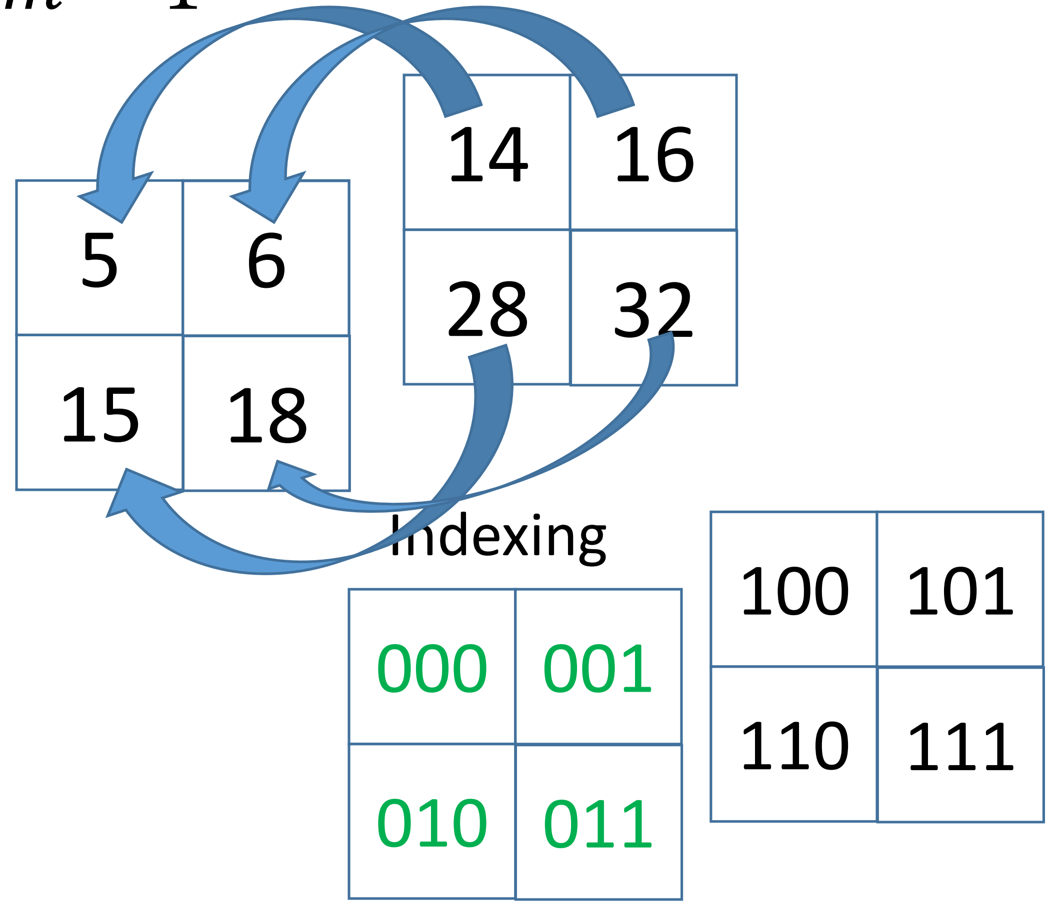
$n = 2, q = 1$
$m = 1$



Parallel Matrix Multiplication on Hypercube

Algorithm HYPERCUBE MATRIX MULTIPLICATION (A, B, C)

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
   (i) $A_{r^{(m)}} \leftarrow A_r$
   (ii) $B_{r^{(m)}} \leftarrow B_r$
  end for
 end for
(1.2) **for** $m = q - 1$ **downto** 0 **do**
  for all $r \in N(r_m = r_{2q+m})$ **do in parallel**
   $A_{r^{(m)}} \leftarrow A_r$
  end for
 end for
(1.3) **for** $m = 2q - 1$ **downto** q **do**
  for all $r \in N(r_m = r_{q+m})$ **do in parallel**
   $B_{r^{(m)}} \leftarrow B_r$
  end for
 end for

Step 2: **for** $r = 0$ **to** $N - 1$ **do in parallel**
 $C_r \leftarrow A_r \times B_r$
end for

Step 3: **for** $m = 2q$ **to** $3q - 1$ **do**
 for all $r \in N(r_m = 0)$ **do in parallel**
  $C_r \leftarrow C_r + C_{r^{(m)}}$
 end for
end for. ■

Use $N = n^3 = 2^{3q}$ processors.

$n = 2, q = 1$
$m = 1$

19	22
43	50

14	16
28	32

Indexing

000	001
010	011

100	101
110	111

Parallel Matrix Multiplication on Hypercube

Algorithm HYPERCUBE MATRIX MULTIPLICATION (A, B, C)

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
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  end for
 end for
 (1.2) **for** $m = q - 1$ **downto** 0 **do**
  for all $r \in N(r_m = r_{2q+m})$ **do in parallel**
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 (1.3) **for** $m = 2q - 1$ **downto** q **do**
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  for all $r \in N(r_m = 0)$ **do in parallel**
    $C_r \leftarrow C_r + C_{r^{(m)}}$
  end for
 end for. ■

<u>Analysis</u>:

Steps 1.1, 1.2, 1.3, and step 3 require q iterations.

Step 2 is in constant time.

Time complexity is $O(q) = O(\log n)$. (how?)

$P_n = n^3$, so cost is $n^3 \cdot \log n$. This is not cost optimal because the straightforward RAM algorithm takes $O(n^3)$ multiplications.

Parallel Matrix Multiplication on Hypercube

- The next slides will illustrate the $O(\log n)$ nature of data distribution for 2×2, 4×4, and 8×8 matrices.
- Only step 1.1 will be visualized, but a similar concept applies for steps 1.2, 1.3, and step 3.
- Observe that the distance between the senders and receivers is halved each iteration.
- Processors holding useful data are shaded in blue. Some processor send data even though they don’t hold useful data yet, but this is not a problem.

Parallel Matrix Multiplication on Hypercube (log n distribution example)

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
    (i) $A_{r^{(m)}} \leftarrow A_r$
    (ii) $B_{r^{(m)}} \leftarrow B_r$
  end for
end for

For all where the mth bit is 0

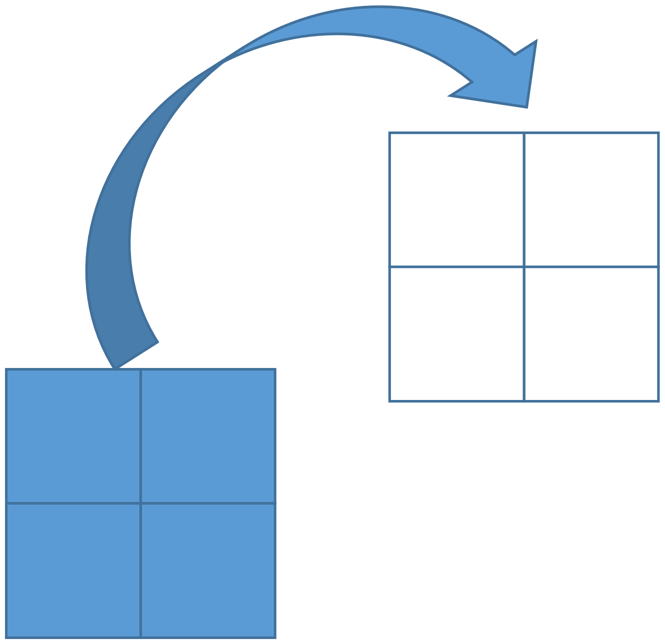
Send to neighboring processor by flipping bit m

2×2 distribution

q = 1

m starts at $3q - 1 = 2$

m ends at $2q = 2$



Binary coordinates:

100 101

110 111

Binary coordinates:

000 001

010 011

Parallel Matrix Multiplication on Hypercube (log n distribution example)

For all where the mth bit is 0

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
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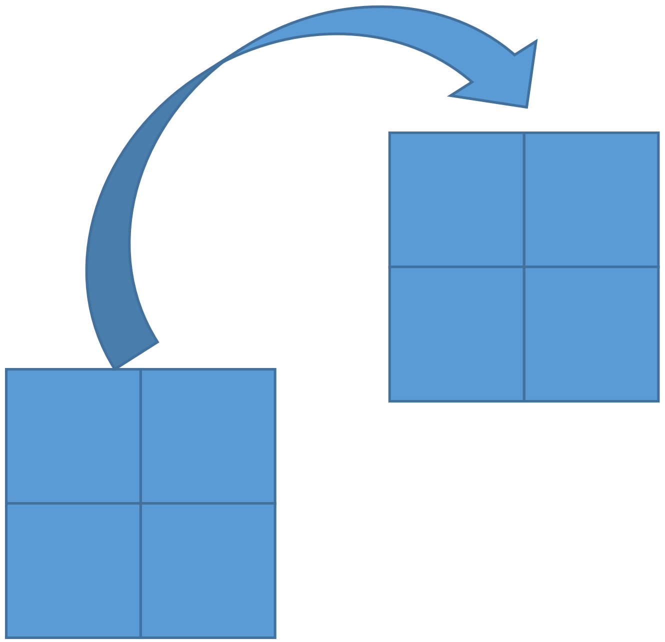


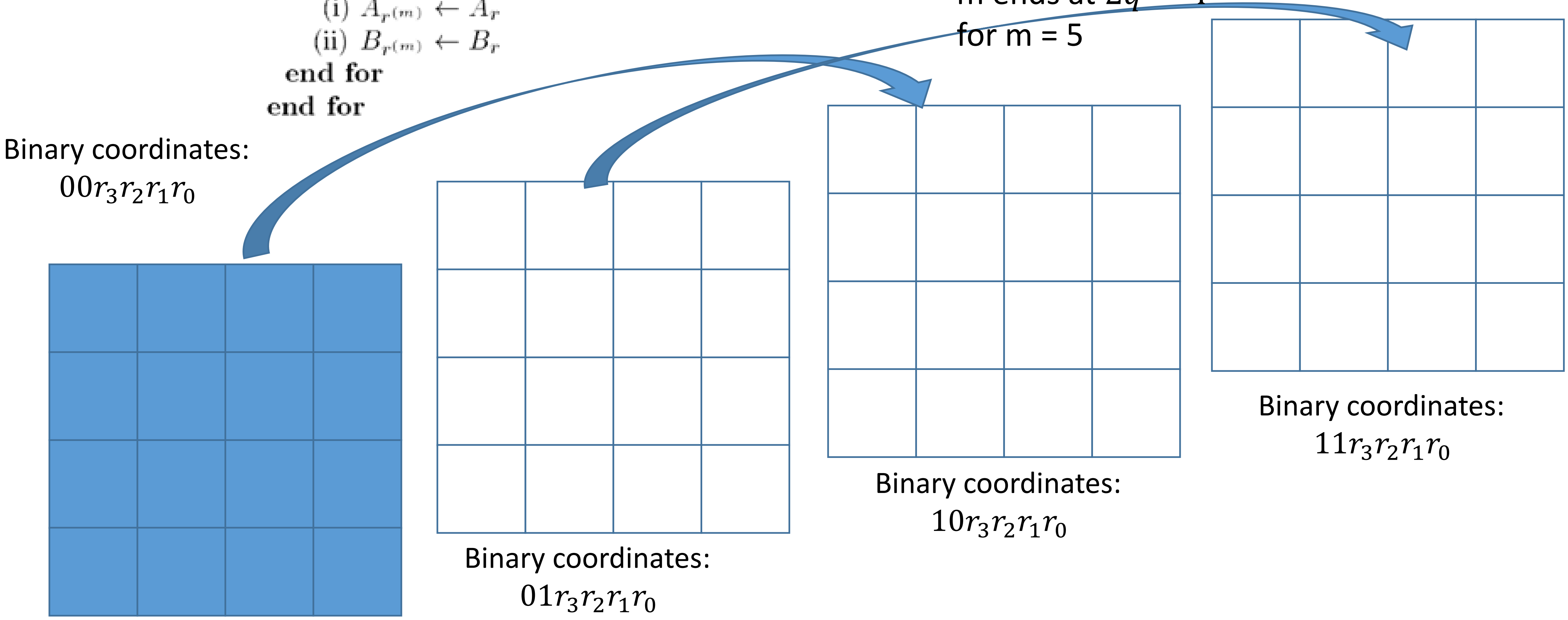
Binary coordinates:
000 001
010 011

Binary coordinates:
100 101
110 111

Parallel Matrix Multiplication on Hypercube (log n distribution example)

4×4 distribution
q = 2
m starts at $3q - 1 = 5$
m ends at $2q = 4$
for m = 5

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
    (i) $A_{r^{(m)}} \leftarrow A_r$
    (ii) $B_{r^{(m)}} \leftarrow B_r$
  end for
end for



Parallel Matrix Multiplication on Hypercube (log n distribution example)

4×4 distribution
q = 2
m starts at $3q - 1 = 5$
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Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
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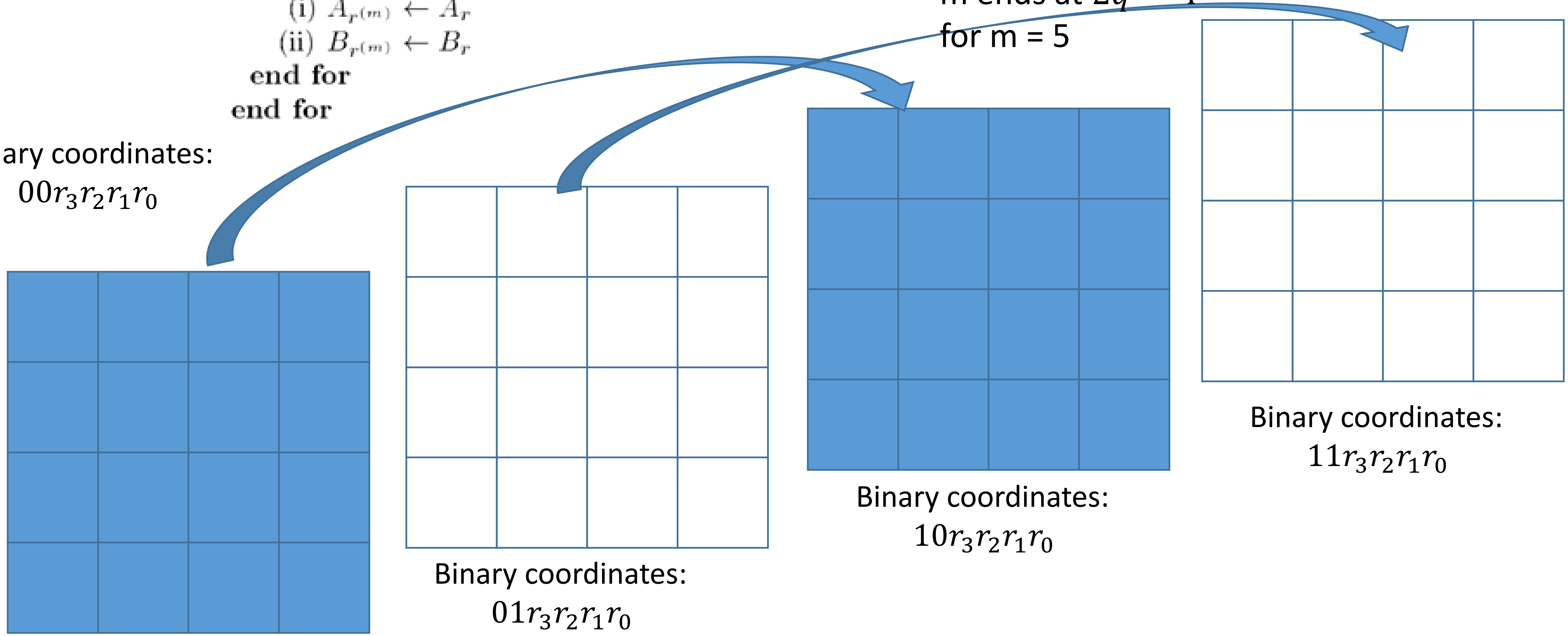

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4×4 distribution
q = 2
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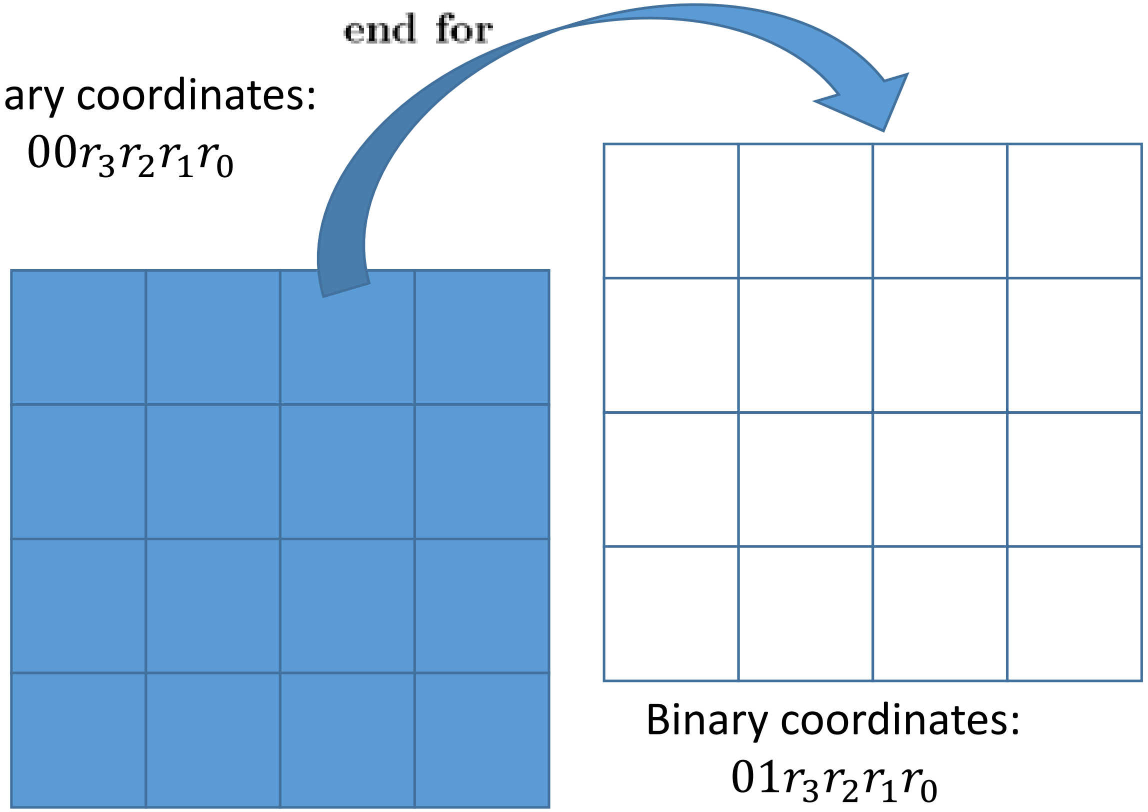
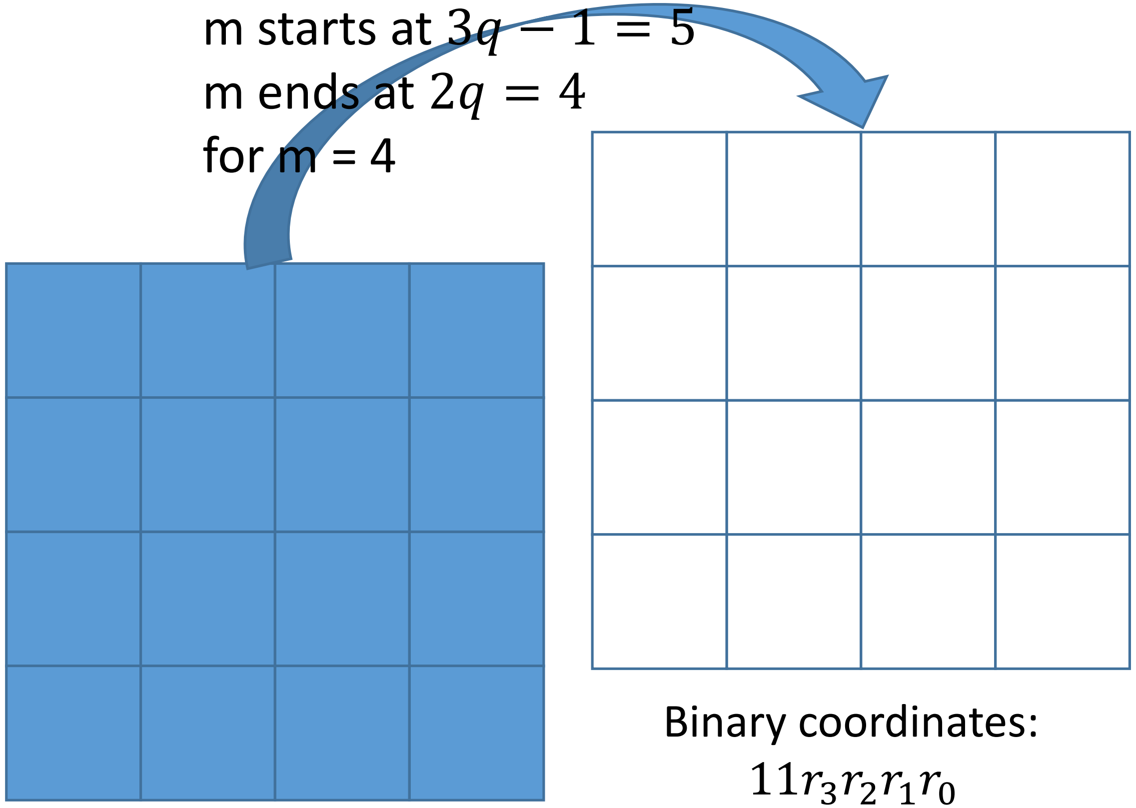
m ends at $2q = 4$
for m = 4



Binary coordinates:
$00r_3r_2r_1r_0$

Binary coordinates:
$01r_3r_2r_1r_0$



Binary coordinates:
$10r_3r_2r_1r_0$

Binary coordinates:
$11r_3r_2r_1r_0$

Parallel Matrix Multiplication on Hypercube (log n distribution example)

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
  for all $r \in N(r_m = 0)$ **do in parallel**
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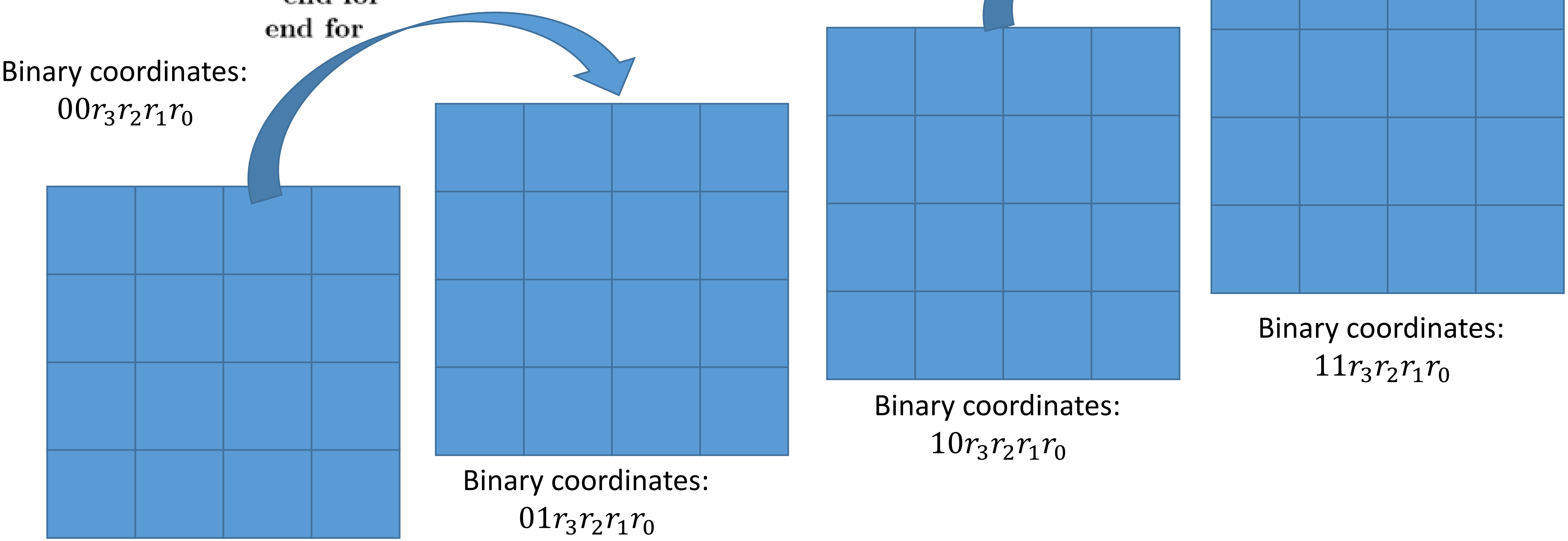

Parallel Matrix Multiplication on Hypercube (log n distribution example)

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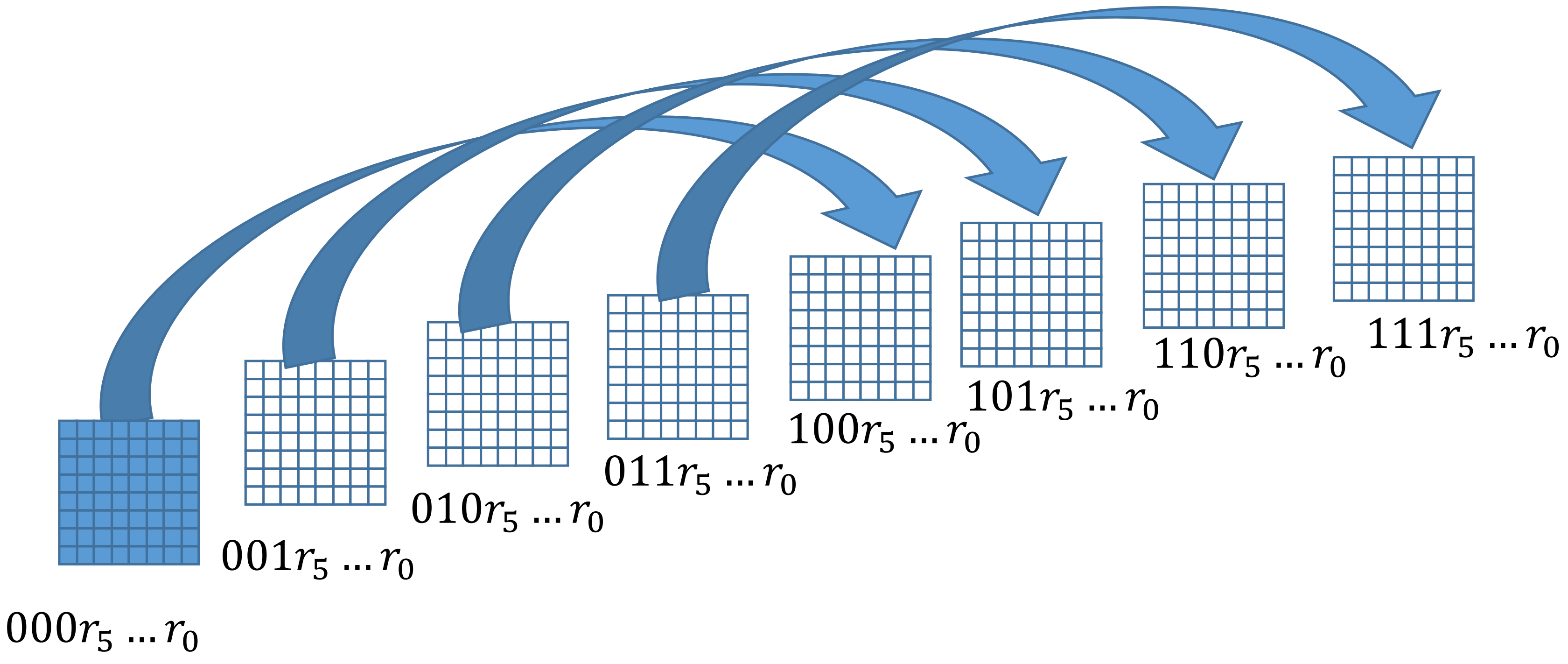
    (ii) $B_{r^{(m)}} \leftarrow B_r$
  end for
end for

8×8 distribution
q = 3
m starts at $3q - 1 = 8$
m ends at $2q = 6$
for m = 8



Parallel Matrix Multiplication on Hypercube (log n distribution example)

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
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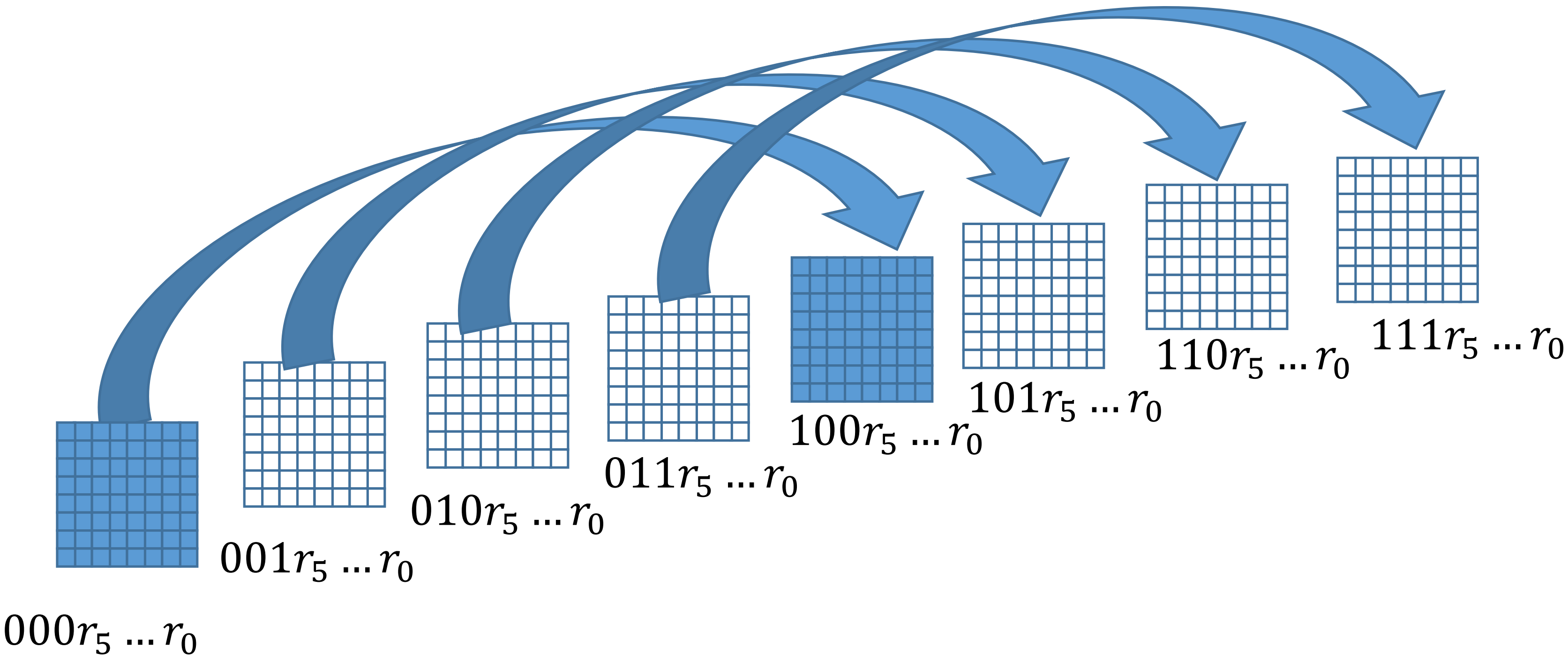

Parallel Matrix Multiplication on Hypercube (log n distribution example)

Step 1: (1.1) **for** $m = 3q - 1$ **downto** $2q$ **do**
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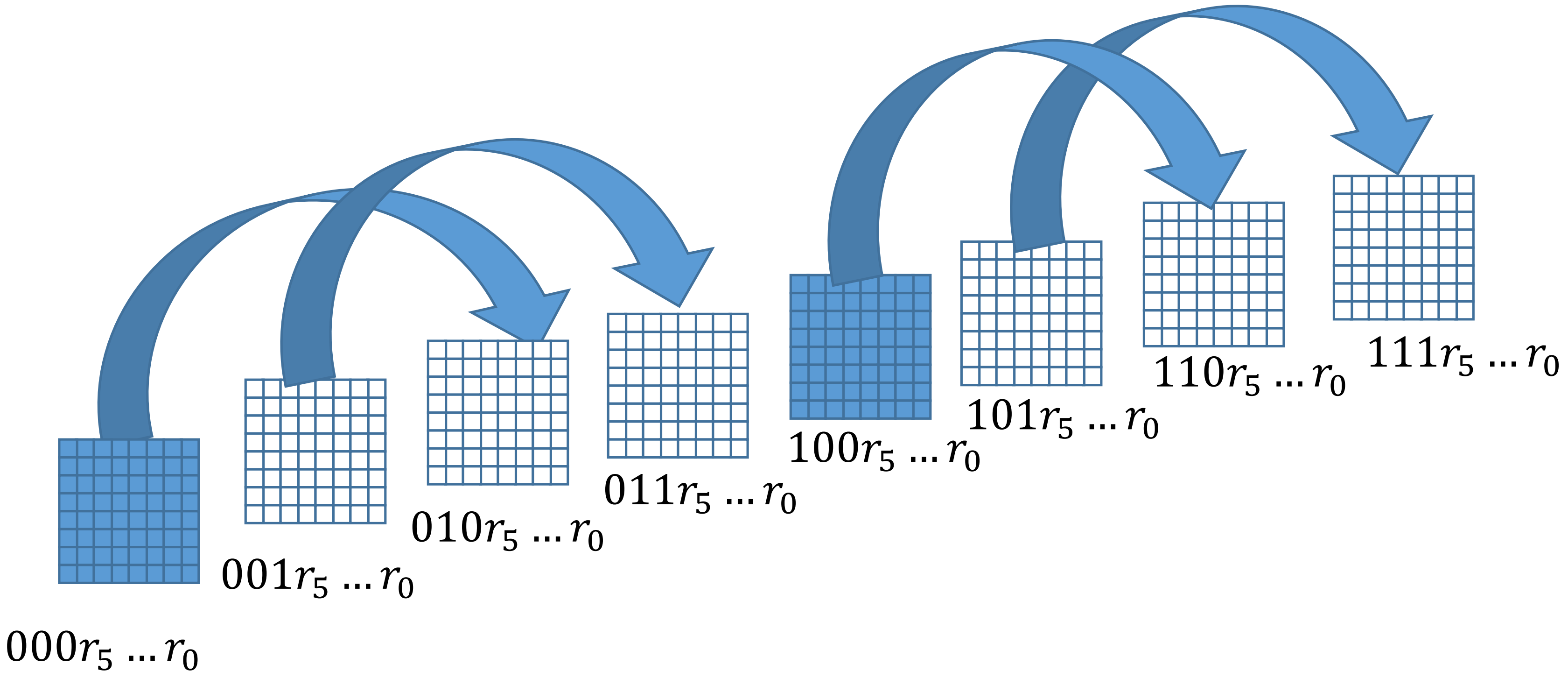
   (ii) $B_{r^{(m)}} \leftarrow B_r$
  end for
end for

8×8 distribution
q = 3
m starts at $3q - 1 = 8$
m ends at $2q = 6$
for m = 7



Parallel Matrix Multiplication on Hypercube (log n distribution example)

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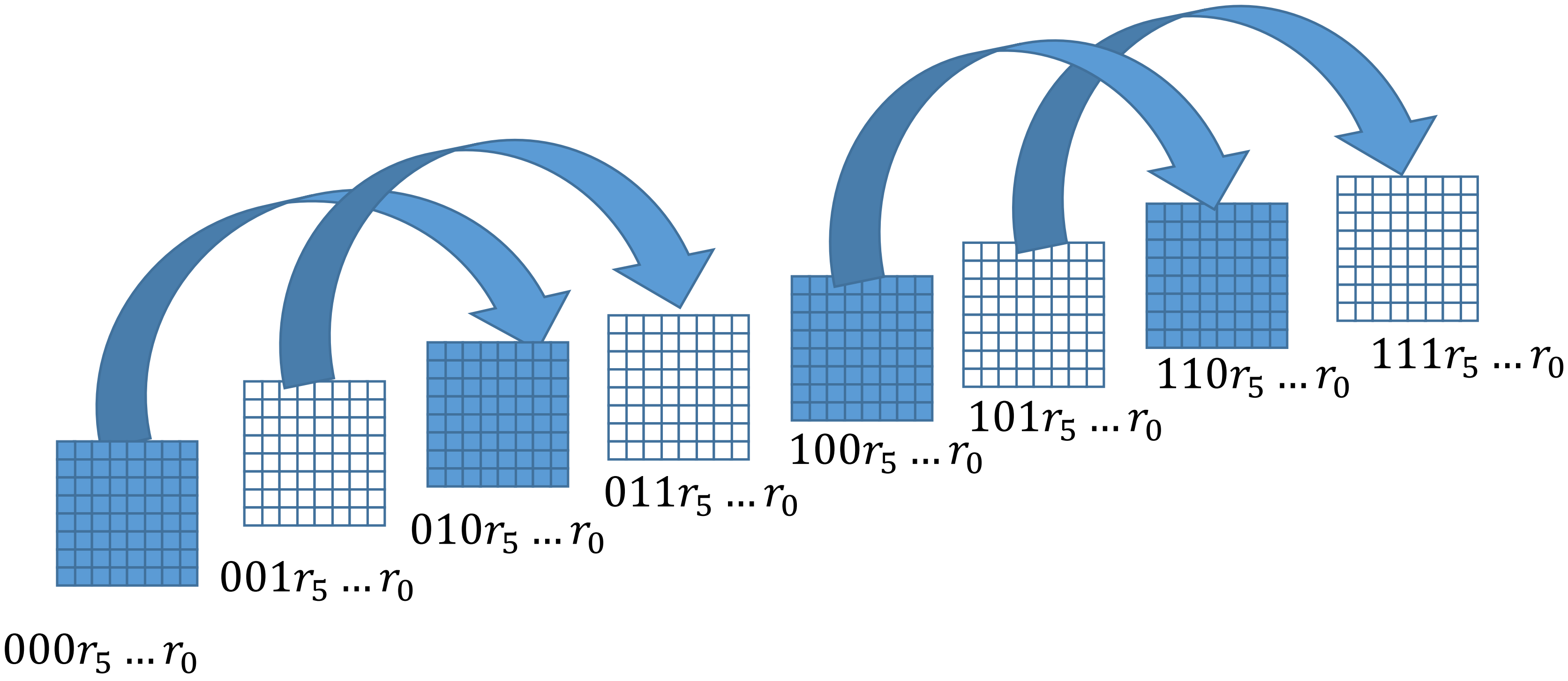

Parallel Matrix Multiplication on Hypercube (log n distribution example)

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8×8 distribution
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m starts at $3q - 1 = 8$
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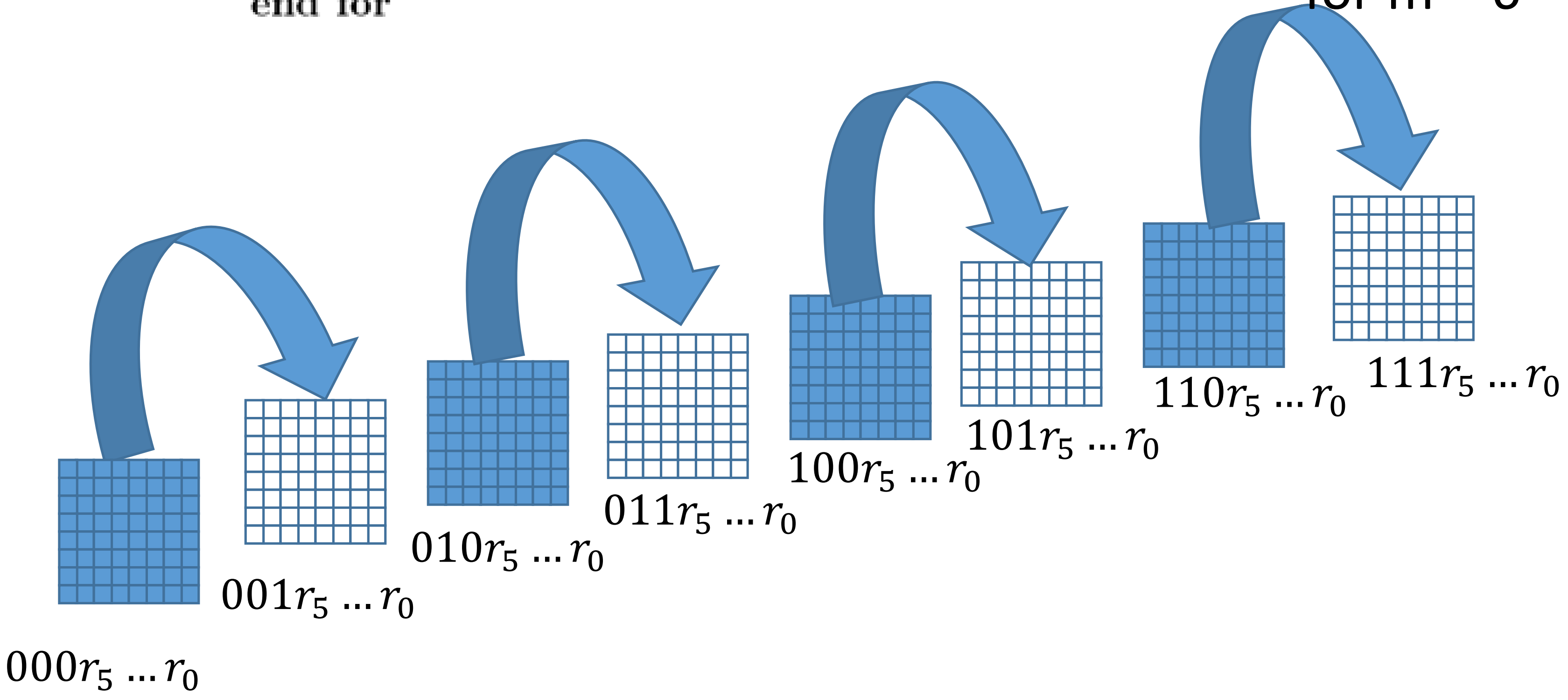

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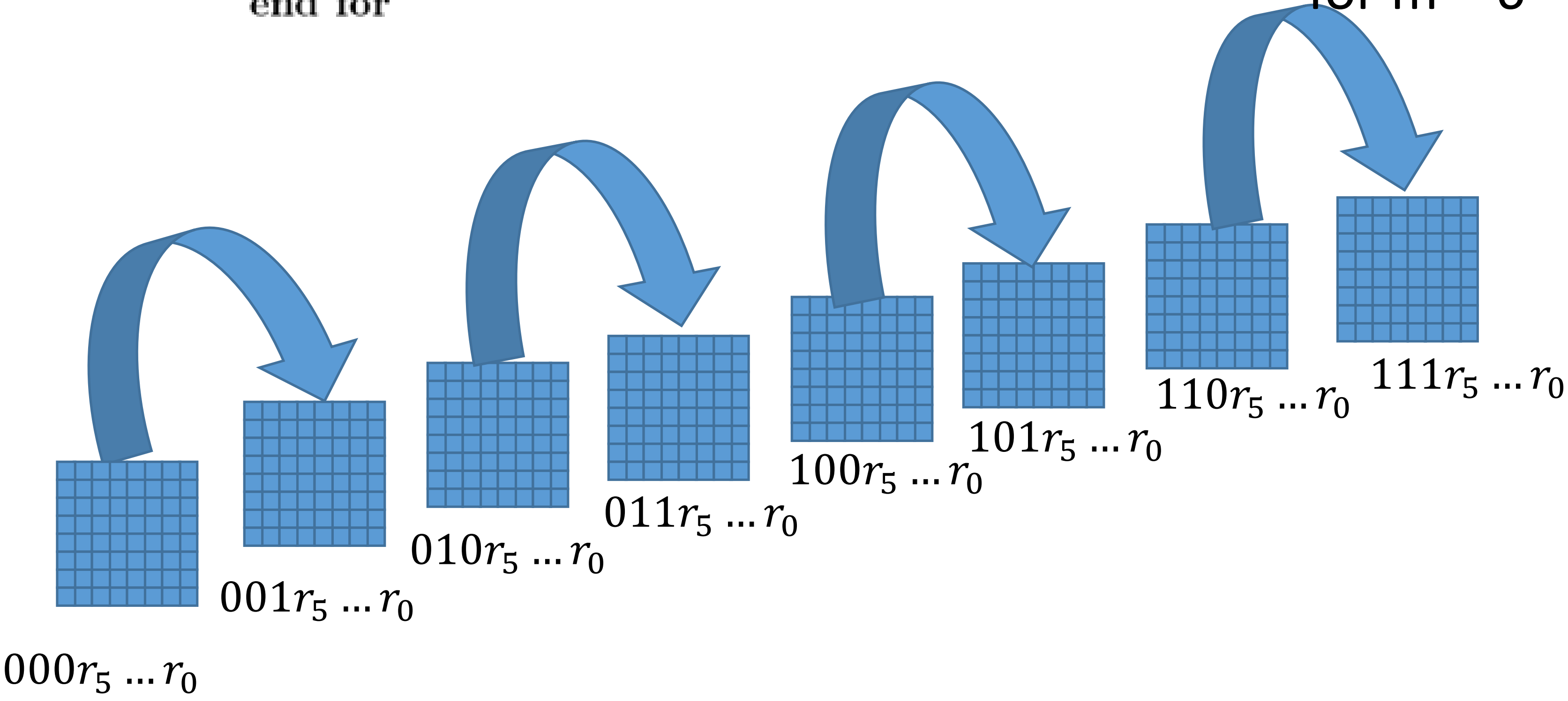

Hypercube Matrix Transpose

- A is an $n \times n$ matrix whose transpose is A^T.
  - Each element a_{ji}^T of A^T is such that $a_{ji}^T = a_{ij}$.
- The transpose can be computed on a hypercube with $N = n^2 = 2^{2q}$ processors.
- Can visualize the processors as being arranged in an $n \times n$ array in row-major order.
- Processor P_r holds position (i, j), where $r = in + j$ and $0 \leq i, j \leq n - 1$.

Hypercube Matrix Transpose

- Initially, every element a_{ij} of A is in processor P_r in a register A_r, where $r = in + j$. After the matrix transpose algorithm, element a_{ij} of A will be in processor P_s, in register A_s, where $s = jn + i$
- The binary representations of r and s are $r_{2q-1}r_{2q-2} \ldots r_q r_{q-1} \ldots r_1 r_0$ and $s_{2q-1}s_{2q-2} \ldots s_q s_{q-1} \ldots s_1 s_0$.
  - $r_{2q-1}r_{2q-2} \ldots r_q$ is the binary representation for i.
  - $r_{q-1}r_{q-2} \ldots r_1 r_0$ is the binary representation for j.
  - $s_{2q-1}s_{2q-2} \ldots s_q$ is the binary representation for j.
  - $s_{q-1}s_{q-2} \ldots s_1 s_0$ is the binary representation for i.
- Thus, $r_{2q-1}r_{2q-2} \ldots r_q = s_{q-1}s_{q-2} \ldots s_1 s_0$
- and $r_{q-1}r_{q-2} \ldots r_1 r_0 = s_{2q-1}s_{2q-2} \ldots s_q$
- Thus, an element a_{ij} can be routed from P_r to P_s in at most $2q$ steps.

Hypercube Matrix Transpose

- A_u of P_u is assumed to hold initially element a_{kl} of A, where $u = kn + l$.
- When the algorithm is done, A_u holds a_{kl}^T.
- An additional register B_u is used by P_u for routing data sent to it by other processors.

Algorithm HYPERCUBE MATRIX TRANSPOSE (A)

for $m = 2q - 1$ **downto** q **do**
  for $u = 0$ **to** $N - 1$ **do in parallel**
    (1) **if** $u_m \neq u_{m-q}$
      then $B_{u^{(m)}} \leftarrow A_u$
      end if
    (2) **if** $u_m = u_{m-q}$
      then $A_{u^{(m-q)}} \leftarrow B_u$
      end if
  end for
end for. ■

Hypercube Matrix Transpose

- How it works:
- Suppose the $n \times n$ matrix is subdivided into four $(n/2) \times (n/2)$ submatrices.
- The elements of the bottom left submatrix are swapped with the corresponding elements of the top right submatrix. The other two submatrices are untouched.
- Next, the same step is applied recursively to each of the four $(n/2) \times (n/2)$ matrices. Each of the $(n/2) \times (n/2)$ matrices, are subdivided into four $(n/4) \times (n/4)$ matrices.

Hypercube Matrix Transpose (The idea)

- Example: Transposing a 4×4 matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

- Divide into four $(n/2) \times (n/2)$ or 2×2 submatrices.
- Swap the elements of the top right submatrix with the elements of the bottom left submatrix.

Hypercube Matrix Transpose (The idea)

- Example: Transposing a 4×4 matrix

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Hypercube Matrix Transpose (The idea)

- Example: Transposing a 4×4 matrix

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- Recursively divide each submatrix into four submatrices.
- Swap the elements of the top right submatrix with the elements of the bottom left submatrix.

Hypercube Matrix Transpose
(The idea)

- Example: Transposing a 4×4 matrix

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- Recursively divide each submatrix into four submatrices.
- Swap the elements of the top right submatrix with the elements of the bottom left submatrix.

Hypercube Matrix Transpose

- With the algorithm: $N = n^2 = 2^{2q}$, so $q = 2$.

Algorithm HYPERCUBE MATRIX TRANSPOSE (A)

for $m = 2q - 1$ **downto** q **do**
  for $u = 0$ **to** $N - 1$ **do in parallel**
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  end for
end for. ■

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

m=3

m-q=1

0000 0001 0010 0011
0100 0101 0110 0111
1000 1001 1010 1011
1100 1101 1110 1111

Hypercube Matrix Transpose

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Hypercube Matrix Transpose

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Algorithm HYPERCUBE MATRIX TRANSPOSE (A)

```
for m = 2q − 1 downto q do
  for u = 0 to N − 1 do in parallel
    (1) if u_m ≠ u_{m−q}
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        end if
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  end for
end for. ■
```

$$\begin{bmatrix} 1,9 & 2,10 & 3 & 4 \\ 5,13 & 6,14 & 7 & 8 \\ 9 & 10 & 11,3 & 12,4 \\ 13 & 14 & 15,7 & 16,8 \end{bmatrix}$$

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m=3

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0000 0001 0010 0011
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end for. ■

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m=3

m-q=1

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Hypercube Matrix Transpose

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m=2

m-q=0

0000 **0001** 0010 **0011**
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1000 **1001** 1010 **1011**
1100 1101 **1110** 1111

Hypercube Matrix Transpose

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    (1) **if** $u_m \neq u_{m-q}$
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$$\begin{bmatrix} 1 & 2 & 9 & 10 \\ 5 & 6 & 13 & 14 \\ 3 & 4 & 11 & 12 \\ 7 & 8 & 15 & 16 \end{bmatrix}$$

m=2

m-q=0

0000 **0001** 0010 **0011**
0100 0101 **0110** 0111
1000 **1001** 1010 **1011**
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Hypercube Matrix Transpose

- With the algorithm: $N = n^2 = 2^{2q}$, so $q = 2$.

Algorithm HYPERCUBE MATRIX TRANSPOSE (A)

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<u>Analysis</u>:

There are q constant time iterations. The run time is $O(q) = O(\log n)$.

$P_n = n^2$, so the cost is $n^2 \cdot \log n$. Not cost optimal because the RAM algorithm only needs $n(n-1)/2$ operations.

References

[1] E. Dekel, D. Nassimi and S. Sahni, "Parallel Matrix and Graph Algorithms," *SIAM Journal on Computing,* vol. 10, no. 4, pp. 657-819, 1981.

[2] S. G. Akl, Parallel Computation: Models and Methods, Prentice Hall, 1997.