I. DEFINITION OF TERMS & FORMULAS

SYSTEMS OF NON-LINEAR EQUATIONS

LINEAR EQUATION – an equation that is in the *first degree*, meaning the highest power of the variable in the equation is **1**.

NON-LINEAR EQUATIONS – the equation is in the *second degree* or higher, meaning the highest power of the variable in the equation is greater than **1**.

SUBSTITUTION METHOD – solving one equation for one variable and then substituting it in the other equation.

ELIMINATION METHOD – transforming one or both equations so that you can eliminate one of the variables by combining the equations together.

SEQUENCE AND SERIES

SEQUENCE – a function whose domain is the set of $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ of positive integers and whose range is the set $\mathbb{R}$ of real numbers.

- For convenience, in a sequence, the function value at n is denoted by a_n, which is also called the $\boldsymbol{n^{th}}$ term of the sequence. In this instance, the sequence is denoted by $\{a_n\}$.

ARITHMETIC SEQUENCE

- A sequence whose term after the first term is obtained by **adding a constant number** to the preceding term. The constant number is called the common difference (d).
- 25, 23, 21, 19, 17, … d = -2
- 3, 7, 11, 15, … d = 4

GEOMETRIC SEQUENCE

- A sequence whose term after the first term is obtained by **multiplying a constant number** to the preceding term. The constant is called the common ratio (r).
- 1, 2, 4, 8, … r = 2
- 80, 20, 5, … r = ¼

SERIES – refers to the sum of the terms of a sequence.

- *Example*: 1 + 3 + 5 + 7 + 9 + 11 +, ….
- $\boldsymbol{S_2 = 1 + 3 = 4}$

ARITHMETIC SEQUENCE

FORMULAS	
Finding n^{th} term	$a_n = a_1 + (n-1)d$
Finding a_1	$a_1 = a_n + (n-1)d$
Finding the sum of the series	$S_n = \left(\frac{a_1 + a_n}{2}\right)n$

GEOMETRIC SEQUENCE

FORMULAS	
Finding n^{th} term	$a_n = a_1(r)^{n-1}$
Finding the a_1	$a_1 = \frac{a_n}{r^{n-1}}$
Finding the sum of the series	$S_n = \frac{a_1(1-r^n)}{1-r}$
The infinite sum of the geometric series	$S_\infty = \frac{a}{1-r}$

THE SUMMATION NOTATION

SUMMATION NOTATION – also known as *Sigma Notation*, this is a writing form using the capital Greek letter Sigma (Σ) to **represent the concise sum** of the sequence of numbers or terms.

$$\sum_{n=1}^{10}(4n-2)$$

On the left side is an example of a summation notation.

PARTS OF A SUMMATION NOTATION

In the given example above, the following are the parts:

- **SUMMATION SIGN**: This is the sigma symbol (Σ).
- **LOWER LIMIT** (the numbers below the summation sign)

- The n is known as the *index of summation*. It can be represented by any letter.
- The 1 is known as the *starting index*. This is where the sequence starts and counts up from this number.

- **UPPER LIMIT** (the numbers above the summation sign)
  - The 10 represents the last term or where the sequence stops counting, it is also known as the *last index value*.

- **FORMULA** (located beside the summation sign)
  - The formula in the given is $(4n - 2)$ wherein the variable n will be substituted to get the series.

PRINCIPLE OF MATHEMATICAL INDUCTION

MATHEMATICAL INDUCTION – a method of **proving that a statement is true** for every natural number n, that is, that the infinitely many cases can hold. This is done by first proving a simple case, then also showing if we assume the claim is true for a given case, then the next case is also true.

PROOF BY INDUCTION HAS **THREE** PARTS:

- **BASE CASE** – it proves that the statement can hold for $n = 1$ without assuming knowledge of other cases.
- **INDUCTION HYPOTHESIS** – an assuming step where we assume that the statement can hold for $n = k$, which is some natural numbers. This assumption is then used for the next case, the *inductive step*.
- **INDUCTION STEP** – this case proves that *if* the statement holds for any given case $n = k$, then it must also hold for the next case $n = k + 1$.

II. EXAMPLE PROBLEMS

SEQUENCE AND SERIES

Example 1: Find the 20th term of the arithmetic sequence: 5, 9, 13, 17, …

Step 1: Identify the given. In this given, it is an arithmetic sequence and we will look for a_{20}.

$$a_n = a_1 + (n-1)d$$

Since this is an arithmetic sequence, to find the common difference (d), simply subtract a term from its preceding term, for example, 9 – 5 = **4**. (d=4)

Step 2: Substitute 20 for n, and 4 for d. Then substitute 5 for a_1 which is the *first term*. Solve.

$$a_{20} = a_1 + (20-1)4$$
$$a_{20} = 5 + (19)4$$
$$a_{20} = 5 + 76$$
$$\boldsymbol{a_{20} = 81}$$

Therefore, the 20th term is **81**.

Example 2: Find the sum of the first 30 terms of the sequence: 7, 11, 15, 19, …

Step 1: Identify the given.

- It is an <u>arithmetic sequence</u> with a common difference of 4 (d).
- The first term is 7.
- N = 30, A_1 = 7.

Step 2: Solve first for a_{30}.

$$a_{30} = 7 + (30-1)4$$
$$a_{30} = 7 + (29)4$$
$$a_{30} + 7 + 116$$
$$a_{30} = 123$$

Step 3: Use the formula for finding the sum of the series, substitute the values & solve.

$$S_n = \left(\frac{a_1 + a_n}{2}\right)n$$
$$S_{30} = \left(\frac{7 + 123}{2}\right)30$$
$$S_{30} = \left(\frac{130}{2}\right)30$$
$$S_{30} = (65)30$$

$$\underline{\boldsymbol{S_{30} = 1950}}$$

Alternate Formula: Alternatively, you may use the formula to solve for the sum of an arithmetic sequence:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Step 1: Substitute the values.

$$S_{30} = \frac{30}{2}(2(7) + (30-1)4)$$
$$S_{30} = 15(14 + (29)4)$$
$$S_{30} = 15(14 + 116)$$
$$S_{30} = 15(130)$$

$$\underline{\boldsymbol{S_{30} = 1950}}$$

Example 3: Find the 10th term of the geometric sequence: 3, 6, 12, 24, …

Step 1: Analyze the given.

- The first term (a_1) is **3**. The n = 10, and the common ratio (r) is **2**.

Step 2: Write the formula and substitute the values, and solve.

$$a_n = a_1(r)^{n-1}$$
$$a_{10} = 3(2)^{10-1}$$
$$a_{10} = 3(2)^9$$
$$a_{10} = 3(512)$$

$$\underline{\boldsymbol{a_{10} = 1536}}$$

Example 4: Find the sum of the first 8 terms of the geometric series: 2 + 6 + 18 + 54 + …

Step 1: Identify the given. The first term (a_1) is 2, the common ratio (r) is 3, and n is 8.

Step 2: Write a working formula, then substitute the values.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_8 = \frac{2(1-3^8)}{1-3}$$
$$S_8 = \frac{2(1-6561)}{1-3}$$
$$S_8 = \frac{2(-6560)}{-2}$$

$$\underline{\boldsymbol{S_8 = 6560}}$$

Example 5: Given the geometric series 8 + 4 + 2 + 1 + … Find the infinite sum*.

Step 1: Write a working solution and substitute the values as needed.

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$
$$S_\infty = \frac{8}{1-\frac{1}{2}}$$
$$S_\infty = \frac{8}{\frac{1}{2}}$$

$$\underline{\boldsymbol{S_\infty = 16}}$$

TRY IT GIVENS!

- The first term of an arithmetic sequence is 12, and the 15th term is 72. Find the sum of the first 15 terms.
  - $\underline{\boldsymbol{S_{15} = 630}}$
- The 5th term of a geometric sequence is 243. With a common ratio of 3, solve for a_1. $\underline{\boldsymbol{a_1 = 3}}$

THE SUMMATION NOTATION

Example 1: Expand $\sum_{k=1}^{5} k$.

In this given case, we will start substituting from **1** (the starting index) up to **5** (the last index).

- FOR 1: $k = (1)$
- FOR 2: $k = (2)$

By following this until 5, the expanded form should be:

$$1+2+3+4+5$$

The simplified value, if needed would be the total, which is **15**.

Example 2: Expand $\sum_{n=2}^{4}(2n+1)$.

In this given case, we will start substituting from **2** (the starting index) up to **4** (the last index).

Therefore, from 2, 3, and 4, substitute each with the n.

$2(2)+1$	$2(3)+1$	$2(4)+1$
$4+1$	$6+1$	$8+1$
$\underline{\boldsymbol{=5}}$	$\underline{\boldsymbol{=7}}$	$\underline{\boldsymbol{=9}}$

Thus, the answer should be:

$$5+7+9$$

The simplified value, if need would be the total, which is **21**.

SYSTEMS OF NON-LINEAR EQUATIONS

Example 1: Substitution Method

- $y = -x + 2$ (1)
- $x^2 + y = 2$ (2)

Since y is already transformed, we solve by *substituting* the value of y to the y in the second equation.

Step 1: Substitute and equate to zero.

$$x^2 + (-x+2) = 2$$

$$x^2 - x + 2 - 2 = 0$$
$$x^2 - x = 0$$

Step 2: Find two solutions to x through factoring and the zero-product property.

$$x(x-1) = 0$$

$$\underline{\boldsymbol{x = 0}} \qquad \begin{gathered} x - 1 = 0 \\ \underline{\boldsymbol{x = 1}} \end{gathered}$$

Step 3: Substitute the two values of *x* to find the *y* variable values.

$$\begin{gathered} y = -x + 2 \\ y = -(0) + 2 \\ \underline{\boldsymbol{y = 2}} \\ \underline{\boldsymbol{(0, 2)}} \end{gathered} \qquad \begin{gathered} y = -x + 2 \\ y = -(1) + 2 \\ \underline{\boldsymbol{y = 1}} \\ \underline{\boldsymbol{(1, 1)}} \end{gathered}$$

Therefore: $\underline{\{\boldsymbol{(0,2)(1,1)}\}}$ TWO SOLUTIONS.

Example 2: Substitution Method

- $x^2 + y^2 = 10$ (1)
- $x - 3y = -10$ (2)

Step 1: Fix the *x* variable of the 2nd equation so that we can substitute it with the 1st.

$$x - 3y = -10$$
$$x = -10 + 3y$$

Step 2: Substitute the new *x* value of the second equation with the first.

$$x^2 + y^2 = 10$$
$$(-10 + 3y)^2 + y^2 = 10$$

- *Foil Method*

$$(-10 + 3y)(-10 + 3y)$$
$$100 - 30y - 30y + 9y^2$$
$$100 - 60y + 9y^2$$

$$100 - 60y + 9y^2 + y^2 = 10$$

Step 3: Rewrite and combine like terms, equate to zero, and solve.

$$10y^2 - 60y + 100 - 10 = 0$$

$$10y^2 - 60y + 90 = 0$$

Step 4: Simplify.

$$\frac{\cancel{10}y^2 - 60y + 90}{\cancel{10}}$$
$$y^2 - 6y + 9 = 0$$
$$(y-3)(y-3) = 0$$

$$\begin{gathered} y - 3 = 0 \\ \underline{\boldsymbol{y = 3}} \end{gathered} \qquad \begin{gathered} y - 3 = 0 \\ \underline{\boldsymbol{y = 3}} \end{gathered}$$

Step 5: Substitute the two values of *y* to find the *x* variable values of the second equation.

- In this particular example, since both *y* are 3, then there is only one solution in technicality.

$$x - 3y = -10$$
$$x - 3(3) = -10$$
$$x - 9 = -10$$
$$x = -10 + 9$$

$$\underline{\boldsymbol{x = -1}}$$
$$\underline{\{\boldsymbol{(-1, 3)}\}}$$

1 SOLUTION.

Example 3: Substitution Method

- $x = 27 - 5y$ (1)
- $x^2 + y^2 - 8x - 4y = -7$ (2)

Step 1: Substitute the *x* value of the first equation with the second equation to find the *y* values. Simplify the equation and solve.

$$x^2 + y^2 - 8x - 4y = -7$$
$$(27 - 5y)^2 + y^2 - 8(27 - 5y) - 4y = -7$$

- *Foil Method*

$$(27 - 5y)(27 - 5y)$$
$$729 - 135y - 135y + 25y^2$$
$$729 - 270y + 25y^2$$

$$729 - 270y + 25y^2 + y^2 - 216 + 40y - 4y + 7 = 0$$

Step 2: Rewrite and re-arrange. Also, combine like terms and simplify further.

$$26y^2 - 234y + 520 = 0$$
$$\frac{\cancel{26}y^2 - 234y + 520}{\cancel{26}} = 0$$

$$y^2 - 9y + 20 = 0$$
$$(y-4)(y-5) = 0$$

Step 3: After finding the factoring, equate both to zero to find the y values.

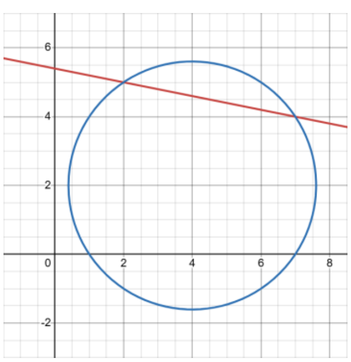
$$y - 4 = 0 \qquad\qquad y - 5 = 0$$
$$\underline{\boldsymbol{y = 4}} \qquad\qquad \underline{\boldsymbol{y = 5}}$$

Step 4: Solve for the *x* values.

$$x = 27 - 5y \qquad\qquad x = 27 - 5y$$
$$x = 27 - 5(4) \qquad\qquad x = 27 - 5(5)$$
$$x = 27 - 20 \qquad\qquad x = 27 - 25$$

$$x = 7 \qquad\qquad x = 2$$
$$\underline{\boldsymbol{(7,4)}} \qquad\qquad \underline{\boldsymbol{(2,5)}}$$

TWO SOLUTIONS.



Example 4: Elimination Method

$$\begin{cases} 2x^2 + 5y^2 = 98 & (1) \\ 2x^2 - y^2 = 2 & (2) \end{cases}$$

Step 1: Multiply both sides of the second equation by -1 to make it negative for the subtraction.

$$\cancel{2x^2} + 5y^2 = 98$$
$$-\cancel{2x^2} + y^2 = -2$$

$$\frac{\cancel{6}y^2}{\cancel{6}} = \frac{96}{6}$$
$$\sqrt{y^2} = \sqrt{16}$$
$$\underline{\boldsymbol{y = \pm 4}}$$

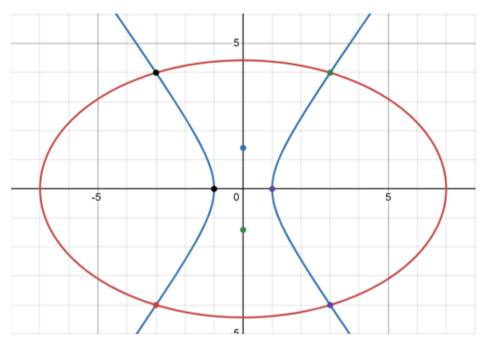
Step 2: Solve for the x values using the second equation (in this case, the first may also be used, it will result in same answer).

$$2x^2 - y^2 = 2$$
$$2x^2 - 16 = 2$$
$$2x^2 = 2 + 16$$
$$\frac{\cancel{2}x^2}{\cancel{2}} = \frac{18}{2}$$
$$\sqrt{x^2} = \sqrt{9}$$
$$\underline{\boldsymbol{x = \pm 3}}$$

Step 3: Identify the solutions. There are four (4) solutions with this one.

$$\underline{\boldsymbol{\{(3,4)(-3,4)(3,-4)(-3,-4)\}}}$$

FOUR SOLUTIONS.



Example 5: Elimination Method

- $x^2 - 6x + y^2 + 2y = 15$ (1)
- $x^2 - 6x - 2y = 3$ (2)

Step 1: Multiply both sides of the second equation to -1 to make it opposite sign for elimination.

$$\cancel{x^2} - \cancel{6x} + y^2 + 2y = 15$$
$$-\cancel{x^2} + \cancel{6x} + 2y = -3$$

$$y^2 + 4y = 12$$

Step 2: Transpose 12, and factor.

$$y^2 + 4y - 12 = 0$$
$$(y + 6)(y - 2) = 0$$

$y + 6 = 0$ $\qquad$ $y - 2 = 0$

$\underline{\boldsymbol{y = -6}}$ $\qquad$ $\underline{\boldsymbol{y = 2}}$

Step 3: Substitute the different y values with one of the equations. (In this case, the second is chosen, though using the first will result in the same answer).
Do not forget to pair the x value attained with the proper y value used to obtain it.

$$x^2 - 6x - 2y = 3$$
$$x^2 - 6x - 2(-6) = 3$$
$$x^2 - 6x + 12 - 3 = 0$$

$$x^2 - 6x + 9 = 0$$
$$(x - 3)(x - 3) = 0$$

$x - 3 = 0$ $\qquad$ $x - 3 = 0$

$\underline{\boldsymbol{x = 3}}$ $\qquad$ $\underline{\boldsymbol{x = 3}}$

In this case, since both are same, only one solution for this equation is recorded, which will be $(\mathbf{3}, -\mathbf{6})$.

$$x^2 - 6x - 2y = 3$$
$$x^2 - 6x - 2(2) = 3$$
$$x^2 - 6x - 4 - 3 = 0$$

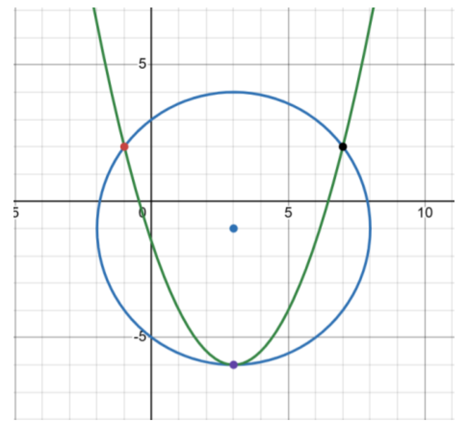
$$x^2 - 6x - 1 = 0$$
$$(x - 7)(x + 1) = 0$$

$x - 7 = 0$ $\qquad$ $x + 1 = 0$

$\underline{\boldsymbol{x = 7}}$ $\qquad$ $\underline{\boldsymbol{x = -1}}$

From this equation, two solutions are derived, which would be paired both to the same y used to obtain them, giving $\underline{(\mathbf{7}, \mathbf{2})}$ and $(-\mathbf{1}, \mathbf{2})$.

<u>THREE SOLUTIONS:</u> $\underline{\{(\mathbf{3}, -\mathbf{6})(\mathbf{7}, \mathbf{2})(-\mathbf{1}, \mathbf{2})\}}$



TIPS TO EASILY IDENTIFY THE EQUATION'S CONIC SECTION

- **CIRCLE** – if both x^2 and y^2 appear and $\underline{A = C}$
- **ELLIPSE** – if both x^2 and y^2 appear and $\underline{A \neq C}$ (same sign).
- **PARABOLA** – if only <u>one squared term</u> appears (x^2 or y^2, but not both)
- **HYPERBOLA** – if both x^2 and y^2 appear and A and C have <u>opposite signs</u>.

MATHEMATICAL INDUCTION*

DISCLAIMER: THE FOLLOWING IS BASED ON CERTAIN YOUTUBE TUTORIALS. PLEASE STUDY WITH CAUTION.

Example 1: Prove the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Step 1: Base Case

- In the base case, $n = 1$. Copy the first term, which happens to be 1. Note: The base case is 1, since it pairs the first term, but the first term in the statement is not always 1.
- Here, we equate the first term, 1, to the sum of the statement. Then, substitute 1 for every n.

Base Case: $n = 1$

$$1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(\cancel{2})}{\cancel{2}}$$

$$\underline{\mathbf{1 = 1}}$$

Thus, the statement holds for the base case.

Step 2: Induction Hypothesis

- In this step, we simply assume that the statement will hold for some numbers (k).
- Simply copy the given statement and substitute every instance of *n* with *k*.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Step 3: Induction Step

- In the final step, take note, this is a bit tricky. In some YouTube tutorials, the solutions are done on separate manners, which may be less confusing, however for this reviewer, we will do a straight method.
- For the *left side*, duplicate the *k* term (or whatever term with k), and then for the new *k* term, substitute every instance of *k*, with *(k + 1)*.
- For the *right side*, substitute every instance of *k* with *(k + 1)*.

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k[(k+1)+1]}{2}$$

- Replace the terms from 1 to *k*, with the $\frac{k(k+1)}{2}$ from the previous step, before the substitutions took place, and add to the (k + 1) on the left side, not included beforehand.

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

- Simplify both sides.

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2}$$

$$\boldsymbol{\frac{k^2 + 3k + 2}{2} = \frac{k^2 + 3k + 2}{2}}$$

Therefore, if the answer is equal, then the statement is proven and holds for $n = k + 1$.

"Trust in the LORD with all your heart, on your own intelligence do not rely; In all your ways be mindful of him, and he will make straight your paths."
Proverbs 3:5-6