PHYS-330 - Classical Mechanics - Fall 2017

Homework 6

Due: 26th October 2017 by the start of class. Anything later will be considered late. Late assignments will be subjected to 10% deduction each day late. The late day starts by the end of class when the assignment is due.

Instructions: Complete all of the questions below. You are encouraged to use Jupyter Notebooks to complete any numerical work and written. While the use of python is encouraged, you can use any programming language you want. You can either email me your assignment or provide me with a hard copy in class.

1. Consider a bead sliding on a wire in the shape of a parabola, $z = k\rho^2$ (ignore frictional effects and use cylindrical coordinates). The parabola is spinning with a constant angular velocity of ω about the z axis.

a) Write out the Lagrangian for the motion in terms of the generalized coordinate ρ.

b) Are there any positions of equilibrium (i.e. ρ is constant). Discuss the stability of any equilibrium positions.

2. Consider the two-body problem for a massless particle positioned at $x = (x_1, x_2) \in \mathbb{R}^2$ around a central body with gravitational parameter μ such that

$$\ddot{\mathbf{x}} + \mu\frac{\mathbf{x}}{r^3} = 0, \quad r = |\mathbf{x}|,$$

where dots denote derivatives with respect to time t. Without loss of generality, we can consider $\mathbf{x} = x_1 + ix_2 \in \mathbf{C}$ and the above equation is still true.

a) Consider transforming time from t to a fictitious time τ via the transformation $dt = r{\cdot}d\tau$. Show that the above equation of motion can be written as

$$r\mathbf{x}'' - r'\mathbf{x}' + \mu\mathbf{x} = 0$$

where a prime denotes a derivative with respect to τ.

b) Show that the energy of the particle can be written as

$$\frac{1}{2r^2}|\mathbf{x}'|^2 - \frac{\mu}{r} = -h$$

where $h = \mu/(2a)$ and a is the semi-major axis.

c) Now consider the transformation,

$$\mathbf{x} = \mathbf{u}^2,$$

where $\mathbf{u} \in \mathbb{C}$. (note $r = |\mathbf{x}| = |\mathbf{u}|^2 = \mathbf{u}\bar{\mathbf{u}}$). Show that the equation of motion (from part a) under this transformation becomes,

$$2r\mathbf{u}'' + \mu\mathbf{u} - 2|\mathbf{u}'|^2\mathbf{u} = 0,$$

and the energy (from part b) is

$$2|\mathbf{u}'|^2 = \mu - rh.$$

d) Show that by combining the two equation in part c, we obtain the differential equation

$$\mathbf{u}'' + \omega^2\mathbf{u} = 0$$

where ω is a constant. What is ω equal to?

3.

a) Show that circular orbits for a central force of the form $f(r) = \gamma r^n$ are unstable when $n = -3$.
Now Consider a particle with mass m and angular momentum l in the field of a central force $F = -k/r^{8/3}$.

b) Find the value r_0 such that U_{eff} is a minimum and make a plot of U_{eff} showing the interesting part of the curve.

c) Find r_{min} when the total energy is $E = -0.075$.

d) When $r = r_{\text{min}}$ when $\phi = 0$ solve the numerically solve the differential equation for this force for $0 \leq \phi \leq 20$. Discuss the orbit and discuss if if the orbit is periodic.

4. 8.21 from Taylor
5. 8.11 from Taylor