Functions

A FUNCTION is a rule which assigns each element in a set A to one and only one element in a set B.

A is called the domain of the function

B is called the range of the function

Denote a function by a letter. $f(x)$ x is an element in the domain

For our purposes A and B comprise of real numbers.

Ex i) $f(x) = x^2 + 3$ Domain: all real number
Range: $y \geq 3$

ii) $g(x) = \sqrt{x-1}$ require $x - 1 \geq 0$
$x \geq 1$ Domain

Function can be combined with the usual operations of addition, subtraction, multiplication and division.

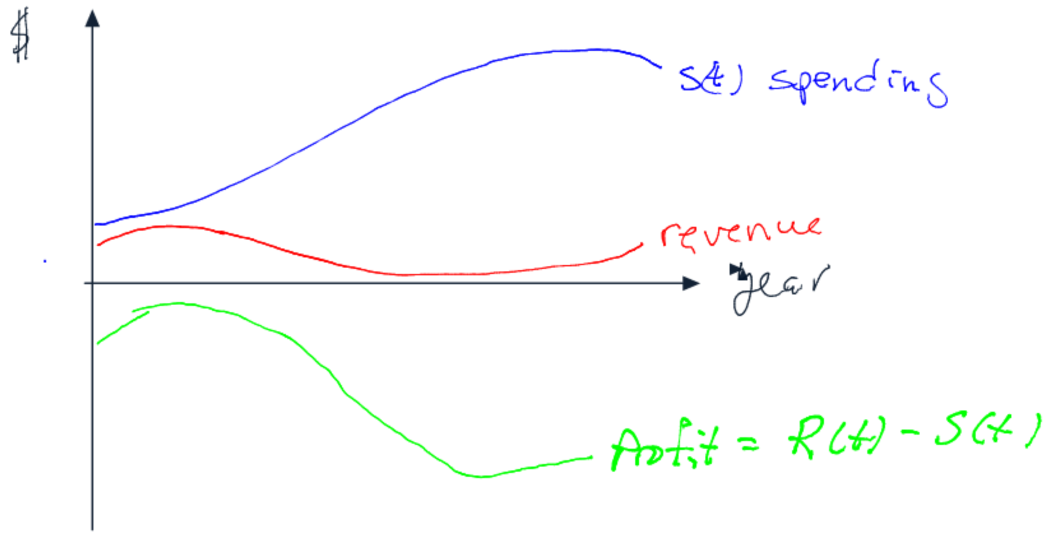
Ex $f(x) = x^2 + 1$ $g(x) = 2x$ $h(x) = \frac{1}{x}$

$f(x) + g(x) = x^2 + 1 + 2x$ $\frac{f(x)}{h(x)} = \frac{x^2+1}{x}$

$g(x) \cdot h(x) = 2x \cdot \frac{1}{x} = 2$

$f(x) \mp g(x) = (f \mp g)(x)$

$f(x) \cdot g(x) = fg(x)$



Composition of two functions

consider $g(x) = \sqrt{x}$ $\quad f(x) = x^2+1$

$$h(x) = \sqrt{x^2+1}$$

$$h(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{x^2+1}$$

g composed of f $\quad g \circ f = g(f(x))$

Ex $\quad g(x) = \sqrt{x} + 1 \quad f(x) = x^2 - 1$

$$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} + 1 = \sqrt{x^2-1} - 1$$

$$(f \circ g)(x) = f(g(x)) = g^2 - 1 = (\sqrt{x} + 1)^2 - 1$$

$$= x + 2\sqrt{x} + 1 - 1$$

$$= x + 2\sqrt{x}$$

Ex find functions f and g such that $h = g \circ f$

$$h(x) = \frac{1}{\sqrt{2x+1}} + \sqrt{2x+1} \qquad g = \frac{1}{x} + x \quad f = \sqrt{2x+1}$$

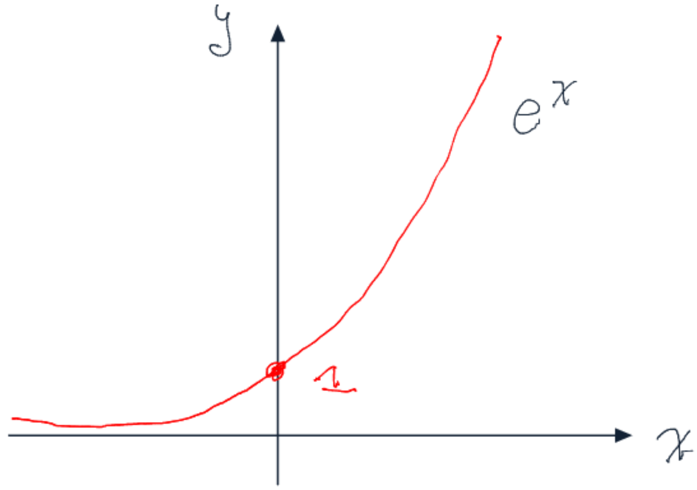
$$h(x) = \frac{1}{(3x^2+2)^{3/2}} \Rightarrow \qquad g = \frac{1}{x^{3/2}} \quad f = 3x^2+2$$

Ex Evaluate $f(a+h) - f(a)$ when $f(x) = 4 - x^2$

$$f(a+h) - f(a) = (4 - (a+h)^2) - 4 + a^2$$
$$= \cancel{4} - \cancel{a^2} - 2ah - h^2 - \cancel{4} + \cancel{a^2}$$
$$= -2ah - h^2$$

Ex $f(x) = x^2$ $g(x) = e^x$ $h(x) = (f \circ g)(x) = e^{f(x)}$
$$= e^{x^2}$$

$g(x) = e^x$ - exponential function.



domain all reals
range $(0, \infty)$
$g(0) = e^0 = 1$
$e = 2.7182818.$

Logorithm

$y = \log_b x$ if and only if $x = b^y$ $b > 0$ $b \neq 1$, $x > 0$

Ex $\log_{10} 100 = 2$ $10^2 = 100$

$\log_5 125 = 3$ $5^3 = 125$

Ex Solve for x

$\log_3 x = 4 \Rightarrow 3^4 = x = 81$

$$\log_{16} 4 = x \Rightarrow 16^x = 4 \Rightarrow x = \frac{1}{2}$$

$$\log_x 8 = 3 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

Rules

Exponential

$$a^x b^y = a^{x+y}$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$(b^x)^n = b^{nx}$$

$$b^0 = 1$$

$$b^1 = b$$

Log

$$\log_b(mn) = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

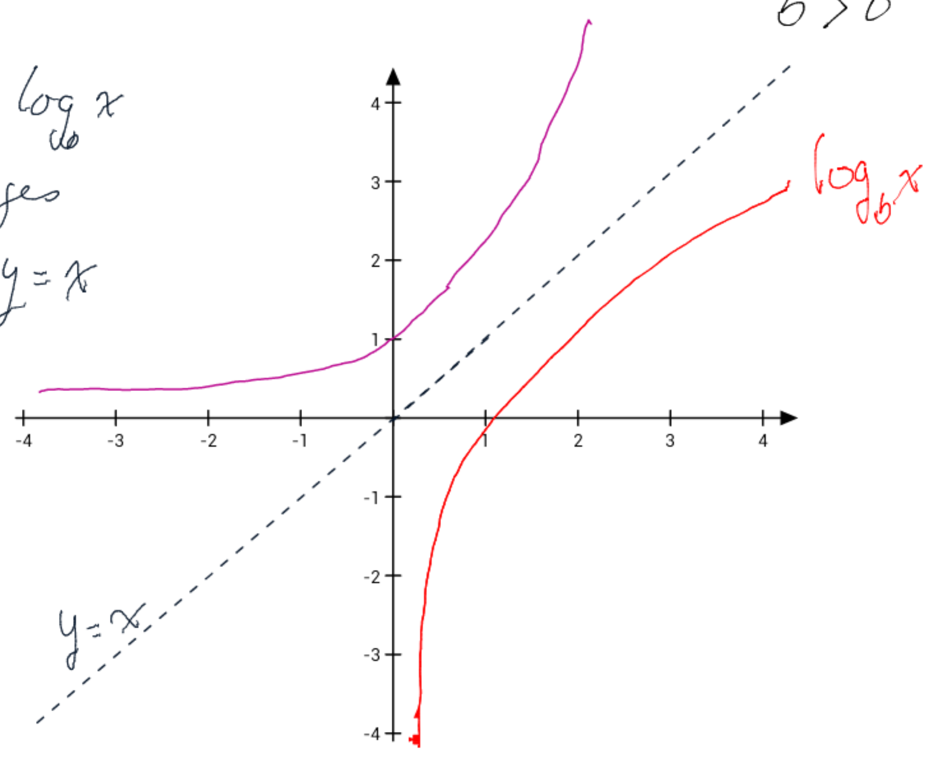
$$\log_b(m^n) = n \log_b m$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$f(x) = \log_b x$ domain $x > 0$ range $(-\infty, \infty)$ $b \neq 1$, $b > 0$

the graphs of b^x $\log_b x$ are mirror images across the line $y = x$



Special Cases

$b = 10$

$\log_{10} x = \log x$

Log base 10

$b = e$

$\log_e x = \ln x$

natural log of x.

Ex/ $\log_3 x^2 y^3 = \log_3 x^2 + \log_3 y^3 = 2\log_3 x + 3\log_3 y$

$$\ln \frac{x^2\sqrt{x^2-1}}{e^x} = \ln\left(x^2\sqrt{(x^2-1)}\right) - \ln e^x$$
$$= \ln x^2 + \ln\sqrt{x^2-1} - x$$
$$= 2\ln(x) + \tfrac{1}{2}\ln(x^2-1) - 1$$

Solve for t

$$e^{0.4t} = 8$$
$$\ln e^{0.4t} = \ln(8)$$
$$0.4t = \ln(8)$$
$$t = \tfrac{5}{2}\ln(8)$$

$$\frac{200}{1+3e^{-0.3t}} = 100$$
$$2 = 1 + 3e^{-0.3t}$$
$$\frac{1}{3} = e^{-0.3t}$$
$$\ln\left(\tfrac{1}{3}\right) = -0.3t$$
$$\ln(3) = 0.3t$$
$$t = \frac{10}{3}\ln(3)$$

$$\frac{A}{1+Be^{t/2}} = C$$
$$\frac{A}{C} = 1 + Be^{t/2}$$
$$\frac{A}{BC} - \frac{1}{B} = e^{t/2}$$
$$-\ln B + \ln\left(\tfrac{A}{C} - 1\right) = t/2$$
$$t = 2\ln\left(\tfrac{A}{C} - 1\right) - 2\ln B$$

Richter Scale

The mgnitude R of an earthquake is given by

$$R = \log \frac{I}{I_0}$$

I - intensity

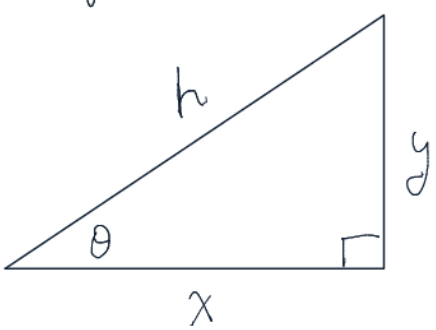
I_0 - standard measure of intensity.

$$R = 5 \Rightarrow 5 = \log\frac{I}{I_0}$$
$$10^5 = \frac{I}{I_0} \Rightarrow I = I_0 10^5$$

$R=8 \Rightarrow 10^8 = \frac{I}{I_0} \quad I = I_0 10^8$

1000 times bigger than $R=5$

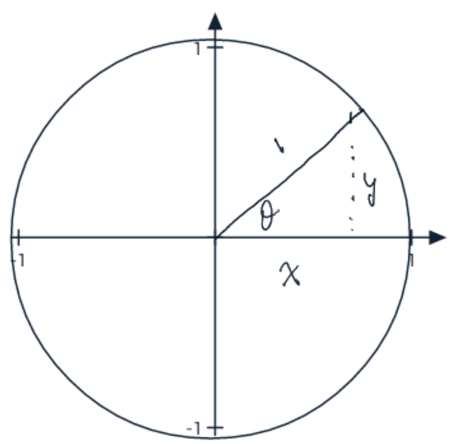
Trigonometric Functions.



$\sin\theta = \frac{y}{h} \qquad \csc\theta = \frac{h}{y}$

$\cos\theta = \frac{x}{h} \qquad \sec = \frac{h}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot = \frac{x}{y}$



$\cos\theta = \frac{x}{1} = \cos(\theta + 2\pi)$

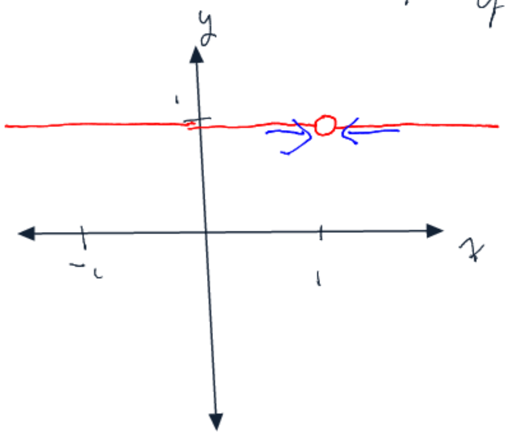
$\sin\theta = \frac{y}{1} = \sin(\theta + 2\pi)$

$\sin\theta$ and $\cos\theta$ are 2π periodic

Limits

Let $f(x) = \dfrac{x-1}{x-1}$ $\qquad f(1) = \dfrac{0}{0}$??? undefined.

$= 1$ if $x \neq 1$



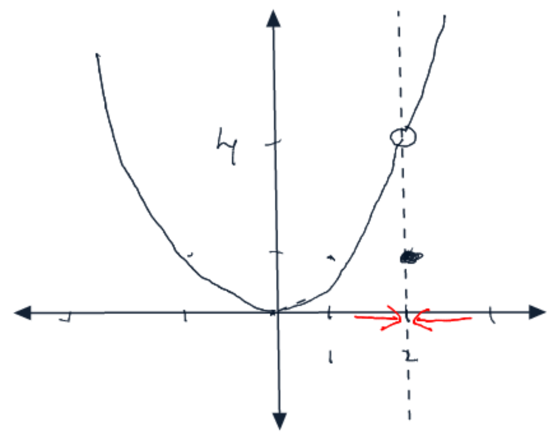
as x gets closer to 1

$f(x)$ is 1

$$\lim_{x\to 1} f(x) = 1$$

$$g(x) = \begin{cases} x^2, & x \neq 2 \\ 1 & x = 2 \end{cases}$$

discontinuity at $x = 2$



$g(2) = 1 \qquad \lim_{x\to 2} g(x) = 4$

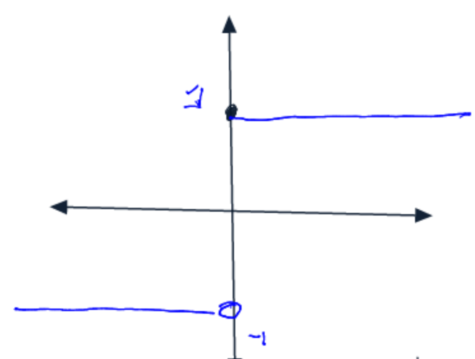
The function $f(x)$ has the limit, L, as x approaches a

$$\lim_{x\to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number L by taking x sufficiently close to (but not equal to) a.

Example

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



from the left $x < 0$

$\lim_{x\to 0} f = -1$

but from the right $\lim_{x\to 0} f(x) = 1$

So is the limit -1 or $+1$?

Neither. we say the limit does not xist

Right hand limit

$$\lim_{x \to a^+} f(x)$$

left handed limit

$$\lim_{x \to a^-} f(x)$$

Theorem

Let f be a function that is defined for all values of x close to (except possibly a). Then

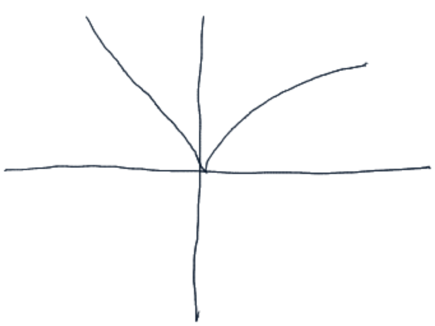
$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^-} f(x) = L$$

Ex
$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = 0$$

$$\lim_{x \to 0^-} f(x) = 0$$

$$\Rightarrow \lim_{x \to 0} f(x) = 0$$



Ex
$$\lim_{x \to 2} 2x + 1 = 2(2) + 1 = 5$$

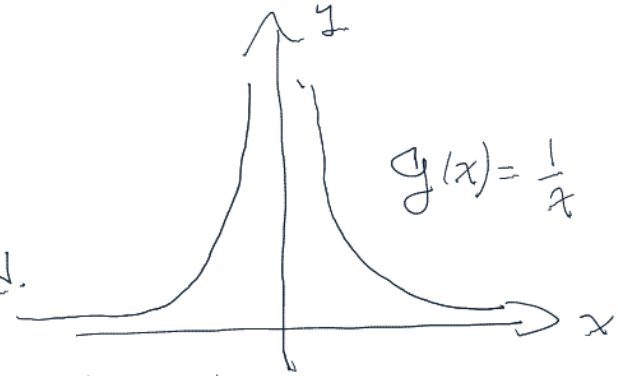
* if the function does not have a discontinuity the limit is just the function evaluated at $x = a$ *

$$\lim_{x \to 0} g(x) = ?$$

$$\lim_{x \to 0^+} g(x) = \infty$$

$$\lim_{x \to 0^-} g(x) = \infty$$

$=$ undefined.



g increases without bount as $x \to 0$

Properties of limits if $\lim_{x \to a} f(x) = L$ $\lim_{x \to a} g(x) = M$

1) $\lim_{x \to a} [f(x)]^r = [\lim_{x \to a} f(x)]^r = L^r$ $r \Rightarrow$ constan

2) $\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x) = cL$ c a real number

3) $\lim_{x \to a} [f(x) \mp g(x)] = \lim_{x \to a} f(x) \mp \lim_{x \to a} g(x) = L \pm M$

4) $\lim_{x \to a} [f(x) g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)] = [L][M] = LM$

5) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ $M \neq 0$

Ex/ $\lim_{x \to 2} x^3 = [\lim_{x \to 2} x]^3 = [2]^3 = 8$, $\lim_{x \to 3} 2x^3 \sqrt{x^2+7}$

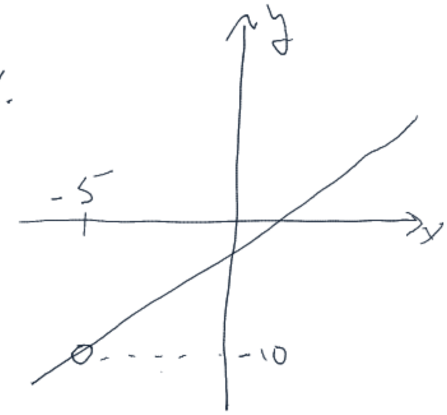
$= 2[\lim_{x \to 3} x]^3 \cdot [\lim_{x \to 3} x^2+7]^{1/2}$

$= 2(3)^3 \cdot [16]^{1/2}$

$= 2 \cdot [27] \, 4 = 216$

Ex $\lim\limits_{x\to -5} \dfrac{x^2-25}{x+5} = \dfrac{0}{0}$ indeterminant.

$= \lim\limits_{x\to -5} \dfrac{(x-5)\cancel{(x+5)}}{\cancel{(x+5)}} = \lim\limits_{x\to -5} x-5 = -10$



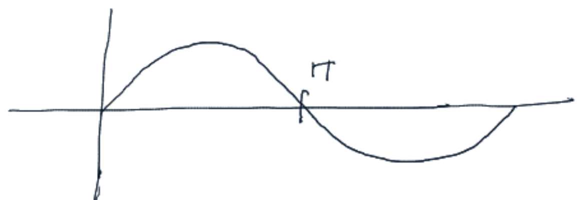
For indeterminant forms.

① relace function with an appropriate one that takes on the same values as the original ~~as~~ everywhere except a.

② evaluate lim $x \to a$.

Ex $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2} = \lim\limits_{x\to 4} \dfrac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}{\cancel{\sqrt{x}-2}} = \lim\limits_{x\to 4} \sqrt{x}+2 = 4$

$\lim\limits_{x\to \pi} \sin(x)$



$= \lim\limits_{x\to \pi} \sin(x) = \sin(\pi) = 0$

$\lim\limits_{x\to \frac{\pi}{4}} \cos(x) = \cos\left(\frac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

$\lim\limits_{x\to \pi} \tan(x) = \lim\limits_{x\to \pi} \dfrac{\sin(x)}{\cos(x)} = \dfrac{0}{\cos(\pi)} = \dfrac{0}{-1} = 0$

$\lim\limits_{x\to \frac{\pi}{2}} \tan(x) = \lim\limits_{x\to \pi} \dfrac{\sin(\bar{x})}{\cos(x)} = \dfrac{1}{0} \Rightarrow$ undefined.

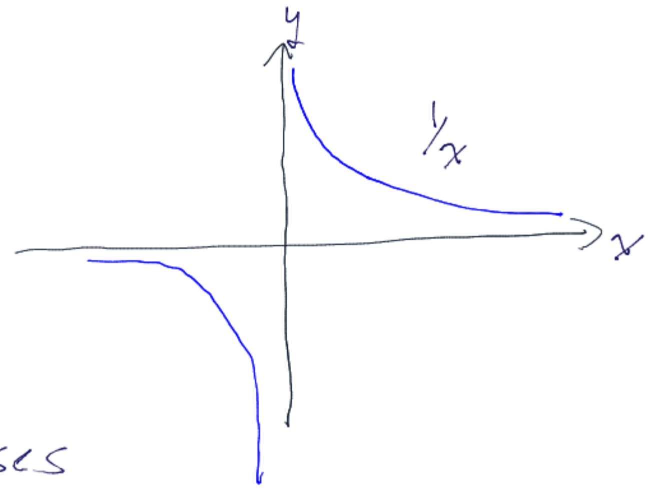
$\lim\limits_{x\to \frac{\pi}{2}} \cot(x) = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos(x)}{\sin(x)} = \dfrac{0}{1} = 0$

Ex $\lim_{x\to 2} \sqrt{\dfrac{2x^3+4}{x^2+1}}$ $\lim_{x\to 3} \dfrac{x\sqrt{x^2+7}}{2x-\sqrt{2x+3}}$

Ex $\lim_{x\to 0} \dfrac{x^2-x}{x}$ $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$

∞ limits

Consider $f(x) = \frac{1}{x}$



what happens as x gets large?

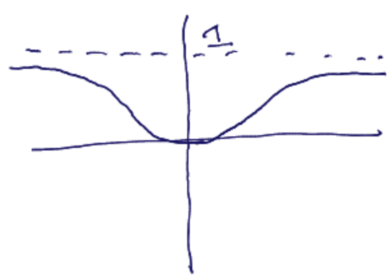
as x increases $\frac{1}{x}$ decreases

$x\to\infty$ $\frac{1}{x}\to 0$

define the ∞ limit $\lim_{x\to\infty} \frac{1}{x} = 0$

what if $x\to-\infty$ $\lim_{x\to-\infty} \frac{1}{x} = 0$

$f(x) = \dfrac{x^2}{x^2+1}$



$\lim_{x\to\infty} \dfrac{x^2}{x^2+1} = 1$

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{x^2\cdot\frac{1}{x^2}}{x^2+1\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0}$$

$= 1$

Ex $\lim_{x \to \infty} \dfrac{3x^3 + x^2 + 1}{x^3 + 1}$

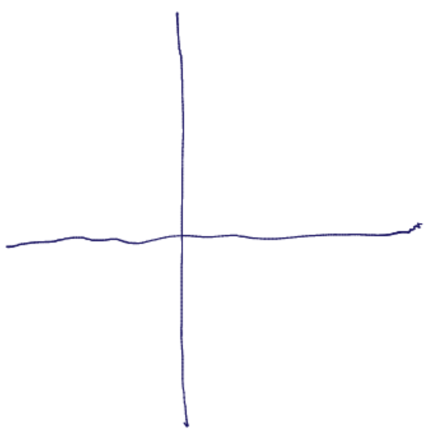
$\lim_{x \to \infty} \dfrac{2x^2 - 1}{x^3 + x^2 + 1}$

Ex $\lim_{x \to 0} \dfrac{\sqrt{1+h} - 1}{h}$

$$= \lim_{x \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h} + 1)}$$

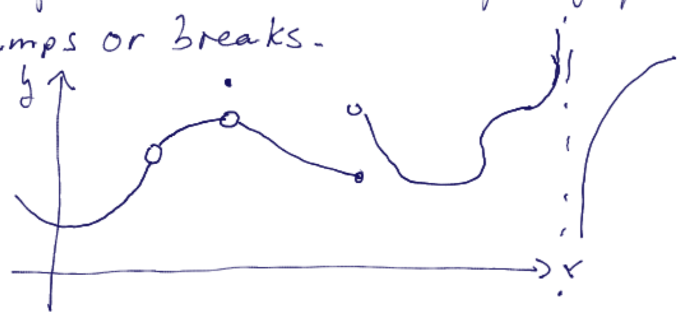
$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1}$$

$$= \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$



Continuous Functions

a function is continuous if its graph is devoid of holes, gaps jumps or breaks.



Continuity of a function at $x = a$

① $f(a)$ is defined ② $\lim_{x \to a} f(x)$ exists $\lim_{x \to a} f(x) = f(a)$