

# **Applications of Analytic and Geometric Methods to Nonlinear Differential Equations**

Edited by

**Peter A. Clarkson**

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# **Applications of Analytic and Geometric Methods to Nonlinear Differential Equations**

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# **Applications of Analytic and Geometric Methods to Nonlinear Differential Equations**

edited by

**Peter A. Clarkson**

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Exeter, U.K.



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## PREFACE

The Advanced Research Workshop, held on the St. Luke's Campus at the University of Exeter, U.K., between 14th and 19th July 1992, brought together sixty scientists from fourteen countries; researchers involved with geometric and analytical aspects of integrable and near-integrable systems and their applications. This book is the proceedings of this workshop.

In the context of integrable systems, two different approaches have attracted considerable attention during the past twenty years or so. The Inverse Scattering Transform (IST), using complex function theory, has been employed to solve many physically significant equations, namely the "soliton" equations such as the Korteweg-de Vries, nonlinear Schrödinger, Kadomtsev-Petviashvili, Davey-Stewartson, Toda Molecule and Painlevé equations. On the other hand twistor theory, using differential geometry, has been used to solve the Self-dual Yang-Mills (SDYM) equations, a four-dimensional system having important applications in Mathematical Physics. Both soliton equations and the SDYM equations have rich algebraic structures which have been extensively studied. Recently, Ward has conjectured that in some sense all soliton equations arise as special cases of the SDYM equations. Subsequently many have been discovered as both exact and asymptotic reductions of the SDYM equations, including all those mentioned above and the Chazy equation, a third order ordinary differential equation possessing solutions with movable natural boundaries in the complex plane which arises in various branches of mathematics, including number theory. Consequently what seems to be emerging is that a natural, physically significant system such as the SDYM equations provides the basis for a unifying framework underlying the class of integrable systems, that is, "soliton" systems. One of the principal objectives of the workshop was to stimulate further investigations on the reduction of the SDYM equations to soliton equations and the relationship between the IST and Twistor methods. It was particularly pleasing that groups of researchers intermingled easily, showing genuine interest in the work of other groups and many speakers avoided using "jargon" as much as possible so as to make their talks more accessible to a wider audience.

The majority of nonlinear evolution equations are non-integrable and so asymptotic, numerical, perturbation and symmetry reduction techniques are often used to study such equations. A second objective of the workshop was to stimulate interactions between those involved with integrable and non-integrable systems. In particular, there has been great interest in low dimensional systems which exhibit chaotic dynamics. It turns out that many near-integrable soliton systems possess similar characteristics. In fact perturbed soliton (that is integrable) systems appear to be driven by only a finite number of participating modes and the analogy with finite dimensional systems is striking.

The close relationship between nonlinear PDES which possess the so-called Painlevé property and those solvable by inverse scattering, first observed by Ablowitz and Segur in 1977, is of great current interest. Painlevé analysis provides a valuable first test as to whether a given PDE is solvable by inverse scattering and also yields other useful information about the PDE such as the existence of soliton solutions, an infinite number of symmetries and conservation laws, Bäcklund transformations, Lax pairs, rational solutions. Though not entirely foolproof Painlevé analysis has also yielded important information about non-integrable equations, such as special solutions.

The use of symmetry reduction techniques to many mathematical and physical applications provides a method for obtaining exact analytical and special solutions of nonlinear partial differential, ordinary differential and differential-difference equations and has a long history dating back to Sophus Lie in the nineteenth century. Symmetry reductions are usually obtained either by seeking a solution in a special form or more generally, by exploiting symmetries of the equation, and are typically obtained in terms of solutions of lower dimensional equations. Symmetry methods do not depend upon whether or not the equation is "integrable", in any sense of the word.

The theme of the first day of the Workshop was reductions of the SDYM equations and twistor theory; this is the subject matter of the first chapter of these proceedings. The theme of the second and third days of the Workshop was soliton theory and near-integrable equations which is subject matter of the second chapter of these proceedings. The theme of the final day of the Workshop was applications of Painlevé analysis and symmetries of nonlinear partial differential equations which are the subject matters of third and fourth chapter of these proceedings. The final session of the Workshop was a general discussion on "Open Problems" led by Professor M.D. Kruskal (Rutgers University).

I would like to thank NATO (Scientific Affairs Division, grant ARW 911078) and the University of Exeter Research Committee for their generous support of the workshop. I also thank the other Members of the International Scientific Committee, Mark Ablowitz (University of Colorado at Boulder, U.S.A.) and Paolo Santini (University of Rome, Italy). For their considerable assistance with the local organization of the workshop I thank the other members of the Organizing Committee, Andrew Bassom and Stephen Maskell (both of the University of Exeter, U.K.), and Andrew Hicks, Simon Hood, David Ludlow and Alice Milne (all of the University of Exeter, U.K.) for their help during the workshop. Finally, I'm grateful to Andrew Bassom and Elizabeth Mansfield (both of the University of Exeter, U.K.) for their assistance with the production of these proceedings and to the Program in Applied Mathematics, University of Colorado at Boulder for their hospitality, where they were completed.

Peter Clarkson,  
Boulder, April 1993

# SDYM HIERARCHIES AND CLASSICAL SOLITON SYSTEMS

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**ABSTRACT.** We introduce a hierarchy of integrable equations in which the self-dual Yang-Mills equation is embedded naturally. This hierarchy reduces to the known soliton hierarchies in 1+1 and 2+1 dimensions when appropriate coordinate symmetries are imposed. Examples are given to illustrate the reduction scheme.

## 1. Introduction

The self-dual Yang-Mills (SDYM) equation being a completely integrable system in four dimensions admits many special features [1] observed in other integrable systems in 1+1 and 2+1 dimensions. This note discusses the construction of an infinite sequence of integrable equations of which the SDYM equation is the first member. All members of this sequence have the same underlying linear spectral problem, and the higher flows can be derived from the infinite sequence of nonlocal conservation laws associated with the original SDYM equation. Therefore this system is to be regarded as a SDYM hierarchy in the same spirit as other soliton hierarchies in lower dimensions. To emphasize this issue, we impose symmetry reduction to this hierarchy and derive from it some of the well known integrable hierarchies in 1+1 and 2+1 dimensions. The rest of the paper is organized in four short sections. In §2 we briefly outline some geometrical features of self-duality which motivates the existence of such hierarchies. In §3 we construct the SDYM hierarchy explicitly and discuss some of its properties. §§4 and 5 are devoted to the symmetry reductions of this hierarchy to 1+1 and 2+1 dimensions respectively. In both cases we consider the  $N$ -wave hierarchy as an illustrative example.

## 2. Preliminaries

We recall briefly some standard features of the SDYM equations in a complex four-manifold  $\mathbb{C}^4$ . Let us introduce coordinates  $y, \bar{y}, z, \bar{z}$  in  $\mathbb{C}^4$  and a metric  $ds^2 = dy d\bar{y} + dz d\bar{z}$ . (The real Euclidean metric is obtained by choosing a real section of  $\mathbb{C}^4$  where  $\bar{y}, \bar{z}$  are complex conjugates of  $y$  and  $z$  respectively). Let  $\mathcal{A}$  be the connection associated with the Yang-Mills vector bundle with structure group  $G$  over  $\mathbb{C}^4$ .  $\mathcal{A}$  is a  $g$ -valued one-form,  $g$  being the Lie algebra of  $G$  and the curvature two-form is given by  $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ . Introducing a

Hodge-star operator  $*$  with respect to the metric, the SDYM equations can be written as

$$\mathcal{F} = * \mathcal{F}$$

In terms of the coordinates in  $\mathbb{C}^4$  they take the explicit form

$$\mathcal{F}_{yz} = \mathcal{F}_{\bar{y}\bar{z}} = \mathcal{F}_{y\bar{y}} + \mathcal{F}_{z\bar{z}} = 0 \quad (1)$$

Equation (1) can be interpreted as the vanishing of the curvature  $F$  restricted to the anti self-dual two-planes in  $\mathbb{C}^4$  defined by:  $\omega_1 = \bar{z} - \lambda y = \text{constant}$  and  $\omega_2 = \bar{y} + \lambda z = \text{constant}$  where  $\lambda \in \mathbb{CP}^1$  is a complex parameter. In terms of the coordinates on the planes by  $\bar{\omega}_1$  and  $\bar{\omega}_2$ , complimentary to  $\omega_1$  and  $\omega_2$ , (1) takes the form of a single zero curvature condition, namely  $\mathcal{F}_{\bar{\omega}_1\bar{\omega}_2} = 0$ . This condition implies that the  $YM$  vector bundle over the space of the anti self-dual two-planes can be trivialized. That is, there exist functions  $\psi(\mathbf{x}, \lambda)$ , local sections of the vector bundle, such that

$$D_1\psi = A_1\psi \quad (2a)$$

$$D_2\psi = A_2\psi \quad (2b)$$

where  $D_1 \equiv D_{\bar{\omega}_1} = \partial_y + \lambda \partial_{\bar{z}}$ ,  $D_2 \equiv D_{\bar{\omega}_2} = \partial_z - \lambda \partial_{\bar{y}}$  (subscripts denoting partial derivatives) and  $A_1 \equiv A_{\bar{\omega}_1} = A_y + \lambda A_{\bar{z}}$ ,  $A_2 \equiv A_{\bar{\omega}_2} = A_z - \lambda A_{\bar{y}}$ . This linear system [2] can be treated as the Lax pair for SDYM equation since their compatibility conditions yield (1). We need at least two solutions of (2) since a single function  $\psi(\mathbf{x}, \lambda)$  is not analytic in the entire  $\mathbb{CP}^1$ . Let us choose a covering of  $\mathbb{CP}^1$  given by the open sets  $U_0(\lambda \neq \infty)$  and  $U_\infty(\lambda \neq 0)$  and let  $\psi_0$  and  $\psi_\infty$  be two solutions of (2) which are analytic in  $U_0$  and  $U_\infty$  respectively. Then the transition function given by the ratio

$$P = \psi_\infty^{-1} \psi_0 \quad (3)$$

for  $\lambda \in U_0 \cap U_\infty$  satisfies  $D_1 P = D_2 P = 0$  so that  $P$  is a function of  $\omega_1, \omega_2$  and  $\lambda$ , the coordinates labeling the anti self-dual two-planes.

Conversely, given  $P(\omega_1, \omega_2, \lambda)$  over  $U_0 \cap U_\infty$ , the triviality condition of the  $YM$  bundle is equivalent to the factorization of  $P$  as in (3). Furthermore, since  $P$  is annihilated by the operators  $D_1$  and  $D_2$ , we obtain the  $g$ -valued functions  $A_i(\mathbf{x}, \lambda)$ ,  $i = 1, 2$ , defined over all of  $\mathbb{CP}^1$  by

$$A_1 = D_1 \psi_0 \psi_0^{-1} = D_1 \psi_\infty \psi_\infty^{-1}$$

$$A_2 = D_2 \psi_0 \psi_0^{-1} = D_2 \psi_\infty \psi_\infty^{-1}$$

$A_i$  is regular in  $U_0$  whereas  $\lambda^{-1} A_i$  is regular in  $U_\infty$ , so by an extension of Liouville's theorem the  $A_i$ 's are linear in  $\lambda$ . Thus we recover the SDYM gauge potentials given in (2). This is a brief review of essentially the twistor construction for the SDYM equation [3] apart from some mathematical formalities.

### 3. Hierarchies

The hierarchies are obtained by a generalization of the twistor construction discussed in the earlier section. We consider again a set of two-planes in  $\mathbb{C}^4$  defined by:  $\omega_1 = \bar{z} - \lambda y = \text{constant}$  and  $\omega_k = \bar{y} + \lambda^{k-1} z_k = \text{constant}$  for  $k \geq 2$ . (Note that for  $k = 2$ ,  $z_2 = z$ , these are

precisely the anti self-dual planes of §2.) The space of all such two-planes labeled by  $\omega_1, \omega_k$  and  $\lambda$  forms a complex line bundle of chern class  $k \geq 2$  over  $\mathbb{CP}^1$ . The condition that a holomorphic vector bundle (cf. the YM vector bundle of §2) is trivial over this space implies that the transition function  $P(\omega_1, \omega_k, \lambda)$  is factorizable as in (3). The  $g$ -valued functions  $\mathcal{A}_1$  and  $\mathcal{A}_k$  can be constructed in a similar way since  $P(\omega_1, \omega_k, \lambda)$  is anihilated by the operators  $D_1 = \partial_y - \lambda \partial_{\bar{z}}$  and  $D_k = \partial_{z_k} - \lambda^{k-1} \partial_{\bar{y}}$  so that

$$\mathcal{A}_1 = D_1 \psi_0 \psi_0^{-1} = D_1 \psi_\infty \psi_\infty^{-1}$$

$$\mathcal{A}_k = D_k \psi_0 \psi_0^{-1} = D_k \psi_\infty \psi_\infty^{-1}$$

are defined over all of  $\mathbb{CP}^1$ . By an extension of Liouville's theorem  $\mathcal{A}_1$  is linear in  $\lambda$  whereas  $\mathcal{A}_k$  is a polynomial in  $\lambda$  of order  $k - 1$  with  $g$ -valued coefficients. Alternatively, one can start with the zero curvature condition  $\mathcal{F}_{\bar{\omega}_1 \bar{\omega}_k} = 0$ , where  $\bar{\omega}_1$  and  $\bar{\omega}_k$  are the coordinates of the planes defined above. This is equivalent to the fact that the sections of the holomorphic vector bundle are covariantly constant with respect to the connections  $\mathcal{A}_1, \mathcal{A}_k$ :

$$D_1 \psi = \mathcal{A}_1 \psi \tag{2a'}$$

$$D_k \psi = \mathcal{A}_k \psi \tag{2b'}$$

for  $k \geq 2$ . The linear system (2') is the Lax pair of the SDYM hierarchy whose compatibility conditions

$$\mathcal{F}_{\bar{\omega}_1 \bar{\omega}_k} \equiv D_1 \mathcal{A}_k - D_k \mathcal{A}_1 - [\mathcal{A}_1, \mathcal{A}_k] = 0 \tag{4}$$

are the generalized SDYM equations.

REMARK 1.

- (a) Equation (4) is a polynomial of order  $\lambda^k$  whose coefficients are PDEs in the variables  $y, \bar{y}, \bar{z}$  and  $z_k$  for each  $k \geq 2$ .
- (b) The “time” variable  $z_k$  enters only in the coefficient of  $\lambda^0$  as  $\partial_{z_k} \mathcal{A}_y = \partial_y \mathcal{A}_k^{(0)} - [\mathcal{A}_y, \mathcal{A}_k^{(0)}]$  where  $\mathcal{A}_k^{(0)}$  is the constant coefficient in the polynomial  $\mathcal{A}_k(\lambda)$ .
- (c) The remaining equations in the expansion of (4) suggest that  $\mathcal{A}_k$  is related to a certain function  $\Phi(\mathbf{x}, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} \Phi_n(\mathbf{x})$ , as  $\mathcal{A}_k = (\lambda^{k-1} \Phi)_+$  where “+” indicate the polynomial part of the expansion in  $\lambda$ . Additionally, the function  $\Phi$  satisfies

$$D_1 \Phi = [\mathcal{A}_1, \Phi] - \partial_{\bar{y}} \mathcal{A}_1 \tag{5}$$

and is related to the solution  $\psi(\mathbf{x}, \lambda)$  of (2') via a “intertwining” or “dressing” transformation.

A brief outline of this procedure is now given (see [4] for details). Consider a solution  $\psi(\mathbf{x}, \lambda)$  of (2') which admits the factorization  $\psi = \mathbf{W} \psi_0$  where  $\mathbf{W} = \mathbf{I} + \sum_{n=1}^{\infty} \lambda^{-n} \mathbf{W}_n(\mathbf{x})$  and  $\psi_0$  satisfies  $D_1 \psi_0 = \lambda \tilde{\mathcal{A}}_1 \psi_0$ ,  $D_k \psi_0 = -\lambda^{k-1} \tilde{\mathcal{A}}_k \psi_0$  for  $g$ -valued functions  $\tilde{\mathcal{A}}_1$  and  $\tilde{\mathcal{A}}_k$ , which are constant with respect to the coordinates in  $\mathbb{C}^4$  and  $[\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_k] = 0$ .  $\tilde{\mathcal{A}}_1$  and  $\tilde{\mathcal{A}}_k$  are trivial solutions for the generalized SDYM equations since the corresponding curvature  $\tilde{\mathcal{F}} = D_1(-\lambda^{k-1} \tilde{\mathcal{A}}_k) - D_k(\lambda \tilde{\mathcal{A}}_1) + \lambda^k [\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_k]$  is identically zero (cf. (4)). In particular, when  $k = 2$  one can make an appropriate gauge choice for the SDYM connection components such that  $\tilde{\mathcal{A}}_1 = \mathcal{A}_{\bar{z}}$  and  $\tilde{\mathcal{A}}_2 = \mathcal{A}_{\bar{y}}$ . Substituting the above factorization for  $\psi(\mathbf{x}, \lambda)$  in (2')

and using the equations for  $\psi_0$  one can express the nontrivial connection components  $\mathcal{A}_1$  and  $\mathcal{A}_k$  as

$$\begin{aligned}\mathcal{A}_1 &= \mathbf{W}(\lambda \tilde{\mathcal{A}}_1) \mathbf{W}^{-1} + D_1 \mathbf{W} \mathbf{W}^{-1}, \\ \mathcal{A}_k &= \mathbf{W}(-\lambda^k \mathcal{A}_k) \mathbf{W}^{-1} + D_k \mathbf{W} \mathbf{W}^{-1}.\end{aligned}$$

Hence  $\mathcal{A}_1$  and  $\mathcal{A}_k$  are  $\lambda$ -dependent gauge transforms of the trivial connection components  $\tilde{\mathcal{A}}_1$  and  $\tilde{\mathcal{A}}_k$  respectively. Furthermore,  $\mathbf{W}(\mathbf{x}, \lambda)$  plays the role of an “intertwiner” between the trivial and nontrivial potentials. Since  $\mathcal{A}_k$  is a polynomial in  $\lambda$ , its expression given above can be decomposed into Laurent and polynomial parts in  $\lambda$  to yield  $\mathcal{A}_k = (\lambda^{k-1} \Phi)_+$  and  $\partial_{z_k} \mathbf{W} \mathbf{W}^{-1} = -(\lambda^{k-1} \Phi)_-$  where

$$\Phi = \sum_{n=1}^{\infty} \frac{\Phi_n(\mathbf{x})}{\lambda^n} = -\partial_{\bar{y}} \mathbf{W} \mathbf{W}^{-1} + \mathbf{W} \tilde{\mathcal{A}}_k \mathbf{W}^{-1} \quad (6)$$

with  $\Phi_0 = \tilde{\mathcal{A}}_k$  where the subscripts “+” and “−” denote polynomial and Laurent parts respectively. It can be easily verified that  $\Phi(\mathbf{x}, \lambda)$  defined in (6) satisfies (5) as claimed in Remark 1.

#### REMARK 2.

- (a) The construction of such generalized SDYM hierarchies is not unique and several such extensions of the SDYM equations are possible. The distinguishing feature of these equations is that the first equation of the linear system (2') is *common* to all members of the hierarchy. Moreover, only the “time” variable  $z_k$  is different from one member of the hierarchy to the next. This particular hierarchy is motivated from the soliton type hierarchies in 1 + 1 and 2 + 1 dimensions where all members share the same spectral (eigenvalue) problem.
- (b) It is possible [4] to define a common wave function  $\psi(y, \bar{y}, \bar{z}, z_2, z_3, \dots, \lambda)$  depending on multi “time” variables for the hierarchy satisfying  $D_k \psi = \mathcal{A}_k \psi$  for all  $k \geq 1$ . The corresponding flows commute in the sense that  $[D_k - \mathcal{A}_k, D_\ell - \mathcal{A}_\ell] = 0$  for  $k, \ell \geq 1$ .
- (c) The dressing procedure outlined earlier provides a recipe to recover formally the generalized SDYM potentials  $\mathcal{A}_k$  from the intertwiner  $W$  or equivalently from  $\Phi(\mathbf{x}, \lambda)$ . Substituting the expansion for  $\Phi(\mathbf{x}, \lambda)$  in (5) one obtains the following recurrence relations for the coefficients  $\Phi_n$ ,  $n \geq 0$

$$(\partial_{\bar{z}} - \text{ad} \tilde{\mathcal{A}}_1) \Phi_{n+1} = -(\partial_y - \text{ad} \mathcal{A}_y) \Phi_n - (\partial_{\bar{y}} \mathcal{A}_y) \delta_{n,0}.$$

Thus, for a given trivial solution  $\tilde{\mathcal{A}}_1$  and  $\tilde{\mathcal{A}}_k$ , these can be solved recursively with the initial condition  $\Phi_0 = \tilde{\mathcal{A}}_k$  and the  $\Phi_n$ 's can be expressed as functions, in general nonlocal, of  $\mathcal{A}_1$  and its derivatives. Once we have solved for  $\Phi$ , the  $\mathcal{A}_k$ 's can be obtained via  $\mathcal{A}_k = (\lambda^{k-1} \Phi)_+$ . The nonlocal expression

$$\Phi_{n+1} = -(\partial_{\bar{z}} - \text{ad} \tilde{\mathcal{A}}_1)^{-1} (\partial_y - \text{ad} \mathcal{A}_y)(\Phi_n)$$

can be regarded as a formal recursion operator for the SDYM heirarchy. However upon reduction by imposing further coordinate symmetries, this formal operator leads to the appropriate recursion operators for 1 + 1- and 2 + 1-dimensional soliton systems.

#### 4. 1+1-dimensional Reductions

The symmetry reductions of the SDYM equations to integrable systems in one space and one time dimensions have been widely studied over the past few years. Although a systematic analysis is still lacking, significant work by several authors [5,6] has revealed that essentially all the well known 1 + 1-dimensional soliton type equations are embedded in the SDYM equations. These equations are obtained from the SDYM equations by proper choices of the signature of the underlying metric, the gauge group and by factoring out the appropriate coordinate symmetries. This reduction scheme can be extended to obtain the entire hierarchy of integrable equations. Specifically, if we apply the same reduction procedure which yields any 1 + 1-dimensional soliton equation (e.g.,  $N$ -wave, KdV equations etc.) to the generalized SDYM hierarchy introduced in §3, then the resulting system will be precisely the corresponding hierarchy. We illustrate this result with an example.

##### 4.1. THE $N$ -WAVE EQUATIONS

Let us first briefly review [7] the SDYM reduction to the 1 + 1-dimensional  $N$ -wave system. Consider the SDYM equations with the gauge group  $SL(n)$  where the connection is independent of the variables  $\bar{y}$  and  $\bar{z}$ . In addition, let us also choose a gauge where  $A_{\bar{z}} \equiv A$ ,  $A_{\bar{y}} \equiv -B$  are constant and diagonal and where  $A_y \equiv U$ ,  $A_z \equiv V$  are off-diagonal  $sl(n)$  matrices. Then the linear system (2) takes the form

$$\begin{aligned}\partial_y \psi &= (U + \lambda A)\psi \\ \partial_z \psi &= (V + \lambda B)\psi\end{aligned}$$

whose compatibility conditions are given by

$$\begin{aligned}[A, V] &= [B, U] \\ \partial_z U &= \partial_y V - [U, V]\end{aligned}$$

The first condition implies that  $U$  and  $V$  are related as  $V = L(U)$  and  $L = (\text{ad}B)^{-1}(\text{ad}A)$  where  $\text{ad}X(Y) \equiv [X, Y]$ . Then the second equation yields the “time” evolution for the matrix  $U$

$$\partial_z U = (\partial_y - \text{ad}U)L(U)$$

Finally, if  $U \in su(n)$ , then the above equation determines the nonlinear interaction between  $N = \frac{1}{2}n(n-1)$  “elementary waves”. We will now apply this reduction to the SDYM hierarchy.

Since the  $\bar{y}$  derivative drops out of (6),  $\Phi(x, \lambda)$  becomes simply  $\Phi = WBW^{-1} = B + \sum_{n=1}^{\infty} \lambda^{-n} \Phi_n$  where we have chosen  $\tilde{A}_k = B$  to be a diagonal matrix for simplicity. Next from (5) one can easily set up the recurrence relations among the coefficients  $\Phi_n$  ( $\Phi_0 = B$ ) as

$$(\partial_y - \text{ad}U)\Phi_n = \text{ad}A(\Phi_{n+1}), \quad n \geq 0$$

To obtain the coefficients explicitly from this recurrence relation one has to invert the operator  $\text{ad}A$  appropriately. Separating the above equation into diagonal and off-diagonal parts, one finds that  $(\text{ad}A)^{-1}$  determines the off-diagonal part of  $\Phi_{n+1}$  whereas the diagonal

part of  $\Phi_{n+1}$  is obtained by applying the operator  $\partial_y^{-1}\text{adU}$  to its off-diagonal part. This results in the  $N$ -wave recursion operator  $\Lambda$  such that  $\Phi_{n+1} = \Lambda(\Phi_n)$ . The  $\Phi_n$ 's are local functions expressible in terms of the potential  $\mathbf{U}$  and its  $y$  derivatives. Hence the operator  $\partial_y^{-1}$  appearing in  $\Lambda$  acts on total derivatives and can be canonically inverted. The local nature of the  $\Phi_n$ 's can be easily seen if we decompose  $\mathbf{W} = (\mathbf{I} + \mathbf{M})\exp(\mathbf{Z})$ , where  $\mathbf{M}$  is off-diagonal and  $\mathbf{Z}$  is diagonal. Then a similar analysis as above shows that  $\mathbf{M}(y, \lambda) = \sum_{n=1}^{\infty} \lambda^{-n} \mathbf{M}_n(y)$  is local in  $y$ , whence it follows that  $\Phi = \mathbf{W}\mathbf{B}\mathbf{W}^{-1} = (\mathbf{I} + \mathbf{M})\mathbf{B}(\mathbf{I} + \mathbf{M})^{-1}$  is also local in  $y$ .

Finally, we construct the potential  $\mathcal{A}_k = (\lambda^{k-1}\Phi)_+$  and the  $k^{\text{th}}$   $N$ -wave flow by

$$[\partial_{z_k} - \mathcal{A}_k, \partial_y - \mathbf{U} - \lambda \mathbf{A}] = 0.$$

The evolution equation can be written explicitly from the  $\lambda^0$  coefficient of the above equation

$$\partial_{z_k} \mathbf{U} = \partial_y \Phi_{k-1} - [\mathbf{U}, \Phi_{k-1}] = (\partial_y - \text{adU}) \cdot \Lambda^{k-1}(\Phi_0).$$

These are precisely the  $N$ -wave hierarchy (for  $k \geq 2$ ) for  $U \in \text{su}(n)$ .

In conclusion, we point out that all other important features of the integrable systems such as the infinite number of symmetries and conservation laws can also be derived from the SDYM hierarchy by this reduction procedure. Besides the  $N$ -wave system, this scheme also works for other 1 + 1-dimensional soliton systems, e.g., the NLS and KdV hierarchies [4].

## 5. 2+1-dimensional Reductions

The  $N$ -wave hierarchies considered in the previous section along with other 1+1-dimensional examples admit a natural generalization to the case of two spatial and one time variables. This gives rise to the well known equations like the 2 + 1-dimensional  $N$ -wave interaction, the Kadomtsev-Petviashvili (KP) and the Davey-Stewartson (DS) equations. However this extra variable is to be treated differently from the original one and this difference is central to the inverse scattering methods for the 2 + 1-dimensional equations [8]. This feature is also manifest in the SDYM reduction procedure for these systems.

Most of the 1 + 1-dimensional integrable equations obtained as SDYM reductions correspond to a finite dimensional Lie algebra. But to the best of our knowledge, such reduction schemes do not yield any of the 2 + 1-dimensional systems mentioned above. This additional variable does not originate from the SDYM coordinate variables, but from an infinite dimensional Lie algebra associated with the SDYM equations. We briefly outline an approach that allows us to obtain 2 + 1-dimensional integrable hierarchies as SDYM reductions with a special choice of an infinite dimensional Lie algebra. Let  $g$  now be the Lie algebra of formal pseudodifferential operators with respect to an auxiliary variable  $s$ , with matrix coefficients. Thus,

$$g = \left\{ \sum_{n=-\infty}^{n=\infty} a_n \partial_s^n \right\}$$

with  $a_n$  being  $n \times n$  matrix valued functions of  $y, \bar{y}, \bar{z}, z_k$  and  $s$ . The algebra  $g$  is associative with respect to the standard composition rules for pseudodifferential operators and has a natural Lie algebra structure with the commutator defined as  $[\alpha, \beta] \equiv \alpha \cdot \beta - \beta \cdot \alpha$  for  $\alpha, \beta \in g$ .

Moreover it admits the decomposition:  $g = g_+ \oplus g_-$  where  $g_+ = \left\{ \sum_{\ell=0}^k a_\ell \partial_s^\ell, \ k \text{ finite} \right\}$  and  $g_- = \left\{ \sum_{n=1}^\infty a_n \partial_s^{-n} \right\}$ . We take the SDYM potential  $\mathcal{A}_k$ 's in  $g_+$  and illustrate the reduction procedure by means of the following example.

### 5.1. THE $2 + 1$ -DIMENSIONAL $N$ -WAVE EQUATIONS

The reduction is essentially the same as in  $1 + 1$ -dimensional  $N$ -wave case and the resulting SDYM equations take the same form

$$[\partial_y - \mathbf{U} - \lambda \mathbf{B}, \partial_z - \mathbf{V} - \lambda \mathbf{A}] = 0$$

where  $\mathbf{A}, \mathbf{B}$  are constant, diagonal matrices as before. But  $\mathbf{U}$  and  $\mathbf{V}$  are now differential operators in  $g_+$  of the form  $\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 \partial_s$ ,  $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1 \partial_s$  with matrix coefficients  $\mathbf{U}_0, \mathbf{U}_1, \mathbf{V}_0$  and  $\mathbf{V}_1$ . If one sets  $\mathbf{U}_1 = \mathbf{A}$  and  $\mathbf{V}_1 = \mathbf{B}$  the resulting equations become

$$\partial_z \mathbf{U}_0 = \partial_y \mathbf{L}(\mathbf{U}_0) + \partial_s \mathbf{M}(\mathbf{U}_0) - [\mathbf{U}_0, \mathbf{L}(\mathbf{U}_0)]$$

where  $\mathbf{L}$  and  $\mathbf{M}$  are linear maps over the space of  $n \times n$  matrices defined as  $\mathbf{L} \equiv (\text{ad}\mathbf{B})^{-1}(\text{ad}\mathbf{A})$  and  $\mathbf{M}(\mathbf{U}_0) \equiv \mathbf{B}\mathbf{U}_0 - \mathbf{A}\mathbf{L}(\mathbf{U}_0)$ . When  $\mathbf{U}_0 \in \text{su}(n)$  as before, one obtains the evolution equations for  $N = \frac{1}{2}n(n-1)$  elementary waves in two spatial dimensions.

**REMARK 3.** The above choice of SDYM connection components for the  $N$ -wave case suggests that one can eliminate from the linear systems (2) the term linear in  $\lambda$  by the exponential transformation  $\psi = \hat{\psi} e^{-\lambda s}$ . It is simpler to work with the reduced linear system

$$\begin{aligned} \partial_y \hat{\psi} &= (\mathbf{U}_0 + \mathbf{A} \partial_s) \hat{\psi}, \\ \partial_z \hat{\psi} &= (\mathbf{V}_0 + \mathbf{B} \partial_s) \hat{\psi}. \end{aligned}$$

Comparing this to the linear system for  $1 + 1$ -dimensional  $N$ -wave in §4 one finds that the spectral parameter is formally replaced by the differential operator  $\partial_s$ . This is a standard feature in the transition from  $1 + 1$ -dimensional to  $2 + 1$ -dimensional systems. Next we briefly discuss the construction of the hierarchies.

It is convenient to work with the reduced linear system. We introduce the intertwiner  $\hat{\mathbf{W}} = \mathbf{I} + \sum_{n=1}^\infty \mathbf{W}_n \partial_s^{-n}$  via the dressing of the trivial potentials

$$\begin{aligned} \hat{\mathbf{W}} \cdot (\partial_y - \mathbf{A} \partial_s) \cdot \hat{\mathbf{W}}^{-1} &= \partial_y - \mathbf{U}_0 - \mathbf{A} \partial_s \\ \hat{\mathbf{W}} \cdot (\partial_{z_k} - \mathbf{B} \partial_s^{k-1}) \cdot \hat{\mathbf{W}}^{-1} &= \partial_{z_k} - \mathbf{V}_k \end{aligned}$$

for  $k \geq 2$ . The potential is constructed from the operator

$$\Phi^{(k)} \equiv \hat{\mathbf{W}} \cdot (\mathbf{B} \partial_s^{k-1}) \cdot \hat{\mathbf{W}}^{-1} = \sum_{n=0}^{\infty} \Phi_n^{(k)} \partial_s^{k-1-n},$$

$\Phi_0 = \mathbf{B}$  as  $\mathbf{V}_k = (\Phi^{(k)})_+$  where “+” is the differential part of the operator. The coefficients  $\Phi_n^{(k)}$  are obtained from the recurrence relations which follow from the equation satisfied by  $\Phi^{(k)}$ , namely

$$\partial_y \Phi^{(k)} = [\mathbf{U}_0 + \mathbf{A} \partial_s, \Phi^{(k)}].$$

Note that in contrast to the  $1 + 1$ -dimensional case the operators  $\Phi^{(k)}$  are different for each  $k$  even though they satisfy the same equation. But all other essential features have clear analogues to the  $1 + 1$ -dimensional case. The  $k^{\text{th}}$  member of the hierarchy is given by the operator equation

$$[\partial_{z_k} - \mathbf{V}_k, \partial_y - \mathbf{U}_0 - \mathbf{A}\partial_s] = 0$$

or more specifically by the coefficient of  $\partial_s^0$  as

$$\partial_{z_k} \mathbf{U}_0 = [\mathbf{B}, \Phi_k^{(k)}].$$

#### REMARK 4.

- (a) The higher members of the hierarchy ( $k > 2$ ) are nonlocal in spatial variables  $y$  and  $s$  because the coefficients  $\Phi_n^{(k)}$  are nonlocal for  $k > 2$ . This follows from a detailed analysis of the recurrence relations and will be omitted here.
- (b) One can pursue this reduction scheme involving the pseudodifferential operators further to obtain the symmetries, conservation laws and recursion operators for the  $N$ -wave hierarchy. The procedure is similar to the  $1 + 1$ -dimensional case, only the calculations become lengthier.
- (c) The reduction scheme can be applied successfully to other  $2 + 1$ -dimensional cases as well. For example [4], one obtains the KP and DS hierarchies as reduction of SDYM equations and its hierarchy when the underlying Lie algebra is infinite dimensional.

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# TWISTOR THEORY, SELF-DUALITY AND INTEGRABILITY

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**ABSTRACT.** This article describes an overview of completely integrable systems based on the self-dual Yang-Mills equations and its twistor correspondence. It also gives a brief survey of results obtained in the area. The Ward construction is briefly described. A classification of integrable systems in 2-dimensions as reductions of the self-dual Yang-Mills equations in 4-dimensions with two translational symmetries is briefly reviewed. Applications of the twistor constructions to the theory of such integrable systems are summarized and further problems are discussed.

## 1. Introduction

A basic aim of twistor theory is to obtain correspondences between solutions of physical field equations such as the Einstein vacuum equations of general relativity, and deformations of structures on twistor space, an auxiliary complex manifold. Such correspondences only exist in completely satisfactory form for the self-dual Einstein and the self-dual Yang-Mills equations. Important features of these correspondences are that each degree of freedom of the structures on twistor space corresponds to a degree of freedom of the physical field and vice versa, and that the structures on twistor space can be described by effectively free functions for general local solutions on space-time.

The original hope was that the twistor description should be the correct framework for describing basic physics, and would perhaps lead the way to the theory that unites quantum mechanics and general relativity. This is still a long way from being realized, but there have been many spin-offs from the mathematical framework. One particular spin-off, the subject of this contribution, is that the existing twistor constructions that motivated the above programme provide a paradigm for integrability and lead to a substantial unification of the theory of completely integrable systems. This overview has been developed over a number of years by many workers (Ward 1985, 1986; Hitchin 1987; Woodhouse & Mason 1988; Mason & Sparling 1989, 1992 — see also Witten 1992).

The purpose of this article is to give a brief overview of the state of play with regards to this new overview of integrable systems arising from twistor theory and self-duality. I have left out most of the technical details in what follows, but have attempted to give sufficient references to fill in the gaps.

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## 2. The twistor correspondence for the self-dual Yang-Mills equations

The self-dual Yang Mills equations are defined on  $\mathbb{R}^4$ , with coordinates  $x^a$ ,  $a = 0, 1, 2, 3$ , and metric  $ds^2 = dx^0 \cdot dx^3 + dx^1 \cdot dx^2$ . They are equations on a connection

$$D_a = \partial_a - A_a,$$

where  $\partial_a = \partial/\partial x^a$  and  $A_a = A_a(x^b) \in l$ ;  $l$  is the Lie algebra of some fixed group  $G$  which is usually a finite dimensional group of matrices. The connection  $A_a$  is defined modulo the gauge freedom:

$$A_a \rightarrow g^{-1} A_a g - g^{-1} \partial_a g, \quad g(x) \in G.$$

The self-dual Yang-Mills equations are

$$[D_0, D_2] = 0, \quad [D_1, D_3] = 0, \quad [D_0, D_3] + [D_1, D_2] = 0$$

which are the compatibility conditions

$$[D_0 + \lambda D_1, D_2 + \lambda D_3] = 0$$

for the linear system of equations

$$(D_0 + \lambda D_1)s = (D_2 + \lambda D_3)s = 0$$

where  $\lambda \in \mathbb{C}$  and  $s$  is an  $n$ -component column vector.

The Ward construction provides a 1–1 correspondence between gauge equivalence classes of solutions of the self-dual Yang-Mills equations and *holomorphic vector bundles* on an auxiliary complex manifold, a twistor space denoted  $\mathbf{PT}$ . This complex manifold is 3-dimensional and can be given complex local coordinates  $(\lambda, z_1, z_2)$ . Globally, it is a region in  $\mathbb{CP}^3$  and one generally needs at least one more coordinate chart,  $(\lambda', z'_1, z'_2) = (1/\lambda, z_1/\lambda, z_2/\lambda)$  away from  $\lambda = 0, \infty$ .

Points of space-time correspond to Riemann spheres,  $\mathbb{CP}^1$ 's embedded in  $\mathbf{PT}$  given by  $\lambda \rightarrow (\lambda, x^1 - \lambda x^0, x^3 - \lambda x^2)$  where  $\lambda$  is an affine coordinate on the Riemann sphere  $\mathbb{C} \cup \infty$  and is allowed to take on the value  $\infty$ .

Holomorphic vector bundles are geometric objects and have a variety of different descriptions. Perhaps the two most common descriptions are due to Čech and Dolbeault. In the Čech description the bundle can be characterized, in this case, by a holomorphic  $SL(N, \mathbb{C})$  valued function  $P(\lambda, z_1, z_2)$  holomorphic away from  $\lambda = 0$ . In the Dolbeault description it can be characterized by a Lie algebra-valued function  $A(\lambda, \bar{\lambda}, z_1 + \bar{z}_2)$  which deforms the  $\bar{\partial}$ -operator of the trivial holomorphic vector bundle  $\bar{\partial} = dz^1 \partial_{\bar{z}} + d\bar{z}^2 \partial_{\bar{z}^2} + d\bar{\lambda} \partial_{\bar{\lambda}}$  to  $\bar{\partial}_E = \bar{\partial} - A(\lambda, \bar{\lambda}, z_1 + \bar{z}_2) d\bar{\lambda}$ . Both  $P$  and  $A$  are effectively freely prescribable functions of three variables and are both subject to gauge freedom. They contain the data for the general local solution.

When the bundle is described in terms of a patching function,  $P$ , the solution  $A_a(x^b)$  on space-time is reconstructed purely by differentiation of the solution  $G_{\pm}$  of a parametrized Riemann-Hilbert problem

$$G_+(x^a, \lambda) = P(\lambda, x^1 - \lambda x^0, x^3 - \lambda x^2) G_-(x^a, \lambda)$$

where  $G_+$  is holomorphic on  $|\lambda| > 0$  and bounded as  $\lambda \rightarrow \infty$  and  $G_-$  on  $|\lambda| < \infty$ . (Note that the patching function  $P$  is only defined for  $\infty > |\lambda| > 0$ .)

Alternatively, when the bundle is defined by a  $\bar{\partial}$ -operator the connection  $\mathcal{A}_a$  is reconstructed from the (parametrized) solution of a linear  $\bar{\partial}$ -equation on the sphere. See for example Ward (1977) for a description in terms of patching functions, Atiyah, Hitchin & Singer (1978) (or Atiyah 1979) for a  $\bar{\partial}$ -operator description appropriate to Euclidean signature and Sparling (1991) for a  $\bar{\partial}$ -operator description in line with the above or Chakravarty, Mason & Newman (1987) for a comparison.

It is perhaps worth mentioning that this construction had a precursor, the nonlinear graviton construction for the self-dual Einstein equations, Penrose 1976.

### 3. The connection with integrable systems

The main connection arises from the fact that perhaps all completely integrable ODEs, PDEs in 1 + 1 and 2-dimensions and many PDEs in 3 dimensions are symmetry reductions of the self-dual Yang-Mills equations. There are, however, exceptions. In particular, some systems in 2 + 1-dimensions, such as the KP equations are very likely not reductions of the self-dual Yang-Mills equations. It is nevertheless the case that ‘most’ integrable systems seem to be obtainable in this way.

Integrable systems in 1 and 2 dimensions (and some in 3 dimensions) can therefore be classified as reductions of SDYM by choice of the various ingredients arising in reduction which are:

- a) Signature of metric on  $\mathbb{R}^4$ , (2,2) versus (4,0),
- b) Subgroup of the conformal group (the symmetry group of SDYM) with  $(4 - n)$ -dimensional orbits for a system in  $n$ -dimensions,
- c) Gauge group,
- d) Certain constants of integration that arise in the reduction process when one or more of the symmetries is essentially null.

How does this classification work in practice? This is a very large scheme with many subcases. To illustrate the framework I shall consider the case of 2-dimensional translation subgroups.

#### 3.1. REDUCTIONS BY 2-TRANSLATIONS.

These reductions are classified partially by the signature of the metric restricted to the 2-plane spanned by the translations:

1) **Nondegenerate cases:**

- i) (+,+) signature yields

$$\partial_x(\mathcal{J}^{-1}\partial_x\mathcal{J}) + \partial_y(\mathcal{J}^{-1}\partial_y\mathcal{J}) = 0$$

where  $\mathcal{J} = \mathcal{J}(x, y)$ .  $\mathcal{J}$  describes a harmonic map from  $\mathbb{R}^2$  into:

a)  $G_C/G$  for (++++) signature

b)  $G$  for (+ + --) signature

- ii) (+,-) signature yields

$$\partial_t(\mathcal{J}^{-1}\partial_t\mathcal{J}) - \partial_x(\mathcal{J}^{-1}\partial_x\mathcal{J}) = 0$$

where  $\mathcal{J} = \mathcal{J}(x, t) \in G$ . This is the Chiral model or ‘wave map’.

- 1a) As above, but for  $G = \text{SL}(n)$ , impose a  $\mathbb{Z}_n$  rotational symmetry in the 2-plane spanned by the translations and require also that the generator of the  $\mathbb{Z}_n$  symmetry acts on  $\mathcal{A}_a$  at the axis by conjugation with

$$\text{diag} \left( 1, \omega, \dots, \omega^{n-1} \right)$$

where  $\omega^n = 1$ . Then we obtain the extended Toda Lattice (and sine-Gordon for  $n = 2$ ).

See Ward (1985, 1986) and Hitchin (1987) for details on these reductions.

## 2) Partially degenerate case

In this case the metric on the 2-plane spanned by translations has rank 1 which requires that the signature for  $\mathbb{R}^4$  be  $(++--)$ .

Choose coordinates so that the metric is

$$ds^2 = dx^2 - dy^2 + dv \cdot dt$$

and the symmetries are  $\partial_y$  and  $\partial_v$  so that the latter is null.

The reduced Lax pair, or linear system, can then be put in the form

$$(\partial_x - A_1 + \lambda\Phi)s = 0$$

$$(\partial_t - A_2 + \lambda\partial_x)s = 0$$

Two of the field equations that arise from the compatibility conditions lead to ‘constants of integration’. These are

- i)  $\partial_x\Phi = 0$  which implies that  $\Phi$  is a function of  $t$  only. However, the components of this ( $l$ -valued) function of  $t$  persist into the final reduced equations and we must actually demand that they be  $t$ -independent if we want the reduced equation to be independent of  $t$ . So we shall assume that  $\partial_t\Phi = 0$  and regard  $\Phi$  as a constant of integration.
- ii)  $\partial_x A_1 = [\Phi, A_2]$  which implies that the image of  $A_1$  in  $l/[\Phi, l]$  is independent of  $x$ . The image of  $A_1$  in  $l/[\Phi, l]$  will also be assumed constant following the same considerations as above. Take a constant representative  $\Psi$  in  $l$  of the equivalence class determined by  $A_1$  in  $l/[\Phi, l]$ .

Reductions of this form are classified by pairs  $(\Phi, \Psi)$  modulo conjugation by the gauge group  $G$  together with the transformation  $\Psi \mapsto \Psi + [a, \Phi], a \in l$ . The two quantities can naturally be considered together as an element of the loop algebra  $\hat{l}$  of  $l$ . Write

$$\Lambda = \Phi + \lambda^{-1}\Psi \in \hat{l}.$$

Various examples are as follows:

- i)  $G = \text{SL}(2)$ ,  $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  yields the non-linear Schrödinger equations (Mason & Sparling 1989).
- ii)  $G = \text{SL}(2)$ ,  $\Lambda = \begin{pmatrix} 0 & 1/\lambda \\ 1 & 0 \end{pmatrix}$  yields the Korteweg-de Vries equations (Mason & Sparling 1989).
- iii)  $G$  general,  $\Lambda = \text{Coxeter element of } \hat{l}$  yields parts of the Drinfeld-Sokolov hierarchies (Drinfeld & Sokolov 1985; Mason & Singer, in preparation).

More general choices of  $\Lambda$  lead to generalizations of the Drinfeld & Sokolov hierarchies such as those discovered by Kac & Wakimoto (1989).

In the  $\text{SL}(2)$  case, if the constants of integration are allowed to depend on  $t$ , one can still eliminate the functions and reduce to standard KdV and non-linear Schrödinger equations if one is allowed to perform a nonlinear Galilean coordinate transformation (Mason & Sparling 1992).

## 3) Signature $(0,0)$ , A.S.D.

There are two distinct types of 2-planes on which the metric restricts to zero, self-dual and anti-self-dual. (Again this can only happen in  $(2,2)$  signature.) In this A.S.D. case the Lax pair now reduces to:

$$(\partial_x - A_1 + \lambda\Phi_1)s = 0$$

$$(\partial_x - \mathcal{A}_2 + \lambda \Phi_2) \mathbf{s} = 0.$$

- a) Use gauge freedom to set  $\mathcal{A}_1 = \mathcal{A}_2 = 0$  yields the Wess-Zumino-Witten equations (Strachan 1992).
- b) Use  $[\Phi_1, \Phi_2] = 0$  to put  $\Phi_1$  and  $\Phi_2$  both into simultaneous normal form under conjugation using the gauge freedom.
  - (i)  $\Phi_1, \Phi_2$  diagonal, assume constant and assume  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are skew yields the  $N$ -wave equations (see for example Ablowitz & Clarkson 1991 and references therein).
  - (ii)  $\Phi_1, \Phi_2$  nilpotent, assumed constant yields other parts of the Drinfeld-Sokolov hierarchies (Mason & Singer, in preparation).
- 4) **Signature (0, 0), S.D.**  
No known applications.

#### REMARKS

1. Each equation in each hierarchy of all these equations can be obtained using higher order symmetries of the self-dual Yang-Mills equations. However to do so in a uniform manner requires the use the Bogomolny hierarchy and generalizations (see Mason & Sparling 1989, 1992; Mason & Singer preprint).
2. Non-translation symmetry groups generally lead to equations that are not translation invariant although they may have some other residual symmetry such as the group preserving a hyperbolic metric (Atiyah 1987).
3. ODEs obtained by the Adler, Kostant & Symes procedure (see for example the description in the last chapter of Fadeev & Takhtajan 1987) are reductions of the Bogomolny hierarchy (Wong & Mason, preprint).
4. The various Painlevé ODEs are reductions of the  $SL(2)$  self-dual Yang-Mills equations with different choices of 3-dimensional abelian subgroups of the conformal group in 4-dimensions (Mason & Woodhouse 1992, 1993).

#### 4. Twistor theory and the theory of integrable systems

One can impose symmetries on the Ward construction for SDYM and encode the constants of integration to obtain a reduced twistor correspondence for solutions of each integrable equation. This provides a  $1 - 1$  correspondence between solutions of the equations and holomorphic vector bundles perhaps satisfying some additional conditions (associated with the constants of integration, the chosen reality structure and boundary conditions etc.) on a reduced twistor space.

These have been investigated by many authors. For one translational symmetry see Hitchin (1982, 1987) and for one rotational symmetry see Atiyah (1987), see also Jones & Tod (1985). For two translations see Mason & Sparling (1989, 1992) and Mason & Singer (preprint) for the case of 2 translations spanning a degenerate 2-plane. See Ward (1983) and Woodhouse & Mason (1989) for the case of one rotation and one translation.

After one has the abstract geometrical correspondence, there is still work to be done. Holomorphic vector bundles are geometric objects that have many different possible descriptions and in order to describe solutions using this method one must focus on the most suitable description and adapt it to the problem at hand.

There is too much material to be reviewed here in any detail, but the following general features have emerged:

- Riemann-Hilbert problems for obtaining solutions can always be obtained from the twistor correspondence via the presentation of bundles on twistor space using patching functions. The inverse scattering transform can be understood (perhaps rather naively in this context—see later) as a particular normal form for the patching function on twistor space (Mason & Sparling 1992).
- The  $\bar{\partial}$  formulations correspond to Dolbeault presentation of bundles on twistor space and lead to a solution generation procedure that requires the parametrized global solution of a  $\bar{\partial}$  equation on a sphere (see Mason, Chakravarty & Newman 1987 and references therein for a description of this method in terms of the Sparling equation).
- Neither the Riemann-Hilbert problem, nor the  $\bar{\partial}$  problem, are explicitly soluble in general. However, it is possible to write down a class of ansatze that are explicitly soluble (Ward 1981) and dense in the solution space (Ivancovitch, Mason & Newman, 1990)
- Hidden loop group symmetries can be realized as pointwise matrix multiplication of the element of the loop group with patching functions (Woodhouse 1987; Mason, Chakravarty & Newman 1988; Woodhouse & Mason 1988).
- It is often possible to obtain a good description of the complete solution space via its description as a moduli space of holomorphic vector bundles on some reduced twistor space (see for example Atiyah, Drinfeld &c. 1978, Hitchin 1982; Atiyah 1987; Woodhouse & Mason 1988; Mason & Singer, preprint).
- Certain aspects of the theory of integrable equations become perhaps less obscure in the twistor framework. A couple of examples from the theory of the KdV equations follow:

1. The  $\tau$  function  $\tau(x, t)$  for the KdV equations is described as an infinite determinant in Sato theory. It can be understood as the determinant of the  $\bar{\partial}$ -operator of the holomorphic vector bundle on twistor space restricted to the  $S^2$  in twistor space corresponding to the point  $(x, t)$  in space-time.
2. The Krichever construction of solutions in terms of flows on Jacobians of Riemann surfaces correspond to the representation of the bundle on twistor space via the process of *abelianization*. That is, the bundle is obtained as a push down of a line bundle (an abelian object) on a branched cover of twistor space.

See Mason & Singer, preprint for details.

- **The inverse scattering transform and solitonic versus radiating/dispersive solutions**

As mentioned above, the scattering data for integrable systems can be inserted into an ansatze for the patching function (Mason & Sparling 1992). However, this is not a very satisfactory understanding as patching functions are subject to gauge freedom whereas the scattering data is not. The scattering data in fact gives a precise parametrization of the solution space of the equations which is much better. This feature is tied in to the fact that one is incorporating rapidly decreasing boundary conditions.

It emerges that one can obtain much insight into this from twistor theory in the context of global solutions of the self-dual Yang-Mills equations in 4-dimensions in  $(2, 2)$  signature on  $S^2 \times S^2$ . Two special classes of global solutions are available in 4-dimensions. One is the  $(2, 2)$  signature analogue of the instanton solution. These correspond to global holomorphic vector bundles on  $\mathbb{CP}^3$ . The other is obtained from a nonlinear analogue of the Radon transform that generates a solution from a general (smooth) map from  $RP^3$  to the gauge group  $G$ . It is possible to show that the general solution

is a kind of nonlinear superposition of these two types of solution (Mason preprint). It is expected that under dimensional reduction, the instantons reduce to the soliton type solutions, while the solutions obtained from the nonlinear analogue of the Radon transform reduce to the radiative/dispersive solutions. However, in different cases, the various boundary conditions for the reduced systems may not correspond precisely to globality on  $S^2 \times S^2$ . See Mason (1992) for more details.

## 5. Conclusions and outlook

The self-dual Yang-Mills equations and twistor theory provide an effective synthesis of the theory of integrable systems in dimensions 1 and 2.

However, in  $2+1$ -dimensions and above there are many systems that are not reductions of the self-dual Yang-Mills equations. One rather trivial reason for this is that many such equations, such as various equations associated with hyper-Kahler geometry or indeed simultaneous solutions of hierarchies, involve more than 4 independent variables. Many of these do, however, have a perfectly straightforward generalization of the Ward construction. Indeed many of these equations were first written down with that in mind (Ward 1986). This leads to the following point of view:

*it is the twistor constructions that are central to the concept of integrability, not the self-dual Yang-Mills equations.*

It should also be mentioned that an important role is played by the self-dual Einstein equations and its twistor construction. This is thought to be completely integrable, but the twistor construction for it is qualitatively different. In particular, there is an ODE reduction of the self-dual Einstein equations, the Chazy equations (see Ablowitz, Chakravarty & Clarkson 1991) which is probably not obtainable from the self-dual Yang-Mills equations with a finite dimensional gauge group, owing to the fact that its solutions have movable natural boundaries. Nevertheless there are close connections. On the one hand all integrable systems in two dimensions with  $2 \times 2$  linear systems can be obtained as reductions of the self-dual Einstein equations (Mason 1990). On the other hand the self-dual Einstein equations are a reduction of the self-dual Yang-Mills equations if you allow yourself the use of an infinite dimensional gauge group (Ashtekar, Jacobson & Smolin 1988; Mason & Newman 1989).

Even so, the Kadomtsev-Petviashvili equations and the Davey-Stewartson equations, amongst others, do not appear to have a twistor correspondence of the standard type. There is still a possibility that they will be obtainable as reductions of the self-dual Einstein equations, but this may well turn out not to be possible. So, in order to extend this point of view to these systems we must develop new generalized twistor constructions based on their inverse scattering transform. One may hope that these new constructions may eventually provide new tools for realizing the original aspirations of twistor theory and provide new insight into basic physics.

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## TWISTOR THEORY AND THE SCHLESINGER EQUATIONS

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**ABSTRACT.** We show that solutions of the Schlesinger equations correspond to holomorphic vector bundles on (subsets of)  $\mathbb{CP}_n$  that are invariant under the action of the diagonal subgroup of  $GL(n+1, \mathbb{C})$ . As an application, we demonstrate that the Schlesinger equations are a reduction of the hyper-Kähler equations.

### 1. Introduction

Ablowitz and Clarkson (1991) have reviewed the important part played by the six Painlevé transcendental equations in the theory of integrability. One way in which they arise is through the isomonodromy condition for a family of linear ordinary differential equations. For example, in the case of ( $P_{VI}$ ), the linear equations are of the form

$$\frac{dy}{d\zeta} = \mathbf{F}(\zeta)y, \quad (1)$$

where  $\mathbf{F}$  is a  $2 \times 2$  matrix of rational functions of  $\zeta$  with four simple poles. By a Möbius transformation of  $\zeta$ , one can arrange that the poles are at  $\zeta = 0, 1, t, \infty$ . The problem then is to determine the dependence of  $\mathbf{F}$  on  $t$  so that the monodromy at each of the poles remains constant as  $t$  varies: this comes down to  $P_{VI}$ , which is a second-order ordinary differential equation in the independent variable  $t$  (a classic result; the details are given, for example, by Jimbo and Miwa 1981).

The Schlesinger equations are a direct generalization: the linear equation has  $n+1$  poles at  $\zeta = x_1, \dots, x_n$  and at  $\zeta = x_0 = \infty$ , and can be written in the form

$$\frac{dy}{d\zeta} = \left( \sum_{\alpha=1}^n \frac{\mathbf{A}_\alpha}{\zeta - x_\alpha} \right) y \quad (2)$$

where the  $\mathbf{A}_\alpha$  are  $N \times N$  matrices that depend on the  $x_\alpha$ , but not on  $\zeta$ . The presence of the pole at  $\infty$  is seen by replacing  $\zeta$  by  $\zeta' = \zeta^{-1}$ , so that

$$\frac{dy}{d\zeta'} = \left( \frac{\mathbf{A}_0}{\zeta'} + \mathbf{F}(\zeta') \right) y, \quad (3)$$

where  $\mathbf{A}_0 = -\mathbf{A}_1 - \dots - \mathbf{A}_n$  and  $\mathbf{F}$  is holomorphic at  $\zeta' = 0$ .

Provided that no two eigenvalues of  $\mathbf{A}_\alpha$  differ by an integer (for  $\alpha = 0, 1, \dots, n$ ), the isomonodromy condition is equivalent to the Schlesinger equations

$$\frac{\partial \mathbf{A}_\alpha}{\partial x_\beta} = \frac{[\mathbf{A}_\alpha, \mathbf{A}_\beta]}{x_\alpha - x_\beta} \quad (\alpha \neq \beta), \quad \frac{\partial \mathbf{A}_0}{\partial x_\alpha} = 0 \quad (4)$$

( $\alpha, \beta = 1, 2, \dots, n$ ). In the present context, we shall want to allow  $x_0$  to move so that all the poles are on the same footing. The equations then become

$$\frac{\partial \mathbf{A}_\alpha}{\partial x_\beta} = \frac{[\mathbf{A}_\alpha, \mathbf{A}_\beta]}{x_\alpha - x_\beta}, \quad (\alpha \neq \beta), \quad \mathbf{A}_0 + \mathbf{A}_1 + \dots + \mathbf{A}_n = 0 \quad (5)$$

( $\alpha, \beta = 0, 1, \dots, n$ ), as can be seen from the previous formulation by adding an extra pole at a finite point and then removing the pole at infinity by setting its residue to zero. This is the form in which we shall use the Schlesinger equations. That they are sufficient for isomonodromy is clear from the fact that they imply that the  $n+2$  operators

$$\frac{\partial}{\partial \zeta} - \sum_0^n \frac{\mathbf{A}_\alpha}{\zeta - x_\alpha} \quad \text{and} \quad \frac{\partial}{\partial x_\alpha} + \frac{\mathbf{A}_\alpha}{\zeta - x_\alpha} \quad (6)$$

on  $\mathbb{C}^N$ -valued functions commute in pairs. The converse is proved by assuming the existence of a family of solutions  $\mathbf{Y}(\zeta, x)$  to (2) such that each  $\mathbf{B}_\alpha = (\partial \mathbf{Y} / \partial x_\alpha) \mathbf{Y}^{-1}$  is single-valued as a function of  $\zeta$ , and then showing that  $\mathbf{B}_\alpha$  has a single simple pole.

It is known that the six Painlevé equations also arise as reductions of the self-dual Yang-Mills (SDYM) equations (Ablowitz and Clarkson 1991, Mason 1993, Ward 1990). We have shown elsewhere that they are, in effect, *equivalent* to the  $\text{SL}(2, \mathbb{C})$  SDYM equations with a three-dimensional conformal symmetry group of a certain type (Mason and Woodhouse 1992). We derived the equivalence from a twistor translation of the isomonodromy problem: an isomonodromic family of differential equations determines, and is determined by, a holomorphic bundle on twistor space ( $\mathbb{CP}_3$ ) which is symmetric under the action of a three-dimensional abelian subgroup of the projective group  $\text{PGL}(4, \mathbb{C})$ . Such a bundle determines a SDYM field with the corresponding conformal symmetries, by the Ward transform (Ward 1977). We showed that there is an analogous construction in the general case which takes invariant holomorphic bundles on  $\mathbb{CP}_n$  to isomonodromic families of linear equations with any number of poles of any orders. We also indicated that, in the case that all the poles are simple, there is an inverse transform that maps a solution of the Schlesinger equations with  $n+1$  poles to an invariant bundle on  $\mathbb{CP}_n$ . This suggests the possibility of making interesting links between the Schlesinger equations and the hyper-Kähler equations, by following a method similar to our treatment of the Painlevé equations. Our purpose here is to explain the details of the construction.

## 2. The Schlesinger equations

It follows from (5) that

$$\frac{\partial \mathbf{A}_\alpha}{\partial x_\alpha} = - \sum_{\beta \neq \alpha} \frac{[\mathbf{A}_\alpha, \mathbf{A}_\beta]}{x_\alpha - x_\beta} \quad (7)$$

(here and below the subscripts  $\alpha, \beta, \dots$  run over the range  $0, 1, \dots, n$ , but without the summation convention). Hence, after a short calculation,

$$\sum_{\beta=0}^n \frac{\partial \mathbf{A}_\alpha}{\partial x_\beta} = 0, \quad \sum_{\beta=0}^n x_\beta \frac{\partial \mathbf{A}_\alpha}{\partial x_\beta} = 0, \quad \sum_{\beta=0}^n x_\beta^2 \frac{\partial \mathbf{A}_\alpha}{\partial x_\beta} = [\mathbf{C}, \mathbf{A}_\alpha], \quad (8)$$

where  $\mathbf{C} = \sum x_\beta \mathbf{A}_\beta$ . The first two equations imply that the  $\mathbf{A}_\alpha$  are unchanged when the poles are dragged along the two vector fields  $\partial/\partial\zeta$  and  $\zeta\partial/\partial\zeta$ , the generators of the affine transformations of the  $\zeta$  plane. Hence for any constant  $\lambda \neq 0$  and  $\mu$

$$\mathbf{A}_\alpha(\lambda\mathbf{x} + \mu) = \mathbf{A}_\alpha(\mathbf{x}), \quad (9)$$

where  $\mathbf{x} = (x_0, \dots, x_n)$  and we use the notation  $\lambda\mathbf{x} + \mu$  to denote  $(\lambda x_0 + \mu, \dots, \lambda x_n + \mu)$ . The third equation determines the behaviour of the  $\mathbf{A}_\alpha$  when the poles are dragged along  $\zeta^2\partial/\partial\zeta$ , which generates the infinitesimal Möbius transformation

$$\zeta \mapsto \zeta' = \frac{\zeta}{1 - \epsilon\zeta} + O(\epsilon^2). \quad (10)$$

To the first order in  $\epsilon$ ,

$$\mathbf{A}_\alpha \mapsto A'_\alpha = \exp\{-\epsilon\mathbf{C}\} \mathbf{A}_\alpha \exp\{\epsilon\mathbf{C}\}. \quad (11)$$

One can check directly that (5) is invariant under the combination of (10) and (11).

We conclude that the Schlesinger equations are invariant under affine transformations of the  $\zeta$  plane; and that, since the three vector fields together span the Lie algebra of the Möbius group, they are invariant under general Möbius transformations provided that the transformation of the  $\zeta$  coordinate in (2) is accompanied by an appropriate linear ‘gauge transformation’  $\mathbf{y} \mapsto \mathbf{y}' = \mathbf{g}\mathbf{y}$  of the dependent variable (the matrix  $\mathbf{g}$  depends on the  $x_\alpha$ , but is independent of  $\zeta$ ). We can thus think of the linear equation (2) as a constraint on a map  $\mathbf{y}$  from the Riemann sphere, less  $n+1$  points, to an abstract  $N$ -dimensional vector space  $S$ . Given a choice for the point at  $\infty$ , the constraint takes the explicit form (2). Alternatively, we can think of the linear equation as defining a flat connection  $D$  on the trivial bundle  $(\mathbb{CP}_1 - \{x_0, \dots, x_n\}) \times S$ : its covariantly constant sections are the solutions of (2). The Schlesinger equations on  $D$  are the condition that the holonomy of the connection should remain constant as the  $x_\alpha$  vary.

**REMARK.** The holonomy of a flat connection on a complex bundle  $S$  of fibre dimension  $N$  over a connected manifold  $M$  is the homomorphism  $\pi_1(M) \rightarrow \mathrm{GL}(N, \mathbb{C})$  generated by parallel transport around the loops in  $M$ . It depends on a choice of base point  $p \in M$  and of basis in the fibre  $S_p$ , but only up to conjugation by a constant element of  $\mathrm{GL}(N, \mathbb{C})$ . When we say that the holonomy of the connection is unchanged as the connection varies, we mean that it is unchanged to within this freedom. Given an isoholonomic family of connections on  $S$  and a base point  $p \in M$ , it is possible to choose a basis in  $S_p$  for each connection so that the holonomy is actually constant, but these choices are not canonical.

### 3. Forward transform

The forward transform maps a holomorphic vector bundle  $E$  over a subset  $U \subset \mathbb{CP}_n$  to an isomonodromic family of linear differential equations on  $\mathbb{CP}_1$ , less  $n+1$  points. Let  $G \subset \mathrm{PGL}(n+1, \mathbb{C})$  be the  $n$ -dimensional abelian subgroup represented by the diagonal elements in  $\mathrm{GL}(n+1, \mathbb{C})$ . We require that

- (i)  $U$  should contain a neighbourhood of a line;
- (ii) the restriction of  $E$  to the general line  $L \subset U$  should be trivial; and
- (iii)  $U$  and  $E$  should be invariant under the action of  $G$  on  $\mathbb{CP}_n$ .

Let  $z_0, z_1, \dots, z_n$  be homogeneous coordinates on  $\mathbb{CP}_n$ . The action of  $G$  on  $\mathbb{CP}_n$  is generated by the  $n+1$  commuting vector fields

$$X_0 = z_0 \frac{\partial}{\partial z_0}, \quad X_1 = z_1 \frac{\partial}{\partial z_1}, \quad \dots, \quad X_n = z_n \frac{\partial}{\partial z_n} \quad (12)$$

(one is redundant since their sum vanishes). The invariance condition (iii) is that these should lift to vector fields  $X'_0, \dots, X'_n$  on  $E$  such that the flows of the  $X'_\alpha$  generate an action of the covering group of  $G$  on  $E$  by holomorphic bundle automorphisms. In a local trivialization,

$$X'_\alpha = X_\alpha + \sum_1^N e_i A_{\alpha j i}(z) \frac{\partial}{\partial e_j} \quad (13)$$

where the  $e_i$  are linear coordinates in the fibres,  $N$  is the fibre dimension, and the  $A_\alpha$  are  $N \times N$  matrix-valued functions of  $z$  such that  $\sum A_\alpha = 0$ .

It is in fact enough that  $E$  and the vector fields  $X'_\alpha$  should be given on a neighbourhood of a line  $L \subset \mathbb{CP}_n$ , and that the restriction  $E_L$  should be trivial. Then  $E$  has a trivial restriction to nearby lines and can be extended to an invariant open set by the action of the (covering group) of  $G$ .

The vector fields  $X_\alpha$  span the tangent space to  $\mathbb{CP}_n$  except on the coordinate hyperplanes  $z_0 = 0, z_1 = 0, \dots, z_n = 0$ . Let  $H$  be the union of these hyperplanes. The group action determines a flat holomorphic connection  $D$  on  $E_{U-H}$  such that its horizontal subspaces are spanned by the vector fields  $X'_\alpha$ . In the local trivialization,

$$Dy = \left( \frac{\partial}{\partial z_\alpha} - \frac{A_\alpha}{z_\alpha} \right) y dz_\alpha \quad (14)$$

where  $y$  is a section. Thus  $D$  has poles at points of  $H$ .

Let  $L$  be a general line  $U$ . Then  $L - H$  is a copy of the Riemann sphere, less  $n+1$  points. The restriction of  $D$  is a flat connection  $D_L$  on the trivial bundle  $E_{L-H}$ . Since the holonomy of  $D_L$  coincides with that of  $D$ , it must remain constant as  $L$  varies. Therefore  $D_L$  is an isoholonomic family of connections.

To be more explicit, first fix a point  $[b] \in U - H$  with homogeneous coordinates  $b_\alpha$ ; and pick a basis in the fibre  $E_{[b]}$ . Suppose that  $L$  passes through  $[b]$ , and that  $[a]$  is some other point of  $L$ . Then on  $L$ ,

$$z_\alpha = a_\alpha - \zeta b_\alpha, \quad (15)$$

where  $\zeta$  is a stereographic coordinate. The vector field

$$\frac{d}{d\zeta} = - \sum_0^n \frac{b_\alpha}{a_\alpha - \zeta b_\alpha} X_\alpha \quad (16)$$

is tangent to  $L$ . Therefore in a local trivialization of  $E$ ,

$$D_L y = \left( \frac{d}{d\zeta} + \sum_0^n \frac{b_\alpha A_\alpha}{a_\alpha - \zeta b_\alpha} \right) y. \quad (17)$$

Thus the connection 1-form of  $D_L$  has simple poles at the points  $\zeta = x_\alpha$ , where  $x_\alpha = a_\alpha/b_\alpha$ .

The basis in  $E_{[b]}$  determines a global trivialization of  $E_L$  for every line that passes through  $[b]$ . In such a trivialization, the covariantly constant sections of  $E_{L-H}$  are the solutions of a linear differential equation

$$\frac{dy}{d\zeta} + \sum_0^n \frac{b_\alpha A_\alpha}{a_\alpha - \zeta b_\alpha} y = 0 \quad (18)$$

where the  $A_\alpha$  depend  $x_\alpha$ , but not on  $\zeta$ , by Liouville's theorem. Since the  $D_L$  are isoholonomic, the linear equations are isomonodromic, and consequently the  $A_\alpha$ 's satisfy the Schlesinger equations (in general, i.e. provided that the  $A_\alpha$  satisfy the condition on their eigenvalues). A different choice of  $[b]$  gives the same solution because the symmetry group  $G$  acts transitively on the complement of  $H$ .

#### 4. The inverse transform

We shall now show that every solution  $A_\alpha(x)$  of the Schlesinger equations arises from this construction. Suppose that  $A_\alpha$  is defined on some open set  $R \subset \mathbb{C}^{n+1}$  which does not intersect any of the hyperplanes  $x_\alpha = x_\beta$  ( $\alpha \neq \beta$ ). Because the  $A_\alpha$  satisfy (9), there is no loss of generality in assuming that  $R$  contains  $\lambda x + \mu$  whenever it contains  $x$ , for any complex  $\lambda, \mu$  with  $\lambda \neq 0$ . By the following steps, we shall construct from these data a holomorphic bundle  $E$  on a subset of  $\mathbb{CP}_n$  that generates  $A_\alpha$ .

(A) *The double fibration.* Let  $a, bb \in \mathbb{C}^{n+1}$  and put

$$\pi_1(a, b) = \left( \frac{a_0}{b_0}, \dots, \frac{a_n}{b_n} \right) \in \mathbb{C}^{n+1}, \quad \pi_2(a, b) = [a] \in \mathbb{CP}_n. \quad (19)$$

Let  $A_\alpha(a, b) = A_\alpha(\pi_1(a, b))$ , let  $B$  be a ball in  $\mathbb{C}^{n+1}$  that does not intersect any of the coordinate hyperplanes, and let

$$W = \{(a, b) \in \mathbb{C}^{2n+2} \mid \pi_1(a, b) \in R, b \in B\}. \quad (20)$$

Finally, let  $U = \pi_2(W) \subset \mathbb{CP}_n$ . By (9),  $A_\alpha(\lambda a + \mu b, b) = A_\alpha(a, b)$  for any complex  $\lambda, \mu$  with  $\lambda \neq 0$ .

Suppose that  $(a, b) \in W$ . Then  $[\lambda a + \mu b] \in U$  provided that  $\lambda \neq 0$ . However,  $W$  is open, so  $(a, b + \epsilon a) \in W$  for sufficiently small  $\epsilon > 0$ , and therefore  $[\lambda a + \mu(b + \epsilon a)] \in U$  for any  $\mu$  and any nonzero  $\lambda$ . It follows that  $[\lambda a + \mu b] \in U$  whenever  $\lambda$  and  $\mu$  do not both vanish. Therefore  $U$  contains the line joining  $[a]$  to  $[b]$  whenever  $(a, b) \in W$ .

(B) *The construction of  $E$ .* Define  $2n + 2$  operators on  $\mathbb{C}^N$ -valued functions on  $W$  by

$$D_\alpha = a_\alpha \frac{\partial}{\partial a_\alpha} - A_\alpha(a, b) \quad \text{and} \quad D'_\beta = b_\beta \frac{\partial}{\partial b_\beta} + A_\beta(a, b) \quad (21)$$

where  $\alpha, \beta = 0, 1, 2, \dots, n$ , without summation. These commute as a consequence of the Schlesinger equation. Note that

$$\sum_0^n D_\alpha = \sum_0^n a_\alpha \frac{\partial}{\partial a_\alpha} \quad \text{and} \quad \sum_0^n D'_\alpha = \sum_0^n b_\alpha \frac{\partial}{\partial b_\alpha} \quad (22)$$

since  $\sum A_\alpha = 0$ .

We now construct  $E \rightarrow U$  by taking the fibre  $E_{[a]}$  above  $[a] \in U$  to be the space of simultaneous solutions of

$$\sum_0^n D_\alpha y = 0 \quad \text{and} \quad D'_\beta y = 0 \quad (\beta = 0, \dots, n) \quad (23)$$

on  $\pi_2^{-1}([a])$ . The equations make sense since the vectors fields  $\sum a_\alpha \partial/\partial a_\alpha$  and  $b_\beta \partial/\partial b_\beta$  are tangent to  $\pi_2^{-1}([a])$ . Because  $B$  is convex, and does not contain any points at which  $b_\alpha = 0$  for some  $\alpha$ , the solutions are single-valued. They are determined uniquely by their values at any given point of this set. Therefore  $E_{[a]}$  is  $N$ -dimensional.

(C) *Triviality on lines.* Let  $(a, c) \in W$  and let  $L$  be the line joining  $[a]$  to  $[c]$ . We know that  $L \subset U$ . We need to show that the restriction of  $E$  to  $L$  is trivial as a holomorphic bundle, which we shall do by constructing local trivializations. Let  $V$  be the complement of  $[c]$  in  $L$ : it consists of points  $[z]$  such that  $z_\alpha = a_\alpha + \zeta c_\alpha$ ,  $\zeta \in \mathbb{C}$ . We trivialize  $E_L$  by evaluating  $y$  in (23) at  $(a, c)$ .

Now choose  $\epsilon > 0$  such that  $\tilde{c} = \epsilon a + b \in B$ , and take  $\tilde{V}$  to be the complement of  $[\tilde{c}]$  in  $L$ . We can trivialize  $E_{\tilde{V}}$  in the same way by evaluating  $y$  at  $(a, \tilde{c})$ . The transition matrix  $G(\zeta)$  between these two trivializations is given solving for each fixed  $\zeta$

$$b_\alpha \frac{\partial M}{\partial b_\alpha} + A_\alpha(a + \zeta c, b)M = 0 \quad (24)$$

for a matrix-valued function of  $b$  subject to  $M = 1$  when  $b = c$ , and by putting  $G(\zeta) = M(\tilde{c})$ . By putting  $b_\beta = c_\beta + \tau a_\beta$ , this is equivalent to solving the ordinary differential equation

$$\left( \frac{d}{d\tau} + \sum_{\alpha=0}^n \frac{a_\alpha A_\alpha(a + \zeta c, \tau a + c)}{c_\alpha + \tau a_\alpha} \right) M(\tau) = 0, \quad (25)$$

subject to  $M(0) = I$ , and putting  $G(\zeta) = M(\epsilon)$ . But

$$\begin{aligned} A_\alpha(a + \zeta c, \tau a + c) &= A_\alpha((\pi_1(a + \zeta c, \tau a + c))) \\ &= A_\alpha(\zeta + (1 - \tau\zeta)x(\tau)) \\ &= A_\alpha(x(\tau)), \end{aligned} \quad (26)$$

by (9), where  $x_\beta(\tau) = a_\beta/(\tau a_\beta + c_\beta)$ . It follows that  $G(\zeta)$  is in fact independent of  $\zeta$ , and hence that  $E_L$  is trivial.

(D) *Invariance.* A local section of  $E$  is represented by a  $\mathbb{C}^N$ -valued function on a neighbourhood in  $W$  such that eqns (23) hold. We define the action of  $G$  in such a way that  $y$  is a local invariant section if, in addition,  $D_\alpha y = 0$ . Another way to put this is that  $E$  is the quotient of the trivial  $\mathbb{C}^N$  bundle over  $W$  by the flows of the vector fields

$$\sum a_\alpha \frac{\partial}{\partial a_\alpha} \quad \text{and} \quad b_\beta \frac{\partial}{\partial b_\beta} + \sum_{i,j} A_{\beta ij} y_j \frac{\partial}{\partial y_i}, \quad (27)$$

where the  $y_i$  are linear coordinates in the fibres. The  $X'_\alpha$  are the projections under the quotient map of the commuting vector fields

$$X''_\beta = a_\beta \frac{\partial}{\partial a_\beta} - \sum_{i,j} A_{\beta ij} y_j \frac{\partial}{\partial y_i}. \quad (28)$$

In this way we construct from the solution of the Schlesinger equations an invariant vector bundle on  $U$ , which we know is trivial on the open set of lines joining  $[a]$  to  $[b]$  for each  $(a, b) \in W$ . By the remarks in the previous section, we can extend  $E$  to an invariant bundle on an invariant subset of  $\mathbb{CP}_n$ , which is trivial on the general line.

If we now apply the forward transform, then we recover the original solution of the Schlesinger equations. To see this, fix  $b \in B$ . Let  $L$  be a line in  $U$  joining  $[a]$  to  $[b]$ . Then  $E_L$  is trivialized by evaluating  $y$  at  $b$  and in this trivialization, the linear equation on  $L$  constructed from  $E$  is simply (18), where  $\mathbf{A}_\alpha$  is the given solution.

**REMARK** Let  $(a, b) \in W$ . The restriction of  $E$  to a line joining  $[a]$  to  $[b]$  is the trivial bundle  $\mathbb{CP}_1 \times S$  that we considered in the first section. The role of the point  $[b]$  is to pick out the point  $\zeta = \infty$  and thus to determine a trivialization. The  $n + 1$  distinguished points on the line (the poles of the connection) are its intersections with the hyperplanes  $z_\alpha = 0$ .

## 5. The hyper-Kähler equations

The link between the isomonodromy problem and the hyper-Kähler equations is a consequence of the fact that the twistor space of the (complex)  $4k$ -dimensional flat hyper-Kähler space  $\mathbb{H}_k$  is  $\mathbb{CP}_{2k+1}$  (see, for example, Hitchin *et al.* 1987). The compactification of  $\mathbb{H}_k$  is the space of lines in  $\mathbb{CP}_{2k+1}$ . By an extension of the Ward transform, a holomorphic vector bundle on  $U \subset \mathbb{CP}_{2k+1}$  such that conditions (i) and (ii) hold corresponds to a solution of the hyper-Kähler (HK) equations on  $\mathbb{H}_k$  (these are a generalization of the SDYM equations). Consequently a solution of the Schlesinger equations with an even number of poles (i.e. with  $n$  odd) can be regarded as a solution of the HK equations with  $k = \frac{1}{2}(n - 1)$ : in the even case, the Schlesinger equations are a reduction of the HK equations, just as  $P_{VI}$  is a reduction of the SDYM equations.

Real hyper-Kähler space is a real  $4k$ -dimensional vector space on which there is given an action by the quaternions. That is, it is the space  $\mathbb{H}^k \simeq \mathbb{R}^{4k}$ . The complexification is  $\mathbb{H}_k$ , which we shall represent as  $\mathbb{C}^{2k} \otimes \mathbb{C}^2$ : in the complex, the quaternion action extends to an action by  $\text{SL}(2, \mathbb{C})$  on the second factor. The points of  $\mathbb{H}_k$  have coordinates  $w^{AA'}$  where  $A' = 0', 1'$  and  $A = 2, \dots, 2k + 1$ , and the complexified quaternion action is

$$w^{AA'} \mapsto M^{A'}_{B'} w^{AB'}, \quad (29)$$

where  $M \in \text{SL}(2, \mathbb{C})$ .<sup>1</sup> The reason for the eccentric choice for the range of  $A$  will become apparent.

The HK equations are a constraint on a connection  $D$  on a complex vector bundle on  $\mathbb{H}_k$ . They are

$$F_{AB(A'B')} = 0 \quad (30)$$

where  $F_{ABA'B'} = [D_{AA'}, D_{BB'}]$  is the curvature of  $D$ . They are the compatibility conditions  $[L_A, L_B] = 0$  for the  $2k$  linear operators

$$\mathbb{L}_A = \pi^{A'} D_{AA'} \quad (31)$$

<sup>1</sup> Both  $\mathbb{C}^2$  and  $\mathbb{C}^{2k}$  carry natural symplectic forms,  $\epsilon_{A'B'}$  and  $\epsilon_{AB}$ , which together determine the metric  $g_{AA'BB'} = \epsilon_{A'B'} \epsilon_{AB}$  on  $\mathbb{H}_k$ . The real hyper-Kähler space is recovered as the set of fixed points of an antilinear involution  $\mathbb{H}_k \rightarrow \mathbb{H}_k$ : this is the tensor product of the involutions of  $\mathbb{C}^{2k}$  and  $\mathbb{C}^2$  induced by the natural unitary structures. Neither of these on its own has nonzero fixed points. The metric and the real slice will play no part in what follows.

The symplectic form  $\epsilon_{AB}$  does not appear in the HK equations, so the equations have a larger symmetry group than the underlying hyper-Kähler geometry: of the symmetries involved in the reduction, only a  $(k + 2)$ -dimensional subgroup preserves the conformal class of the metric.

where  $\pi^A \in \mathbb{C}^2$ . In other words, with  $(\pi^{0'}, \pi^{1'}) = (1, \lambda)$ , they are the integrability conditions for the linear system

$$\mathcal{L}_A = \partial_{A0'} + \Phi_{A0'} + \lambda(\partial_{A1'} + \Phi_{A1'}), \quad (32)$$

where  $\Phi$  is the connection 1-form (each component  $\Phi_{AA'}$  is a matrix-valued function of the coordinates on  $\mathbb{H}_k$ ).

A point of  $\mathbb{H}_k$  corresponds to a line in  $\mathbb{CP}_n$ , where  $n = 2k+1$ . In homogeneous coordinates  $(\omega^A, \pi_{A'})$  on  $\mathbb{C}^{n+1}$ , the line is given by the familiar ‘incidence relations’

$$\omega^A = w_{A'}^{AA'} \quad (33)$$

of twistor theory (Penrose and Rindler 1986). The only difference is that  $A$  runs over  $2, \dots, n$ , rather than over  $0, 1$ . In the new coordinates,  $G$  is generated by the  $n+1$  vector fields

$$\pi_{0'} \frac{\partial}{\partial \pi_{0'}}, \quad \pi_{1'} \frac{\partial}{\partial \pi_{1'}}, \quad \omega^2 \frac{\partial}{\partial \omega^2}, \quad \dots, \quad \omega^n \frac{\partial}{\partial \omega^n} \quad (34)$$

on  $\mathbb{CP}_n$ . Again one is redundant since their sum is zero.

These generate vector fields on  $\mathbb{H}_k$ . To demonstrate the reduction of the HK equations to the Schlesinger equations, we want to construct a solution of the Schlesinger equations from a solution  $\Phi_{AA'}$  of the HK equations which is invariant under their flows.

A function which is constant along the vector fields on  $\mathbb{H}_k$  depends only on the ratios of the  $2k$  functions

$$t_A = \frac{w^{A1'}}{w^{A0'}}, \quad A = 2, \dots, n. \quad (35)$$

Consequently, in an appropriate gauge, we can write an invariant HK potential as

$$\Phi = \sum_{A=2}^n \left( \frac{\mathbf{P}_A}{w^{A0'}} dw^{A0'} + \frac{\mathbf{Q}_A}{w^{A1'}} dw^{A1'} \right) \quad (36)$$

where  $\mathbf{P}_A$  and  $\mathbf{Q}_A$  are matrices depending only on the ratios of  $t_2, \dots, t_n$ .

As in four-dimensional twistor theory (Penrose and Rindler 1986; Ward and Wells 1990), a section of the corresponding bundle on  $\mathbb{CP}_n$  can be represented by a vector-valued function  $\mathbf{y}$  on the ‘incidence space’ on which  $w^{AA'}$  and  $\pi_{B'}$  are coordinates;  $\mathbf{y}$  must be homogeneous of degree zero in  $\pi_{B'}$  and must satisfy linear equations along the fibration over  $\mathbb{CP}_n$  (the linear system  $\mathcal{L}_A \mathbf{y} = 0$ ). By writing  $(\pi_{0'}, \pi_{1'}) = (1, \lambda)$ , we can express  $\mathbf{y}$  as a function of  $w^{AA'}$  and  $\lambda$ . If  $\mathbf{y}$  is an invariant section, then, in addition to satisfying the linear system, it must be constant along

$$\lambda \frac{\partial}{\partial \lambda} - \sum_2^n w^{A1'} \frac{\partial}{\partial w^{A1'}} \quad \text{and} \quad w^{B0'} \frac{\partial}{\partial w^{B0'}} + w^{B1'} \frac{\partial}{\partial w^{B1'}} \quad (37)$$

( $B = 2, \dots, n$ , without summation). Consequently, an invariant section is a function of  $t_2, \dots, t_n, \lambda$  such that

$$(\lambda - t_B) \frac{\partial \mathbf{y}}{\partial t_B} + \left( \mathbf{P}_B + \frac{\lambda \mathbf{Q}_B}{t_B} \right) \mathbf{y} = 0, \quad \lambda \frac{\partial \mathbf{y}}{\partial \lambda} - \sum_{A=2}^n t_A \frac{\partial \mathbf{y}}{\partial t_A} = 0 \quad (38)$$

( $B = 2, \dots, n$ , without summation). The reduced HK equations are the compatibility condition for these  $n$  equations. By substituting from the first  $n - 1$  equations, the last can be rewritten as a linear equation in the independent variable  $\lambda$

$$\frac{\partial \mathbf{y}}{\partial \lambda} = \left( -\frac{1}{\lambda} \sum_2^n \mathbf{P}_A + \sum_2^n \frac{\mathbf{P}_A + \mathbf{Q}_A}{\lambda - t_A} \right) \mathbf{y}. \quad (39)$$

The compatibility of this with the other equations implies isomonodromy as the poles at  $t_2, \dots, t_n$  are moved.

Finally, we can put  $\zeta = (x_1 + \lambda x_0)/(1 + \lambda)$  where  $x_0$  and  $x_1$  are additional variables. Then (39) becomes

$$\frac{\partial \mathbf{y}}{\partial \zeta} = \left( \sum_{\alpha=0}^n \frac{\mathbf{A}_\alpha}{\zeta - x_\alpha} \right) \mathbf{y}, \quad (40)$$

where

$$\mathbf{A}_0 = - \sum_{B=2}^n \mathbf{Q}_B, \quad \mathbf{A}_1 = - \sum_{B=2}^n \mathbf{P}_B, \quad \mathbf{A}_C = \mathbf{P}_C + \mathbf{Q}_C \quad (41)$$

and  $x_C = (x_0 + x_1 t_C)/(1 + t_C)$  for  $C = 2, \dots, n$ . The  $\mathbf{A}_\alpha$  ( $\alpha = 0, \dots, n$ ) satisfy the Schlesinger equations (5) as functions of  $x_0, x_1, \dots, x_n$ .

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## SOLITON EQUATIONS AND CONNECTIONS WITH SELF-DUAL CURVATURE.

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**ABSTRACT.** This article describes how soliton equations of ‘zero curvature’ type can be used to satisfy the self-dual Yang-Mills equations; this produces connections which need not possess translationally invariant self-dual curvature. This construction is particularly clear for those solutions in the ‘dressing orbit of the trivial solution’.

One of the stated aims of this workshop was to investigate the relationship between the self-dual Yang-Mills (SDYM) equations and soliton equations. The evidence of there being some relationship seems to originate with the announcements by Mason and Sparling [1,2] that one could obtain either the Korteweg de Vries (KdV) or nonlinear Schrödinger (NLS) equations by a symmetry reduction of the self-dual Yang-Mills conditions with respect to a metric of signature (2, 2). They have shown that not only can one use these equations to construct  $SU_2$  connections with self-dual curvature on a trivial vector bundle over  $\mathbb{R}^4$ , but that every such connection possessing a particular pair of translational symmetries (*i.e.* the curvature only depends upon two variables) corresponds to a solution of either the KdV or the NLS equations.

This raises the question: which other soliton equations can be used to satisfy the self-duality conditions, and must the curvature always possess translational symmetry? I claim that all soliton equations of ‘zero curvature’ type can be used to construct self-dual connections with non-trivial curvature and these need not possess any translational symmetry.

Here I have a very precise meaning for ‘zero curvature equations’ which is explained in section 1. For the moment it is enough to know that this definition includes all the equations in both the AKNS hierarchy (cf. [3]) and the Drinfel’d-Sokolov hierarchy [4] for any choice of complex semisimple Lie group. So for  $SL_2$  we are talking about the NLS hierarchy and the *modified* KdV hierarchy. We mention in passing the scalar Lax equations described by Drinfel’d and Sokolov but they do not rightfully belong to the family of zero curvature equations in the sense used here. Rather, they may be obtained after a gauge transformation which implements the generalized Miura transformation [4].

I make no claims that there is an identification between certain soliton equations and self-duality conditions. The relationship described here simply allows one to construct, using

solutions of soliton equations, SDYM ‘gauge fields’ with non-trivial curvature on a trivial vector bundle over some open subset of  $\mathbb{C}^4$ . By complexifying everything we can work with the SDYM equations for either Euclidean or type (2, 2) metric, since they are equivalent under complex changes of variable. Because we know a great deal about the solutions of these soliton equations it is possible to construct these connections explicitly. Regrettably this has no bearing on the problems of constructing self-dual gauge fields over  $\mathbb{R}^4$  associated with a *compact* group unless we choose the signature of the metric to be of type (2, 2).

This article has three short sections. The first shows how to use zero curvature equations to satisfy the self-duality conditions. I deal explicitly with two cases, the NLS hierarchy and the mKdV hierarchy, and present an outline of the general case, but the idea is elementary. Thanks to the observations of Ward [5,6] and Zakharov & Shabat [7], the SDYM conditions may be encapsulated inside a single zero curvature equation by introducing a complex parameter; this single equation is of a type which implicitly occurs in the study of soliton equations.

In the next section I explain how these observations fit in with the class of solutions, for example to the AKNS equations, known as the ‘dressing orbit of the trivial solution’. If  $(\partial_x - \mathbf{v}_0 - z\Lambda)\Psi = 0$  is the ‘linear scattering problem’ associated with a soliton equation where  $\mathbf{v}_0$  is the ‘potential’,  $\Lambda$  is a constant diagonalisable matrix and  $\Psi$  is a Lie group-valued function of  $x$  and  $z$ , then  $\Psi$  admits a formal series description as  $\Psi = (\mathbf{I} + O(z^{-1}))\exp(xz\Lambda)$ . The dressing orbit of the trivial solution is essentially that class of potentials  $\mathbf{v}_0$  for which this series actually converges for  $z$  sufficiently large. This class contains all the familiar solutions to our soliton equations, namely the soliton and  $\theta$ -function solutions. The dressing construction involves the solution of a factorisation problem for a matrix-valued function on the circle, often called a ‘Riemann-Hilbert’ problem, as does Ward’s construction of self-dual curvature fields. Both of these have a geometric interpretation in terms of families of complex vector bundles over the Riemann sphere. From this point of view the relationship described in §1 is transparent.

Finally in §3 I have made a brief remark concerning reality conditions. These concern the passage from a complex semisimple Lie group to its compact form. It turns out that although it is not possible to use soliton equations to satisfy the appropriate reality conditions in the Euclidean signature, we can satisfy them in signature (2, 2). As a result one can find a globally analytic  $SU_2$ -connection over a real 4-torus which satisfies the self-duality conditions in this signature.

## 1. Zero curvature equations

Each equation in the AKNS hierarchy for  $sl_2$ , which may be thought of as the complexified nonlinear Schrödinger hierarchy (cf. for example [3]), has a *zero curvature* representation

$$[\partial_{t_k} - \mathbf{V}_k, \quad \partial_x - \mathbf{V}_1] = 0 \tag{1.1}$$

where

$$\mathbf{V}_1 = z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & p \\ -r & 0 \end{pmatrix} \tag{1.2}$$

and

$$\mathbf{V}_k = z^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{j=0}^{k-1} z^j \mathbf{v}_{k-j}. \tag{1.3}$$

Here  $z$  is a formal parameter independent of the variables  $t_0, t_1, \dots, t_k, \dots$  (with  $x = t_1$ ), and  $p, r$  are the dependent variables in each equation. Each  $\mathbf{v}_{k-j}$  is a matrix whose entries are homogeneous polynomials in  $p, r, p_x, r_x, \dots$  of degree  $k-j$ , where  $p, r$  each have degree 1 and  $\partial_x$  increases the degree by 1. The polynomial  $\mathbf{V}_k(z)$  may be characterised as  $\mathbf{V}_k = (z^k \mathbf{V})_+$ , the polynomial part of a formal series  $z^k V$ , where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{j=1}^{\infty} z^{-j} \mathbf{v}_j \quad (1.4)$$

is uniquely determined by the two properties:

- (i)  $[\partial_x - \mathbf{V}_1, \mathbf{V}] = 0$ ,
- (ii) the entries of each matrix  $\mathbf{v}_j$  are homogeneous differential polynomials of degree  $j$ .

Equation (1.1) $_k$  is the compatibility condition for the pair of equations

$$\begin{aligned} (\partial_x - \mathbf{V}_1)\Psi &= 0, \\ (\partial_{t_k} - \mathbf{V}_k)\Psi &= 0, \end{aligned} \quad (1.5)$$

where  $\Psi$  is an  $SL_2$ -valued function of  $z$  and all the  $t_j$ . It follows from (1.4) that

$$(\partial_{t_{k+1}} - \mathbf{v}_{k+1})\Psi = z\partial_{t_k}\Psi \quad (1.6)$$

since  $\mathbf{V}_{k+1} = \mathbf{v}_{k+1} + z\mathbf{V}_k$ . Thus (1.1) $_{k+1}$  may be rewritten as

$$[\partial_{t_{k+1}} - \mathbf{v}_{k+1} - z\partial_{t_k}, \partial_{t_1} - \mathbf{v}_1 - z\partial_{t_0}] = 0. \quad (1.7)$$

Now let us consider  $t_0, t_1, t_k, t_{k+1}$  as complex variables and therefore coordinates for  $\mathbb{C}^4$ . For each solution  $p, r$  of (1.7) valid in some open region  $\mathbb{R} \subset \mathbb{C}^4$  we can use the operators  $\partial_{t_0}, \partial_{t_1} - \mathbf{v}_1, \partial_{t_k}, \partial_{t_{k+1}} - \mathbf{v}_{k+1}$  to define a connection on the trivial  $\mathbb{C}^2$ -bundle  $\mathbb{C}^2 \times \mathbb{R}$ . Equation (1.7) says that the curvature for this connection vanishes over each plane of constant  $t_0 + zt_1$ , and  $t_k + zt_{k+1}$ . It is a simple matter to interpret these as the ‘anti-self dual’ planes in  $\mathbb{C}^4$  and thus to conclude that the curvature for this connection is self-dual.

To see this explicitly, set

$$x_0 = \frac{1}{2}(t_1 - t_k), \quad x_1 = \frac{1}{2}i(t_0 - t_{k+1}), \quad x_2 = -\frac{1}{2}(t_0 + t_{k+1}), \quad x_3 = \frac{1}{2}i(t_1 + t_k). \quad (1.8)$$

In terms of these new coordinates on  $\mathbb{C}^4$  the connection is governed by the covariant derivatives  $\partial_\mu + \mathcal{A}_\mu$  where  $\partial_\mu \equiv \partial/\partial x_\mu$  and

$$\mathcal{A}_0 = i\mathcal{A}_3 = -\mathbf{v}_1, \quad \mathcal{A}_2 = i\mathcal{A}_1 = -\mathbf{v}_{k+1}. \quad (1.9)$$

One easily sees that the components  $\mathcal{F}_{\mu\nu}$  of the curvature in these coordinates satisfy the self-duality conditions

$$\mathcal{F}_{01} = \mathcal{F}_{23}, \quad \mathcal{F}_{02} = -\mathcal{F}_{13}, \quad \mathcal{F}_{03} = \mathcal{F}_{12}, \quad (1.10)$$

where  $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$ .

EXAMPLE 1. The first three terms in the series  $V$  above are given by

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + z^{-1} \begin{pmatrix} 0 & p \\ -r & 0 \end{pmatrix} + \frac{1}{2}z^{-2} \begin{pmatrix} pr & p_x \\ r_x & -pr \end{pmatrix} + \dots$$

and therefore we have, as an example of (1.7),

$$[\partial_{t_2} - \frac{1}{2} \begin{pmatrix} pr & px \\ rx & -pr \end{pmatrix} - z\partial_{t_1}, \partial_{t_1} - \begin{pmatrix} 0 & p \\ -r & 0 \end{pmatrix} - z\partial_{t_0}] = 0.$$

The reader can easily check that this contains the first three flows in the AKNS hierarchy. The  $t_0$ -flow is simply  $\partial_{t_0}p = -2p$ ,  $\partial_{t_0}r = 2r$ ; the  $t_1$ -flow identifies  $t_1$  with ‘ $x$ ’ and the  $t_2$ -flow is the complexified NLS equation.

#### REMARKS 1.

- (i) For AKNS systems the connection corresponding to the equations (1.7) always possesses one translational symmetry. To see this observe that a gauge transformation by  $\exp(t_0\Lambda)$ , where in this example  $\Lambda$  is the diagonal matrix  $\text{diag}(1, -1)$ , removes the  $t_0$ -dependence. In particular, in example 1 above where  $k = 1$ , the connection possesses two translational symmetries and is precisely the system found in [1]. However, we can always construct connections which do not possess any translational symmetry, using the identity

$$[\partial_{t_{k+1}} - \mathbf{v}_{k+1} - z\partial_{t_k}, \partial_{t_{j+1}} - \mathbf{v}_{j+1} - z\partial_{t_j}] = 0$$

provided we choose  $k \neq j, j + 1$ .

- (ii) Observe that in the  $t$ -coordinates the matrix coefficient of  $dt_0 \wedge dt_1$  in the curvature form is given by  $\partial_{t_0}\mathbf{v}_1$ . From this example we see that this has determinant  $4pr$ , which is a gauge invariant. So a necessary condition for two solutions of (1.1) to yield the same connection up to gauge equivalence is that this determinant be the same in both cases. However it seems unlikely that two distinct solutions of (1.1)  $p, r$  and  $\hat{p}, \hat{r}$  have  $pr = \hat{p}\hat{r}$  so it would seem that this construction yields gauge inequivalent connections for each solution of (1.1).

Notice that the only property we used of the hierarchy is that  $\mathbf{V}_k = (z^k V)_+$ . There is a standard way of constructing hierarchies of zero curvature equations for any choice of (complex) semisimple Lie algebra  $g$  which takes as its starting point an operator of the form

$$\partial_x - \mathbf{v}_1 - z\Lambda = \partial_x - \mathbf{V}_1, \quad (1.11)$$

where  $\Lambda$  is a diagonalisable element of  $g$  and  $\mathbf{v}_1$  is a ‘matrix’ of dependent variables (cf. [4]). Then for each  $\Lambda_l \in g$  in the centre of the centralizer of  $\Lambda$  we may construct a unique formal series

$$\mathbf{V}_l = \Lambda_l + \sum_{j=1}^{\infty} z^{-j} \mathbf{v}_{lj} \quad (1.12)$$

possessing the properties (i) and (ii) above.

It follows that all of these hierarchies may be related to self dual connections on the trivial  $g$ -bundle over some open subset of  $\mathbb{C}^4$ . Amongst these hierarchies we find the AKNS hierarchy for  $sl_{n+1}$  and the vector NLS hierarchy for  $sl_{n+1}$  (cf. [8]).

It is also true that the mKdV hierarchy and generalised mKdV hierarchies of Drinfel'd and Sokolov [4] fit into this construction. However, it is not immediately clear that we can perform the same manipulation to obtain equations of the type (1.7). The reason for this is that in the construction of  $\mathbf{V}_l$  in (1.12) an additional constraint is made, namely, that

$\mathbf{V}_l$  have a particular symmetry. This is easiest to describe for the case  $g = \mathfrak{sl}_2$ , that is, the mKdV hierarchy.

In this case

$$\mathbf{V}_1 = z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -q & 0 \\ 0 & q \end{pmatrix} \quad (1.13)$$

and

$$\mathbf{V} = z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{j=0}^{\infty} z^{-j} \mathbf{v}_{j+1}. \quad (1.14)$$

This formal series in  $z$  has the symmetry

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{V}(-z) \quad (1.15)$$

which forces  $\mathbf{v}_{j+1}$  to be diagonal for  $j$  odd and off-diagonal for  $j$  even. Consequently we only get a sensible equation of the type  $(1.1)_k$  for  $k = 2j+1$ . Thus we take  $\mathbf{V}_{k+1} = (z^{2k}\mathbf{V})_+$  to get the mKdV hierarchy. Of course this means that  $\mathbf{V}_{k+1} = \mathbf{v}_{2k+1} + z\mathbf{v}_{2k} + z^2\mathbf{V}_k$ , which doesn't permit us to rewrite the equations in the mKdV hierarchy in the form (1.7).

Fortunately we can get around this obstacle by appreciating that the equations  $(1.1)_k$  are independent of the realisation of  $\mathbf{V}_k$ . By this I mean that we can identify  $\mathbf{V}_k$  with another polynomial  $\mathbf{W}_k$  in  $z$ , this time possessing no symmetry, such that

$$[\partial_{t_1} - \mathbf{W}_1, \partial_{t_k} - \mathbf{W}_k] = 0 \quad (1.16)$$

still presents us with the  $k^{\text{th}}$  equation in the mKdV hierarchy.

To see this let us denote by  $L\mathfrak{sl}_2$  the Lie algebra  $\mathbb{C}[z^{-1}, z] \otimes \mathfrak{sl}_2$  of Laurent polynomials in  $z$  with coefficients in  $\mathfrak{sl}_2$ . We denote by  $L(\mathfrak{sl}_2, \nu)$  the subalgebra of those Laurent polynomials possessing the symmetry  $\nu \circ f(z) = f(-z)$  where  $\nu$  is the automorphism of  $\mathfrak{sl}_2$  given by conjugating elements by the diagonal matrix  $\text{diag}(1, -1)$ . It is quite easy to see that in fact  $L\mathfrak{sl}_2$  and  $L(\mathfrak{sl}_2, \nu)$  are isomorphic as Lie algebras. The isomorphism is given by

$$\begin{pmatrix} a(z^2) & z b(z^2) \\ z^{-1} c(z^2) & -a(z^2) \end{pmatrix} \rightarrow \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}. \quad (1.17)$$

It is common to think of these two Lie algebras as being realisations of an abstract Lie algebra. Indeed this abstract Lie algebra is the quotient of the Kac-Moody Lie algebra of type  $A_1^{(1)}$  by its one-dimensional centre. Usually,  $L\mathfrak{sl}_2$  is called ‘the standard realisation’ and  $L(\mathfrak{sl}_2, \nu)$  is called ‘the principal realisation’.

So when we pass to the standard realisation we replace  $\mathbf{V}$  by

$$\mathbf{W} = z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{j=0}^{\infty} z^{-j} \mathbf{w}_{j+1} \quad (1.18)$$

and consequently  $\mathbf{V}_{k+1}$  is replaced by

$$\mathbf{W}_{k+1} = z^{k+1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \sum_{j=1}^k z^j \mathbf{w}_{k+1-j} + \tilde{\mathbf{w}}_{k+1}, \quad (1.19)$$

where  $\tilde{\mathbf{w}}_{k+1}$  is the upper triangular part of  $\mathbf{w}_{k+1}$ . Hence

$$\mathbf{W}_{k+1} = \tilde{\mathbf{w}}_{k+1} + z\hat{\mathbf{w}}_k + z\mathbf{W}_k, \quad (1.20)$$

where  $\hat{\mathbf{w}}_k$  is the strictly lower triangular part of  $\mathbf{w}_k$ . Consequently, for each  $k \geq 1$  the system

$$(\partial_{t_{k+1}} - \mathbf{W}_{k+1})\Psi = 0 \quad (1.21)$$

may be rewritten as

$$(\partial_{t_{k+1}} - \tilde{\mathbf{w}}_{k+1})\Psi = z(\partial_{t_k} - \hat{\mathbf{w}}_k)\Psi. \quad (1.22)$$

Further, we can include another equation amongst this family, namely the two dimensional Toda lattice for  $sl_2$ , otherwise known as the complexified sine-Gordon equation. This takes the form, in the principal realisation, of (1.1) with  $k = 0$ , where

$$\mathbf{V}_0 = z^{-1} \begin{pmatrix} 0 & e^u \\ e^{-u} & 0 \end{pmatrix}, \quad u_x = q. \quad (1.23)$$

It is well known (cf. [9]) that this equation is compatible with all the equations in the mKdV hierarchy in the sense that there are common solutions  $\Psi$  to each system of the form (1.5) including the case  $k = 0$ . It follows that

$$(\partial_{t_1} - \mathbf{v}_1)\Psi = z\Lambda\Psi = z^2 z^{-1}(\Lambda - \mathbf{V}_0 + \mathbf{V}_0)\Psi = z^2 [\partial_{t_0} - (\mathbf{V}_0 - \Lambda)z^{-1}] \Psi \quad (1.24)$$

in the principal realisation, where  $\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . When we pass to the standard realisation, (1.24) is replaced by

$$[\partial_{t_1} - \tilde{\mathbf{w}}_1 - z(\partial_{t_0} - \mathbf{w}_0)]\Psi = 0 \quad (1.25)$$

with

$$\tilde{\mathbf{w}}_1 = \begin{pmatrix} -q & 1 - e^u \\ 0 & q \end{pmatrix}, \quad \mathbf{w}_0 = \begin{pmatrix} 0 & 0 \\ 1 - e^{-u} & 0 \end{pmatrix}. \quad (1.26)$$

Hence we have the zero curvature equations

$$[\partial_{t_{k+1}} - \tilde{\mathbf{w}}_{k+1} - z(\partial_{t_k} - \hat{\mathbf{w}}_k), \partial_{t_1} - \tilde{\mathbf{w}}_1 - z(\partial_{t_0} - \mathbf{w}_0)] = 0 \quad (1.27)$$

for each  $k \geq 1$ .

EXAMPLE 2. Since

$$\mathbf{V}_2 = z^2 \begin{pmatrix} -q & z \\ z & q \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 2q^3 - q_{xx} & 2(q_x - q^2)z \\ -2(q_x + q^2)z & q_{xx} - 2q^3 \end{pmatrix}$$

it follows that

$$\mathbf{W}_2 = z\mathbf{W}_1 + \frac{1}{4} \begin{pmatrix} 2q^3 - q_{xx} & 2(q_x - q^2) \\ -2(q_x + q^2)z & q_{xx} - 2q^3 \end{pmatrix} \quad \text{where} \quad \mathbf{W}_1 = \begin{pmatrix} -q & 1 \\ z & q \end{pmatrix}$$

in which case

$$\tilde{\mathbf{w}}_2 = \frac{1}{4} \begin{pmatrix} 2q^3 - q_{xx} & 2(q_x - q^2) \\ 0 & q_{xx} - 2q^3 \end{pmatrix}.$$

Now, to include the sine-Gordon equation we set

$$\mathbf{w}_1 = \begin{pmatrix} -q & 1 - e^u \\ -\frac{1}{2}(q_x + q^2) & q \end{pmatrix}.$$

If we substitute these choices into (1.27) we find that

$$\partial_{t_1} u = 2q, \quad \partial_{t_0} q = 2(e^u - e^{-u}), \quad \partial_{t_1} q = q_x, \quad \partial_{t_2} q = \frac{1}{4}(q_{xxx} + 6q^2 q_x).$$

It is now a straightforward matter to interpret (1.27) as the self-duality condition for some connection on the trivial  $\mathbb{C}^2$ -bundle over  $\mathbb{C}^4$ ; I will leave it to the reader to find the ‘gauge potentials’  $\mathcal{A}_\mu$  corresponding to the coordinates given by (1.8).

I now claim that we can use the same construction as that above to produce self-dual connections on the trivial  $g$ -bundle over some open  $R \subset \mathbb{C}^4$  from the hierarchies described by Drinfel'd and Sokolov in [4]. The key to this relationship lies in passing from the principal realisation to the standard realisation in order to achieve equations of the form (1.27). This is quite elementary and I will not present the details here.

In a somewhat less satisfactory sense, the equations in the KdV hierarchy may also be extracted from (1.27) since it is well-known that there is a gauge transformation which takes the mKdV hierarchy into the KdV hierarchy (cf. [4]). Indeed a similar fact is true of each hierarchy described by Drinfel'd and Sokolov in [4]. We will see in the next section that this relationship makes the most sense for the class of solutions known as ‘the dressing orbit of the trivial solution’.

**REMARK 2.** The connection constructed using (1.27) does not possess any translational invariance unless we choose  $k = 1$  as in example 2. However, we can eliminate the  $t_0$ -flow from the picture by replacing (1.25) with  $(\partial_{t_1} - \mathbf{W}_1)\Psi = 0$ , so that, if in (1.27)  $\mathbf{w}_0$  is replaced by the matrix  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ , these equations will be satisfied if  $\partial_{t_0}$  is replaced by, say,  $\partial_y$  where there is no  $y$ -dependence amongst any of the variables. In this case the sine-Gordon equation no longer appears and in example 3 we will have only the mKdV equation. The corresponding connection has two translational symmetries and is gauge equivalent to the connection found by Sparling & Mason [1] to correspond to the KdV equation.

## 2. The dressing orbit of the trivial solution.

In this section I will explain how the observations of §1 tie together the ‘dressing orbit of the trivial solution’ and Ward’s method for constructing self-dual curvature fields. We will restrict our attention to the case of AKNS for  $SL_2$ , but at almost any point in the discussion  $SL_2$  could be replaced by any semisimple Lie group  $G$ . To begin with I will briefly recall both the dressing transformation and Ward’s construction.

First we recall from Zakharov & Shabat [7] that given any solution  $\Psi$  of (1.5) we can construct other solutions in the following way. Let  $g(z)$  be a real-analytic  $SL_2$ -valued function on the circle  $S^1$  which factorises into a product  $\mathbf{g}_-(z^{-1})\mathbf{g}_+(z)$  of  $SL_2$ -valued functions whose Fourier series have the form  $\mathbf{g}_-(z^{-1}) = \mathbf{I} + O(z^{-1})$  and  $\mathbf{g}_+(z) = \mathbf{g}_0 + O(z)$ , where

$\mathbf{I}$  is the identity matrix and  $\mathbf{g}_0 \in \mathrm{SL}_2$ . Then  $\Psi \mathbf{g} \Psi^{-1}$  also possesses such a factorisation for values of  $x$  and  $t_k$  near  $0 \in \mathbb{C}$  and  $(\Psi \mathbf{g} \Psi^{-1})^{-1} \Psi$  is again a solution of (1.5) wherever it is defined. This transformation is called the dressing transformation. Now suppose we look at the solution to (1.5) which corresponds to the trivial solution  $p, r = 0$ . By exponentiation we see that

$$\Psi^{(0)} = \exp \left( \sum_{k=0}^{\infty} t_k z^k \Lambda \right), \quad \text{with } \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.1)$$

satisfies (1.5) for each  $k$ . Given  $\mathbf{g} = \mathbf{g}_- \mathbf{g}_+$  as above it can be shown that  $\mathbf{X}(t) = \Psi^{(0)} \mathbf{g} \Psi^{(0)-1}$  also factorises, but this time for almost all complex sequences  $\mathbf{t} = (t_k)$  for which  $\Psi^{(0)}$  is real-analytic in  $z$ . This fact is not immediately obvious (cf. [8]). So we may construct from each  $\Psi = \mathbf{X}_-^{-1} \Psi^{(0)}$  a possibly new solution to (1.1) for each  $k$ . We think of this as ‘dressing’ the ‘bare operator’  $\partial_{t_k} - z^k \Lambda$  into  $\mathbf{X}_-^{-1}(\partial_{t_k} - z^k \Lambda)\mathbf{X}_-$ . The class of all these solutions is called the dressing orbit of the trivial solution and its properties have been studied in [8]. In particular it contains all the soliton and  $\theta$ -function solutions to the AKNS hierarchy for  $\mathrm{SL}_2$ . In fact the dressing orbit of the trivial solution can be given the structure of an infinite dimensional manifold in such a way that the equations (1.1) correspond to the action of an infinite dimensional abelian group on this manifold. The solutions we understand how to write down correspond to the finite dimensional orbits of this group action cf. [8].

In the same paper Zakharov & Shabat [7] describe how to adapt the dressing construction to handle the SDYM equations. Independently, Ward [5] described essentially the same construction but also made clear the link with Penrose’s twistor correspondence. Leaving the latter aside for the moment, let us simple recall the dressing construction according to Ward [5]. Here one fixes coordinates  $x_{ij}$  on  $\mathbb{C}^4$  (with  $i, j$  either 1 or 2) and makes any choice of  $\mathrm{SL}_2$ -valued function  $\mathbf{X}(x_{ij}, z)$  possessing the following properties:

- (i)  $\mathbf{X}$  is only a function of  $\alpha_j = x_{j2} + zx_{j1}$  and  $z$ , analytic in these variables,
- (ii)  $\mathbf{X}(0, z)$  factorises into  $\mathbf{X}_- \mathbf{X}_+$  in the sense described above.

It follows that near  $0 \in \mathbb{C}^4$  we have  $\mathbf{X}(\alpha_j, z) = \mathbf{X}_-(x_{ij}, z^{-1}) \mathbf{X}_+(x_{ij}, z)$ . Now we observe that self-dual curvature fields may be derived from dressing the bare operator: we construct

$$\mathbf{X}_-^{-1}(\partial_{j1} - z\partial_{j2})\mathbf{X}_- = (\partial_{j1} - \mathcal{A}_{j1}) - z(\partial_{j2} - \mathcal{A}_{j2}) \quad (2.2)$$

which should be interpreted as a covariant derivative along the planes of constant  $\alpha_j$  in  $\mathbb{C}^4$ . In the literature it is usual to transform into the coordinates  $x_\mu$  in which the self-duality conditions take the familiar form (1.10) (cf. [5,10,11]).

Now that we have compared the two constructions, a natural question to ask is what happens if we use  $\mathbf{X} = \Psi^{(0)} \mathbf{g} \Psi^{(0)-1}$  from the former discussion in the formula (2.2), after identifying

$$t_0 = x_{12}z, \quad t_1 = x_{11}, \quad t_k = x_{22}, \quad t_{k+1} = x_{21}. \quad (2.3)$$

The answer is that we construct self-dual curvature fields over an open dense region of  $\mathbb{C}^4$  whose gauge potentials are given by (1.9). Explicitly, we have

$$\mathbf{X}_-^{-1}(\partial_{t_{k+1}} - z\partial_{t_k})\mathbf{X}_- = \partial_{t_{k+1}} - \mathbf{v}_{k+1} - z\partial_{t_k}. \quad (2.4)$$

REMARK 3. Of course the construction given in §1 holds for all analytic solutions of the equations (1.7), not just those in the dressing orbit of the trivial solution. The difference

is that while every solution  $\Psi$  of (1.5) possesses a formal expansion of the form  $(\mathbf{I} + O(z^{-1})) \exp(\sum t_k z^k \Lambda)$  in a neighbourhood of  $z^{-1} = 0$ , this expansion is only convergent for those solutions given by the dressing transformation above. An analytic solution  $\Psi$  of (1.5) is only equivalent to a solution of

$$(\partial_{t_{k+1}} - \mathbf{v}_{k+1} - z\partial_{t_k})\Psi = 0$$

up to right multiplication by an  $SL_2$ -valued function of the form  $a(t_k + zt_{k+1}, z)$ . So Ward's construction does not imply that the solution belongs to the dressing orbit of the trivial solution.

In practice one does not proceed by choosing  $g(z)$  and attempting to factorise this function  $\mathbf{X} = \Psi^{(0)} \mathbf{g} \Psi^{(0)-1}$ . Instead one attempts to find solutions to the equations (1.5) of the form  $(\mathbf{I} + O(z^{-1}))\Psi^{(0)}$ : in the soliton literature such solutions are called (matrix) Baker-Akhiezer functions. Notice that we can replace  $\mathbf{X}_-^{-1}$  in (2.4) by  $\mathbf{X}_-^{-1}\Psi^{(0)}$  to get the same result. Roughly speaking we only understand one type of Baker-Akhiezer function, which can be derived using function theory on a Riemann surface. In the soliton literature this Riemann surface is called the ‘spectral curve’. The reader who wishes to see examples of the construction of such functions can consult, for example, Previato [12] or Dickey [13] who treat the AKNS systems for  $SL_2$  and  $SL_n$  respectively. I believe it is easier to see how this Riemann surface fits into the picture by adopting the geometric point of view available through the twistor correspondence and Ward's theorem. So let me review these as they apply to the construction of self-dual curvature fields for the trivial  $\mathbb{C}^2$  bundle  $\mathcal{E}$  over some open region  $R$  in  $\mathbb{C}^4$ .

By Ward's theorem I mean the assertion, due to Ward (see [5,6]), that to each self-dual, holomorphic  $SL_2$ -connection on  $\mathcal{E}$  there corresponds a unique holomorphic  $\mathbb{C}^2$ -bundle  $\mathcal{F}$  over some open region  $U \subset \mathbb{P}^3$ . This subset  $U$  is determined by  $R$  via the twistor (or Klein) correspondence between  $\mathbb{P}^3 = \text{Gr}(1, 4)$  and  $\text{Gr}(2, 4)$  (cf. [6]). According to Penrose we view  $\mathbb{C}^4$  as an affine subvariety of  $\text{Gr}(2, 4)$  so that we have a restricted Klein correspondence

$$\mathbb{P}^3 - \mathbf{P}_\infty \xleftarrow{\mu} \mathbb{C}^4 \times \mathbf{P} \xrightarrow{\nu} \mathbb{C}^4 \quad (2.5)$$

Here  $\mathbf{P}_\infty$  is some projective line (‘the line at infinity’) in  $\mathbb{P}^3$  and  $\mathbb{C}^4 \times \mathbf{P}$  should be thought of as an open subset of the flag manifold  $\text{Fl}(1, 2, 4)$ . The map  $\nu : \mathbb{C}^4 \times \mathbf{P} \rightarrow \mathbb{C}^4$  is simply projection onto the first factor, while the fibres of  $\mu : \mathbb{C}^4 \times \mathbf{P} \rightarrow \mathbf{P}_3 - \mathbf{P}_\infty$  are copies of  $\mathbb{C}^2$ . Under  $\nu$  these fibres, map onto the ‘anti-self-dual planes’ in  $\mathbb{C}^4$ . In terms of our earlier coordinates these are the planes of constant  $\alpha_j$ ; we think of  $z$  as a rational coordinate on  $\mathbf{P}$ . Given (2.5) the open subset  $U$  is  $\mu \circ \nu^{-1}(R)$ .

Ward's theorem enables us to construct self-dual connections on  $\mathcal{E}$  from holomorphic bundles  $\mathcal{F} \rightarrow U$  which are trivial over each projective line in  $U$ . Here we are making certain assumptions about  $R$ , in particular we assume it is convex (cf. [6]). This condition is sufficient to guarantee that  $\mathcal{E} = \nu_* \mu^* \mathcal{F}$  is well-defined; it follows that  $\mathcal{E}$  possesses a connection derived from the relative differential  $d_\mu$  along the fibres of  $\mu$  which necessarily has zero curvature in the ‘anti-self-dual planes’. Moreover if the first Chern class of  $\mathcal{F}$  is trivial this will be an  $SL_2$ -connection.

However, from a practical point of view it is the intermediate space  $\mathbb{C}^4 \times \mathbf{P}$  where everything happens. Every holomorphic  $\mathbb{C}^2$ -bundle over  $\mathbb{C}^4 \times \mathbf{P}$  can be constructed from a transition function  $\mathbf{X}$  between the trivial bundles over  $U_0 \times \mathbb{C}^4$  and  $U_\infty \times \mathbb{C}^4$ . Here  $(U_0, U_\infty)$  is an open cover of  $\mathbf{P}$  and  $\mathbf{X} : (U_0 \cap U_\infty) \times \mathbb{C}^4 \rightarrow GL_2$  is a holomorphic function. We restrict

its values to  $\mathrm{SL}_2$  to get an associated  $\mathrm{SL}_2$ -bundle. To fit in with our earlier discussion we will choose  $U_0 \cap U_\infty$  to be an annular neighbourhood of the unit circle  $|z| = 1$ . Let us denote by  $\hat{E}$  the  $\mathbb{C}^2$ -bundle corresponding to  $\mathbf{X}(x_{ij}, z)$ . Then the condition that  $\hat{E}$  is trivial over  $R \times \mathbb{P}$  is that  $\mathbf{X}$  factorises into  $\mathbf{X}(x_{ij}, z) = \mathbf{X}_-(x_{ij}, z^{-1})\mathbf{X}_+(x_{ij}, z)$  for each  $(x_{ij}) \in R$ , where each factor is holomorphic in the variables shown. The condition that  $\hat{E}$  is the pullback of a holomorphic bundle  $\mathcal{F} \rightarrow \mathbb{P}^3 - \mathbb{P}_\infty$  is that  $\mathbf{X}$  can be written as a function of  $\alpha_j$  and  $z$  only.

One way of constructing a trivial  $\mathbb{C}^2$ -bundle over  $R \times \mathbb{P}$  is by direct image (cf. for example, [14]) from a  $\mathbb{C}$ -bundle  $\mathcal{L}$  over the double covering  $R \times X$  where  $X$  is a Riemann surface possessing a double covering  $\pi : X \rightarrow \mathbb{P}$ . Let us denote by  $\mathcal{L}(x_{ij})$  the restriction of  $\mathcal{L}$  to a copy of  $X$ . Then the condition that its direct image (which will be a  $\mathbb{C}^2$ -bundle over  $\mathbb{P}$ ) is trivial is that  $\mathcal{L}(x_{ij})$  has degree  $g_X + 1$ , where  $g_X$  is the genus of  $X$ , and possesses a pair of independent, globally holomorphic sections.

In the particular case of the AKNS systems for  $\mathrm{SL}_2$  we require  $X$  to have two points lying over  $\infty$ . If we equip  $\mathcal{L}(x_{ij})$  with a trivialisation  $\phi$  over these two points this induces a trivialisation  $\pi_*\phi$  of  $\pi_*\mathcal{L}(x_{ij})$  over a neighbourhood of  $\infty$  (which we may assume, without much loss of generality, extends as far as  $U_\infty$ ). Then there is a unique global frame for  $\pi_*\mathcal{L}(x_{ij})$  which, over  $U_\infty$ , has the form  $I + O(z^{-1})$  when viewed using  $\pi_*\phi$ : this gives  $\mathbf{X}_-^{-1}$ .

To get  $\mathbf{X}_- = (\Psi^{(0)}\mathbf{g}\Psi^{(0)-1})_-$  it can be shown (cf. [8]) that the thing to do is to use a natural homomorphism  $L$  from the abelian group generated by  $\Psi^{(0)}$  onto the Jacobi variety of  $X$ , so that for each  $k$  we have a 1-parameter family  $L(t_k z^k)$  of line bundles over  $X$  of degree zero. Then  $\mathbf{X}$  depends only upon  $\alpha_j, z$  when  $\mathcal{L}(x_{ij}) \cong \mathcal{L}_0 \otimes L(\alpha_1 + z^k \alpha_2)$  for some fixed  $\mathcal{L}_0$  with the variables  $x_{ij}$  and  $t_k$  identified as earlier. We may think of this as a family of line bundles over  $X$  which vary linearly over its Jacobi variety. The corresponding solutions to the AKNS equations are written down in terms of the  $\theta$ -functions of  $X$ . If we want instead the soliton solutions we amend the discussion above by replacing the Riemann surface by a complete rational curve whose only singularities are ordinary double points (cf. for example, [8] or [12]).

#### REMARKS 4.

- (i) It has been widely recognised that the factorisation problems encountered in soliton theory and in Ward's construction are related. However when thinking about soliton theory the temptation is to think of that Riemann-Hilbert problem which one encounters when trying to solve the initial value problem for real-valued 'asymptotically flat' potentials  $p, r$  (see in particular [2,7]). In that case the function  $\mathbf{X}_-$  is generally not holomorphic for  $z$  large but has a discontinuity across the imaginary  $x$ -axis and a number of poles in the  $z$ -plane off this axis (cf. for example [15]). This led Zakharov & Shabat to formulate their Riemann-Hilbert problem using this axis as 'the circle'. However, when we are dealing with the dressing orbit of the trivial solution, which corresponds to the case where all the scattering data is contained in a bounded region of the  $z$ -plane *i.e.*  $\mathbf{X}_-$  converges for  $z$  large, we can take 'the circle' to be a circle about infinity and, since rescaling  $z$  only changes the scale of  $x$ , we may as well rescale to get the unit circle. This is the reason why in the discussion above we do not need to resort to a 'Riemann-Hilbert problem with zeroes' to find the soliton solutions.
- (ii) To obtain instead the self-dual connections corresponding to equations in the mKdV

hierarchy, *i.e.* the operators appearing in (1.22), we choose

$$\Psi^{(0)} = \exp \left\{ \sum_{k=0}^{\infty} t_k z^k \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \right\} \quad (2.6)$$

where once again  $g(z)$  factorizes. This time we require that  $g_-(z^{-1})$  has the form  $g_0 + 0(z^{-1})$  where  $g_0$  is lower triangular with 1's along the diagonal. We make the same requirement of  $\mathbf{X}_-(t_k, z^{-1})$  for each  $t_k$  and discover

$$\mathbf{X}_-^{-1}(\partial_{t_{k+1}} - z\partial_{t_k})\mathbf{X}_- = \partial_{t_{k+1}} - \tilde{\mathbf{w}}_{k+1} - z(\partial_{t_k} - \tilde{\mathbf{w}}_k) \quad (2.7)$$

Thus if we identify  $t_0, t_1, t_k, t_{k+1}$  with  $x_{ij}$  as in (2.3) we obtain a connection for the bundle  $E$  which has self-dual curvature over almost all of  $\mathbb{C}^4$ . As we noted earlier, there is a unique choice of gauge for which the self duality conditions in this case contain amongst them an equation in the KdV hierarchy.

### 3. Reality conditions

The real slice of  $\mathbb{C}^4$  which is usually taken corresponds to requiring all the  $x_\mu$  to be real-valued. In this case there is a condition on  $\mathbf{X}$  which guarantees that the gauge potentials are skew-Hermitian in some gauge, or more generally, that they take values in the appropriate compact form of the Lie algebra. The condition is  $\mathbf{X}(x_{ij}, z)^* = \mathbf{X}(\bar{x}_{ij}, -\bar{z}^{-1})$  (cf. [16]). It is not difficult to see that this is not compatible with any of the soliton hierarchies since it requires that  $\Psi^{(0)}$  depend upon  $z^{-1}$ . However, the advantage of working in  $\mathbb{C}^4$  is that we can also obtain solutions of the SDYM equations corresponding to a metric with signature (2,2). In this case it is possible to use the AKNS equations to find solutions of the SDYM equations in signature (2,2) for any choice of compact Lie group. I will demonstrate this for the group  $SU_2$ .

The SDYM equations in signature (2,2) are equivalent to the equations

$$\mathcal{F}_{01} = -\mathcal{F}_{23}, \quad \mathcal{F}_{02} = -\mathcal{F}_{13}, \quad \mathcal{F}_{03} = \mathcal{F}_{12}, \quad (3.1)$$

where the curvature components  $\mathcal{F}_{\mu\nu}$  are defined as before. Here we have used the metric  $\text{diag}(1, 1, -1, -1)$  and then transposed the indices 2 and 3. We can use solutions of (1.10) to obtain solutions of (3.1) simply by replacing  $\partial_\mu + \mathcal{A}_\mu$  with  $i(\partial_\mu + \mathcal{A}_\mu)$  for  $\mu = 2, 3$ . In this case we have to replace the identifications (1.8) and (1.9) by

$$t_0 = -i(x_1 + x_2), \quad t_1 = x_0 + x_3, \quad t_k = i(x_0 - x_3), \quad t_{k+1} = x_1 - x_2, \quad (3.2)$$

and

$$\mathcal{A}_0 = \mathcal{A}_3 = -\mathbf{v}_1, \quad \mathcal{A}_1 = -\mathcal{A}_2 = -\mathbf{v}_{k+1}. \quad (3.3)$$

In principle we may construct solutions of (3.1) using the dressing construction given in §2. Moreover we can ensure that the gauge potentials are skew-Hermitian by taking all the  $x_\mu$  to be real and setting

$$\begin{aligned} \Psi^{(0)} = \exp \left\{ -i(x_1 + x_2) + (x_0 + x_3)z + i(x_0 - x_3)z^k \right. \\ \left. + (x_1 - x_2)z^{k+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \end{aligned} \quad (3.4)$$

for  $k$  even. We now let  $\mathbf{X} = \Psi^{(0)}\mathbf{g}(z)\Psi^{(0)-1}$  where  $\mathbf{g}^{*-1}(z) = \mathbf{g}(-\bar{z})$ . It follows that  $\mathbf{X}^{*-1}(z) = \mathbf{X}(-\bar{z})$  which ensures that the term in  $\partial_\mu \mathbf{X}_+ \cdot \mathbf{X}_+^{-1}$  which is independent of  $z$  is skew-Hermitian. This term is easily seen to be  $\mathcal{A}_\mu$ .

An interesting example of this is given by taking  $k = 2$ . In that case the explicit forms of the potentials  $\mathcal{A}_\mu$  are given by (3.3) with

$$\mathbf{v}_1 = \begin{pmatrix} 0 & p \\ -\bar{p} & 0 \end{pmatrix} \text{ and } \mathbf{v}_3 = -\frac{1}{4} \begin{pmatrix} \bar{p}p_x - \bar{p}_x p & p_{xx} + 2p^2\bar{p} \\ -\bar{p}_{xx} - 2p\bar{p}^2 & \bar{p}p_x - \bar{p}_x p \end{pmatrix}.$$

The  $t_2$ -flow in this case is simply the ‘focusing’ case of the NLS equation; since  $t_2$  is pure imaginary we may set  $t = it_2$  and we obtain the equation  $ip_t = \frac{1}{2}p_{xx} + p^2\bar{p}$  (where  $x = t_1$ ).

It is a consequence of Previato’s work [12] that we can find solutions  $p$  and  $r$  of the equations (1.1) which, for  $t_0, t_2$  imaginary,  $t_1, t_3$  real, satisfy the reality condition  $r = \bar{p}$  and are *globally* analytic. The function  $p$  may be written down in terms of a ratio of  $\theta$ -functions together with an exponential factor. These  $\theta$ -functions correspond to a hyperelliptic Riemann surface  $X$  whose branch points are not real but occur in complex conjugate pairs. Before I reproduce Previato’s formula here, let me simply explain the underlying geometry. From [12] we know that the first  $g + 1$   $t_k$ -flows span the tangent space of a generalized Jacobian, corresponding to the singularisation of  $X$  obtained by identifying its two points ‘at infinity’. This is complex manifold of dimension  $g + 1$ , where  $g$  is the genus of  $X$ . When we impose the reality conditions above we single out a real  $g + 1$ -dimensional torus in this Jacobian along which the reality condition  $r = \bar{p}$  is maintained. In particular, if we choose  $g = 3$  we obtain a globally analytic solution  $p$  on a real 4-torus. Thus it is possible to compute globally analytic  $SU_2$ -connections on a 4-torus which satisfy the self-duality conditions in signature  $(2, 2)$ .

To finish let us write down (somewhat incompletely) this formula for the function  $p$  which satisfies our reality conditions. Let us look only at the case of a genus 3 surface, which we take to be the Riemann surface  $X$  for the real polynomial  $y^2 = \prod_1^4(z^2 + k_i^2)$  where each  $k_i$  is real. To define the classical Riemann  $\theta$ -function of  $X$  we need to fix a homology basis of  $a$ -cycles and  $b$ -cycles with the standard intersection properties (cf. for example, [17]). This uniquely determines a basis  $\omega_1, \omega_2, \omega_3$  of holomorphic differentials on  $X$  normalized over the  $a$ -cycles. It also fixes a differential  $\omega_0$  of the third kind on  $X$  whose only poles are simple ones at the two points  $\infty_1$  and  $\infty_2$  lying at infinity:  $\omega_0$  is normalized so that its integral around any  $a$ -cycle vanishes and its residues at infinity are  $\pm 1$ . Having fixed these things, the formula for  $p$  is

$$p = C \exp \left( -2t_0 + \sum_j \alpha_j t_j \right) \frac{\theta \left( \sum_j \mathbf{W}_j t_j + \Delta - A(R_+ + \infty_1 - \infty_2) \right)}{\theta \left( \sum_j \mathbf{W}_j t_j + \Delta - A(R_+) \right)}$$

where  $j$  runs over 1,2,3. Here  $C$  is some constant whose explicit determination is a little complicated, cf. [12] and the other symbols are defined as follows:

1.  $A$  is the Abel mapping with one of the ramification points as basepoint and  $R_+$  is the divisor of those four ramification points lying over the upper half  $z$ -plane,
2.  $\Delta$  is the vector of Riemann constants,
3. the vectors  $\mathbf{W}_j \in \mathbb{C}^3$  are given by the residue formulae  $\mathbf{W}_j = \text{res}_{\infty_1} z^j \omega - \text{res}_{\infty_2} z^j \omega$  where  $\omega$  is the vector of holomorphic differentials,
4. the numbers  $\alpha_j$  are given by the residue formulae  $\alpha_j = \text{res}_{\infty_1} z^j \omega_0 - \text{res}_{\infty_2} z^j \omega_0$ .

With  $t_0, t_2$  imaginary and  $t_1, t_3$  real the theory of [12] ensures that  $p$  and  $r = \bar{p}$  are globally analytic solutions of the first four AKNS flows and may be thought of as functions on a real 4-torus.

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# SMOOTH STATIC SOLUTIONS OF THE EINSTEIN/YANG-MILLS EQUATIONS

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**ABSTRACT.** This talk will consider the Einstein/Yang-Mills equations in  $3 + 1$  space-time dimensions and show how one can prove rigorously the existence of a globally defined smooth static solution, with the associated Einstein metric being asymptotically flat and the total mass finite. Physically this means that the Yang-Mills repulsive force can balance the gravitational attractive force and prevent the formation of singularities in space-time.

## 1. Introduction

It is well known that there are no non-singular, radially symmetric, static solutions of Einstein's equations in free space. The equations can be explicitly solved, and the solution, the so-called Schwarzschild metric, has a singularity at  $r = 0$ , the centre of the spherical symmetry [1]. Physically, this occurs because there is nothing to balance out the gravitational attraction.

If one attempts to obtain a balance by introducing electro-magnetic forces, coupling Einstein's equations with Maxwell's equations, there is still no smooth, radially symmetric, static solution. The only solution, the Reissner-Nordström metric, is again singular at the origin [1]. Even if one couples the Einstein equations with the Yang-Mills equations, the result is the same in  $2 + 1$  space-time dimensions [2], but this talk will show that the situation changes when we move to  $3 + 1$  space-time dimensions.

The work described is work done in conjunction with J.A. Smoller, A.G. Wasserman and S.-T. Yau [3], and also with W.C. Troy at the University of Pittsburgh.

## 2. Derivation of the equations

The derivation of the equations will be brief. For more details the reader is referred to [3].

To solve the Einstein/Yang-Mills equations, we have to find a metric

$$ds^2 = -T^2 dt^2 + R^2 dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.1)$$

where  $T, R$  are functions of  $r$  alone. (This form of the metric is dictated by the requirements that the solution be static and spherically symmetric.) The Einstein/Yang-Mills equations

are then obtained as the variational equations for the functional

$$\int (-\mathcal{R} + |\mathcal{F}|^2) \sqrt{|g|} dx, \quad (2.2)$$

where  $\mathcal{R}$  is the scalar curvature associated with the metric (2.1),  $g$  is the metric tensor, and  $\mathcal{F}$  is the Yang-Mills curvature. Using gauge invariance,  $\mathcal{F}$  can be written in the form

$$\mathcal{F} = w' \tau_1 dr \wedge d\theta + w' \tau_2 dr \wedge (\sin \theta d\phi) - (1 - w^2) \tau_3 d\theta \wedge (\sin \theta d\phi), \quad (2.3)$$

where  $w = w(r)$  is a function to be determined,  $w' = dw/dr$ , and

$$\tau_1 = \frac{1}{2}i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{2}i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \frac{1}{2}i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From (2.3) we can readily calculate

$$|\mathcal{F}|^2 = \frac{2(w')^2}{r^2} + \frac{(1 - w^2)^2}{r^4}.$$

The Yang-Mills curvature  $\mathcal{F}$  must satisfy the Yang-Mills equations, usually written  $d^* \mathcal{F} = 0$ , which in the current context reduce simply to

$$\left( \frac{w'}{RT} \right)' + \frac{R}{r^2 T} (1 - w^2) = 0.$$

From the variational equations for (2.2) we obtain two further equations for  $w, R, T$ , from which we can eliminate  $T$ , leaving us finally with two equations for  $w, R$ . These can be written in the form, with  $1 - \frac{m}{r} = R^{-2}$ ,

$$r^2 \left( 1 - \frac{m}{r} \right) w'' + \left[ m - \frac{(1 - w^2)^2}{r} \right] w' + w(1 - w^2) = 0, \quad (2.4)$$

$$m' = 2 \left( 1 - \frac{m}{r} \right) (w')^2 + \frac{(1 - w^2)^2}{r^2}. \quad (2.5)$$

It should be noted that the variable  $m$  is related to the mass, so that  $m(\infty)$  is the total mass of the system.

Some boundary conditions have to be added to the system (2.4)–(2.5). Specifically, we require

$$w(0) = 1, \quad w'(0) = 0, \quad m(0) = 0. \quad (2.6)$$

(The second condition arises from the spherical symmetry, and if there is to be no singularity in  $m$  and  $w$  at the origin, then we need  $w(0) = 1$ . The condition  $m(0) = 0$  is clear from the interpretation of  $m$  as mass.)

It might seem that these three initial conditions for a third-order system will specify the solution entirely, but if we expand  $m$  and  $w$  in power series, we find that the first terms are

$$w = 1 - \frac{\lambda^2}{r} + \dots, \quad m = \lambda^2 r^3 + \dots,$$

where  $\lambda = -w''(0)$  is arbitrary. Thus, in order to have the desired behaviour at  $\infty$ , we have to choose  $\lambda$  so that

$$w''(0) = -\lambda \quad (2.7)$$

implies that the solution exists for all  $r$  with

$$w' < 0, \quad w(\infty) = -1, \quad 1 - \frac{m}{r} > 0. \quad (2.8)$$

(The middle condition corresponds to demanding that the solution is asymptotically flat, as we require; the third is necessary to avoid singularities in (2.4), and the first is a desirable characteristic which the proof gives us almost for free.)

With a simple scaling we can write (2.4)–(2.5) in a slightly different form which is sometimes useful. Thus, setting

$$w(r) = z(t), \quad \alpha m(r) = \mu(t), \quad \beta r = t,$$

where

$$\alpha\beta = 1, \quad \beta^2 = \lambda,$$

we have

$$t^2 \left(1 - \frac{\lambda\mu}{t}\right) z'' + \lambda \left(\mu - \frac{(1-z^2)^2}{t}\right) z' + z(1-z^2) = 0, \quad (2.9)$$

$$\mu' = 2 \left(1 - \frac{\lambda\mu}{t}\right) (z')^2 + \frac{(1-z^2)^2}{t^2}, \quad (2.10)$$

with

$$z(0) = 1, \quad z'(0) = 0, \quad z''(0) = -1, \quad \mu(0) = 0, \quad (2.11)$$

$$z' < 0, \quad z(\infty) = -1, \quad 1 - \lambda\mu/t > 0. \quad (2.12)$$

### 3. Existence of a solution

What we have to do is to show that there exists some  $\lambda > 0$  such that there is a solution to (2.4)–(2.8) or alternatively to (2.9)–(2.12). We note first the following basic lemma which we give without proof, which is standard.

**LEMMA 1.** *For each value of  $\lambda$  there exists one and only one solution of (2.4)–(2.7) which is analytic in  $r$  for  $|r|$  sufficiently small. Further, this solution, and all its derivatives, are continuous in  $\lambda$ . (This solution will not, of course, in general satisfy the conditions at  $\infty$ .)*

**LEMMA 2.** *If  $\lambda (> 0)$  is sufficiently small, then, in (2.9)–(2.11),  $z(t)$  (as  $t$  increases from 0) descends to and crosses the value  $-1$  strictly before either  $z'(t) = 0$  or  $p \equiv 1 - \lambda\mu/t = 0$ .*

*Proof.* One can state this theorem either for (2.4)–(2.7) or for (2.9)–(2.11), but its proof is more immediately transparent in the latter case. For if we take  $\lambda = 0$ , then equation (2.9) no longer involves  $\mu$ , and becomes just

$$t^2 z'' + z(1-z^2) = 0.$$

Setting

$$x = \log t, \quad Z(x) = z(t),$$

we have

$$Z'' - Z' + Z(1 - Z^2) = 0,$$

so that, multiplying by  $Z'$  and integrating and using (2.11),

$$\frac{1}{2}(Z')^2 = \int_{-\infty}^x (Z')^2(\xi) d\xi + \frac{1}{4}(1 - Z^2)^2,$$

which certainly implies that  $Z' \neq 0$ . Thus the lemma is true for  $\lambda = 0$ , and continuous dependence ensures that it remains true for  $\lambda > 0$  and sufficiently small.

Let us define

$$\Lambda = \{\lambda : \text{the solution } z(t, \lambda) \text{ of (2.9)–(2.11) has the property of Lemma 2}\}.$$

**LEMMA 3.** *The set  $\Lambda$  is open.*

*Proof.* If the solution for some  $\lambda_0$  takes some value below  $-1$  strictly before  $z' = 0$  or  $p = 0$ , then the same is true, from continuity in  $\lambda$ , for  $\lambda$  sufficiently close to  $\lambda_0$ . The set  $\Lambda$  is therefore open.

**LEMMA 4.** *If  $\lambda$  is sufficiently large, then  $\lambda \notin \Lambda$ .*

We refer the reader to [3] for details of the proof. We merely remark that it is now better to use the equations (2.4)–(2.5), and that it is a matter of careful estimates on the solution.

**THEOREM 1.** *There exists some  $\lambda > 0$  such that the equations (2.9)–(2.12) have a solution.*

*Proof.* The set  $\Lambda$  cannot be closed, since if it were it would contain all  $\lambda > 0$  (since it is also open) and this would contradict Lemma 4. So, as we increase  $\lambda$ , there exists some first  $\lambda_o$  such that  $\lambda_o \notin \Lambda$ . What happens to the solution (2.9)–(2.11) as  $\lambda \uparrow \lambda_o$ ?

There are clearly just three possibilites. Let  $t(\lambda)$ , for  $\lambda < \lambda_0$ , denote the first zero of  $z + 1$ .

CASE 1. As  $\lambda \uparrow \lambda_0$ ,  $t(\lambda) \rightarrow \infty$  and, for  $\lambda = \lambda_0$ , none of  $z + 1, z', 1 - \lambda\mu/t$  have zeros.

CASE 2. As  $\lambda \uparrow \lambda_0$ ,  $t(\lambda) \rightarrow \infty$  and, for  $\lambda = \lambda_o$ , either  $z'$  or  $1 - \lambda\mu/t$  has a zero at a finite value of  $t$ .

CASE 3. As  $\lambda \uparrow \lambda_0$ ,  $t(\lambda) \rightarrow t(\lambda_0)$ ,  $t(\lambda_0)$  finite, and, for  $\lambda = \lambda_0$ , either  $z'$  or  $1 - \lambda\mu/t$  has a zero at or before  $t(\lambda_0)$ .

If Case 1 holds, then  $z + 1 > 0$  and  $z' < 0$  imply that  $z(\infty)$  exists, and it is not a difficult deduction from (2.4) that the limit must be  $z(\infty) = -1$ . (Again, the reader is referred to [3] for full details.)

In order to complete the proof of the theorem, all we need do is to show that Cases 2 and 3 are impossible. It is easy to see that they can occur only if  $1 - \lambda\mu/t$  has a zero at or before  $t(\lambda_0)$  (at a finite value in Case 2). For if  $1 - \lambda\mu/t$  does not have a zero, then  $z'$  must have, and since, for  $\lambda < \lambda_0$ ,  $z' < 0$ , we deduce that at the zero when  $\lambda = \lambda_0$  we have both  $z' = 0, z'' = 0$ , which implies  $z = 0$  or  $z = -1$  from (2.4), and then  $z \equiv 0$  or  $z \equiv -1$ , both of which are impossible.

Thus  $1 - \lambda\mu/t$  must vanish, and it is then a matter of investigating carefully what happens as  $\lambda \uparrow \lambda_0$  and  $1 - \lambda\mu/t \rightarrow 0$  at  $t_0$ , say. The final contradiction is reached only after a rather laborious series of *reductio ad absurdum* arguments, which it is difficult to motivate. The one interesting comment is that it appears to be necessary to consider what happens not only on the real  $t$ -axis, but also in the complex  $t$ -plane.

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## EXTENDED STRUCTURES IN $(2+1)$ DIMENSIONS

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**ABSTRACT.** We study extended structure solutions of the  $S^2$  sigma model and the corresponding Skyrme models in  $(2+1)$  dimensions. We review some results reported earlier and concentrate our attention on the process of annihilation of two such structures – one corresponding to a soliton like object and another to an antisoliton. We find that the process of annihilation proceeds in three stages; the initial approach, then rapid annihilation of the soliton cores followed slow annihilation of the soliton tails.

### 1. Introduction

Over the past few years classical field theories in  $(2+1)$  dimensions have become an increasingly important area of research. These models can be considered as low dimensional toy models describing various properties of lump-like structures which in  $(3+1)$  dimensions are thought to provide a good description of many physical objects such as nucleons or monopoles.

One of these models is the  $(2+1)$  dimensional  $S^2$  sigma model which possess stable extended structure solutions that can be characterised by a topological quantity [1]. This model is described by a three component vector field  $\phi$  lying on a sphere of radius 1, that is,  $\phi$  satisfies  $\phi \cdot \phi = 1$ . The Lagrangian of the model is given by

$$\mathcal{L}_t = (\partial^\mu \phi) \cdot (\partial_\mu \phi), \quad (1)$$

where  $\mu$  is the Lorentz index running from 0 to 2 for respectively  $t$ ,  $x$  and  $y$ .

The finite energy static configurations of the model correspond to the mappings of the two dimensional sphere into itself and so these field configurations are characterised by an integer-valued topological charge describing such mappings. As the time evolution corresponds to a continuous transformation, the topological charge remains constant and so is conserved. Thus we can think of our extended structures as providing us with  $(2+1)$  dimensional analogues of, say, monopoles and antimonopoles or protons and antiprotons.

### 2. The $S^2$ Sigma Model

All finite energy static solutions of the model (1) can be presented in an explicit form [2]. They correspond to a number of lump-like configurations where the number of lumps is determined by the value of the topological charge. The existence of the multi-lump solutions

is due to the fact that the lumps do not interact with each other. Moreover, as a consequence of the conformal symmetry, the lumps can have any size and be placed at an arbitrary distance from each other.

To have the explicit form of these lump-like solutions it is convenient to introduce another formulation of the model. Instead of using the  $\phi$  fields we can express all the dependence on  $\phi$  in terms their stereographic projection onto the complex plane  $W$ . The  $\phi$  fields are then

$$\phi^1 = \frac{W + W^*}{1 + |W|^2}, \quad \phi^2 = i \frac{W - W^*}{1 + |W|^2}, \quad \phi^3 = \frac{1 - |W|^2}{1 + |W|^2}, \quad (2)$$

where  $*$  denotes the complex conjugation.

The  $W$  formulation is very useful, because it is in this formulation that the static solutions take their simplest form. As originally shown by Belavin and Polyakov [3] and Woo, [4] they are given by  $W$  being any rational function of either  $z = x + iy$  or of  $\bar{z} = x - iy$ . It is easy to see that the topological charge of these solutions is respectively a positive or a negative integer. By convention the first case corresponds to solitons and the other to antisolitons.

The simplest nontrivial solution corresponds to one soliton and is given by

$$W = \lambda \frac{z - a}{z - b}, \quad (3)$$

where  $z = x + iy$  and  $a, b$  and  $\lambda$  are arbitrary complex numbers.

It is easy to calculate the energy density,  $E$ , corresponding to the static solution (3). We find

$$E = \frac{8|\lambda|^2 |a - b|^2}{(|z - b|^2 + |\lambda|^2 |z - a|^2)^2} \quad (4)$$

and so we see that this structure has a bell-like shape, with its position and size determined by

$$\frac{a|\lambda|^2 + b}{|\lambda|^2 + 1} \quad \text{and} \quad \frac{|\lambda|^2 |a - b|^2}{|\lambda|^2 + 1}$$

respectively. In Figure 1 we present a picture of the energy density of such a configuration.

It is easy to see that the field configurations describing larger numbers of solitons are described by more complicated expressions for  $W$ ;  $n$  solitons are described by a  $W$  field which is a ratio of polynomials of degree  $n$  in  $z$ .

To study the time evolution of systems involving many solitons and those involving solitons and antisolitons we performed numerical calculations of the equations of motion corresponding to our Lagrangian (1). These equations are given by

$$\partial^\mu \partial_\mu \phi + (\partial^\mu \phi \bullet \partial_\mu \phi) \phi = 0. \quad (5)$$

To specify the problem completely we have to state the boundary conditions on  $\phi$ . Using polar coordinates  $(r, \theta)$  we take

$$\phi(r, \theta, t) \rightarrow \phi_0(t) \quad \text{as} \quad r \rightarrow \infty, \quad (6)$$

where  $\phi_0$  is independent of the polar angle  $\theta$ . Then we can translate the problem onto a finite dimensional lattice and perform numerical simulations of the time evolution using a convenient numerical simulation method such as an appropriate Runge-Kutta scheme. To perform numerical simulations we need the initial conditions for which we can take the lattice versions of the appropriate initial  $W$  field and its time derivative.

We have performed many such simulations, involving many processes and using many computers, some in Durham and some at Los Alamos. Most of our simulations were performed using a 4th order Runge-Kutta method working in double precision. The machines used included the Los Alamos Connection Machine, various Sun and HP work stations and the Durham Vax. Almost all our simulations were performed on fixed lattices which varied from  $200 \times 200$  to  $512 \times 512$ , with lattice spacing  $\delta x = \delta y = 0.02$ . The time step was 0.01.

To perform any simulation we must face the problem of how to deal with the constraint  $\phi \cdot \phi = 1$ . We decided to perform the simulations of all three components of  $\phi$  independently. Even though the initial configuration satisfies  $\phi \cdot \phi = 1$ , due to the unavoidable numerical truncation errors introduced at various stages of the calculations  $\phi$  gradually moves away from the unit sphere and the constraint is no longer satisfied. To overcome this problem, we rescaled all fields

$$\phi \rightarrow \phi' = \phi / \sqrt{\phi \cdot \phi} \quad (7)$$

every few iterations. In fact, just before the rescaling of  $\phi$ , we evaluated  $\mu_e = \phi \cdot \phi - 1$  at each lattice point. If we treat  $\mu_{smax} \equiv \max|\mu_e|$  as a measure of the numerical errors we have found that in general,  $\mu_{smax} \approx 10^{-5} \sim 10^{-9}$ , depending on the process in question. We checked that when we used an unsound numerical procedure such as taking the time step in the Runge-Kutta procedure larger than the lattice spacing, this has always reflected itself in the rapid growth of  $\mu_{smax}$ . Hence we believe  $\mu_{smax}$  to be a good estimate of the overall numerical accuracies and the error associated with this procedure is of the order of the accuracy of our calculations.

In these simulations, the results of which have been reported in several papers [1] [5], we have studied the stability of solitons, their scattering properties, their interactions with pure waves and the interactions of solitons with antisolitons. The results have shown that the solitons do not have stable size. This instability could be generated by any perturbation. In particular the addition of a small background noise could give the system enough kinetic energy to cause the shrinking of the solitons. This instability is due to the conformal invariance of the model in (2+0) dimensions. Due to this invariance, the solitons can have any size and so their change of size can be induced with very little extra energy.

We have observed that when two solitons are sent towards each other at a zero impact parameter, they collide, form a ring and then scatter at  $90^\circ$ . After the collision, the two solitons that emerge from the ring shrink or spread out as they move away from each other, thus exhibiting their instability. This instability prevents us from following the time evolution too far. Luckily, for slow initial velocities, the instability does not set in too fast and we can still talk about the trajectories after the scattering. In fact the shrinking or spreading out of solitons can be cured by adding extra terms to the Lagrangian (1). We need at least two such terms; one with more and one with fewer derivatives than two. In [6] we added

$$\mathcal{L}_l = -\frac{1}{4} \left( \theta_1 \left( (\partial^\mu \phi \cdot \partial_\mu \phi)^2 + (\partial^\mu \phi \cdot \partial^\nu \phi)(\partial_\mu \phi \cdot \partial_\nu \phi) \right) + \theta_2 (1 + \phi^3)^4 \right), \quad (8)$$

where  $\theta_1$  and  $\theta_2$  are two new (real) parameters of the model. It is clear that the model based on the Lagrangian with these terms is still Lorentz invariant and for positive values of  $\theta$  its Hamiltonian is positive definite. As we argued in [6] the first term of (8), which is a  $(2+1)$  analogue of the more familiar  $(3+1)$  dimensional “Skyrme” term [7], is unique as long as we restrict ourselves to only four derivatives. The second term is not unique. With the choice made above it is easy to check that [6]

$$W = \lambda(z - a) \quad (9)$$

is a static solution of the full equation of motion derived from the sum of both Lagrangians if

$$\lambda = \lambda_0 = \left( \frac{\theta_2}{2\theta_1} \right)^{1/4}. \quad (10)$$

Hence we can compare the scattering properties of the same solitons in both models. This was reported in detail in [6] where it was found that the effect of the extra terms, apart from stabilising the solitons, corresponds to the introduction of small additional repulsive forces between solitons. These forces lead to the emergence of a critical velocity below which the solitons scatter back to back in the head-on collisions, and above which they behave very much as in the original model except that the additional terms prevent them from shrinking.

When two solitons are sent towards each other at a zero impact parameter, they progressively slow down, and if their speed is large enough, they collide, form a ring and scatter at  $90^\circ$ . After the scattering the two solitons emerge with their initial sizes. If the speed is too small, the two solitons slow down, stop and then move back to back. The critical speed that differentiates between the back to back and the  $90^\circ$  scattering increases with the increase of  $\theta_1$  and  $\theta_2$ . We have also observed that the scattering time is longer when the speed is close to this critical velocity. When the speed is very close to this value the total energy density has the form of a nearly perfect ring, whereas the kinetic energy density exhibits an oscillatory motion. This motion generates a ring structure from which two peaks emerge at  $90^\circ$ , come back to form a new ring, scatter along the original direction and emerge again at  $90^\circ$  for a few oscillations after which the two solitons eventually scatter either back to back or at  $90^\circ$ . This behaviour is very similar to the formation of a resonance seen in some dynamical systems.

For the scattering with non zero impact parameters, the results have been very similar. In every simulation the two solitons have scattered at an angle dependent on the incoming speed. At small speeds the solitons have scattered at a small angle, whereas at larger speeds, the scattering angle has been larger than  $90^\circ$ ; the larger the impact parameter the larger the angle.

As these results have been discussed in detail in some of our previous publications [5,6] we will not discuss them further here. Instead we will report some of our new results on the annihilation of solitons and antisolitons.

### 3. Solitons and Antisolitons

A system involving solitons and antisolitons corresponds to  $W$  being a function of  $z$  and  $\bar{z}$ . In [6] we analysed in some detail various properties of a system described by the initial

configuration

$$W = \frac{(z - a)(\bar{z} + a)}{2\mu a}. \quad (11)$$

In particular, we considered this configuration for different values of  $\theta_2$  and  $\theta_1$ . As (11) is not a solution of the equation of motion the system is clearly unstable and when started at rest the soliton and the antisoliton approach each other and annihilate into pure radiation. During their approach the soliton and antisoliton attract each other and so accelerate while moving towards each other. As a result of this attraction the kinetic energy flows between them, distorting their shape. We have observed that right up to their annihilation, the soliton and the antisoliton preserve their identity. After the interaction the system represents pure radiation waves. What is interesting is that their maxima flow at  $90^\circ$  to the original direction of motion. The outgoing structures represent pure radiation waves and are not distorted solitons and antisolitons. To be convinced of this we observe that these waves move with the velocity of light and that their topological charge density gradually vanishes. In fact, the topological density of these waves is quite complicated. We can see four peaks in the density distribution (two maxima and two minima), all four decreasing in magnitude. Clearly, the outgoing waves are mixtures of the positive and negative topological charge densities and we interpret our results as being due to the attractive forces between solitons and antisolitons. Moreover, as we have reported in [8], our simulations have shown little dependence on the presence of the additional “Skyrme-like” and potential terms  $\mathcal{L}_1$ .

However (11) is not the most general configuration involving solitons and antisolitons. Clearly, (11) involves solitons whose phases are related to each other, that is,  $W$  close to  $z - a \sim 0$  is related to  $W$  around  $\bar{z} + a \sim 0$ . To consider more general configurations we have to go beyond (11).

This problem was discussed further in [9] where we considered

$$W = \frac{(z + a)(\bar{z} - a)\sqrt{1 + r^2}}{\mu[2a + ir(\dot{z} + \bar{z})]}, \quad (12)$$

where  $r$  is a real number. We observed that if we varied  $r$  we generated a phase between the solitons. The introduction of nonzero values of  $r$  did not alter the scattering process too much, it only affected the initial motion of the solitons. In fact for larger values of  $r$ , the initial net force between the soliton and the antisoliton was repulsive. At the same time they started rotating in the internal space into an attractive channel and, once there, the scattering proceeded as for  $r = 0$ . The results depended a little on whether  $\theta_s$  was zero or not except that for nonzero  $\theta_s$  and for larger  $r_s$ , when the whole process took too long, the solitons shrank before they were in the attractive channel and the numerical procedure broke down. Otherwise, the results were qualitatively the same.

In our previous papers [8,9] we have not discussed the time dependence of the annihilation process. We use this opportunity to present new results on this topic.

We have reanalysed the process of annihilation for a series of simulations, all with  $r = 0$  and with vanishing  $\theta_s$ , as the previous simulations have shown that there is no qualitative difference between taking these parameters zero or nonzero. These simulations have been performed for many values of  $\mu$  in (11), that is, for different values of the soliton sizes. The results are all qualitatively the same. In Figure 2 we present the pictures of the energy density at three values of time: initial, close to the annihilation, and at some time after the

annihilation. In Figure 3 we present the pictures of the topological charge density at the corresponding times. We see that although the annihilation is rapid it is not complete. The topological charge density after the annihilation is not identically zero but seems to be going to zero (notice the decreasing value of the maximum by reading the value shown on the axis.) Moreover, even though the maxima of the radiation waves after the scattering move at  $90^\circ$  to the direction of motion of the soliton and antisoliton just before their annihilation, the maxima and minima of the topological density do not seem to be related to the direction of the waves.

To study this problem further we decided to look at the contour plots of the energy density and of the topological charge and also to look at the time dependence of the total topological charge in the half plane which initially had contained, say, the antisoliton. This last case has involved looking at the total charge in the region  $x < 0$ .

Note in our pictures that the topological charge density never changes from being positive to negative or vice versa. In Figure 4 we present a series of contour plots of the energy density and of the topological charge. These plots support the claim that the topological charge density does not change its sign. They also show that the topological charge densities of the waves have a dipole-like character thus explaining the two maxima and two minima seen in the pictures. The strength of the peaks decreases rapidly with time. The dipoles follow the maxima of the waves and the distance between the minimum and the maximum increases slowly. This decrease is associated with the process of annihilation of the original solitons and antisolitons, at first of their cores and later of their tails. The reason for this is that although our solitons are well localised this localisation is only power-like and so, strictly speaking, the solitons extend all way to infinity.

The dipole-like structure of the topological charge density of the waves is seen quite clearly when one studies its angular distribution. In Figure 5 we present the angular distribution of the energy and the topological charge density of the initial configuration involving a soliton and an antisoliton, and of the wave configuration some time after the annihilation but before the wave has reached the boundary of our grid. The distribution is around the point  $x = 0, y = 0$  and its angle called theta on the graphs is normalised to 1 (*i.e.* it is really  $2\pi \times$  the value on the graph.)

We observe that initially both the maxima of the energy density and of the topological charge density are in the region  $(-0.25, 0.25)$ , and corresponding to the antisoliton,  $(0.25, 0.75)$ . After the scattering the maxima are in the regions  $(0.0, 0.5)$  and  $(0.5, 1.0)$  while the topological charge, as initially, is positive in the region  $(-0.25, 0.25)$ . In the region  $(0.0, 0.5)$ , the topological charge has both positive and negative contributions and is zero at the midpoint  $(0.25)$ . Looking at the time dependence of the rapidly decreasing maxima and minima of the total angular charge distribution, we see a slight movement towards 0.25 and 0.75, towards the directions of the maxima of the energy density. This effect is very small and is in agreement with what we have seen for the motion of the actual maxima and minima.

In Figure 6 we have plotted the trajectories in the  $x, y$  plane of the maxima of the total energy and topological charge densities. The horizontal parts of the trajectories describe the initial solitons and antisolitons. Only solitons contribute to the maximum of the topological charge density, the remaining parts (basically vertical) describe the radiation waves. We observe two maxima of the topological charge density which, together with the corresponding minima, clearly demonstrate the dipole nature of the radiation waves.

In Figure 7 we have plotted the time dependence of the value of the maximum of the

total energy and topological charge densities. We see that the decrease of the maximum of the topological charge density is rapid. In fact, most of this decrease takes place more or less at the time of the annihilation, which as can be seen from Figure 8a, takes place at  $t \sim 2$ .

Even more important is the rate of the decrease of the total topological charge in one hemisphere, say for theta in the region (0.25, 0.75). We have performed several such simulations in which we varied  $\mu$  in (11). We took  $\mu$  to be 0.225, 0.23, 0.3 and 0.4. We have plotted the total charge for  $x < 0$ , in the region initially containing the antisoliton. As the solitons with smaller  $\mu$  are better localised the attractive forces between them are weaker and hence they take more time to attract each other and annihilate. This is seen quite clearly in Figure 8 where we have plotted the time dependence of the distance from the origin of the maximum of the energy density for two such simulations, the  $\mu = 0.4$  case described before and also the  $\mu = 0.225$  case. Incidentally, looking at the slope of these curves we see that the radiation travels with the velocity of light (In our units,  $c = 1$ .)

In Figure 9 we present the curves of the time dependence of the total topological charge in the region  $x < 0$ . We find that after the initial motion, which depends on the value of  $\mu$ , the process of annihilation is the same. That is, the sections of the curves which describe the annihilation process are the same. Looking at the curves we see that once the annihilation starts going it proceeds quite rapidly and then it tails off. The main bulk of the annihilation, from  $\sim -0.9$  to  $\sim -0.2$ , takes about 1 unit of time, which as seen from the contour plots corresponds roughly to the soliton sizes just before the scattering. Hence the annihilation proceeds with the velocity of light.

Looking at the latter parts of the curves we see some differences. However, these differences look like lattice artifacts. To check this we decided to vary the lattice sizes and lattice steps. All our previous studies were performed on the  $201 \times 201$  lattice with the lattice step being given by  $dx = dy = 0.05$ . In Figure 10 we present our curves for 4 simulations with  $\mu = 0.4$ ; three with  $dx = dy = 0.05$  and with the lattice sizes  $201 \times 201$ ,  $251 \times 251$  and  $401 \times 401$  respectively, and one with  $dx = dy = 0.06$  and lattice size  $401 \times 401$ . As the results obtained with bigger lattice sizes are more reliable since the boundary is further away, we find that the “wiggles” towards the ends of the curves are indeed the finite lattice artifacts and should be disregarded, and that the decrease is smooth and monotonic as seen on bigger lattices. Of course the slow decrease part of the curve is due to the tails of the solitons themselves. As solitons extend all the way to infinity the complete annihilation can only be asymptotic.

#### 4. Some Conclusions

In this paper we have studied the annihilation of a soliton and an antisoliton, concentrating our study on the time evolution and on the properties of the final product of such scatterings. The time evolution of the annihilation process was essentially the same in all our simulations, irrespective of the initial size of the solitons. The only parameter that influenced it was the initial distance between the soliton and the antisoliton. As the attractive force between the soliton and the antisoliton decreases with the distance, the further they are away from each other the longer it takes them to move into the annihilation region. Once they get there the annihilation takes place very quickly and produces 2 waves of radiation that emerge mainly at  $90^\circ$  to the direction of motion just before the annihilation.

We have also observed that the waves carry non-zero topological densities whose struc-

tures look very much like those of a dipole field. However, the strength of this field rapidly decreases with time.

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## Figure captions

1. Picture of the energy density of a field configuration describing one solitons *ie* given by (9) with  $\lambda = 4$ .
2. Pictures of the energy densities as seen in the simulation with the initial configuration given by (11) (and  $\frac{\partial W}{\partial t} = 0$ ), with  $a = 1.0$ ,  $\mu = 0.4$ , for  $t = 0.0$ ,  $t = 2.0$  and  $t = 3.875$
3. As Figure 2 except that it shows the topological charge density.
4. Contour plots of the total energy density (on the left) and of the topological charge densities (on the right) obtained in the simulation as in Figures 2 and 3. The five plots correspond to  $t = 2.5$ ,  $t = 3$ ,  $t = 3.5$ ,  $t = 4$  and  $t = 5$
5. The angular distribution (the angle theta normalised to  $1$  ( $2\pi \rightarrow 1$ )) of the total energy density (on the left) and of the topological charge densities (on the right) seen in the same simulation for  $t = 0.0$  and  $t = 3.875$
6. Trajectories of the maxima of the total energy density (on the left) and of the topological charge density (on the right) seen in the same simulation.
7. The time dependence of the value of the maxima shown in Figure 6 (of the total energy density on the left and and of the topological charge density on the right).
8. Time dependence of the distance from the origin of the maxima of the energy density for the simulation shown before and also for the simulation in which  $\mu = 0.225$
9. Time dependence of the total topological charge in the region  $x < 0$ . The four cases correspond to the simulations with  $\mu = 0.225, 0.23, 0.3$  and  $0.4$ .
10. The dependence on the grid size and the number of points used in the simulation. The plots as in Figure 9 for  $dx = dy = 0.05$  for the lattice sizes  $201 \times 201$ ,  $251 \times 251$ ,  $401 \times 401$  and for  $dx = dy = 0.06$  for  $401 \times 401$ .

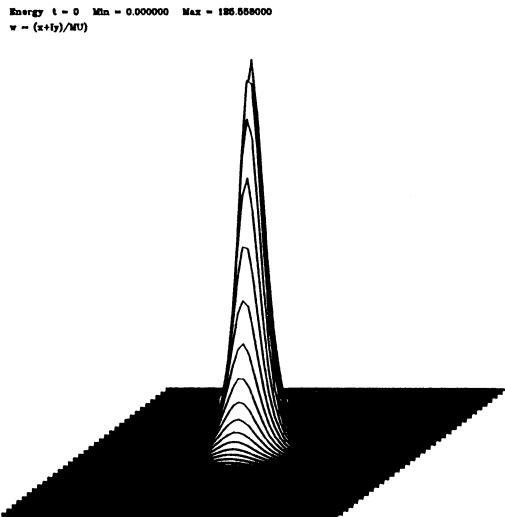


Figure 1

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bi= 0.0000 0.0000 bd= 0.0000 0.0000
mu= 0.4000 0.0000 vr= 0.0000 0.0000
gheb1eb = 0.00000E+00 gheb2eb = 0.00000E+00
```

```
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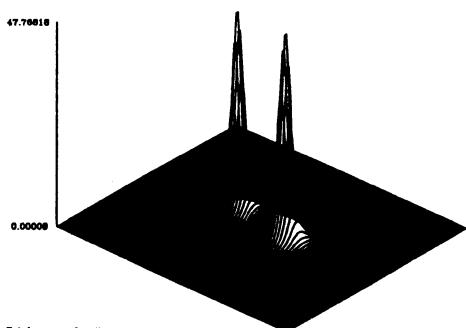


Figure 2a)

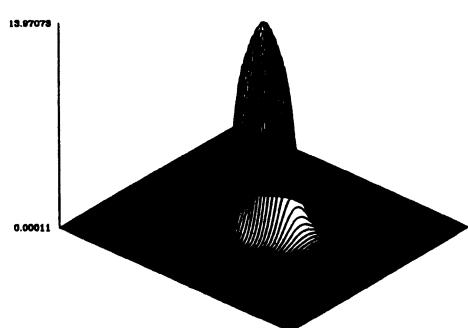
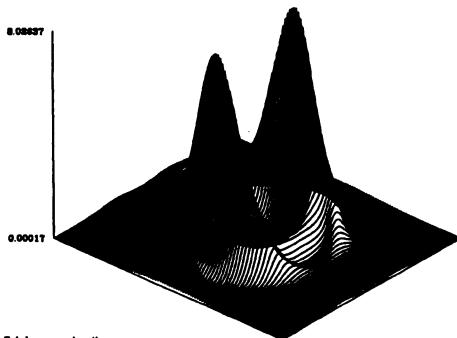


Figure 2b)

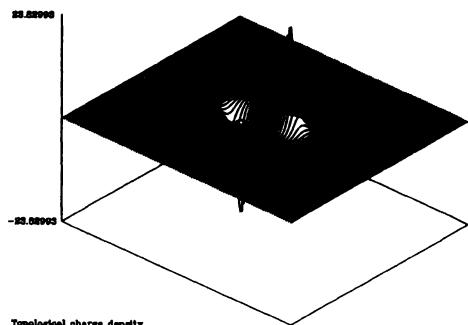
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ghab1eb = 0.00000E+00 ghab2eb = 0.00000E+00
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Total energy density

Figure 2c)

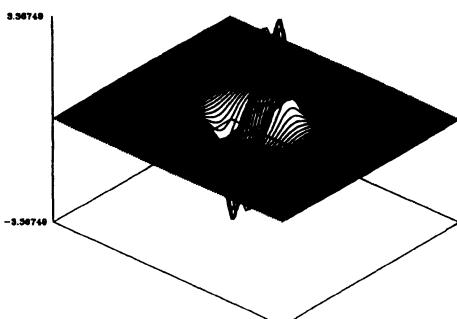
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Topological charge density

Figure 3a)

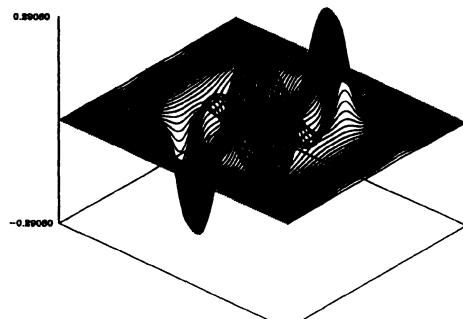
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bi= 0.0000 0.0000 bd= 0.0000 0.0000
mu= 0.4000 0.0000 vr= 0.0000 0.0000
ghab1eb = 0.00000E+00 ghab2eb = 0.00000E+00
```



Topological charge density

Figure 3b)

```
T = -3.8760 E= -0.19968174E+01 Zmax= -0.80979605E+00
ai= 1.0000 0.0000 ab= 1.0000 0.0000
bi= 0.0000 0.0000 bd= 0.0000 0.0000
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ghab1eb = 0.00000E+00 ghab2eb = 0.00000E+00
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Topological charge density

Figure 3c)

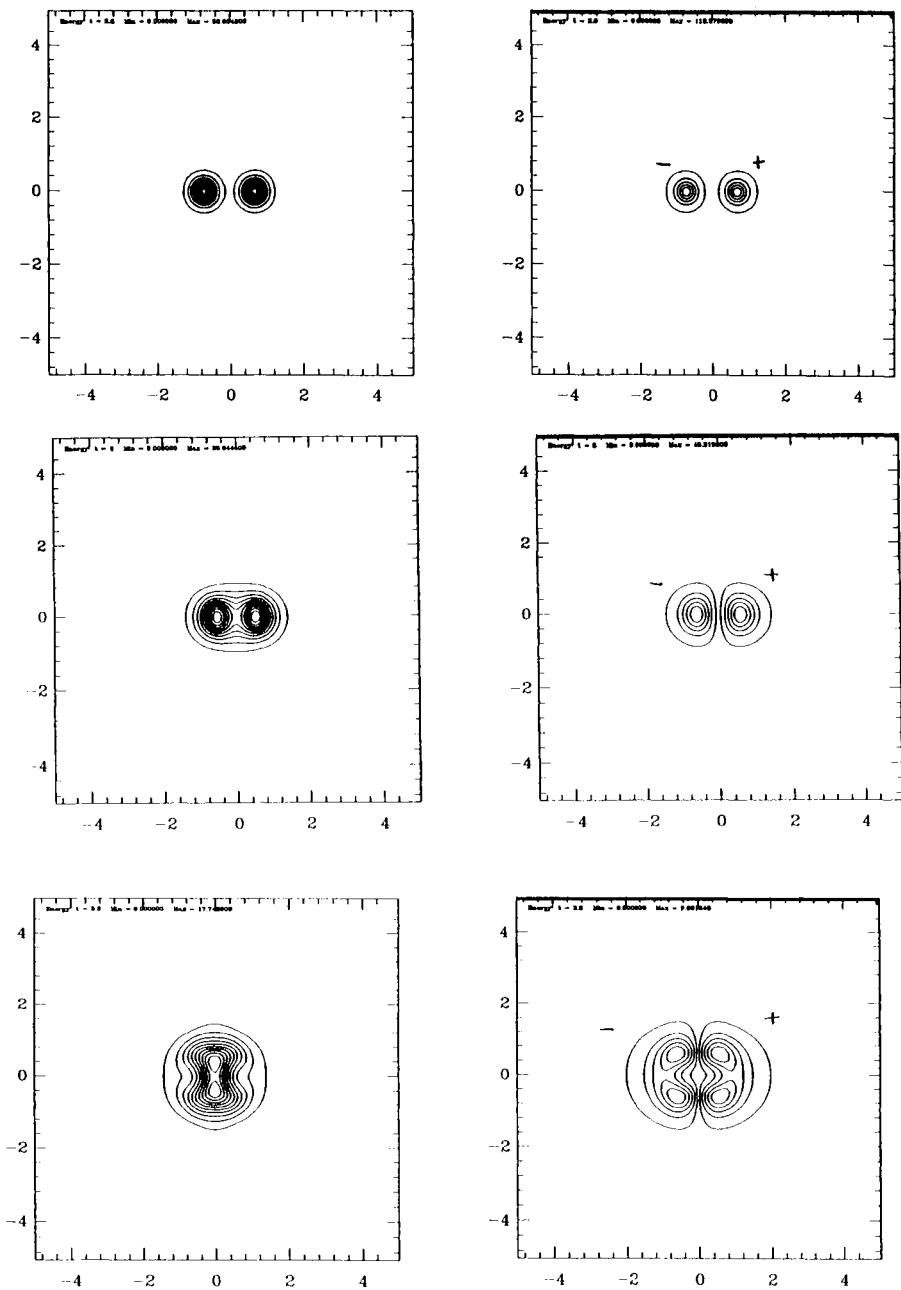


Figure 4

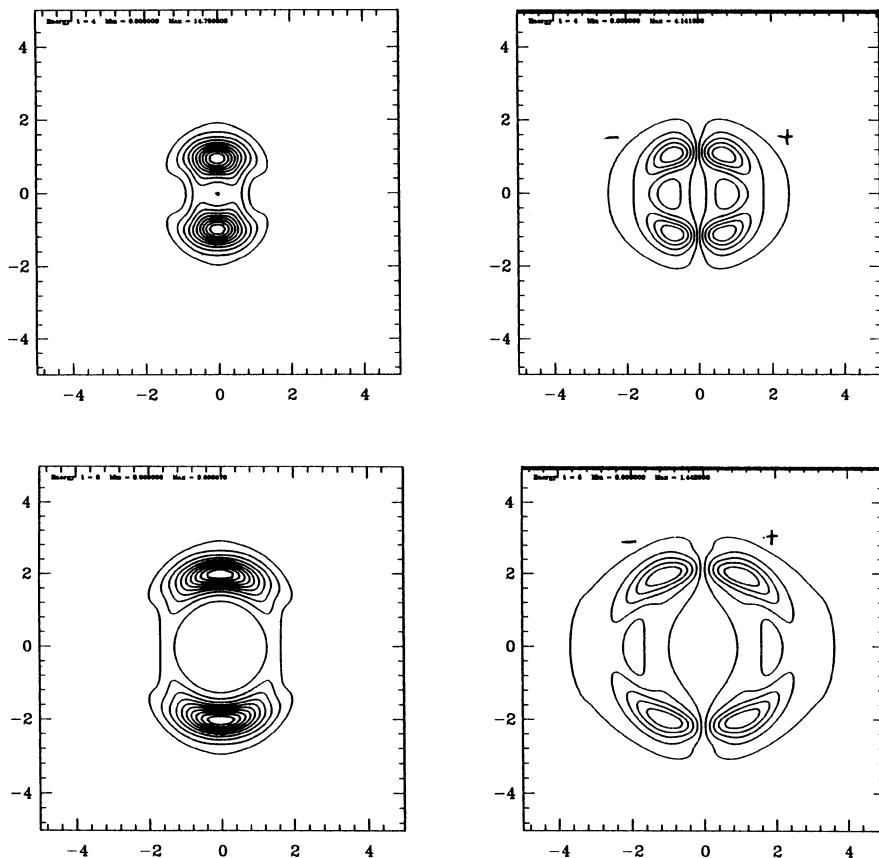


Figure 4 (cont.)

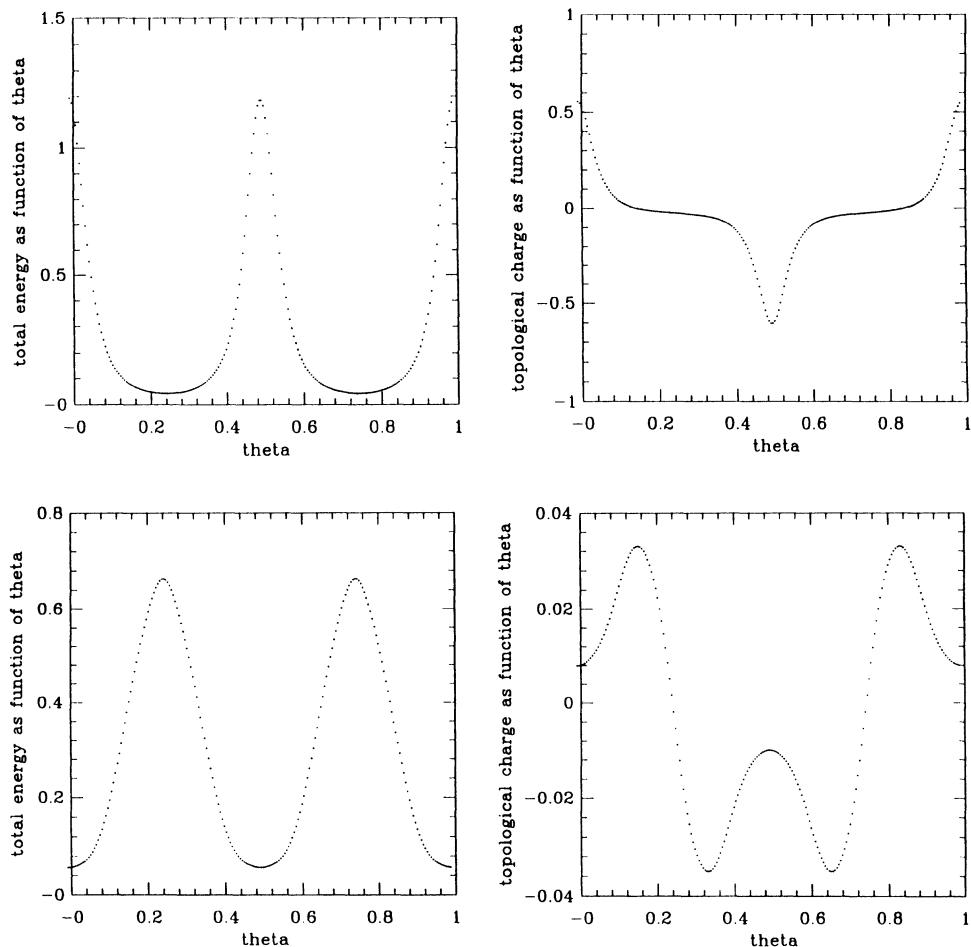


Figure 5

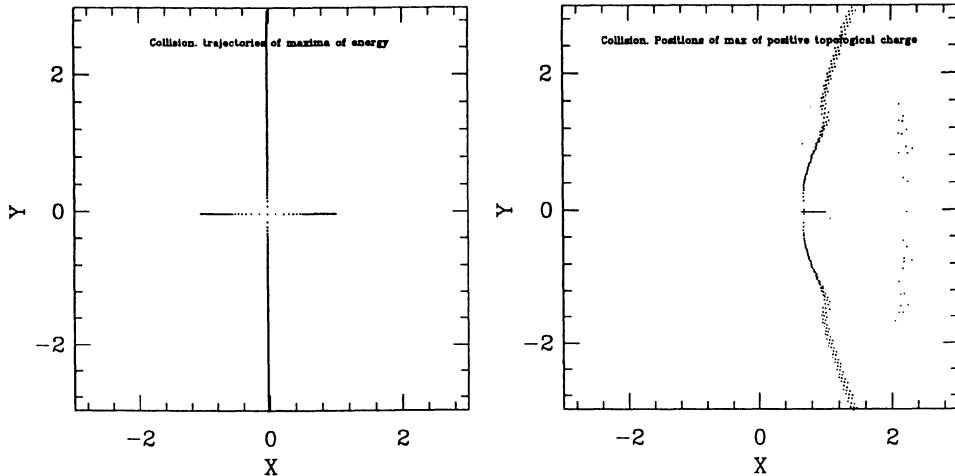


Figure 6

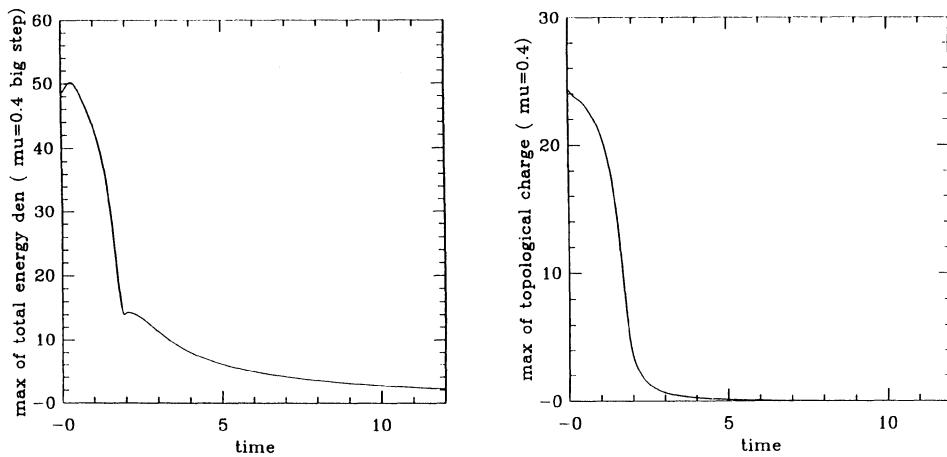


Figure 7

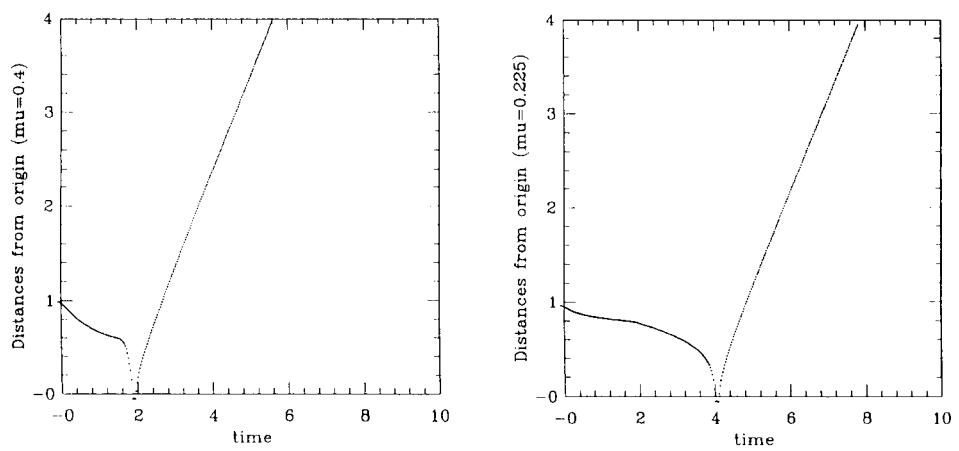


Figure 8

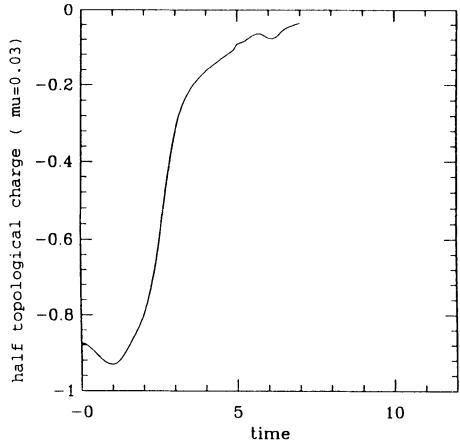


Figure 9a)

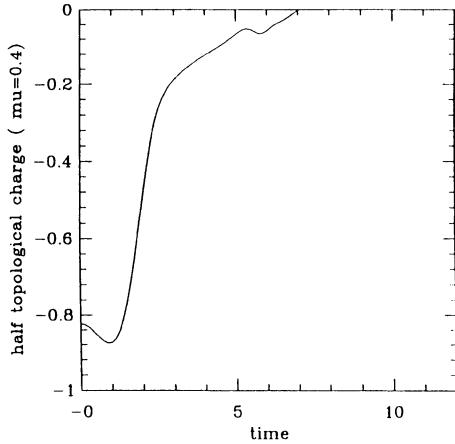


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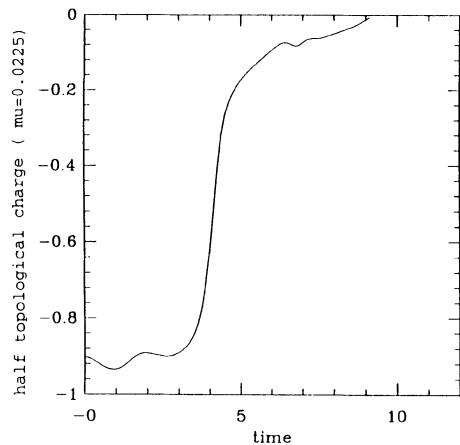


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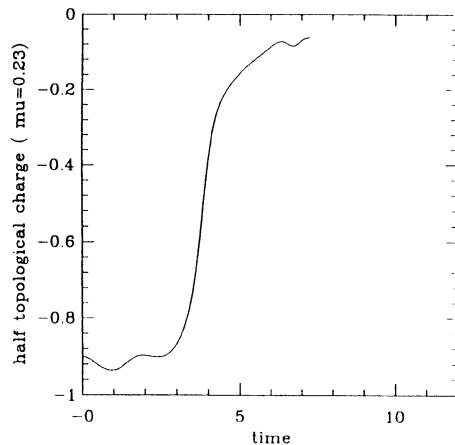


Figure 9d)

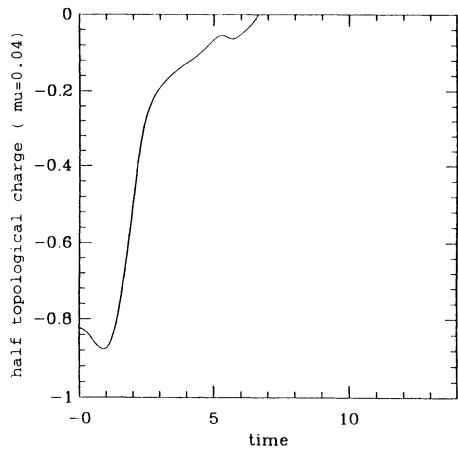


Figure 10a)

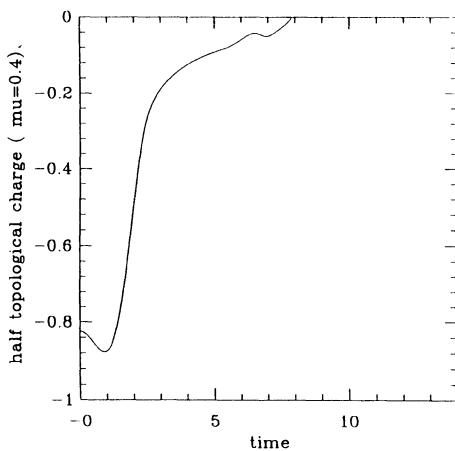


Figure 10b)

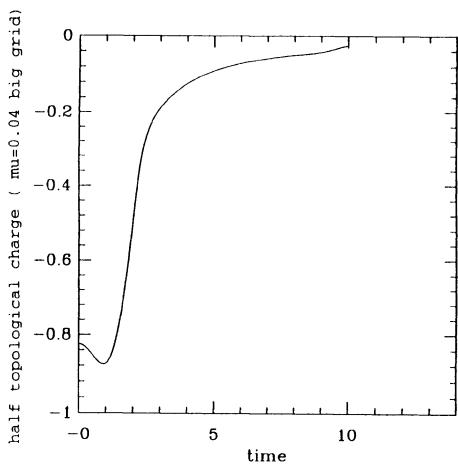


Figure 10c)

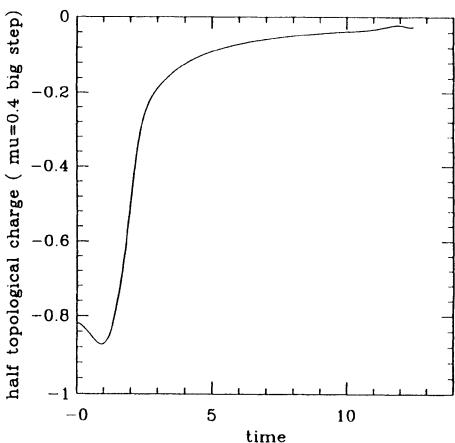


Figure 10d)

# NULL REDUCTIONS OF THE YANG-MILLS SELF-DUALITY EQUATIONS AND INTEGRABLE MODELS IN (2 + 1)-DIMENSIONS

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**ABSTRACT.** The self-dual Yang-Mills equations and their generalizations are known to contain many examples of lower dimensional integrable systems via a process of dimensional reduction. Here the self-duality equations are studied under a single null translational symmetry, resulting in (2 + 1)-dimensional integrable systems. The simplest soliton solutions are constructed, with the property of depending on arbitrary functions as well as arbitrary constants, and have non-trivial trajectories.

## 1. Introduction

One of the earliest, and still one of the most important, reductions of the Yang-Mills self-duality equations on  $\mathbf{R}^4$  is to impose a single translational symmetry. Explicitly, let the metric on  $\mathbf{R}^4$  be given by

$$ds^2 = \begin{cases} dz d\bar{z} + dw d\bar{w}, & z = x_1 + ix_2, \\ dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, & w = x_3 + ix_4, \end{cases}$$

with which the self-dual Yang-Mills equations  $F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}^{\alpha\beta}F_{\alpha\beta}$  are the integrability conditions for the otherwise overdetermined linear system

$$\begin{aligned} [(\partial_z - A_z) - \xi(\partial_{\bar{w}} - A_{\bar{w}})]\Phi &= 0, \\ [(\partial_w - A_w) + \xi(\partial_{\bar{z}} - A_{\bar{z}})]\Phi &= 0, \end{aligned}$$

where the gauge fields belong to the Lie algebra of the gauge group  $\mathcal{G}$ . Assuming that the gauge potentials are independent of one of the coordinates, say  $x_4$ , the self-duality equations reduce to an equation on  $\mathbf{R}^3$  with metric  $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ , namely

$$D_\mu\Psi \equiv (\partial_\mu - A_\mu)\Psi = \frac{1}{2}\varepsilon_\mu^{\alpha\beta}F_{\alpha\beta},$$

where  $\Psi = A_4$ . Solutions to this equation are known as magnetic monopoles, and have been much studied [1,2].

However, the actual construction does not depend critically on the choice of spacetime signature (though in  $\mathbf{R}^{3+1}$  there are no non-trivial real solutions for groups such as  $SU(N)$ ). Thus performing an identical reduction in  $\mathbf{R}^{2+2}$  yields the equation

$$D_\mu\Psi = \frac{1}{2}\varepsilon_\mu^{\alpha\beta}F_{\alpha\beta}, \tag{1}$$

defined on  $\mathbf{R}^{2+1}$  with metric  $ds^2 = -dt^2 + dx^2 + dy^2$ . Formally this may be thought of as defining a (1) on  $\mathbf{R}^{2+1}$ , rather than on  $\mathbf{R}^3$ .

On writing, where  $\mathbf{J} : \mathbf{R}^{2+1} \rightarrow \mathcal{G}$ ,

$$\begin{aligned}\mathcal{A}_t &= \mathcal{A}_y = \frac{1}{2}\mathbf{J}^{-1}(\partial_t\mathbf{J} + \partial_y\mathbf{J}), \\ \mathcal{A}_x &= -\Psi = \frac{1}{2}\mathbf{J}^{-1}\partial_x\mathbf{J},\end{aligned}$$

equation (1) becomes an integrable chiral model with torsion in  $(2+1)$ -dimensions [3,4]

$$\eta^{\mu\nu}\partial_\mu(\mathbf{J}^{-1}\partial_\nu\mathbf{J}) + \mathbf{V}_\alpha\varepsilon^{\alpha\mu\nu}\partial_\mu(\mathbf{J}^{-1}\partial_\nu\mathbf{J}) = 0, \quad \mathbf{V}_\alpha = (0, 1, 0).$$

This model has been much studied, and has the following properties:

- Conserved energy functional [3]
- Soliton solutions [3,5]
- Wave solutions [6]
- Non-trivial scattering of solitons [7].

The symmetry which reduces the  $(2+2)$ -dimensional self-duality equation to this  $(2+1)$ -dimensional model is a non-null Killing vector. In  $\mathbf{R}^{2+2}$ , unlike in  $\mathbf{R}^4$ , there is a further possibility, that of reducing by a symmetry generated by a single null Killing vector.

## 2. Null reductions of the wave equation

That null reductions differ from non-null reductions is apparent from looking at the wave equation on  $\mathbf{R}^{2+2}$  (sometimes called the ultrahyperbolic equation [8]). Let the metric on  $\mathbf{R}^{2+2}$  be given by

$$ds^2 = \begin{cases} dx_0^2 + dx_1^2 - dx_3^2 - dx_4^2, \\ du\,dv + dw\,dz, \end{cases}$$

the wave equation being  $\square\phi = 0$ .

Consider first a non-null reduction of the wave-equation, i.e.  $\partial_{x_4}\phi = 0$ . This implies that  $\phi$  is independent of  $x_4$ , and hence the wave equation reduces to

$$(\partial_{x_0}^2 + \partial_{x_1}^2 - \partial_{x_2}^2)\phi = 0.$$

However, if the same procedure is applied to a null symmetry, say  $\partial_z$ , so  $\partial_z\phi = 0$ , then the resulting equation is degenerate

$$\partial_u\partial_v\phi = 0;$$

the dependence on  $w$  being free.

More generally, one could look for a solution with the symmetry  $\partial_z\phi = \kappa\phi$ ,  $\kappa \neq 0$ , so  $\phi = e^{\kappa z}\tilde{\phi}(u, v, w)$ , obtaining the null-reduced wave equation

$$(\partial_u\partial_v + \kappa\partial_w)\tilde{\phi} = 0. \tag{2}$$

Thus to obtain a non-trivial solution to the null-reduced wave equation one has to look for a solution with a specific (and non-trivial) dependence on the null coordinate, rather than for a solution that has no dependence on the null coordinate.

The reasons for looking at such an apparently trivial example are two-fold: firstly it shares some of the same characteristic features as the null reduction of the non-Abelian –

and hence non-linear – self-duality equations (the wave equation itself is just the self-duality equation for an Abelian group), and secondly methods that have been developed to solve the non-linear self-duality equations (such as the Corrigan-Fairlie-t’Hooft-Wilczek ansatz [9], or more generally the Atiyah-Ward ansätze [10]) use solutions to linear equations – such as the wave equation – to construct solutions to the non-linear equations. Indeed, the soliton solutions to the non-linear equations that will be constructed in §3 occur exactly from the solutions of the null-reduced wave equation discussed above.

### 3. Null reductions of the $SU(2)$ self-dual Yang-Mills equation

The Yang-Mills self-duality equations are known to be a particularly rich source of integrable systems, these being obtained by the process of dimensional reduction (for a review see [11]). In particular, in  $\mathbf{R}^{2+2}$  with an  $SU(2)$  gauge group and under a non-null and an null translational symmetry, the self-duality equations reduce to the Non-Linear Schrödinger equation [12,13]. However, reducing by a single translation will also result in an integrable system, but now in  $(2+1)$ -dimensions. In §1 it was shown how Ward’s integrable chiral model came from a non-null reduction of the self-dual Yang-Mills equations, and in this section a reduction via a single null translation symmetry will be studied.

The form of the linear system is motivated by the requirement that the system be autonomous and reduces to the Non-Linear Schrödinger equation under a further non-null symmetry [12,13]. Thus on  $\mathbf{R}^{2+2}$  with metric  $ds^2 = dx dy - dt du$ , the self-duality equations are

$$\begin{aligned} [(\partial_x - \mathcal{A}_x) - \xi(\partial_u - \mathcal{A}_u)]\Phi &= 0, \\ [(\partial_t - \mathcal{A}_t) - \xi(\partial_y - \mathcal{A}_y)]\Phi &= 0, \end{aligned} \quad (3)$$

and with the gauge potentials

$$\begin{aligned} \mathcal{A}_u &= -\frac{1}{2}i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{A}_x = \begin{pmatrix} 0 & \psi \\ -\bar{\psi} & 0 \end{pmatrix}, \\ \mathcal{A}_t &= i \begin{pmatrix} \partial_x^{-1} \partial_y |\psi|^2 & \partial_y \psi \\ \partial_y \bar{\psi} & -\partial_x^{-1} \partial_y |\psi|^2 \end{pmatrix}, \quad \mathcal{A}_y = \mathbf{0}, \end{aligned} \quad (4)$$

where  $\psi$  is a complex-valued function of  $x, y$  and  $t$ , one obtains the  $(2+1)$ -dimensional integrable system

$$\begin{aligned} i\partial_t \psi &= \partial_{xy} + V\psi, \\ \partial_x V &= 2\partial_y |\psi|^2. \end{aligned} \quad (5)$$

This system has been considered by Zakharov [14], though not in the context of a reduction of the self-duality equations. That this was such a reduction was shown by Ward [15] and independently by Schiff [16].

In §5 solutions to this equation will be constructed, but before this some generalizations will be considered.

### 4. Generalizations

Two generalizations will be considered here. The first will generate the hierarchies of which (5) is the simplest example, and the second will involve higher dimensional gauge groups.

#### 4.1. HIERARCHIES

To construct the hierarchies of which (5) is the lowest lying member from a geometrical point of view, one generalizes the underlying complex manifold, known as twistor space. For the self-duality equations this space is  $\mathbf{CP}^3 - \mathbf{CP}^1 \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ . A simple generalization is to replace  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  by  $T_{m,n} \cong \mathcal{O}(m) \oplus \mathcal{O}(n)$ . The Ward construction then leads to the generalized self-duality equations [17], which are are integrability condition for the linear system

$$K_k \Phi = 0 \quad k = 1, \dots, m,$$

$$L_l \Phi = 0 \quad l = 1, \dots, n,$$

where  $K_k$  and  $L_l$  are defined by

$$\begin{aligned} K_k &= \xi \left[ \frac{\partial}{\partial x_{k-1}} - \mathcal{A}_{k-1} \right] - \left[ \frac{\partial}{\partial x_k} - \mathcal{D}_k \right], \\ L_l &= \xi \left[ \frac{\partial}{\partial t_{l-1}} - \mathcal{C}_{l-1} \right] - \left[ \frac{\partial}{\partial t_l} - \mathcal{B}_l \right]. \end{aligned} \quad (6)$$

The gauge potential  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  are Lie algebra-valued functions of the coordinates  $x_i, i = 1, \dots, m$  and  $t_j, j = 1, \dots, n$ .

By performing an analogous reduction to the one outlined in §3 one is left with the system

$$\begin{aligned} \partial_x \Phi &= \left\{ \xi^m \mathcal{A} + \sum_{i=0}^{m-1} \xi^i \mathcal{D}_{m-i} \right\} \Phi, \\ \partial_t \Phi &= \left\{ \xi^n \partial_y + \sum_{j=0}^n \xi^j \mathcal{B}_{n-j} \right\} \Phi, \end{aligned}$$

where  $\mathcal{A}$  is a normal form for the gauge algebra, which for  $SU(2)$  is  $\mathcal{A} = -\frac{1}{2}i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

The integrability conditions for this system yields the equation in question.

Thus for  $m = n = 1$  (and  $\mathcal{B}_0 = 0$ ) one recovers the original system (3) with potentials (4) and hence the equation (5). With  $m = 1$  and  $n$  arbitrary one obtains a  $(2+1)$ -dimensional AKNS hierarchy, and with  $m = 2$  and  $n$  even one obtains a  $(2+1)$ -dimensional DNLS hierarchy. With the further symmetry  $\partial_x - \partial_y = 0$  the standard forms of the hierarchies is recovered (for general  $m$  and  $n$  these reduced hierarchies have an interesting structure [18]).

#### 4.2. HIGHER RANK GAUGE GROUPS

Increasing the dimension of the gauge group results in multicomponent generalizations of (5). When the algebra is one associated to an Hermitian symmetric space the structure is particularly elegant [19], and leads to the system

$$aq_t^\alpha = q_{xy}^\alpha + \sum_{\beta, \gamma, \delta} R_{\beta\gamma-\delta}^\alpha q^\beta V^{\gamma\delta},$$

$$-ap_t^\alpha = p_{xy}^\alpha + \sum_{\beta, \gamma, \delta} R_{-\beta-\gamma\delta}^{-\alpha} p^\beta V^{\gamma\delta},$$

$$V_x^{\gamma\delta} = (p^\gamma q^\delta)_y.$$

Here  $R_{\beta\gamma-\delta}^\alpha$  is the curvature tensor of the underlying Hermitian symmetric space.

Of course, the integrability does not depend on the particular gauge group, and other  $(2+1)$ -dimensional systems may be obtained with different groups. It might also be interesting to look at infinite dimensional groups.

## 5. Solution generating techniques

This section outlines some of the methods that may be used to solve the system (5).

### 5.1. DRESSING

As the system (5) is a reduction of the self-dual Yang-Mills equations it is not surprising that the methods used to solve the duality equations may be adapted to solve the null-reduced system.

The dressing method (sometimes called the “Riemann Problem with zeros” method [20]) is a way to construct a new solution from an old solution via the ansatz

$$\Phi_{\text{new}} = \left( 1 + \frac{R(x, y, t, u)}{\xi - \xi_0(x, y, t, u)} \right) \Phi_{\text{old}},$$

where  $\Phi_{\text{old}}$  is a known solution of (3). For  $\Phi_{\text{new}}$  to be a new solution (with a new set of gauge potentials) implies differential equations for the matrix  $R$  and the scalar  $\xi_0$  which may be easily solved in terms of some freely specified functions. For  $\Phi_{\text{new}}$  to be a solution to the null-reduced self-duality equations imposes simple constraints on these otherwise free functions (such as  $\xi_0$  being a constant), and thus new solutions may be constructed from old [17].

In terms of the geometrical picture solutions correspond to bundles over the underlying complex manifold, and these may be characterised by a so-called patching matrix [21]. A class of such patching matrices from which the solution may be easily extracted is given by the Atiyah-Ward ansätze [10]. As shown by Tafel [22,23], the patching matrix may be found from the data in the dressing method, and the above dressing procedure corresponds to a transform from the Atiyah-Ward class  $\mathcal{A}_N$  to the class  $\mathcal{A}_{N+1}$ .

The patching matrices  $\mathbf{g}$  in the simplest such class,  $\mathcal{A}_1$ , are given by matrices of the form

$$\mathbf{g} = \begin{pmatrix} \xi & \Gamma \\ 0 & \xi^{-1} \end{pmatrix},$$

and these correspond to the one soliton solution (more generally, the class  $\mathcal{A}_N$  corresponds to an  $N$ -soliton solution). The off-diagonal element  $\Gamma$  gives rise to a solution to the null-reduced wave equation (2), as indicated at the end of §2.

In §2 it was shown that to obtain a non-trivial solution to the null reduced wave equation one had to look for solutions with a specific dependence on the null symmetry. Similar comments now apply to the non-Abelian case. The non-null translational Killing vector corresponds to a non-vanishing holomorphic vector field  $V$  on the twistor space  $T$ , and so it is possible to quotient out by the vector field and just consider bundles over the quotient manifold  $T/V$ , known as minitwistor space. This was originally studied by Hitchin in the construction of monopoles [1,2], and may also be used in the construction of a solution to Ward’s integrable chiral model [24]. The holomorphic vector field corresponding to a null

translational Killing vector has, in contrast, zeros and so it is not possible to quotient out by this symmetry. Thus one cannot entirely remove the dependence on the symmetry, and one has to look for solutions with a specific dependence on the null coordinate, as in the case of the null-reduced wave equation studied in §2. Here the role of the constant  $\kappa$  is played by the constant matrix  $A$ . If this were zero the resulting equations would be essentially trivial.

## 5.2. HIROTA'S BILINEAR METHOD

One of the aims of the Exeter conference was to promote the study of the connection between the inverse scattering transform and twistorial methods. Perhaps one could also add the connection between Hirota's method and the more geometrical approaches. In this section soliton solutions to the system (5) will be constructed using Hirota's bilinear approach [25]. There are tantalizing similarities between this approach and that outlined in the above section, though the connection between these methods is at present unknown.

On writing  $\psi = g/f$  and  $V = h\bar{h}/f^2$  (where  $g, h$  are complex valued functions and  $f$  is a real valued function) and using the identity

$$\partial_x \left[ \frac{D_y D_x(f \bullet f)}{f^2} \right] = \partial_y \left[ \frac{D_x^2(f \bullet f)}{f^2} \right]$$

the equation (5) may be written in the form

$$(iD_t + D_x D_y)(g \bullet f) = 0, \\ D_x^2(f \bullet f) = 2g\bar{g}, \quad (7)$$

where  $D_\mu$  is the Hirota bilinear operator, defined by

$$D_x^m \dots D_y^n(f \bullet g) = (\partial_x - \partial_{x'})^m \dots (\partial_y - \partial_{y'})^n f(x, \dots, y) g(x', \dots, y')|_{x=x', \dots, y=y'}.$$

The construction of solutions to these equations is standard (for a review see, for example, [26]). One expands the functions  $f$  and  $g$  as a series

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \dots, \\ g = \varepsilon g_1 + \varepsilon^3 g_3 + \dots$$

Substituting these expressions into (7) and equating coefficients of  $\varepsilon$  yields

$$(i\partial_t + \partial_x \partial_y)g_1 = 0, \\ (i\partial_t + \partial_x \partial_y)g_3 = -(iD_t + D_x D_y)(g_1 \bullet f_2), \\ \vdots \\ (i\partial_t + \partial_x \partial_y)g_{2n+1} = - \sum_{r+s=n} (iD_t + D_x D_y)(g_{2r+1} \bullet f_{2s}), \quad (8)$$

and

$$\partial_x^2 f_2 = g_1 \bar{g}_1, \\ \partial_x^2 f_4 = (g_1 \bar{g}_3 + g_3 \bar{g}_1) - \frac{1}{2} D_x^2(f_2 \bullet f_2),$$

$$\begin{aligned} & \vdots \\ \partial_x f_{2n} &= \sum_{r+s=n} g_{2r-1} \bar{g}_{2s+1} - \frac{1}{2} D_x^2(f_{2r} \bullet f_{2s}). \end{aligned}$$

The crucial equation is the one for  $g_1$ ,

$$(i\partial_t + \partial_x \partial_y)g_1 = 0.$$

With the ansatz  $g_1 = (a + \bar{a}) \exp(ax + by + ct)$  (where  $a, b$  and  $c$  are complex constants) one obtains the dispersion relation  $c = iab$  and hence the familiar looking one-soliton solution

$$\psi = \frac{a + \bar{a}}{2} \left\{ \frac{\exp\{i[x\Im(a) + y\Im(b) + t\Im(c)]\}}{\cosh[x\Re(a) + y\Re(b) + t\Re(c)]} \right\}, \quad c = iab.$$

However, a more general solution is given by the ansatz

$$g_1 = \exp(ax)\phi(y, t), \quad a \in \mathbb{C},$$

where  $\phi$  satisfies the equation

$$(i\partial_t + a\partial_y)\phi(y, t) = 0, \tag{9}$$

that is,  $\phi(y, t) = \phi(y + iat)$ .

From this one proceeds to calculate  $f_2$ :

$$f_2 = \partial_x^{-2} e^{(a+\bar{a})x} |\phi|^2 = \frac{e^{(a+\bar{a})x}}{(a+\bar{a})^2} |\phi|^2.$$

For this to give a solution one requires the inhomogeneous term in the definition of  $g_3$  to vanish, so

$$\begin{aligned} (iD_t + D_x D_y)(g_1 \bullet f_2) &= \frac{1}{(a+\bar{a})^2} (iD_t + D_x D_y)(e^{ax}\phi \bullet e^{(a+\bar{a})x}\phi\bar{\phi}), \\ &= \frac{1}{(a+\bar{a})^2} e^{(2a+\bar{a})x} (iD_t - \bar{D}_y)(\phi \bullet \phi\bar{\phi}), \\ &= \frac{1}{(a+\bar{a})^2} e^{(2a+\bar{a})x} (-i\partial_t + \bar{a}\partial_y)\bar{\phi}, \\ &= 0, \end{aligned}$$

using the ansatz (9).

The one-soliton solution to (5) is thus

$$\psi = \frac{a + \bar{a}}{2} \left\{ \frac{\exp\{i[\Im(a)x + \Im h(y, t)]\}}{\cosh[\Re(a)x + \Re h(y, t)]} \right\}, \quad a \in \mathbb{C},$$

where  $\phi = (a + \bar{a}) \exp[h(y + iat)]$ . For fixed  $(y, t)$ ,  $|\psi| \rightarrow 0$  as  $x \rightarrow \pm\infty$ . The wave front itself is defined by the equation

$$\Re[ax + h(y + iat)] = 0.$$

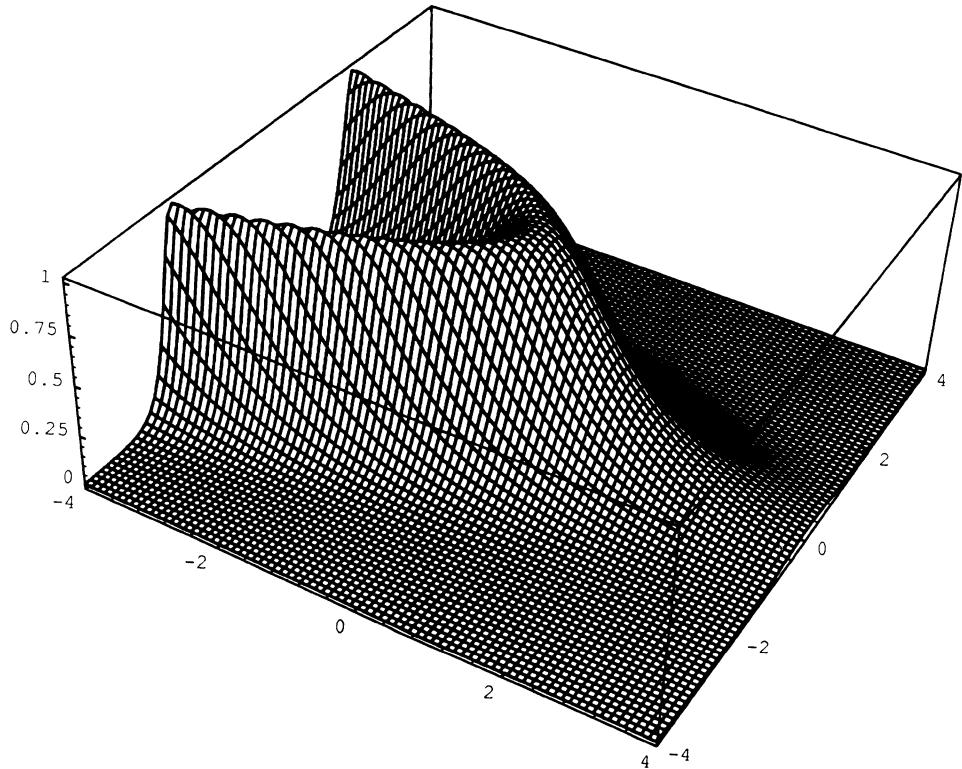


Figure 1. A 1-soliton solution, with  $h(\omega) = \omega^2$ .

An example, corresponding to the choice  $h(\omega) = \omega^2$  is given in Figure 1 (the data being  $a = 1 + i$  and  $t = 0$ ). The wave front moves rigidly with the  $(x, y)$  velocity being given by  $(v_x, v_y) = (2\Re(a)t, \Im(a)t)$ . This is summarised (for  $a = 1 + i$ ) in Figure 2.

The construction on the  $N$ -soliton solution is standard [26]. One takes

$$g_1 = \sum_{r=1}^N \exp[a_r + h_r(y + a_r t) + \eta_r^0],$$

(where  $a_r$  and  $\eta_r^0$  are arbitrary complex constants and  $h_r$  is an arbitrary function), and substitutes this into (8) and (9), and eventually the inhomogeneous term of  $g_{2N+1}$  will vanish, even with the arbitrary functions being present.

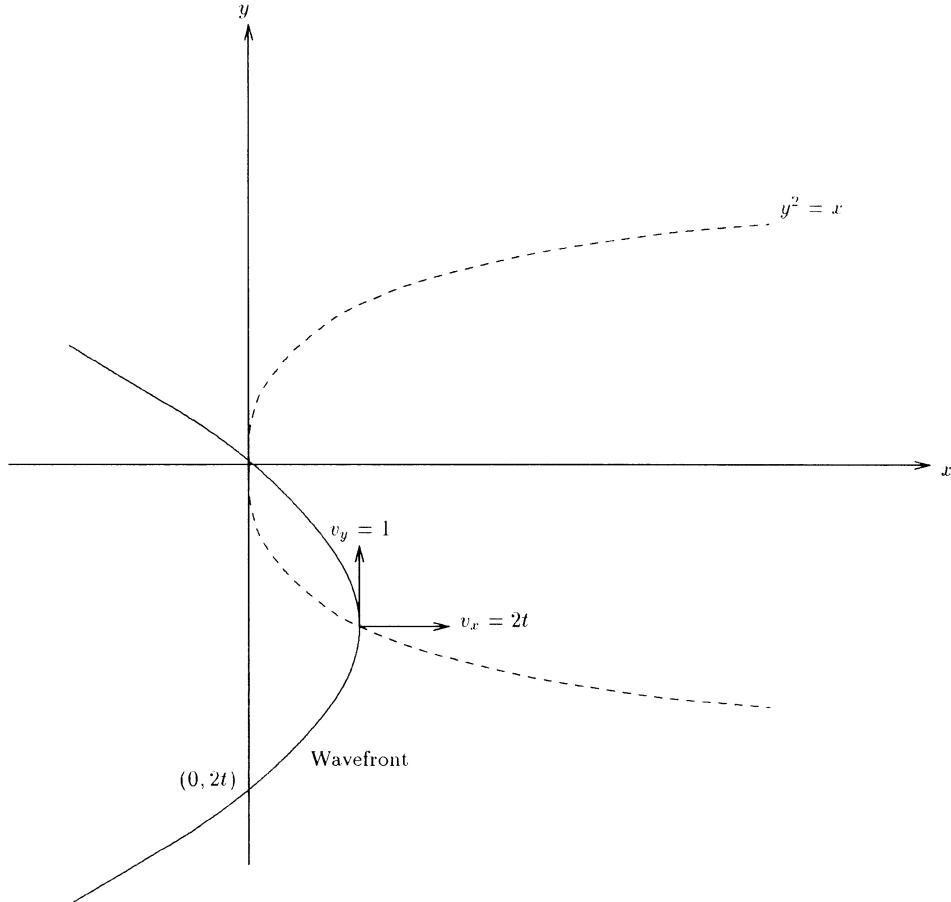


Figure 2. The geometry of the 1-soliton configuration.

Figure 3 shows the solution of a 2-soliton system, corresponding to the data

- (i)  $h_1(\omega) = 1, \quad a_1 = 2, \quad \eta_1^o = 0$
- (ii)  $h_2(\omega) = \omega^2, \quad a_2 = 1, \quad \eta_2^o = 0,$  (10)

at time  $t = 0$ .

The system (7) was found by Hietarinta to have a 3-soliton solution, which is often taken to be an indication that the system is integrable. However the corresponding Lax pair (which guarantees both integrability and the existence of multi-soliton solutions) was not previously known for this system, nor the arbitrary function solution.

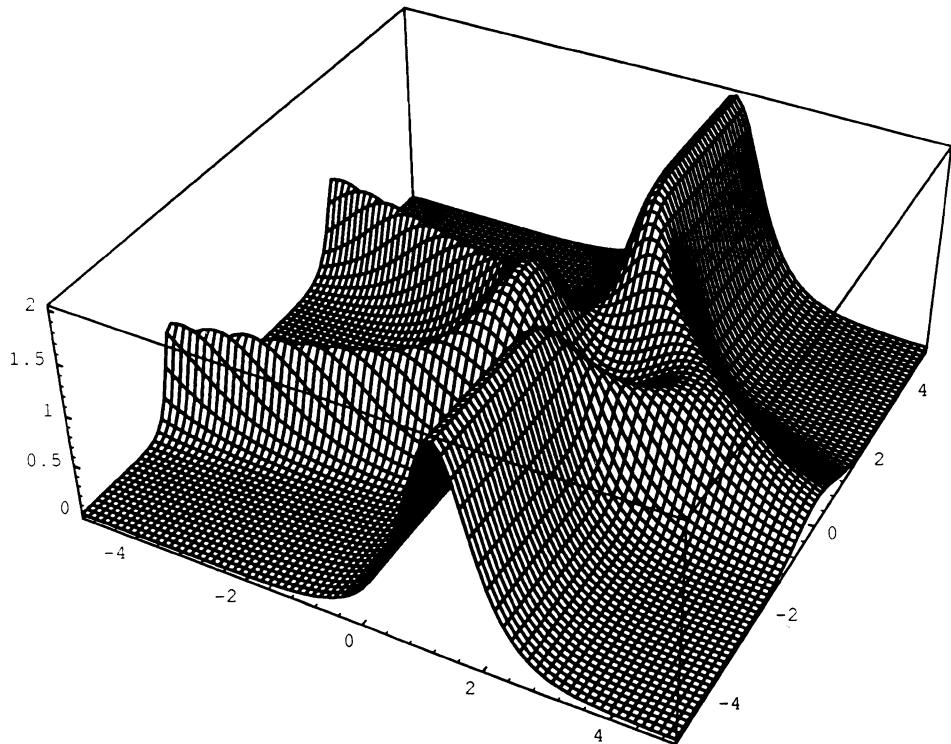


Figure 3. A 2-soliton interaction, corresponding to the data given by (10).

## 6. Conclusion

In this paper some  $(2+1)$ -dimensional integrable models have been studied which possess a geometrical description in terms of a twistor space construction. While systems such as the KP and DS equations may be written as the integrability condition for a self-dual system with an infinite rank gauge group, the corresponding twistor geometry for such infinite rank bundles has not, at yet, been discovered, though it is widely believed that such a description must exist.

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## CONTINUOUS AND DISCRETE SDYM, AND REDUCTIONS

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**ABSTRACT.** This talk reviews an integrable chiral equation in 2+1 dimensions, a reduction of Self-Dual Yang-Mills, which admits non-trivial soliton dynamics. It also describes a strategy for obtaining integrable discrete versions of SDYM and its reductions, involving the use of an algebra of lattice functions on which finite-difference operators are derivations.

### 1. An integrable chiral equation in 2 + 1 dimensions

Let me begin by describing an integrable hyperbolic partial differential equation in 3-dimensional space-time, a dimensional reduction of the self-dual Yang-Mills equation in 2 + 2 dimensions. It admits multi-soliton solutions which can interact or scatter either trivially or non-trivially.

The chiral field is a  $2 \times 2$  unitary matrix function  $\varphi(x^\mu) \in \mathrm{SU}(2)$ , where  $x^\mu = (t, x, y)$  are the space-time coordinates. The equation is

$$(\eta^{\mu\nu} + \varepsilon^{\mu\nu})\partial_\mu(\varphi^{-1}\partial_\nu\varphi) = 0, \quad (1)$$

where  $\partial_\mu = \partial/\partial x^\mu$ ,  $\eta^{\mu\nu}$  is the space-time metric,  $\varepsilon^{\mu\nu}$  is a constant skew tensor, and the Einstein summation convention applies. So (1) represents a hyperbolic system of three coupled partial differential equations, since  $\mathrm{SU}(2)$  is 3-dimensional. Note that if  $\varphi$  is restricted to be a diagonal matrix  $\mathrm{diag}[\exp(if), \exp(if)]$ , then (1) reduces to the wave equation for  $f$ . So (1) may be thought of as a non-linear generalization of the wave equation in 2 + 1 dimensions. There is a conserved energy functional, which is the integral over  $x$  and  $y$  of the energy density

$$E = |\varphi^{-1}\partial_t\varphi|^2 + |\varphi^{-1}\partial_x\varphi|^2 + |\varphi^{-1}\partial_y\varphi|^2; \quad (2)$$

here  $|\mathbf{M}|^2$  denotes the trace of the matrix  $\mathbf{M}^2$  (Ward 1988a). The boundary condition on  $\varphi$  at spatial infinity is that its energy should be finite. The finite-energy static solutions of (1) are all known explicitly, and a plot of  $E$  reveals that they look like  $n$  localized solitons (lumps) in the  $xy$ -plane, where  $n$  can be any positive integer.

A “Painlevé test” indicates that (1) is integrable if and only if  $\varepsilon^{\mu\nu}\varepsilon_{\mu\nu} = -2$  (Ward 1988b), and it is precisely for these values of  $\varepsilon^{\mu\nu}$  that (1) is a reduction of the self-duality equations. If  $\varepsilon^{\mu\nu}$  is zero, then the lump solutions referred to above are unstable. In the

integrable case, however, numerical experiments indicate that they are stable (Sutcliffe 1992a), although an analytic proof is as yet lacking. From now on, I shall restrict to the integrable case, by setting  $\varepsilon^{20} = 1 = -\varepsilon^{02}$  (other components zero).

The equations (1) arise as the consistency conditions for a pair of linear equations, which is what is meant by integrability in the present context. They are

$$(\zeta \partial_x - \partial_t - \partial_y) \Phi = \mathbf{A} \Phi, \quad (3a)$$

$$(\zeta \partial_t - \zeta \partial_y - \partial_x) \Phi = \mathbf{B} \Phi. \quad (3b)$$

Here  $\zeta$  is a complex parameter, and  $\mathbf{A}, \mathbf{B}, \Phi$  are  $2 \times 2$  matrix functions of  $x^\mu$ , with  $\Phi$  additionally dependent on  $\zeta$ . If we take  $\Phi$  to have  $n$  simple poles in  $\zeta$ , whose location does not depend on  $x^\mu$ , then the corresponding solutions of (1) are easily understood (Ward 1988a). Such a solution corresponds to  $n$  solitons, each moving at constant velocity and experiencing no scattering, not even a phase shift, when it interacts. For these solutions, the field  $\varphi$  is a rational function of  $t, x, y$ .

In principle, one should be able to predict and understand all the qualitative features of soliton solutions of (1) by exploiting the linear system (3). This might use the geometrical language of twistor theory (see Ward 1989, 1990 for some preliminary work in this direction), or it might use the more analytical approach of inverse scattering. However, it is not clear that one can carry out such a theoretical analysis in practice. For example, numerical work (see below) has demonstrated the existence of another sort of soliton solution, for which  $\Phi$  would presumably have a more complicated singularity structure.

Since  $\varphi$  consists of three real-valued functions, the field has internal degrees of freedom which do not show up in the energy density  $E$ . In other words, one may have two different sets of initial data, for which  $E$  was initially the same. Sutcliffe (1992b) carried out some numerical simulations involving the head-on collision of two solitons, using various different sets of initial data. By choosing data corresponding to one of the exact solutions mentioned above, he reproduced the trivial scattering that was expected which provided some evidence that his numerical procedure was reliable. But when he used different initial data, he found that the solitons scattered at right angles after colliding.

So the indication seems to be that the internal degrees of freedom in the chiral field  $\varphi$  allow the behaviour of its solitons to be rather rich. In the absence of a theoretical understanding, more numerical evidence would be welcome. The  $(2+1)$ -dimensional numerical simulations referred to above are somewhat tedious, and an alternative approach would be preferable. One possibility might be to find a way of making the variables  $x, y$  and possibly also  $t$  discrete, while maintaining the integrability of the equation. In other words, one might look for an integrable lattice model of the original PDE. The next section describes a search for such a model.

## 2. Integrable lattice SDYM and reductions

The problem is to find an acceptable lattice version of a linear system like (3). To start with, and to illustrate this problem, let us examine a simpler case, namely the chiral equation in  $1+1$  dimensions. The relevant linear system is a reduction of (3), in which the functions are independent of the variable  $y$ :

$$(\zeta \partial_x - \partial_t) \Phi = \mathbf{A} \Phi, \quad (4a)$$

$$(\zeta \partial_t - \partial_x) \Phi = \mathbf{B} \Phi. \quad (4b)$$

In terms of the characteristic coordinates  $\xi = \frac{1}{2}(x + t)$  and  $\sigma = \frac{1}{2}(x - t)$ , the system (4) may also be written in the form

$$\partial_\xi \Phi = (\zeta - 1)^{-1} \mathbf{A}' \Phi, \quad (5a)$$

$$\partial_\sigma \Phi = (\zeta + 1)^{-1} \mathbf{B}' \Phi. \quad (5b)$$

Now one can discretize either  $\xi$  or  $\sigma$ , or both, by replacing the appropriate partial derivative  $\partial$  in (5) with (say) the corresponding forward difference operator  $\Delta$ . The consistency condition for the discrete linear system is then an integrable differential/difference equation, which is a discrete version of the  $(1+1)$ -dimensional chiral equation in characteristic coordinates. There are many examples in the literature of discrete integrable versions of equations such as the NLS or KdV equations, obtained in this sort of way.

But this procedure is in fact rather limited. Suppose, for example, that we wished to discretize the spatial variable  $x$  in (4), while keeping time  $t$  continuous. If we simply replace  $\partial_x$  in (4) by a partial difference operator  $\Delta_x$ , then the linear system becomes unacceptable because its consistency requires  $\mathbf{A}$  and  $\mathbf{B}$  not to depend on  $x$  at all. In other words, one gets extra constraints in addition to the discrete version of the original chiral equation.

The same difficulty arises if we try to discretize one or more of the independent variables in (3), or for that matter in the linear system for the four-dimensional SDYM equations. In a sense, this difficulty is due to the failure of the Leibniz rule for finite differences. One way of preserving the Leibniz rule

$$\Delta(f * g) = f * (\Delta g) + (\Delta f) * g. \quad (6)$$

is to allow the multiplication operator  $*$  to be non-local, rather than pointwise. Some possibilities along these lines were analysed by Bouguenaya & Fairlie (1986) and Ward (1992); the following gives a flavour of what is involved.

Each variable can be dealt with separately, so for simplicity let us consider functions of a discrete variable  $x$  (implicitly also depending on the continuous variables  $t, y, \dots$ ). Take the  $x$ -derivative operator to be the forward difference

$$(\Delta f)(x) = h^{-1}[f(x + h) - f(x)]; \quad (7)$$

here  $h$  is a fixed positive number (the lattice spacing). This is just the simplest possibility; one could also take  $\Delta$  to be more non-local than (7).

We now consider the action of  $\Delta$  on polynomials in  $x$ . The identity function  $x$  satisfies  $\Delta x = 1$ . The product  $x * x$  must satisfy, in order for the Leibniz rule (6) to hold,

$$\Delta(x * x) = 2x; \quad (8)$$

the solution of (8) is

$$x^{[2]} := x * x = x(x - h) + c_1, \quad (9)$$

where  $c_1$  is a constant. Continuing in this way, we obtain an expression for the  $*$ -product  $x^{[n]}$  of  $n$  copies of  $x$ , involving  $n - 1$  arbitrary constants. In other words, the operator  $\Delta$  is a derivation on the algebra of (lattice) functions spanned by  $\{1, x, x^{[2]}, x^{[3]}, \dots\}$ , with  $*$  as its multiplication. This structure has the correct continuum limit as  $h \rightarrow 0$ , provided that the constants  $c_k$  vanish in this limit. The  $*$ -product of two functions  $f$  and  $g$  on a finite interval

$0 \leq x \leq Nh$  is defined by expanding them in “Taylor series” in terms of  $1, x, \dots, x^{[N]}$ , and  $*$ -multiplying their components together according to

$$x^{[k]} * x^{[m]} = x^{[k+m]}. \quad (10)$$

Using this algebra enables one to discretize the  $x$ -variable in a linear system such as (4), at the cost of replacing the products on the right-hand side by the corresponding  $*$ -products. One obtains an integrable lattice (a system of coupled ordinary differential equations), and any continuum solution, such as a soliton, has its discrete counterpart. The example of sine-Gordon is described in Ward (1992) using the  $*$ -product in which all of the constants  $c_k$  vanish. This particular discrete version of sine-Gordon turns out to be somewhat unsatisfactory, since the boundary behaviour at  $x = \infty$  is rather pathological.

### 3. Conclusion

In order to obtain integrable discrete (lattice) versions of SDYM and its reductions, one may use an algebra of functions on the lattice, for which the appropriate finite-difference operators are derivations (i.e., satisfy the Leibniz rule). This involves the multiplication of functions being non-local rather than pointwise. There are many possible choices of algebras which achieve this. One needs to choose the partial difference operators, and the constants  $c_k$  which determine the multiplication rule, in such a way that the appropriate boundary conditions can be imposed consistently. Whether such a choice is possible for any particular equation remains to be seen.

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## RAPIDLY FORCED BURGERS EQUATION

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**ABSTRACT.** A forced Burgers equation is considered. Using an analysis on the whole line, earlier results found by semiline methods are recovered. In the case where the time dependence of the forcing is rapidly varying an asymptotic expansion of the solution of Burgers equation is obtained.

In recent years there has been considerable interest in forced nonlinear evolution equations of physical significance (see for example [1,2]). In this context the Burgers equation may be considered as a prototypical equation modelling the time evolution of one-dimensional nonlinear diffusive systems under the influence of an external driver [3].

In [2] we obtained the solution of the initial value problem on the whole line for the forced Burgers equation

$$u_t = (u_x + u^2)_x + \delta(x)F(t) \quad (1)$$

where  $u \equiv u(t, x)$  and  $\delta(x)$  is the usual Dirac delta function. Here  $F(t)$  is a given function of time which is assumed to be continuous and bounded.

In [2] the main result was the reduction of the initial value problem for (1) to a linear integral equation of Volterra type in  $t$ .

In this contribution we consider as a particular application the case where the forcing is rapidly varying in time. Namely we put in (1)

$$F(t) = \sin(t/\epsilon) \quad (2)$$

with  $\epsilon$  a small parameter.

We obtain for the solution  $u(t, x)$  of (1) an asymptotic (high frequency) expansion in powers of  $\epsilon$ , where not only integer, but also semi-integer powers of  $\epsilon$  appear. Moreover, in

in the present contribution we obtain the solution  $u(t, x)$  via the whole line linearization of (1), without employing the semiline theory [4] that we used in [2].

We shall begin by outlining the solution of the initial value problem for (1) corresponding to a generic (continuous and bounded) function  $F(t)$ . We show that the problem reduces to a linear Volterra integral equation equivalent to the one obtained in [2]. Subsequently we consider the case of a rapid forcing, corresponding to the form (2) for  $F(t)$ . We obtain the asymptotic (high frequency) behaviour of the solution  $u(t, x)$  as a power expansion in the small parameter  $\epsilon$ . Such an expansion is characterized by the presence of integer and semi-integer powers of  $\epsilon$ , which are a consequence of the Dirac delta function appearing in the forcing term of (1).

For simplicity we shall restrict our consideration to a zero initial datum  $u(0, x) = u_0(x) = 0$ . The formulae for the general case  $u_0(x) \neq 0$  are more cumbersome but may be obtained in a straightforward manner.

We start by introducing the Hopf-Cole transformation [5]

$$u(t, x) = v_x(t, x)/v(t, x) \quad (3a)$$

or

$$v(t, x) = \exp \left\{ \int_{-\infty}^x dx' u(t, x') \right\}. \quad (3b)$$

This transformation maps on (1) into the linear equation

$$v_t = v_{xx} + H(x)F(t)v, \quad (4a)$$

where  $H(x)$  is the Heaviside step function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \quad (4b)$$

We denote by  $u_R(t, x)$  and  $u_L(t, x)$  the two solutions of (1) in the quarter planes  $\{x > 0, t \geq 0\}$  and  $\{x < 0, t \geq 0\}$  respectively.  $u_L(t, x)$  and  $u_R(t, x)$  are taken to satisfy the continuity condition

$$u_L(t, 0) = u_R(t, 0) \quad (5)$$

as  $x \rightarrow 0_-$  and  $x \rightarrow 0_+$ .

Integrating (1) across the discontinuity on the  $x$  axis and imposing the above condition yields

$$u_{R,x}(t, 0) - u_{L,x}(t, 0) = -F(t). \quad (6)$$

In terms of the solution of the linear problem (4), relations (5) and (6), via (3), imply that both the functions  $v(t, x)$  and the  $x$ -derivative  $v_x(t, x)$  are continuous at  $x = 0$

$$v_L(t, 0) = v_R(t, 0) \quad (7a)$$

and

$$v_{L,x}(t, 0) = v_{R,x}(t, 0), \quad (7b)$$

while from (4) the second derivative has a discontinuity given by

$$v_{R,xx}(t, 0) - v_{L,xx}(t, 0) = -F(t)v(t, 0). \quad (7c)$$

The initial value problem for the forced Burgers equation (1) with initial datum

$$u(0, x) = u_0(x) = 0, \quad (8)$$

is now solved through the following algorithm

1. compute the initial datum for  $v(t, x)$

$$v(0, x) = v_0(x) = 1; \quad (9)$$

2. evaluate  $v(t, x)$  from (4) and (9);

3. recover  $u(t, x)$  from  $v(t, x)$  and  $v_x(t, x)$  via (3a).

In order to evaluate  $v(t, x)$  it is convenient to make the change of dependent variable

$$v(t, x) = 1 + w(t, x), \quad (10)$$

which, together with (4) and (9) implies

$$w_t = w_{xx}, \quad x < 0 \quad (11a)$$

$$w_t = w_{xx} + F(t)w + F(t), \quad x > 0 \quad (11b)$$

$$w(0, x) = w_0(x) = 0. \quad (11c)$$

We denote by  $w_L(t, x)$  ( $w_R(t, x)$ ) the solution of the linear equation (11a) ((11b)) with initial datum (11c).

By introducing the Laplace transform

$$p(s, x) = \int_0^\infty dt e^{-st} w_L(t, x) \quad (12)$$

we obtain from (11a) and (11c) the equation

$$p_{xx} - sp = 0. \quad (13)$$

Solving this equation and requiring that  $p(s, x)$  be bounded as  $x \rightarrow -\infty$ , we get the solution

$$p(s, x) = A(s) \exp\left(s^{1/2}x\right) \quad (14a)$$

with

$$A(s) = \int_0^\infty dt e^{-st} w_L(t, 0). \quad (14b)$$

In order to obtain  $w_R(t, x)$ , it is convenient to put

$$\tilde{w}_R(t, x) = w_R(t, x) \exp\left\{-\int_0^t dt' F(t')\right\}, \quad (15a)$$

$$\tilde{F}(t) = F(t) \exp\left\{\int_0^t dt' F(t')\right\}. \quad (15b)$$

In terms of the Laplace transforms

$$\tilde{q}(s, x) = \int_0^\infty dt e^{-st} \tilde{w}_R(t, x), \quad (16a)$$

$$\tilde{R}(s) = \int_0^\infty dt e^{-st} \tilde{F}(t). \quad (16b)$$

We obtain from (11b) and (11c) the equation

$$\tilde{q}_{xx} - s\tilde{q} = -\tilde{R}(s), \quad (17)$$

which, after requiring that  $\tilde{q}(s, x)$  be bounded as  $x \rightarrow +\infty$ , admits the solution

$$\tilde{q}(s, x) = B(s) \exp(-s^{1/2}x) + \frac{\tilde{R}(s)}{s}. \quad (18)$$

By inverting the Laplace transform, we get from (14a) and (18)

$$w_L(t, x) = -\frac{x}{2\sqrt{\pi}} \int_0^t dt' \frac{A(t')}{(t-t')^{3/2}} \exp\left\{-\frac{x^2}{4(t-t')}\right\}, \quad (19a)$$

$$\tilde{w}_R(t, x) = \frac{x}{2\sqrt{\pi}} \int_0^t dt' \frac{B(t')}{(t-t')^{3/2}} \exp\left\{-\frac{x^2}{4(t-t')}\right\} + \int_0^t dt' \tilde{F}(t'), \quad (19b)$$

where  $A(t)$  and  $B(t)$  denote the inverse Laplace transform conditions (7a) and (7b). These, together with (10), allow us to determine the unknown functions  $A(t)$  and  $B(t)$ . In fact, we get

$$w_L(t, 0) = w_R(t, 0) = \tilde{w}_R(t, 0) \exp\left\{\int_0^t dt' F(t')\right\}, \quad (20a)$$

$$w_{L,x}(t, 0) = w_{R,x}(t, 0) = \tilde{w}_{R,x}(t, 0) \exp\left\{\int_0^t dt' F(t')\right\}, \quad (20b)$$

which in turn imply, via (19a) and (19b)

$$A(t) = \left\{B(t) + \int_0^t dt' \tilde{F}(t')\right\} \exp\left\{\int_0^t dt' F(t')\right\} \quad (21a)$$

and

$$\frac{d}{dt} \int_0^t dt' \frac{A(t')}{(t-t')^{1/2}} = -\frac{d}{dt} \int_0^t dt' \frac{B(t')}{(t-t')^{1/2}} \exp\left\{\int_0^t dt' F(t')\right\}, \quad (21b)$$

where (15a) and (15b) have also been used.

Substituting (21a) in (21b) we get

$$\frac{d}{dt} \int_0^t dt' \frac{B(t')G(t')}{(t-t')^{1/2}} + \frac{d}{dt} \int_0^t dt' \frac{G(t')-1}{(t-t')^{1/2}} = -G(t) \frac{d}{dt} \int_0^t dt' \frac{B(t')}{(t-t')^{1/2}} \quad (21c)$$

where

$$G(t) = \exp\left\{\int_0^t dt' F(t')\right\}. \quad (21d)$$

From (21c), after integrating by parts, we obtain

$$\begin{aligned} & \int_0^t dt' \frac{1}{(t-t')^{1/2}} \frac{d}{dt'} [B(t')G(t')] \\ &= G(t) \int_0^t dt' \frac{1}{(t-t')^{1/2}} \frac{dB}{dt'}(t') \int_0^t dt' \frac{1}{(t-t')^{1/2}} \frac{dG}{dt'}(t'), \end{aligned} \quad (21e)$$

which can be considered as an Abel equation for the function  $\frac{d}{dt'} [B(t')G(t')]$ .

From (21e) we finally obtain the evolution equation for  $B(t)$

$$B(t) = -\left(1 - \frac{1}{G(t)}\right) + \int_0^t dt' H(t, t') B(t'), \quad (22a)$$

where

$$H(t, t') = \frac{1}{\pi G(t)} \frac{d}{dt'} \int_{t'}^t dt_2 \frac{G(t_2)}{(t-t_1)^{1/2}(t_2-t')^{1/2}}. \quad (22b)$$

Equation (22a) is a linear Volterra integral equation in  $t$ , which admits a unique continuous solution under the assumption that  $F(t)$  is a continuous, bounded function of its argument.

It can be shown that (22a) can be put in the equivalent form

$$B(t) = h(t) + \int_0^t dt' K(t, t') B(t') \quad (23a)$$

where

$$h(t) = -\frac{1}{2\pi} \int_0^t dt_1 \int_0^{t_1} dt_2 \frac{1}{(t-t_1)^{1/2}(t_1-t_2)^{1/2}} \frac{d}{dt_2} \left\{ \exp \left( \int_{t_1}^{t_2} dy F(y) \right) - 1 \right\} \quad (23b)$$

$$K(t, t') = -\frac{1}{4\pi} \int_0^t dt_2 \frac{1}{(t-t_2)^{1/2}(t_2-t')^{1/2}} \left\{ \exp \left( \int_{t_2}^{t'} dy F(y) \right) - 1 \right\}. \quad (23c)$$

Equations (23) were previously obtained in [2]. The proof of the equivalence between (22) and (23) involves a comparison between the two different techniques of linearization of the forced problem [1] employed here and in [2]. We shall report it elsewhere.

Here we only note that (23) can be obtained from (21e) by solving this integral equation with respect to  $dB/dt$  and then using integration by parts.

In the following we prefer to compute  $B(t)$  through (23a), which allows a more straightforward computation than (22a), due to the form (23c) of the kernel. We solve (23a) by successive iterations.

Once  $B(t)$  is known, the solutions  $v_L(t, x)$  and  $v_R(t, x)$  of the linear problem can be obtained according to

$$v_L(t, x) = 1 - \frac{x}{2\sqrt{\pi}} \int_0^t dt' \frac{A(t')}{(t-t')^{3/2}} \exp \left\{ -\frac{x^2}{4(t-t')} \right\} \quad (24a)$$

$$v_R(t, x) = 1 + \frac{x}{2\sqrt{\pi}} \int_0^t dt' \frac{B(t')}{(t-t')^{3/2}} \exp \left\{ -\frac{x^2}{4(t-t')} + \int_0^t dt' F(t') \right\} + \int_0^t dt' \tilde{F}(t'), \quad (24b)$$

where (19a), (19b), (15) and (10) have been used.

In the case of a rapidly varying forcing,  $F(t)$  is given by (2), and we obtain the following asymptotic expansions in the small parameter  $\epsilon$

$$\exp \left\{ \int_0^t dt' F(t') \right\} \simeq 1 + \epsilon (1 - \cos(t/\epsilon)) + \dots \quad (25a)$$

$$\int_0^t dt' F(t') \exp \left\{ \int_0^t dt' F(t') \right\} \simeq \epsilon (1 - \cos(t/\epsilon)) + \dots \quad (25b)$$

Expressions (25a) and (25b) imply, via (23b) and (23c)

$$h(t) \approx -\frac{1}{2}\epsilon (1 - \cos(t/\epsilon)) + \dots \quad (26a)$$

and

$$k(t, t') \simeq -\frac{\epsilon^{1/2}}{4[\pi(t-t')]^{1/2}} [\cos(t'/\epsilon) + \sin(t'/\epsilon)] + O(\epsilon^{3/2}) \quad (26b)$$

and (23a) then gives

$$B(t) \simeq -\frac{1}{2}\epsilon [1 - \cos(t/\epsilon)] - \frac{\epsilon^{3/2} t^{1/2}}{8\pi^{1/2}} + O(\epsilon^2). \quad (27)$$

The above expansions when substituted back into (24a) and (24b), give

$$\begin{aligned} v(t, x) &\simeq 1 + H(x)\epsilon (1 - \cos(t/\epsilon)) - \frac{1}{2}\epsilon \left[ 1 - \operatorname{erf} \left( \frac{x}{2t^{1/2}} \right) \right] \\ &- \frac{x\epsilon^{3/2}}{16\pi} \int_0^t dt' \frac{t'^{1/2}}{(t-t')^{3/2}} \exp \left[ -x^2/4(t-t') \right] + O(\epsilon^2), \end{aligned} \quad (28)$$

$$v(t, x) \simeq \delta(x)\epsilon [1 - \cos(t/\epsilon)] + \frac{\epsilon}{2(\pi t)^{1/2}} \exp \left( -\frac{x^2}{4t} \right) + \frac{x\epsilon^{3/2}}{16\pi} \left[ 1 - \operatorname{erf} \left( \frac{x}{2\sqrt{t}} \right) \right] + O(\epsilon^2). \quad (29)$$

Substituting (28) and (29) in (3a), we finally obtain the asymptotic expansion for the solution  $u(x, t)$  of the forced problem (1)

$$\begin{aligned} u(t, x) &\simeq \delta(x)\epsilon [1 - \cos(t/\epsilon)] + \frac{\epsilon}{2(\pi t)^{1/2}} \exp \left( -\frac{x^2}{4t} \right) \\ &+ \frac{\epsilon^{3/2}}{16} \left[ 1 - \operatorname{erf} \left( \frac{x}{2\sqrt{t}} \right) \right] + O(\epsilon^2). \end{aligned} \quad (30)$$

We note that in the above expansion there appear not only integer but also semi-integer powers of the small parameter  $\epsilon$ . This is due to the impulsive forcing considered here, namely to the presence of the Dirac delta function in the forcing term of (1). On the other hand, impulsive forcing such as that considered here is one of the few cases where the analytic solution can be obtained; the asymptotic expansion of such solution then gives (30).

A different approach (cf. [6]) based on a standard perturbation method in this case reproduces exactly only the first order term of (30) but would require care at higher orders, due to the presence of higher order derivatives of the Dirac delta function.

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# NONLINEAR EVOLUTION EQUATIONS FROM AN INVERSE SPECTRAL PROBLEM

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**ABSTRACT.** It is known that a number of nonlinear evolution equations can be related to an inverse spectral problem and, then, (in principle) linearized by solving a linear integral equation. In this lecture the opposite path is followed, i.e. we start from a linear inverse spectral problem and deduce the possible nonlinear evolution equations which can be solved by using it. In order to fix the most general possible time evolution and space dependence of the spectral data in the considered spectral problem, only consistency requirements are used. Then a related nonlinear evolution equation for a matrix field is derived. In general, this matrix field satisfies some additional nonlinear differential constraints. The main advantage of the method is that it allows a systematic search of all possible nonlinear evolution equations of given order which can be solved by using the considered inverse spectral problem. New classes of first and second order evolution equations are obtained, in particular a new Davey–Stewartson system.

## 1. Introduction

We study applications of a spectral method recently described by one of us [1–3]. For a complete exposition and details see reference [4].

The input is an integral equation which is restricted here to the form

$$\Psi(k, \mathbf{x}, t) = \mathbf{I} + \int \frac{d\sigma(\lambda, \Lambda)}{-k + \Lambda} \mathbf{T}(\lambda, \Lambda, \mathbf{x}, t) \Psi(\lambda, \mathbf{x}, t) \quad (1.1)$$

where  $k, \lambda$ , and  $\Lambda \in \mathbb{C}$ ,  $\mathbf{x} \in \mathbb{C}^M$ ,  $t \in \mathbb{C}$ ,  $\Psi$  and  $\mathbf{T}$  have their values in the set  $\mathcal{M}_N$  of complex valued  $N \times N$  matrices,  $M \leq N$ ,  $N > 1$ ,  $d\sigma$  is a measure in  $\mathbb{C}^2$  and  $\mathbf{I}$  is the identity matrix in  $\mathcal{M}_N$ . We call  $\mathbf{T}$  the spectral data,  $k$ ,  $\lambda$  and  $\Lambda$  the spectral parameters,  $\mathbf{x}$  the spatial variables and  $t$  the time. Any special dependence of  $\mathbf{T}$  on the spatial and time variables is called a dispersion law.

This integral equation fixes the departure from analyticity in  $k = k_R + ik_I$  of  $\Psi$  according to the following equation

$$\bar{\partial}\Psi(k) = -\pi \int d\sigma(\lambda, \Lambda) \delta(k - \Lambda) \mathbf{T}(\lambda, \Lambda) \Psi(\lambda) \quad (1.2)$$

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where  $\bar{\partial} \equiv \frac{1}{2} \left( \frac{\partial}{\partial k_R} + i \frac{\partial}{\partial k_I} \right)$  is the so called  $\bar{\partial}$ -operator. Therefore, the integral equation (1.1) is also called an integral  $\bar{\partial}$ -equation and the problem of finding  $\Psi$ , once given  $\mathbf{T}$ , is called a  $\bar{\partial}$ -problem, or more generically an inverse spectral problem (see [5–8] and the review article [9]).

In the following, we will often use a shorthand for equation (1.1) in the form:

$$\Psi = \mathbf{I} + \oint \mathbf{T} \Psi. \quad (1.3)$$

The essential assumption of the method is that, for the chosen spectral data  $\mathbf{T}$  and measure  $d\sigma$ , the matrix function  $\Psi$  is uniquely determined by (1.1). Since  $\mathbf{T}$  depends on  $\mathbf{x}, t$  so does  $\Psi$ .

If a linear differential operator  $H$ , with fixed coefficients in  $\mathcal{M}_N$  and depending on the spectral parameter  $k$ , acts on  $\Psi$ , it yields on the right hand side of (1.1) a new free term  $h$ , and, inside the integral,  $\mathbf{T}H\Psi$  plus additional terms called *parasite terms*. A differential operator  $H$  which cancels parasite terms and generates a free term  $h$  independent of the spectral parameter  $k$  for a given dependence of  $\mathbf{T}$  on the space-time variables  $\mathbf{x}$  and  $t$  is said to be *compatible with the dispersion law* of  $\mathbf{T}$ .

In such a case we have

$$H\Psi = h + \oint \mathbf{T}H\Psi \quad (1.4)$$

where  $\mathbf{h}$  is a functional of  $\Psi$  depending only on the space-time variables, i.e.  $\mathbf{h} = \mathbf{h}[\Psi](\mathbf{x}, t)$ .

Now, it is obvious that multiplying both sides of (1.1) by a matrix  $h(\mathbf{x}, t)$  is freely allowed on the right. Comparing the result with (1.4) and using the uniqueness assumption we find that (1.4) has the unique solution

$$H\Psi = \Psi\mathbf{h}. \quad (1.5)$$

In the cases we consider,  $\mathbf{h}$  depends only on the first term  $\Psi^{(1)}(\mathbf{x}, t)$  (and eventually on the space derivatives of the first term  $\Psi^{(1)}(\mathbf{x}, t)$ ) of the asymptotic expansion in the spectral parameter  $k$  of  $\Psi$ , which, according to (1.1), should be

$$\Psi = \mathbf{I} + k^{-1}\Psi^{(1)} + k^{-2}\Psi^{(2)} + o(k^{-2}). \quad (1.6)$$

In the method we use, in a way which is reminiscent of the Lax method that introduces a pair of differential operators, the so called Lax pair, we introduce a pair of compatible operators. The first one,  $F$ , is composed of derivatives on  $\mathbf{x}$  and  $t$  and therefore we call it the evolution operator. The second one,  $G$ , acts on  $\mathbf{x}$  only and we call it the spectral operator. They are both compatible with the same (set of) dispersion relations involving differential operators on  $\mathbf{x}$  only (i.e. space dispersion relations). However, to the same  $G$  one can associate different  $F$  compatible with different time dispersion relations, i.e. different dispersion relations involving differential operators on  $\mathbf{x}$  and  $t$ .

According to the previous remarks and with a notation which is self explanatory we get the integral equations

$$F\Psi = \mathbf{f} + \oint \mathbf{T}F\Psi \quad (1.7)$$

$$G\Psi = g + \oint \mathbf{T}G\Psi \quad (1.8)$$

which admit the unique solutions

$$F\Psi = \Psi f(\Psi^{(1)}) \quad (1.9)$$

$$G\Psi = \Psi g(\Psi^{(1)}). \quad (1.10)$$

By inserting the asymptotic expansion (1.6) into these two equations, we get a set of coupled differential equations in  $\Psi^{(1)}$  and  $\Psi^{(2)}$ . In general, they can be solved with respect to  $\Psi^{(1)}$  yielding a nonlinear evolution equation for  $\Psi^{(1)}$  and, eventually, some nonlinear relations for  $\Psi^{(1)}$  not containing the partial derivative with respect to  $t$ , which are called *constraints* (see [10,11] for nonlinear evolution equations with constraints, studied in a different context).

The two linear partial differential equations of  $\Psi$  (1.9) and (1.10) must satisfy the compatibility condition

$$[F, G]\Psi = F(\Psi g(\Psi^{(1)})) - G(\Psi f(\Psi^{(1)})). \quad (1.11)$$

In general, by inserting in it the asymptotic expansion (1.6) and by using the nonlinear evolution equation obtained for  $\Psi^{(1)}$ , one gets a nonlinear additional constraint for  $\Psi^{(1)}$ . If no additional constraint is found one recovers from (1.11) the Lax representation of the nonlinear evolution equation in the weak form as proposed in [12,13].

Finally, one has to take into account that for some dispersion relations of  $T$  there exist compatible linear operators  $A$  independent on the spectral parameter  $k$  that cancel the free term

$$A\Psi = \oint T A\Psi. \quad (1.12)$$

Then  $\Psi$  satisfies the linear differential equation

$$A\Psi = 0 \quad (1.13)$$

and, consequently, because  $A$  is  $k$  independent,  $\Psi^{(1)}$  satisfies the same linear equation. One usually wishes to discard linear constraints, which we call *ghost equations*, and therefore one has to discard those dispersion relations of  $T$  which allow them.

In S2 we give a way of constructing the most general linear operator of fixed order which admits a compatible dispersion relation. In SS3 and 4 we examine evolution operators  $F$  of order 1 with respect to  $x$  and  $t$  and spectral operators  $G$  of order 1 with respect to  $x$ . A large class of new nonlinear evolution equations is obtained, which needs to be explored carefully. For a special choice of the parameters we recover the  $n$ -wave equation in  $2 + 1$  dimensions (i.e. two space and one time dimension) [14,15]. The  $n$ -wave equation in  $M + 1$  dimensions ( $M > 2$ ) can be recovered only by considering dispersion relations for  $T$  allowing the existence of ghost equations, in accordance with the well known fact that, for  $M > 2$ , they are not true  $M + 1$  dimensional equations [16]. In SS5 and 6 we consider evolution operators  $F$  of order 2 with respect to  $x$  and of order 1 with respect to  $t$  and a spectral operator  $G$  of the same type already considered. In this case also, a large class of new nonlinear evolution equations is derived. The spectral operator  $G$  (as in the first order case) defines a spectral problem, which generalizes the multidimensional Zakharov–Shabat spectral problem. In the case of the  $2 + 1$  dimensional Zakharov–Shabat spectral problem, we find to our surprise, considering how much this problem has been studied, another integrable Davey–Stewartson system which differs from the so called DSI and DSII systems [8,14,17–21]. Its properties are studied in detail in the lecture by F. Pempinelli in these proceedings [2\*2].

In the above we may consider the special case of a measure  $d\sigma(\lambda, \Lambda)$  containing  $\delta(\lambda - \Lambda)$  as a factor. This case is considered in S7. We consider, as a special example, instead of a pair of compatible operators  $F$  and  $G$ , a couple of first order evolution operators  $F$  and  $F'$  with different time and space variables  $t, x$  and  $t', x'$ . For  $M = 1$  and  $M' = 1$  we recover the self-dual Yang-Mills equation as a  $(1+1) + (1+1)$  dimensional evolution equation [23].

The method furnishes a way of classifying the integrable nonlinear evolution equations by looking at all possible dispersion relations for  $T$  which admit a couple of compatible operators  $F$  and  $G$ . It can be extended to consider  $n^{\text{th}}$  order differential operators (work in progress) and differential-difference operators [24].

As regards previous attempts to derive integrable evolution equations starting from a  $\bar{\partial}$ -problem one may see, in particular, references [25–30]. For an updated and exhaustive exposition on solitons and related topics one may consult the book by Ablowitz and Clarkson [31].

## 2. A Bit of Algebra

We write down the most general linear differential operator of a given order acting on a  $N \times N$  matrix. It should be made by composing differential operators with linear endomorphisms of  $C_{N^2}$  into  $C_{N^2}$ , written by means of  $N \times N$  matrices. The general form of an endomorphism  $\Psi \rightarrow \Phi$  is

$$\Phi = \sum_{ij} \mathbf{A}_{ij} \Psi \tau_{ij} \quad (2.1)$$

where  $\Psi, \mathbf{A}_{ij} \in \mathcal{M}_N$  and  $\tau_{ij}$  is defined in  $\mathcal{M}_N$  as

$$(\tau_{ij})_{hk} = \delta_{ih} \delta_{jk}. \quad (2.2)$$

The null endomorphism is of course that which maps any matrix  $\Psi$  into  $\mathbf{0}$ . According to this definition, the endomorphism (2.1) is null if and only if all the matrices  $\mathbf{A}_{ij}$  vanish. We call any endomorphism which maps the identity matrix of  $\mathcal{M}_N$  into 0, a *pseudonull endomorphism*. For such an endomorphism

$$\sum_{ij} \mathbf{A}_{ij} \tau_{ij} = \mathbf{0}. \quad (2.3)$$

Our differential operators  $F, G$  and  $A$  will be obtained by composing elementary differential operators with endomorphisms. Since we shall not study in this paper orders larger than 2, the parasite terms and the free terms can always be derived by means of the following catalogue of formulas:

$$\mathbf{B} \frac{\partial \Psi}{\partial x} = \oint \mathbf{T} \mathbf{B} \frac{\partial \Psi}{\partial x} + \oint \left\{ [\mathbf{B}, \mathbf{T}] \frac{\partial \Psi}{\partial x} + \mathbf{B} \frac{\partial \mathbf{T}}{\partial x} \Psi \right\} \quad (2.4)$$

$$\mathbf{B} \frac{\partial^2 \Psi}{\partial x \partial y} = \oint \mathbf{T} \mathbf{B} \frac{\partial^2 \Psi}{\partial x \partial y} + \oint \left\{ [\mathbf{B}, \mathbf{T}] \frac{\partial^2 \Psi}{\partial x \partial y} + \mathbf{B} \frac{\partial \mathbf{T}}{\partial x} \frac{\partial \Psi}{\partial y} + \mathbf{B} \frac{\partial \mathbf{T}}{\partial y} \frac{\partial \Psi}{\partial x} + \mathbf{B} \frac{\partial^2 \mathbf{T}}{\partial x \partial y} \Psi \right\} \quad (2.5)$$

$$k \mathbf{B} \frac{\partial \Psi}{\partial x} = \mathbf{B} \frac{\partial \Psi^{(1)}}{\partial x} + \oint \lambda \mathbf{T} \mathbf{B} \frac{\partial \Psi}{\partial x} + \oint \left\{ \Lambda \mathbf{B} \mathbf{T} \frac{\partial \Psi}{\partial x} - \lambda \mathbf{T} \mathbf{B} \frac{\partial \Psi}{\partial x} + \Lambda \mathbf{B} \frac{\partial \mathbf{T}}{\partial x} \Psi \right\} \quad (2.6)$$

$$\sum_{ij} \mathbf{A}_{ij} \Psi \tau_{ij} = \oint \mathbf{T} \sum_{ij} \mathbf{A}_{ij} \Psi \tau_{ij} + \oint \sum_{ij} [\mathbf{A}_{ij}, \mathbf{T}] \Psi \tau_{ij} \quad (2.7)$$

$$k \sum_{ij} \mathbf{A}_{ij} \Psi \tau_{ij} = \sum_{ij} \mathbf{A}_{ij} \Psi^{(1)} \tau_{ij} + \oint \lambda \mathbf{T} \sum_{ij} \mathbf{A}_{ij} \Psi \tau_{ij} + \oint \sum_{ij} \{\Lambda \mathbf{A}_{ij} \mathbf{T} \Psi - \lambda \mathbf{T} \mathbf{A}_{ij} \Psi\} \quad (2,8)$$

where

$B$  is an arbitrary fixed matrix,  $x, y$  are two components of  $\mathbf{x}$ , and  $\{\mathbf{A}_{ij}\}$  defines a pseudonull endomorphism.

The aim will be to find the most general dispersion relations of  $\mathbf{T}$  that kill the parasite terms for any  $\Psi$  and for any solution  $\mathbf{T}$  of the dispersion relations.

### 3. Linear Differential Operators for Problems of Order 1

#### 3.1. THE EVOLUTION OPERATOR AND THE DISPERSION RELATIONS

In order to write down the most general dispersion relations (linear in the spectral variables  $\Lambda$  and  $\lambda$ ) for problems of order 1 we start by considering the most general evolution operator  $F$  that is of first order in the space variables and which is linear in  $k$ :

$$F\Psi = \sum_{i,j=1}^N \left\{ -\Theta_{ij} \frac{\partial \Psi}{\partial t} + \sum_{l=1}^M \left( \mathbf{C}_{ij}^l + k \mathbf{G}_{ij}^l \right) \frac{\partial \Psi}{\partial X_l} + (\mathbf{D}_{ij} + k \mathbf{E}_{ij}) \Psi \right\} \tau_{ij} \quad (3.1)$$

where  $\Theta_{ij}$ ,  $\mathbf{C}_{ij}^l$ ,  $\mathbf{G}_{ij}^l$ ,  $\mathbf{D}_{ij}$  and  $\mathbf{E}_{ij}$  are  $N \times N$  matrices.

The free term generated by the term with  $\Psi$  is

$$\sum_{ij} \mathbf{D}_{ij} \tau_{ij} + k \sum_{ij} \mathbf{E}_{ij} \tau_{ij} + \sum_{ij} \mathbf{E}_{ij} \Psi^{(1)} \tau_{ij}. \quad (3.2)$$

The  $k$  independence of the free term requires that  $\{\mathbf{E}_{ij}\}$  defines a pseudonull endomorphism

$$\sum_{ij} \mathbf{E}_{ij} \tau_{ij} = 0. \quad (3.3)$$

Since the two equations of the form (1.9) derived from the two operators  $F\Psi$  and  $F'\Psi = F\Psi - \Psi \sum_{ij} \mathbf{D}_{ij} \tau_{ij}$  coincide we can, without loss of generality, require also that

$$\sum_{ij} \mathbf{D}_{ij} \tau_{ij} = 0. \quad (3.4)$$

The parasite terms are

$$\begin{aligned} PT[F] &= \sum_{ij} \left\{ -[\Theta_{ij}, \mathbf{T}] \frac{\partial \Psi}{\partial t} - \Theta_{ij} \frac{\partial \mathbf{T}}{\partial t} \Psi \right\} \tau_{ij} \\ &\quad + \sum_{ijl} \left\{ [\mathbf{C}_{ij}^l, \mathbf{T}] \frac{\partial \Psi}{\partial X_l} + \mathbf{C}_{ij}^l \frac{\partial \mathbf{T}}{\partial X_l} \Psi \right\} \tau_{ij} \\ &\quad + \sum_{ijl} \left\{ \Lambda \mathbf{G}_{ij}^l \mathbf{T} \frac{\partial \Psi}{\partial X_l} - \lambda \mathbf{T} \mathbf{G}_{ij}^l \frac{\partial \Psi}{\partial X_l} + \Lambda \mathbf{G}_{ij}^l \frac{\partial \mathbf{T}}{\partial X_l} \Psi \right\} \tau_{ij} \\ &\quad + \sum_{ij} \{[\mathbf{D}_{ij}, \mathbf{T}] \Psi + (\Lambda \mathbf{E}_{ij} \mathbf{T} - \lambda \mathbf{T} \mathbf{E}_{ij}) \Psi\} \tau_{ij}. \end{aligned} \quad (3.5)$$

In the case of two independent spectral variables  $\Lambda$  and  $\lambda$ , which we study first, cancelling the parasite terms readily enforces  $\Theta_{ij}$  and  $\mathbf{C}_{ij}^l$  to be flat and  $\mathbf{G}_{ij}^l$  to vanish:

$$\Theta_{ij} = \theta_{ij}\mathbf{I}, \quad \mathbf{C}_{ij}^l = c_{ij}^l\mathbf{I}, \quad \mathbf{G}_{ij}^l = 0. \quad (3.6)$$

Let  $c_{ij}$  be the vector  $\{c_{ij}^l\}$  and let  $\bullet$  denote the scalar product in  $\mathbb{R}^N$ . Then  $F\Psi$  reduces to

$$F\Psi = \sum_{ij} \left\{ -\theta_{ij} \frac{\partial \Psi}{\partial t} + c_{ij} \bullet \nabla \Psi + (\mathbf{D}_{ij} + k\mathbf{E}_{ij})\Psi \right\} \tau_{ij}. \quad (3.7)$$

It is compatible with the set of  $N^2$  dispersion relations

$$-\theta_{ij} \frac{\partial \mathbf{T}}{\partial t} + c_{ij} \bullet \nabla \mathbf{T} + [\mathbf{D}_{ij}, \mathbf{T}] + \Lambda \mathbf{E}_{ij} \mathbf{T} - \lambda \mathbf{T} \mathbf{E}_{ij} = 0 \quad (3.8)$$

and has the free term

$$\sum_{ij} \mathbf{E}_{ij} \Psi^{(1)} \tau_{ij}. \quad (3.9)$$

It is necessary to perform a careful analysis of the dispersion relation (3.8), which takes into account that the spectral data  $\mathbf{T}$  do not depend on the indices  $i, j$ . The result is that we can rewrite the evolution operator and the dispersion relations as follows:

$$F\Psi = -\theta \frac{\partial \Psi}{\partial t} + \nabla \Psi \bullet \mathbf{N} - [\mathbf{H}, \Psi] \bullet \mathbf{N} + [\mathbf{D}, \Psi] + k\Gamma \bullet [\mathbf{N}, \Psi] \quad (3.10)$$

$$\nabla \mathbf{T} = [\mathbf{H}, \mathbf{T}] + \Lambda \Gamma \mathbf{T} - \lambda \mathbf{T} \Gamma \quad (3.11)$$

$$\theta \frac{\partial \mathbf{T}}{\partial t} = [\mathbf{D}, \mathbf{T}] + \Lambda \Gamma \bullet \mathbf{N} \mathbf{T} - \lambda \mathbf{T} \Gamma \bullet \mathbf{N} \quad (3.12)$$

where the constant matrix valued vectors  $\mathbf{N}$ ,  $\mathbf{H}$  and  $\Gamma$  can be chosen arbitrarily. If we assume (as we do in the following) that the second order derivatives of  $\mathbf{T}$  are continuous, the dispersion relations can be explicitly solved provided that  $\mathbf{N}$ ,  $\mathbf{H}$  and  $\Gamma$  satisfy the following integrability conditions:

$$[\mathbf{H}^h, \mathbf{H}^l] = 0 \quad (3.13)$$

$$[\Gamma^h, \Gamma^l] = 0 \quad (3.14)$$

$$[\mathbf{H}^h, \mathbf{D}] = 0 \quad (3.15)$$

$$[\Gamma^h, \Gamma \bullet \mathbf{N}] = 0 \quad (3.16)$$

$$[\mathbf{H}^h, \Gamma^l] = [\mathbf{H}^l, \Gamma^h] \quad (3.17)$$

$$[\Gamma^h, \mathbf{D}] = [\Gamma \bullet \mathbf{N}, \mathbf{H}^h]. \quad (3.18)$$

### 3.2. THE SPECTRAL OPERATOR

In order to deduce the nonlinear evolution equation satisfied by the first term  $\Psi^{(1)}$  in the asymptotic expansion of  $\Psi$  we need to find the most general first order linear differential operator  $G$  in the space variable  $x$  which is compatible with the dispersion relations found for  $\mathbf{T}$ .

One could also consider the case of an operator  $G$  containing linear differential operators in some additional space variables  $\mathbf{y}$  and/or time  $t$ . An interesting example will be given in the section dealing with the case  $\Lambda = \lambda$ . For simplicity, we exclude this possibility here.

The most general first order linear differential operator we can write is

$$G\Psi = \sum_{i,j=1}^N \sum_{l=1}^M \mathbf{B}_{ij}^l \frac{\partial \Psi}{\partial x_l} \tau_{ij} + \sum_{i,j=1}^N (\mathbf{K}_{ij} + k\mathbf{L}_{ij}) \Psi \tau_{ij} \quad (3.19)$$

where  $\{\mathbf{K}_{ij}\}$  and  $\{\mathbf{L}_{ij}\}$  define pseudonull endomorphisms. Cancelling the coefficient of  $\frac{\partial \Psi}{\partial x_l}$  in the parasite terms requires that  $\mathbf{B}_{ij}$  is flat; say  $\mathbf{B}_{ij} = b_{ij}\mathbf{I}$ . With this reduction the parasite terms are

$$PT[G] = \sum_{i,j=1}^N \left\{ \sum_{l=1}^M b_{ij}^l \frac{\partial \mathbf{T}}{\partial x_l} + [\mathbf{K}_{ij}, \mathbf{T}] + \Lambda \mathbf{L}_{ij} \mathbf{T} - \lambda \mathbf{T} \mathbf{L}_{ij} \right\} \Psi \tau_{ij}. \quad (3.20)$$

By requiring that they vanish we have that the most general spectral operator  $G$  compatible with the space dispersion relations (3.11) is

$$G\Psi = \nabla \Psi \bullet \mathbf{B} - [\mathbf{H}, \Psi] \bullet \mathbf{B} - k\Gamma \Psi \bullet \mathbf{B}, \quad (3.21)$$

where the matrix valued vector  $\mathbf{B}$  can be freely chosen provided the constraint

$$\Gamma \bullet \mathbf{B} = 0 \quad (3.22)$$

is satisfied.

### 3.3. THE GHOST OPERATOR

The most general possible ghost operator of first order  $A$  is of the form

$$A\Psi = \sum_{i,j=1}^N \left\{ -\eta_{ij} \theta \frac{\partial \Psi}{\partial t} + \sum_l \beta_{ij}^l \frac{\partial \Psi}{\partial x_l} + \mathbf{F}_{ij} \Psi \right\} \tau_{ij} \quad (3.23)$$

where  $\eta_{ij}\mathbf{I}$  and  $\beta_{ij}\mathbf{I}$  are flat matrices and  $\{\mathbf{F}_{ij}\}$  is a pseudonull endomorphism. We have to find the conditions that must be satisfied by the dispersion relations of  $\mathbf{T}$  in (3.11) and (3.12) in order to ensure that the only ghost operator  $A$  compatible with them is identically zero.

The parasite terms of  $A$  are

$$PT[A] = \sum_{ij} \left\{ -\eta_{ij} \theta \frac{\partial \mathbf{T}}{\partial t} + \beta_{ij} \bullet \nabla \mathbf{T} + [\mathbf{F}_{ij}, \mathbf{T}] \right\} \Psi \tau_{ij}. \quad (3.24)$$

By using the requirement that they vanish we deduce that the ghost operator  $A$  is identically zero if and only if

$$\text{rank } \mathbf{M} = M + 1 \quad (3.25)$$

where  $\mathbf{M}$  is the  $N^2 \times (M + 1)$  matrix

$$\mathbf{M}^{hl'} = \begin{cases} (\Gamma^l)_{pq} & \text{for } l' = l \leq M \\ (\Gamma \bullet \mathbf{N})_{pq} & \text{for } l' = M + 1. \end{cases} \quad (3.26)$$

If  $\Gamma^I$  and  $\Gamma \bullet N$  are normal matrices (i.e. if they commute with their adjoint), and according to the integrability condition (3.16) they commute between them, they can be diagonalized in the same reference frame. In this reference frame the  $p, q$  matrix elements vanish unless  $p = q$ , and we can redefine  $M$  as an  $N \times (M + 1)$  matrix with  $h = p = q \in [1, N]$ . We deduce that the anti-ghost condition can be satisfied only for  $M < N$ .

#### 4. Nonlinear Evolution Equations of Order 1

In listing all possible evolution equations that can be obtained starting from the integral  $\bar{\partial}$ -equation (1.1), we have to take into account that it is invariant under the similarity transformation

$$\mathbf{T} \longrightarrow \mathbf{S}\mathbf{T}\mathbf{S}^{-1} \quad (4.1)$$

$$\Psi \longrightarrow \mathbf{S}\Psi\mathbf{S}^{-1} \quad (4.2)$$

where  $\mathbf{S}$  is an arbitrary (possibly  $x, t$  dependent) matrix which does not depend on the spectral parameters.

Since the  $\bar{\partial}$ -problem (1.1) is supposed to have a unique solution  $\Psi$ , the evolution equations and constraints generated by two dispersion relations for  $\mathbf{T}$  related by a similarity transformation must be related by the same similarity transformation and can be considered equivalent.

Finally, let us remark that one can substantially simplify the computation needed to deduce the evolution equations and their constraints by choosing a convenient representation of the matrices which are used to define the dispersion equations. For instance, in the special case of (3.11) and (3.12) in the previous section, one can freely choose different  $N$  provided that  $\Gamma \bullet N$  is not changed.

In the following, we need to consider the diagonal and the off-diagonal part of a matrix  $M$ . We note them, respectively,  $M_D$  and  $M_F$ .

##### 4.1. THE EVOLUTION EQUATIONS

We consider the dispersion equations (3.11) and (3.12), the corresponding evolution operator  $F$  in (3.10) and the spectral operator  $G$  in (3.21). They define the evolution equation

$$F\Psi = \Psi f(\Psi^{(1)}) \quad (4.3)$$

with

$$F\Psi = -\theta \frac{\partial \Psi}{\partial t} + \nabla \Psi \bullet N - [\mathbf{H}, \Psi] \bullet N + [\mathbf{D}, \Psi] + k\Gamma \bullet [N, \Psi] \quad (4.4)$$

$$f(\Psi^{(1)}) = \Gamma \bullet [N, \Psi^{(1)}] \quad (4.5)$$

and the spectral equation

$$G\Psi = \Psi g(\Psi^{(1)}) \quad (4.6)$$

with

$$G\Psi = \nabla \Psi \bullet B - [\mathbf{H}, \Psi] \bullet B - k\Gamma \Psi \bullet B \quad (4.7)$$

$$g(\Psi^{(1)}) = -\Gamma \Psi^{(1)} \bullet B \quad (4.8)$$

where

$$\Gamma \bullet B = 0. \quad (4.9)$$

From now on, we suppose that  $\Gamma^l$  and  $\Gamma \bullet N$  are normal. Then, since they commute (see (3.16)), we can exploit the invariance property of the integral  $\bar{\partial}$ -equation and diagonalize them via a similarity transformation. Moreover, since the dispersion relations (3.11) and (3.12) depend on  $\Gamma \bullet N$  and not directly on  $N$  we can consider  $N$  to be diagonal.

In addition, it is convenient to consider the similarity transformation

$$T \longrightarrow \exp\{H_D \bullet x + D_D(t/\theta)\} T \exp\{-H_D \bullet x - D_D(t/\theta)\}. \quad (4.10)$$

This transformation does not change  $\Gamma$  and  $N$  and its only effect is to kill, in the dispersion relations (3.11) and (3.12), the diagonal part of  $H$  and  $D$ .

We conclude that the dispersion relations

$$\nabla T = [H, T] + \Lambda \Gamma T - \lambda T \Gamma, \quad (4.11)$$

$$\theta \frac{\partial T}{\partial t} = [D, T] + \Lambda \Gamma \bullet N T - \lambda T \Gamma \bullet N, \quad (4.12)$$

can be considered with  $\Gamma^l$  and  $N^l$  diagonal, and with  $H^l$  and  $D$  off-diagonal.

Then, let

$$\Gamma^l = \sum_{i=1}^N g_i^l \eta_i, \quad N^l = \sum_{i=1}^N c_i^l \eta_i \quad (4.13)$$

where  $g_i$  and  $c_i$  are arbitrary vectors in  $\mathbb{C}^M$ . The anti-ghost condition (3.25) is then

$$\text{rank} \begin{pmatrix} g_1^1 & g_2^1 & \cdots & g_N^1 \\ \vdots & \vdots & & \vdots \\ g_1^M & g_2^M & \cdots & g_N^M \\ g_1 \bullet c_1 & g_2 \bullet c_2 & \cdots & g_N \bullet c_N \end{pmatrix} = M + 1. \quad (4.14)$$

If  $H^l$  and  $D$  are all zero the integrability conditions (3.13)–(3.18) are satisfied. For non-vanishing  $H^l$  and  $D$  the vectors  $g_i$  and  $c_i$  must satisfy the algebraic constraints

$$\frac{g_i^l - g_k^l}{g_j^l - g_k^l} = \frac{g_i^h - g_k^h}{g_j^h - g_k^h} = \frac{g_i \bullet c_i - g_k \bullet c_k}{g_j \bullet c_j - g_k \bullet c_k} \quad (4.15)$$

for  $l, h = 1, 2, \dots, M$  and  $i, j, k = 1, 2, \dots, N$  while  $H^l, D$  can be cast in the form

$$H^l = [\Gamma^l, K], \quad D = [\Gamma \bullet N, K] \quad (4.16)$$

with  $K$  an off-diagonal matrix.

One can prove that the algebraic constraints (4.15) imply that the maximum possible value of the rank of the matrix in (4.14) is 2. Therefore, the anti-ghost condition can be satisfied only for  $M = 1$ . In this case, however,  $\Gamma$  is regular and from (4.9) it follows that the spectral operator  $G$  vanish.

The most general  $B$  satisfying  $\Gamma \bullet B = 0$  can be written as

$$B = \sum_{i=1}^N \omega_i \eta_i B_i \quad (4.17)$$

where the  $\mathbf{B}_i$  are arbitrary  $N \times N$  matrices and the vectors  $\omega_i$  satisfy  $\omega_i \cdot g_i = 0$  ( $i = 1, 2, \dots, N$ ).

Hence the spectral equation (4.6) can be rewritten in the equivalent form

$$\nabla \Psi \bullet \Omega - [\mathbf{H}, \Psi] \bullet \Omega - k \Gamma \Psi \bullet \Omega + \Psi \Gamma \Psi^{(1)} \bullet \Omega = 0 \quad (4.18)$$

where

$$\Omega = \sum_{i=1}^N \omega_i \eta_i, \quad \omega_i \cdot g_i = 0. \quad (4.19)$$

We can distinguish two cases:

**Case  $\mathbf{K} = \mathbf{0}$ .** Due to (4.16)  $\mathbf{H} = 0$ ,  $\mathbf{D} = 0$  and this spectral equation is the transposed of the multidimensional Zakharov–Shabat spectral problem, where the matrix field  $\Psi^T$  plays the role of the eigenmatrix and the off-diagonal matrix  $(\Gamma \Psi^{(1)} \bullet \Omega)^T$  the role of the potential.

**Case  $\mathbf{K} \neq \mathbf{0}$ .** As far as we know, this case has not been considered previously.

Let us insert the asymptotic expansion (1.6) for  $\Psi$  into the spectral equation (4.18) and into the evolution equation (4.3) and compute separately the diagonal and the off-diagonal parts. We have from the spectral equation (4.18) that

$$\nabla \Psi_F^{(1)} \bullet \Omega - [\mathbf{H}, \Psi^{(1)}]_F \bullet \Omega - \Gamma \Psi_F^{(2)} \bullet \Omega + (\Psi^{(1)} \Gamma \Psi_F^{(1)} \bullet \Omega)_F = 0 \quad (4.20)$$

$$\nabla \Psi_D^{(1)} \bullet \Omega - [\mathbf{H}, \Psi_F^{(1)}]_D \bullet \Omega + (\Psi_F^{(1)} \Gamma \Psi_F^{(1)} \bullet \Omega)_D = 0 \quad (4.21)$$

$$\nabla \Psi_D^{(2)} \bullet \Omega - [\mathbf{H}, \Psi_F^{(2)}]_D \bullet \Omega + (\Psi_F^{(2)} \Gamma \Psi_F^{(1)} \bullet \Omega)_D = 0 \quad (4.22)$$

and from the evolution equation that

$$\begin{aligned} \theta \frac{\partial \Psi_F^{(1)}}{\partial t} &= \nabla \Psi_F^{(1)} \bullet \mathbf{N} - [\mathbf{H}, \Psi^{(1)}]_F \bullet \mathbf{N} + [\mathbf{D}, \Psi^{(1)}]_F + \Gamma \bullet [\mathbf{N}, \Psi_F^{(2)}] \\ &\quad - (\Psi^{(1)} \Gamma \bullet [\mathbf{N}, \Psi_F^{(1)}])_F \end{aligned} \quad (4.23)$$

$$\theta \frac{\partial \Psi_D^{(1)}}{\partial t} = \nabla \Psi_D^{(1)} \bullet \mathbf{N} - [\mathbf{H}, \Psi_F^{(1)}]_D \bullet \mathbf{N} + [\mathbf{D}, \Psi_F^{(1)}]_D - (\Psi_F^{(1)} \Gamma \bullet [\mathbf{N}, \Psi_F^{(1)}])_D. \quad (4.24)$$

By inserting  $\Gamma \Psi_F^{(2)} \bullet \Omega$  from (4.20) into (4.23) we get a coupled set of nonlinear evolution equations for  $\Phi \equiv \Psi_F^{(1)}$  and  $\mathbf{A} \equiv \Psi_D^{(1)}$

$$\begin{aligned} \theta \frac{\partial}{\partial t} (\Gamma \Phi \bullet \Omega) &= \Gamma (\nabla \Phi \bullet \mathbf{N}) \bullet \Omega + \Gamma \bullet [\mathbf{N}, (\nabla \Phi \bullet \Omega)] - \Gamma \bullet [\mathbf{N}, ([\mathbf{H}, \Phi + \mathbf{A}] \bullet \Omega)] \\ &\quad - \Gamma ([\mathbf{H}, \Phi + \mathbf{A}] \bullet \mathbf{N}) \bullet \Omega + \Gamma [\mathbf{D}, \Phi + \mathbf{A}] \bullet \Omega + \Gamma \bullet [\mathbf{N}, \Phi (\Gamma \Phi \bullet \Omega)] \\ &\quad - \Gamma \Phi (\Gamma \bullet [\mathbf{N}, \Phi]) \bullet \Omega \end{aligned} \quad (4.25)$$

$$\theta \frac{\partial \mathbf{A}}{\partial t} = \nabla \mathbf{A} \bullet \mathbf{N} - [\mathbf{H}, \Phi]_D \bullet \mathbf{N} + [\mathbf{D}, \Phi]_D - (\Phi \Gamma \bullet [\mathbf{N}, \Phi])_D \quad (4.26)$$

$$\nabla \mathbf{A} \bullet \Omega = [\mathbf{H}, \Phi]_D \bullet \Omega - (\Phi \Gamma \Phi \bullet \Omega)_D. \quad (4.27)$$

We show in the following section that in the case  $M = 2$  and  $N$  arbitrary ( $N > M$ ) there are no other constraints in addition to that in (4.27), while in the case  $M > 2$  there are additional constraints.

**Case  $\mathbf{K} = \mathbf{0}$ .** Since  $\mathbf{H} = \mathbf{0}$  and  $\mathbf{D} = 0$ , the two fields  $\Phi$  and  $\mathbf{A}$  decouple and in the case  $M = 2$  we recover the  $n$ -wave interacting equations.

**Case  $\mathbf{K} \neq \mathbf{0}$ .** The two off-diagonal matrices  $\mathbf{H}$  and  $\mathbf{D}$  do not vanish and the system of coupled nonlinear evolution equations (4.25)–(4.27) is, as far as we know, new. For  $M = 2$ , as already noted, there are no additional constraints. Recall, however, that, for  $M \geq 2$ , the algebraic constraints (4.15) satisfied by the vectors  $\mathbf{g}_i$  and  $\mathbf{c}_i$ , which we obtained in order to ensure the integrability of the dispersion laws of  $\mathbf{T}$ , imply that the solution satisfies a ghost equation.

#### 4.2. THE CONSTRAINTS

According to the general discussion in the introduction, if the compatibility condition (1.11) is not a consequence of the evolution and spectral equations for  $\Psi$  defined by the couple of operators  $F$  and  $G$ , then the fields  $\Phi$  and  $\mathbf{A}$  satisfy an additional constraint.

By substituting into (1.11) the evolution equation (4.3) for  $\Psi$  and (4.25) for  $\Phi$  one gets the following differential equation of first order in the space variables

$$(\nabla\Psi - [\mathbf{H}, \Psi] - k\Gamma\Psi + \Psi\Gamma\Phi) \bullet \Xi = 0 \quad (4.28)$$

where

$$\Xi = [\mathbf{N}, (\Gamma\Phi \bullet \Omega)] - (\Gamma \bullet [\mathbf{N}, \Phi])\Omega + [\mathbf{N}, (\mathbf{H} \bullet \Omega)] - [\Omega, (\mathbf{H} \bullet \mathbf{N})] + [\Omega, \mathbf{D}]. \quad (4.29)$$

Note that  $\Gamma \bullet \Xi = 0$ , in accordance with the result obtained in the previous section that the most general first order differential operator  $G$  is of the form (3.21), with  $B$  satisfying  $\Gamma \bullet B = 0$ .

Substituting the asymptotic expansion (1.6) of  $\Psi$  at large  $k$  into (4.28) we get that  $\Phi$  and  $\mathbf{A}$  satisfy the constraint

$$(\nabla(\Phi + \mathbf{A}) - [\mathbf{H}, \Phi + \mathbf{A}] - \Gamma\Psi_F^{(2)} + (\Phi + \mathbf{A})\Gamma\Phi) \bullet \Xi = 0, \quad (4.30)$$

where  $\Psi_F^{(2)}$  can be written, by using (4.21), as

$$\Gamma\Psi_F^{(2)} \bullet \Omega = \nabla\Phi \bullet \Omega - [\mathbf{H}, \Phi + \mathbf{A}]_F \bullet \Omega + ((\Phi + \mathbf{A})\Gamma\Phi \bullet \Omega)_F. \quad (4.31)$$

The equation (4.28) furnishes no constraint for  $\Phi$  and  $\mathbf{A}$  if and only if it is a consequence of the spectral equation (4.18), i.e. if and only if there exists a matrix  $M$  such that

$$\Xi = \Omega M. \quad (4.32)$$

After a detailed analysis one can conclude that there are no first order integrable evolution equations in more than 2 space dimensions free of constraints and ghost equations. This result does not contradict the existence of  $n$ -wave interacting equations in  $M+1$  dimensions with  $M > 2$ , which are not true  $M+1$  dimensional equations since they admit ghost equations.

## 5. Linear Differential Operators for Problems of Order 2

### 5.1. THE EVOLUTION OPERATOR AND THE DISPERSION RELATIONS

We consider the following evolution operator  $F$  of second order in the space variables and linear in  $k$

$$F\Psi = \sum_{i,j=1}^N \left\{ -\Theta_{ij} \frac{\partial \Psi}{\partial t} - \frac{1}{2} \sum_{l,m=1}^M A_{ij}^{lm} \frac{\partial^2 \Psi}{\partial x_l \partial x_m} + \sum_{l=1}^M (H_{ij}^l + k\Gamma_{ij}^l) \frac{\partial \Psi}{\partial x_l} \right\} \tau_{ij} \quad (5.1)$$

where  $\Theta_{ij}$ ,  $A_{ij}^{lm}$ ,  $H_{ij}^l$  and  $\Gamma_{ij}^l$  are  $N \times N$  matrices. Since we are considering functions which are at least twice differentiable we avoid redundancy by setting

$$A_{ij}^{lm} = A_{ij}^{ml} \quad (5.2)$$

and moreover, we assume without loss of generality, that the matrices  $A_{ij}^{ml}$  are real.

The most general evolution operator  $F$  is obtained by adding the term

$$\sum_{i,j=1}^N (D_{ij} + kE_{ij}) \Psi \tau_{ij}.$$

This would be equivalent to adding to the related nonlinear evolution equations a term of first order in the space variables of the form obtained in the previous section and, therefore, we would not get new interesting information.

The parasite terms read

$$\begin{aligned} PT[F] = & \sum_{ij} \left\{ -[\Theta_{ij}, T] \frac{\partial \Psi}{\partial t} - \Theta_{ij} \frac{\partial T}{\partial t} \Psi \right\} \tau_{ij} \\ & - \frac{1}{2} \sum_{ijlm} \left\{ [A_{ij}^{lm}, T] \frac{\partial^2 \Psi}{\partial x_l \partial x_m} + 2A_{ij}^{lm} \frac{\partial T}{\partial x_m} \frac{\partial \Psi}{\partial x_l} + A_{ij}^{lm} \frac{\partial^2 T}{\partial x_l \partial x_m} \Psi \right\} \tau_{ij} \\ & + \sum_{ijl} \left\{ [H_{ij}^l, T] \frac{\partial \Psi}{\partial x_l} + H_{ij}^l \frac{\partial T}{\partial x_l} \Psi \right\} \tau_{ij} \\ & + \sum_{ijl} \left\{ \Lambda \Gamma_{ij}^l T \frac{\partial \Psi}{\partial x_l} - \lambda T \Gamma_{ij}^l \frac{\partial \Psi}{\partial x_l} + \Lambda \Gamma_{ij}^l \frac{\partial T}{\partial x_l} \Psi \right\} \tau_{ij}. \end{aligned} \quad (5.3)$$

The cancelling of the coefficients of  $\frac{\partial \Psi}{\partial t}$  and  $\frac{\partial^2 \Psi}{\partial x_l \partial x_m}$  requires that the matrices  $\Theta_{ij}$  and  $A_{ij}^{lm}$  be flat:

$$\Theta_{ij} = \theta_{ij} I, \quad A_{ij}^{lm} = a_{ij}^{lm} I. \quad (5.4)$$

The cancelling of the coefficient of  $\frac{\partial \Psi}{\partial x_l}$  yields the space dispersion relation

$$\sum_{m=1}^M a_{ij}^{lm} \frac{\partial T}{\partial x_m} = [H_{ij}^l, T] + \Lambda \Gamma_{ij}^l T - \lambda T \Gamma_{ij}^l. \quad (5.5)$$

By taking into account that  $\mathbf{T}$  does not depend on the indices  $i, j$  and by making a convenient change of variable, we can write down the standard forms of  $\mathbf{F}$  and the dispersion relations by using, for simplicity, the same name for new variables and new coefficients as the old ones:

$$\mathbf{F}\Psi = -\theta \frac{\partial \Psi}{\partial t} - \frac{1}{2} \Delta \Psi + \mathbf{Q}(k) \cdot \nabla \Psi + \boldsymbol{\kappa} \cdot \nabla \Psi \quad (5.6)$$

$$\nabla \mathbf{T} = \mathbf{Q}(\Lambda) \mathbf{T} - \mathbf{T} \mathbf{Q}(\lambda) \quad (5.7)$$

$$\theta \frac{\partial \mathbf{T}}{\partial t} = \frac{1}{2} (\mathbf{Q}(\Lambda) \cdot \mathbf{Q}(\Lambda) \mathbf{T} - \mathbf{T} \mathbf{Q}(\lambda) \cdot \mathbf{Q}(\lambda)) + \boldsymbol{\kappa} \cdot \mathbf{Q}(\Lambda) \mathbf{T} - \mathbf{T} \boldsymbol{\kappa} \cdot \mathbf{Q}(\lambda) \quad (5.8)$$

where

$$\mathbf{Q}(k) = \mathbf{H} + k\Gamma. \quad (5.9)$$

The matrix valued constant matrices  $\mathbf{H}$  and  $\Gamma$  can be chosen arbitrarily provided that

$$[\mathbf{Q}^l(k), \mathbf{Q}^m(k)] = 0 \quad (5.10)$$

or, equivalently, that

$$[\Gamma^m, \Gamma^l] = 0 \quad (5.11)$$

$$[\mathbf{H}^m, \mathbf{H}^l] = 0 \quad (5.12)$$

$$[\mathbf{H}^m, \Gamma^l] = [\mathbf{H}^l, \Gamma^m] \quad (5.13)$$

for any  $l, m = 1, 2, \dots, M$ .

The free term of  $\mathbf{F}$  is

$$f(\Psi^{(1)}) = \boldsymbol{\Gamma} \cdot \nabla \Psi^{(1)}. \quad (5.14)$$

## 5.2. THE SPECTRAL AND THE GHOST OPERATOR

The space dispersion equation (5.7) has the same form as in the general first order case (see (3.11)) and also the spectral operator  $\mathbf{G}$  is the same; that is, it is given by formulae (3.21) and (3.22) in the previous section.

The ghost operator  $\mathbf{A}$  has been already written in its more general form in (3.23). The parasite terms, after substituting in the dispersion equations (5.7) and (5.8), become as follows:

$$\begin{aligned} PT[\mathbf{A}] = \sum_{ij} & \left\{ -\eta_{ij} \left( \frac{1}{2} \mathbf{Q}(\Lambda) \cdot \mathbf{Q}(\Lambda) \mathbf{T} - \frac{1}{2} \mathbf{T} \mathbf{Q}(\lambda) \cdot \mathbf{Q}(\lambda) + \boldsymbol{\kappa} \cdot \mathbf{Q}(\Lambda) \mathbf{T} - \mathbf{T} \boldsymbol{\kappa} \cdot \mathbf{Q}(\lambda) \right) \right. \\ & \left. + \beta_{ij} \cdot \mathbf{Q}(\Lambda) \mathbf{T} - \mathbf{T} \beta_{ij} \cdot \mathbf{Q}(\lambda) + [\mathbf{F}_{ij}, \mathbf{T}] \right\} \Psi \tau_{ij}. \end{aligned} \quad (5.15)$$

By demanding that the parasite terms vanish we get the following anti-ghost condition

$$\text{rank}\{\boldsymbol{\Gamma}^l\} = M. \quad (5.16)$$

By a similar analysis one can show that the same anti-ghost condition (5.16) is sufficient to prevent the existence of ghost operators  $\mathbf{A}$  of second order.

## 6. Nonlinear Evolution Equations of Order 2

### 6.1. THE EVOLUTION EQUATIONS

We suppose in the following that the matrices  $\Gamma^l$  are normal. Since by (5.11) they commute, they are diagonalized by the same similarity transformation. We can write the diagonal representation as

$$\Gamma^l = \sum_{i=1}^N g_i^l \eta_i, \quad l = 1, 2, \dots, M \quad (6.1)$$

Thanks to the anti-ghost condition (5.16) the vectors  $\mathbf{g}^i$  satisfy the condition

$$\text{rank}(\mathbf{g}^1 \mathbf{g}^2 \cdots \mathbf{g}^N) = M. \quad (6.2)$$

Note that in the case  $M = 1$ , requiring (4.9) forces the spectral operator  $G$  to be zero.

By using the invariance properties of the integral  $\bar{\partial}$ -equation (1.1) in a way similar to the previous section, we can reduce the dispersion relations (5.7) and (5.8) to the case in which the  $\Gamma^l$ 's and the  $\mathbf{H}^l$ 's are diagonal and off-diagonal matrices respectively. Moreover, since the terms with  $\kappa^l$  in (5.8) generate in the evolution equations a trivial linear term in the space derivatives, we choose  $\kappa^l = 0$ .

If  $\mathbf{H} \neq 0$ , the integrability conditions (5.12) and (5.13) impose on the vectors  $\mathbf{g}_i$  the additional algebraic constraints

$$\frac{g_i^l - g_k^l}{g_i^h - g_k^h} = \frac{g_j^l - g_k^l}{g_j^h - g_k^h} \quad (6.3)$$

for  $l, h = 1, 2, \dots, M$  and  $i, j, k = 1, 2, \dots, N$ . Then the  $\mathbf{H}^l$  matrices can be represented by the equation

$$\mathbf{H}^l = [\Gamma^l, \mathbf{K}] \quad (6.4)$$

with  $\mathbf{K}$  an off-diagonal matrix. This representation includes the case  $\mathbf{H} = 0$  for  $\mathbf{K} = 0$ , but then one has to take into account that the constraints (6.3) are no longer necessary. Note that the constraints (6.3) imply that the maximum possible value of the rank of the matrix in (6.2) is 2. Therefore in the case  $\mathbf{K} \neq 0$ , the anti-ghost condition can be satisfied only if  $M = 2$ .

On obtaining the nonlinear evolution equations we proceed in the same way as in the first order case. We substitute the asymptotic expansion (1.6) of  $\Psi$  into the spectral equation  $G\Psi = \Psi\mathbf{g}(\Psi^{(1)})$  and into the evolution equation  $F\Psi = \Psi\mathbf{f}(\Psi^{(1)})$ . The equations resulting from the first relation have been already written down in (4.20)–(4.22). From the second equation we obtain

$$\begin{aligned} \theta \frac{\partial \Psi_F^{(1)}}{\partial t} &= -\frac{1}{2} \Delta \Psi_F^{(1)} + (\mathbf{H} \bullet \nabla \Psi^{(1)})_F + \Gamma \bullet \nabla \Psi_F^{(2)} \\ &\quad - (\Psi^{(1)} \Gamma \bullet \nabla \Psi^{(1)})_F \end{aligned} \quad (6.5)$$

$$\begin{aligned} \theta \frac{\partial \Psi_D^{(1)}}{\partial t} &= -\frac{1}{2} \Delta \Psi_D^{(1)} + (\mathbf{H} \bullet \nabla \Psi_F^{(1)})_D + \Gamma \bullet \nabla \Psi_D^{(2)} \\ &\quad - (\Psi^{(1)} \Gamma \bullet \nabla \Psi^{(1)})_D. \end{aligned} \quad (6.6)$$

Let us call  $\Psi_F^{(1)}$ ,  $\Psi_D^{(1)}$  and  $\Psi_D^{(2)}$ , respectively,  $\Phi$ ,  $\mathbf{A}$  and  $\mathbf{B}$  for short. In the set of equations (4.20)–(4.22) and (6.5), (6.6) the equation (4.20) defines the off-diagonal part  $\Psi_F^{(2)}$  of  $\Psi^{(2)}$  as an explicit functional of  $\Phi$  and  $\mathbf{A}$ ,

$$\Gamma \Psi_F^{(2)} \bullet \Omega = \nabla \Phi \bullet \Omega - [\mathbf{H}, \Phi + \mathbf{A}]_F \bullet \Omega + ((\Phi + \mathbf{A}) \Gamma \Phi \bullet \Omega)_F, \quad (6.7)$$

while the other ones are coupled equations in the fields  $\Phi$ ,  $\mathbf{A}$  and  $\mathbf{B}$ . From (6.5) we get an evolution equation for  $\Phi$  which is coupled to  $\mathbf{A}$ ,

$$\begin{aligned} \theta \Gamma \frac{\partial \Phi}{\partial t} \bullet \Omega &= -\frac{1}{2} \Gamma \Delta \Phi \bullet \Omega + \Gamma (\mathbf{H} \bullet \nabla (\Phi + \mathbf{A})) \bullet \Omega + \Gamma \bullet \nabla (\nabla \Phi \bullet \Omega) \\ &\quad - \Gamma \bullet \nabla ([\mathbf{H}, \Phi + \mathbf{A}]_F \bullet \Omega) + \Gamma \bullet \nabla ((\Phi + \mathbf{A}) \Gamma \Phi \bullet \Omega)_F \\ &\quad - \Gamma ((\Phi + \mathbf{A}) \Gamma \bullet \nabla (\Phi + \mathbf{A})) \bullet \Omega \end{aligned} \quad (6.8)$$

and (4.21) yields the constraint equation for  $\mathbf{A}$ ,

$$\nabla \mathbf{A} \bullet \Omega = [\mathbf{H}, \Phi]_D \bullet \Omega - (\Phi \Gamma \Phi \bullet \Omega)_D. \quad (6.9)$$

The remaining two equations (6.6) and (4.22) furnish additional information on  $\mathbf{A}$ . In fact from (6.6) we get an evolution equation for  $\mathbf{A}$  coupled with  $\mathbf{B}$ ,

$$\theta \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{2} \Delta \mathbf{A} + (\mathbf{H} \bullet \nabla \Phi)_D + \Gamma \bullet \nabla \mathbf{B} - (\Phi \Gamma \bullet \nabla \Phi)_D - A \Gamma \bullet \nabla \mathbf{A} \quad (6.10)$$

while (4.22) gives a constraint equation for  $\mathbf{B}$

$$\nabla \mathbf{B} \bullet \Omega - [\mathbf{H}, \Psi_F^{(2)}]_D \bullet \Omega + (\Psi_F^{(2)} \Gamma \Phi \bullet \Omega)_D = 0. \quad (6.11)$$

These equations however do not exhaust the information on  $\mathbf{A}$ . In fact, if we expand  $\Psi$  in powers of  $1/k$  and insert this larger expansion into the spectral and evolution equations, the result is that  $\mathbf{B} \equiv \Psi_D^{(2)}$  is coupled to  $\Psi_D^{(3)}$  in the asymptotic expansion, and so forth.

This result seems to indicate that the coupled equations (6.8) and (6.9) can be solved by using the spectral problem (1.1) only for a special functional class of solutions  $\mathbf{A}$  of the constraint equation (6.9). As we show in a following subsection, for  $M = N = 2$  and for a special choice of the parameters, one can obtain from (6.8) and (6.9) the Davey–Stewartson equations I and II. The fact, discovered here, that they allow too much freedom for  $\mathbf{A}$  with respect to the constraints imposed by the spectral theory, could explain the (so far) insurmountable difficulties found in trying to build the spectral transform for special choices of  $\mathbf{A}$  as in the so-called Hamiltonian case.

## 6.2. THE CONSTRAINTS

Let us consider the compatibility condition (1.11). If we assume that  $\Psi$  admits continuous derivatives up to the third order and we recall the integrability condition  $[\mathbf{Q}^l, \mathbf{Q}^h] = 0$  satisfied by the matrix coefficients of the dispersion laws, we get easily that  $[F, G]\Psi = 0$ , and consequently that

$$F(\Psi g(\Psi^{(1)})) - G(\Psi f(\Psi^{(1)})) = 0. \quad (6.12)$$

By considering the evolution equation  $F\Psi = \Psi f(\Psi^{(1)})$  for  $\Psi$  and the evolution equation (6.8) for  $\Psi_F^{(1)} = \Phi$  we obtain

$$(\nabla \Psi - [\mathbf{Q}, \Psi] + \Psi \Gamma \Phi) \bullet \Xi = 0 \quad (6.13)$$

where

$$\Xi = \nabla(\Gamma\Psi^{(1)} \bullet \Omega) - (\Gamma \bullet \nabla\Psi^{(1)})\Omega. \quad (6.14)$$

Note that in accordance with the general form of a first order compatible operator in (3.21) and (3.22),  $\Gamma \bullet \Xi = 0$ . Since  $\Psi$  satisfies the spectral equation

$$(\nabla\Psi - [\mathbf{Q}, \Psi] + \Psi\Gamma\Phi) \bullet \Omega = 0 \quad (6.15)$$

the diagonal part in the second term of  $\Xi$  can be cancelled and we can rewrite  $\Xi$ :

$$\Xi = \nabla(\Gamma\Phi \bullet \Omega) - (\Gamma \bullet \nabla\Phi)\Omega. \quad (6.16)$$

If we note

$$\Omega^{(i)} = \omega_i \eta_i, \quad i = 1, 2, \dots, N \quad (6.17)$$

and

$$\Xi^{(i)} = \nabla(\Gamma\Phi \bullet \Omega^{(i)}) - (\Gamma \bullet \nabla\Phi)\Omega^{(i)}, \quad i = 1, 2, \dots, N \quad (6.18)$$

by substituting the asymptotic expansion (1.6) of  $\Psi$  in (6.13), we get the following  $N$  constraint equations ( $i = 1, 2, \dots, N$ )

$$(\nabla(\Phi + \mathbf{A}) - [\mathbf{H}, \Phi + \mathbf{A}] - \Gamma\Psi_F^{(2)} + (\Phi + \mathbf{A})\Gamma\Phi) \bullet \Xi^{(i)} = 0, \quad (6.19)$$

where  $\Psi_F^{(2)}$  has to be expressed in terms of  $\Phi$  and  $\mathbf{A}$  by using (4.31). These equations do no furnish any new additional constraint if they are a trivial consequence of the spectral equation. This is the case for  $M = 2$ . In the case  $M > 2$  the equations (6.19) are effective additional constraints.

### 6.3. SOME RELEVANT EXAMPLES

*6.3.1. Case  $K = 0, M = N$ .* The dispersion relations (5.7) and (5.8) do depend on the “length”  $\mathbf{g}_l \bullet \mathbf{g}_l \equiv \alpha_l$  of the vectors  $\mathbf{g}_l$  but not on their relative scalar product  $\mathbf{g}_l \bullet \mathbf{g}_{l'}$  ( $l \neq l'$ ). Therefore, we can choose the vectors  $\mathbf{g}_l$  to be orthogonal. Moreover, since the evolution equations and the constraints are invariant under a scale transformation we can choose the  $\mathbf{g}_l$ ’s to satisfy

$$\mathbf{g}_l \bullet \mathbf{g}_{l'} = \alpha_l \delta_{ll'}, \quad \alpha_l^2 = 1. \quad (6.20)$$

Consider the change of variables

$$x_l \longrightarrow \sum_{l'} g_l^{l'} x_{l'} \quad (6.21)$$

and introduce a potential field  $\phi$  as follows:

$$\partial_l \phi = 2\mathbf{A}_{ll}. \quad (6.22)$$

Let us finally choose  $\theta = t \text{fr}12i$  and define

$$Q_{ll'} = 2\Phi_{ll'}. \quad (6.23)$$

We obtain the nonlinear evolution equations

$$\begin{aligned} i\partial_t Q_{ll'} + (\alpha_l \partial_l^2 - \alpha_{l'} \partial_{l'}^2) Q_{ll'} + [(\alpha_l \partial_l^2 - \alpha_{l'} \partial_{l'}^2)\phi] Q_{ll'} \\ + \frac{1}{2} \sum_{\substack{m \neq l \\ m \neq l'}} \alpha_m [(\partial_m Q_{lm}) Q_{ml'} - Q_{lm} (\partial_m Q_{ml'})] = 0, \quad l \neq l' \end{aligned} \quad (6.24)$$

which are coupled to the field  $\phi$  via the equation

$$2\partial_l \partial_{l'} \phi = -Q_{ll'} Q_{l'l}, \quad l \neq l', \quad (6.25)$$

and the constraints

$$2\partial_l Q_{l'l'} = -Q_{ll'} Q_{l'l'}, \quad l \neq l', \quad l \neq l'', \quad l' \neq l''. \quad (6.26)$$

These equations satisfy the reduction

$$Q_{ll'} = \epsilon_{ll'} \bar{Q}_{ll}, \quad l \neq l' \quad (6.27)$$

where

$$\epsilon_{ll'} = \epsilon_{l'l}, \quad \epsilon_{ll'}^2 = 1, \quad l \neq l' \quad (6.28)$$

$$\epsilon_{lm} \epsilon_{m'l'} = \epsilon_{ll'}, \quad l \neq l', \quad l \neq m, \quad l' \neq m, \quad (6.29)$$

and the overbar means the complex conjugation.

A similar procedure can be followed in the other cases for casting the nonlinear evolution equations and the constraints into the simplest possible form.

**6.3.2. Case  $K = 0, M = N = 2$  or the Davey-Stewartson systems.** We can distinguish four cases according to the sign of  $\alpha = -\alpha_1 \alpha_2$  and the symmetry properties of the space variables with respect to complex conjugation, i.e. according to the two cases,  $x_1 = \bar{x}_1$  and  $x_2 = \bar{x}_2$ , or  $x_1 = \bar{x}_2$ . Let us define, for short,

$$q = Q_{12}, \quad r = Q_{21}. \quad (6.30)$$

In the case

$$x_1 = v \in \mathbf{R}, \quad x_2 = u \in \mathbf{R}, \quad (6.31)$$

we get

$$\begin{aligned} iq_t + \alpha q_{uu} + q_{vv} + (\alpha \phi_{uu} + \phi_{vv})q = 0 \\ - ir_t + \alpha r_{uu} + r_{vv} + (\alpha \phi_{uu} + \phi_{vv})r = 0 \\ 2\phi_{uv} = -qr. \end{aligned} \quad (6.32)$$

In the case

$$x_1 = x + iy, \quad x_2 = x - iy \quad (6.33)$$

we get, for  $\alpha = 1$ , the system

$$\begin{aligned} iq_t + \frac{1}{2}(q_{xx} - q_{yy}) + \frac{1}{2}(\phi_{xx} - \phi_{yy})q = 0 \\ - ir_t + \frac{1}{2}(r_{xx} - r_{yy}) + \frac{1}{2}(\phi_{xx} - \phi_{yy})r = 0 \\ \phi_{xx} + \phi_{yy} = -2qr \end{aligned} \quad (6.34)$$

and, for  $\alpha = -1$ ,

$$\begin{aligned} q_t - q_{xy} - \phi_{xy}q &= 0 \\ r_t - r_{xy} - \phi_{xy}r &= 0 \\ \phi_{xx} + \phi_{yy} &= -2qr \end{aligned} \tag{6.35}$$

which can be transformed into (6.34) by a complex rescaling of  $t$ ,  $t \rightarrow it$ , and a rotation of the space coordinates,  $x \rightarrow \frac{1}{2}\sqrt{2}(x+y)$ ,  $y \rightarrow \frac{1}{2}\sqrt{2}(x-y)$ .

We obtain from (6.32), for  $\alpha = 1$ , the so called Davey–Stewartson I system (DSI) in characteristic coordinates and from (6.34) the Davey–Stewartson II system (DSII). It is well-known that they can be linearized using the  $\bar{\partial}$ -method, and this was first shown almost a decade ago [32].

The Davey–Stewartson system obtained for  $\alpha = -1$  in (6.32) was not known to be integrable, as far as we know. We call this system DSIII.

## 7. Case of a Local Inverse Spectral Problem

If the measure  $d\sigma(\lambda, \Lambda)$  in the integral equation (1.1) is a distribution in one variable, say the  $\lambda$  variable, and has support on single points whose location in the  $\lambda$ -plane depend on  $\Lambda$ , the  $\bar{\partial}$ -equation (1.2) becomes local and we are dealing with a local inverse spectral problem.

We consider here the simple case

$$d\sigma(\lambda, \Lambda) = d\sigma(\Lambda)d\lambda\delta(\lambda - \Lambda). \tag{7.1}$$

With respect to the cases considered in the previous sections, the requirement that the parasite terms of the evolution and spectral operators  $F$  and  $G$  cancel becomes less stringent since the two variables  $\lambda$  and  $\Lambda$  are no longer independent and, consequently, the class of possible  $F$  and  $G$  operators becomes larger and its study more complex. In fact, the local  $\bar{\partial}$ -problem is typical of the integrable nonlinear evolution equations in  $1+1$  dimensions and it is well known that this class is much richer than the corresponding class in  $2+1$  dimensions where the  $\bar{\partial}$ -problem is nonlocal.

In order to show how the method works in the local case we consider as an example the self dual Yang–Mills equation. This equation is especially interesting because, in order to obtain it, instead of considering a pair of  $F$  and  $G$  operators we consider a pair of  $F$  operators. Thus it suggests a possible generalization of the method outlined in this paper.

Let us consider the following evolution operator  $F$  of first order in the space variable  $x$  and linear in  $k$

$$F\Psi = -\frac{\partial\Psi}{\partial t} + k\frac{\partial\Psi}{\partial x} \tag{7.2}$$

where  $\Psi \in \mathcal{M}_N$ . The parasite terms are zero if and only if

$$\frac{\partial\mathbf{T}}{\partial t} = \Lambda\frac{\partial\mathbf{T}}{\partial x}. \tag{7.3}$$

The evolution equation for  $\Psi$  reads

$$F\Psi = \Psi\frac{\partial\Psi^{(1)}}{\partial x}. \tag{7.4}$$

We consider another evolution operator  $F'$  of the same form as  $F$  but depending on a different time and space variables  $x'$  and  $t'$ .

$$F'\Psi = -\frac{\partial \Psi}{\partial t'} + k \frac{\partial \Psi}{\partial x'}, \quad (7.5)$$

and then the Spectral Transform  $T$  must satisfy the additional dispersion law

$$\frac{\partial T}{\partial t'} = \Lambda \frac{\partial T}{\partial x'}. \quad (7.6)$$

We consider the operators  $F$  and  $F'$  as a pair of associated operators and as usual, we substitute the asymptotic expansion (1.6) for  $\Psi$  into the equation (7.4) and the corresponding one for  $F'$ , viz

$$F'\Psi = \Psi \frac{\partial \Psi^{(1)}}{\partial x'}, \quad (7.7)$$

obtaining

$$-\frac{\partial \Psi^{(1)}}{\partial t} + \frac{\partial \Psi^{(2)}}{\partial x} - \Psi^{(1)} \frac{\partial \Psi^{(1)}}{\partial x} = 0 \quad (7.8)$$

$$-\frac{\partial \Psi^{(1)}}{\partial t'} + \frac{\partial \Psi^{(2)}}{\partial x'} - \Psi^{(1)} \frac{\partial \Psi^{(1)}}{\partial x'} = 0. \quad (7.9)$$

On eliminating  $\Psi^{(2)}$  we obtain the following evolution equation for  $\Psi^{(1)}$ :

$$\frac{\partial^2 \Psi^{(1)}}{\partial x \partial t'} - \frac{\partial^2 \Psi^{(1)}}{\partial x' \partial t} + \frac{\partial \Psi^{(1)}}{\partial x} \frac{\partial \Psi^{(1)}}{\partial x'} - \frac{\partial \Psi^{(1)}}{\partial x'} \frac{\partial \Psi^{(1)}}{\partial x} = 0. \quad (7.10)$$

One can easily verify that the compatibility condition (1.11) is satisfied and that, consequently, there are no constraints.

For variables  $t, t'$  and  $x, x'$  such that

$$\bar{x}' = t \quad \bar{x} = -t' \quad (7.11)$$

this evolution equation is an alternative form of the self-dual Yang-Mills equation. In fact, it is the compatibility equation for the solvability of the coupled set of equations

$$\frac{\partial \mathbf{J}^T}{\partial t} = \frac{\partial \Psi^{(1)}}{\partial x} \mathbf{J}^T \quad (7.12)$$

$$\frac{\partial \mathbf{J}^T}{\partial t'} = \frac{\partial \Psi^{(1)}}{\partial x'} \mathbf{J}^T \quad (7.13)$$

where a superscript  $T$  indicates the transpose, defining the matrix field  $\mathbf{J} \in \mathcal{M}_N$ . From these two equations it is an immediate consequence that  $\mathbf{J}$  satisfies the following nonlinear equation in  $(1+1)+(1+1)$  dimensions

$$\frac{\partial}{\partial x'} \left( \mathbf{J}^{-1} \frac{\partial \mathbf{J}}{\partial t} \right) - \frac{\partial}{\partial x} \left( \mathbf{J}^{-1} \frac{\partial \mathbf{J}}{\partial t'} \right) = 0 \quad (7.14)$$

which, in the case (7.11), is the version due to Pohlmeyer [33] of the self-dual Yang-Mills equation.

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## UNIVERSAL INTEGRABLE NONLINEAR PDEs

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**ABSTRACT.** The results reported at the Exeter meeting are briefly reviewed. Some results obtained during that meeting, motivated by findings of which I became aware there, are reported. In particular it is noted that the "universal"  $S$ -integrable equation which describes in the limit of weak nonlinearity and in appropriately rescaled time and space variables, the amplitude modulation of the solution representing a single dispersive wave of the "Konopelchenko-Rogers" equation, includes the so-called DS-III system, whose  $S$ -integrability has recently been demonstrated by Boiti, Pempinelli and Sabatier.

### 1. Introduction

At the Exeter meeting I reviewed the findings that provide a heuristic explanation of the remarkable fact that certain nonlinear partial differential equations are both widely applicable and integrable [1]. The basic concept underlying this notion is the existence of "universal" nonlinear PDEs, namely PDEs that are associated via some limiting procedure to a large class of nonlinear PDEs. If the limiting procedure is "physically reasonable", namely if it corresponds to phenomenological circumstances likely to arise in several contexts, then the universal nonlinear PDE yielded by this procedure will be widely applicable, since the large class of nonlinear PDEs from which the universal PDE is obtained certainly contains — because it is large — many instances of equations having applicable relevance. If moreover the limiting procedure is in some (perhaps asymptotic) sense exact, and therefore generally such to preserve integrability, namely such as to yield, whenever it is applied to an equation having some character of integrability, an equation at least as integrable, then it is justifiable to expect that a universal equation obtained by applying such a limiting procedure to a large class of nonlinear PDEs be integrable, since in order for this to happen it is sufficient that the large class of nonlinear PDEs contain just one integrable PDE.

As reported in Exeter, this philosophy is actually rather fruitful to identify nonlinear PDEs which are quite interesting, since their universality generally entails a simplicity having an aesthetic appeal and, perhaps more importantly, possessing both properties, wide applicability and integrability.

In particular, a large class of nonlinear PDEs being fruitfully explored in this context is that characterized by a linear part which is dispersive and a nonlinear part which is largely

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arbitrary (perhaps, “weakly analytic”); correspondingly, a limiting process being considered is that evidencing the *amplitude modulation* of a single, or a few, dispersive waves, in the context of *weak fields* (regime of “weak nonlinearity”). The strategy to obtain *integrable universal* nonlinear PDEs, that characterize the evolution in appropriately rescaled variables of the amplitudes of one, or a few, dispersive waves, takes as point of departure for the application of the limiting process, some *integrable* equation contained in this large class, and the outcome of this approach is to identify integrable PDEs of “*nonlinear Schrödinger type*” or “*N-wave interaction type*” [1,2].

In this context the different roles were noted of integrable nonlinear PDEs which are, respectively, “*S-integrable*” (i.e., integrable via the *Spectral transform* or *inverse Scattering method*) or “*C-integrable*” (i.e., integrable via an appropriated *Change of variables*). The potential interest of *C-integrable* nonlinear PDEs was emphasized (in spite — or perhaps because — of their mathematical simplicity).

An obvious and appealing direction of research suggested by these findings is to use this approach to identify nonlinear *C-integrable* PDEs in  $N + 1$  dimensions that have a *universal* character, in the sense outlined above, and are therefore likely to be of interest from the theoretical, as well as the applied, points of view. The technique to obtain such results involves, firstly, the manufacture of some *C-integrable* nonlinear PDEs in  $N + 1$  dimensions with dispersive linear part, and, secondly, the derivation from such equations of *universal C-integrable nonlinear PDEs in  $N + 1$  dimensions*. Such findings were indeed the main focus of my presentation in Exeter, whose title was “Universal *C-integrable PDEs in  $N + 1$  dimensions*”, and whose Abstract read:

“The notions of “universal”, “applicable” and “integrable” (“*S-integrable*” and “*C-integrable*”) nonlinear PDEs are outlined, and their interconnections are reviewed, thereby recalling a heuristic explanation of the fact that certain nonlinear PDEs are both widely applicable and integrable.

“A nonlinear ( $N + 1$ )-dimensional *C-integrable* PDE is presented, which has some characteristics of “universability” and is a natural ( $N + 1$ )-dimensional extension of the ( $1 + 1$ )-dimensional *C-integrable* Eckhaus equation.”

[In reporting this Abstract I have retained the lovely spelling mistake made by an unknown typist, which resulted in the creation of a new term (“universability”) which deserves a future].

However, since results corresponding to the first part of the two-stage program outlined above are already in print [3], and results corresponding to the second part have already been submitted for publication [4], it was felt appropriate to limit the written report of these findings here to the descriptive presentation given above, referring the interested reader to the literature [1-4] for a more detailed treatment (with formulas in addition to words).

An additional motivation for making this choice is my preference to report here instead some results which were actually obtained *during* the Exeter meeting, motivated by findings of which I became aware by listening to presentations at that meeting.

## 2. Results

A *S-integrable* equation much talked about at the Exeter meeting was the  $2 + 1$  generalization of the Sine-Gordon equation, recently discovered by Konopelchenko and Rogers [5]. This equation comes in several avatars [6]. Here we report the beautiful form that was

displayed by Nimmo [7] in Exeter:

$$u_{xyt} + u_x v_{yt} + u_y v_{xt} = 0, \quad (1a)$$

$$v_{xy} = u_x u_y. \quad (1b)$$

Hereafter we refer to this nonlinear  $S$ -integrable PDE in  $2 + 1$  dimensions as the Konopelchenko-Rogers (KR) equation.

It is amusing to note that the KR equation admits the class of solutions

$$u(x, y, t) = \alpha \ln[f_1(x) + f_2(y) + f_3(t)], \quad (2a)$$

$$v(x, y, t) = \alpha u(x, y, t) + f_4(x) + f_5(y) + f_6(t), \quad (2b)$$

with  $\alpha^2 = -1$ , and the  $f_j$  six arbitrary functions. Other solutions can be obtained from this one by applying standard techniques, exploiting the large symmetry of (2.1). Of course the solution (2.2), for all its six arbitrary functions of a single variable, falls quite short of providing the general solution of (2.1), which should contain an arbitrary function of two variables.

For our purpose it is now convenient to set

$$u(x, y, t) = \epsilon U(x, y, t), \quad (3a)$$

$$v(x, y, t) = ayt + bxt + \epsilon V(x, y, t), \quad (3b)$$

with  $a$  and  $b$  arbitrary constants and  $\epsilon$  an arbitrary (“small”) parameter. The equations satisfied by  $U$  and  $V$  then read:

$$U_{xyt} + aU_x + bU_y = -\epsilon(U_x V_{yt} + U_y V_{xt}), \quad (4a)$$

$$V_{xy} = \epsilon U_x U_y. \quad (4b)$$

In the (extreme) weak field limit that is obtained by setting  $\epsilon = 0$ , these equations, which are then linear, admit the special solution

$$U(x, y, t) = C \exp(iz) + c.c., \quad (5a)$$

$$V(x, y, t) = 0, \quad (5b)$$

$$z \equiv kx + py - \omega t. \quad (5c)$$

This solution represents a single dispersive wave of constant amplitude  $C$ . The “wave-numbers”  $k$  and  $p$  in (5c) are two arbitrary real constant parameters which will be kept fixed hereafter (as well as the two arbitrary real constants  $a$  and  $b$ ). The “frequency”  $\omega$  is given in terms of  $k$  and  $p$  by the formula

$$\omega = -a/p - b/k. \quad (6)$$

Let us recall that associated to this dispersive wave is the “group velocity”, having the  $x$ - and  $y$ -components

$$c_1 = \frac{d\omega}{dk} = \frac{b}{k^2}, \quad c_2 = \frac{d\omega}{dp} = \frac{a}{p^2}. \quad (7)$$

If the parameter  $\epsilon$  is small, but not quite zero, then there must be a solution of (4) which is close to (5). The following ansatz is convenient to analyze such a solution:

$$U(x, y, t) = \psi(\xi, \eta, \tau) \exp(iz) + c.c. + \dots, \quad (8a)$$

$$V(x, y, t) = \epsilon^{-1} \varphi(\xi, \eta, \tau) + \epsilon[\chi(\xi, \eta, \tau) \exp(2iz) + c.c.] + \dots, \quad (8b)$$

with the “coarse-grained” and “slow” variables  $\xi, \eta$  and  $\tau$  defined as follows:

$$\xi = \epsilon(x - c_1 t), \quad \eta = \epsilon(y - c_2 t), \quad \tau = \epsilon^2 t. \quad (8c)$$

The dots in the right-hand side of (8) indicate additional “small” corrective terms, which can be consistently neglected. Note that, while the first term in the right hand side of (8) becomes large when  $\epsilon \rightarrow 0$ , all the second derivatives of  $V$  which enter into (4) are small for small  $\epsilon$  and vanish as  $\epsilon \rightarrow 0$  (see (A.2)).

Insertion of the *ansatz* (8) in (4) yields, in the  $\epsilon \rightarrow 0$  limit, the following equation of Davey-Stewartson type, which we write for neatness in appropriately rescaled variables (see details in the Appendix):

$$i\psi_\tau + L\psi + \mu\psi = 0, \quad (9a)$$

$$\mu_{\xi\eta} = s_3 L|\phi|^2, \quad (9b)$$

where

$$L = s_1 \partial_{\xi\xi} + s_2 \partial_{\eta\eta}, \quad (9c)$$

$$s_1 = -\text{sign}(b/k), \quad s_2 = -\text{sign}(a/p), \quad s_3 = \text{sign}(kp), \quad (9d)$$

and of course  $\psi \equiv \psi(\xi, \eta, \tau)$ ,  $\mu \equiv \mu(\xi, \eta, \tau)$ .

Note that all possible choices of the 3 signs  $s_1, s_2$  and  $s_3$  can be obtained, since the choice of the 4 constants  $a, b, k$  and  $p$  is our privilege.

Since these equations of Davey-Stewartson type have been obtained from the  $S$ -integrable KR equation (1), they are certainly themselves  $S$ -integrable. Their  $S$ -integrability could be derived from the integrability of (1), by appropriately extending the technique which has been applied here only to the nonlinear PDEs to extract (9) from (1) and also from the mathematical machinery [5,6] that underlies the integrability of (1) (the effectiveness of this methodology was first demonstrated, in different contexts, by Zakharov and Kuznetsov [8]; see also [9,1]). But this need not be done here, since the integrability of (9) is already known. It is, however, amusing to note that while for some choices of the signs  $s_j$  the  $S$ -integrability of (9) was already demonstrated long ago, for other choices it appears to have been demonstrated only recently [10], as reported at the Exeter meeting both by Boiti [11] and by Pempinelli [12]. Incidentally, a version of what is referred to in [10] as DS-III — namely, (9) with  $s_1 s_2 = -1$  — already appears in [13], where however no integrability technique is given.

As a final remark we note that, while the equation we have obtained here describes the evolution of the single complex field  $\psi$  coupled to the auxiliary field  $\mu$  in (9), it would be easy to derive equations for two coupled fields analogous to those reported in [10] (see in particular the subsection 6.3.2 of that paper), by starting from a complex, rather than a real solution of (1), namely by replacing (8) with the more general *ansatz*

$$U(x, y, t) = \psi_1(\xi, \eta, \tau) \exp(iz) + \psi_2(\xi, \eta, \tau) \exp(-iz) + \dots \quad (10a)$$

$$V(x, y, t) = \epsilon^{-1} \varphi(\xi, \eta, \tau) + \epsilon[\chi_1(\xi, \eta, \tau) \exp(2iz) + \chi_2(\xi, \eta, \tau) \exp(-2iz)] + \dots \quad (10b)$$

## Appendix

From the *ansatz* (8) (with (5e), (6) and (7)) we obtain:

$$U_x = (ik\psi + \epsilon\psi_\xi) \exp(iz) + c.c. + \dots \quad (\text{A.1a})$$

$$U_y = (ip\psi + \epsilon\psi_\eta) \exp(iz) + c.c. + \dots \quad (\text{A.1b})$$

$$\begin{aligned} U_{xyt} &= \{-i(ak + bp)\psi - \epsilon(a\psi_\xi + b\psi_\eta) \\ &\quad - \epsilon^2[kp\psi_\tau + i(bp/k^2)\psi_{\xi\xi} + i(ak/p^2)\psi_{\eta\eta}]\} \exp(iz) + c.c. + \dots, \end{aligned} \quad (\text{A.1c})$$

$$V_{xy} = \epsilon\phi_{\xi\eta} - \epsilon[4kp\chi \exp(2iz) + c.c.] + \dots \quad (\text{A.2a})$$

$$V_{xt} = -\epsilon(c_1\phi_{\xi\xi} + c_2\psi_{\xi\eta}) + \epsilon[4kp\omega\chi \exp(2iz) + c.c.] + \dots \quad (\text{A.2b})$$

$$V_{yt} = -\epsilon(c_2\phi_{\eta\eta} + c_1\psi_{\xi\eta}) + \epsilon[4p\omega\chi \exp(2iz) + c.c.] + \dots \quad (\text{A.2c})$$

Note that again, in these formulas, we only write the leading terms. Other terms which are smaller for small  $\epsilon$  and which do not affect the final result are absorbed in the omitted part represented by the dots.

Insertion of these expressions in (4) yields the 3 equations (exact in the  $\epsilon \rightarrow 0$  limit)

$$i\psi_\tau + M\psi = -\psi\{M\varphi - [\omega/(kp)]\varphi_{\xi\eta} + 8\omega\chi\}, \quad (\text{A.3a})$$

$$\varphi_{\xi\eta} = 2kp|\psi|^2, \quad (\text{A.3b})$$

$$\chi = \frac{1}{4}|\psi|^2, \quad (\text{A.3c})$$

with

$$M = -(b/k^3)\partial_{\xi\xi} - (a/p^3)\partial_{\eta\eta}. \quad (\text{A.4})$$

Insertion of (A.3b) and (A.3c) in (A.3a) yields moreover a cancellation of the last two terms in the right-hand side. of (A.3a), so that this equation can be rewritten in the simpler form

$$i\psi_\tau + M\psi + \psi M\varphi = 0. \quad (\text{A.5})$$

It is finally convenient to perform the “cosmetic” rescaling

$$\xi \rightarrow |b/k^3|^{1/2}\xi, \quad \eta \rightarrow |a/p^3|^{1/2}\eta, \quad \psi \rightarrow |kp/(4ab)|^{1/4}\psi,$$

so that (A.5) and (A.3b) read

$$i\psi_\tau + L\psi + \psi L\varphi = 0, \quad (\text{A.6a})$$

$$\varphi_{\xi\eta} = s_3|\psi|^2, \quad (\text{A.6b})$$

with  $L$  and  $s_j$  defined by (9c,d). The final position

$$\mu = L\varphi \quad (\text{A.7})$$

puts the equations in the canonical form (9).

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## CONSTRUCTION OF REFLECTIONLESS POTENTIALS WITH INFINITELY MANY DISCRETE EIGENVALUES

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### 1. Introduction

We report here some results on the relation between a real function  $q(x)$  of a real variable  $x \in \mathbb{R}$  and the spectral data associated with it through the Schrödinger equation

$$\psi_{xx} + [q(x) + k^2]\psi = 0, \quad \psi = \psi(x, k). \quad (1)$$

(where a subscripted variable means partial differentiation). Further details and results are given in [1]. This relation is of enormous relevance in many different contexts such as applied inverse problems, extension to ODEs other than (1) and solution of nonlinear PDEs, and does not need to be emphasized here.

If  $\int_{-\infty}^{+\infty} dx |q(x)| < \infty$ , the Fourier transform of  $q(x)$  is well defined, and if the equation (1) has no discrete eigenvalue, which is the case if  $q(x) < 0$ , its spectral transform via (1), which is the reflection coefficient  $R(k)$  (see below), is well defined despite possibly having a singularity at  $k = 0$ . Moreover, if  $q(x)$  is small enough, the mapping  $q(x) \rightarrow R(k)$  linearizes via the Fourier transformation. However, if equation (1) has a discrete spectrum, the spectral transform of  $q(x)$ , in addition to the nonlinear Fourier component, has a discrete part which has no linear analogue. Indeed, in the linear approximation, the discrete part would bring into play the unbounded exponentials  $\exp(ikx)$  for  $k$  off the real axis. Therefore, if  $q(x) \in L_1(\mathbb{R})$ , the problem of the discrete spectral component is inevitably nonlinear. This problem has been dealt with by various techniques, for example the Riemann-Hilbert or D-Bar method, or Bäcklund transformations, which put the problem in an algebraic setting if

the number of discrete eigenvalues is finite. A sufficient condition for the number of discrete eigenvalues to be finite is  $\int_{-\infty}^{+\infty} dx(1+|x|)|q(x)| < \infty$ . Among such functions, those whose nonlinear Fourier component vanishes, the reflectionless or transparent potentials, are well characterized and understood.

On the other hand, if there are infinitely many discrete eigenvalues, the corresponding class of reflectionless potentials is not yet well characterized or understood. The natural approach based on the Gel'fand-Levitan-Marchenko equation, in the limit in which the kernel  $M(x+y)$  is given by the convergent series  $M(x) = \sum_{n=1}^{\infty} \rho_n \exp(-p_n x)$ , does not prove to be practical, since the kernel is not in the Hilbert-Schmidt class, and the Neumann series does not provide even the large  $x$  behaviour of the corresponding potential. A more fruitful direction has been recently indicated by one of the present authors [2], which relies on an infinite chain of Bäcklund transformations (see also [3]). Here we limit ourselves to giving a method of construction of a special class of reflectionless potentials with an infinite discrete spectrum, which are everywhere bounded,  $L_1$  functions. In this case, the Jost solutions are well defined for  $k \neq 0$ , and therefore the discrete eigenvalues can accumulate only at the origin  $k = 0$ .

For convenience, we give here a few well-known formulae of the spectral theory (cf. for example [4]). The Jost solutions  $f^{(\pm)}(x, k)$  of (1) are defined by the asymptotic conditions

$$\lim_{x \rightarrow \pm\infty} [\exp(\mp ikx)f^{(\pm)}(x, k)] = 1, \quad (2)$$

for any nonvanishing real  $k$ . The relations

$$T(k)f^{(-)}(x, k) = f^{(+)}(x, -k) + R^{(+)}(k)f^{(+)}(x, k), \quad (3a)$$

$$T(k)f^{(+)}(x, k) = f^{(-)}(x, -k) + R^{(-)}(k)f^{(-)}(x, k), \quad (3b)$$

define the transmission,  $T(k)$ , and reflection,  $R^{(\pm)}(k)$ , coefficients, with the implication that  $R^{(\pm)}(k)R^{(\pm)}(-k) + T(k)T(-k) = 1$  and  $R^{(+)}(k)T(-k) + R^{(-)}(-k)T(k) = 0$ . Since  $q(x) \in L_1$ , the discrete spectrum, if it occurs, generically has infinitely many eigenvalues at  $k = ip_n$ ,  $p_0 > p_1 > \dots > p_n > 0$ , to each of which there corresponds an  $L_2$  solution  $\varphi_n(x)$  of (1), with

$$\|\varphi_n\|^2 = \int_{-\infty}^{+\infty} dx \varphi_n^2(x) = 1. \quad (4)$$

The transmission coefficient  $T(k)$  is analytic for  $\Im k > 0$ , with the exception of  $k = ip_n$  where it has a simple pole with residue

$$r_n = \lim_{k \rightarrow ip_n} [(k - ip_n)T(k)]. \quad (5)$$

Two normalization parameters  $\rho_n^{(\pm)}$  are associated with each eigenvalue  $k = ip_n$ , as defined by the limits

$$\rho_n^{(\pm)} = \lim_{x \rightarrow \pm\infty} [\exp(\pm 2p_n x)\varphi_n^2(x)]; \quad (6)$$

they are not both independent from the discrete eigenvalues since they are related to the residue (5) by the equation

$$\rho_n^{(+)}\rho_n^{(-)} = -r_n^2. \quad (7)$$

In the following, we use either one of the two correspondences  $q(x) \rightarrow \{R^{(+)}(k); p_n, \rho_n^{(+)}\}$  and  $q(x) \rightarrow \{R^{(-)}(k); p_n, \rho_n^{(-)}\}$  to support our arguments and results. Finally, we report also some relevant formulae regarding two transformations which are basic in our construction, namely the dilation and Bäcklund transformations (cf. for example [4]).

For any positive  $p$ , the dilation transformation  $\delta(p)$  is defined in  $L_1$  by the following rescaling of both the independent and dependent variables:

$$\delta(p)[q(x)] = p^2 q(px). \quad (8)$$

For future reference, we note that, for given  $p$ , the function  $q(x) = x^{-2}a(\ln x)$ , where  $a(z)$  is periodic with period  $|\ln p|$ , is a fixed point of the map  $\delta(p)$ . With abuse of notation, we denote by  $\delta(p)$  also the corresponding transformation of the spectral data of  $q(x)$ , which reads

$$\delta(p)[f^{(\pm)}(x, k)] = f^{(\pm)}(px, k/p), \quad (9a)$$

$$\delta(p)[T(k)] = T(k/p), \quad (9b)$$

$$\delta(p)[R^{(\pm)}(k)] = R^{(\pm)}(k/p), \quad (9c)$$

$$\delta(p)[p_0, p_1, \dots, p_n, \dots] = [pp_0, pp_1, \dots, pp_n, \dots], \quad (9d)$$

$$\delta(p)[\varphi_n(x)] = p^{1/2}\varphi_n(px) \quad (9e)$$

$$\delta(p)[\rho_n^{(\pm)}] = p\rho_n^{(\pm)}. \quad (9f)$$

Next we consider a Bäcklund transformation on  $q(x)$ . This originates from a Darboux transformation of the solution  $\psi$  of (1), which is

$$B[\psi(x, k)] = \psi_x(x, k) - f(x)\psi(x, k), \quad (10)$$

where  $f(x) = h_x(x)/h(x)$  is the logarithmic derivative of a fixed solution  $h(x)$  of (1) with  $k = \bar{k}$ , i.e.  $h_{xx} + [q + \bar{k}^2]h = 0$ . Then the new function  $B[\psi]$  satisfies the Schrödinger equation (1) with the new potential

$$B[q(x)] = q(x) + 2f_x(x) = -q(x) - 2[f^2(x) + \bar{k}^2]. \quad (11)$$

This formula defines the Bäcklund transformation of  $q(x)$ , which goes via the following steps: first fix  $\bar{k}$  and solve the Riccati equation

$$f_x(x) + f^2(x) + q(x) + \bar{k}^2 = 0, \quad (12)$$

with  $f(0) = \bar{f}$ , and then insert  $f(x)$  into (11) and (10) to transform the potential and, respectively, the solutions of the Schrödinger equation for any value of the spectral variable  $k$ . It is therefore plain that a Bäcklund transformation depends on two parameters, say  $\bar{k}$  and  $\bar{f}$ . The crucial issue here is the choice of the particular solution of (12), which does not introduce in the new potential double poles in the variable  $x$  (see (11)). Since  $f(x) = h_x(x)/h(x)$ , this amounts to asking when the solution  $h(x)$  has no zero on the real axis. A necessary and sufficient condition for the existence of a solution  $h(x)$  with no zero is that  $\bar{k} = i\bar{p}$  with  $\bar{p} \geq p_0$ , where  $p_0$  is the largest discrete eigenvalue. Indeed, if  $\bar{p} > p_0$ , then the Bäcklund transformation (11) generically add the eigenvalue  $\bar{p}$  to the spectrum of  $q(x)$ .

In this case, the transformation of the spectral data, which corresponds to (11), reads

$$B[f^{(\pm)}(x, k)] = \pm[f_x^{(\pm)}(x, k) - f(x)f^{(\pm)}(x, k)]/(ik - \bar{p}) \quad (13a)$$

$$B[T(k)] = [(k + i\bar{p})/(k - i\bar{p})]T(k), \quad (13b)$$

$$B[R^{(\pm)}(k)] = -[(k + i\bar{p})/(k - i\bar{p})]R^{(\pm)}(k), \quad (13c)$$

$$B[p_0, p_1, \dots, p_n, \dots] = [\bar{p}, p_0, p_1, \dots, p_n, \dots], \quad (13d)$$

$$B[\varphi_n(x)] = (\bar{p}^2 - p_n^2)^{-1/2}[\varphi_{nx}(x) - f(x)\varphi_n(x)]. \quad (13e)$$

$$B[\rho_n^{(\pm)}] = [(\bar{p} + p_n)/(\bar{p} - p_n)]\rho_n^{(\pm)}. \quad (13f)$$

Of course, associated with the new eigenvalue  $\bar{p}$  of the potential  $B[q(x)]$  there exist the normalized eigenfunction  $\bar{\varphi}(x)$  and the normalization parameters  $\bar{\rho}^{(\pm)}$ . Their expressions in terms of the Bäcklund parameter  $\bar{f}$  are not explicit, as shown by the following formulae:

$$\bar{\rho}^{(\pm)} = (2\bar{p}E^{\mp 1})T(i\bar{p}), \quad \bar{\varphi}(x) = 2\bar{p}(\bar{\rho}^{(+)})^{-1/2}/[f^{(+)}(x, i\bar{p}) + Ef^{(-)}(x, i\bar{p})], \quad (14a)$$

$$E \equiv \exp\left\{\int_0^{+\infty} dx[f(x) - \bar{p}] + \int_{-\infty}^0 dx[f(x) + \bar{p}]\right\}. \quad (14b)$$

## 2. The potential and its construction

Our potential is transparent, so that  $R(k) = 0$ , and its (infinite) discrete spectrum is

$$p_n = p_0 p^n, \quad n = 0, 1, 2, \dots, \quad (15)$$

with normalization constants

$$\rho_n^{(\pm)} = \rho_0^{(\pm)} p^n \prod_{m=1}^n \frac{1 + p^m}{1 - p^m}, \quad n = 0, 1, 2, \dots, \quad \left(\prod_{m=1}^0 \equiv 1\right). \quad (16)$$

The potential  $R$  therefore depends on three parameters, namely  $p_0 > 0$  (which could be eliminated by rescaling, see (9d)),  $p$  where  $0 < p < 1$ , and  $\rho_0^{(+)} > 0$  or equivalently  $\rho_0^{(-)}$ , since

$$\rho_0^{(+)}\rho_0^{(-)} = 4p_0^2 \prod_{m=1}^{\infty} \left(\frac{1 + p^m}{1 - p^m}\right)^2.$$

Note that the restriction that  $p$  be less than one implies the spectrum accumulates at zero and that  $\rho_n^{(\pm)}$  vanishes as  $n \rightarrow \infty$ . Corresponding to these spectral properties, the potential  $q(x)$  is an everywhere smooth function (with some cautions and specifications, see below) which indeed is analytic at  $x = 0$  (i.e. it has there a Taylor power expansion), and asymptotically oscillates, on a logarithmic scale, with an amplitude decreasing as  $1/x^2$ . This large  $x$  behaviour is described by the equation

$$q(x) = (1/x^2)\eta_{\pm}(\ln|x|) + O(x^{-3}), \quad x \rightarrow \pm\infty \quad (17)$$

where the functions  $\eta_{\pm}(z)$  are periodic,  $\eta_{\pm}(z + b) = \eta_{\pm}(z)$ , and the period  $b$  is related to the spectrum (15),

$$b = -\ln p. \quad (18)$$

Moreover, the potential can be numerically computed for all  $x$ , and its plot, corresponding to two particular values of its parameters, is shown in Fig.1.

Before proceeding to a brief description of the way this potential can be investigated and computed, we deem it useful to show that the asymptotic behaviour (17) can be predicted by simple arguments. Each eigenvalue  $p_n$ , with the normalization parameter  $\rho_n^{(+)}$ , contributes to the asymptotic behaviour of the potential with the exponential  $4p_n\rho_n^{(+)}\exp(-2p_nx)$ , as  $x \rightarrow +\infty$ ; since, in our case,  $p_n = p_0 p^n$  and  $\rho_n^{(+)}$  is proportional to  $p^n$  for large  $n$ , we do expect the potential to behave, as  $x \rightarrow +\infty$ , as an infinite sum of exponential tails of the form  $\sigma(x) = \sum_{n=0}^{\infty} p^{2n} \exp(-2p_0 p^n x)$ . It is easily seen that this sum satisfies the scaling equation  $p^2 \sigma(px) = \sigma(x) - \exp(-2p_0 x)$ , and this implies that, for  $x \rightarrow +\infty$ ,  $\sigma(x)$  converges to the fixed point of the dilation transformation  $p^2 \sigma(px) = \sigma(x)$  (see(8)), whose general expression is  $x^{-2}a(\ln x)$ , with  $a(z)$  periodic (cf (18)), this being precisely of the form of the leading term in equation (17). This reasoning suggests the method we follow below.

Let  $P$  be a transformation on a  $L_1$  function which is obtained by first applying a dilation (8) with scaling parameter  $p$ , ( $0 < p < 1$ ), and then a Bäcklund transformation  $B_0$  with parameter  $\bar{k} = ip_0$ , where  $p_0 > 0$ . Our starting point is to look for a potential  $q(x)$  which is invariant under the action of  $P$ , namely,

$$B_0[\delta(p)[q(x)]] \equiv P[q(x)] = q(x). \quad (19)$$

Let us first consider this equation in the space of spectral data by assuming that such a potential exists in  $L_1$ . It is a simple matter to find immediately that if  $q(x)$  has a discrete spectrum then it must be infinite, and be given precisely by the power law (15). Indeed, the transformation  $\delta(p)$  annihilates the first eigenvalue  $p_0$ , and the Bäcklund transformation  $B_0$  reinstates the same eigenvalue. By using first (9f) and then (13f), it is easily verified that the expression (16) for the normalization constants  $\rho_n^{(\pm)}$  provides the fixed-point solution. On the continuum spectrum, we have the equation

$$R^{(\pm)}(k) = - \left( \frac{k + ip_0}{k - ip_0} \right) R^{(\pm)}(k/p), \quad (20)$$

for the reflection coefficient (see (9c) and (14c)). Repeated application of the right hand side transformation yields the formula

$$R^{(\pm)}(k) = (-)^n \prod_{m=0}^{n-1} \left( \frac{k + ip_0 p^m}{k - ip_0 p^m} \right) R^{(\pm)}(k/p^n), \quad (21)$$

where  $k$  is real and  $n$  is any positive integer. Since the assumption that  $R^{(\pm)}(k)$  corresponds to a  $L_1$  potential implies that  $\lim_{k \rightarrow \infty} R^{(\pm)}(k) = 0$ , the equation  $|R^{(\pm)}(k)| = |R^{(\pm)}(k/p^n)|$ , which follows from (21), together with the condition  $p < 1$ , consequently yields in the limit as  $n \rightarrow \infty$  that

$$R^{\pm}(k) = 0. \quad (22)$$

In other words, the vanishing potential  $q(x) = 0$  is the unique  $L_1$  solution of (19) with no discrete spectrum. Using (9b) and (14b), we have by the same arguments the transmission

coefficient,

$$T(k) = \prod_{m=0}^{n-1} \left( \frac{k + ip_0 p^m}{k - ip_0 p^m} \right) T(k/p^n), \quad (23)$$

which, as  $n \rightarrow \infty$ , gives the expression

$$T(k) = \prod_{m=0}^{\infty} \left( \frac{k + ip_0 p^m}{k - ip_0 p^m} \right), \quad (24)$$

taking into account that  $T(k) \rightarrow 1$  as  $k \rightarrow \infty$ .

Let us now turn our attention to the fixed point equation (19). Due to the assumed spectrum of  $q(x)$ ,  $p_0$  is not an eigenvalue of the rescaled potential  $\delta(p)[q(x)]$ . Therefore there exists a non-singular solution  $f(x)$  of the Riccati equation

$$f_x(x) + f^2(x) + \delta(p)[q(x)] - p_0^2 = 0, \quad (25)$$

which we now use to perform the Bäcklund transformation  $B_0$ . Using the condition (19) on  $q(x)$  and the formula (11), we have

$$q(x) = \delta(p)[q(x)] + 2f_x(x), \quad (26)$$

which implies that  $f(x)$  satisfies the equation

$$-f_x(x) + f^2(x) + q(x) - p_0^2 = 0. \quad (27)$$

By applying the operator  $\delta(p)$  to this equation, and subtracting the result from (25), so as to eliminate the potential, we obtain finally the basic equation

$$f_x(x) + p^2 f_x(px) + f^2(x) - p^2 f^2(px) - p_0^2(1 - p^2) = 0. \quad (28)$$

This is a sort of functional-Riccati equation which has been introduced by one of us [2] while investigating the  $N$ -soliton solution as  $N \rightarrow \infty$ , as a self-similar solution of the infinite system of coupled equations

$$f_x^{(n)}(x) + f_x^{(n+1)}(x) + f^{(n)^2}(x) - f^{(n+1)^2}(x) + p_{n+1}^2 - p_n^2 = 0, \quad n = 0, 1, 2, \dots \quad (29)$$

(a dressing dynamical system). Indeed, it is easily seen that  $f^{(n)}(x) = p^n f(p^n x)$ , where  $f(x)$  solves the functional-Riccati equation (28), is a solution of the infinite chain (29) with  $p_n = p_0 p^n$ . It is plain that once the equation (28) has been solved with a given initial condition

$$f(0) = f_0, \quad (30)$$

then the potential is given by (27), and all its properties can be recovered from those of  $f(x)$ . Thus, for instance, the relation (26) shows that if  $f(x)$  has a singularity, then so does  $q(x)$ . Moreover, the expression of the normalization constants  $\rho_0^{(\pm)}$  which appear in (16) may be given in terms of  $f(x)$  by the formula

$$\rho_0^{(\pm)} = 2p_0 E^{\mp 1} \prod_{m=1}^{\infty} \frac{1 + p^m}{1 - p^m} \quad (31a)$$

$$E = \exp \left\{ \int_0^{+\infty} dx [f(x) - p_0] + \int_{-\infty}^0 dx [f(x) + p_0] \right\}. \quad (31b)$$

which follows from (14). In the following, whenever convenient, we will limit our analysis to the semiaxis  $x \geq 0$ . This does not imply any loss of generality because, if we denote by  $f(x; f_0)$  the solution of (28) with the initial condition (30), then  $-f(-x; -f_0)$  is also solution of the same problem, and since the solution is unique [1], we have  $f(-x; f_0) = -f(x; -f_0)$ . In particular, if  $f_0 = 0$  then the solution  $f(x)$  of (28) is odd, and therefore the potential is an even function satisfying the inequalities [2]

$$p_0[(1-p^2)/(1+p^2)]^{1/2} \tanh\{p_0[(1-p^2)/(1+p^2)]^{1/2}x\} < f(x) < p_0 \tanh(p_0 x). \quad (32)$$

If  $f_0 \neq 0$ , this nice result may not apply. In fact, in the limit case  $p = 0$ , the equation (28) reduces to the Riccati equation  $f_x + f^2 - p_0^2 = 0$ , and it is well-known that singularity-free solutions on the real axis exist if  $-p_0 \leq f_0 \leq p_0$ . A similar situation may occur also for the functional-Riccati equation (say  $0 < p < 1$ ), for which, however, the analysis of the singularities of its solutions in the complex  $x$ -plane is still an open problem. In contrast with global features, local properties can be more easily investigated, and they are reported below (for proofs, see[1]).

**THEOREM 1.** *The solution of (28) with (30) is analytic at  $x = 0$  and the radius of convergence  $R$  of its Taylor series expansion satisfies the inequality*

$$R \geq \min\{1/|f_0|, [(1+p^2)/(1-p^2)]^{1/2}/|p_0^2 - f_0^2|^{1/2}\}. \quad (33)$$

This result not only asserts local uniqueness and existence, but proves to be a relevant computational tool, as discussed below. The next theorem regards the behaviour of the solution of (28) at large  $x$ .

**THEOREM 2.** *If the solution  $f(x)$  of (28) tends, as  $x \rightarrow +\infty$  ( $x \rightarrow -\infty$ ), to a positive (negative) constant, then it has the following asymptotic expansion*

$$f(x) = \pm p_0 + \sum_{m=1}^n x^{-m-1} \varphi_m^\pm(z) + O(x^{-n-2}), \quad x \rightarrow \pm\infty, \quad (34)$$

where  $\varphi_n^\pm(z)$  are bounded periodic functions of the variable  $z = \ln|x|$ , with period (18). Moreover, these functions are all given in terms of the first  $\varphi_1^\pm$  by the recurrence relation

$$\varphi_{n+1}^\pm = \frac{1}{2p_0} \left\{ \left( \frac{1+p^n}{1-p^n} \right) [\varphi_{n+1}^\pm - (n+1)\varphi_n^\pm] - \sum_{m=1}^{n-1} \varphi_m^\pm \varphi_{n-m}^\pm \right\}, \quad n = 1, 2, \dots \quad (35)$$

Of course, this expansion, together with (27), implies that the asymptotic formula (17) holds with  $\eta_\pm(z) = \mp 2p_0 \varphi_1^\pm(z)$ . This theorem is remarkably sharp evidence of the difference between the usual ODE-Riccati equation, and the functional-Riccati equation (28). Indeed, while in the ODE case the exponential approach to the asymptotic constant value comes from the balance between the derivative and the linear term, in our case the derivative does not contribute to the asymptotic oscillations, which are due only to the balance between the two linear terms.

These two theorems can be easily proved because  $x = 0$  and  $x = \infty$  are the only points which are scale invariant. Finding the behaviour of the solution  $f(x)$  which matches the Taylor expression at  $x = 0$  and the asymptotic formula (34) is a less simple task, and is still to be completed. The important task in this respect is to obtain the singularities of

the solution  $f(x)$  in the complex  $x$ -plane, as functions of the initial value  $f_0$ . Moreover, it should be noted that the functional-Riccati equation defines a mapping between  $f_0$  and the asymptotic periodic functions  $\varphi_1^\pm(z) = \varphi_1^\pm(z; f_0)$ , whose singularities in the complex  $f_0$ -plane are also of interest.

At present, we have good numerical evidence that, for  $f_0 = 0$ , the solution  $f(x)$  is a smooth function of  $x$  for  $0 < p < 1$ . This has been obtained by first computing  $f(x)$  through its Taylor expansion at  $x = 0$ , for  $0 \leq x \leq x_0$ , with  $x_0$  not larger than the lower bound (33) of the radius of convergence  $R$ . Knowing the solution  $f(x)$  between  $px_0$  and  $x_0$ , the ordinary Riccati equation (see(28))

$$f_x(x) + f^2(x) = p_0^2(1 - p^2) + p^2[f^2(px) - f_x(px)], \quad (36)$$

has been integrated in the interval  $x_0 \leq x \leq x_0/p$  by standard Runge-Kutta methods, using the initial value  $f(x_0)$ . This process has been iterated so as to reach the asymptotic oscillating behaviour given by (34). The result of this numerical scheme is shown in Fig.1, where the plot of  $f(x)$  is partially given for  $p = 0.20$  and  $p = 0.30$  (with  $p_0 = 1$ ). The first few sine and cosine Fourier coefficients of the asymptotic periodic function  $\varphi_1^+(z)$  are displayed there. We finally note that the solution  $f(x)$  and its corresponding periodic function  $\varphi_1^+(z)$  as defined by (34), are related to each other by the linear equation

$$\varphi_1^+(z) = \sum_{n=-\infty}^{+\infty} F(z + nb), \quad (37)$$

where  $z = \ln x$  and  $F(z) = -(x^2/p_0)f_x(x)$ . A similar formula holds, of course, for  $\varphi_1^-(z)$ . This equation follows directly from (28) and (34), [1], and can be used to check the solution  $f(x)$  obtained by numerical integration.

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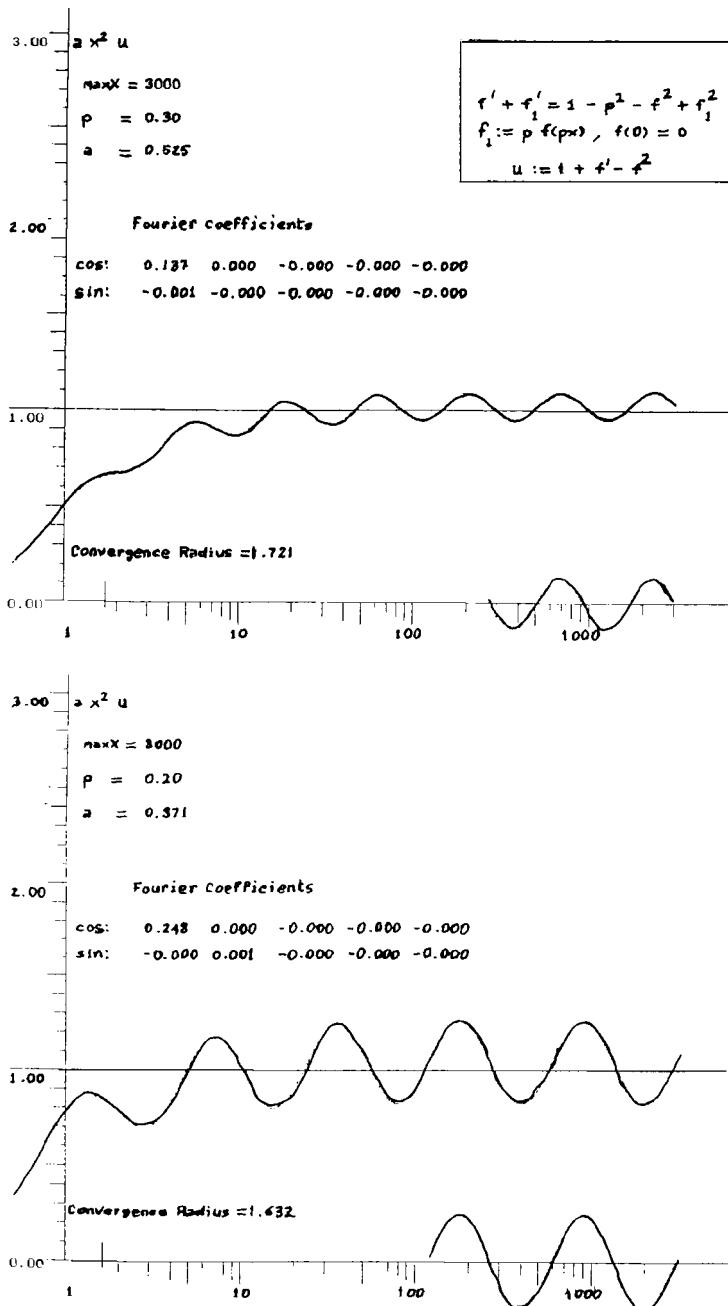


Figure 1

## COUPLING OF COMPLETELY INTEGRABLE SYSTEMS: THE PERTURBATION BUNDLE

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**ABSTRACT.** We introduce a canonical Lie algebra in the direct sum of vector fields and 1-1-tensors, the perturbation bundle. This Lie algebra is extended to a full tensor structure and it is related to the Lie algebra obtained by coupling linear systems to nonlinear ones. Using Lie algebra isomorphisms from the original structure to the abstract perturbation bundle, new completely integrable systems are obtained. The formalism of Lax pairs is found to be a special case of the new structure.

### 1. Introduction

The starting point for this paper was the question of how a coupling between linear equations and integrable nonlinear ones has to be done such that the resulting two-component flow again is integrable. To avoid misinterpretation, we should mention that this question is nontrivial, even for ordinary differential equations. Otherwise, for example, if all such couplings were integrable then finding explicitly the eigenvectors of the Schrödinger operator would be possible for all cases where the potential fulfills an integrable ordinary differential equation; this certainly is not the case.

However, in section 4.3 we present a simple and direct method to construct such integrable couplings. The essential point, which makes the underlying construction possible, is that in the direct sum of vector fields and 1-1-tensors (perturbation bundle) we can find two different Lie algebras. One of these, the concrete one, simply results from the vector field Lie algebra of dynamical systems like

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K(u) \\ \Omega(u)v \end{pmatrix} \quad (1.1)$$

where  $\Omega$  is a linear operator on the tangent bundle. The other one is an abstract algebra, not depending on any differential geometric assumptions, like affine connections which are needed for the concrete algebra. The surprising observation is that there is a one-to-one correspondence between the commuting pairs in both algebras, surprising insofar as both algebras are quite different with respect to the differential geometric set-up needed for their construction.

The correspondence between commuting pairs in both algebras opens the possibility to start from known commuting vector fields, then apply to them Lie algebra homomorphisms

in the abstract algebra, and to interpret the resulting commuting pairs in the concrete algebra, thus obtaining commuting pairs of equations like (1.1).

Other applications of the interplay between the two algebras are the existence of embeddings of the usual Virasoro algebras for integrable systems into the perturbation bundle. There the second components are those operators which usually occur in the isospectral-nonisospectral Lax pair formulations for these Virasoro algebras.

Another application, just for the amusement of the reader, is the construction of very many absolutely meaningless Lax pair formulations for arbitrary evolution equations.

In order to carry out all constructions which are needed in the paper, we first have to show that tensor structures can be built up over Lie algebra representations without assuming that these arise from Lie modules; this is done in the first section of the paper.

## 2. Tensors

First we present the essential requirements needed to build up a *tensor structure*. In the subsequent construction the important point is that we do not need to start from a Lie algebra module. This, although only a slight generalization, will be needed later on when the perturbation bundle is treated.

Let a Lie algebra  $\mathcal{L}$  be given, furthermore a vectorspace  $\mathcal{F}$  (called *scalar fields*) and a representation of  $\mathcal{L}$  acting on  $\mathcal{F}$ . That means for each  $K \in \mathcal{L}$  we have a linear map  $L_K : \mathcal{F} \rightarrow \mathcal{F}$  such that the map  $K \rightarrow L_K$  is a Lie algebra isomorphism, i.e.

$$L_{K_1} L_{K_2} - L_{K_2} L_{K_1} = L_{[K_1, K_2]} \quad (2.1)$$

for all  $K_1, K_2 \in \mathcal{L}$ .

From here we build up the *tensor structure*. Let  $f \in \mathcal{F}$  then we denote by  $\nabla f$  the linear map  $\mathcal{L} \rightarrow \mathcal{F}$  given by

$$\langle \nabla f, K \rangle := L_K f \text{ for all } K \in \mathcal{L}. \quad (2.2)$$

The map  $\nabla f$  is called the *gradient* of  $f$ . The set of gradients we denote by  $\mathcal{L}^*$ . The space  $\mathcal{L}^*$  separates the elements of  $\mathcal{L}$  since  $K \rightarrow L_K$ , being an isomorphism, is injective. An  $\mathcal{F}$ -valued multilinear form  $T$  on  $(\otimes \mathcal{L})^n \otimes (\otimes \mathcal{L}^*)^r$  is called an  $n$ -times covariant and  $r$ -times contravariant tensor. Observe that the elements of  $\mathcal{L}$  and  $\mathcal{L}^*$  are embedded in the 1-times covariant and 1-times contravariant tensors, respectively.

Now we first extend the maps  $L_K$  to maps on  $\mathcal{L}$  by defining for all  $G \in \mathcal{L}$

$$L_K G := [K, G]. \quad (2.3)$$

For  $G^* \in \mathcal{L}^*$  we then define  $L_K G^*$  by its action on the elements  $G \in \mathcal{L}$

$$\langle L_K G^*, G \rangle := L_K \langle G^*, G \rangle - \langle G^*, L_K G \rangle. \quad (2.4)$$

This we then extend to suitable maps on all tensors by defining for an arbitrary ( $n$ -times covariant and  $r$ -times contravariant) tensor  $T$

$$\begin{aligned} L_K(T)(G_1, \dots, G_n, G_1^*, \dots, G_r^*) &:= L_K(T(G_1, \dots, G_n, G_1^*, \dots, G_r^*)) \\ &\quad - \sum_{i=1}^r T(G_1, \dots, G_n, G_1^*, \dots, L_K G_i^*, \dots, G_r^*) \\ &\quad - \sum_{j=1}^n T(G_1, \dots, L_K G_j, \dots, G_n, G_1^*, \dots, G_r^*) \end{aligned} \quad (2.5)$$

where  $G_1^*, \dots, G_r^*$  and  $G_1, \dots, G_n$  are arbitrary elements of  $\mathcal{L}^*$  and  $\mathcal{L}$ , respectively. This extension fulfills

$$L_{K_1} L_{K_2} - L_{K_2} L_{K_1} = L_{[K_1, K_2]}. \quad (2.6)$$

$L_K$  is said to be the *Lie derivative* with respect to  $K$ . Purely covariant tensors are called *forms* ( $n$ -forms if  $n$ -times covariant). The elements of  $\mathcal{F}$  are called *zero-forms*.

Let  $\alpha$  be some  $n$ -form and  $K \in \mathcal{L}$ , then by  $\alpha \bullet K$  we denote the form where  $K$  is inserted as first entry into  $\alpha$ . If  $\alpha$  is a zero-form then we use the convention  $\alpha \bullet K := 0$ . Now, we define an *exterior derivative*  $d$  on forms by

$$d(0) = 0 \quad (2.7)$$

and for arbitrary  $n$ -forms  $\alpha$  recursively by

$$(d\alpha) \bullet K := L_K(\alpha) - d(\alpha \bullet K) \text{ for all } K \in \mathcal{L}. \quad (2.8)$$

This derivative  $d$  maps  $n$ -forms into  $(n+1)$ -forms. On the elements of  $\mathcal{F}$  it coincides with the operation of taking the gradient.  $d$  commutes with any Lie derivative and we have

$$d \cdot d = 0. \quad (2.9)$$

A form  $\alpha$  is said to be *closed* if  $d\alpha = 0$ . Gradients are closed one-forms because of (2.9).

A tensor  $T$  is said to be  *$K$ -invariant* if  $L_K(T) = 0$ . Observe that, by (2.8) a closed covariant tensor  $\alpha$  is  $K$ -invariant if and only if  $\alpha \bullet K$  is again closed.

Sometimes it is useful not to consider all possible tensors, but rather a suitable substructure. Therefore, any substructure of the set of all tensors

- (i) containing  $\mathcal{F}$ ,  $\mathcal{L}$  and  $\mathcal{L}^*$
- (ii) being closed against the operations  $\otimes$ ,  $\bullet$  as well as against all Lie derivatives and the operations of inserting any variable from  $\mathcal{L}$  and  $\mathcal{L}^*$  into elements of this substructure
- (iii) and being closed against forming new tensors by taking linear sums of tensors of equal type or by interchanging entries of equal type is called a *tensor structure* over  $\mathcal{L}$  and  $\mathcal{F}$ .

For the following considerations we restrict our attention to a suitable tensor structure. The basics of hamiltonian mechanics are:

Consider some antisymmetric linear operator  $\Theta$  from the one-forms into  $\mathcal{L}$ . Define for arbitrary one-forms  $\gamma_1, \gamma_2$  the brackets

$$\{\gamma_1, \gamma_2\} := L_{(\Theta\gamma_1)}\gamma_2 - L_{(\Theta\gamma_2)}\gamma_1 + d\langle\gamma_1, \Theta\gamma_2\rangle. \quad (2.10)$$

Then  $\{\ , \}$  defines a Lie algebra structure in the one-forms if and only if for all one-forms  $\gamma_1, \gamma_2$  we have

$$\Theta\{\gamma_1, \gamma_2\} = [\Theta\gamma_1, \Theta\gamma_2]. \quad (2.11)$$

In that case the  $\{\ , \}$  are called *Poisson brackets* (with respect to  $\Theta$ ) and  $\Theta$  is said to be an *imprectic operator* (or Poisson operator). For the proof of this crucial fact compare [15].

**OBSERVATION 2.1:** (Noethers Theorem) *Let  $\Theta$  be an imprectic operator and let  $K = \Theta df$ . Then  $L_K(\Theta\gamma) = \Theta L_K(\gamma)$  for all one-forms  $\gamma$ . In particular:  $\Theta$  and  $f$  are invariant with*

respect to  $K$ .

*Proof.* Using (2.8) we can rewrite

$$\{\gamma_1, \gamma\} = L_{(\Theta\gamma_1)}\gamma - (d\gamma_1) \bullet (\Theta\gamma). \quad (2.12)$$

Now using (2.11) and (2.12) we find for  $\gamma_1 = df$  with (2.9)

$$\begin{aligned} L_K\Theta\gamma &= [K, \Theta\gamma] = [\Theta df, \Theta\gamma] = \Theta\{df, \gamma\} \\ &= \Theta L_K\gamma - (d \cdot d f) \bullet (\Theta\gamma) = \Theta L_K\gamma. \end{aligned} \quad (2.13)$$

This also shows the invariance of  $\Theta$ . The invariance of  $f$  is a trivial consequence of the antisymmetry of  $\Theta$ .

We complete this section by some additional remarks. We observe first that conveniently we can represent 1-1-tensors  $T$  as linear operators  $\Omega_T : \mathcal{L} \rightarrow \mathcal{L}$  via

$$\langle G^*, \Omega_T K \rangle := T(K, G^*) \text{ for all } G^* \in \mathcal{L}^*, K \in \mathcal{L}. \quad (2.14)$$

Then, in this notation, for the Lie derivative of  $\Omega_T$  we have

$$L_K(\Omega_T G) = (L_K \Omega_T)G + \Omega_T L_K G \quad (2.15)$$

which yields

$$L_K(\Omega_T) = L_K \circ \Omega_T - \Omega_T \circ L_K. \quad (2.16)$$

This is a special case of the product rule for Lie-derivatives, which, by use of the Jacobi identity, holds in general with respect to arbitrary tensor products. As a consequence we remark that for arbitrary "scalars"  $\epsilon$  the map  $\exp(\epsilon L_K)$  is a Lie algebra homomorphism. Here, for the definition of  $\exp(\epsilon L_K)$  we use the application of the Taylor series of the exponential function to arbitrary tensors  $T$

$$\exp(\epsilon L_K)T = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} L_K^n(T)$$

and we assume that  $K \in \mathcal{L}$  is such that this sum converges, which, for example, is the case when the series truncates (master symmetries). Since  $\exp(\epsilon L_K)$  is a homomorphism it follows in particular for arbitrary 1-1-tensors  $\Omega$  and  $G \in \mathcal{L}$  that

$$\exp(\epsilon L_K)(\Omega G) = (\exp(\epsilon L_K)(\Omega))(\exp(\epsilon L_K)G). \quad (2.17)$$

For the generation of invariant elements in  $\mathcal{L}$  the notion of *hereditary operator*<sup>1</sup> [10,11,24] is important. A linear  $\Phi : \mathcal{L} \rightarrow \mathcal{L}$  is said to be hereditary if

$$\Phi L_G(\Phi) = L_{\Phi G}(\Phi) \text{ for all } G \in \mathcal{L}. \quad (2.19)$$

The use of this notion is well-known (see [11])

**THEOREM 2.2:** *If a hereditary  $\Phi$  is invariant w.r.t.  $K$ , i.e.  $L_K(\Phi) = 0$ , then the Lie algebra generated by the*

$$K_n = \Phi^n K, \quad n = 0, 1, 2, \dots \quad (2.20)$$

<sup>1</sup> Using the product rule one easily sees that this is equivalent to the usual definition

$$\Phi^2[A, B] + [\Phi A, \Phi B] = \Phi\{[\Phi A, B] + [A, \Phi B]\} \text{ for all } A, B \quad (2.18)$$

is abelian.

Homomorphisms preserve properties like hereditariness or the implectic property.

REMARK 2.3: Let  $H$  be some Lie-algebra homomorphism, then if  $\Phi$  is hereditary the operator  $H(\Phi)$  is again hereditary. Likewise, if  $\Theta$  is implectic, then  $H(\Theta)$  is again implectic.

At the end of this section we would like to mention a case where dropping the assumption that  $(\mathcal{L}, \mathcal{F})$  must be a Lie module is essential, namely in nonlinear quantum mechanics (see [16] for details). Another case is presented in the forthcoming sections.

### 3. Perturbation tensor bundle

Consider a tensor structure over the Lie algebra  $\mathcal{L}$  with scalars  $\mathcal{F}$ . Denote by  $\mathcal{T}_{(n,m)}$  the set of its  $n$ -times covariant and  $m$ -times contravariant tensors. Again, as before, the elements  $\Omega$  of  $\mathcal{T}_{(1,1)}$  are considered as linear operators  $\Omega : \mathcal{L} \rightarrow \mathcal{L}$ . Observe that on the corresponding operator algebra for any Lie derivative  $L_K$  the product rule (with respect to operator multiplication) does hold. We introduce direct sums

$$\hat{\mathcal{L}} := \mathcal{L} \oplus \mathcal{T}_{(1,1)}, \quad \hat{\mathcal{F}} := \mathcal{F} \oplus \mathcal{T}_{(1,0)}. \quad (3.1)$$

In  $\hat{\mathcal{L}}$  we define a bracket by

$$[(K_1, \Omega_1), (K_2, \Omega_2)] := ([K_1, K_2], L_{K_1}(\Omega_2) - L_{K_2}(\Omega_1) + \Omega_1 \Omega_2 - \Omega_2 \Omega_1) \quad (3.2)$$

for  $K_1, K_2 \in \mathcal{L}$  and  $\Omega_1, \Omega_2 \in \mathcal{T}_{(1,1)}$ . Observe that from  $\hat{\mathcal{L}}$  we have a one-to-one correspondence

$$(K, \Omega) \rightarrow L_K + \Omega \quad (3.3)$$

into (nonlocal) linear operators on  $\mathcal{L}$  and that by this correspondence the second component of the bracket (3.2) goes over into the bracket given by operator commutation. The reason that this is one-to-one comes from the fact that, with respect to the manifold variable,  $\Omega$  is a local linear operator on  $\mathcal{L}$ , whereas  $L_K$  is nonlocal; and splitting up linear operators on  $\mathcal{L}$  into local and nonlocal parts is a unique operation. Hence, (3.2) must define a Lie product.

On  $\hat{\mathcal{F}}$  we define a representation of this Lie product by:

$$\hat{L}_{(K, \Omega)}(f, \gamma) := (L_K(f), L_K(\gamma) - \gamma \circ \Omega) \quad (3.4)$$

where  $K \in \mathcal{L}$ ,  $f \in \mathcal{F}$ ,  $\Omega \in \mathcal{T}_{(1,1)}$  and  $\gamma \in \mathcal{T}_{(1,0)}$ . To see that this indeed is a representation of the Lie algebra structure we compute

$$\begin{aligned} \hat{L}_{(K_1, \Omega_1)} \hat{L}_{(K_2, \Omega_2)}(f, \gamma) &= (L_{K_1} L_{K_2}(f), L_{K_1} L_{K_2}(\gamma) - L_{K_1}(\gamma \circ \Omega_2) \\ &\quad + \gamma \circ \Omega_2 \circ \Omega_1 - L_{K_2}(\gamma) \circ \Omega_1) \\ &= (L_{K_1} L_{K_2}(f), L_{K_1} L_{K_2}(\gamma) - L_{K_1}(\gamma) \circ \Omega_2 \\ &\quad - L_{K_2}(\gamma) \circ \Omega_1 - \gamma \circ L_{K_1}(\Omega_2) + \gamma \circ \Omega_2 \cdot \Omega_1). \end{aligned}$$

Hence we have

$$\begin{aligned} &\left( \hat{L}_{(K_1, \Omega_1)} \hat{L}_{(K_2, \Omega_2)} - \hat{L}_{(K_2, \Omega_2)} \hat{L}_{(K_1, \Omega_1)} \right)(f, \gamma) \\ &= \left( L_{[K_1, K_2]}(f), L_{[K_1, K_2]}(\gamma) - \gamma \circ \{L_{K_1}(\Omega_2) - L_{K_2}(\Omega_1) + \Omega_1 \Omega_2 - \Omega_2 \Omega_1\} \right) \\ &= \hat{L}_{[(K_1, \Omega_1), (K_2, \Omega_2)]}(f, \gamma) \end{aligned}$$

which shows that  $\hat{L}$  defines a Lie-derivative with respect to the Lie algebra structure in  $\hat{\mathcal{L}}$ .

If now, in the considerations of section 2, we replace  $(\mathcal{L}, \mathcal{F})$  by  $(\hat{\mathcal{L}}, \hat{\mathcal{F}})$ , then all assumptions which were needed to build up tensor calculus are fulfilled, hence we can build up a suitable tensor structure based on  $(\hat{\mathcal{L}}, \hat{\mathcal{F}})$ . The corresponding tensor bundle is called the *perturbation bundle*. In this bundle  $\hat{T}_{(n,m)}$  denotes the  $n$ -times covariant,  $m$ -times contravariant tensors.

We demonstrate that this new tensor bundle, in modification, naturally arises when perturbations of flows are considered. Take some path-connected  $C^\infty$ -manifold  $\mathcal{M}$  (eventually infinite dimensional). Let an affine connection on  $\mathcal{M}$  be given, and assume that this connection has vanishing torsion and curvature. Denote by  $\nabla_B$  the *covariant derivative* in the direction of the vectorfield  $B$ . Let  $\mathcal{L}$  now be the Lie algebra of  $C^\infty$ -vector fields on  $\mathcal{M}$ , and  $\mathcal{F}$  shall correspondingly denote the  $C^\infty$ -scalar fields. Recall that vanishing torsion means that for all vector fields  $A, B$  we have

$$\nabla_A B - \nabla_B A = [A, B]. \quad (3.5)$$

The curvature is defined by ( $A, B$  arbitrary vector fields)

$$R(A, B) := \nabla_A \nabla_B - \nabla_B \nabla_A - \nabla_{[A, B]}. \quad (3.6)$$

Since curvature is assumed to vanish, parallel transport of tensors around closed loops leaves them unchanged (see [23]), hence parallel transport from one point of the manifold to another does not depend on the path taken.

Denote by  $u$  the manifold variable. If  $v$  is a tangent vector at  $u_0 \in \mathcal{M}$  then by  $\Pi_{(u_0, u)}(v)$  we denote the tangent vector at  $u$  obtained by parallel transport of  $v$  from  $u_0$  to  $u$ . If no confusion can arise we denote  $\Pi_{(u_0, u)}(v)$  simply by  $v$ . The same notation is chosen for tensors, if  $T(u)$  is a tensor field then  $\Pi_{(u_0, u)}(T(u_0))$  denotes the parallel transport of  $T(u_0)$  from  $u_0$  to  $u$ . Again, we simply write  $T(u)$  instead.

The fields just introduced are exactly the constant tensors, i.e. those tensors having a vanishing covariant derivative.

One should observe that the assumptions on which this geometric situation is based are not too restrictive. Locally, this situation can be established for any manifold, one only has to take a  $C^\infty$ -smooth parametrization by a vector space and to define parallel transport by translation in the parameter space.

Now, consider a new manifold  $\hat{\mathcal{M}}$  consisting of all pairs  $(u, v)$ , where  $u \in \mathcal{M}$  and where  $v$  is a constant vector field. On this new manifold we consider flows of the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K(u) \\ \Omega(u)v \end{pmatrix} \quad (3.7)$$

where  $K(u)$  is a vector field and  $\Omega(u)$  is a 1-1-tensor. These flows are to be understood as two-component systems where a second component has been coupled linearly to the non-linear flow  $u_t = K(u)$ . Particular examples are the linearizations (i.e  $\Omega(u)v = \nabla_v K$ ) of the original equation  $u_t = K(u)$ .

Taking the commutator of the infinitesimal generators of these flows we obtain

$$\left[ \begin{pmatrix} K_1(u) \\ \Omega_1(u)v \end{pmatrix}, \begin{pmatrix} K_2(u) \\ \Omega_2(u)v \end{pmatrix} \right] = \begin{pmatrix} [K_1, K_2] \\ (\nabla_{K_1}(\Omega_2) - \nabla_{K_2}(\Omega_1) + [\Omega_2, \Omega_1])v \end{pmatrix} \quad (3.8)$$

where  $\llbracket A, B \rrbracket := AB - BA$  denotes operator commutators.

Introducing for vector fields  $K$  the 1-1-tensors  $\nabla K$  defined by  $(\nabla K)G := \nabla_G K$  we claim that we can rewrite the right-hand side of (3.8) as

$$\begin{pmatrix} [K_1, K_2] \\ -\llbracket L_{K_1} + \nabla K_1 - \Omega_1, L_{K_2} + \nabla K_2 - \Omega_2 \rrbracket v \end{pmatrix} \quad (3.9)$$

*Proof of the claim.* By the product rule for covariant derivatives we have

$$\llbracket \nabla_K, \Omega \rrbracket = \nabla_K(\Omega). \quad (3.10)$$

From (3.5) we obtain

$$(L_K + \nabla K)G = \nabla_K G. \quad (3.11)$$

Hence

$$\begin{aligned} & -\llbracket L_{K_1} + \nabla K_1 - \Omega_1, L_{K_2} + \nabla K_2 - \Omega_2 \rrbracket v \\ &= \llbracket \nabla_{K_2} - \Omega_2, \nabla_{K_1} - \Omega_1 \rrbracket v \\ &= (\llbracket \nabla_{K_2}, \nabla_{K_1} \rrbracket - \nabla_{K_2}(\Omega_1) + \nabla_{K_1}(\Omega_2) + \llbracket \Omega_2, \Omega_1 \rrbracket) v \\ &= (\nabla_{[K_2, K_1]} - \nabla_{K_2}(\Omega_1) + \nabla_{K_1}(\Omega_2) + \llbracket \Omega_2, \Omega_1 \rrbracket) v \end{aligned} \quad (3.12)$$

Here the last identity came from vanishing curvature (3.6). Since  $v$  is constant the term  $\nabla_{[K_1, K_2]}$  vanishes and the right-hand side of (3.12) clearly equals the second line of the right side of (3.8).■

Looking back at the definition (3.2) we find that (3.8) and the Lie algebra (3.2) are related since for  $\Omega \rightarrow \nabla K - \Omega$  the second component of (3.9) is, up to a change of sign, equal to the second component of (3.2). We denote this new Lie algebra by  $\hat{\mathcal{L}}^T$  and write its elements as

$$\begin{pmatrix} K \\ \Omega \end{pmatrix} \quad (3.13)$$

instead of  $(K, \Omega)$  as they were denoted in  $\hat{\mathcal{L}}$ .

We may summarize now:

OBSERVATION 3.1: *In  $\hat{\mathcal{L}}^T$  we have a Lie-algebra defined by*

$$\left[ \begin{pmatrix} K_1 \\ \Omega_1 \end{pmatrix}, \begin{pmatrix} K_2 \\ \Omega_2 \end{pmatrix} \right] := \begin{pmatrix} [K_1, K_2] \\ \nabla_{K_1}(\Omega_2) - \nabla_{K_2}(\Omega_1) + \llbracket \Omega_2, \Omega_1 \rrbracket \end{pmatrix}. \quad (3.14)$$

The Lie algebras in  $\hat{\mathcal{L}}$  and  $\hat{\mathcal{L}}^T$  are related in the following way: When

$$\left[ \begin{pmatrix} K_1 \\ \Omega_1 \end{pmatrix}, \begin{pmatrix} K_2 \\ \Omega_2 \end{pmatrix} \right] = \begin{pmatrix} [K_1, K_2] \\ \Omega \end{pmatrix} \quad (3.15)$$

then

$$\llbracket (K_1, \nabla K_1 - \Omega_1), (K_2, \nabla K_2 - \Omega_2) \rrbracket = ([K_1, K_2], \nabla_{[K_1, K_2]} - \Omega) \quad (3.16)$$

Hence commuting pairs of  $\hat{\mathcal{L}}$

$$[(K_1, \Omega_1), (K_2, \Omega_2)] = 0 \quad (3.17)$$

correspond uniquely to commuting pairs in  $\hat{\mathcal{L}}^T$

$$\left\| \left[ \begin{pmatrix} K_1 \\ \nabla K_1 - \Omega_1 \end{pmatrix}, \begin{pmatrix} K_2 \\ \nabla K_2 - \Omega_2 \end{pmatrix} \right] \right\| = 0. \quad (3.18)$$

On  $\hat{\mathcal{F}}^T$  a representation for  $[\ , \ ]$  is easily found:

$$\tilde{L}_{(\Omega)} \begin{pmatrix} f \\ \gamma \end{pmatrix} := \begin{pmatrix} \nabla_K(f) \\ \nabla_K(\gamma) + \gamma \circ \Omega \end{pmatrix} \quad (3.19)$$

Only the statement about the representation is not yet proved, but this is exactly the same proof as before, this time based on the fact that the connection is curvature free. Examples for well known notions where implicitly the Lie algebra structure of the perturbation bundle is involved are easily found.

EXAMPLE 3.2:

(1) Consider in  $\hat{\mathcal{L}}$  the commuting pair

$$[(K, \nabla K + B), (0, \Lambda)] = 0 \quad (3.20)$$

then by using

$$L_K(\Lambda) = \nabla_K \Lambda + [\Lambda, \nabla K] \quad (3.21)$$

we see that relation (3.20) is equivalent to

$$\nabla_K \Lambda = [\Lambda, B] \quad (3.22)$$

hence to  $(\Lambda, B)$  being a Lax pair for  $u_t = K(u)$ .

(2) By the same argument we see that

$$[(K, 0), (0, \Phi)] = 0 \quad (3.23)$$

is equivalent to  $L_K \Phi = 0$  hence to  $\Phi$  being a recursion operator for  $u_t = K(u)$ .

(3) Via observation 3.1 we see furthermore that when  $(\Lambda, B)$  is a Lax pair then the flows

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K(u) \\ -B(u)v \end{pmatrix} \quad (3.24)$$

and

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ -\Lambda(u)v \end{pmatrix} \quad (3.25)$$

do commute. This generalizes the well known fact that when  $v$  evolves according to  $v_t = -B(u)v$  then the spectral decomposition of  $v(t)$  with respect to  $\Lambda(u(t))$  is independent of time.

## 4. Applications

### 4.1. LINEARIZED EQUATIONS

Obviously the Lie-algebra  $\mathcal{L}$  can be embedded isomorphically into any of the perturbation algebras in  $\hat{\mathcal{L}}$ . However, this fact does not mean that automatically the tensor structure based on  $(\mathcal{L}, \mathcal{F})$  can be embedded isomorphically into  $(\hat{\mathcal{L}}, \hat{\mathcal{F}})$ .

However, in case of the existence of a torsion-free and curvature-free affine connection we give a simple proof that for the Lie product  $[ , ]$  such an embedding indeed can be

achieved. To see this we choose a matrix  $\epsilon \neq 0$  with  $\epsilon^2 = 0$  (say a two-by-two-matrix). Then as coefficients in front of elements from  $\mathcal{L}$  and  $\mathcal{F}$  we admit linear combinations of the unit matrix  $I$  and  $\epsilon$ . To this new structure  $(\mathcal{L}^*, \mathcal{F}^*)$  we extend the Lie-algebra structure, and hence the tensor structure, in the obvious way. Furthermore we embed  $\mathcal{L}$  and  $\mathcal{F}$  via  $K \rightarrow IK$ ,  $f \rightarrow IK$  into this new tensor structure.

So we may consider the tensor structure  $(\mathcal{L}, \mathcal{F})$  as the  $\epsilon$ -free substructure of  $(\mathcal{L}^*, \mathcal{F}^*)$ . We choose now an arbitrary constant vector field  $v$  in  $\mathcal{L}$ , and we consider the isomorphism  $\exp(\epsilon L_v)$ . Observe that because of  $\epsilon^2 = 0$  all terms higher than first order cancel in the Taylor series of this exponential function. We now consider the isomorphic image of the tensor structure  $(\mathcal{L}, \mathcal{F})$  under  $\exp(\epsilon L_v)$ . We claim that this is isomorphic to a tensor structure of  $(\hat{\mathcal{L}}, \hat{\mathcal{F}})$  (perturbation bundle). To see this we write  $\binom{A}{B}$  instead of  $IA + \epsilon B$ . Then we can write for  $f \in \mathcal{F}$  and  $K \in \mathcal{L}$

$$\exp(\epsilon L_v)K = \begin{pmatrix} K \\ \nabla K \cdot v \end{pmatrix} \quad (4.1)$$

$$\exp(\epsilon L_v)f = \begin{pmatrix} f \\ \langle \nabla f, v \rangle \end{pmatrix} \quad (4.2)$$

and the Lie bracket coming out of that isomorphism is exactly the one considered in (3.8), which if rewritten leads to the Lie-algebra (3.14). Now, in order to identify elements of  $(\mathcal{L}^*, \mathcal{F}^*)$  as elements in the tensor structure over  $\hat{\mathcal{L}}^T$  we write

$$\begin{pmatrix} f \\ \nabla f \end{pmatrix} \text{ and } \begin{pmatrix} K \\ \nabla K \end{pmatrix} \text{ instead of } \begin{pmatrix} f \\ \langle \nabla f, v \rangle \end{pmatrix} \text{ and } \begin{pmatrix} K \\ \nabla K \cdot v \end{pmatrix}. \quad (4.3)$$

Using

$$L_{(\exp(\epsilon L_v)K)}(\exp(\epsilon L_v)f) = \exp(\epsilon L_v)(L_K f) \quad (4.4)$$

we find that this rewritten as an element of  $\hat{\mathcal{F}}$  yields

$$\begin{aligned} \hat{L}_{\binom{K}{\nabla K}} \begin{pmatrix} f \\ \nabla f \end{pmatrix} &= \begin{pmatrix} L_K f \\ \nabla(L_K f) \end{pmatrix} = \begin{pmatrix} \nabla_K f \\ \nabla(\nabla_K f) \end{pmatrix} \\ &= \begin{pmatrix} \nabla_K f \\ \nabla_K(\nabla f) + \nabla f \circ \nabla K \end{pmatrix} \end{aligned} \quad (4.5)$$

which is a special case of the representation given by (3.19). Using this map we have found

OBSERVATION 4.1: All tensors which are invariant under the flow  $u_t = K(u)$  are mapped by  $\exp(\epsilon L_v)$  onto invariant tensors of

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K(u) \\ \nabla K(u)v \end{pmatrix}. \quad (4.6)$$

As a consequence, complete integrability of  $u_t = K(u)$  yields complete integrability of the coupling between the original equation and its linearization. These arguments can be applied to any order of perturbation, say  $n$ -th order. For this one only has to use instead of  $\epsilon$  another nilpotent matrix  $\epsilon$  with  $\epsilon^{n-1} \neq 0$  and  $\epsilon^n = 0$ . Certainly this result is not so surprising, but usually in the literature, a fair amount of computation is needed in order to prove this observation even for special cases (see for example [25] in case of the KdV).

In case  $u_t = K(u)$  admits a recursion operator  $\Phi$  then the corresponding operator for (4.6) is easily found. Application of  $\exp(\epsilon L_v)$  yields that

$$\hat{\Phi} = \begin{pmatrix} \Phi & [\nabla, \Phi] \\ 0 & \Phi \end{pmatrix} \quad (4.7)$$

must be that recursion operator. Here  $[\nabla, \Phi]$  means the operator mapping each vector field  $G \in \mathcal{L}$  onto the linear operator

$$\nabla(\Phi G) - \Phi \nabla(G). \quad (4.8)$$

In the case  $\Phi$  is hereditary, then  $\hat{\Phi}$  has the same property.

#### 4.2. LAX PAIR HIERARCHIES

As we have seen, the recursion operator of a hierarchy of commuting flows can be understood as a new symmetry (with vanishing first component) for the canonical embedding of the hierarchy into the perturbation bundle. The same viewpoint can be adopted for Lax pairs. However there, not the trivial embedding but a more sophisticated one is needed. Furthermore an affine connection is necessary since all constructions have to be carried out in  $\hat{\mathcal{L}}^T$ .

We consider a Virasoro algebra of vector fields (i.e. an algebra of symmetries and mastersymmetries, or a hereditary algebra, see [4,12–14]). The commutation relations of such a Virasoro algebra are

$$[K_n, K_m] = 0 \quad (4.9)$$

$$[\tau_n, K_m] = (m + \rho) K_{n+m} \quad (4.10)$$

$$[\tau_n, \tau_m] = (m - n) \tau_{n+m} \quad (4.11)$$

where  $\rho$  is a fixed number (depending on the hierarchy under consideration), and  $m, n$  run from either 0 or 1 to infinity.<sup>2</sup> Let furthermore a Lax pair  $(\Lambda, B_1)$ , say for  $u_t = K_1(u)$ , be given. Then for almost all completely integrable systems a sequence of related isospectral and nonisospectral equations can be found in the literature (see [3,5–8,18–22]). Looking at

<sup>2</sup> There are also meaningful cases where the  $m, n$  run from  $-\infty$  to  $+\infty$  (see [27]).

those results, and reformulating them in the purely Lie algebraic setup of this paper, one discovers that there are sequences of operators  $A_m, B_n, m, n \in \mathbb{B}$  such that

$$\left[ \begin{pmatrix} K_n \\ A_n \end{pmatrix}, \begin{pmatrix} 0 \\ \Lambda \end{pmatrix} \right] = 0 \quad (4.12)$$

$$\left[ \begin{pmatrix} \tau_n \\ B_n \end{pmatrix}, \begin{pmatrix} 0 \\ \Lambda \end{pmatrix} \right] = \Lambda^{n+1}. \quad (4.13)$$

Now, using the Jacobi identity, one discovers that, modulo parts commuting with  $\Lambda$ , these vector fields must fulfill in  $\hat{\mathcal{L}}^T$

$$\left[ \begin{pmatrix} K_n \\ A_n \end{pmatrix}, \begin{pmatrix} K_m \\ A_m \end{pmatrix} \right] = 0 \quad (4.14)$$

$$\left[ \begin{pmatrix} \tau_n \\ B_n \end{pmatrix}, \begin{pmatrix} K_m \\ A_m \end{pmatrix} \right] = (m + \rho) \begin{pmatrix} K_{m+n} \\ A_{m+n} \end{pmatrix} \quad (4.15)$$

$$\left[ \begin{pmatrix} \tau_n \\ B_n \end{pmatrix}, \begin{pmatrix} \tau_m \\ B_m \end{pmatrix} \right] = (m - n) \begin{pmatrix} \tau_{n+m} \\ B_{n+m} \end{pmatrix}. \quad (4.16)$$

Implicitly, almost the same relation can be found in the important paper [20] of Wen-Xiu Ma. Indeed, the uncertainty with respect to the parts commuting with  $\Lambda$  is easily excluded by scaling arguments and thus the relations (4.14) to (4.16) are fully established. Hence we have found an extension of the original Lie algebra to a nontrivial Virasoro algebra in  $\hat{\mathcal{L}}^T$ . This then gives rise to a new integrable hierarchy, where the  $(K_m, A_m)^T$  correspond to the action variables and the  $(\tau_m, B_m)^T$  to the angle variables. As a consequence all flows

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K_n(u) \\ A_n(u)v \end{pmatrix} \quad (4.17)$$

do commute, a result which for the case of the KdV was already observed by Degasperis [9] (using spectral methods).

One should observe that having established the Virasoro relations for this algebra, we now have at our disposal a powerful computational tool, since such a Virasoro algebra is finitely generated. Consequently we only need to know the elements

$$\begin{pmatrix} K_1 \\ A_1 \end{pmatrix}, \begin{pmatrix} \tau_1 \\ B_1 \end{pmatrix} \text{ and } \begin{pmatrix} \tau_2 \\ B_2 \end{pmatrix}. \quad (4.18)$$

Then the others are simply computed by recursion

$$(n + \rho)A_{n+1} := \nabla_{\tau_1}(A_n) - \nabla_{K_n}(B_1) + [A_n, B_1] \quad (4.19)$$

$$(n - 1)B_{n+1} := \nabla_{\tau_1}(B_n) - \nabla_{\tau_n}(B_1) + [B_n, B_1]. \quad (4.20)$$

Other applications of this kind are possible. So for example, when one equation is obtained by another one via a series of Lie isomorphisms (in the vector field Lie algebra), then the corresponding Lax pairs can be transferred from the original equation by the same isomorphisms (canonically extended). For example, using the Lie isomorphism derivation of the cylindrical KdV from the KdV (see [17]) one easily obtains the Lax pair formulation for the cylindrical KdV.

### 4.3. COUPLING

Linearizations and isospectral pairs are examples of cases where a linear field has been coupled integrably to a nonlinear integrable evolution equation.

This can be generalized: Consider a sequence  $K_n$  of commuting vector fields. Embed these isomorphically by

$$K_n \rightarrow \hat{K}_n := (K_n, 0) \quad (4.21)$$

into  $\hat{\mathcal{L}}$ . They again do commute. Now, we take an arbitrary element in  $\hat{\mathcal{L}}$  of the form

$$H = (0, \Omega) \quad (4.22)$$

and we apply the Lie algebra isomorphism  $\exp(-\lambda \hat{L}_H)$  to the algebra generated by the  $\hat{K}_n$ . The result again is a commuting algebra. The interesting point about this isomorphism is that its application does not change the local part because the first component of  $H$  is equal to zero. This we see from the formula

$$\exp(-\lambda \hat{L}_H)(A, 0) = \left( A, \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n (-\text{ad}_{\Omega})^{n-1} (\nabla_A(\Omega) + [\![\Omega, \nabla A]\!]) \right) \quad (4.23)$$

where

$$\text{ad}_{\Omega}(B) := [\![\Omega, B]\!]. \quad (4.24)$$

Of course, in order to avoid convergence difficulties we should restrict our attention to those  $\Omega$  such that the series in (4.23) truncates. Now we use the one-to-one correspondence between commuting pairs of  $\hat{\mathcal{L}}$  and  $\hat{\mathcal{L}}^T$  to see that all the flows

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K_n(u) \\ \nabla K_n - \left( \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n (-\text{ad}_{\Omega})^{n-1} (\nabla_{K_n}(\Omega) + [\![\Omega, \nabla K_n]\!]) \right) v \end{pmatrix} \quad (4.25)$$

do commute.

**EXAMPLE 4.2:** Take the KdV hierarchy, and take for  $\Omega$  any polynomial in  $u$  and  $x$  (not containing derivatives of  $u$ ). Then all these series do truncate. For  $\Omega = u$  and for  $K := u_{xxx} + 6u_x u$  we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n (-\text{ad}_{\Omega})^{n-1} (\nabla_K(\Omega) + [\![\Omega, \nabla K]\!]) \\ = -3\lambda u_x D^2 + (-3\lambda u_{xx} - 6\lambda^2 u_x^2) D - 6\lambda^2 u_x u_{xx} - 6\lambda^3 u_x^3. \end{aligned}$$

Hence, the flow

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_{xxx} + 6u_x u \\ v_{xxx} + 6(uv)_x + 3\lambda u_x v_{xx} + (3\lambda u_{xx} + 6\lambda^2 u_x^2)v_x + 6\lambda^2 u_x u_{xx} v + 6\lambda^3 u_x^3 v \end{pmatrix} \quad (4.26)$$

belongs to a completely integrable hierarchy. This construction can be continued in many different ways.

In this construction one should observe, that in order to obtain nontrivial pairs, the trick that an isomorphism in  $\hat{\mathcal{L}}$  (instead of  $\hat{\mathcal{L}}^T$ ) is used, is essential. If instead an isomorphism in  $\hat{\mathcal{L}}^T$  is taken, then the pairs obtained are rather trivial because then there is a unique way to relate solutions of  $u_t = K(u)$  with those for the two-component system obtained by this procedure.

#### 4.4. LAX PAIRS GALORE

In [2] the authors show that there are lots of meaningless Lax pairs. This observation for first order equations was already made in [26] when commenting on [1].

In the sequel, we show how to construct impressively looking Lax pairs, which are nevertheless meaningless.

Consider an arbitrary evolution equation

$$u_t = K(u) \quad (4.27)$$

where the vector field is supposed to depend on  $u$  and arbitrary derivatives of  $u$  with respect to  $x$ . Then for any differential operator  $P = P(x, D)$  which does not depend on  $u$  we obviously have

$$\nabla_K P = [\![P, 0]\!] \quad (4.28)$$

i.e.

$$[(K, 0), (0, P)] = 0. \quad (4.29)$$

Take again

$$H = (0, \Omega) \quad (4.30)$$

where now  $H$  is a multiplication operator depending on  $x$  and  $u$ , but not on any derivatives of  $u$ . Then apply  $\exp(-\hat{L}_H)$  to (4.29) to obtain a nontrivial, but nevertheless fake Lax pair. Because of

$$\left[ \left( \begin{pmatrix} K \\ \nabla K - \sum_{n=1}^{\infty} \frac{1}{n!} (-\text{ad}_{\Omega})^{n-1} (\nabla_K(\Omega + [\![\Omega, \nabla K]\!])) \end{pmatrix}, \begin{pmatrix} 0 \\ \exp(-\text{ad}_{\Omega})P \end{pmatrix} \right) \right] = 0 \quad (4.31)$$

the flow  $u_t = K(u)$  must be an isospectral flow for the operator

$$\Lambda = \exp(-\text{ad}_{\Omega})P. \quad (4.32)$$

Since now this operator really depends on the field variable  $u$  it is, for general cases, far from obvious that this Lax formulation is absolutely meaningless. Carrying out the computation for the simple case  $P = D^2$  and  $\Omega = u$  we find the operator

$$\Lambda = \exp(-\text{ad}_{\Omega})P = D^2 + u_{xx} + 2u_xD + 2u_x^2 \quad (4.33)$$

looking almost like a decent Lax operator. That this really is not the case follows however from the fact that for any  $K(u)$  of the above type we find a  $B$  such that

$$\nabla_K \Lambda = [\![B, \Lambda]\!]. \quad (4.34)$$

Therefore the question by what kind of conditions a Lax formulation is made meaningful deserves some attention.

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## THE NEGATIVE WEIGHT KP HIERARCHY

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**ABSTRACT.** The equations in the positive weight KP hierarchy are well known. In this article we investigate some of the equations formed by including negative weight variables in the hierarchy. A pictorial representation of these equations is shown and reductions to some known systems are discussed.

### 1. Introduction

The Kadomtsev Petviashvili (KP) equation has been studied by many people. Of particular interest are the works of Jimbo and Miwa [1] and Sato and Sato [2]. They have looked at the Lie algebra structure of the KP hierarchy and related hierarchies such as the modified KP hierarchy and the BKP hierarchy. In their work they look at the representations on a function space of an infinite dimensional Lie algebra and show that the defining equations on this function space are the soliton equations. In this approach the equations from the various hierarchies are written in Hirota bilinear form. In this article we shall be investigating the ‘negative weight’ hierarchy (or perhaps more aptly the ‘mixed weight’ hierarchy) which consists of equations formed by introducing integral operators into the system, in addition to the standard differential operators.

The main objective here is to generate some new systems of equations with KP soliton type solutions. This will also enable us to see where some known equations arise from. Although here we shall only talk about the KP hierarchy, i.e., equations which admit wronskian type solutions, a similar type of analysis can be used on other hierarchies such as the BKP or the two component hierarchies — where further interesting results should arise.

### 2. The Solutions to the KP Hierarchy Equations

When working in Hirota derivative form [3] the solutions to the equations from the KP hierarchy can be written as wronskians, typically a solution will be of the form [4]

$$\tau_s(\mathbf{x}) = \det(\mathbf{M}_s^0), \quad (1)$$

where

$$\mathbf{M}_s^k(\mathbf{x}) = (\psi^{(s)} \psi^{(s+1)} \dots \psi^{(s+N+k-1)}). \quad (2)$$

$\psi^{(s)}$  is a column vector length  $(N + k)$  where  $(s)$  means each term in this vector is differentiated  $s$ -times with respect to  $x_1$ , so we suppress the reference to the functions  $\psi$  in the matrix and use a more compact notation

$$\tau_s = (s, s+1, s+2 \dots s+N-1). \quad (3)$$

Often  $\tau_0$  is abbreviated further to

$$\tau_0 = (0, 1, 2 \dots N-1) = (\widehat{N-1}), \quad (4)$$

where  $(\widehat{N-1})$  represents  $N$  consecutive columns,  $\psi$  and all its derivatives up to the  $(N-1)^{\text{th}}$ . Typically a function  $\psi$  depends upon several independent variables labeled  $(x_1, x_2, \dots)$  such that

$$\frac{\partial \psi_i}{\partial x_j} = \frac{\partial^j \psi_i}{\partial x_1^j}. \quad (5)$$

If the  $\psi$ 's are sums of exponentials this usually gives soliton type solutions; the exact form will depend upon the actual application. If the  $\psi$ 's are chosen as polynomials this will give rational solutions.

### 3. Group Characters and Young Diagrams

Because we are taking our solution or ‘ $\tau$ -function’ to be a wronskian, differentiation is quite straightforward. Let us introduce the notation

$$\partial_\lambda = \frac{\partial^n}{\partial x_{\lambda_1} \partial x_{\lambda_2} \dots \partial x_{\lambda_n}}, \quad (6)$$

where  $\lambda = (\lambda_1 \lambda_2 \dots \lambda_n)$  is a partition. Partitions are always ordered so that the largest numbers are first. Differentiating a  $\tau$ -function (for convenience we shall look at  $\tau_0$ ) we get

$$\begin{aligned} \tau_0 &= (0, 1, 2, \dots, N-2, N-1) \\ \partial_1 \tau_0 &= (0, 1, 2, \dots, N-2, N) = S_1 \tau_0 \end{aligned} \quad (7)$$

$$\begin{aligned} \partial_2 \tau_0 &= (0, 1, \dots, N-2, N+1) - (0, 1, \dots, N-3, N-1, N) \\ &= (S_2 - S_{(12)}) \tau_0 \end{aligned} \quad (8)$$

$$\begin{aligned} \partial_1^2 \tau_0 &= (0, 1, \dots, N-2, N+1) + (0, 1, \dots, N-3, N-1, N) \\ &= (S_2 + S_{(12)}) \tau_0, \end{aligned} \quad (9)$$

where the  $S_\lambda$  are ‘shift’ operators (sometimes they are called Schur functions). They have the effect of shifting the last column of the wronskian up by  $\lambda_1$  the second to last column by  $\lambda_2$  etc, until all the  $\lambda_i$ 's are used up. It is always possible to write a general derivative  $\partial_\lambda$  in terms of a sum of Schur functions using group characters

$$\partial_\lambda = \sum_\mu \chi_{\lambda\mu} S_\mu, \quad (10)$$

where the sum is over partitions of length  $|\lambda|$  and the  $\chi$  are the group characters of the symmetric group [6]. Due to the near orthogonality of the group characters the inverse relation is easily generated and is given by

$$S_\lambda = \sum_\mu z_\lambda^{-1} \chi_{\lambda\mu} \partial_\lambda, \quad (11)$$

where  $z_\lambda = \prod_{i=1}^r m_i! i^{m_i}$  for partition  $\lambda = (r^{m_r} \dots i^{m_i} \dots 1^{m_1})$ . Thus, up to scaling factors,  $\chi^{-1}$  is given by  $\chi^T$ . These shift operators acting on a  $\tau$ -function can be expressed pictorially in terms of a Young diagram. The operator  $S_\lambda = S_{(\lambda_1, \dots, \lambda_m)}$  acting on  $\tau$  is represented by a diagram with  $\lambda_1$  boxes in the first row,  $\lambda_2$  boxes in the second row, up to  $\lambda_m$  boxes in the last row. So

$$S_{(6421)}\tau = \begin{array}{|c|c|c|c|c|c|c|} \hline & \square & \square & \square & \square & \square & \square \\ \hline \square & & \square & \square & \square & \square & \square \\ \hline \square & \square & & \square & \square & \square & \square \\ \hline \square & \square & \square & & \square & \square & \square \\ \hline \end{array} \quad (12)$$

Differentiation with respect to  $x_k$  is achieved by adding  $k$  boxes to the diagram, and gives a sum of terms which obey certain simple rules;

1. The new boxes added in one operation must be properly connected.
  2. Boxes are added to either the right hand end of an existing row or below an existing row to create new rows.
  3. Boxes can only be added so that a lower row does not overlap a higher row.
  4. In one operation you cannot add a set of boxes containing a  $2 \times 2$  block, although after a number of operations you may well have blocks of  $2 \times 2$  in your diagram!
  5. The sign associated with the terms formed is + if the added block is odd in height and is - if it is even in height.

As an example if we wanted to differentiate a wronskian with respect to  $x_4$  we would get the following terms

$$\partial_4(\cdot) = \square\square\square\square - \begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} - \begin{array}{|c|}\hline \square \\ \hline \end{array}, \quad (13)$$

where  $(\cdot)$  means a wronskian with no shifts. So the sum consists of all permitted layouts of the four boxes (note that the  $2 \times 2$  block is not permitted).

Once the rules for differentiating pictorially are mastered it becomes quite simple to use!

#### 4. The Positive Weight KP Hierarchies

The first result I wish to talk about (due to Nimmo [7]) is connected with an operator called the ‘bilinear shift operator’  $\tilde{S}$

$$\tilde{S}_\mu = \sum_\lambda z_\lambda^{-1} \chi_{\lambda\mu} D_\lambda, \quad (14)$$

where  $D_\mu$  is a Hirota derivative. The structure of this operator is essentially the same as the normal shift operator except that the partial derivatives have been replaced by Hirota derivatives. The operator can also be expressed in terms of Schur functions [6].

$$\tilde{S}_\lambda = \sum_\mu (-1)^{|\mu|} S_\mu / \bar{S}_{\lambda/\mu} \Big|_{\bar{x}=x}, \quad (15)$$

where  $\mu'$  is the conjugate partition to  $\mu$ . The  $\bar{S}_{\lambda/\mu}$  is a skew Schur function which is some linear combination of ordinary Schur functions, obtained by using the Littlewood-Richardson rule [8], i.e., for some structure constants  $c_{\lambda\mu}^\beta$

$$S_{\lambda/\mu} = \sum_{\beta} c_{\lambda\mu}^\beta S_\beta. \quad (16)$$

In this notation the basic KP equation can be represented by a single bilinear shift operator

$$\tilde{S}_{(2^2)} = \frac{1}{12}[D_{14} + 3D_{22} - 4D_{31}], \quad (17)$$

the expansion in terms of equation (15) is given by

$$\tilde{S}_{(2^2)}\tau \bullet \tau = 2 \left[ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right] (\bullet) - \left[ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right] \square + \left[ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right] \square. \quad (18)$$

Thus, by doing this expansion we get a Plucker relation. An alternative way to generate the KP equation is to do a Laplace expansion by  $N \times N$  minors on a  $2N \times 2N$  determinant. Consider

$$\Delta_{s_1 s_2}^{k_1 k_2}(\mathbf{x}, \mathbf{x}') = \begin{vmatrix} M_{s_1}^{k_1}(\mathbf{x}) & 0 \\ 0 & M_{s_2}^{k_2}(\mathbf{x}') \end{vmatrix}, \quad (19)$$

when  $k_1 \neq k_2$  this  $\Delta_{s_1 s_2}^{k_1 k_2} = 0$ . Thus, if we take  $k_1 = -k_2 = k \neq 0$  the determinant obtained by adding the bottom  $N$  rows to the top  $N$  rows and setting  $x = x'$  is also identically zero,

$$\Delta_{s_1 s_2}^{-kk}(\mathbf{x}, \mathbf{x}) = \begin{vmatrix} M_{s_1}^{-k}(\mathbf{x}) & M_{s_2}^k(\mathbf{x}) \\ 0 & M_{s_2}^k(\mathbf{x}) \end{vmatrix} = 0. \quad (20)$$

By choosing  $k = 2$  the Hirota form of the KP equation is obtained

$$\Delta_{0,0}^{-2,2} = \tilde{S}_{(2^2)} = \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right]. \quad (21)$$

This gives us the basic ‘seed’ equation for the hierarchy. To generate the other equations in the hierarchy we need to define a generalised Hirota derivative  $\tilde{D}_i$  [9]

$$P(\tilde{D}_1, \tilde{D}_2, \dots) \Delta(\mathbf{x}, \mathbf{x}) := P(\partial_1 - \partial'_1, \dots) \Delta(\mathbf{x}, \mathbf{x}')|_{x=x'} = 0. \quad (22)$$

Although the formalism is rather tedious to set up we are now in a position to describe the equations in the hierarchy pictorially:

$$\Delta_{0,0}^{-2,2} = \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] \quad (23)$$

$$\tilde{D}_1 \Delta_{0,0}^{-2,2} = \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] - \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] \quad (24)$$

$$\tilde{D}_2 \Delta_{0,0}^{-2,2} = \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] - \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] + \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] \quad (25)$$

$$\tilde{D}_1 \tilde{D}_1 \Delta_{0,0}^{-2,2} = \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] + \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] - 2 \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] + \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right] + \left[ \begin{array}{|c|c|} \hline \widetilde{\square} & \widetilde{\square} \\ \hline \widetilde{\square} & \widetilde{\square} \\ \hline \end{array} \right]. \quad (26)$$

As we act more generalised Hirota derivatives on the ‘seed’ solution we get the equations from higher in the hierarchy. The rules for adding the boxes to the tilded diagrams is very similar to the original Young diagram rules, the only difference is if the boxes are added to the bottom of a diagram, then there is an additional change of sign associated with such a term, if boxes are added to the RH side of the diagram there is no additional change of sign. This change of sign can be thought as being caused by working with Hirota derivatives as opposed to ordinary derivatives. This approach accounts correctly for all the equations in the hierarchy up to weight nine (see appendix of [1]) at which point there is an ambiguity as to whether the new boxes are being added to the side or the bottom of the diagram.

$$\begin{aligned}\tilde{D}_5 \Delta_{0,0}^{-2,2} &= \begin{array}{|c|c|}\hline \tilde{\square} & \tilde{\square} \\ \hline \end{array} + 5 \text{ boxes} \\ &= \dots + \begin{array}{|c|c|c|c|}\hline \tilde{\square} & \tilde{\square} & \tilde{\square} & \tilde{\square} \\ \hline \end{array} + \dots\end{aligned}\tag{27}$$

This can be overcome by not allowing new boxes to be added on the leading diagonal.

$$\begin{array}{|c|c|}\hline \tilde{\square} & \tilde{\square} \\ \hline \end{array}\tag{28}$$

#### BANNED REGION

To reach this banned region new ‘seeds’ are needed. The tables below show pictorially the ‘seeds’ for the KP hierarchy. The ‘seeds’ for the associated hierarchies can also be expressed in terms of tilded Young diagrams, some of which are also pictured below.

#### KP ‘seeds’

Trivial cases; for the KP case when there are an odd amount of boxes the equations turn out to be trivial,

$$\begin{array}{ccc}\tilde{\square} & \begin{array}{|c|c|}\hline \tilde{\square} & \tilde{\square} \\ \hline \end{array} & \begin{array}{|c|c|c|c|}\hline \tilde{\square} & \tilde{\square} & \tilde{\square} & \tilde{\square} \\ \hline \end{array}\end{array}\tag{29}$$

Non-trivial cases;

$$\begin{array}{ccc}\begin{array}{|c|c|}\hline \tilde{\square} & \tilde{\square} \\ \hline \end{array} & \begin{array}{|c|c|c|c|}\hline \tilde{\square} & \tilde{\square} & \tilde{\square} & \tilde{\square} \\ \hline \end{array} & \begin{array}{|c|c|c|c|c|c|}\hline \tilde{\square} & \tilde{\square} & \tilde{\square} & \tilde{\square} & \tilde{\square} \\ \hline \end{array}\end{array}\tag{30}$$

modified KP ‘seeds’modified modified KP ‘seeds’

The ‘seeds’ plus the equations generated by successive application of the generalised Hirota derivatives give us the equations in the hierarchy. As with the KP case there is also a banned region for the mKP and mmKP cases. Boxes cannot be added in any position for which both the row and the column are beyond the original seed, for example



## BANNED REGION

It appears from the results that Laplace type expansions give us all the equations in the hierarchy. This can be shown explicitly for the KP hierarchy by using a counting argument. The number of equations at weight  $k$  is  $p^o(k) - p(k - 1)$  [1] where  $p(k)$  is the number of partitions of  $k$  and  $p^o(k)$  is the number of partitions of  $k$  with an odd number of elements. Our method gives

$$p(k - 4) - p(k - 9) + p(k - 16) - p(k - 25) \dots = \sum_i (-)^i p(k - i^2), \quad (34)$$

where  $p(k) = 0$  for  $k$  negative and  $p(0) = 1$ . The terms  $+p(k - (2i)^2)$  give the number of equations of weight  $k$  generated from the non trivial seeds of size  $(2i)^2$  not taking in to account the banned region. The terms  $p(k - (2i + 1)^2)$  are theoretically, the number of equations of weight  $k$  generated from the trivial seeds, but since these are all trivial, we need to take them away from the sum of the non trivial ones. The number in (34) can be shown to be equal to  $p^o(k) - p(k - 1)$  using an identity of Jacobi [10].

**5. Negative Weights**

So far we have only talked about positive weight equations. To introduce negative weights we require another set of independent variables labeled  $(x_{-1}x_{-2} \dots x_{-i} \dots)$ . The effect of acting a negative weight derivative on a wronskian is that it shifts the zeroth derivative column to be a  $(-1)^{th}$  derivative column. One can think of having functions containing exponentials of the form

$$\exp(\dots + p^{-r}x_{-r} \dots + p^{-1}x_{-1} + p^1x_1 \dots + p^rx_r + \dots) := \exp(p, \underline{x}) \quad (35)$$

differentiating with respect to  $x_{-1}$  produces  $\exp(p, \underline{x})/p$  which is exactly what is obtained when integrating with respect to  $x_1$ .

Using only ‘negative derivatives’, the shifts on the wronskians will have exactly the same structure as the positive case except that the shifts are in the other direction: only when you actually mix negative and positive derivatives will you really get the possibility of different equations. Pictorially we shall represent the negative derivatives by shaded boxes  , by convention we shall add them to the bottom LH corner of the diagrams. As an example a wronskian with negative shifts  $(-4, -2, -1)$  would be represented pictorially by

$$\begin{array}{c} \square \square \square \\ \square \quad \square \\ \square \end{array} = S_{(-4, -2, -1)}\tau. \quad (36)$$

Differentiation with respect to the variables  $x_{-i}$  is achieved by adding the relevant number of shaded boxes to the left hand corner of the diagrams, using the same sort of rules as for the positive case. This again is best illustrated by an example

$$\partial_{(-3)} \left[ \begin{array}{c} \square \square \\ \square \end{array} \right] = \begin{array}{c} \square \square \square \square \square \\ \square \quad \square \end{array} - \begin{array}{c} \square \square \square \\ \square \quad \square \end{array} - \begin{array}{c} \square \square \square \\ \square \quad \square \end{array} + \begin{array}{c} \square \square \\ \square \quad \square \\ \square \end{array}. \quad (37)$$

To act derivatives of both positive and negative weights we need to combine the two types of diagram together. Acting a positive weight derivative to a mixed diagram does two things. First, you get terms with positive boxes added according to the original rules and second you must also have terms where negative boxes are taken away from the diagram. Similarly acting a negative weight derivative to a mixed diagram also does two things: first, you obtain diagrams where negative boxes have been added according to the original rules and, second there are additional terms where positive boxes are taken away from the diagram. As an example with overall weight 1 take;

$$\partial_{(21-1^2)}\tau = \partial_{(-1^2)}\partial_{(21)}\tau \quad (38)$$

$$= \partial_{(-1^2)} \left( \begin{array}{c} \square \square \square \\ \square \quad \square \end{array} - \begin{array}{c} \square \\ \square \\ \square \end{array} \right) \quad (39)$$

$$= \partial_{(-1)} \left( \begin{array}{c} \square \square \\ \square \quad \square \end{array} + \begin{array}{c} \square \square \square \square \\ \square \quad \square \end{array} - \begin{array}{c} \square \\ \square \end{array} - \begin{array}{c} \square \square \\ \square \quad \square \\ \square \end{array} \right) \quad (40)$$

$$= \left( \begin{array}{c} \square + 2 \begin{array}{c} \square \square \square \\ \square \quad \square \end{array} + \begin{array}{c} \square \square \square \square \square \\ \square \quad \square \end{array} + \begin{array}{c} \square \square \square \square \\ \square \quad \square \end{array} - \begin{array}{c} \square \\ \square \end{array} \right. \\ \left. - 2 \begin{array}{c} \square \square \\ \square \quad \square \\ \square \end{array} - \begin{array}{c} \square \square \square \\ \square \quad \square \\ \square \end{array} - \begin{array}{c} \square \square \\ \square \quad \square \\ \square \end{array} \right) \quad (41)$$

$$= \left( 2 \begin{array}{c} \square \square \square \\ \square \quad \square \end{array} + \begin{array}{c} \square \square \square \square \square \\ \square \quad \square \end{array} + \begin{array}{c} \square \square \square \square \\ \square \quad \square \end{array} \right. \\ \left. - 2 \begin{array}{c} \square \square \\ \square \quad \square \\ \square \end{array} - \begin{array}{c} \square \square \square \\ \square \quad \square \\ \square \end{array} - \begin{array}{c} \square \square \\ \square \quad \square \\ \square \end{array} \right) \quad (41)$$

At this stage it is quite possible to work just with various Laplace expansions. We shall come on to this in a moment, but first we should complete the overall picture by extending the ‘bilinear shift operator’ to the mixed case

$$\tilde{S}_\lambda = \sum_{\mu} (-1)^{|\mu| + |\mu| - |\lambda|} S_{\mu'} \bar{S}_{\lambda/\mu} |_{\bar{x}=x}. \quad (42)$$

Here  $\lambda$  is a partition with both a positive part  $\lambda_+$  and a negative part  $\lambda_-$ . The sum over  $\mu$  is over all possible subdiagrams of  $\lambda$ . As with the non mixed case we need to know what our mixed ‘seed’ equations look like. The other equations will be built up from these seeds to form a mixed weight hierarchy. The mixed hierarchy equations usually involve more than one  $\tau$ -function. To indicate which  $\tau$ -functions a particular diagram relates to we have added subscripts to the diagrams, for instance,

$$\begin{array}{c} \square \\ \square \end{array} \quad _2 = S_{(2,1;-1)} \tau_2 \quad (43)$$

$$\begin{array}{c} \square \\ \square \end{array} \quad _{(12)} = \tilde{S}_{(2,1;-1)} \tau_1 \bullet \tau_2 \quad (44)$$

Listed below are some of the known mixed seeds

$$\begin{aligned} \Delta_{2,0}^{-2,2}(x, x) &= \begin{vmatrix} M_2^{-2}(x) & M_0^2(x) \\ 0 & M_0^2(x) \end{vmatrix} = (\square \square)_1 (\bullet)_1 - (\square)_1 (\square)_1 + \tau_0 \tau_2 \\ &= \widetilde{\square \square}_{(11)} + \widetilde{(\bullet)}_{(02)} = 0 \end{aligned} \quad (45)$$

$$\begin{aligned} \Delta_{2,0}^{-3,3}(x, x) &= \begin{vmatrix} M_2^{-3}(x) & M_0^3(x) \\ 0 & M_0^3(x) \end{vmatrix} \\ &= \begin{array}{c} \square \\ \square \end{array} \quad _{(11)} + \begin{array}{c} \square \square \square \\ \square \end{array} \quad _{(02)} = 0 \end{aligned} \quad (46)$$

$$\begin{aligned} \Delta_{2,0}^{-4,4}(x, x) &= \begin{vmatrix} M_2^{-4}(x) & M_0^4(x) \\ 0 & M_0^4(x) \end{vmatrix} \\ &= \begin{array}{c} \square \\ \square \end{array} \quad _{(11)} + \begin{array}{c} \square \square \square \square \\ \square \end{array} \quad _{(02)} = 0 \end{aligned} \quad (47)$$

The second case is a trivial one. The third case is nontrivial but it does not depend on  $\tau_0$  or  $\tau_2$  since the second term is identically zero. There is the possibility of more complicated cases — you may need more than the standard three  $\tau$ -functions, for instance

$$\begin{aligned} \Delta_{4,0}^{-4,4}(x, x) &= \begin{vmatrix} M_4^{-4}(x) & M_0^4(x) \\ 0 & M_0^4(x) \end{vmatrix} \\ &= \begin{array}{c} \square \square \square \square \\ \square \square \end{array} \quad _{(22)} + \begin{array}{c} \square \square \square \square \\ \square \end{array} \quad _{(13)} + \widetilde{(\bullet)}_{(04)} = 0 \end{aligned} \quad (48)$$

## 6. Negative Weight Equations and Reductions

By using Laplace type expansions on determinants we can generate various ‘seed’ equations. Although in their pictorial form they appear quite simple they often contain many terms. To see precisely what equations these pictures give we must first turn them back into Hirota bilinear equations and then, by suitable change of variables, into nonlinear equations. Turning the equations into bilinear form is moderately straightforward and can be done by computer. The second stage, turning the bilinear equations in to nonlinear differential equations, is not a unique process since any change of variables can be taken, but it seems sensible to choose a change of variables similar to the standard change of variables already used in the KP equation and known reductions from the KP hierarchy. We will take

$$Q = 2 \log \tau_1, \quad U = \tau_2 / \tau_1, \quad U^* = \tau_0 / \tau_1, \quad (49)$$

where  $*$  does not necessarily mean complex conjugate. Let us look at some of the low weight equations involved with  $\Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x})$

$$\Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow (D_1 D_{-1} - 2) \tau_1 \bullet \tau_1 + 2 \tau_0 \bullet \tau_2 = 0 \quad (50)$$

$$\widetilde{D}_1 \widetilde{D}_{-1} \Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow (D_2 D_{-2} + 2 D_1 D_{-1} - 8) \tau_1 \bullet \tau_1 - (2 D_1 D_{-1} - 6) \tau_0 \bullet \tau_2 = 0 \quad (51)$$

$$\widetilde{D}_1 \Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow (D_2 D_{-1}) \tau_1 \bullet \tau_1 - (2 D_1) \tau_0 \bullet \tau_2 = 0 \quad (52)$$

$$\widetilde{D}_{-1} \Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow (D_1 D_{-2}) \tau_1 \bullet \tau_1 + (2 D_{-1}) \tau_0 \bullet \tau_2 = 0 \quad (53)$$

$$\begin{aligned} \widetilde{D}_2 \Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow & (6 D_1^2 - 2 D_3 D_{-1} - D_1^3 D_{-1}) \tau_1 \bullet \tau_1 \\ & + (6 D_2) \tau_0 \bullet \tau_2 = 0 \end{aligned} \quad (54)$$

$$\begin{aligned} \widetilde{D}_1^2 \Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow & (6 D_1^2 + 4 D_3 D_{-1} - D_1^3 D_{-1}) \tau_1 \bullet \tau_1 \\ & + (6 D_1^2) \tau_0 \bullet \tau_2 = 0 \end{aligned} \quad (55)$$

$$\begin{aligned} \widetilde{D}_3 \widetilde{D}_{-1} \Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow & (12 D_1^2 - 3 D_4 D_{-2} - 3 D_2 D_1^2 D_{-2}) \tau_1 \bullet \tau_1 \\ & + (12 D_2 + 12 D_1^2 + 8 D_3 D_{-1} \\ & - 6 D_2 D_1 D_{-1} - 2 D_1^3 D_{-1}) \tau_0 \bullet \tau_2 = 0 \end{aligned} \quad (56)$$

The above equations are all equations from the seed  $\Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x})$ . Starting with different seeds gives us different equations. Because of the negative weight variables there are an infinity of equations at each weight level, which ones are useful will depend on the application in mind.

One system which falls out very easily as a reduction from these equations is the ‘dark’ form of the Davey-Stewartson equation. In bilinear form the DS equations are;

$$(D_x D_y - 2) F \bullet F + 2 G G^* = 0, \quad (i D_t + D_x^2 + D_y^2) G \bullet F = 0 \quad (57)$$

where  $*$  does mean complex conjugate here. If we look at some of the mixed weight KP hierarchy equations  $\Delta_{1,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0$ ,  $\Delta_{2,1}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0$ ,  $\Delta_{3,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0$ ,  $\Delta_{4,1}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0$  and  $\Delta_{2,0}^{-2,2}(\mathbf{x}, \mathbf{x}) = 0$ , in terms of bilinear forms these equations give;

$$(D_2 + D_1^2) \tau_0 \bullet \tau_1 = 0 \quad (58)$$

$$(D_2 + D_1^2) \tau_1 \bullet \tau_2 = 0 \quad (59)$$

$$(D_{-2} - D_{-1}^2)\tau_0 \bullet \tau_1 = 0 \quad (60)$$

$$(D_{-2} - D_{-1}^2)\tau_1 \bullet \tau_2 = 0 \quad (61)$$

$$(D_1 D_{-1} - 2)\tau_1 \bullet \tau_1 + 2\tau_0 \tau_2 = 0. \quad (62)$$

If we make the identifications  $F = \tau_1$ ,  $G = \tau_0$ ,  $G^* = \tau_2$  for the dependent variables and  $x = x_1$ ,  $y = x_{-1}$  and  $t = i(x_2 - x_{-2})/2$  for the independent variables, these equations from the mixed weight hierarchy can be identified with the Davey-Stewartson equations [11].

## 7. Conclusions

We have shown how to generate the equations in the KP and some related hierarchies by using pictorial methods. It also appears possible to produce mixed and negative weight equations pictorially by introducing negative shift functions. By turning these equations back in to Bilinear form or into nonlinear differential equations there is an abundance of ‘new’ equations. Most of the low order equations occur in systems which are already known about. To obtain systems such as the Davey Stewartson system often some kind of reduction of the equations in the hierarchy is required.

The method of Laplace expansion to produce these equations is very simple to use and guarantees that the solution to such an equation is a wronskian. In some ways this technique is rather a back to front approach in that it starts with the solutions and generates the equations, hopefully this method will enable us to find new and interesting systems within this framework.

There are other hierarchies of equations which are less well studied than the KP hierarchy, such as the multicomponent KP hierarchy and the BKP hierarchy. Similar methods to the approach used above could be applied to these. The multi component case would be a small extension of the standard KP case presented here in that it would require looking at identities between multicomponent wronskians as opposed to single component ones. The approach for the BKP case would be somewhat different as it will involve looking at pfaffian identities.

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# THE COMPLETE SOLUTION TO THE CONSTANT QUANTUM YANG-BAXTER EQUATION IN TWO DIMENSIONS

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ABSTRACT. We describe how the complete solution of the two-dimensional constant quantum Yang-Baxter equation was found.

## 1. Introduction

In this talk we discuss the solving of the constant quantum Yang-Baxter equation (YBE) in two dimensions. The results were announced in [1], a detailed account of the solution process will be published elsewhere [2], see also [3].

First let us recall the spectral parameter dependent form of YBE:

$$R_{j_1 j_2}^{k_1 k_2}(u) R_{k_1 j_3}^{l_1 k_3}(u+v) R_{k_2 k_3}^{l_2 l_3}(v) = R_{j_2 j_3}^{k_2 k_3}(v) R_{j_1 k_3}^{k_1 l_3}(u+v) R_{k_1 k_2}^{l_1 l_2}(u). \quad (1)$$

Here summation over the repeated  $k$  indices is understood. The constant quantum YBE

$$R_{j_1 j_2}^{k_1 k_2} R_{k_1 j_3}^{l_1 k_3} R_{k_2 k_3}^{l_2 l_3} = R_{j_2 j_3}^{k_2 k_3} R_{j_1 k_3}^{k_1 l_3} R_{k_1 k_2}^{l_1 l_2} \quad (2)$$

is obtained from (1) with  $u = v = 0$  or  $u = v = \pm\infty$ .

The YBE appears in many physical contexts, here is a short list:

- It was first derived in the study of solvable vertex models in statistical mechanics as the condition of commuting transfer matrices (Yang, Baxter).
- Another derivation follows from the factorization of the  $S$ -matrix in 1 + 1-dimensional Quantum Field Theory (Zamolodchikov).
- The YBE is also essential in Quantum Inverse Scattering Method for integrable systems as developed by the Leningrad school.
- The more abstract setting is in terms of Sklyanin algebras and quantum groups (Drinfel'd: related to Hopf-algebras).
- More recently the connection to braid groups and knot theory has been studied. Here the spectral parameter independent form arises naturally.

For a nice annotated collection of basic papers, see [4].

For applications we then need solutions of YBE. Many solutions have been found before, either by Lie-algebraic methods [5], or by using a specific ansatz [6]. In general YBE has  $N^6$  cubic equations for  $N^4$  unknowns. Even in the simplest, i.e. two-dimensional case, one

has 64 equations for 16 unknowns. Such a set of equations is certainly too complicated for a brute force approach, thus one must simplify the problem by all means possible (without sacrificing generality). This is done using the inherent symmetries of the system.

In writing out the two induced object  $\mathbf{R}$  we use the usual matrix notation. In two dimensions we have four  $2 \times 2$  blocks, the second index pair denotes the location of the block:

$$\mathbf{R} = \begin{pmatrix} R_{11}^{11} & R_{11}^{21} & R_{11}^{12} & R_{11}^{22} \\ R_{21}^{11} & R_{21}^{21} & R_{21}^{12} & R_{21}^{22} \\ R_{12}^{11} & R_{12}^{21} & R_{12}^{12} & R_{12}^{22} \\ R_{22}^{11} & R_{22}^{21} & R_{22}^{12} & R_{22}^{22} \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ f & g & h & j \\ k & l & m & n \\ p & q & u & v \end{pmatrix}. \quad (3)$$

## 2. Symmetries

The quantum YBE is invariant under the following continuous group of transformations

$$\mathbf{R} \rightarrow \kappa(\mathbf{Q} \otimes \mathbf{Q})\mathbf{R}(\mathbf{Q} \otimes \mathbf{Q})^{-1}, \quad (4)$$

where  $\mathbf{Q}$  is a nonsingular  $N \times N$  matrix and  $\kappa$  a nonzero number. In the solution process we will use up this rotational freedom in a specific order, using the following parametrization:

$$\mathbf{Q} = \begin{pmatrix} A & 0 \\ 0 & 1/A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}. \quad (5)$$

In particular note the scalings with  $\kappa$  and a diagonal  $\mathbf{Q}$ . By scaling weight let us denote the power of  $A$  that a matrix element of  $\mathbf{R}$  picks up under (5) (with  $B = C = 0$ ), then we have

$$\text{the scaling weights of } \mathbf{R} = \begin{pmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix}. \quad (6)$$

Two nonzero elements of  $\mathbf{R}$  with different weights can be scaled to one using  $A$  and  $\kappa$ .

The YBE has also discrete symmetries related to index changes:

$$R_{ij}^{kl} \rightarrow R_{kl}^{ij}, \quad (7)$$

$$R_{ij}^{kl} \rightarrow R_{i+n,j+n}^{k+n,l+n} \text{ (indices mod } N\text{)}, \quad (8)$$

$$R_{ij}^{kl} \rightarrow R_{ji}^{lk}. \quad (9)$$

In two dimensions and using the matrix notation (3): equation (7) corresponds to a reflection across the diagonal (i.e., the usual transposition P reflection); equation (8) with  $n = 1$  and followed by (7) corresponds to a reflection across the secondary diagonal (C reflection); and equation (9) corresponds to a reflection among the two central rows and among the two central columns (T reflection).

### 3. Breakup into smaller parts

Since the system under study is so large we must first split it by hand into several smaller subproblems. When we refer to specific equations, we use the following numbering:

$$E_{l_3+2l_2+4l_1+8j_3+16j_2+32j_1-62} = R_{j_1 j_2}^{k_1 k_2} R_{k_1 j_3}^{l_1 k_3} R_{k_2 k_3}^{l_2 l_3} - R_{j_2 j_3}^{k_2 k_3} R_{j_1 k_3}^{k_1 l_3} R_{k_1 k_2}^{l_1 l_2} \quad (10)$$

The equations are written out explicitly in [2,3].

CASE (i). First we analyzed the equations by counting how many times each variable appears in them. It turned out that the corner elements  $d$  and  $p$  appeared most frequently. Thus it seemed to be good idea to use part of the rotational freedom to transform  $d$  to 0 with  $B$  in  $\mathbf{Q}$  (5). The different cases that one encounters are as follows: If  $d = 0$  already take  $B = 0$ , if  $d \neq 0$ ,  $p = 0$  take the transpose of  $\mathbf{R}$  and  $B = 0$ . If  $p \neq 0$  then the  $B$  part of transformation  $\mathbf{Q}$  yields

$$\begin{aligned} d_{new} := & B^4 p + B^3(f + k - q - u) + B^2(a - g - h - l - m + v) \\ & + B(-b - c + j + n) + d. \end{aligned} \quad (11)$$

Since  $p \neq 0$  we can always find a  $B$  so that  $d_{new} = 0$ .

CASE (ii). At this point we had  $d = 0$  and to keep it that way we take  $B = 0$  in subsequent transformations. When looking at the equations we found that one of them had a nice form

$$E_{22} := bc(f - k) + jn(q - u) = 0, \quad (12)$$

and thus the problem would factorize into three parts, if we could transform so that  $f - k = 0$  or  $q - u = 0$ . This was accomplished as follows:

If  $f = k$  already there is nothing to do, if  $f \neq k$  but  $q = u$  use C reflection to put  $f = k$ , and in both case take  $C = 0$  in  $\mathbf{Q}$  (5). If both  $f \neq k$  and  $q \neq u$  we have after transforming with the  $C$  part

$$\begin{aligned} (f - k)_{new} := & C^2(j - n) + C(-g - h + l + m) + f - k = 0, \\ (q - u)_{new} := & C^2(b - c) + C(g - h + l - m) + q - u = 0. \end{aligned} \quad (13)$$

We can now solve for  $C$  in one of the these (and use reflection (8) if necessary), except if  $j = n$ ,  $b = c$ ,  $h = l$ ,  $g = m$ ,  $f \neq k$ ,  $q \neq u$ , which will become case C.

CASE (iii). When we use  $d = 0$ ,  $f = k$  in  $E_{22}$  we get

$$jn(q - u) = 0. \quad (14)$$

The problem splits therefore into two big cases A:  $q = u$  and B:  $n = 0$ ,  $q \neq u$ . (The case  $j = 0$  can be T reflected (9) into  $n = 0$ .)

CASE (iv). When we substitute  $q = u$  in case A it turns out that  $E_{57}$  factorizes as

$$-p(k + u)(g - m) = 0, \quad (15)$$

and we get the subcases A1:  $k = -u$  and A2:  $g = m$ ,  $k + u \neq 0$ . The third possibility  $p = 0$  yields  $E_{58} + E_{61} = -2u^2(g - m)$  and  $E_{25} + E_{49} = -2k^2(g - m)$ , and therefore leads to one of the previous cases.

CASE (v). In Case B,  $n = 0$  implies through equations  $E_{8,16,24,32}$  that either B1:  $j = 0$ , B2:  $j \neq 0$ ,  $c = 0$ ,  $m = 0$  or B3:  $j \neq 0$ ,  $b = 0$ ,  $l = 0$  and  $c \neq 0$  or  $m \neq 0$ .

Thus using the two rotational degrees of freedom we were able to split the problem into six simpler cases (A1, A2, B1, B2, B3, C).

CASE (vi). The two remaining continuous freedoms are related to scalings by  $A$  and  $\kappa$  (4,5). If there are two nonzero elements with different weights they can both be scaled to unity. If we have elements of equal weight only one of them can still be scaled to unity. With the scalings we can split the problem further into a total of 33 subcases.

#### 4. Computer solution

The best way to analyze sets of polynomial equations is by using Gröbner bases. This is a systematic approach to sets of equations and defines a canonical form in a given ordering of variables. The basic idea in constructing a Gröbner basis is that after fixing a polynomial one eliminates multiples of it from other polynomials in the set. One also forms combinations eliminating leading terms. This will be repeated systematically (Buchberger algorithm) as long as possible.

Since we just need solutions to a set of equations it is a good strategy to factorize the polynomials when possible and split the problem into smaller ones.

Thus for each of the 33 subsubcases we computed the factorized Gröbner basis using the ‘groebner’-package written by Melenk, Möller and Neun [7] for the REDUCE 3.4 [8]. The raw output contained repeats and subcases which were eliminated by a separate program. (In the newest version they are eliminated automatically). In the end we had 96 solutions to analyze.

#### 5. Canonical form

Since many of the solutions obtained above could be transformed into each other it is important to bring them into a canonical form (using the continuous and discrete symmetries discussed before) for a final comparison.

In order to define a reasonable canonical form let us consider the trace matrices of  $\mathbf{R}$ :

$$r_i^k = R_{ij}^{kj}, \quad r_j^l = R_{ij}^{il}, \quad \lambda_i^l = R_{ij}^{jl}, \quad t_j^k = R_{ij}^{ki}. \quad (16)$$

Under (4) all of these transform according to

$$\mathbf{r} \rightarrow \kappa \mathbf{Q} \mathbf{r} \mathbf{Q}^{-1}. \quad (17)$$

The basic definition of the proposed canonical form is that the above trace matrices are in the Jordan canonical form. Since the trace matrices do not necessarily commute it is possible that they cannot be brought to the canonical form simultaneously, we will therefore work in the above order. If this requirement is not enough to fix the rotational freedom completely, we must look at individual matrix elements of  $\mathbf{R}$ . For a detailed algorithm for constructing the canonical form, see [1,2].

Using these ideas we were able to combine the 96 solutions into 23 cases using homogeneous parametrization [1].

#### 6. A comment on Krichever’s method

In [9] Krichever proposed a method of solving the spectral parameter dependent equation (1) in general. This method relies heavily on algebraic geometry, and the complex analytic properties of  $\mathbf{R}$  as a function of the spectral parameter play an important role. It is therefore not really designed for finding constant solutions, but let us anyway see how it starts.

Krichever’s method begins as follows: Let us consider so called ‘vacuous’ vectors, i.e. tensor products that stay tensor products under the action of  $\mathbf{R}$ :

$$R_{jb}^{ia} X_i U_a = h Y_j V_b \quad (18)$$

(The notational correspondence with [9] is:  $R(\text{here})_{ij}^{kl} = \mathcal{L}([9])_{ij}^{kl} = R([9])_{ji}^{kl}$ , furthermore we only consider the two-dimensional case.) Krichever normalizes the vectors by  $X_2 = 1$ ,  $V^t = (v, 1)$ ,  $U^t = (u, 1)$ .

Now, after multiplying (17) by  $\tilde{V}^t = (1, -v)$  we get

$$LX = 0, \text{ where } L_j^i = \tilde{V}^b R_{jb}^{ia} U_a. \quad (19)$$

For this to have nontrivial solution we must have

$$P(u, v) := \det L = 0 \quad (20)$$

From this point on Krichever considers only the generic situation, i.e. the case where 1) for almost all  $u$  the roots  $v_i$  are different, 2) the genus of the curve determined by  $P(u, v) = 0$  is 1.

When the above is applied to the constant solutions obtained in [1,2] it turns out that the polynomial  $P(u, v)$  is irreducible only in one case: for  $R_{H1,1}$  we find

$$P = (q^2 - 1)^2 u^2 v^2 - (q^4 - 2q^3 - 2q - 1)u^2 - (q^4 + 2q^3 + 2q - 1)v^2 + (q^2 - 1)^2. \quad (21)$$

In all other cases  $P$  factorizes. Sometimes one gets two quadratic algebraic curves, sometimes the factors are linear, and for the most singular solutions  $P$  vanishes identically. Thus it seems that what is generic in (19) is quite specific among the constant solutions.

Above we were looking at what happens when the spectral parameter vanishes or is infinite. However, if we want solutions with interesting dependence on the spectral parameter, then the curve  $P(u, v) = 0$  will often have the ‘generic’ form for generic values of the spectral parameter. For many such solutions one nevertheless has a quite simple  $R(0)$  and  $R(\infty)$ . But if the problem is to find constant solutions then Krichever’s method is not really applicable.

## 7. Conclusions

With the work [1,2] we know all solutions of constant quantum YBE in 2 dimensions. Most of them fit into the 8-vertex ansatz, but not all. One interesting observation is that all nonsingular solutions are either upper triangular, or have the property  $R_{ij}^{kl} = 0$  unless  $i + j = k + l \pmod{2}$ .

There are now many things that we can do with the results. For example, what kind of algebras do we get with the new solutions? And what kind of applications are related to them?

As for extensions, next one could search for solutions with a spectral parameter. Another interesting problem is to go to higher dimensions.

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# STIMULATED RAMAN SCATTERING: AN INTEGRABLE SYSTEM WHICH BLOWS UP IN FINITE TIME

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**ABSTRACT.** We present an extension of the spectral transform theory for solving initial-boundary values problems — describing the resonant interaction of radiation with matter — for *arbitrary* boundary values. As it has been published elsewhere [1], the general method is only sketched and it is illustrated here on a particularly representative example: the Stimulated Raman Scattering equations. We give the general solution for arbitrary boundary values. The the *natural* boundary values (asymptotic state of the radiation) are shown to lead to a solution which *blows up in finite time* for any localized initial profile of the acoustic wave (initial state of the matter).

## 1. Introduction

The spectral transform method has been revealed as a most useful tool for solving *initial value* problems for integrable nonlinear evolutions such as the nonlinear Schrödinger and Korteweg-de Vries equations. In that case the physical problem is that of the time evolution of an initial profile or disturbance. One of the main results of the theory which has deeply influenced the approach of nonlinear processes is that any localized initial profile eventually evolves into a set of nonlinear coherent structures (solitons) travelling on a background of vanishing quasi-linear oscillations (radiation) with everything being completely determined from the initial datum.

Here we are interested in the interaction of radiation with matter, and it is clear that boundary values will come into play (e.g. the input radiation value). Then, in that context the problem to consider is a nonlinear *boundary value problem* for coupled waves. The main mathematical property of systems of coupled waves is the existence of a *singular dispersion relation* ( $\omega(k)$  is a non-analytic function of the complex variable  $k$ ), and Lamb Jr. discovered that some of such systems are integrable by an extension of the spectral transform theory [2]. His work on the equations of self-induced-transparency have been generalized in [3] and set up on a general basis in [4]. We have developed a systematic approach to integrable systems of coupled waves [5] by use of the  $\bar{\delta}$ -formulation of the spectral transform and have applied it to different situations [6–8] for which the relevant problem is an initial value problem.

We have recently proved that the method can be extended to a more general evolution of the spectral transform which allows for solving *boundary value problems* (with arbitrary

boundary values) for those integrable systems having a singular dispersion relation [1]. Depending on the problem the time evolution of the spectral transform can well be nonlinear, but still explicitly solvable in the physically interesting cases.

We will not give here any detail on this extension of the spectral transform theory but merely discuss an explicit example, the Stimulated Raman Scattering (SRS) equations of Chu and Scott [9] (see also [10]). This example is quite representative of the novelties contained in the method: the freedom of the boundary values may induce very unsuspected and interesting behaviours. We have described some other physical situations with different behaviours in [1].

Let us first define some convenient notation. For a complex valued function  $f(k) = a(k) + ib(k)$  of the complex variable  $k = \zeta + i\eta$ , we note

$$\bar{f}(k) = a(\zeta + i\eta) - ib(\zeta + i\eta), \quad (1)$$

$$f^*(k) = a(\zeta - i\eta) - ib(\zeta - i\eta) = \bar{f}(\bar{k}). \quad (2)$$

We will use also the distributions  $\delta^\pm(\eta)$  given by

$$\int_{-\infty}^{+\infty} d\eta f(\zeta + i\eta) \delta^\pm(\eta) = f(\zeta \pm i0) \quad (3)$$

which have the property

$$\delta^+(\eta) = \bar{\delta}^-(\eta). \quad (4)$$

## 2. General Integrable Problem

The results of [1] can be summarized as follows: the system of coupled equations for the fields  $q(x, t)$ ,  $a_j(k, x, t)$

$$q_t = \int_{-\infty}^{+\infty} g dk a_1 \bar{a}_2, \quad (5)$$

$$a_{1,x} = qa_2, \quad a_{2,x} - 2ika_2 = \sigma \bar{q}a_1 \quad (6)$$

(with  $\sigma = \pm 1$ ,  $x \in \text{Re } , t > 0$  and  $g = g(k, t)$  an arbitrary function in  $L^2$ ), is integrable for arbitrary boundary values, say

$$a_1 \xrightarrow[x \rightarrow +\infty]{} J_1(k, t), \quad a_2 \xrightarrow[x \rightarrow -\infty]{} J_2(k, t) \exp(2ikx). \quad (7)$$

and arbitrary initial datum  $q(x, 0)$  in  $L^1$ . We briefly recall here the method of solution.

Given  $q(x, 0)$  and  $r(x, 0) = \sigma \bar{q}(x, 0)$ , we first calculate the spectral transform by solving the Zakharov-Shabat spectral problem for the  $2 \times 2$  matrix  $M(k, x, 0)$

$$M_x + ik[\sigma_3, M] = QM, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \quad (8)$$

The two fundamental solutions of (8), say  $M^\pm$ , are determined by

$$\begin{pmatrix} \mu_{11}^+(k, x) \\ \mu_{21}^+(k, x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \begin{array}{c} - \int_x^\infty d\xi q(\xi) \mu_{21}^+(k, \xi) \\ \int_{-\infty}^x d\xi r(\xi) \mu_{11}^+(k, \xi) \exp(2ik(x - \xi)) \end{array} \right)$$

$$\begin{aligned} \begin{pmatrix} \mu_{12}^+(k, x) \\ \mu_{22}^+(k, x) \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left( \int_x^\infty d\xi q(\xi) \mu_{22}^+(k, \xi) \exp(-2ik(x - \xi)) \right. \\ &\quad \left. \int_x^\infty d\xi r(\xi) \mu_{12}^+(k, \xi) \right) \\ \begin{pmatrix} \mu_{11}^-(k, x) \\ \mu_{21}^-(k, x) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left( \int_x^\infty d\xi q(\xi) \mu_{21}^-(k, \xi) \right. \\ &\quad \left. \int_x^\infty d\xi r(\xi) \mu_{11}^-(k, \xi) \exp(2ik(x - \xi)) \right) \\ \begin{pmatrix} \mu_{12}^-(k, x) \\ \mu_{22}^-(k, x) \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left( \int_{-\infty}^x d\xi q(\xi) \mu_{22}^-(k, \xi) \exp(-2ik(x - \xi)) \right. \\ &\quad \left. - \int_x^\infty d\xi r(\xi) \mu_{12}^-(k, \xi) \right) \end{aligned} \tag{9}$$

The first column vector  $\mu_1^+$  of the matrix  $\mathbf{M}^+$  is meromorphic in  $\text{Im}(k) > 0$  where it has a finite number  $N^+$  of poles  $k_n^+$  (assumed to be simple). The second vector  $\mu_2^+$  is holomorphic in  $\text{Im}(k) > 0$ . The vector  $\mu_1^-$  is holomorphic in  $\text{Im}(k) < 0$  while the second one  $\mu_2^-$  is meromorphic in  $\text{Im}(k) < 0$  where it has a finite number  $N^-$  of poles  $k_n^-$  (simple). One can check directly that

$$\text{Res}_{k=k_n^+} \mu_1^+(k) = iC_n^+ \mu_2^+(k_n^+), \tag{10}$$

$$\text{Res}_{k=k_n^-} \mu_2^-(k) = -iC_n^- \mu_1^-(k_n^-), \tag{11}$$

which define the *normalization coefficients*  $C_n^\pm$ .

The function  $\mathbf{M}(k)$  defined as  $\mathbf{M}^+$  in the upper half plane and  $\mathbf{M}^-$  in the lower is then discontinuous on the real  $k$ -axis. Its discontinuity can be expressed simply in terms of  $\mathbf{M}$  itself as

$$\begin{aligned} \mu_1^+ - \mu_1^- &= \exp(2ikx) \alpha^+ \mu_2^+, \\ \mu_2^+ - \mu_2^- &= -\exp(-2ikx) \alpha^- \mu_1^-, \end{aligned} \tag{12}$$

which define the *reflection coefficients*  $\alpha^\pm(k)$ :

$$\alpha^+(k) = \int_{-\infty}^{+\infty} d\xi r(\xi) \mu_{11}^+(k, \xi) \exp(-2ik\xi), \tag{13}$$

$$\alpha^-(k) = \int_{-\infty}^{+\infty} d\xi q(\xi) \mu_{22}^-(k, \xi) \exp(2ik\xi). \tag{14}$$

For future use we define also the transmission coefficients  $\beta^\pm$

$$\beta^+(k) = 1 - \int_{-\infty}^{+\infty} d\xi q(\xi) \mu_{21}^+(k, \xi), \tag{15}$$

$$\beta^-(k) = 1 - \int_{-\infty}^{+\infty} d\xi r(\xi) \mu_{12}^-(k, \xi). \tag{16}$$

The integral equations give the following behaviours at large  $x$ :

$$\begin{pmatrix} \beta^+ & -\exp(-2ikx) \alpha^- / \beta^- \\ 0 & 1/\beta^+ \end{pmatrix} \xleftarrow{-\infty} \mathbf{M}^+ \xrightarrow{+\infty} \begin{pmatrix} 1 & 0 \\ \exp(2ikx) \alpha^+ & 1 \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} 1/\beta^- & 0 \\ -\exp(2ikx) \alpha^+ / \beta^+ & \beta^- \end{pmatrix} \xleftarrow{-\infty} \mathbf{M}^- \xrightarrow{+\infty} \begin{pmatrix} 1 & \exp(-2ikx) \alpha^- \\ 0 & 1 \end{pmatrix} \tag{18}$$

We have also the unitarity relation

$$\alpha^+ \alpha^- + \beta^+ \beta^- = 1. \quad (19)$$

Solving the direct scattering problem consists of solving the integral equations for given  $\mathbf{Q}(x)$  and then calculating the *spectral data*  $\mathcal{S}$ :

$$\mathcal{S} = \{\alpha^\pm(k); k_n^\pm, C_n^\pm, n = 1, \dots, N^\pm\}. \quad (20)$$

In [1] we have proved that the following evolution of these spectral data

$$\begin{aligned} \alpha_t^+ &= 2\omega\alpha^+ - 2im_0^+, \quad \alpha_t^- = -2\omega\alpha^- + 2im_0^-, \\ C_{n,t}^+ &= 2\omega(k_n^+)C_n^+, \quad C_{n,t}^- = -2\omega(k_n^-)C_n^-, \end{aligned} \quad (21)$$

implies that the set  $\{\mathbf{M}^\pm, q\}$  obeys the following

$$q_t = \frac{2i}{\pi} \int_{-\infty}^{+\infty} d\lambda [2ip(\mu_{11}^+ \mu_{12}^+ + \mu_{11}^- \mu_{12}^-) + m_0^-(\mu_{11}^-)^2 \exp(-2i\lambda x) - m_0^+(\mu_{12}^+)^2 \exp(2i\lambda x)]. \quad (22)$$

This eq. is coupled to (8) that is (remember  $\sigma = \pm 1$ )

$$\mu_{11,x} = q\mu_{21}, \quad \mu_{21,x} - 2ik\mu_{21} = \sigma\bar{q}\mu_{11}, \quad \mu_{12}^\pm = \sigma\bar{\mu}_{21}^\mp. \quad (23)$$

Hereabove, we have defined  $p(k, t)$  from  $\omega(k, t)$  through

$$\omega(k, t) = -\frac{2i}{\pi} P \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} p(\lambda, t), \quad (24)$$

where  $P$  denotes the principal valued integral.

The problem now consists of finding the relation between the functions  $\mathbf{M}^\pm(k, x, t)$  and the vector  $(a_1(k, x, t), a_2(k, x, t))$ . By comparison of the behaviours (7) to the above behaviours of  $\mathbf{M}^\pm$  (valid for any value of the external parameter  $t$ ), we deduce

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = J_1 \mu_1^+ + \beta^+ J_2 \mu_2^+ \exp(2ikx). \quad (25)$$

Then we have to determine the entries  $\omega(k, t)$  and  $m_0^\pm(k, t)$  for which the evolutions (6) and (22) are *identical*. This is done by first calculating the quantity  $a_1 \bar{a}_2$  in terms of the  $\mu_{ij}$ 's out of (25), and then by expressing this result in terms of the quantities  $(\mu_{11}^+ \mu_{12}^+ + \mu_{11}^- \mu_{12}^-)$ ,  $(\mu_{11}^-)^2 \exp(-2ikx)$  and  $(\mu_{12}^+)^2 \exp(2ikx)$ . This is possible by using repeatedly the Riemann-Hilbert problem (12) and we obtain [1]

$$\begin{aligned} \omega(k, t) &= \frac{1}{4} i P \int \frac{d\lambda}{\lambda - k} g \left[ \sigma |J_1|^2 (1 + \alpha^+ \alpha^-) + |J_2|^2 \beta^+ \beta^- \right. \\ &\quad \left. + J_1 \bar{J}_2 \alpha^+ \beta^- + \bar{J}_1 J_2 \sigma \alpha^- \beta^+ \right], \end{aligned} \quad (26)$$

$$\begin{aligned} m_0^+(k, t) &= \frac{1}{2} \pi i g \left[ \sigma \bar{J}_1 J_2 \beta^+ (1 - \frac{1}{2} \alpha^+ \alpha^-) - \frac{1}{2} \beta^- (\alpha^+)^2 J_1 \bar{J}_2 \right. \\ &\quad \left. + \frac{1}{2} \alpha^+ \beta^+ \beta^- (\sigma |J_1|^2 - |J_2|^2) \right], \end{aligned} \quad (27)$$

$$\begin{aligned} m_0^-(k, t) &= -\frac{1}{2} \pi i g \left[ J_1 \bar{J}_2 \beta^- (1 - \frac{1}{2} \alpha^+ \alpha^-) - \frac{1}{2} \sigma \beta^+ (\alpha^-)^2 \bar{J}_1 J_2 \right. \\ &\quad \left. + \frac{1}{2} \alpha^- \beta^+ \beta^- (\sigma |J_1|^2 - |J_2|^2) \right]. \end{aligned} \quad (28)$$

In summary, the initial-boundary value problem quoted at the beginning of this section is solved in the following way:

- i) for given  $q(x, 0)$  solve the integral equations (9) to determine the set of spectral data (20) at time zero.
- ii) compute the spectral data at time  $t$  by solving the differential equations (21) where the quantities  $\omega$  and  $m_0^\pm$  are given by (27).
- iii) reconstruct the solution  $q(x, t)$ ,  $a_j(k, x, t)$  of (6,7) by solving the inverse problem, that is by solving the integral Cauchy-Green equation:

$$\mathbf{M}(k) = \mathbf{1} + \frac{1}{2}i\pi \iint \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} \mathbf{M}(\lambda) \mathbf{R}(\lambda), \quad (29)$$

with

$$\begin{aligned} \mathbf{R}(k) = \frac{1}{2}i & \begin{pmatrix} 0 & -\alpha^-(k)\delta^-(k_I) \\ \alpha^+(k)\delta^+(k_I) & 0 \end{pmatrix} \exp(2ik\sigma_3 x) \\ & + 2\pi \sum_{n=1}^{N^\pm} \begin{pmatrix} 0 & C_n^- \delta(k - k_n^-) \\ C_n^+ \delta(k - k_n^+) & 0 \end{pmatrix} \exp(2ik\sigma_3 x). \end{aligned} \quad (30)$$

The solution is then given by (25) and

$$\mathbf{Q} = i[\sigma_3, \mu^{(1)}]. \quad (31)$$

### 3. Stimulated Raman Scattering

Since the work of Chu and Scott [9] considerable interest has been devoted to the so-called SRS equations [10] which we write with the notation of [9]

$$\begin{aligned} Y_\tau - i\delta Y &= -iA_1\bar{A}_2 \exp(-i\Delta\kappa\zeta), \\ A_{1,\zeta} &= -iA_2Y \exp(i\Delta\kappa\zeta), \\ A_{2,\zeta} &= -iA_1\bar{Y} \exp(-i\Delta\kappa\zeta). \end{aligned} \quad (32)$$

Here,  $A_1$  is the scaled slowly varying amplitude of the incident (pump) electromagnetic wave,  $A_2$  is the scaled scattered (Stokes) electromagnetic wave and  $Y$  is the scaled amplitude of the scattering (acoustic) wave. The variables denote the rest frame of the electromagnetic waves (which propagate in the same direction) and  $\Delta\kappa$  is the mismatch wave number.

Let us consider the above equations as being the sharp line limit  $\gamma(\Delta\kappa) \rightarrow \delta(\Delta\kappa)$  of the following more realistic system

$$\begin{aligned} Y_\tau - i\delta Y &= -i \int_{-\infty}^{+\infty} d(\Delta\kappa) \gamma(\Delta\kappa) A_1 \bar{A}_2 \exp(-i\Delta\kappa\zeta), \\ A_{1,\zeta} &= -iA_2Y \exp(i\Delta\kappa\zeta), \\ A_{2,\zeta} &= -iA_1\bar{Y} \exp(-i\Delta\kappa\zeta). \end{aligned} \quad (33)$$

The physical meaning of  $\gamma(\Delta\kappa)$  is that the resonant interaction of waves does not occur strictly at the value  $\Delta\kappa = 0$ , but is distributed around this value. In other words,  $\gamma(\Delta\kappa)$  represents the spreading of the resonant band frequency.

Our system (6) now maps into this system through the following transformation:

$$\sigma = -1, \quad k = \frac{1}{2}\Delta\kappa, \quad g(k) = -2\gamma(\Delta\kappa), \quad x = \zeta, \quad t = \tau, \quad (34)$$

$$q = -iY \exp(-i\delta\tau), \quad a_1 = A_1 \exp(-i\delta\tau), \quad a_2 = A_2 \exp(i\Delta\kappa\zeta). \quad (35)$$

The natural boundary values to associate with this system consist in a normalized pump wave at, say  $+\infty$  ( $A_1 \rightarrow 1$ ) and no Stokes wave coming from the other end ( $-\infty$  where  $A_2 \rightarrow 0$ ), hence

$$a_1 \xrightarrow[x \rightarrow +\infty]{} \exp(-i\delta t), \quad a_2 \xrightarrow[x \rightarrow -\infty]{} 0. \quad (36)$$

The solution of (33) with the above boundary values will then be obtained by setting in the evolution (21)

$$J_1 = \exp(-i\delta t), \quad J_2 = 0. \quad (37)$$

This evolution can then be solved exactly and we have, setting  $E = |\alpha^+|^2 \equiv -\alpha^+ \alpha^-$ ,

$$E(k, t) = \frac{E(k, 0)}{[1 + E(k, 0)] \exp(\pi g t) - E(k, 0)}. \quad (38)$$

At this point it is essential to note that  $g(k)$  is strictly negative [ $g(k) = -2\gamma(\Delta\kappa)$ ]. Hence the solution is valid up to the time when  $E$  becomes singular, that is for  $t < t_s$  with

$$t_s = \frac{1}{2} \gamma \log \left( 1 + \frac{1}{E(k, 0)} \right). \quad (39)$$

As  $t$  approaches  $t_s$ , the modulus of the spectral transform  $\alpha^+(k, t)$  goes to infinity. Then for  $t > t_s$ , the spectral transform does not exist anymore. The question is now: what is the behaviour of the solution  $q(x, t)$  (or  $Y(\zeta, \tau)$ ) as  $t$  approaches  $t_s$ ? The answer is given by computing the first conservation law associated with (6), namely

$$\frac{\partial}{\partial t} |q|^2 = \frac{\partial}{\partial x} \int d\lambda g(\lambda) |a_1|^2. \quad (40)$$

This expression can be integrated on the  $x$ -axis and, thanks to the asymptotic behaviours of  $a_1$  as  $x \rightarrow \pm\infty$  given by (25) and (17,18), it can be also explicitly integrated in time. The result reads

$$\int_{-\infty}^{+\infty} dx \left[ |q(x, t)|^2 - |q(x, 0)|^2 \right] = -\frac{1}{\pi} \int d\lambda \log [1 + E(\lambda, 0) \{1 - \exp(-\pi g t)\}]. \quad (41)$$

This expression proves that

$$t \rightarrow t_s \Rightarrow |q(x, t)| \rightarrow \infty \quad (42)$$

and hence the solution *blows up in finite time*.

Note that this is an ideal case where the system is conservative: when the initial profile of the acoustic wave is made of *pure solitons*. But this is in no way physical: any tiny little deviation (even numerical noise) from the ideal situation produces a finite time singularity. What we actually have is the result that  $t_s$  becomes larger as the initial background noise becomes smaller.

#### 4. Comments

In [1], we have also examined the case of the interaction of a laser beam with a two component plasma (in the fluid approximation) [8]. We have obtained an exact model to explain the total reflexivity due to the stimulated Brillouin scattering of the electromagnetic wave with the acoustic wave.

Then we recovered the results of self-induced transparency [11,2,3], when the physical situation is that of the interaction of an electromagnetic radiation (laser pulse) with a two-level system of atoms or molecules initially at rest. We showed that the system is highly sensitive to the boundary values: if the initial state contains some atoms in the excited state, then the related evolution of the spectral transform is dramatically different and possibly becomes unsolvable.

In the case when the atoms are initially in the excited state (Laser pulse amplification) [12,13], we have proved that the existence of a radiative part in the spectrum (as small as we want) dramatically modifies the asymptotic state of the system (far in the medium). Since physically the input pulse can never have a strictly vanishing radiative part, we obtained a generic behaviour for the penetration of a laser pulse in an excited medium. In particular the positive velocity of the solitons is compatible only with a non zero initial radiative part.

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# INTEGRABLE QUANTUM MAPPINGS AND QUANTIZATION ASPECTS OF INTEGRABLE DISCRETE-TIME SYSTEMS

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**ABSTRACT.** We study a quantum Yang-Baxter structure associated with non-ultralocal lattice models, and discuss the canonical structure of a class of integrable quantum mappings, i.e. canonical transformations preserving the basic commutation relations. As a particular class of solutions we present two examples of quantum mappings associated with the lattice analogues of the KdV and MKdV equations, together with their exact quantum invariants.

## 1. Introduction

Discrete integrable models, in which the spatial dimension is discretized, but the time is continuous, have traditionally played an important role in mathematics and physics, both in the classical as well as in the quantum regime. On the quantum level the algebraic structure of integrable systems is discussed in terms of quantum groups [1]–[3]. The discretized version of such models has played a particular role in this respect, e.g. in the quantum inverse scattering method [4]. The models, in which the time-flow is discretized as well (i.e. integrable lattices or partial difference equations), have been considered on the classical level in a number of papers [5,6]. Recently, they have become of interest in connection with the construction of integrable *mappings*, i.e. finite-dimensional reductions of these integrable lattice equations [7,8]. Their integrability is to be understood in the sense that the discrete time-flow is the iterate of a canonical transformation preserving a suitable symplectic structure, leading to invariants which are in involution with respect to this symplectic form. A theorem à la Liouville then tells us in analogy with the continuous-time situation that one can linearize the discrete-time flow on a hypertorus, which is the intersection of the level sets of the invariants [9]. Integrable mappings have been considered from a slightly different perspective also in the recent literature, cf. [10]–[13].

Integrable two-dimensional lattices arise, both on the classical as well as on the quantum level, as the compatibility conditions of a discrete-time ZS (Zakharov-Shabat) system

$$\mathbf{L}'_n(\lambda) \bullet \mathbf{M}_n(\lambda) = \mathbf{M}_{n+1}(\lambda) \bullet \mathbf{L}_n(\lambda), \quad (1.1)$$

in which  $\lambda$  is a spectral parameter,  $\mathbf{L}_n$  is the lattice translation operator at site  $n$ , and the prime denotes the discrete time-shift corresponding to a translation in the second lattice direction. As  $\mathbf{L}$  and  $\mathbf{M}$ , in the quantum case, depend on operators, the question of operator ordering becomes important. Throughout this paper we impose in the quantum case as a normal order the order which is induced by the lattice enumeration, with  $n$  increasing from the left to the right. Finite-dimensional mappings are obtained from (1.1) by imposing a periodicity condition

$$\mathbf{L}_n(\lambda) = \mathbf{L}_{n+P}(\lambda), \quad \mathbf{M}_n(\lambda) = \mathbf{M}_{n+P}(\lambda), \quad (1.2)$$

for some  $P \in \mathbb{B}$ .

In a recent paper [14], we introduced a novel quantum structure that is appropriate for obtaining an integrable quantization of mappings of the so-called KdV-type, i.e. mappings derived from a lattice version of the KdV equation, cf. [7]. In this paper we will review the construction of such integrable quantum mappings and their quantum invariants and we consider also the quantum mappings associated with the lattice version of the MKdV equation. As was indicated in [8], it turns out that these mappings and their underlying integrable lattices are — on the classical level — symplectic with respect to a so-called *non-ultralocal* Poisson structure [8],[15]. In the continuous-time case such (classical) non-ultralocal  $\mathbf{r}$ -matrix structures have been studied in a number of papers, cf. [16]–[20]. The discrete version of the non-ultralocal Poisson bracket structure reads, cf. [20],

$$\begin{aligned} \{\mathbf{L}_{n,1}, \mathbf{L}_{m,2}\} = & -\delta_{n,m+1} \mathbf{L}_{n,1} \mathbf{s}_{12}^+ \mathbf{L}_{m,2} + \delta_{n+1,m} \mathbf{L}_{m,2} \mathbf{s}_{12}^- \mathbf{L}_{n,1} \\ & + \delta_{n,m} [\mathbf{r}_{12}^+ \mathbf{L}_{n,1} \mathbf{L}_{m,2} - \mathbf{L}_{n,1} \mathbf{L}_{m,2} \mathbf{r}_{12}^-], \end{aligned} \quad (1.3)$$

Throughout this paper we adopt the usual convention that the subscripts  $1, 2, \dots$  in (1.3) denote the factors in a matricial tensor product, i.e.  $\mathbf{A}_{i_1, i_2, \dots, i_M} = \mathbf{A}_{i_1, i_2, \dots, i_M}(\lambda_1, \lambda_2, \dots, \lambda_M)$  denotes a matrix acting nontrivially only on the factors labeled by  $i_1, i_2, \dots, i_M$  of a tensor product  $\otimes_\alpha V_\alpha$ , of vector spaces  $V_\alpha$  and trivially on the other factors, cf. [4,15]. For example, in (1.3), the subscripts  $\alpha, \beta = 1, 2, \dots$  for the operator matrices  $\mathbf{L}_{n,\alpha}$  denote the corresponding factor on which this  $\mathbf{L}_n$  acts (acting trivially on the other factors), i.e.  $\mathbf{L}_{n,1} = \mathbf{L}_n(\lambda_1) \otimes \mathbf{1}$ ,  $\mathbf{L}_{n,2} = \mathbf{1} \otimes \mathbf{L}_n(\lambda_2)$ . We suppress the explicit dependence on the spectral parameter  $\lambda = \lambda_1$  (respectively  $\lambda_2$ ), assuming that each value accompanies its respective factor in the tensor product.

Equation (1.3) defines a proper Poisson bracket provided that the following relations hold for  $\mathbf{s}^\pm = \mathbf{s}^\pm(\lambda_1, \lambda_2)$  and  $\mathbf{r}^\pm = \mathbf{r}^\pm(\lambda_1, \lambda_2)$ :

$$\mathbf{s}_{12}^-(\lambda_1, \lambda_2) = \mathbf{s}_{21}^+(\lambda_2, \lambda_1), \quad \mathbf{r}_{12}^\pm(\lambda_1, \lambda_2) = -\mathbf{r}_{21}^\pm(\lambda_2, \lambda_1), \quad (1.4)$$

to ensure the skew-symmetry, and

$$[\mathbf{r}_{12}^\pm, \mathbf{r}_{13}^\pm] + [\mathbf{r}_{12}^\pm, \mathbf{r}_{23}^\pm] + [\mathbf{r}_{13}^\pm, \mathbf{r}_{23}^\pm] = \mathbf{0}, \quad (1.5)$$

$$[\mathbf{s}_{12}^\pm, \mathbf{s}_{13}^\pm] + [\mathbf{s}_{12}^\pm, \mathbf{r}_{23}^\pm] + [\mathbf{s}_{13}^\pm, \mathbf{r}_{23}^\pm] = \mathbf{0}, \quad (1.6)$$

to ensure that the the Jacobi identities hold for the Poisson bracket (1.3). The relation (1.5) for  $\mathbf{r}^\pm$  is nothing but the usual classical Yang-Baxter equation (CYBE). As a consequence of the ZS system (1.1) we have on the classical level a complete family of invariants of the mapping constructed by introducing the associated monodromy matrix  $\mathbf{T}(\lambda)$ . This matrix is obtained by gluing the elementary translation matrices  $\mathbf{L}_j$  along a line connecting the sites 1 and  $P + 1$  over one period  $P$ , namely

$$\mathbf{T}(\lambda) \equiv \overleftarrow{\prod}_{n=1}^P \mathbf{L}_n(\lambda). \quad (1.7)$$

In order to be able to integrate (1.3) to obtain Poisson brackets for the monodromy matrix we need in addition to these relations the extra relation

$$\mathbf{r}_{12}^+ - \mathbf{s}_{12}^+ = \mathbf{r}_{12}^- - \mathbf{s}_{12}^-. \quad (1.8)$$

In the classical case the traces of powers of the monodromy matrix are invariant under the mapping as a consequence of

$$\mathbf{T}'(\lambda) = \mathbf{M}_{P+1}(\lambda)\mathbf{T}(\lambda)\mathbf{M}_1^{-1}(\lambda) \quad (1.9)$$

and the periodicity condition  $\mathbf{M}_{P+1} = \mathbf{M}_1$ , thus leading to a sufficient number of invariants which are obtained by expanding the traces in powers of the spectral parameter  $\lambda$ . The involution property of the classical invariants follows from the Poisson bracket

$$\{\text{tr } \mathbf{T}(\lambda), \text{tr } \mathbf{T}(\lambda')\} = 0 \quad (1.10)$$

which can be derived from (1.3).

For the quantum mappings we will use the structure of [14,21] which is the quantum analogue of this non-ultralocal Poisson structure. In the continuous-time case such a novel quantization scheme was proposed in [22], in connection with the quantum Toda theory. Similar structures with continuous time flow have been introduced also for the quantum Wess-Zumino-Novikov-Witten (WZNW) theory with discrete spatial variable, cf. [23,24]. When considering discrete-time flows some interesting new features arise, as was indicated in [14,21]. In fact, the conventional point of view that the  $\mathbf{M}$ -part of the Lax equations does not need to be considered explicitly in order to construct quantum invariants, is no longer true. Therefore one needs to establish the complete quantum algebra, containing commutation relations between the  $\mathbf{L}$ -operators as well as between the  $\mathbf{L}$ - and the  $\mathbf{M}$ -operators, and between the  $\mathbf{M}$ -operators themselves. As a consequence, we will find in the quantum mappings under consideration non-trivial quantum corrections in the quantum invariants of the mappings. From an algebraic point of view, the basic algebraic relations for the monodromy matrices which are relevant in the context of non-ultralocal models, are the algebras of currents introduced in various papers in different contexts [25]–[27]. Interestingly enough, the relations between the monodromy matrix and the time-part of the Lax representation are very similar to the relations associated with the description of the cotangent bundle of a quantum group  $(T^*G)_q$  [28].

The outline of this paper is as follows. In §2 we introduce the basic ingredients of the non-ultralocal quantum  $\mathbf{R}, \mathbf{S}$ -structure. In §3 we investigate the canonical structure of quantum gauge- or similarity transformations, leading to (integrable) quantum mappings. This leads to a ‘full’ Yang-Baxter structure including the discrete-time part of the Lax representation. In §4 we present two examples of this structure: quantum mappings associated

with the lattice KdV and with the lattice MKdV equation. In order to establish the quantum integrability of these mappings, we then develop in §5 the ‘full’ quantum structure for the monodromy matrix, and show how to construct commuting families of exact quantum invariants for these mappings.

## 2. Non-ultralocal Yang-Baxter structure

We now define a quantum Yang-Baxter structure that is adequate for the mappings in this paper, that is, discrete-time systems arising (both on the classical as well as quantum level) from compatibility equations of the form of (1.1).

We introduce the *quantum L*-operator  $\mathbf{L}_n(\lambda)$  at each site  $n$  of a one-dimensional lattice, which is a matrix whose entries are quantum operators acting on some properly chosen Hilbert space. The operators  $\mathbf{L}_n(\lambda)$  are supposed to have non-trivial commutation relations between themselves on the same and nearest-neighbour sites only, as follows:

$$\mathbf{R}_{12}^+ \mathbf{L}_{n,1} \bullet \mathbf{L}_{n,2} = \mathbf{L}_{n,2} \bullet \mathbf{L}_{n,1} \mathbf{R}_{12}^-, \quad (2.1)$$

$$\mathbf{L}_{n+1,1} \bullet \mathbf{s}_{12}^+ \mathbf{L}_{n,2} = \mathbf{L}_{n,2} \bullet \mathbf{L}_{n+1,1}, \quad (2.2)$$

$$\mathbf{L}_{n,1} \bullet \mathbf{L}_{m,2} = \mathbf{L}_{m,2} \bullet \mathbf{L}_{n,1}, \quad |n - m| \geq 2. \quad (2.3)$$

These relations are the quantum analogue of the *non-ultralocal* Poisson bracket (1.3). We will show in §4 that the quantum mappings provide examples of such a non-ultralocal quantum  $\mathbf{R}$ -matrix structure.

The compatibility relations of the equations (2.1)-(2.3) lead to the following consistency conditions on  $\mathbf{R}^\pm$  and  $\mathbf{S}$ :

$$\mathbf{R}_{12}^\pm \mathbf{R}_{13}^\pm \mathbf{R}_{23}^\pm = \mathbf{R}_{23}^\pm \mathbf{R}_{13}^\pm \mathbf{R}_{12}^\pm, \quad (2.4)$$

$$\mathbf{R}_{23}^\pm \mathbf{S}_{12}^\pm \mathbf{S}_{13}^\pm = \mathbf{S}_{13}^\pm \mathbf{S}_{12}^\pm \mathbf{R}_{23}^\pm, \quad (2.5)$$

where  $\mathbf{S}_{12}^+ = \mathbf{S}_{21}^-$ . Equation (2.4) is the quantum Yang-Baxter equation (QYBE’s) for  $\mathbf{R}^\pm$  coupled with an additional equation (2.5) for  $\mathbf{S}^\pm$ . They were given for the first time explicitly in [14], but they are implicit in the previous literature [25]–[27], where the relations (2.8) presented below were given in more special situations. For a derivation of (2.4), (2.5) see appendix A.

In order to establish that the structure given by the commutation relations (2.1), (2.3) allows for suitable commutation relations for the monodromy matrix, we need to impose in addition to (2.4), (2.5) that

$$\mathbf{R}_{12}^\pm \mathbf{S}_{12}^\pm = \mathbf{S}_{12}^\mp \mathbf{R}_{12}^\mp. \quad (2.6)$$

Using these relations it is easy to establish that each sign of equations (2.4,2.5) can be combined into a single equation as follows

$$\mathbf{R}_{12}^\pm (\mathbf{R}_{13}^\pm \mathbf{S}_{13}^\pm) \mathbf{S}_{12}^\pm (\mathbf{R}_{23}^\pm \mathbf{S}_{23}^\pm) = (\mathbf{R}_{23}^\pm \mathbf{S}_{23}^\pm) \mathbf{S}_{12}^\mp (\mathbf{R}_{13}^\pm \mathbf{S}_{13}^\pm) \mathbf{R}_{12}^\mp. \quad (2.7)$$

At this point it is useful to introduce the following decomposition of the monodromy matrix (1.6)

$$\mathbf{T} = \mathbf{T}_n^+ \bullet \mathbf{T}_n^-, \quad (2.8)$$

in which

$$\mathbf{T}_n^+(\lambda) = \overleftarrow{\prod}_{j=n+1}^P \mathbf{L}_j(\lambda), \quad \mathbf{T}_n^-(\lambda) = \overleftarrow{\prod}_{j=1}^n \mathbf{L}_j(\lambda), \quad (2.9)$$

First, one derives for the monodromy matrices  $\mathbf{T}_n^+$  and  $\mathbf{T}_n^-$  the following set of relations

$$\mathbf{R}_{12}^+ \mathbf{T}_{n,1}^\pm \bullet \mathbf{T}_{n,2}^\pm = \mathbf{T}_{n,2}^\pm \bullet \mathbf{T}_{n,1}^\pm \mathbf{R}_{12}^-, \quad (2.10)$$

$$\mathbf{T}_{n,1}^+ \bullet \mathbf{S}_{12}^+ \mathbf{T}_{n,2}^- = \mathbf{T}_{n,2}^- \bullet \mathbf{S}_{12}^- \mathbf{T}_{n,1}^+, \quad (2.11)$$

for  $1 \leq n \leq P - 1$ .

Next, taking into account the periodic boundary conditions we obtain for the monodromy matrix the commutation relations

$$\mathbf{R}_{12}^\pm \mathbf{T}_1 \bullet \mathbf{S}_{12}^\pm \mathbf{T}_2 = \mathbf{T}_2 \bullet \mathbf{S}_{12}^- \mathbf{T}_1 \mathbf{R}_{12}^-. \quad (2.12)$$

Some details of the derivation of (2.7,2.8) are presented in appendix B.

### REMARKS

- (a) From the theory of quantum groups [3], we know that the relations (2.7) are the defining relations for a quasi-triangular Hopf algebra  $\mathcal{A}^\pm$ , generated by the entries of  $\mathbf{T}_n^\pm$  and the unit, with defining relations (2.8). The co-algebra structure on  $\mathcal{A}$  is defined with the following coproduct

$$\Delta(\mathbf{T}_n^\pm) = \mathbf{T}_n^\pm \dot{\otimes} \mathbf{T}_n^\pm, \quad (2.13)$$

in which the dot denotes a matrix product. The algebra generated by the unit and  $\mathbf{T}$  is no longer a Hopf algebra; instead  $\mathbf{T}$  and the unit  $\mathbf{1}$  generate a Hopf-ideal  $\mathcal{A}$  [27]

$$\Delta(\mathcal{A}) \subset \mathcal{A}^\pm \otimes \mathcal{A}. \quad (2.14)$$

In fact, the coproduct on the generators  $\mathbf{T}$  is defined as

$$\Delta(\mathbf{T}) = (\mathbf{T}^+ \otimes \mathbf{1}) \bullet (\mathbf{1} \otimes \mathbf{T}) \bullet (\mathbf{T}^- \otimes \mathbf{1}), \quad (2.15)$$

as a consequence of (2.6) and (2.13).

- (b) The classical limit of the quantum structure (2.1), (2.2) is easily obtained by considering the quasi-classical expansion

$$\begin{aligned} \mathbf{S}_{12}^\pm &= \mathbf{1} \otimes \mathbf{1} - h \mathbf{S}_{12}^\pm + \mathcal{O}(h^2), \\ \mathbf{R}_{12}^\pm &= \mathbf{1} \otimes \mathbf{1} + h \mathbf{R}_{12}^\pm + \mathcal{O}(h^2). \end{aligned} \quad (2.16)$$

In this limit the quantum commutation relations (2.1) yield the non-ultralocal Poisson bracket structure given in equations (1.3–1.5).

### 3. Quantum Mappings

We are interested in the canonical structure of discrete-time integrable systems, that is, systems for which the time evolution is given by an iteration of mappings. If the mapping contains quantum operators, the commutation relations with the monodromy or Lax matrices become nontrivial and it is not a priori clear in this case that the Yang-Baxter structure is preserved. Furthermore, the traces of powers of the monodromy matrix are no longer trivially invariant as the cyclic property of the traces is no longer true for operator-valued arguments.

Let us comment why in the discrete-time case the spatial part of the Yang-Baxter equations is not sufficient for quantum integrable systems. When dealing with Yang-Baxter

structures for continuous-time systems, cf. [4,15], it is not necessary to consider explicitly the commutation relations (Poisson brackets on the classical level) involving the  $M$ -operator, i.e. the time-dependent part of the Lax pair. For instance, knowing the commutation relations for the spatial part, i.e. the  $L$ -operator, allows one to construct all relevant objects, namely the (conserved) Hamiltonians of the system as well as the corresponding  $M$ -operators, cf. [15] and [29] for the quantum case. In doing so, one may use the fact that the time evolution is governed by a canonical transformation which leaves the commutation relations between operators invariant. In other words one can reconstruct the allowed integrable continuous-time flows. In the discrete-time case this is no longer true. Firstly, on the lattice there is not a preferred ‘spatial’ versus ‘time-like’ direction, as both components appear on the same footing, and hence, there is no natural impetus to introduce reconstruction formulae expressing one of the Lax operators in terms of the other one. Secondly, the process of integrating an integrable continuous-time flow to a finite-step discrete-time flow is a highly non-trivial procedure, in which – a priori – it is not at all clear that the integrated time flow will lead to nice ‘local’ partial difference equations described by explicit closed-form expressions. Furthermore, in the time-discrete case the symplectic property is not automatically guaranteed, it is not a priori clear that the commutation properties of operators are invariant under the mapping. On the other hand, the symplectic property is a necessary ingredient for mappings to be integrable [9].

However, the explicit investigations of mappings obtained by reduction from integrable partial difference equations reveal that this is actually the case. Probably there is a highly nonlinear self-consistent mechanism at work in these systems. In order to get a grasp on this mechanism we feel that it is necessary to take the  $M$ -part of the Lax or ZS system into account, and investigate the *full* quantum structure involved in these systems, consisting of commutation relations between the  $L$ -part as well as of the  $M$ -part of the Lax pair.

In [14] and [21] we have introduced such a full Yang-Baxter structure taking account of the spatial as well as the time part of the Lax pair. The structure is obtained by supplying in addition to (2.1) the following equations:

$$M_{n+1,1} \bullet S_{12}^+ L_{n,2} = L_{n,2} \bullet M_{n+1,1}, \quad (3.1)$$

$$L'_{n,2} \bullet S_{12}^- M_{n,1} = M_{n,1} \bullet L'_{n,2}, \quad (3.2)$$

and

$$R_{12}^+ M_{n,1} \bullet M_{n,2} = M_{n,2} \bullet M_{n,1} R_{12}^-, \quad (3.3)$$

$$M'_{n,1} \bullet S_{12}^+ M_{n,2} = M_{n,2} \bullet M'_{n,1}. \quad (3.4)$$

The trivial commutation relations are the following

$$M_{n,1} \bullet L_{m,2} = L_{m,2} \bullet M_{n,1}, \quad |n - m| \geq 2, \quad (3.5)$$

$$M_{n+1,1} \bullet L'_{m,2} = L'_{m,2} \bullet M_{n+1,1}, \quad |n - m| \geq 2, \quad (3.6)$$

$$M_{n,1} \bullet M_{m,2} = M_{m,2} \bullet M_{n,1}, \quad |n - m| \geq 2, \quad (3.7)$$

in combination with

$$M''_{n,1} \bullet M_{n,2} = M_{n,2} \bullet M''_{n,1}, \quad (3.8)$$

$$M'_{n+1,1} \bullet M_{n,2} = M_{n,2} \bullet M'_{n+1,1}, \quad (3.9)$$

$$M'_{n+1,1} \bullet L_{n,2} = L_{n,2} \bullet M'_{n+1,1}, \quad (3.10)$$

for multiple applications of the mapping. We shall not specify other commutation relations, as they do not belong to the Yang-Baxter structure. More precisely, one may notice in the explicit examples of §4 that the commutation relations

$$[\mathbf{L}_n \otimes \mathbf{M}_n], \quad [\mathbf{L}_{n+1} \otimes \mathbf{M}_n], \quad [\mathbf{M}_{n+1} \otimes \mathbf{L}'_n], \quad [\mathbf{M}_{n+1} \otimes \mathbf{L}'_{n-1}], \quad [\mathbf{M}_{n+1} \otimes \mathbf{M}_n],$$

are nontrivial, and they depend on the details of the system satisfying the Yang-Baxter equations. However, in order for the Yang-Baxter structure to be preserved under the mapping, we do not need information on these latter commutation relations. Let us now make a more explicit statement on the invariance of the non-ultralocal Yang-Baxter system. The full Yang-Baxter structure consists of two sets of nontrivial commutation relations: (i), equations (2.1), (2.2) and (3.1); and (ii), equations (3.2), (3.3) and (3.4).

Imposing the first set of equations for all  $n$ , and the second set of equations for a fixed value of  $n$  but for all iterates of the mapping and using also trivial commutation relations, it follows that the first set of equations, and in particular the commutation relations between the matrices  $\mathbf{L}_n$ , is invariant under the mapping

$$\mathbf{L}_n \rightarrow \mathbf{L}'_n = \mathbf{M}_{n+1} \mathbf{L}_n \mathbf{M}_n^{-1} \quad (3.11)$$

(see appendix C for some details).

For the quantum mappings under consideration here the operator  $\mathbf{L}_n$  has a composite structure, i.e.

$$\mathbf{L}_n = \mathbf{V}_{2n} \bullet \mathbf{V}_{2n-1} \quad (3.12)$$

and the commutation relations of the Yang-Baxter structure involving the  $\mathbf{L}_n$  can be inferred from the commutation relations among the  $\mathbf{V}_n$  themselves, as well as the commutation relations between the  $\mathbf{V}_n$  and  $\mathbf{M}_m$ . In fact, imposing the commutation relations

$$\mathbf{V}_{n+1,1} \bullet \mathbf{S}_{12}^+(n) \mathbf{V}_{n,2} = \mathbf{V}_{n,2} \bullet \mathbf{V}_{n+1,1}, \quad (3.13)$$

$$\mathbf{R}_{12}^+ \mathbf{V}_{n,1} \bullet \mathbf{V}_{n,2} = \mathbf{V}_{n,2} \bullet \mathbf{V}_{n,1} \mathbf{R}_{12}^-, \quad (3.14)$$

$$\mathbf{V}_{n,1} \bullet \mathbf{V}_{m,2} = \mathbf{V}_{m,2} \bullet \mathbf{V}_{n,1}, \quad |n - m| \geq 2, \quad (3.15)$$

we obtain the relations (2.1) as can be easily verified.

In (3.13) the  $\mathbf{S}_{12}^+(2n)$  is independent of  $n$  and is equal to the  $\mathbf{S}_{12}^+$  occurring in (2.2). For odd values of  $n$ ,  $\mathbf{S}_{12}^+(n)$  may be a different solution of equations (2.1), (2.2) and (2.3). The proof of equation (2.1) from equation (3.14) is essentially the same as the proof in appendix B showing how equation (2.10) is obtained from equations (2.1), (2.2) and (2.3).

Next we impose the commutation relations between the operators  $\mathbf{V}_n$  and  $\mathbf{M}_n$ . The only nonvanishing commutation relations involving  $\mathbf{M}_n$  are taken to be the following ones

$$\begin{array}{ccc} \mathbf{V}_{2n+1} & & \mathbf{V}'_{2n-1} \\ \mathbf{V}_{2n} & \leftrightarrow \mathbf{M}_n \leftrightarrow & \mathbf{V}'_{2n-2} \\ \mathbf{V}_{2n-1} & & \mathbf{V}'_{2n-3} \\ \mathbf{V}_{2n-2} & & \mathbf{V}'_{2n-4} \end{array} \quad (3.16)$$

and in addition we impose simple commutation relations between  $\mathbf{M}_n$  and  $\mathbf{V}_{2n-2}$  and  $\mathbf{V}'_{2n-1}$ , respectively

$$\mathbf{M}_{n+1,1} \bullet \mathbf{S}_{12}^+ \mathbf{V}_{2n,2} = \mathbf{V}_{2n,2} \bullet \mathbf{M}_{n+1,1}, \quad (3.17)$$

$$\mathbf{V}'_{2n-1,2} \mathbf{S}_{12}^- \bullet \mathbf{M}_{n,1} = \mathbf{M}_{n,1} \bullet \mathbf{V}'_{2n-1,2}. \quad (3.18)$$

With the use of (3.8), (3.9) and (3.6) it is straightforward to derive (3.1) and (3.3). Equation (3.10) can also be shown by replacing  $L_n$  by  $M_{n+1}^{-1} L'_n M_n$  and taking account of the invariance of commutation relations under the mapping. The relations (3.7), (3.8) and (3.9) are satisfied by the quantum mappings which will be considered in §4. In §5 we construct commuting families of quantum invariants on the basis of the full Yang-Baxter structure given above.

#### 4. The Quantum Lattice KdV and MKdV System

Here we consider two examples of integrable quantum mappings coming from the lattice analogues of the KdV and MKdV equations.

EXAMPLE A. The first example of a concrete integrable family of quantum mappings that exhibit the structure outlined above is the mapping of the KdV type, that is, mappings arising from the periodic initial value problem of lattice versions of the KdV equation [7]. These are rational mappings of the form  $\mathbb{R}^{2P} \rightarrow \mathbb{R}^{2P} : (\{v_j\}) \mapsto (\{v'_j\})$  where

$$v'_{2j-1} = v_{2j}, \quad v'_{2j} = v_{2j+1} + \frac{\epsilon\delta}{v_{2j}} - \frac{\epsilon\delta}{v_{2j+2}} \quad j = 1, \dots, P, \quad (4.1)$$

imposing the periodicity condition  $v_{i+2P} = v_i$ . The mapping (4.1) has the Casimirs

$$\sum_{j=1}^P v_{2j} = \sum_{j=1}^P v_{2j-1} = c, \quad (4.2)$$

where  $c$  is chosen to be invariant under the mapping, in which case we obtain a  $(2P - 2)$ -dimensional generalization of the McMillan mapping [10].

The mapping (4.1) is obtained by reduction from the partial difference equation

$$u(n, m+1) - u(n+1, m) = q - p + \frac{p^2 - q^2}{p + q + u(n, m) - u(n+1, m+1)} \quad (4.3)$$

for fields  $u$  defined at the sites  $(n, m)$  of a two-dimensional lattice,  $n, m \in \mathbf{Z}$ . Equation (4.3) is completely integrable in the sense that solutions can be obtained via the direct linearization method, that is, by solving a linear integral equation with arbitrary measure and contour [6].

To investigate equation (4.3) one can choose initial data on a staircase consisting of alternating horizontal and vertical steps, i.e.

$$a_{2j} = u(j, j), \quad a_{2j+1} = u(j+1, j), \quad (4.4)$$

and the solution above and below the staircase can be calculated from (4.3) by completing elementary squares. In the case of periodic initial data

$$a_j = a_{j+2P}, \quad P \in \mathbf{B}, \quad (4.5)$$

the complete solution of (4.3) satisfies the periodicity property

$$u(n, m) = u(n+P, m+P). \quad (4.6)$$

The periodic solutions of (4.3) can be obtained via a  $2P$ -dimensional mapping which is defined in terms of the vertical shift:

$$u(n, m) \rightarrow u'(n, m) = u(n, m+1). \quad (4.7)$$

In terms of the reduced variables

$$v_j = p + q + a_j - a_{j+2}, \quad (4.8)$$

the  $2P$ -dimensional mapping is given by (4.1).

To obtain the Yang-Baxter structure it is worthwhile to note that equation (4.1) arises as the compatibility condition of a ZS system (1.1) with

$$\mathbf{L}_j = \mathbf{V}_{2j} \bullet \mathbf{V}_{2j-1}, \quad \mathbf{M}_j = \begin{pmatrix} u_j & 1 \\ \lambda_{2j} & 0 \end{pmatrix}, \quad \mathbf{V}_i = \begin{pmatrix} v_i & 1 \\ \lambda_i & 0 \end{pmatrix}, \quad (4.9)$$

in which  $\lambda_{2j} = k^2 - q^2$ ,  $\lambda_{2j+1} = k^2 - p^2$  and  $\epsilon\delta = p^2 - q^2$ . In fact, from the ZS condition (1.1) one obtains

$$u_j = v_{2j-1} - \epsilon\delta/v_{2j} \quad (4.10)$$

as well as the mapping (4.1). The corresponding classical invariants, obtained by expanding the trace of the monodromy matrix (1.6) in powers of  $k^2$ , are in involution, cf. (1.9), with respect to the Poisson structure [7]

$$\{v_j, v_{j'}\} = \delta_{j+1, j'} - \delta_{j, j'+1}, \quad (4.11)$$

which was obtained using a Legendre transformation on an appropriately chosen Lagrangian [7]. This ensures that the mapping (4.1) is symplectic, i.e. the same Poisson brackets hold also for the primed variables  $v'_j$ . This property can also be checked easily by direct computation. On the basis of this a canonical transformation to action-angle variables can be found [9], thereby showing complete integrability in the sense of Liouville [7,8]. In the quantum case the variables  $v_j$  become hermitian operators on which we impose the following Heisenberg type of commutation relations, as a natural quantization of the Poisson relations (4.11), cf. [14,21],

$$[v_j, v_{j'}] = h(\delta_{j, j'+1} - \delta_{j+1, j'}). \quad (4.12)$$

It is easy to check that the quantum mapping (4.1) is a canonical transformation with respect to these commutation relations.

The special solution of the quantum relations (2.2), (2.3), which constitutes the **R**, **S**-matrix structure for the quantum mapping (4.1), together with the commutation relation (4.12), is given by

$$\begin{aligned} \mathbf{R}_{12}^+ &= \mathbf{R}_{12}^- - \mathbf{S}_{12}^+ + \mathbf{S}_{12}^-, \\ \mathbf{R}_{12}^- &= \mathbf{1} \otimes \mathbf{1} + h \mathbf{P}_{12}/(\mu_1 - \mu_2), \\ \mathbf{S}_{12}^+ &= \mathbf{1} \otimes \mathbf{1} - \frac{h}{\mu_2} \mathbf{F} \otimes \mathbf{E}, \quad \mathbf{S}_{12}^- = \mathbf{S}_{21}^+, \end{aligned} \quad (4.13)$$

in which  $\mu_\alpha = k_\alpha^2 - q^2$ ,  $\alpha = 1, 2$ , and the permutation operator  $\mathbf{P}_{12}$  and the matrices  $\mathbf{E}$  and  $\mathbf{F}$  are given by

$$\mathbf{P}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (4.14)$$

We mention the useful identity

$$\mathbf{R}_{12}^+ = \Lambda_1 \Lambda_2 \mathbf{R}_{12}^- \Lambda_1^{-1} \Lambda_2^{-1}, \quad (4.15)$$

where  $\Lambda_\alpha = \mu_\alpha \mathbf{F} + \mathbf{E}$  ( $\alpha = 1, 2$ ), from which it is evident that it is not strictly necessary to introduce two different  $\mathbf{R}$ -matrices  $\mathbf{R}^\pm$ .

The complete Yang-Baxter structure can now be derived from the mapping (4.1), the relation (4.10) for  $u_j$  and the commutation relation (4.12). In fact, from (4.12) one obtains immediately (3.13) with

$$\mathbf{S}_{12}^+(n) = \mathbf{1} \otimes \mathbf{1} - \frac{h}{\lambda_{n,2}} \mathbf{F} \otimes \mathbf{E} \quad (4.16)$$

with  $\lambda_{2j,2} = k_2^2 - q^2$ ,  $\lambda_{2j-1,2} = k_2^2 - p^2$ , and also (3.14) and (3.15). These relations are at the basis of the  $\mathbf{L}$  part, i.e. (2.1), of the Yang-Baxter structure. To derive the commutation relations (3.16), (3.9) one first checks by explicit calculation that the only nonvanishing commutation relations between the matrix  $\mathbf{M}_n$  and the matrices  $\mathbf{V}_m$ ,  $\mathbf{V}'_m$  are indeed given by (3.16). Furthermore one has the commutation relations

$$[\mathbf{M}_{n+1} - \mathbf{V}_{2n+1} \otimes \mathbf{V}_{2n}] = 0, \quad [\mathbf{M}_{n+1} - \mathbf{V}'_{2n} \otimes \mathbf{V}'_{2n+1}] = 0 \quad (4.17)$$

which with (3.13) and its counterpart in terms of the primed operators immediately yield (3.9). Finally the nontrivial commutation relations (3.2) follow from

$$[\mathbf{M}_n \otimes \mathbf{M}_n] = 0, \quad [\mathbf{M}'_n - \mathbf{V}'_{2n-1} \otimes \mathbf{M}_n] = 0 \quad (4.18)$$

together with (3.9). The trivial commutation relations can be checked in a similar way.

Thus, the mapping (4.1) and its ZS system (4.9) with the commutation relation (4.12) satisfy the complete Yang-Baxter structure treated in §§2 and 3.

#### REMARK

(a). The KdV mappings considered here are the discrete-time analogue of the quantum Volterra system treated in [30]. Such systems are of interest, in connection with discretizations of the Virasoro algebra [31]–[34].

(b). We now consider the example of the MKdV mappings. Another two-parameter family of  $\mathbf{R}$ ,  $\mathbf{S}$ -matrices is generated by

$$\mathbf{R}_{12}(x) = \begin{pmatrix} qx - 1 & 0 & 0 & 0 \\ 0 & x - 1 & q - 1 & 0 \\ 0 & x(q - 1) & q(x - 1) & 0 \\ 0 & 0 & 0 & qx - 1 \end{pmatrix}, \quad \mathbf{S}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (4.19)$$

It is straightforward to check that the matrices of (4.19) obey for spectral parameter  $x = \lambda_1/\lambda_2$  the relations

$$\mathbf{R}_{12} \Lambda_1 \mathbf{S}_{21} \Lambda_2 = \Lambda_2 \mathbf{S}_{12} \Lambda_1 \mathbf{R}_{12}, \quad \mathbf{R}_{12} \Lambda_1^{-1} \mathbf{S}_{12} \Lambda_2^{-1} = \Lambda_2^{-1} \mathbf{S}_{21} \Lambda_1^{-1} \mathbf{R}_{12}, \quad (4.20)$$

in which  $\Lambda_1 = \Lambda(\lambda_1) \otimes \mathbf{1}$ ,  $\Lambda_2 = \mathbf{1} \otimes \Lambda(\lambda_2)$  and

$$\Lambda(\lambda) = \begin{pmatrix} a & b \\ \lambda & d \end{pmatrix}. \quad (4.21)$$

Equation (4.20) then yields a solution of the Yang-Baxter relations (2.2), (2.3) with

$$\begin{aligned}\mathbf{R}_{12}^- &\equiv \mathbf{R}_{12}^-(\lambda_1, \lambda_2) = \mathbf{R}_{12}(\lambda_1/\lambda_2), \\ \mathbf{R}_{12}^+ &= \Lambda_1 \Lambda_2 \mathbf{R}_{12}(\Lambda_1 \Lambda_2)^{-1} = \mathbf{R}_{12}^- - \mathbf{S}_{12}^+ + \mathbf{S}_{12}^-, \\ \mathbf{S}_{12}^+ &= \Lambda_2 \mathbf{S}_{12} \Lambda_2^{-1}, \\ \mathbf{S}_{12}^- &= \mathbf{S}_{12}^+ = \Lambda_1 \mathbf{S}_{12} \Lambda_1^{-1}.\end{aligned}\quad (4.22)$$

As an example of a quantum mapping associated with this solution of the Yang-Baxter equation we consider

$$\begin{aligned}\varphi'_{2n-1} &= \varphi_{2n}, \\ \exp \{\varphi'_{2n}\} &= \frac{(p_{2n} - r) + (p_{2n+1} + r) \exp \{\varphi_{2n+2}\}}{(p_{2n+1} - r) + (p_{2n} + r) \exp \{\varphi_{2n+2}\}} \exp \{\varphi_{2n+1}\} \\ &\quad \times \frac{(p_{2n-1} - r) + (p_{2n} + r) \exp \{\varphi_{2n}\}}{(p_{2n} - r) + (p_{2n-1} + r) \exp \{\varphi_{2n}\}},\end{aligned}\quad (4.23)$$

for  $n = 1, \dots, 2P$ . On the classical level this mapping is obtained by reduction from the lattice version of the MKdV equation [6]:

$$\begin{aligned}(p - r) \frac{w(n, m + 1)}{w(n + 1, m + 1)} - (q - r) \frac{w(n + 1, m)}{w(n + 1, m + 1)} \\ = (p + r) \frac{w(n + 1, m)}{w(n, m)} - (q + r) \frac{w(n, m + 1)}{w(n, m)}\end{aligned}\quad (4.24)$$

Choosing periodic initial values on a staircase of alternating horizontal and vertical steps, i.e.

$$b_{2j} = w(j, j), \quad b_{2j+1} = w(j + 1, j), \quad b_{j+2P} = b_j, \quad (4.25)$$

and defining the mapping in terms of the vertical shift, cf. (4.7) with  $u \rightarrow w$ , we obtain (4.23) in terms of the reduced variables  $\phi_j$  defined by

$$\exp \{\phi_j\} = b_j/b_{j-2}. \quad (4.26)$$

The MKdV mapping (4.23) arises as the compatibility condition of a ZS system (1.1) with  $\mathbf{L}_n = \mathbf{V}_{2n} \bullet \mathbf{V}_{2n-1}$ , cf. (3.6), and

$$\mathbf{V}_n = \Lambda_n \bullet \bar{\mathbf{V}}_n, \quad \bar{\mathbf{V}}_n = \begin{pmatrix} 1 & 0 \\ 0 & \exp \{\varphi_n\} \end{pmatrix}, \quad \Lambda_n = \begin{pmatrix} p_n - r & 1 \\ \lambda & p_n + r \end{pmatrix}. \quad (4.27)$$

in which  $\lambda = k^2 - r^2$ ,  $p_{2n-1} = p$ ,  $p_{2n} = q$  and

$$\mathbf{M}_n = \Lambda_{2n} \bullet \begin{pmatrix} 1 & 0 \\ 0 & \exp \{\gamma_n\} \end{pmatrix}. \quad (4.28)$$

In fact, working out (1.1) with (3.6) and (4.27), (4.28) one finds the mapping (4.23) and

$$\exp \{\gamma_n\} = \frac{(p_{2n} - r) + (p_{2n-1} + r) \exp \{\varphi_{2n}\}}{(p_{2n-1} - r) + (p_{2n} + r) \exp \{\varphi_{2n}\}} \exp \{\varphi_{2n-1}\}. \quad (4.29)$$

At this stage it is worthwhile to note that the  $v_j$  in (4.27) and in (4.9) are related via a gauge transformation of the type

$$\begin{pmatrix} p_j - r & \exp \{\phi_j\} \\ k^2 - r^2 & (p_j + r) \exp \{\phi_j\} \end{pmatrix} = \begin{pmatrix} (p_j - r)b_{j-1} & b_j \\ (k^2 - r^2)b_{j-1} & 0 \end{pmatrix} \begin{pmatrix} v_j & 1 \\ k^2 - p_j^2 & 0 \end{pmatrix} \times \begin{pmatrix} (p_{j-1} - r)b_{j-2} & b_{j-1} \\ (k^2 - r^2)b_{j-2} & 0 \end{pmatrix}^{-1} \quad (4.30)$$

and the Miura transformation relating the KdV and MKdV mappings is given by

$$v_j = (p_{j-1} - r) \frac{b_{j-2}}{b_{j-1}} + (p_j + r) \frac{b_j}{b_{j-1}}. \quad (4.31)$$

In the classical case the mapping is completely integrable with  $P$  integrals in involution with respect to the (invariant) Poisson bracket

$$\{\phi_j, \phi_{j'}\} = \delta_{j', j+1} - \delta_{j', j-1} \quad (4.32)$$

cf. (1.9) and the expansion of  $\text{tr } \mathbf{T}(\lambda)$  in powers of  $k^2$ .

In the quantum case we have the commutation relation

$$[\phi_j, \phi_{j'}] = \hbar (\delta_{j, j'+1} - \delta_{j, j'-1}) \quad (4.33)$$

implying in particular that

$$\exp \{\phi_n\} \exp \{\phi_{n+1}\} = q \exp \{\phi_{n+1}\} \exp \{\phi_n\}, \quad q = e^{-\hbar}. \quad (4.34)$$

Starting from (4.33) we find the commutation relations (3.7), in which the  $\mathbf{R}_{12}^\pm, \mathbf{S}_{12}^\pm$  are given by (4.22) with

$$q = e^{-\hbar}, \quad x = (k_1^2 - r^2)/(k_2^2 - r^2) \quad (4.35)$$

and  $\Lambda_n$  given by (4.27) with  $\lambda = k^2 - r^2$ .

From the commutation relation (4.33), together with the explicit expression (4.29) for  $\exp \{\gamma_n\}$  it is straightforward to derive the remaining relations of the Yang-Baxter structure, i.e. (3.8), (3.9) and (3.2), completely analogously to the case of the KdV-type of mappings. The trivial commutation relations can also be checked directly. Thus with the MKdV-type of mappings we have another example of the complete Yang-Baxter structure presented in §§2 and 3, but here the  $\mathbf{R}_{12}^\pm$  and  $\mathbf{S}_{12}^\pm$  correspond to different (trigonometric) solutions of the Yang-Baxter equations.

## 5. Quantum Invariants

In the classical case the trace of the monodromy matrix yields a sufficient number of invariants which are in involution. In the quantum case the trace is no longer invariant and we have to consider more general families of commuting operators.

Following the treatment of [25], a commuting parameter-family of operators is obtained by taking (for details, cf. appendix D)

$$\tau(\lambda) = \text{tr} (\mathbf{T}(\lambda)\mathbf{K}(\lambda)), \quad (5.1)$$

for any family of numerical matrices  $\mathbf{K}(\lambda)$  obeying the relations

$$\mathbf{K}_1(\mathbf{S}_{12}^-)^{-1} \mathbf{K}_2 \mathbf{R}_{12}^+ = \mathbf{R}_{12}^- \mathbf{K}_2(\mathbf{S}_{12}^+)^{-1} \mathbf{K}_1. \quad (5.2)$$

Expanding (5.1) in powers of the spectral parameter  $\lambda$ , we obtain a set of commuting observables of the quantum system in terms of which we can find a common basis of eigenvectors in the associated Hilbert space. We note that a matrix  $\mathbf{K}(\lambda)$  is commonly introduced in connection with quantum boundary conditions other than periodic ones [25], but in relation to the quantum mappings of the present paper it is essential in the periodic case as well.

Furthermore the Yang-Baxter equations of §3 lead to the following commutation relations between  $\mathbf{M} \equiv \mathbf{M}_{n=1}$  and the monodromy matrix  $\mathbf{T}$ ,

$$\mathbf{T}_1 \bullet \mathbf{M}_1^{-1} \bullet \mathbf{S}_{12}^+ \mathbf{M}_2 = \mathbf{M}_2 \bullet \mathbf{S}_{12}^- \mathbf{T}_1 \bullet \mathbf{M}_1^{-1}. \quad (5.3)$$

Here we use the notation  $\mathbf{M}_1 = \mathbf{M} \otimes \mathbf{1}$ ,  $\mathbf{M}_2 = \mathbf{1} \otimes \mathbf{M}$  as usual for the factors 1 and 2 in the matricial tensor product. The derivation of (5.3) is based on the commutation relation

$$\mathbf{M}_{n,1} \bullet \left( \mathbf{L}_{n+1} \bullet \mathbf{L}_n \bullet \mathbf{M}_n^{-1} \right)_2 = \left( \mathbf{L}_{n+1} \bullet \mathbf{L}_n \bullet \mathbf{M}_n^{-1} \right)_2 \bullet \mathbf{S}_{12}^- \mathbf{M}_{n,1} \quad (5.4)$$

which is easily checked by noting that

$$\mathbf{L}_{n+1} \bullet \mathbf{L}_n \bullet \mathbf{M}_n^{-1} = \mathbf{M}_{n+2}^{-1} \bullet \mathbf{L}'_{n+1} \bullet \mathbf{L}'_n \quad (5.5)$$

and using the commutation relation (3.1b) and the trivial relations (3.3b,c). Then with the use of (3.1a) it is found that

$$\begin{aligned} \mathbf{M}_1 \bullet \mathbf{S}_{12}^+ \left( \mathbf{L}_P \bullet \mathbf{L}_{P-1} \bullet \cdots \bullet \mathbf{L}_1 \bullet \mathbf{M}^{-1} \right)_2 &= \mathbf{L}_{P,2} \bullet \mathbf{M}_1 \bullet \left( \mathbf{L}_{P-1} \bullet \cdots \bullet \mathbf{L}_1 \bullet \mathbf{M}^{-1} \right)_2 \\ &= (\mathbf{L}_P \bullet \cdots \bullet \mathbf{L}_3)_2 \bullet \left( \mathbf{L}_2 \bullet \mathbf{L}_1 \bullet \mathbf{M}^{-1} \right)_2 \bullet \mathbf{S}_{12}^- \bullet \mathbf{M}_1 \end{aligned} \quad (5.6)$$

which is just (5.3).

The commutation relation (2.8) for the monodromy matrices is invariant. This can be shown noting that (2.1) is invariant under the mapping and by repeating the derivation of (2.8), but now with the updated variables  $\mathbf{L}_j'$ . In addition, it follows directly from (5.7) and the commutation relation (5.3). In fact,

$$\begin{aligned} \mathbf{R}_{12}^+ \mathbf{T}'_1 \bullet \mathbf{S}_{12}^+ \mathbf{T}'_2 &= \mathbf{R}_{12}^+ \mathbf{M}_1 \bullet \mathbf{T}_1 \bullet \mathbf{M}_1^{-1} \bullet \mathbf{S}_{12}^+ \mathbf{M}_2 \bullet \mathbf{T}_2 \bullet \mathbf{M}_2^{-1} \\ &= \mathbf{R}_{12}^+ \mathbf{M}_1 \bullet \mathbf{M}_2 \bullet \mathbf{S}_{12}^- \mathbf{T}_1 \bullet \mathbf{M}_1^{-1} \bullet \mathbf{T}_2 \bullet \mathbf{M}_2^{-1} \\ &= \mathbf{M}_2 \bullet \mathbf{M}_1 \bullet \mathbf{S}_{12}^+ \mathbf{R}_{12}^+ \mathbf{T}_1 \bullet \mathbf{M}_1^{-1} \bullet \mathbf{T}_2 \bullet \mathbf{M}_2^{-1} \\ &= \mathbf{M}_2 \bullet \mathbf{M}_1 \bullet \mathbf{S}_{12}^+ \mathbf{R}_{12}^+ \mathbf{T}_1 \bullet \mathbf{S}_{12}^+ \mathbf{T}_2 \bullet \mathbf{M}_2^{-1} \bullet \mathbf{M}_1^{-1} \mathbf{S}_{12}^{-1} \\ &= \mathbf{M}_2 \bullet \mathbf{M}_1 \bullet \mathbf{S}_{21}^- \mathbf{T}_2 \bullet \mathbf{S}_{12}^- \mathbf{T}_1 \bullet \mathbf{R}_{12}^- \mathbf{M}_2^{-1} \bullet \mathbf{M}_1^{-1} \mathbf{S}_{12}^{-1} \\ &= \mathbf{T}'_2 \bullet \mathbf{S}_{21}^+ \mathbf{M}_1 \bullet \mathbf{M}_2 \bullet \mathbf{S}_{12}^- \mathbf{T}_1 \mathbf{M}_1^{-1} \bullet \mathbf{M}_2^{-1} \mathbf{S}_{12}^{+1} \mathbf{R}_{12}^- \\ &= \mathbf{T}'_2 \bullet \mathbf{S}_{12}^- \mathbf{T}_1 \mathbf{R}_{12}^-. \end{aligned} \quad (5.7)$$

Our aim is now to describe the integrability of the quantum mappings of §4 which obey the commutation relations (2.8) and (5.3). For this we need to show that one can find a sufficient family of commuting *invariants* of the mapping. Let us thus use (5.3) to calculate

commuting families of quantum invariants in the case of the KdV and MKdV mappings of §4.

In fact, introducing a tensor

$$\mathbf{K}_{12} = \mathbf{P}_{12}\mathbf{K}_1\mathbf{K}_2, \quad (5.8)$$

where  $\mathbf{P}_{12}$  is the permutation operator satisfying

$$\mathbf{P}_{12}\mathbf{A}_1 = \mathbf{A}_2\mathbf{P}_{12}, \quad \mathbf{P}_{12}\mathbf{A}_2 = \mathbf{A}_1\mathbf{P}_{12}, \quad \mathbf{P}_{12} = \mathbf{P}_{21}, \quad \text{tr}_2 \mathbf{P}_{12} = \mathbf{1}_1 \quad (5.9)$$

for matrices  $\mathbf{A}$  not depending on the spectral parameter, and choosing  $\lambda_1 = \lambda_2$ , we can take the trace over both the left and right hand sides of (5.3) contracting with  $\mathbf{K}_{12}$ . This leads to

$$\text{tr}_{12} \left( \mathbf{K}_{12}(\mathbf{T}\mathbf{M}^{-1})_1 \bullet \mathbf{S}_{12}^+ \mathbf{M}_2 \right) = \text{tr}_2 \left( \mathbf{K}_2 \mathbf{T}_2 M_2^{-1} \text{tr}_1 (\mathbf{P}_{12}\mathbf{K}_2\mathbf{S}_{12}^+) M_2 \right) = \text{tr}(\mathbf{KT}) \quad (5.10)$$

provided that

$$\text{tr}_1(\mathbf{P}_{12}\mathbf{K}_2\mathbf{S}_{12}^+) = \mathbf{1}_2. \quad (5.11)$$

In (5.10)  $\text{tr}_{12} = \text{tr}_1 \text{tr}_2$  denotes the trace over the factors 1 and 2 in the direct product space of matrices, whereas  $\text{tr}_1$  and  $\text{tr}_2$  are restricted to only one of these factors.

Similarly, the right hand side of (5.2) gives, when contracting with  $\text{tr}_{12}\mathbf{K}_{12}$ ,

$$\text{tr}_{12} \left( \mathbf{K}_{12}\mathbf{M}_2 \bullet \mathbf{S}_{12}^-(\mathbf{T}\mathbf{M}^{-1})_1 \right) = \text{tr}_1 \left( \mathbf{K}_1 \mathbf{M}_1 \text{tr}_2 (\mathbf{P}_{12}\mathbf{K}_1\mathbf{S}_{12}^-)(\mathbf{T}\mathbf{M}^{-1})_1 \right) = \text{tr}(\mathbf{KT}') \quad (5.12)$$

provided that

$$\text{tr}_2(\mathbf{P}_{12}\mathbf{K}_1\mathbf{S}_{12}^-) = \mathbf{1}_1. \quad (5.13)$$

In the cases of the KdV and MKdV mappings, it is easy to find common solutions to (5.12) and (5.13). For example, using (4.5) for KdV we find

$$\mathbf{K}(\lambda) = \mathbf{1} + \frac{h}{\lambda} \mathbf{S}_-, \quad \mathbf{S}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (5.14)$$

and from (4.5) and (5.14) it is straightforward to show that the  $\mathbf{K}(\lambda)$  of (5.14) satisfies the condition (5.2) for a commuting parameter family of operators.

In the case of the MKdV mapping with the use of the relation

$$\mathbf{S}_{12}^- = \mathbf{1} \otimes \mathbf{1} + (q - 1) \left( \mathbf{\Lambda}_1 \bullet \mathbf{S}_- \bullet \mathbf{\Lambda}_1^{-1} \otimes \mathbf{S}_- \right) \quad (5.15)$$

cf. (4.19) and (4.20), we find the solution

$$\mathbf{K}(\lambda) = \mathbf{1} + (q^{-1} - 1) \mathbf{S}_- \bullet \mathbf{\Lambda} \bullet \mathbf{S}_- \bullet \mathbf{\Lambda}^{-1}, \quad \mathbf{\Lambda} = \begin{pmatrix} q - r & 1 \\ \lambda & q + r \end{pmatrix}, \quad \lambda = k^2 - r^2 \quad (5.16)$$

and again the  $\mathbf{K}(\lambda)$  in combination with the  $\mathbf{R}_{12}^\pm, \mathbf{S}_{12}^\pm$  of (4.19), (4.20) satisfies (5.2).

Hence, in the case of the KdV and MKdV mappings we have obtained a commuting family of quantum invariants that can be evaluated expanding  $\tau(\lambda) = \text{tr}(\mathbf{K}(\lambda)\mathbf{T}(\lambda))$  in powers of  $k^2$ .

For instance, in the KdV mapping the explicit expression of the invariants can be inferred from

$$\tau(\lambda) =: \left( \prod_{j=1}^{2P} v_j \right) \left[ 1 + \sum_{\substack{1 \leq J_1 < \dots < J_N \leq 2P \\ J_{\nu+1} - J_{\nu} \geq 2, J_1 - J_N + 2P \geq 2}} \prod_{\nu=1}^N \frac{\hat{\lambda}_{J_{\nu}}}{v_{J_{\nu}+1} v_{J_{\nu}}} \right] : \quad (5.17)$$

leading to a full family of commuting invariants. In (5.17)  $::$  denotes the normal ordering of the operators  $v_j$  in accordance with their enumeration, and  $\hat{\lambda}_J = \lambda_J$  for  $J \neq 2P$ ,  $\hat{\lambda}_{2P} = \lambda_{2P} + h$ . Thus the quantum effect is only visible in the boundary terms associated with the factor  $1/(v_1 v_{2P})$ .

As a very simple example we give the quantum invariant of the original McMillan mapping [10], i.e. (4.1) for  $P = 2$ , namely

$$x' = y, \quad y' = -x - 2\epsilon\delta y/(\epsilon^2 - y^2), \quad (5.18)$$

for  $x = v_1 - \epsilon$ ,  $y = v_2 - \epsilon$  (choosing  $c = 2\epsilon$ ) and where  $[y, x] = h$ , having the invariant

$$\mathcal{I} = (\epsilon^2 - y^2)(\epsilon^2 - x^2) - (\epsilon\delta + h)(2yx - \epsilon\delta). \quad (5.19)$$

The invariant  $\mathcal{I}$  can be viewed as a Hamiltonian generating a continuous-time interpolating flow,  $\dot{x} = h^{-1}[\mathcal{I}, x]$ ,  $\dot{y} = h^{-1}[\mathcal{I}, y]$ , whose solutions can be considered to be parametrized in terms of what we could call a quantum version of the Jacobi elliptic functions. More general two-dimensional quantum mappings have been studied in [35].

**REMARK.** The construction of quantum mappings can be generalized to a larger class of models, namely those associated with the lattice Gel'fand-Dikii hierarchy [36] as was explicitly shown in [21]. In this case the **R**, **S**-matrix structure is provided by the following solutions of the equations (2.4,2.5) and (2.3)

$$\begin{aligned} \mathbf{R}_{12}^+ &= \mathbf{R}_{12}^- - \mathbf{S}_{12}^+ + \mathbf{S}_{12}^- = \boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2 \mathbf{R}_{12}^- (\boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_2)^{-1}, \\ \mathbf{R}_{12}^- &= \mathbf{1} + h \frac{\mathbf{P}_{12}}{\lambda_1 - \lambda_2}, \quad \mathbf{S}_{12}^+ = \mathbf{S}_{21}^- = \mathbf{1} - \frac{h}{\lambda_2} \sum_{i=1}^{N-1} \mathbf{E}_{Ni} \otimes \mathbf{E}_{iN}, \\ \mathbf{P}_{12} &= \sum_{i,j=1}^N \mathbf{E}_{i,j} \otimes \mathbf{E}_{j,i}, \quad \boldsymbol{\Lambda}(\lambda) = \lambda \mathbf{E}_{N,1} + \sum_{i=1}^{N-1} \mathbf{E}_{i,i+1}, \end{aligned} \quad (5.20)$$

and  $\mathbf{E}_{i,j}$  being the basic generators of  $GL(n)$ , i.e.  $(\mathbf{E}_{i,j})_{kl} = \delta_{ik}\delta_{jl}$ . The relations of the full Yang-Baxter structure of §§2 and 3 are satisfied for matrices  $\mathbf{L}_n = \mathbf{V}_{2n} \bullet \mathbf{V}_{2n-1}$  and  $\mathbf{M}_n$  of the form

$$\mathbf{V}_n = \boldsymbol{\Lambda}_n \left( \mathbf{1} + \sum_{i>j=1}^N v_{i,j}(n) \mathbf{E}_{i,j} \right), \quad \mathbf{M}_n = \boldsymbol{\Lambda}_{2n} \left( \mathbf{1} + \sum_{i>j=1}^N \mathbf{M}_{i,j}(n) \mathbf{E}_{i,j} \right) \quad (5.21)$$

in which  $\boldsymbol{\Lambda}_n = \boldsymbol{\Lambda}(\lambda_n)$ ,  $\lambda_{2n} = \lambda_{2n+2} = \mu$  and the  $v_{i,j}(n)$  satisfy the following Heisenberg type of commutation relations

$$[v_{i,j}(n), v_{k,l}(m)] = h (\delta_{n,m+1} \delta_{k,j+1} \delta_{i,N} \delta_{l,1} - \delta_{m,n+1} \delta_{i,l+1} \delta_{k,N} \delta_{j,1}). \quad (5.22)$$

A commuting family of quantum invariants is obtained from

$$\tau(\lambda) = \text{tr}(\mathbf{K}(\lambda)\mathbf{T}(\lambda)), \quad \mathbf{K}(\lambda) = \mathbf{1} + (N-1) \frac{h}{\lambda} \mathbf{E}_{N,N} \quad (5.23)$$

but for  $N \geq 3$  the expansion in powers of  $k^2$  does not yield enough invariants to establish complete integrability in the quantum case. For  $N \geq 3$  one needs additional invariants corresponding to higher order commuting families of operators. The construction of these higher order invariants needs the application of the fusion procedure [29,37,38] and is left to a future publication [39].

## Appendix A

In order to derive the compatibility relations for the Yang-Baxter matrices  $\mathbf{R}$  and  $\mathbf{S}$ , i.e. (2.2), from the commutation relations (2.1) for the Lax matrices  $\mathbf{L}$ , we encounter four different types of combinations of matrices  $\mathbf{L}$ . Embedding the  $\mathbf{L}$  matrices in a tensorial product of three copies of the matrix algebra, i.e.  $\mathbf{L}_{n,j}$ ,  $j = 1, 2, 3$  acting on vector spaces  $V \otimes V \otimes V$ , and denoting

$$\mathbf{L}_{n,1} = \mathbf{L}_n \otimes \mathbf{1} \otimes \mathbf{1}, \quad \mathbf{L}_{n,2} = \mathbf{1} \otimes \mathbf{L}_n \otimes \mathbf{1}, \quad \mathbf{L}_{n,3} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{L}_n,$$

we can distinguish the following types of combinations of matrices  $\mathbf{L}$  involving only coinciding and/or neighbouring sites:

*i)*  $\underline{\mathbf{L}_1 \equiv \mathbf{L}_{n+2,1}}$ ,  $\underline{\mathbf{L}_2 \equiv \mathbf{L}_{n+1,2}}$ ,  $\underline{\mathbf{L}_3 \equiv \mathbf{L}_{n,3}}$ . In this case, no conditions on the  $\mathbf{R}$  or  $\mathbf{S}$  matrices will appear, because

$$\mathbf{L}_1 \bullet \mathbf{S}_{12}^+ \mathbf{L}_2 \bullet \mathbf{S}_{23}^+ \mathbf{L}_3 = \mathbf{L}_3 \bullet \mathbf{L}_2 \bullet \mathbf{L}_1, \quad (\text{A.1})$$

independently of the order in which the relation (2.1) is applied.

*ii)*  $\underline{\mathbf{L}_1 \equiv \mathbf{L}_{n+1,1}}$ ,  $\underline{\mathbf{L}_2 \equiv \mathbf{L}_{n+1,2}}$ ,  $\underline{\mathbf{L}_3 \equiv \mathbf{L}_{n,3}}$ . In this case, we have on the one hand

$$\mathbf{R}_{12}^+ \mathbf{L}_1 \bullet \mathbf{L}_2 \bullet \mathbf{S}_{13}^+ \mathbf{S}_{23}^+ \mathbf{L}_3 = \mathbf{L}_2 \bullet \mathbf{L}_1 \bullet \mathbf{R}_{12}^- \mathbf{S}_{13}^+ \mathbf{S}_{23}^+ \mathbf{L}_3, \quad (\text{A.2})$$

whereas on the other hand we find

$$\begin{aligned} \mathbf{R}_{12}^+ \mathbf{L}_1 \bullet \mathbf{S}_{13}^+ \mathbf{L}_2 \bullet \mathbf{S}_{23}^+ \mathbf{L}_3 &= \mathbf{R}_{12}^+ \mathbf{L}_3 \bullet \mathbf{L}_1 \bullet \mathbf{L}_2 \\ &= \mathbf{L}_3 \bullet \mathbf{L}_2 \bullet \mathbf{L}_1 \mathbf{R}_{12}^-, \end{aligned} \quad (\text{A.3})$$

Comparing relations (A.2) and (A.3), we have

$$\mathbf{R}_{12}^- \mathbf{S}_{13}^+ \mathbf{S}_{23}^+ = \mathbf{S}_{23}^+ \mathbf{S}_{13}^+ \mathbf{R}_{12}^-, \quad (\text{A.4})$$

which after relabelling of the vector spaces becomes (2.5).

*iii)*  $\underline{\mathbf{L}_1 \equiv \mathbf{L}_{n+1,1}}$ ,  $\underline{\mathbf{L}_2 \equiv \mathbf{L}_{n,2}}$ ,  $\underline{\mathbf{L}_3 \equiv \mathbf{L}_{n,3}}$ . Take for this case the combination

$$\mathbf{R}_{23}^+ \mathbf{L}_1 \bullet \mathbf{S}_{12}^+ \mathbf{L}_2 \bullet \mathbf{S}_{13}^+ \mathbf{L}_3 = \mathbf{L}_1 \bullet \mathbf{R}_{23}^+ \mathbf{S}_{12}^+ \mathbf{S}_{13}^+ \bullet \mathbf{L}_2 \bullet \mathbf{L}_3, \quad (\text{A.5})$$

and compare this with

$$\begin{aligned} \mathbf{R}_{23}^+ \mathbf{L}_1 \bullet \mathbf{S}_{12}^+ \mathbf{L}_2 \bullet \mathbf{S}_{13}^+ \mathbf{L}_3 &= \mathbf{R}_{23}^+ \mathbf{L}_2 \bullet \mathbf{L}_3 \bullet \mathbf{L}_1 \\ &= \mathbf{L}_3 \bullet \mathbf{L}_2 \bullet \mathbf{L}_1 \mathbf{R}_{23}^- \\ &= \mathbf{L}_1 \bullet \mathbf{S}_{13}^+ \mathbf{S}_{12}^+ \mathbf{L}_3 \bullet \mathbf{L}_2 \mathbf{R}_{23}^- \\ &= \mathbf{L}_1 \bullet \mathbf{S}_{13}^+ \mathbf{S}_{12}^+ \mathbf{R}_{23}^+ \mathbf{L}_2 \bullet \mathbf{L}_3, \end{aligned} \quad (\text{A.6})$$

yielding (2.5) with the + sign.

*iv)*  $\underline{\mathbf{L}_1 \equiv \mathbf{L}_{n,1}}$ ,  $\underline{\mathbf{L}_2 \equiv \mathbf{L}_{n,2}}$ ,  $\underline{\mathbf{L}_3 \equiv \mathbf{L}_{n,3}}$ . In this case we have the standard braiding type of argument to find as a sufficient condition the quantum  $\mathbf{R}$ -matrix relations (2.4) for  $\mathbf{R}^+$  and  $\mathbf{R}^-$ .

## Appendix B

In this appendix we establish the commutation relations between the monodromy matrices  $\mathbf{T}$  and  $\mathbf{T}_n^+, \mathbf{T}_n^-$  of (2.8), using the fundamental commutation relations of the matrices  $\mathbf{L}_n$ .

Using (2.1), (2.2) we can establish

$$\begin{aligned} \mathbf{R}_{12}^+ \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{n,1} \bullet \mathbf{L}_{n+1,2} \bullet \mathbf{L}_{n,2} &= \mathbf{R}_{12}^+ \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{n+1,2} \bullet \mathbf{S}_{21}^+ \mathbf{L}_{n,1} \bullet \mathbf{L}_{n,2} \\ &= \mathbf{L}_{n+1,2} \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{R}_{12}^- \mathbf{S}_{21}^+ \mathbf{L}_{n,1} \bullet \mathbf{L}_{n,2}, \end{aligned} \quad (\text{B.1})$$

which, by imposing the relation (2.6), reduces to

$$\begin{aligned} &= \mathbf{L}_{n+1,2} \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{S}_{12}^+ \mathbf{L}_{n,2} \bullet \mathbf{L}_{n,1} \mathbf{R}_{12}^-, \\ &= \mathbf{L}_{n+1,2} \bullet \mathbf{L}_{n,2} \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{n,1} \mathbf{R}_{12}^-. \end{aligned} \quad (\text{B.2})$$

By repeated application of (B.1) and (B.2) together with (2.6) one shows that

$$\begin{aligned} &\mathbf{R}_{12}^+ \mathbf{L}_{P,1} \bullet \dots \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{P,2} \bullet \dots \bullet \mathbf{L}_{n+1,2} \\ &= \mathbf{R}_{12}^+ \mathbf{L}_{P,1} \bullet \mathbf{L}_{P-1,1} \bullet \mathbf{L}_{P,2} \bullet \mathbf{L}_{P-2,1} \bullet \mathbf{L}_{P-1,2} \bullet \dots \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{n+2,2} \bullet \mathbf{L}_{n+1,2} \\ &= \mathbf{R}_{12}^+ \mathbf{L}_{P,1} \bullet \mathbf{L}_{P,2} \bullet \mathbf{S}_{21}^+ \mathbf{L}_{P-1,1} \bullet \mathbf{L}_{P-1,2} \bullet \mathbf{S}_{21}^+ \bullet \mathbf{L}_{P-2,1} \bullet \dots \bullet \mathbf{L}_{n+2,2} \bullet \mathbf{S}_{21}^+ \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{n+1,2} \\ &= \mathbf{L}_{P,2} \bullet \mathbf{L}_{P,1} \mathbf{S}_{12}^+ \bullet \mathbf{L}_{P-1,2} \bullet \mathbf{L}_{P-1,1} \mathbf{S}_{12}^+ \bullet \mathbf{L}_{P-2,2} \bullet \dots \bullet \mathbf{L}_{n+2,1} \mathbf{S}_{12}^+ \bullet \mathbf{L}_{n+1,2} \bullet \mathbf{L}_{n+1,1} \mathbf{R}_{12}^- \\ &= \mathbf{L}_{P,2} \bullet \mathbf{L}_{P-1,2} \bullet \dots \bullet \mathbf{L}_{n+1,2} \bullet \mathbf{L}_{P,1} \bullet \mathbf{L}_{P-1,1} \bullet \dots \bullet \mathbf{L}_{n+1,1} \mathbf{R}_{12}^-, \end{aligned} \quad (\text{B.3})$$

leading to (2.10) for  $\mathbf{T}_n^+$ . A similar argument can be applied for  $\mathbf{T}_n^-$ . Furthermore (2.11) is derived from (2.2), (2.3) by noting that

$$\begin{aligned} &\mathbf{L}_{P,1} \bullet \dots \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{S}_{12}^+ \mathbf{L}_{n,2} \bullet \dots \bullet \mathbf{L}_{1,2} \\ &= \mathbf{L}_{P,1} \bullet \dots \bullet \mathbf{L}_{n+2,1} \bullet \mathbf{L}_{n,2} \bullet \mathbf{L}_{n+1,1} \bullet \mathbf{L}_{n-1,2} \bullet \dots \bullet \mathbf{L}_{1,2} \\ &= \mathbf{L}_{n,2} \bullet \dots \bullet \mathbf{L}_{2,2} \bullet \mathbf{L}_{P,1} \bullet \mathbf{L}_{1,2} \bullet \mathbf{L}_{P-1,1} \bullet \dots \bullet \mathbf{L}_{n+1,1} \\ &= \mathbf{L}_{n,2} \bullet \dots \bullet \mathbf{L}_{2,2} \bullet \mathbf{L}_{1,2} \mathbf{S}_{21}^+ \mathbf{L}_{P,1} \bullet \dots \bullet \mathbf{L}_{n+1,1}, \end{aligned} \quad (\text{B.4})$$

where in the last step we have used the commutation relation

$$\mathbf{L}_{P,1} \bullet \mathbf{L}_{1,2} = \mathbf{L}_{1,2} \bullet \mathbf{S}_{21}^+ \mathbf{L}_{P,1} \quad (\text{B.5})$$

taking into account the periodic boundary conditions.

Finally, from (2.10)-(2.11) we immediately obtain

$$\begin{aligned} \mathbf{R}_{12}^+ \mathbf{T}_1 \bullet \mathbf{S}_{12}^+ \mathbf{T}_2 &= \mathbf{R}_{12}^+ \mathbf{T}_{n,1}^+ \bullet \mathbf{T}_{n,2}^+ \bullet \mathbf{S}_{21}^+ \mathbf{T}_{n,1}^- \bullet \mathbf{T}_{n,2}^- \\ &= \mathbf{T}_{n,2}^+ \bullet \mathbf{T}_{n,1}^+ \bullet \mathbf{R}_{12}^- \mathbf{S}_{21}^+ \mathbf{T}_{n,1}^- \bullet \mathbf{T}_{n,2}^- \\ &= \mathbf{T}_{n,2}^+ \bullet \mathbf{T}_{n,1}^+ \bullet \mathbf{S}_{12}^+ \mathbf{T}_{n,2}^- \bullet \mathbf{T}_{n,1}^- \mathbf{R}_{12}^- \\ &= \mathbf{T}_2 \bullet \mathbf{S}_{21}^+ \mathbf{T}_1 \mathbf{R}_{12}^-, \end{aligned} \quad (\text{B.6})$$

which is (2.12).

## Appendix C

We prove that (3.1),(3.2), together with (3.4) are sufficient to ensure that the basic commutation relations between the matrices  $\mathbf{L}_n$ , (2.1), are preserved under the mapping

$$\mathbf{L}_n \mapsto \mathbf{L}'_n = \mathbf{M}_{n+1} \bullet \mathbf{L}_n \bullet \mathbf{M}_n^{-1}.$$

In fact,

$$\begin{aligned}
L'_{n+1,1} \bullet S_{12}^+ L'_{n,2} &= L'_{n+1,1} \bullet S_{12}^+ M_{n+1,2} \bullet L_{n,2} \bullet M_{n,2}^{-1} \\
&= M_{n+1,2} \bullet L'_{n+1,1} \bullet L_{n,2} \bullet M_{n,2}^{-1} \\
&= M_{n+1,2} \bullet M_{n+2,1} \bullet L_{n+1,1} \bullet M_{n+1,1}^{-1} \bullet L_{n,2} \bullet M_{n,2}^{-1} \\
&= M_{n+1,2} \bullet M_{n+2,1} \bullet L_{n+1,1} S_{12}^+ L_{n,2} \bullet M_{n+1,1}^{-1} \bullet M_{n,2}^{-1} \\
&= L'_{n,2} \bullet M_{n,2} \bullet L'_{n+1,1} \bullet M_{n,2}^{-1} \\
&= L'_{n,2} \bullet L'_{n+1,1},
\end{aligned} \tag{C.1}$$

and similarly

$$\begin{aligned}
R_{12}^+ L'_{n,1} \bullet L'_{n,2} &= R_{12}^+ M_{n+1,1} \bullet L_{n,1} \bullet M_{n,1}^{-1} \bullet L'_{n,2} \\
&= R_{12}^+ M_{n+1,1} \bullet L_{n,1} \bullet L'_{n,2} \bullet M_{n,1}^{-1} (S_{12}^-)^{-1} \\
&= R_{12}^+ M_{n+1,1} \bullet M_{n+1,2} \bullet S_{12}^- L_{n,1} \bullet L_{n,2} \bullet M_{n,2}^{-1} \bullet M_{n,1}^{-1} (S_{12}^-)^{-1} \\
&= M_{n+1,2} \bullet M_{n+1,1} \bullet S_{12}^+ R_{12}^+ L_{n,1} \bullet L_{n,2} \bullet M_{n,2}^{-1} \bullet M_{n,1}^{-1} (S_{12}^-)^{-1} \\
&= M_{n+1,2} \bullet M_{n+1,1} \bullet S_{12}^+ L_{n,2} \bullet L_{n,1} \bullet M_{n,1}^{-1} \bullet M_{n,2}^{-1} (S_{12}^+)^{-1} R_{12}^- \\
&= M_{n+1,2} \bullet L_{n,2} \bullet M_{n+1,1} \bullet L_{n,1} \bullet M_{n,1}^{-1} \bullet M_{n,2}^{-1} (S_{12}^+)^{-1} R_{12}^- \\
&= L'_{n,2} \bullet M_{n,2} \bullet L'_{n,1} \bullet M_{n,2}^{-1} (S_{12}^+)^{-1} R_{12}^- \\
&= L'_{n,2} \bullet L'_{n,1} R_{12}^-.
\end{aligned} \tag{C.2}$$

Finally, we have that

$$\begin{aligned}
M'_{n+1,1} \bullet S_{12}^+ L'_{n,2} &= M'_{n+1,1} \bullet S_{12}^+ M_{n+1,2} \bullet L_{n,2} \bullet M_{n,2}^{-1} \\
&= M_{n+1,2} \bullet M'_{n+1,1} \bullet L_{n,2} \bullet M_{n,2}^{-1} \\
&= M_{n+1,2} \bullet L_{n,2} \bullet M'_{n+1,1} \bullet M_{n,2}^{-1} \\
&= L'_{n,2} \bullet M'_{n+1,1}.
\end{aligned} \tag{C.3}$$

It is straightforward to check along similar lines that all trivial commutation relations remain so after applying the mapping.

## Appendix D

In order to show that (5.2) provides a commuting family of operators, let us give an argument similar to the one given by Sklyanin in [25]. Denoting by  $\mathbf{T}_i$  and  $\mathbf{K}_i$ , ( $i = 1, 2$ ) the monodromy matrix and the matrix  $\mathbf{K}$  respectively for two different values of the spectral parameter  $\lambda_1$  and  $\lambda_2$ , and acting in two different factors of a matricial tensor product, we have on the one hand, assuming that  $[\mathbf{K}_i \otimes \mathbf{T}_j] = 0$ ,  $i, j = 1, 2$ ,

$$\begin{aligned}
\tau_1 \bullet \tau_2 &= \text{tr}_1 (\mathbf{T}_1 \mathbf{K}_1) \bullet \text{tr}_2 (\mathbf{T}_2 \mathbf{K}_2) = \text{tr}_{12} \left( \mathbf{T}_1 \mathbf{K}_1 \bullet {}^{t_2} \mathbf{T}_2 {}^{t_2} \mathbf{K}_2 \right) \\
&= \text{tr}_{12} \left( {}^{t_2} (\mathbf{T}_1 \bullet S_{12}^+ \mathbf{T}_2) \hat{\iota}_1 ({}^{t_1} \mathbf{K}_1 \bullet {}^{t_{12}} (S_{12}^+)^{-1} {}^{t_2} \mathbf{K}_2) \right) \\
&= \text{tr}_{12} \left( R_{12}^{+1} \mathbf{T}_2 \bullet S_{21}^+ \mathbf{T}_1 R_{12}^- {}^{t_{12}} ({}^{t_1} \mathbf{K}_1 \bullet {}^{t_{12}} (S_{12}^+)^{-1} {}^{t_2} \mathbf{K}_2) \right) \\
&= \text{tr}_{12} \left( \mathbf{T}_2 \bullet S_{21}^+ \mathbf{T}_1 \hat{\iota}_1 ({}^{t_{12}} R_{12}^{+1} {}^{t_1} \mathbf{K}_1 \bullet {}^{t_{12}} (S_{12}^+)^{-1} {}^{t_2} \mathbf{K}_2 {}^{t_{12}} R_{12}^-) \right),
\end{aligned} \tag{D.1}$$

whereas on the other hand we have

$$\begin{aligned}\tau_2 \bullet \tau_1 &= \text{tr}_2(\mathbf{T}_2 \mathbf{K}_2) \bullet \text{tr}_1(\mathbf{T}_1 \mathbf{K}_1) \\ &= \text{tr}_{12} \left( {}^{t_1}(\mathbf{T}_2 \bullet \mathbf{S}_{21}^+ \mathbf{T}_1) \hat{\mathbf{t}}_2 {}^{t_2} \mathbf{K}_2 \bullet {}^{t_{12}}(\mathbf{S}_{21}^+)^{-1} {}^{t_1} \mathbf{K}_1 \right) \\ &= \text{tr}_{12} \left( \mathbf{T}_2 \bullet \mathbf{S}_{21}^+ \mathbf{T}_1 \hat{\mathbf{t}}_{12} {}^{t_2} \mathbf{K}_2 \bullet {}^{t_{12}}(\mathbf{S}_{21}^+)^{-1} {}^{t_1} \mathbf{K}_1 \right)\end{aligned}\quad (\text{D.2})$$

from which it is clear that (D.1) and (D.2) can be identified provided that we have the following condition on the matrices  $\mathbf{K}$

$${}^{t_1} \mathbf{K}_1 {}^{t_{12}}(\mathbf{S}_{12}^+)^{-1} {}^{t_2} \mathbf{K}_2 {}^{t_{12}} \mathbf{R}_{12}^- = {}^{t_{12}} \mathbf{R}_{12}^+ {}^{t_2} \mathbf{K}_2 {}^{t_{12}}(\mathbf{S}_{21}^+)^{-1} {}^{t_1} \mathbf{K}_1. \quad (\text{D.3})$$

Equation (D.3) is a very general condition for operator valued matrices  $\mathbf{K}$  of which the entries commute with the entries of  $\mathbf{T}$ , which is sufficient to ensure that the  $\mathbf{T}(\lambda)$  form a parameter family of commuting operators. For numerical matrices  $\mathbf{K}(\lambda)$  (D.3) leads to the condition (5.2) given in the main text.

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## DARBOUX TRANSFORMATIONS IN (2 + 1)-DIMENSIONS

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**ABSTRACT.** The two-dimensional sine-Gordon equation discovered by Konopelchenko and Rogers is shown to have a broad class of solutions obtained by means of a Darboux-like transformation. These solutions are expressed in terms of pfaffians of the solutions of the associated spectral problem. A related two-dimensional modified Korteweg-de Vries equation is also discussed.

### 1. Introduction

There has been a good deal of interest in recent years in integrable systems in (2 + 1)-dimensions. Such systems have dimensional reductions to known integrable equations in (1 + 1)-dimensions. If the two spatial variables appear on an equal footing and hence allow such reductions in either variable one calls the (2 + 1)-dimensional system a strong generalization of the (1 + 1)-dimensional system. The most well-known example is the Davey-Stewartson (DS) equations [1,2] which strongly generalize the nonlinear Schrödinger equation.

For the Korteweg-de Vries (KdV) equation the most famous (2 + 1)-dimensional generalization, the Kadomtsev-Petviashvili (KP) equation [3], is only a weak generalization. Physically the KP equation arises in situations where one dimensional motion governed by the KdV equation is weakly perturbed in a perpendicular direction. A second, less well studied, generalization of the KdV equation is the Novikov-Veselov (NV) equation [4]

$$u_t + u_{xxx} + u_{yyy} + 3(\Phi_{xx}u)_x + 3(\Phi_{yy}u)_y = 0, \quad u = \Phi_{xy}. \quad (1)$$

On the lines  $y = x$  and  $y = -x$  (1) reduces to the KdV equation (this is equivalent to eliminating one or other variable after a rotation of spatial variables through  $\pi/4$ ). In this way one sees that (1) is a strong generalization of the KdV equation.

The property of strong two-dimensionality seems to be closely related to the existence of localized, exponentially decaying solutions (two-dimensional solitons or dromions). Indeed, this is the key feature which leads to the existence of the underlying plane-wave structures of such solutions in the DS equations [5] and in the NV equations [6,7]. See also [8] for a discussion of this idea in a more general context. Furthermore the apparent lack of such soliton solutions of the (weakly two dimensional) KP equation supports this contention.

In this talk we will describe a technique for constructing a class of solutions to another example of the above type found recently by Konopelchenko and Rogers [9] which is a strong generalization of the sine-Gordon (sG) equation. Despite the preceding remarks it

has not been possible, at least as yet, to determine localized solutions within this class. The basic ingredient in our approach is a classical result due to Moutard [10] which predates, and includes as a reduction, the famous Darboux transformation [11] (see [7] for a more on the interesting history of Moutard's result). This result is needed in place of the Darboux transformation in application of the technique used by Wadati, Sanuki and Konno [12] to obtain a Bäcklund transformation for the sG equation and as a result of this approach it will be seen that, from this particular mathematical viewpoint at least, the Konopelchenko-Rogers (KR) equations appear to be a very natural strong generalization of the sG equation.

By considering the change of dependent variables that is suggested by this approach we will determine the Hirota form of the KR equations. This Hirota form was found to be one of the 'unknown' Hirota forms of modified KdV-type to pass the 'three-soliton test' of Hietarinta [13]. This discovery also led to the realization that another of Hietarinta's unknown Hirota forms was that of another strong two dimensional generalization, this time of the modified KdV equation, which is related to the NV equations by a Miura-like transformation. This system will also be discussed briefly.

## 2. Gauge Transformation

The system of Konopelchenko and Rogers may be written as

$$\left( \frac{\theta_{Xt}}{\sin \theta} \right)_X - \left( \frac{\theta_{Yt}}{\sin \theta} \right)_Y - \frac{\theta_Y \rho_{Xt} - \theta_X \rho_{Yt}}{\sin^2 \theta} = 0, \quad (2)$$

$$\left( \frac{\rho_{Xt}}{\sin \theta} \right)_X - \left( \frac{\rho_{Yt}}{\sin \theta} \right)_Y - \frac{\theta_Y \theta_{Xt} - \theta_X \theta_{Yt}}{\sin^2 \theta} = 0. \quad (3)$$

In the reduction  $\rho \rightarrow 0$ ,  $\theta \rightarrow \theta(X, t)$  or  $\theta(Y, t)$ , and with an appropriate constant of integration, we get the sG equation

$$\theta_{Xt} = \sin \theta \quad \text{or} \quad \theta_{Yt} = \sin \theta.$$

The system (2), (3) was found in [9] as the compatibility conditions for the triad of operators ('Loewner system')

$$\begin{aligned} L_1 &= \partial_X - S\partial_Y, \\ L_2 &= \partial_t\partial_Y - V\partial_Y - W_Y, \\ L_3 &= \partial_t\partial_X - V\partial_X - W_X, \end{aligned}$$

where

$$S = - \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \quad (4)$$

and  $V$  and  $W$  are  $2 \times 2$  matrices with entries depending on  $\theta$  and  $\rho$ , the forms of which are not presented as they will not be used in what follows.

The operators  $L_1$ ,  $L_2$  and  $L_3$  are compatible in the weak sense;

$$[L_1, L_2] = S_Y L_2, \quad [L_1, L_3] = S_X L_3, \quad [L_2, L_3] = 0,$$

when (2), (3) are satisfied.

Here we will not use this formulation of the linear problem, rather we will use a Lax pair which is gauge equivalent to a pair of  $L_1, L_2, L_3$ . To do this we observe that the matrix  $S = S_\theta$  may be written in terms of the rotation matrix

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

and reflection matrix

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$S_\theta = R_\theta J = JR_{-\theta}.$$

The gauge is chosen to be a ‘half-rotation and reflexion’

$$g := S_{\theta/2}$$

for which  $g^2 = I$ . We find that

$$L := g^{-1}L_1g = \partial_X + J\partial_Y + Q$$

in which

$$Q = \frac{1}{2} \begin{pmatrix} 0 & (\theta_X + \theta_Y) \\ -(\theta_X - \theta_Y) & 0 \end{pmatrix}$$

is an operator of two-dimensional ZS/AKNS-type.

If we rotate axes  $(X, Y) \rightarrow (x, y)$  so that  $\partial_x = \partial_X + \partial_Y, \partial_y = \partial_X - \partial_Y$  and define

$$\Theta^{(x)} = -\frac{\rho_y + \theta_{yt} \cos \theta}{2 \sin \theta}, \quad (5)$$

$$\Theta^{(y)} = \frac{\rho_x - \theta_{xt} \cos \theta}{2 \sin \theta}, \quad (6)$$

then we get

$$L = \begin{pmatrix} \partial_x & \frac{1}{2}\theta_x \\ -\frac{1}{2}\theta_y & \partial_y \end{pmatrix},$$

$$M := g^{-1}(L_2 - JL_3)g = \begin{pmatrix} \partial_y \partial_t + \Theta^{(y)} & \frac{1}{2}\theta_y \partial_t \\ -\frac{1}{2}\theta_x \partial_t & \partial_x \partial_t + \Theta^{(x)} \end{pmatrix},$$

and we see that one of the operators  $L_i$  is redundant since

$$g^{-1}(L_2 + JL_3)g = \partial_t L.$$

The requirement that  $L$  and  $M$  commute (on the kernel of  $L$  and  $M$ ) gives  $\Theta_y^{(x)} = \Theta_x^{(y)}$ , and, defining  $\Theta$  so that  $\Theta^{(x)} = \Theta_x$  and  $\Theta^{(y)} = \Theta_y$ , gives

$$\theta_{xyt} + \frac{1}{2}(\theta_x \Theta_y + \theta_y \Theta_x) = 0, \quad (7)$$

and

$$\Theta_{xy} = \frac{1}{4}(\theta_x\theta_{yt} + \theta_{xt}\theta_y). \quad (8)$$

Solving (5), (6) for the derivatives of  $\rho$  we get the integrable pair

$$\rho_x = 2\Theta_x \sin \theta + \theta_{xt} \cos \theta, \quad (9)$$

$$\rho_y = -2\Theta_y \sin \theta - \theta_{yt} \cos \theta, \quad (10)$$

through which (7), (8) are equivalent to (2), (3).

To simplify the notation and to free  $\theta$  for more conventional usage, we rescale  $\theta \rightarrow 2u$  and  $\Theta \rightarrow v_t$ . In these variables the Lax pair is

$$\mathbf{L} = \begin{pmatrix} \partial_x & u_x \\ -u_y & \partial_y \end{pmatrix}, \quad (11)$$

$$\mathbf{M} = \begin{pmatrix} \partial_y \partial_t + v_{yt} & u_y \partial_t \\ -u_x \partial_t & \partial_x \partial_t + v_{xt} \end{pmatrix}, \quad (12)$$

the compatibility conditions for which are

$$u_{xyt} + u_x v_{yt} + u_y v_{xt} = 0 \quad (13)$$

and

$$v_{xy} = u_x u_y. \quad (14)$$

For more details of the above calculation see [14,15]. In [15] a version of the dressing method is used to construct solutions and the initial value problem for (13), (14) is discussed.

### 3. Complex Scalar System

We now generalize a method used in [12] to obtain the Bäcklund transformation for the sG equation. Let

$$\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix},$$

be a common solution of  $\mathbf{L}\phi = \mathbf{M}\phi = 0$  for  $\mathbf{L}$  and  $\mathbf{M}$  given by (11) and (12) respectively. If we write  $\mathbf{L}\phi = 0$  in component form we get

$$\phi_x^1 + u_x \phi^2 = 0, \quad (15)$$

$$\phi_y^2 - u_y \phi^1 = 0. \quad (16)$$

Now we define the complex quantity

$$\psi = \phi^1 + i\phi^2, \quad (17)$$

and take appropriate combinations of the  $y$ - and  $x$ -derivatives of (15) and (16) respectively to obtain the complex scalar equation

$$\psi_{xy} + U\psi = 0, \quad (18)$$

in which the potential is related to that in  $\mathbf{L}\phi = 0$  by

$$U = -iu_{xy} + u_x u_y. \quad (19)$$

This coincides with the calculation performed for the sG equation in [12] in the limit  $y \rightarrow x$ . In this reduction (18) becomes the time independent Schrödinger equation and it was the Darboux transformation for that equation that was utilized in [12]. In the present case, the transformation relating copies of (18) with different potentials is due to Moutard [10] but the mechanism by which solutions are constructed is identical.

The result of Moutard in most conveniently expressed by means of the skew-product

$$S[a, b] = \int_{(x_0, y_0)}^{(x, y)} W_x[a, b] dx - W_y[a, b] dy \quad (20)$$

in which  $W_x$  and  $W_y$  denote the wronskians of their arguments with  $x$ - and  $y$ -derivatives respectively. If we take  $a$  and  $b$  to satisfy (18) then

$$(W_x[a, b])_y = -(W_y[a, b])_x = a_y b_x - a_x b_y,$$

so that the integral in (20) is exact, making  $S[a, b]$  independent of the choice of contour.

The result of Moutard is as follows: let  $\rho$  and  $\psi$  be solutions of (18) and define  $A = \rho$  and  $B = S[\rho, \psi]$ . Then

$$\tilde{\psi} = B/A, \quad (21)$$

satisfies the equation

$$\tilde{\psi}_{xy} + \tilde{U}\tilde{\psi} = 0 \quad (22)$$

in which the new potential is

$$\tilde{U} = U + 2(\ln A)_{xy}. \quad (23)$$

In terms of  $u$  (23) gives

$$\tilde{u}_x \tilde{u}_y - i\tilde{u}_{xy} = u_x u_y - iu_{xy} + 2(\ln A)_{xy},$$

and taking real and imaginary parts we find

$$\tilde{u}_{xy} = u_{xy} + i(\ln A/\bar{A})_{xy}$$

and

$$\tilde{u}_x \tilde{u}_y = u_x u_y + [\ln(A\bar{A})]_{xy}.$$

Finally, using (14) and an integration in which we choose arbitrary functions of integration to vanish, we have

$$\tilde{u} = u + i \ln(A/\bar{A}), \quad (24)$$

$$\tilde{v} = v + \ln(A\bar{A}). \quad (25)$$

The same procedure for the components of  $\mathbf{M}\phi = 0$  yields

$$\psi_{xyt} + U\psi_t + V\psi = 0, \quad (26)$$

where  $\psi$  and  $U$  are as in (18) and

$$V = v_{xy} + i(u_x v_y + u_y v_x). \quad (27)$$

The compatibility condition for (18) and (26) is exceedingly simple;

$$U_t = V, \quad (28)$$

and the real and imaginary parts of (28) give (13) and (14).

#### 4. A Class of Solutions

To determine solutions of (13), (14) we consider iteration of the Moutard transformation. If  $U_0$  is related by (19) to  $u_0$ , a ('seed') solution of (13), (14), and  $u_N$  is similarly related to  $U_N$ , the  $N^{\text{th}}$  iterant of  $U_0$  under the Moutard transformation, then  $u_N$  will also satisfy (13), (14).

It may be shown [6,7] that the quantity  $A$ , that determines the relationship between the  $U_0$  and  $U_N$  as in (23), after such iterations is expressed in terms of pfaffians determined by the skew-product  $S$  defined by (20). The other quantity  $B$  which, through (21), is used to determine the solution of the transformed equation (22) may be expressed similarly but we will not consider that further here. In order to give the definition of  $A$  and exhibit examples we first recall some basic facts about pfaffians.

An  $N \times N$  skew-symmetric matrix  $\mathbf{M}$  is such that  $\det \mathbf{M}$  is a perfect square if  $N$  is even;  $\det \mathbf{M} = (\text{Pfaffian}(\mathbf{M}))^2$  or  $\det \mathbf{M} = 0$  if  $N$  is odd. To be precise, when  $N$  is even,

$$\text{Pfaffian}(\mathbf{M}) = \sum_{\sigma} \epsilon(\sigma)(\sigma(1) \sigma(2))(\sigma(3) \sigma(4)) \dots (\sigma(n-1) \sigma(N)), \quad (29)$$

in which the sum is over those permutations  $\sigma$  of  $\{1, \dots, N\}$ , with parity  $\epsilon(\sigma)$ , which satisfy

$$\begin{aligned} \sigma(1) &< \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(N) \\ \sigma(1) &< \sigma(3) < \dots < \sigma(N-1), \end{aligned}$$

and where  $(i j)$  denotes the  $(i, j)$ th entry of  $\mathbf{M}$ .

Given the skew-product  $S$ , and  $N$  solutions  $\theta_i$  ( $i = 1, \dots, N$ ) of (18), one obtains an  $N \times N$  skew-symmetric matrix  $\mathbf{M} = (S[\theta_i, \theta_j])$ , from which, for  $N$  even, we have  $A = \text{Pfaffian}(\mathbf{M})$ . For  $N$  odd it is necessary to form an augmented matrix

$$\mathbf{M}' = \left( \begin{array}{c|c} & \theta_1 \\ \mathbf{M} & \vdots \\ \hline -\theta_1 & \cdots & -\theta_N & 0 \end{array} \right),$$

and then  $A = \text{Pfaffian}(\mathbf{M}')$ . For  $N = 1$  we get

$$\mathbf{M} = (S[\theta_1, \theta_1]) = (0) \quad \text{and} \quad \mathbf{M}' = \left( \begin{array}{c|c} 0 & \theta_1 \\ -\theta_1 & 0 \end{array} \right),$$

giving  $\det \mathbf{M}' = \theta_1^2$  and  $A = \theta_1$  in agreement with the description of the Moutard transformation in §3. For  $N = 2$

$$A = S[\theta_1, \theta_2],$$

and for  $N = 3$

$$\begin{aligned} A &= \begin{vmatrix} S[\theta_1, \theta_2] & S[\theta_1, \theta_3] & \theta_1 \\ & S[\theta_2, \theta_3] & \theta_2 \\ & & \theta_3 \end{vmatrix} \\ &= S[\theta_1, \theta_2]\theta_3 - S[\theta_1, \theta_3]\theta_2 + \theta_1 S[\theta_2, \theta_3]. \end{aligned}$$

In summary then, we have solutions

$$u = u_0 + i \ln(A/\bar{A}) = u_0 - 2 \tan^{-1}(\Im(A)/\Re(A)), \\ v = v_0 + \ln(A\bar{A}),$$

of (13), (14) for any given solution  $(u_0, v_0)$  of this system and where

$$A = \text{Pfaffian}(S[\theta_i, \theta_j]), \quad \theta_k = \phi_k^1 + i\phi_k^2$$

in which, for  $k = 1, \dots, N$ ,  $\phi_k^1$  and  $\phi_k^2$  satisfy

$$\phi_x^1 + u_{0x}\phi^2 = 0, \quad (30)$$

$$\phi_y^2 - u_{0y}\phi^1 = 0, \quad (31)$$

$$\phi_{yt}^1 + v_{0yt}\phi^1 + u_{0y}\phi_t^2 = 0, \quad (32)$$

$$\phi_{xt}^2 + v_{0xt}\phi^2 - u_{0x}\phi_t^1 = 0. \quad (33)$$

The simplest type of solutions that may be obtained in this way arise by choosing the "vacuum seed"  $(u_0, v_0) = (0, 0)$ . Now (30) and (31) give

$$\phi_k^1 = \phi_k^1(y, t) \quad \text{and} \quad \phi_k^2 = \phi_k^2(x, t),$$

while (32) and (33) give

$$\phi_k^1 = Y_k(y) + T_k^1(t) \quad \text{and} \quad \phi_k^2 = X_k(x) + T_k^2(t),$$

for arbitrary functions  $X_k$ ,  $Y_k$ ,  $T_k^1$  and  $T_k^2$  ( $k = 1, \dots, N$ ). These solutions seem not to describe propagating waves.

The simplest case which permits travelling wave solutions is that in which  $u_0 = 0$  and  $v_0$  is non-zero;

$$v_{0xt} = \lambda \quad \text{and} \quad v_{0yt} = \mu,$$

where  $\lambda$  and  $\mu$  are constants. Now we may choose solutions of (30)–(33) of the form

$$\phi_k^1 = \alpha_k \exp[p_k y - (\mu/p_k)t] \quad \text{and} \quad \phi_k^2 = \beta_k \exp[q_k x - (\lambda/q_k)t],$$

for constants  $p_k$ ,  $q_k$ ,  $\alpha_k$  and  $\beta_k$ , from which we obtain solutions describing the interaction of quasi one-dimensional kinks. For example, if we take  $N = 1$  we get

$$u = -2 \tan^{-1} \left[ \exp \left( \phi^2 / \phi^1 \right) \right] \\ = -2 \tan^{-1} \left\{ \frac{\beta}{\alpha} \exp [qx - py - (\lambda/q - \mu/p)t] \right\}. \quad (34)$$

If we consider (34) at a fixed time and let  $(x, y) \rightarrow \infty$  along a ray in the half-plane defined by

$$qx - py > (\lambda/q - \mu/p)t$$

then  $u \rightarrow -\text{sgn}(\beta/\alpha)\pi$  while if the limit is taken in the other half-plane then  $u \rightarrow 0$ . Thus we see that (34) is a kink orientated parallel to the line  $qx - py = 0$ . Note that here, because of the choice of scaling of  $u$ , a kink has a difference of  $\pi$  between its asymptotic states rather

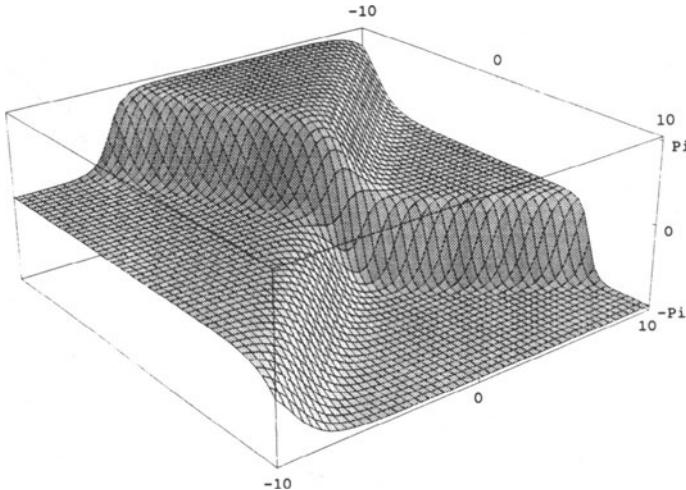


Fig. 1. Interaction of quasi one-dimensional kink solutions

than the more usual  $2\pi$ , and because of the extra degree of freedom the plane affords it is no longer meaningful to distinguish between kinks ('increasing') and anti-kinks ('decreasing').

In the case  $N = 2$ ,  $A$  is given by

$$S[\phi_1^1 + i\phi_1^2, \phi_2^1 + i\phi_2^2] = S[\phi_1^1, \phi_2^1] - S[\phi_1^2, \phi_2^2] + i(S[\phi_1^1, \phi_2^2] + S[\phi_1^2, \phi_2^1]).$$

As stated following its definition, the integral in each term is contour independent and in the above expression the two integrals in the real part are most easily evaluated along contours parallel to the  $x$ - and  $y$ -axes respectively giving

$$\begin{aligned} \Re(A) &= \alpha_1 \alpha_2 \frac{p_2 - p_1}{p_2 + p_1} \exp[(p_1 + p_2)y - \mu(1/p_1 + 1/p_2)t] \\ &\quad - \beta_1 \beta_2 \frac{q_2 - q_1}{q_2 + q_1} \exp[(q_1 + q_2)x - \lambda(1/q_1 + 1/q_2)t]. \end{aligned} \quad (35)$$

The integrands in the imaginary part are total derivatives and we get

$$\begin{aligned} \Im(A) &= \alpha_1 \beta_2 \exp[p_1 y + q_2 x - (\mu/p_1 + \lambda/q_2)t] \\ &\quad + \beta_1 \alpha_2 \exp[q_1 x + p_2 y - (\lambda/q_1 + \mu/p_2)t]. \end{aligned} \quad (36)$$

The resulting expression for  $u = -2 \tan^{-1}[\Im(A)/\Re(A)]$  is continuous provided  $\Re(A)$  is of one sign for all  $(x, y, t)$ . A necessary and sufficient condition for this is that

$$\operatorname{sgn}\left(\alpha_1 \alpha_2 \frac{p_2 - p_1}{p_2 + p_1}\right) = -\operatorname{sgn}\left(\beta_1 \beta_2 \frac{q_2 - q_1}{q_2 + q_1}\right).$$

A typical two-kink solution obtained when the parameters satisfy this constraint is shown in Figure 1.

## 5. Hirota Forms and a modified Novikov-Veselov equation

In the light of the results of the previous sections one sees that a natural change of variables to effect the bilinearization of (13), (14) is

$$u = u_0 + i \ln(G/F), \quad v = v_0 + \ln(GF), \quad (37)$$

where  $G$  and  $F$  are complex conjugates of one another. Introducing this change into (14) we get

$$v_{0xy} - u_{0x}u_{0y} + (FG)^{-1}[D_xD_y - i(u_{0x}D_y + u_{0y}D_x)]G \bullet F = 0. \quad (38)$$

We suppose that  $(u_0, v_0)$  is itself a solution of (13), (14) and so from (38) we find the bilinear equation

$$[D_xD_y - i(u_{0x}D_y + u_{0y}D_x)]G \bullet F = 0. \quad (39)$$

Here  $D_x$ ,  $D_y$  and  $D_t$  as usual denote Hirota derivatives defined by

$$D_a^i D_b^j A \bullet B = \left. \frac{\partial^i}{\partial \alpha^i} \frac{\partial^j}{\partial \beta^j} A(a + \alpha, b + \beta) B(a - \alpha, b - \beta) \right|_{\alpha=\beta=0}.$$

Now considering (13) in a similar way we get

$$\begin{aligned} & u_{0xyt} + u_{0x}v_{0yt} + u_{0y}v_{0xt} \\ & + (FG)^{-1}[i(D_xD_yD_t + v_{0xt}D_y + v_{0yt}D_x) + u_{0x}D_yD_t + u_{0y}D_yD_t]G \bullet F \\ & + (FG)^{-2}(-iD_tG \bullet F)[D_xD_y - i(u_{0x}D_y + u_{0y}D_x)]G \bullet F = 0. \end{aligned}$$

Since we suppose that  $(u_0, v_0)$  satisfies (13), using (39) we get a second bilinear equation

$$[D_xD_yD_t + v_{0xt}D_y + v_{0yt}D_x - i(u_{0x}D_yD_t + u_{0y}D_yD_t)]G \bullet F = 0. \quad (40)$$

The pair (39), (40) are the Hirota form of (13), (14).

For the particular class of solutions discussed at the end of §4 we take  $u_0 = 0$ ,  $v_{0xt} = \lambda$  and  $v_{0yt} = \mu$ . For this choice the Hirota form simplifies to become

$$(D_xD_yD_t + \lambda D_y + \mu D_x)G \bullet F = 0, \quad (41)$$

$$D_xD_yG \bullet F = 0. \quad (42)$$

It is of note that this pair were found earlier by a very different approach. In his search for (possibly) integrable systems by identifying Hirota equations which have three-soliton solutions Hietarinta [13] obtained the system (41), (42) but did not determine the nonlinear system from which it arose. It is now clear where these bilinear equations come from.

In a subsequent paper [16] it was shown that (41), (42) passed a bilinear version of the Painlevé test providing further evidence for the integrability of (13), (14).

The above connection with the work of Hietarinta was discovered during conversations at this NATO workshop and at the same time Hietarinta pointed out the Hirota form (equivalent to)

$$(D_t + D_x^3 + D_y^3)G \bullet F = 0, \quad (43)$$

$$D_xD_yG \bullet F = 0, \quad (44)$$

which passes the three-soliton test [13] and bilinear Painlevé test [16]. Again the nonlinear version of this system was unknown but with knowledge of how to treat the first example it is easy to employ the change of variables (37) (in the reverse direction) to obtain

$$u_t + u_{xxx} + u_{yyy} + 3u_x v_{xx} + 3u_y v_{yy} - u_x^3 - u_y^3 = 0, \quad (45)$$

$$v_{xy} = u_x u_y. \quad (46)$$

It will be shown below that this system is related to the NV equation (1) by a transformation which reduces to the Miura transformation in the limit  $y \rightarrow x$  and for this reason we refer to (45), (46) as modified Novikov-Veselov equations (mNV).

It has been seen that (19) defines a relation between the potential ( $U$ ) in (18) and the potential ( $u$ ) in (15), (16). Since (18) is the spectral problem associated with NV and we suspect that (15), (16) is the spectral problem for (45), (46) it is likely to provide the link between mNV and NV that we seek. Indeed if we define  $U = \Phi_{xy}$ , (c.f. (1)) and use (46), (19) gives

$$\Phi = v - iu. \quad (47)$$

Then if we suppose that  $(u, v)$  satisfy (45), (46) then it is straightforward to verify that  $\Phi$  satisfies

$$\Phi_{xyt} + \Phi_{xxxxy} + \Phi_{yyyyy} + 3(\Phi_{xx}\Phi_{xy})_x + 3(\Phi_{yy}\Phi_{xy})_y = 0,$$

as required and so (47) defines the Miura-like transformation between mNV and NV.

Note that when  $y = x$  (45), (46) gives the *potential* mKdV equation

$$u_t + 2(u_{xxx} + 2u_x^3) = 0, \quad (48)$$

while (1) — with  $U$  replacing  $u$  to avoid a conflict of notation — gives the KdV equation

$$U_t + 2(U_{xxx} + 3(U^2)_x) = 0. \quad (49)$$

In this reduction (47) gives

$$U = u_x^2 - iu_{xx}, \quad (50)$$

which is precisely the Miura transformation linking (48) and (49).

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## DARBOUX THEOREMS AND THE KP HIERARCHY

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**ABSTRACT.** Generalizations of the classical Darboux theorem are established for arbitrary ordinary matrix differential operators. Darboux transformations may be regarded as gauge transformations, where the gauge operator is a first order differential operator parametrized by an eigenfunction. Adjoint Darboux transformations triggered by adjoint eigenfunctions are introduced via transposition of the operators. The composition of a Darboux and an adjoint Darboux transformation leads to the notion of binary transformations, which are triggered by pairs of eigenfunctions and adjoint eigenfunctions. A formalism involving pseudo-differential symbols is used to give a general formulation of these transformations for arbitrary scattering operators. It is shown how exact solutions of the multicomponent KP hierarchy are generated from these transformations.

### 1. Introduction

Since the mid 1970's Darboux transformations (DTs) have been shown to be a powerful tool for studying the nonlinear integrable systems of soliton theory. Darboux's original result [1] is a transformation between Schrödinger equations with different potentials. Crum [2] considered the iteration of the DT, which can be formulated in a compact way involving the Wronskian determinant of eigenfunctions. The key fact leading to the spectacular properties of soliton systems is their connection to linear scattering problems. For the celebrated Korteweg-de Vries (KdV) equation the associated linear problem is the eigenvalue equation for the Schrödinger operator. Hence, it is not surprising that Darboux's result has implications for the KdV. In particular, it was shown by Crum that the DT adds an eigenvalue to the spectrum of the Schrödinger operator. On the level of the nonlinear equation this means that solitons are added by this transformation [3], and Crum's Wronskian formulas for the iterated DT correspond to compact Wronskian representations of the  $n$ -soliton solutions of the KdV [4].

Generalizations of these ideas for other spectral equations and their associated soliton systems have been successfully applied to a variety of systems. This includes spectral problems for ordinary differential operators, which are associated with soliton systems in 1+1-dimensions, as well as scattering problems in several variables and their associated nonlinear equations in higher dimensions. In particular, interesting localized coherent structures (dromions) were constructed using an appropriate Darboux formalism [5]. The recent book [6] by Matveev and Salle provides an excellent overview of various forms of (generalized) DTs and their applications in soliton theory.

In this note we propose a unified formulation of DTs encapsulating many of the examples in [6]. The key is the interpretation of DTs as gauge transformations involving gauge operators parametrized by (adjoint) eigenfunctions. The use of pseudo-differential symbols allows a formal inversion of differential operators and leads to a compact formalism to define the relevant gauge operators. They also lead to the connection with Sato's definition of the (multicomponent KP) hierarchy [7-11], so that compact formulas for the KP solutions generated by DTs can be established.

No proofs are given in this note. For the scalar KP hierarchy the formulation of the classical DT as a pseudo-differential gauge transformation and corresponding results for the modified KP are given and proven in [12,13]. Details and proofs for the DT and its iteration in context of the multicomponent KP hierarchy are given in [14]. Details and proofs for the adjoint and binary transformations for the KP hierarchy and the modified KP hierarchy are to be published elsewhere.

## 2. Motivation: the classical Darboux transformation as a gauge transformation

The Lax equation  $L_t = [M, L]$  for the differential operators

$$L = L(u) = \partial^2 + u, \quad M = M(u) = 4\partial^3 + 6u\partial + 3u_x \quad (2.1)$$

is known to imply the KdV equation

$$u_t = u_{xxx} + 6uu_x \quad (2.2)$$

for the potential  $u$ . It represents the compatibility condition of the spectral problem for the Schrödinger operator with the time evolution of the eigenfunction given by  $M$ , i.e.

$$[L\Phi] = \lambda\Phi, \quad \Phi_t = [M\Phi] \quad (2.3)$$

(where  $[Aa]$  denotes the action of the operator  $A$  on the function  $a$ ).

Darboux's classical observation is that with an eigenfunction  $\Phi$  any other eigenfunction  $\chi$  solving  $[L(u)\chi] = \mu\chi$  is mapped to an eigenfunction of a new Schrödinger operator  $[L(\tilde{u})\tilde{\chi}] = \mu\tilde{\chi}$  by

$$\tilde{\chi} = \Phi(\Phi^{-1}\chi)_x, \quad \tilde{u} = u + 2(\ln(\Phi))_{xx}. \quad (2.4)$$

We observe that the spectral parameter  $\mu$  remains unchanged in this transformation. A direct computation also shows that, if  $\chi_t = [M(u)\chi]$ , then the new eigenfunction  $\tilde{\chi}$  satisfies the time evolution  $\tilde{\chi}_t = [M(\tilde{u})\tilde{\chi}]$  associated with the new potential  $\tilde{u}$ . Hence, via compatibility, one concludes that  $\tilde{u}$  is a new solution of the KdV. This provides the following mechanism to generate exact solutions of the KdV: one needs a first (maybe trivial) solution  $u$  of the KdV. Then one solves the linear problems (2.3) to obtain a new solution  $\tilde{u}$  of the nonlinear KdV.

As the eigenvalue problem is mapped to the new eigenvalue problem with the same spectral parameter, one expects that the transformed Schrödinger operator  $\tilde{L} = L(\tilde{u})$  is obtained from the original operator  $L = L(u)$  via a transformation of the type  $\tilde{L} = TLT^{-1}$  with some suitable operator  $T$ . As  $T$  applied to an eigenfunction  $\chi$  of  $L$  should produce an eigenfunction  $\tilde{\chi}$  of  $\tilde{L}$ , it is natural to consider the operator

$$T = \Phi\partial\Phi^{-1} = \partial - \Phi_x\Phi^{-1}. \quad (2.5)$$

We introduce a formal integration  $\partial^{-1}$  to invert (2.5) via  $T^{-1} = \Phi\partial^{-1}\Phi^{-1}$ . An elementary calculation shows that the operator identities

$$\begin{aligned} TL(u)T^{-1} &= L(\tilde{u}) + [(\Phi_{xx} + u\Phi)/\Phi]_x \partial^{-1}\Phi^{-1} \\ TM(u)T^{-1} + T_t T^{-1} &= M(\tilde{u}) - (\Phi^{-1}\{\Phi_t - [M(u)\Phi]\})_x \end{aligned} \quad (2.6)$$

are satisfied. With (2.3) these (formal) integro-differential operators reduce to the pair  $L(\tilde{u}), M(\tilde{u})$  of differential operators, which represent the new Lax pair associated with the potential  $\tilde{u}$ .

To summarize: Darboux's classical transformation of the Lax pair (2.1) for the KdV may be understood as the gauge transformation

$$L \rightarrow \tilde{L} = TLT^{-1}, \quad M \rightarrow \tilde{M} = TMT^{-1} + T_t T^{-1} \quad (2.7)$$

with the gauge operator (2.5), which is parametrized by an eigenfunction  $\Phi$  of the Lax pair  $L, M$ .

### 3. Background

For a general discussion of DTs we consider Lax hierarchies  $L_{t_A} = [M_A, L]$ , where  $A$  is some index labeling the different flow parameters  $t_A$  of the hierarchy. The differential matrix operators  $M_A$  generating these flows are to satisfy the zero-curvature condition

$$M_{A,t_B} - M_{B,t_A} = [M_B, M_A] \quad (3.1)$$

for all pairs of indices  $A, B$ , so that the corresponding flows are compatible.

We consider gauge transformations

$$L \rightarrow \tilde{L} = TLT^{-1}, \quad M_A \rightarrow \tilde{M}_A = TM_A T^{-1} + T_{t_A} T^{-1}, \quad (3.2)$$

which preserve the zero-curvature condition as well as the Lax formulations. As motivated in the previous section, differential gauge operators  $T$  have to be considered, which need to be inverted by some suitable formalism. In order to give a rigorous meaning to such inverses, we regard differential operators as special elements of the larger algebra

$$g = \left\{ \sum_{-\infty < i < \infty} \mathbf{u}_i \partial^i \right\} \quad (3.3)$$

of pseudo-differential symbols. The coefficients (multiplication operators)  $\mathbf{u}_i = \mathbf{u}_i(x)$  are elements of some matrix algebra,  $\partial = \partial_x$  is the differential operator, and formal integrations given by negative powers of  $\partial$  are defined by the Leibnitz rule

$$\partial^i u = \sum_{j=0}^{\infty} \frac{i!}{j(i-j)!} \frac{d^j u}{dx^j} \partial^{i-j}, \quad (3.4)$$

which for negative  $i$  leads to an infinite Laurent expansion in the symbol  $\partial$ . This, together with  $\partial\partial^{-1} = \partial^{-1}\partial = 1$  yields a well defined associative algebraic structure on  $g$ . The transposition of such symbols is defined in the usual way by

$$\left( \sum_i \mathbf{u}_i \partial^i \right)^* = \sum_i (-1)^i \partial^i \mathbf{u}_i^\dagger, \quad (3.5)$$

where  $\dagger$  is to denote matrix transposition. We use the notation

$$P_{\geq k}(M) = \sum_{i \geq k} u_i \partial^i, \quad P_{< k}(M) = \sum_{i < k} u_i \partial^i, \quad (3.6)$$

for projections of a pseudo-differential symbol  $M = \sum_i u_i \partial^i$  onto its corresponding differential orders. A further important projection is the residue of a pseudo-differential symbol given by  $\text{res}(\sum_i u_i \partial^i) = u_{-1}$ .

Functions will be identified with multiplication operators. Hence, for an operator  $M \in g$  and a function  $\Phi$  we will *always* use the notation  $M\Phi$  for the *operator*, which is the composition of  $M$  with the multiplication operator given by  $\Phi$ . If a differential operator  $M$  is to act on  $\Phi$ , we will denote the resulting function by  $[M\Phi]$ . For a given differential operator  $M$  we consider eigenfunctions and adjoint eigenfunctions, which are defined as solutions of the linear evolution equations

$$\Phi_t = [M\Phi], \quad \Psi_t = -[M^*\Phi]. \quad (3.7)$$

The letter  $\Phi$  will be reserved for eigenfunctions throughout this paper, whereas  $\Psi$  will always denote an adjoint eigenfunction. We observe the important fact that associated with each pair  $\Phi, \Psi$  there is a bi-linear potential  $\Omega(\Psi, \Phi)$  integrating the “squared eigenfunction”  $\Psi^\dagger \Phi$ .

**LEMMA 1.** *For any matrix differential operator  $M$  and any pair of (adjoint) eigenfunctions  $\Phi$  and  $\Psi$  there exists a potential  $\Omega(\Psi, \Phi)$  satisfying*

$$\Omega(\Psi, \Phi)_x = \Psi^\dagger \Phi, \quad \Omega(\Psi, \Phi)_t = \text{res}(\partial^{-1} \Psi^\dagger M \Phi \partial^{-1}). \quad (3.8)$$

This potential will play a crucial role in the (binary) DTs to be considered. It is defined up to an integration constant and may be regarded as a contour integration of the corresponding expressions in the  $(x, t)$ -plane. Solutions  $\Phi, \Psi$  of the linear problems (3.7) may be either vectors (columns) or several vector solutions may be combined to matrix solutions. In the first case the potential  $\Omega$  associated with such a pair is a scalar function, whereas in the latter case  $\Omega$  is a matrix.

#### 4. Darboux theorems

In this section we consider the linear problems (3.7) where

$$M = u_N \partial^N + u_{N-1} \partial^{N-1} + \dots + u_1 \partial + u_0 \quad (4.1)$$

is any matrix differential operator. We consider gauge transformations

$$M \rightarrow \tilde{M} = TMT^{-1} + T_t T^{-1}, \quad \Phi \rightarrow \tilde{\Phi} = [T\Phi], \quad \Psi \rightarrow \tilde{\Psi} = [T^{-1*}\Psi], \quad (4.2)$$

which is given by some time-dependent operator  $T$ . Expressions such as  $[T\Phi]$  are to denote “application of the operator  $T$  to the function  $\Phi$ ”. For a differential operator  $T$  this was introduced as  $[T\Phi]$ . For operators with formal pseudo-differential parts, however, it is a priori not clear what this action is to be, and special care has to be taken to give a rigorous meaning to this operation. Let us assume that one has  $[A(B\xi)] = [(AB)\xi]$ ,  $[A\xi]_t = [A_t\xi] +$

$A\xi]$  for operators  $A, B$  and functions  $\xi$ . Then, formally, the new (adjoint) eigenfunctions  $\tilde{\Phi}, \tilde{\Psi}$  again satisfy the scattering problems (3.7) with  $M$  replaced by  $\tilde{M}$ , as

$$\begin{aligned}\tilde{\Phi}_t &= [\mathcal{T}_t\Phi + T\Phi_t] = [(\mathcal{T}_tT^{-1} + TMT^{-1})T\Phi] = [\tilde{M}\tilde{\Phi}], \\ \tilde{\Psi}_t &= [\mathcal{T}_t^{-1*}\Psi + T^{-1*}\Psi_t] = -[(T^{-1*}\mathcal{T}_t^* + T^{-1*}MT^*)T^{-1*}\Psi] = -[\tilde{M}^*\tilde{\Psi}].\end{aligned}\quad (4.3)$$

This analysis, however, holds rigorously only if all operators involved were purely differential operators, so that  $[.] = [.]$  is well-defined. As already seen in Section 2 for the Schrödinger operator, the inversion of differential operators leads naturally to formal integration symbols, and the definition of the gauge transformed (adjoint) eigenfunctions  $\tilde{\Phi}$  and  $\tilde{\Psi}$  involving  $[.]$  will have to be given rigorously for each case to be considered in the following.

In particular, we have to ensure that the gauge transformed operator  $\tilde{M} = TMT^{-1} + \mathcal{T}_tT^{-1}$  is again a differential operator. This is the main restriction on the transformation operators  $T$  to be considered. In general, we think of  $T$  as a pseudo-differential symbol, which is to be inverted in the pseudo-differential algebra  $g$  given by (3.3). Hence, in general, if we start with a differential operator  $M$ , the resulting operator  $\tilde{M}$  will not be a differential operator any more, but it will have a part defined by formal integration symbols. In order to ensure that no negative differential orders turn up in  $\tilde{M}$ , we impose the condition that the negative orders of the term  $TMT^{-1}$  are cancelled by the additional term  $\mathcal{T}_tT^{-1}$ . Hence, we restrict our considerations to gauge operators satisfying the following “gauge equation”

$$\mathcal{T}_t = -P_{<0}(TMT^{-1})T, \quad (4.4)$$

which –for given  $M$ – may be considered as a non-linear evolution equation for  $T$ . The gauge transformation  $M \rightarrow \tilde{M}$  now can be written as

$$M \rightarrow \tilde{M} = P_{\geq 0}(TMT^{-1}). \quad (4.5)$$

In this construction the use of pseudo-differential symbols becomes a convenient technical device to define the formal operator  $TMT^{-1}$ . The gauge transformed differential operator  $\tilde{M}$  is then given as the projection of the formal integro-differential operator  $TMT^{-1}$  to its purely differential part.

We shall now consider concrete realizations of suitable gauge operators. The crucial observation behind the following DTs is the fact that the *nonlinear* evolution equation (4.4) can be *linearized*, i.e. there exist realizations of  $T$  in terms of objects satisfying linear equations. In particular, eigenfunctions and adjoint eigenfunctions associated with the differential operator  $M$  can be used to parametrize solutions of the gauge equation (4.4).

**THEOREM 1** (The “elementary” transformations). *Let  $\Phi, \Psi$  be solutions of (3.7). Then the following operators*

- a)  $T(\Phi) = \Phi\partial\Phi^{-1}$  (with inverse  $T^{-1} = \Phi\partial^{-1}\Phi^{-1}$ ),
  - b)  $T^{(adj)}(\Psi) = \Psi^{-1\dagger}\partial^{-1}\Psi^\dagger$  (with inverse  $(T^{(adj)})^{-1} = \Psi^{-1\dagger}\partial\Psi^\dagger$ ),
  - c)  $T^{(bin)}(\Phi, \Psi) = 1 - \Phi\Omega(\Psi, \Phi)^{-1}\partial^{-1}\Psi^\dagger$  (with inverse  $(T^{(bin)})^{-1} = 1 + \Phi\partial^{-1}\Omega(\Psi, \Phi)^{-1}\Psi^\dagger$ )
- satisfy equation (4.4).*

Here, for a) and b) one needs to use invertible matrix solutions  $\Phi$  and  $\Psi$ , whereas c) holds both for matrix or vector solutions of (3.7). As mentioned before, in the latter case the potential  $\Omega(\Psi, \Phi)$  of Lemma 1 is a scalar.

Case a) is to be called a DT, as according to Section 2 this gauge operator is connected with Darboux's classical results for the Schrödinger operator. For reasons to be explained soon, case b) shall be called the "adjoint DT". Case c) will turn out to be a composition of the first two cases, following [6] it is to be called a "binary DT".

A crucial feature of DTs is the fact that they provide a mechanism to transform (adjoint) eigenfunctions to new (adjoint) eigenfunctions for the transformed differential operator  $\tilde{M}$ . In principle, the transformation rules (4.2) provide the new solutions of the linear problems. However, the transformation operators and their inverses involve the formal integration  $\partial^{-1}$  and cannot be applied directly to (adjoint) eigenfunctions. It turns out that formal integrations have to be replaced by the potential  $\Omega$  of Lemma 1. The following theorem provides the correct interpretation of the formal transformations (4.2). We note that here as well as in the following Theorems 3-5 the symbol  $\chi$  is reserved for eigenfunctions, whereas  $\eta$  will always refer to an adjoint eigenfunction:

**THEOREM 2.** *Let  $\Phi, \Psi$  be the (adjoint) eigenfunctions triggering the transformations of Theorem 1. Let  $\chi$  and  $\eta$  be a further pair of (adjoint) eigenfunctions. Then*

- a)  $\tilde{\chi} = \Phi(\Phi^{-1}\chi)_x, \tilde{\eta} = -\Phi^{-1}\Omega(\eta, \Phi)^\dagger,$
  - b)  $\tilde{\chi} = \Psi^{-1\dagger}\Omega(\Psi, \chi), \tilde{\eta} = -\Psi(\Psi^{-1}\eta)_x,$
  - c)  $\tilde{\chi} = \chi - \Phi\Omega(\Psi, \Phi)^{-1}\Omega(\Psi, \chi), \tilde{\eta} = \eta - \Psi\Omega(\Psi, \Phi)^{-1\dagger}\Omega(\eta, \Phi)^\dagger,$
- solve  $\tilde{\chi}_t = [\tilde{M}\tilde{\chi}], \tilde{\eta}_t = -[\tilde{M}^*\tilde{\chi}]$  with  $\tilde{M} = P_{\geq 0}(TMT^{-1})$ .

We note that a superposition principle holds for the potentials  $\Omega(\eta, \chi)$  of Lemma 1 associated with a pair of (adjoint) eigenfunctions. A straightforward calculation shows that

$$\begin{aligned} \text{a)} \quad & \tilde{\Omega}(\tilde{\eta}, \tilde{\chi}) = \Omega(\eta, \chi) - \Omega(\eta, \Phi)\Phi^{-1}\chi, \\ \text{b)} \quad & \tilde{\Omega}(\tilde{\eta}, \tilde{\chi}) = \Omega(\eta, \chi) - \eta^\dagger\Psi^{-1\dagger}\Omega(\Psi, \chi), \\ \text{c)} \quad & \tilde{\Omega}(\tilde{\eta}, \tilde{\chi}) = \Omega(\eta, \chi) - \Omega(\eta, \Phi)\Omega(\Psi, \Phi)^{-1}\Omega(\Psi, \chi) \end{aligned} \tag{4.6}$$

are potentials for the new pair  $\tilde{\eta}, \tilde{\chi}$  satisfying

$$\tilde{\Omega}(\tilde{\eta}, \tilde{\chi})_x = \tilde{\eta}^\dagger\tilde{\chi}, \quad \tilde{\Omega}(\tilde{\eta}, \tilde{\chi})_t = \text{res}(\partial^{-1}\tilde{\eta}^\dagger\tilde{M}\tilde{\chi}\partial^{-1}). \tag{4.7}$$

The adjoint DT b) is in fact a direct consequence of the original DT a) via transposition of the operators, which swaps the roles of eigenfunctions and adjoint eigenfunctions. If we start with an operator  $M$  equipped with an adjoint eigenfunction  $\Psi$ , we may consider the adjoint operator  $M' = -M^*$ , which is equipped with the eigenfunction  $\Phi' = \Psi$ . Now a DT with  $\Phi'$  leads to an operator  $\tilde{M}'$ . A final transposition to  $\tilde{M} = -\tilde{M}'^*$  yields  $\tilde{M} = P_{\geq 0}(TM^*T^{-1})^* = P_{\geq 0}(T^{-1*}MT^*)$ , where  $T = T(\Phi') = T(\Psi) = \Psi\partial\Psi^{-1}$  gives the DT from  $M'$  to  $\tilde{M}'$ . Hence,

$$T = T(\Psi)^{-1*} = (\Psi\partial^{-1}\Psi^{-1})^* = -\Psi^{-1\dagger}\partial^{-1}\Psi^\dagger \tag{4.8}$$

is the resulting gauge operator mapping  $M \rightarrow \tilde{M} = P_{\geq 0}(TMT^{-1})$ . Up to the irrelevant minus sign, this is the adjoint DT given by case b) of Theorem 1. The results of Theorem 2.b) are obtained by introducing further (adjoint) eigenfunctions for  $M$  and tracing the

results of Theorem 2.a) in the diagram

$$\begin{array}{ccc}
 \boxed{M, \Psi} & \xrightarrow{\quad T^{(adj)}(\Psi) \quad} & \boxed{\tilde{M}} \\
 \downarrow \begin{matrix} M' \\ \Phi' \end{matrix} = \begin{matrix} -M^* \\ \Psi' \end{matrix} & & \uparrow \tilde{M} = -\tilde{M}'^* \\
 \boxed{M', \Phi'} & \xrightarrow{T(\Phi')} & \boxed{\tilde{M}'}
 \end{array}$$

DIAGRAM 1

Hence, it is seen that the adjoint transformation is indeed the original DT conjugated with transposition of the operators. We remark that one may also regard the adjoint transformation as an inverse DT. We observe that according to Theorem 2.b) under an adjoint DT  $M \rightarrow \tilde{M}$  given by  $T^{(adj)}(\Psi)$  the transformed operator  $\tilde{M}$  inherits an eigenfunction  $\tilde{\chi} = \Psi^{-1\dagger}$ . This is the image of the trivial eigenfunction  $\chi = 0$  for  $M$ , by choosing the potential  $\Omega(\Psi, \chi) = \Omega(\Psi, 0) = 1$ . If we trigger a DT of  $\tilde{M}$  with this  $\tilde{\chi}$ , then the corresponding gauge operator is

$$T(\tilde{\chi}) \quad (4.9)$$

In this sense the adjoint transformation is the inverse of a DT, which is triggered by the eigenfunction  $\tilde{\chi} = \Psi^{-1\dagger}$  of the transformed operator.

The binary DT of Theorem 1c), 2c) is a composition of the cases a) and b). We start with the operator  $M$ , equipped with the eigenfunction  $\Phi$  and the adjoint eigenfunction  $\Psi$ . We first perform a DT triggered by  $\Phi$ , which results in an operator  $M^{(1)}$ , say. The adjoint eigenfunction  $\Psi$  is mapped to the adjoint eigenfunction  $\Psi^{(1)} = -\Phi^{-1\dagger}\Omega(\Psi, \Phi)\Phi^{-1}$  by this DT. A subsequent adjoint DT using  $\Psi^{(1)}$  maps  $M^{(1)}$  to the final operator  $\tilde{M}$ , say. The composition of these two transformations is given by the gauge operator

$$\begin{aligned}
 T^{(adj)}(\Psi^{(1)}) T(\Phi) &= (\Psi^{(1)})^{-1\dagger} \partial^{-1} \Psi^{(1)\dagger} \Phi \partial^{-1} \Phi^{-1} \\
 &= \Phi \Omega(\Psi, \Phi)^{-1} \partial^{-1} \Omega(\Psi, \Phi) \partial \Phi^{-1} \\
 &= 1 - \Phi \Omega(\Psi, \Phi)^{-1} \partial^{-1} \Psi^\dagger,
 \end{aligned} \quad (4.10)$$

which represents the binary transformation of Theorem 1c). We may also start from  $M$  and first perform an adjoint DT with  $\Psi$ , leading to a transformed operator  $M^{(2)}$ , say. The eigenfunction  $\Phi$  is mapped to the eigenfunction  $\Phi^{(2)} = \Psi^{-1\dagger}\Omega(\Psi, \Phi)$  for  $M^{(2)}$  by this transformation. A subsequent DT triggered by  $\Phi^{(2)}$  leads to the composed gauge operator

$$\begin{aligned}
 T(\Phi^{(2)}) T^{(adj)}(\Psi) &= \Phi^{(2)} \partial(\Phi^{(2)})^{-1} \Psi^{-1\dagger} \partial^{-1} \Psi^\dagger \\
 &= \Psi^{-1\dagger} \Omega(\Psi, \Phi) \partial \Omega(\Psi, \Phi)^{-1} \partial^{-1} \Psi^\dagger \\
 &= 1 - \Phi \Omega(\Psi, \Phi)^{-1} \partial^{-1} \Psi^\dagger,
 \end{aligned} \quad (4.11)$$

i.e. again to the binary DT. Hence, the binary transformation may be regarded as the map  $M \rightarrow \tilde{M}$  resulting from the following commuting diagram:

$$\begin{array}{ccc}
& \boxed{M, \Phi, \Psi} & \\
T(\Phi) \swarrow & & \searrow T^{(adj)}(\Psi) \\
\boxed{M^{(1)}, \Psi^{(1)}} & & \boxed{M^{(2)}, \Phi^{(2)}} \\
T^{(adj)}(\Psi^{(1)}) \searrow & & \swarrow T(\Phi^{(2)}) \\
& \boxed{\tilde{M}} &
\end{array}$$

*DIAGRAM 2*

In this derivation we have to assume that the pair  $\Phi, \Psi$  triggering the binary transformation is given by invertible matrices, so that the intermediate (adjoint) DTs can be carried out. However, the final formula for the binary gauge operator also makes sense for vector solutions  $\Phi$  and  $\Psi$  and an associated scalar potential  $\Omega(\Psi, \Phi)$ .

We remark that for a binary DT  $M \rightarrow \tilde{M}$  triggered by the pair  $\Phi, \Psi$  the gauge transformed operator automatically is equipped with the pair

$$\tilde{\chi} = \Phi\Omega(\Psi, \Phi)^{-1}, \quad \tilde{\eta} = \Psi\Omega(\Psi, \Phi)^{-1\dagger}, \quad (4.12)$$

of (adjoint) eigenfunctions, which are the images of the trivial (adjoint) eigenfunctions  $\chi = 0, \eta = 0$  with the choice  $\Omega(\Psi, \chi) = \Omega(\eta, \Phi) = -1$ . Fixing  $\Omega(\eta, \chi) = 0$  one finds the new potential  $\tilde{\Omega}(\tilde{\eta}, \tilde{\chi}) = -\Omega(\Psi, \Phi)^{-1}$  according to Theorem 2c). The binary gauge operator triggered by this new pair is

$$T^{(bin)}(\tilde{\chi}, \tilde{\eta}) = 1 - \tilde{\chi}\tilde{\Omega}(\tilde{\eta}, \tilde{\eta})^{-1}\partial^{-1}\tilde{\eta}^\dagger = 1 + \Phi\partial^{-1}\Omega(\Psi, \Phi)^{-1}\Psi^\dagger = \left(T^{(bin)}(\Phi, \Psi)\right)^{-1}, \quad (4.13)$$

so that a further binary DT triggered by  $\tilde{\chi}, \tilde{\eta}$  maps  $\tilde{M}$  back to the original operator  $M$ .

So far the (adjoint) DTs of Theorem 1.a),1.b) were used to derive the binary DT as a composition of the elementary (adjoint) DTs. Further gauge operators satisfying (4.4) can be constructed from further compositions of the elementary transformations. As an example, we may start with an operator equipped with a collection of matrix eigenfunctions  $\Phi_1, \dots, \Phi_n$ . Performing a first DT with  $\Phi_1$ , say, we may use Theorem 2.a) to generate eigenfunctions  $\tilde{\Phi}_2, \dots, \tilde{\Phi}_n$  for the gauge transformed operator. The eigenfunction  $\Phi_1$  used to trigger the transformation is lost, as it is mapped to zero. After this first transformation a second DT triggered by  $\tilde{\Phi}_2$  is performed, leading again to eigenfunctions which are the images of  $\tilde{\Phi}_3, \dots, \tilde{\Phi}_n$ . Repeating this process one can perform  $n$  DTs until all the eigenfunctions are “used up”. The resulting gauge operator is an  $n$ th order differential operator given as the composition of the first order operators triggering the individual DTs. It turns out that the final result does not depend on the ordering, in which the eigenfunctions are used for the DTs, and the final operator can be written in a compact way in terms of “Wronskian expressions” of the eigenfunctions.

**THEOREM 3** (Iteration of the DT). *Let  $\Phi_1, \dots, \Phi_n$  be a collection of invertible matrix solutions of  $\Phi_t = [M\Phi]$ . Let  $\mathbf{a}_0, \dots, \mathbf{a}_{n-1}$  and  $\mathbf{b}_1, \dots, \mathbf{b}_n$  be the matrix functions determined as the solution of the linear algebraic system*

$$\sum_{i=0}^n \mathbf{a}_i \frac{d^i \Phi_j}{dx^i} = 0, \quad j = 1, \dots, n, \quad \mathbf{a}_n = 1, \quad (4.14)$$

$$\sum_{j=1}^n \frac{d^i \Phi_j}{dx^i} \mathbf{b}_j = \begin{cases} 0 & \text{for } i = 0, \dots, n-2, \\ 1 & \text{for } i = n-1. \end{cases} \quad (4.15)$$

(4.16)

Then the operator

$$T_n = \sum_{i=0}^n \mathbf{a}_i \partial^i \quad (\text{with inverse} \quad T_n^{-1} = \sum_{j=1}^n \Phi_j \partial^{-1} \mathbf{b}_j) \quad (4.17)$$

satisfies (4.4). A pair of (adjoint) eigenfunctions  $\chi$  and  $\eta$  of  $M$  is transformed into the new pair of (adjoint) eigenfunctions

$$\tilde{\chi} = \sum_{i=0}^n \mathbf{a}_i \frac{d^i \chi}{dx^i}, \quad \tilde{\eta} = - \sum_{j=1}^n \mathbf{b}_j^\dagger \Omega(\eta, \Phi_j)^\dagger \quad (4.18)$$

for the transformed operator  $\tilde{M} = P_{\geq 0} (T_n M T_n^{-1})$ .

For scalar operators and scalar eigenfunctions the solutions of the linear system (4.15) can be given by Cramer's rule in terms of Wronskian type determinants, and  $T_n, T_n^{-1}$  can be rewritten by compact Wronskian formulas [14]. In particular, the transformed eigenfunction is given by

$$\tilde{\chi} = \frac{\mathcal{W}(\Phi_1, \dots, \Phi_n, \chi)}{\mathcal{W}(\Phi_1, \dots, \Phi_n)}, \quad (4.19)$$

where  $\mathcal{W}$  is the Wronskian determinant of the indicated arguments.

A similar construction holds for the iteration of the adjoint DT. Starting with a collection  $\Psi_1, \dots, \Psi_n$  of adjoint eigenfunctions for  $M$ , the composition of the  $n$ -fold adjoint transformation is given as follows.

**THEOREM 4** (Iteration of the adjoint DT). *Let  $\Psi_1, \dots, \Psi_n$  be a collection of invertible matrix solutions of  $\Psi_t = -[M^* \Psi]$ . Let  $\mathbf{a}_0, \dots, \mathbf{a}_{n-1}$  and  $\mathbf{b}_1, \dots, \mathbf{b}_n$  be the matrix functions determined as the solution of the linear algebraic system*

$$\begin{aligned} \sum_{i=0}^n \mathbf{a}_i \frac{d^i \Psi_j}{dx^i} &= 0, \quad j = 1, \dots, n, \quad \mathbf{a}_n = 1, \\ \sum_{j=1}^n \frac{d^i \Psi_j}{dx^i} \mathbf{b}_j &= \begin{cases} 0 & \text{for } i = 0, \dots, n-2, \\ 1 & \text{for } i = n-1. \end{cases} \end{aligned} \quad (4.20)$$

Then the operator

$$T_n^{(adj)} = (-1)^{n+1} \sum_{j=1}^n \mathbf{b}_j^\dagger \partial^{-1} \Psi_j^\dagger \quad (4.21)$$

has the inverse

$$(T_n^{(adj)})^{-1} = (-1)^n \left( \sum_{i=0}^n \mathbf{a}_i \partial^i \right)^* \quad (4.22)$$

and satisfies (4.4). A pair of (adjoint) eigenfunctions  $\chi$  and  $\eta$  of  $M$  is transformed into the new pair of (adjoint) eigenfunctions

$$\tilde{\chi} = (-1)^{n+1} \sum_{j=1}^n \mathbf{b}_j^\dagger \Omega(\Psi_j, \chi), \quad \tilde{\eta} = (-1)^n \sum_{i=0}^n \mathbf{a}_i \frac{d^i \eta}{dx^i} \quad (4.23)$$

for the transformed operator  $\tilde{M} = P_{\geq 0} \left( T_n^{(adj)} M (T_n^{(adj)})^{-1} \right)$ .

In a similar way iterations of the binary DT can be formulated. As the transformation involves the potential  $\Omega(\Psi, \Phi)$  of the triggering pair  $\Phi, \Psi$  explicitly, it is useful to have the potentials associated with transformed pairs given by the superposition formula (4.7.c). The result is as follows.

**THEOREM 5** (Iteration of the binary DT). *Let  $\Phi_1, \dots, \Phi_n, \Psi_1, \dots, \Psi_n$  be a collection of vector (or matrix) solutions of (3.7), let  $\Omega_{ij} = \Omega(\Psi_i, \Phi_j)$  be the matrix of associated potentials. Let  $\mathbf{a}_1, \dots, \mathbf{a}_n$  and  $\mathbf{b}_1, \dots, \mathbf{b}_n$  be the vector (or matrix) functions determined as the solution of the linear algebraic system*

$$\sum_{j=1}^n \mathbf{a}_j \Omega_{ji} = \Phi_i, \quad \sum_{j=1}^n \mathbf{b}_j \Omega_{ij}^\dagger = \Psi_i, \quad i = 1, \dots, n. \quad (4.24)$$

Then the operator

$$T_n^{(bin)} = 1 - \sum_{i=1}^n \mathbf{a}_i \partial^{-1} \Psi_i^\dagger \quad (4.25)$$

has the inverse

$$(T_n^{(bin)})^{-1} = 1 + \sum_{i=1}^n \Phi_i \partial^{-1} \mathbf{b}_i^\dagger \quad (4.26)$$

and satisfies (4.4). A pair of (adjoint) eigenfunctions  $\chi$  and  $\eta$  of  $M$  is transformed into the new pair of (adjoint) eigenfunctions

$$\tilde{\chi} = \chi - \sum_{i=1}^n \mathbf{a}_i \Omega(\Psi_i, \chi), \quad \tilde{\eta} = \eta - \sum_{i=1}^n \mathbf{b}_i \Omega(\eta, \Phi_i)^\dagger \quad (4.27)$$

for the transformed operator  $\tilde{M} = P_{\geq 0} \left( T_n^{(bin)} M (T_n^{(bin)})^{-1} \right)$ .

We remark that by choosing trivial (adjoint) eigenfunctions  $\chi = \eta = 0$  and  $\Omega(\Psi_i, \chi) = \Omega(\eta, \Phi_i) = \delta_{ij}$  it is readily seen, that the coefficients  $\mathbf{a}_j$  and  $\mathbf{b}_j$  are eigenfunctions and adjoint eigenfunctions of  $\tilde{M}$ . A further transformation triggered by the  $\mathbf{a}_j$  and  $\mathbf{b}_j$  will map  $\tilde{M}$  back to the original operator  $M$ .

## 5. Applications: Solution Formulas for the KP Hierarchy

A convenient parametrization of zero-curvature matrices  $M_A, M_B$  satisfying (3.1) is given by Sato's construction [7-11] of the (multicomponent) KP hierarchy. One introduces the dressing operator

$$W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + \dots \quad (5.1)$$

and its inverse

$$W^{-1} = 1 - w_1 \partial^{-1} + (w_1^2 - w_2) \partial^{-2} + \dots \quad (5.2)$$

and imposes the following hierarchy of evolution equations

$$W_{t_A} = -P_{<0}(WAW^{-1})W. \quad (5.3)$$

Here  $A$  is taken from some set  $\mathcal{A}$  of commuting constant matrices, and with each  $A$  a different flow-parameter (time)  $t_A$  is associated. The infinite set of nonlinear partial differential equations (5.3) for the matrix coefficients  $w_1, w_2, \dots$  is called the multicomponent KP hierarchy (associated with the collection  $\mathcal{A}$  of “undressed” (or “bare”) operators). Depending on the underlying matrix algebra used for  $\mathcal{A}$  and the coefficients  $w_i$  of  $W$ , various nonlinear equations for finitely many of the fields  $w_i$  can be extracted from the KP hierarchy. These include the KP equation itself for the scalar case as well as the Davey-Stewartson equation for  $2 \times 2$  matrices or the N-wave system for  $N \times N$  matrices [15].

One defines the hierarchy of differential operators

$$M_A = P_{\geq 0}(WAW^{-1}), \quad (5.4)$$

which are labeled by the bare operators  $A \in \mathcal{A}$ . It is readily checked that for all pairs  $A, B \in \mathcal{A}$  the zero-curvature condition (3.1) is satisfied, which is a consequence of the commutativity of the bare operators. Further, for any fixed operator  $C \in \mathcal{A}$ , the pseudo-differential Lax operator  $L = WCW^{-1}$  satisfies the hierarchy of commuting Lax equations

$$L_{t_A} = [M_A, L]. \quad (5.5)$$

The previous DTs can be applied to the differential operators  $M_A$  and lead to solutions formulas of the KP hierarchy. We observe that (5.4) defines a parametrization of the zero-curvature operators in terms of the coefficients  $w_i$  of the dressing operator. This parametrization is preserved under (adjoint,binary) DTs, as they are generated by transformations of the dressing operator  $W$  itself.

**THEOREM 6** (The Darboux Transformations for the dressing operator). *Let  $W$  solve the KP hierarchy (5.3), let  $\Phi$  and  $\Psi$  be (adjoint) eigenfunctions solving*

$$\Phi_{t_A} = [M_A \Phi], \quad \Psi_{t_A} = -[M_A^* \Psi]. \quad (5.6)$$

Then

- a)  $\tilde{W} = T(\Phi)W\partial^{-1},$
- b)  $\tilde{W} = T^{(adj)}(\Psi)W\partial,$
- c)  $\tilde{W} = T^{(bin)}(\Phi, \Psi)W,$

are again solutions of the KP hierarchy (5.3).

These statements are immediate consequences of the gauge equation (4.4) with  $t = t_A$  satisfied by the gauge operators. The multiplication with  $\partial^{-1}$  and  $\partial$  in the cases a) and b) is to renormalize the leading coefficient of  $\tilde{W}$  to 1. The transformation

$$M_A = P_{\geq 0}(WAW^{-1})\tilde{M}_A = P_{\geq 0}(\tilde{W}AW^{-1}) \quad (5.7)$$

is given by  $\tilde{\mathbf{M}}_A = P_{\geq 0}(\mathcal{T}M_A\mathcal{T}^{-1})$ , so that Theorem 6 indeed generates the DTs for the differential operators given by (5.4). Hence, the parametrization of these operators in terms of the coefficients  $w_i$  is left invariant by the (adjoint,binary) DTs.

The iterations of the elementary transformations lead to solutions

- a)  $\tilde{\mathbf{W}} = T_n W \partial^{-n}$ ,
  - b)  $\tilde{\mathbf{W}} = T_n^{(adj)} W \partial^n$ ,
  - c)  $\tilde{\mathbf{W}} = T_n^{(bin)} W$ ,
- (5.8)

where the gauge operators  $T_n, T_n^{(adj)}, T_n^{(bin)}$  of Theorems 3-5 are given in terms of collections of (adjoint) eigenfunctions.

To summarize, a first solution  $W$  of the nonlinear KP hierarchy (5.3) and solutions of the linear problems (5.6) provide a new solution of the KP via (5.9). In particular, starting with the trivial solution  $W = 1$ , one obtains the compact solution formula

$$a) \quad \tilde{\mathbf{W}} = T_n \partial^{-n}, \quad b) \quad \tilde{\mathbf{W}} = T_n^{(adj)} \partial^n, \quad c) \quad \tilde{\mathbf{W}} = T_n^{(bin)} \quad (5.9)$$

given by the gauge operators of Theorems 3-5. In this case the (adjoint) eigenfunctions defining the gauge operators are given as solutions of the *constant coefficient* PDEs  $\Phi_{t_A} = [A\Phi]$  and  $\Psi_{t_A} = -[A^*\Psi]$ .

We finally remark that the DTs can also be applied to reductions of the general KP hierarchy. One type of reduction is given by dressing operators satisfying the constraint

$$W^* SW = S, \quad (5.10)$$

where  $S$  is any given constant symmetric matrix. This constraint is left invariant by those evolutions (5.3), which are generated by bare operators  $A$  satisfying

$$A^* S = -S A. \quad (5.11)$$

The associated differential operators  $M_A = P_{\geq 0}(WAW^{-1})$  satisfy

$$M_A^* S = -S M_A. \quad (5.12)$$

In general, neither the DT nor the adjoint DT will leave the constraint (5.12) invariant. However, for  $\Psi = S\Phi$  the binary transformation operator  $T^{(bin)} = T^{(bin)}(\Phi, S\Phi)$  satisfies

$$(T^{(bin)})^* S T^{(bin)} = S, \quad (5.13)$$

so that  $\tilde{\mathbf{W}} = T^{(bin)}W$  and  $\tilde{\mathbf{M}}_A = P_{\geq 0}\left(T^{(bin)}M_A(T^{(bin)})^{-1}\right)$  again satisfy the constraints (5.10) and (5.12). Hence, starting from the trivial solution  $W = 1$  and iterating the binary transformation, one obtains exact solutions  $\tilde{\mathbf{W}} = T_n^{(bin)}$  of the KP hierarchy subject to the constraint (5.10). The binary gauge operator is parametrized by a set of eigenfunctions  $\Phi_1, \dots, \Phi_n$  only, as the adjoint eigenfunctions  $\Psi_i$  have to be chosen as  $\Psi_i = S\Phi_i$ ,  $i = 1, \dots, n$ , with associated potentials  $\Omega(S\Phi_i, \Phi_j) = \Omega(S\Phi_j, \Phi_i)$ .

Other reductions — connected with 1+1-dimensional soliton equations — are given by constraints of the type  $P_{<0}(WCW^{-1}) = 0$  with some constant matrix  $C \in \mathcal{A}$ . The differential Lax operator  $L = WCW^{-1}$  is parametrized by the fields  $w_1, \dots, w_N$ , if  $N$  is the order of the constant operator  $C$ . In this case the KP hierarchy (5.3) (or the Lax hierarchy

(5.5), respectively) reduces to the hierarchy of isospectral 1+1-dimensional equations for  $w_1, \dots, w_N$  associated with the Lax operator  $L$ . In this situation the (adjoint) eigenfunctions have to satisfy the spectral equations

$$[L\Phi] = \lambda\Phi, \quad [L^*\Psi] = \mu\Psi. \quad (5.14)$$

These conditions guarantee that for the (adjoint) DT  $L \rightarrow \tilde{L} = TLT^{-1}$  the resulting  $\tilde{L}$  is again a differential operator. Due to (5.14) the potential  $\Omega(\Psi, \Phi)$  can be given explicitly by the formula

$$\Omega(\Psi, \Phi) = \frac{\text{res}(\partial^{-1}\Psi^\dagger L\Phi\partial^{-1})}{\lambda - \mu}. \quad (5.15)$$

If we use this particular potential for the binary transformation, then also the binary case produces differential operators  $\tilde{L}$ . Hence, exact solutions of the 1+1-dimensional Lax equations (5.5) can be obtained from (adjoint,binary) DTs by using (adjoint) eigenfunctions satisfying the spectral problems (5.14).

## 6. Concluding remarks

All results presented here can be transformed into DTs for the modified multicomponent KP hierarchy. For the modified case the differential operators of the linear scattering problems are characterized by that fact that no zero order term is present. As a consequence, these scattering problems admit trivial (constant) eigenfunctions. One observes that the gauge operators

$$\tau(\Phi) = \Phi^{-1}, \quad \tau^{(adj)}(\Psi) = \partial^{-1}\Psi^{-1\dagger}, \quad (6.1)$$

satisfy the gauge equation  $T_t = -P_{<1}(TMT^{-1})T$ , where in contrast to (4.4) the zero order term of  $TMT^{-1}$  is included. Gauge transformations  $M \rightarrow \tilde{M} = TMT^{-1} + T_tT^{-1} = P_{\geq 1}(TMT^{-1})$  with these operators yield differential operators with no zero order term and may be regarded as Miura transformations between the KP and the modified KP hierarchy. Compositions of the (adjoint,binary) DTs of Theorem 1 with these transformations lead to (adjoint,binary) DTs of the modified KP hierarchy. Thus, solution formulas such as (5.9)/(5.9) are obtained for the scalar modified KP equation and its multicomponent versions, which include the Ishimori system or the (2+1)-dimensional sine-Gordon system recently introduced by Konopelchenko and Rogers [16,17].

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# LOCALIZED SOLITON SOLUTIONS FOR THE DAVEY-STEWARTSON I AND DAVEY-STEWARTSON III EQUATIONS \*

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**ABSTRACT.** It is shown that a new Davey-Stewartson equation (which we call DSIII), in addition to the so called DS<sub>I</sub> and DS<sub>II</sub> equations, can be linearized. Moreover, the DSIII equation admits localized soliton solutions with properties similar to those of the DS<sub>I</sub> equation. The solution can simulate quantum effects as inelastic scattering, fusion and fission, creation and annihilation.

## 1. Introduction

We study here a relevant application of the spectral method recently proposed by Sabatier in [1,2], further developed by Boiti, Pempinelli and Sabatier in [3] and presented at this workshop by Boiti [4]; for a general exposition on the method the reader can consult [4].

Here, we only recall that the input is an integral equation of the form

$$\Psi(k, \mathbf{x}, t) = \mathbf{I} + \int \frac{d\sigma(\lambda, \Lambda)}{-k + \Lambda} \mathbf{T}(\lambda, \Lambda, \mathbf{x}, t) \Psi(\lambda, \mathbf{x}, t) \quad (1.1)$$

where  $k, \lambda, \Lambda \in \mathbb{C}$ ,  $\mathbf{x} \in \mathbb{C}^M$ ,  $t \in \mathbb{C}$ ,  $\Psi, \mathbf{T}$  have their values in the set  $\mathcal{M}_N$  of complex valued  $N \times N$  matrices,  $M \leq N$ ,  $N > 1$ ,  $d\sigma$  is a measure in  $\mathbb{C}^2$  and  $\mathbf{I}$  is the identity matrix in  $\mathcal{M}_N$ . We call  $\mathbf{T}$  the spectral data or the spectral transform and  $k, \lambda, \Lambda$  the spectral parameters. Any special dependence of  $\mathbf{T}$  on the spatial and time variables is called a dispersion law.

An operator  $H$  is called compatible with a given dispersion law for  $\mathbf{T}$  if it satisfies the equation

$$H\Psi = h + \int \frac{d\sigma}{-k + \Lambda} \mathbf{T}H\Psi \quad (1.2)$$

where  $h$  is a functional of  $\Psi$  depending only on the space-time variables. We are interested in the case where  $h$  depends only on the first term  $\Psi^{(1)}(\mathbf{x}, t)$  of the asymptotic expansion of  $\Psi$  in the spectral parameter  $k$

$$\Psi = \mathbf{I} + k^{-1}\Psi^{(1)} + o(k^{-1}). \quad (1.3)$$

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The method introduces a pair of compatible operators  $F$  and  $G$  (in a way reminiscent of the Lax pair) which satisfy the integral equations

$$F\Psi = f + \int \frac{d\sigma}{-k + \Lambda} TF\Psi \quad (1.4)$$

$$G\Psi = g + \int \frac{d\sigma}{-k + \Lambda} TG\Psi. \quad (1.5)$$

These equations are admitted to have the unique solution

$$F\Psi = \Psi f(\Psi^{(1)}) \quad (1.6)$$

$$G\Psi = \Psi g(\Psi^{(1)}) \quad (1.7)$$

and to satisfy the compatibility condition

$$[F, G]\Psi = F(\Psi g(\Psi^{(1)})) - G(\Psi f(\Psi^{(1)})). \quad (1.8)$$

We study, here, the case where  $F$  and  $G$  have the form

$$F\Psi = -\frac{1}{2}i \frac{\partial \Psi}{\partial t} - \frac{1}{2}\Delta \Psi + k\Gamma \bullet \nabla \Psi \quad (1.9)$$

$$G\Psi = \nabla \Psi \bullet \Omega - k\Gamma \Psi \bullet \Omega \quad (1.10)$$

with  $\Gamma$  and  $\Omega$  diagonal matrix valued vectors satisfying the orthogonality condition

$$\Gamma \bullet \Omega = 0. \quad (1.11)$$

They are compatible with the following dispersion law for  $T$

$$\nabla T = \Lambda \Gamma T - \lambda T \Gamma \quad (1.12)$$

$$\frac{1}{2}i \frac{\partial T}{\partial t} = \Lambda^2 \Gamma \bullet \Gamma T - \lambda^2 T \Gamma \bullet \Gamma. \quad (1.13)$$

We have

$$f(\Psi^{(1)}) = \Gamma \bullet \nabla \Psi^{(1)} \quad (1.14)$$

$$g(\Psi^{(1)}) = -\Gamma \Psi^{(1)} \bullet \Omega. \quad (1.15)$$

We could obtain the nonlinear evolution equation in  $\Psi^{(1)}$  solved by the integral equation (1.1) directly from (1.6) and (1.7) (see [3]), but we prefer, here, to transform them by using a gauge transformation in a such a way to derive a couple  $T_1, T_0$  of Lax operators which commute in the “weak” sense [4,5]

$$T_1 \varphi = 0, \quad [T_1, T_0] \varphi = 0. \quad (1.16)$$

## 2. The Set of Davey–Stewartson equations

Let us make the gauge transformation [2]

$$\Psi = \exp(k\Gamma \bullet x)\tilde{\varphi} \quad (2.1)$$

in (1.6) and (1.7) (the overtilde means transposed).

In the transformed equations we make a change of space variables, choose a convenient representation for  $\Gamma$  and  $\Omega$ , change, for convenience,  $k$  into  $-ik$  and finally get

$$\mathbf{T}_0\varphi \equiv \left\{ \frac{1}{2}i\partial_t + \frac{1}{2} \sum_{i=1}^M \alpha_i^2 \partial_i^2 + \sum_{i=1}^M \alpha_i (\partial_i \Phi) \eta_i \right\} \varphi = -\frac{1}{2}k^2 \varphi \sum_{i=1}^M \alpha_i \eta_i \quad (2.2)$$

$$\mathbf{T}\mathbf{T}^{(l)}\varphi \equiv (1 - \eta_l)(\partial_l \varphi + \Phi \eta_l \varphi) = 0, \quad l = 1, 2, \dots, M \quad (2.3)$$

where

$$\Phi \equiv \tilde{\Psi}^{(1)}, \quad \alpha_i^2 = 1 \quad \partial_l \equiv \frac{\partial}{\partial x_l}, \quad \partial_t \equiv \frac{\partial}{\partial t} \quad (2.4)$$

and  $\eta_i$  are the  $N \times N$  matrix projectors defined as

$$(\eta_i)_{mn} = \delta_{in} \delta_{im}. \quad (2.5)$$

We are interested in the case  $N = 2$  and  $M = 2$ . We introduce

$$\mathbf{T}_1 = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} \quad (2.6)$$

and we get two linear spectral problems for the couple  $\mathbf{T}_1$ ,  $\mathbf{T}_0$  of Lax operators

$$\mathbf{T}_1 \varphi \equiv \begin{pmatrix} \partial_2 & \Phi_{12} \\ \Phi_{21} & \partial_1 \end{pmatrix} \varphi = 0 \quad (2.7)$$

$$\begin{aligned} \mathbf{T}_0 \varphi \equiv & \left\{ \frac{1}{2}i\partial_t + \frac{1}{2}(\alpha_1 \partial_1^2 + \alpha_2 \partial_2^2) + \begin{pmatrix} \alpha_1(\partial_1 \Phi_{11}) & \alpha_1(\partial_1 \Phi_{12}) \\ \alpha_2(\partial_2 \Phi_{21}) & \alpha_2(\partial_2 \Phi_{22}) \end{pmatrix} \right\} \varphi \\ & = -\frac{1}{2}k^2 \varphi \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \end{aligned} \quad (2.8)$$

where  $\Phi_{ij}$  are the matrix elements of  $\Phi$ .

Let us introduce the standard quantities  $\mathbf{Q}$  and  $\mathbf{A}$

$$\mathbf{Q} \equiv \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & \Phi_{12} \\ \Phi_{21} & 0 \end{pmatrix} \quad (2.9)$$

$$\mathbf{A} \equiv \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = 2 \begin{pmatrix} \alpha_1 \partial_1 \Phi_{11} & 0 \\ 0 & \alpha_2 \partial_2 \Phi_{22} \end{pmatrix}. \quad (2.10)$$

We have four cases to consider Case I).  $x_1 = v \in \mathbb{R}$ ,  $x_2 = u \in \mathbb{R}$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = -1$ . in this case the spectral problems (2.7) and (2.8) take the form

$$\mathbf{T}_1 \varphi \equiv \left\{ 2 \begin{pmatrix} \partial_u & 0 \\ 0 & \partial_v \end{pmatrix} + \mathbf{Q} \right\} \varphi = 0 \quad (2.11)$$

$$2\mathbf{T}_0 \varphi \equiv \left\{ i\partial_t + \partial_v^2 - \partial_u^2 + \mathbf{A} + \begin{pmatrix} 0 & -q_u \\ r_v & 0 \end{pmatrix} \right\} \varphi = -k^2 \varphi \sigma_3 \quad (2.12)$$

where  $\mathbf{T}_1$  is the well known hyperbolic Zakharov-Shabat spectral operator in the plane of the characteristic coordinates. The consistency condition (1.8) becomes in terms of the operators  $\mathbf{T}_1$  and  $\mathbf{T}_0$  the “weak” Lax condition

$$\mathbf{T}_1 \varphi = 0, \quad [\mathbf{T}_1, \mathbf{T}_0] \varphi = 0 \quad (2.13)$$

which furnishes the Davey–Stewartson I (DSI) equation

$$i\mathbf{Q}_t + \sigma_3(\mathbf{Q}_{uu} + \mathbf{Q}_{vv}) + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.14)$$

$$\begin{pmatrix} A_{1,u} & 0 \\ 0 & A_{2,v} \end{pmatrix} = -\frac{1}{2}\sigma_3 \begin{pmatrix} (qr)_v & 0 \\ 0 & (qr)_u \end{pmatrix} \quad (2.15)$$

The DSI equation is compatible with the reduction  $q = \epsilon\bar{r}$  ( $\epsilon = \pm 1$ ) and admits exponentially localized soliton solutions [6–8]. The one-soliton solution has the form

$$q = -\frac{2}{D}\lambda_I\eta e^{i\theta}, \quad r = -\frac{2}{D}\mu_I\rho e^{-i\theta} \quad (2.16)$$

where

$$D = 2\gamma(\cosh \xi_1 + \cosh \xi_2) + \exp(\xi_2) \quad (2.17)$$

$$\xi_1 = -\mu_I u - \lambda_I v + 2(\lambda_R\lambda_I + \mu_R\mu_I)t \quad (2.18)$$

$$\xi_2 = \mu_I u - \lambda_I v + 2(\lambda_R\lambda_I - \mu_R\mu_I)t \quad (2.19)$$

$$\theta = \mu_R u + \lambda_R v + (\lambda_I^2 - \lambda_R^2 + \mu_I^2 - \mu_R^2)t \quad (2.20)$$

$$\gamma = \frac{1}{4}\eta\rho \quad (2.21)$$

The complex parameters  $\lambda = \lambda_R + i\lambda_I$ ,  $\mu = \mu_R + i\mu_I$  are the discrete eigenvalue of the associated Zakharov-Shabat spectral problem and  $\rho$ ,  $\eta$  are arbitrary complex constants satisfying the conditions  $\gamma \in \mathbb{R}$  and  $\gamma(1+\gamma) > 0$ . Case II).  $x_1 = z = x+iy$ ,  $x_2 = \bar{z} = x-iy$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = -1$ . Here the spectral problems (2.7) and (2.8) take the form

$$\mathbf{T}_1\varphi \equiv \left\{ 2 \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} + \mathbf{Q} \right\} \varphi = 0 \quad (2.22)$$

$$2\mathbf{T}_0\varphi \equiv \left\{ i\partial_t + \partial_z^2 - \partial_{\bar{z}}^2 + \mathbf{A} + \begin{pmatrix} 0 & -q\bar{z} \\ r_z & 0 \end{pmatrix} \right\} \varphi = -k^2\varphi\sigma_3 \quad (2.23)$$

where  $\mathbf{T}_1$  is the elliptic Zakharov-Shabat spectral operator in the plane. By imposing the “weak” Lax commutativity condition (1.16) we get the Davey–Stewartson (DSII) equation

$$i\mathbf{Q}_t + \sigma_3(\mathbf{Q}_{zz} + \mathbf{Q}_{\bar{z}\bar{z}}) + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.24)$$

$$\begin{pmatrix} A_{1,\bar{z}} & 0 \\ 0 & A_{2,z} \end{pmatrix} = -\frac{1}{2}\sigma_3 \begin{pmatrix} (qr)_z & 0 \\ 0 & (qr)_{\bar{z}} \end{pmatrix}. \quad (2.25)$$

All this can be rewritten by using real variables. We have for the spectral problems

$$\mathbf{T}_1\varphi \equiv (\partial_x + i\sigma_3\partial_y + Q)\varphi = 0 \quad (2.26)$$

$$2\mathbf{T}_0\varphi \equiv \left\{ i\partial_t - i\partial_x\partial_y + \mathbf{A} - \frac{1}{2}(\mathbf{Q}_x + i\sigma_3\mathbf{Q}_y) \right\} \varphi = -k^2\varphi\sigma_3 \quad (2.27)$$

and for the DSII equation

$$i\mathbf{Q}_t + \frac{1}{2}(\mathbf{Q}_{xx} - \mathbf{Q}_{yy}) + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.28)$$

$$2(\partial_x + i\sigma_3\partial_y)\mathbf{A} = -\sigma_3(\partial_x - i\sigma_3\partial_y)\mathbf{Q}^2. \quad (2.29)$$

Note that it is compatible with the reduction  $q = \epsilon\bar{r}$  ( $\epsilon = \pm 1$ ).

Case III).  $x_1 = v \in \mathbb{R}$ ,  $x_2 = u \in \mathbb{R}$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ . In this case  $\mathbf{T}_1$  takes again the form of the hyperbolic Zakharov-Shabat spectral operator in the plane, while the spectral problem (2.8) becomes

$$2\mathbf{T}_0\varphi \equiv \left\{ i\partial_t + \partial_u^2 + \partial_v^2 + \mathbf{A} + \begin{pmatrix} 0 & q_u \\ r_v & 0 \end{pmatrix} \right\} \varphi = -k^2\varphi. \quad (2.30)$$

By imposing the “weak” Lax commutativity condition (1.16) we obtain a new Davey–Stewartson equation, which we call Davey–Stewartson III (DSIII). It reads

$$i\mathbf{Q}_t + \sigma_3(\mathbf{Q}_{vv} - \mathbf{Q}_{uu}) + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.31)$$

$$\begin{pmatrix} A_{1,u} & 0 \\ 0 & A_{2,v} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} (qr)_v & 0 \\ 0 & (qr)_u \end{pmatrix} \quad (2.32)$$

or in standard coordinates  $2x = u + v$ ,  $2y = u - v$

$$i\mathbf{Q}_t - \sigma_3\mathbf{Q}_{xy} + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.33)$$

$$2(\partial_x + \sigma_3\partial_y)\mathbf{A} = -(\partial_x - \sigma_3\partial_y)\mathbf{Q}^2. \quad (2.34)$$

Note that the DSIII equation is compatible with the reduction  $q = \epsilon\bar{r}$  ( $\epsilon = \pm 1$ ). It admits exponentially localized solitons similar in shape to those of the DSI equation but with a different time evolution. The one soliton solution has the form

$$q = -\frac{2}{D}\lambda_I\eta e^{i\theta}, \quad r = -\frac{2}{D}\mu_I\rho e^{-i\theta} \quad (2.35)$$

where

$$D = 2\gamma(\cosh \xi_1 + \cosh \xi_2) + \exp(\xi_2) \quad (2.36)$$

$$\xi_1 = -\mu_I u - \lambda_I v + 2(\lambda_R\lambda_I - \mu_R\mu_I)t \quad (2.37)$$

$$\xi_2 = \mu_I u - \lambda_I v + 2(\lambda_R\lambda_I + \mu_R\mu_I)t \quad (2.38)$$

$$\theta = \mu_R u + \lambda_R v + (\lambda_I^2 - \lambda_R^2 - \mu_I^2 + \mu_R^2)t \quad (2.39)$$

$$\gamma = \frac{1}{4}\eta\rho \quad (2.40)$$

As in the DSI case, the complex parameters  $\lambda = \lambda_R + i\lambda_I$ ,  $\mu = \mu_R + i\mu_I$  are the discrete eigenvalue of the associated Zakharov-Shabat spectral problem and  $\rho$ ,  $\eta$  are arbitrary complex constants satisfying the conditions  $\gamma \in \mathbb{R}$  and  $\gamma(1 + \gamma) > 0$ .

Case IV).  $x_1 = z = x + iy$ ,  $x_2 = \bar{z} = x - iy$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ . In this case the spectral operator  $\mathbf{T}_1$  takes the form of the elliptic Zakharov-Shabat spectral operator in the plane. The spectral problem (2.8) reads in this case

$$2\mathbf{T}_0\varphi = \left\{ i\partial_t + \partial_z^2 + \partial_{\bar{z}}^2 + \mathbf{A} + \begin{pmatrix} 0 & q_{\bar{z}} \\ r_z & 0 \end{pmatrix} \right\} \varphi = -k^2\varphi. \quad (2.41)$$

The “weak” commutativity condition (1.16) gives an equation, which we call Davey–Stewartson IV (DSIV) equation

$$i\mathbf{Q}_t + \sigma_3(\mathbf{Q}_{zz} - \mathbf{Q}_{\bar{z}\bar{z}}) + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.42)$$

$$\begin{pmatrix} A_{1,\bar{z}} & 0 \\ 0 & A_{2,z} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} (qr)_z & 0 \\ 0 & (qr)_{\bar{z}} \end{pmatrix} \quad (2.43)$$

It can be rewritten by using the real variables  $x$  and  $y$

$$i\mathbf{Q}_t - i\sigma_3 \mathbf{Q}_{xy} + [\mathbf{A}, \mathbf{Q}] = 0 \quad (2.44)$$

$$2(\partial_x + i\sigma_3 \partial_y)\mathbf{A} = -(\partial_x - i\sigma_3 \partial_y)\mathbf{Q}^2. \quad (2.45)$$

However, one can easily verify that for  $t \rightarrow -it$ ,  $\sqrt{2}x \rightarrow x + y$ ,  $\sqrt{2}y \rightarrow x - y$ ,  $A_1 \rightarrow iA_2$ ,  $A_2 \rightarrow iA_1$  the DSIV equation transforms into the DSII equation.

### 3. Evolution of the Spectral Transform

The new obtained equation, the DSIII equation, and the DSI equation are related to the same *principal* spectral problem, the hyperbolic Zakharov-Shabat spectral problem in the plane. As a consequence the direct and the inverse problems are the same. Instead they have different *auxiliary* spectral problems. So the evolution of the spectral transform is not the same. However, by taking into account this difference, all the results known for the DSI equation can be extended to the DSIII equation.

Let us recall that the spectral transform of Fokas and Ablowitz [9, 10] for zero boundary conditions can be extended to include the case of non zero boundary conditions (which allow localized solitons) in different ways [11, 7, 12]. The choice of [13] preserves two relevant properties of the  $1+1$  dimensional case

- 1) the discrete part of the spectrum corresponds to solitons and the continuous part to the radiation;
- 2) the time evolution of the spectral transform can be explicitly integrated.

The spectral transform is defined as the measure of the departure of  $\varphi$  from the analyticity

$$\frac{\partial \varphi}{\partial k} = \iint dl \wedge d\bar{l} \varphi(l) R(k, l). \quad (3.1)$$

The spectral transform  $R$  is trivially related to the spectral transform  $T$  as defined in (1.1). According to the usual scheme for a “weak” Lax pair its time evolution can be considered to be fixed by

$$2T_0\varphi = -k^2\varphi\sigma_3, \quad (3.2)$$

for the DSI equation, and by

$$2T_0\varphi = -k^2\varphi \quad (3.3)$$

for the DSIII equation.

Then we have the explicit form of the evolution of the spectral transform

$$R(k, l, t) = \exp(-i\sigma_3 l^2 t) R(k, l, 0) \exp(i\sigma_3 k^2 t) \quad (3.4)$$

for the DSI equation and

$$R(k, l, t) = \exp(-il^2 t) R(k, l, 0) \exp(ik^2 t) \quad (3.5)$$

for the DSIII equation.

One must also choose

a) a solution of the Zakharov-Shabat spectral problem at  $\mathbf{Q} = \mathbf{0}$  (boundary conditions)

$$\begin{pmatrix} \partial_u & 0 \\ 0 & \partial_v \end{pmatrix} \varphi_0 = 0; \quad (3.6)$$

b) its Green operator  $G$

$$\varphi(u, v, t) = \varphi_0(u, v, t) + G\varphi(u, v, t). \quad (3.7)$$

For b) we use the sectionally holomorphic operator  $G$  of Fokas and Ablowitz [9,10]. The integral equation (3.7) is of Volterra type and therefore the singularities of  $\varphi$  in the complex  $k$ -plane are those of  $\varphi_0$  and of the sectionally holomorphic  $G$  operator.

As regards the boundary conditions for DSIII there is a simple modification to be done with respect to DSI. Let us briefly recall the boundary conditions for DSI. According to (2.15) the boundary conditions of the auxiliary field  $A$  can be arbitrarily chosen, for instance, at  $u = -\infty$  for  $A_1$  and at  $v = -\infty$  for  $A_2$ . Precisely we write

$$A_1(u, v, t) = -\frac{1}{2} \int_{-\infty}^u du'(qr)_v + a_1(v, t) \quad (3.8)$$

$$A_2(u, v, t) = \frac{1}{2} \int_{-\infty}^v dv'(qr)_u + a_2(u, t) \quad (3.9)$$

with  $a_1$  and  $a_2$  arbitrarily given functions. The two boundary values  $a_1$  and  $a_2$  fix the solution

$$\varphi_0 = \text{diag}(\varphi_{01}(v, t), \varphi_{02}(u, t)) \quad (3.10)$$

of (3.6) since from (2.12) at  $\mathbf{Q} = \mathbf{0}$  we get

$$[i\partial_t + \partial_v^2 + a_1(v, t)] \phi_1(v, t) = 0 \quad (3.11)$$

and

$$[-i\partial_t + \partial_u^2 - a_2(u, t)] \phi_2(u, t) = 0 \quad (3.12)$$

where

$$\varphi_{01} = \phi_1 \exp(i k^2 t), \quad \varphi_{02} = \phi_2 \exp(-i k^2 t). \quad (3.13)$$

Therefore  $\phi_1$  and  $\phi_2$  must be solutions of two different non stationary Schrödinger equations, which can be considered as principal spectral problems for two Kadomtsev-Petviashvili I (KPI) equations in the space plane, respectively,  $(v, t)$  and  $(u, t)$ .

Let us, now, consider the DSIII equation. From (2.32) we obtain for the auxiliary field  $A$

$$A_1(u, v, t) = -\frac{1}{2} \int_{-\infty}^u du'(qr)_v + a_1(v, t) \quad (3.14)$$

$$A_2(u, v, t) = -\frac{1}{2} \int_{-\infty}^v dv'(qr)_u + a_2(u, t) \quad (3.15)$$

and from (2.30)

$$[i\partial_t + \partial_v^2 + a_1(v, t)] \phi_1(v, t) = 0 \quad (3.16)$$

$$[i\partial_t + \partial_u^2 + a_2(u, t)] \phi_2(u, t) = 0 \quad (3.17)$$

where

$$\varphi_{01} = \phi_1 \exp(i k^2 t), \quad \varphi_{02} = \phi_2 \exp(i k^2 t). \quad (3.18)$$

#### 4. Multisoliton Solutions

We are interested in the discrete part of the spectrum. Therefore, we choose  $a_1$  and  $a_2$  to be wave-solitons of the KPI equation. If  $a_1(v, t)$  and  $a_2(u, t)$  describe, respectively,  $N$  wave-solitons in the plane  $(v, t)$  and  $M$  wave-solitons in the plane  $(u, t)$ , then the two columns of the  $2 \times 2$  eigenmatrix  $\varphi$  have, respectively,  $N$  poles at  $k = \lambda_n$  ( $n = 1, 2, \dots, N$ ) and  $M$  poles at  $k = \mu_m$  ( $m = 1, 2, \dots, M$ ). All the richness of the dynamics of solitons [9] is discovered by choosing the more general  $N$ -soliton solution for KPI given in [14].

Let us define the spectral transforms  $r_i(k, l)$  for  $a_i$  ( $i = 1, 2$ )

$$\frac{\partial \phi_i}{\partial k} = \iint dl \wedge d\bar{l} \phi_i(l) r_i(k, l). \quad (4.1)$$

The most general discrete spectral transforms compatible with the requirement  $a_i = \bar{a}_i$  is

$$r_1(k, l) = \sum_{n,m} r_{nm}^{(1)} \exp[-i(k^2 - l^2)t] \delta(l - \bar{\lambda}_m) \delta(k - \lambda_n) \quad (4.2)$$

$$r_2(k, l) = \sum_{n,m} r_{nm}^{(2)} \exp[i(k^2 - l^2)t] \delta(l - \bar{\mu}_m) \delta(k - \mu_n) \quad (4.3)$$

for  $a_1$  and  $a_2$  boundaries of the DSII equation and

$$r_1(k, l) = \sum_{n,m} r_{nm}^{(1)} \exp[-i(k^2 - l^2)t] \delta(l - \bar{\lambda}_m) \delta(k - \lambda_n) \quad (4.4)$$

$$r_2(k, l) = \sum_{n,m} r_{nm}^{(2)} \exp[-i(k^2 - l^2)t] \delta(l - \bar{\mu}_m) \delta(k - \mu_n) \quad (4.5)$$

for  $a_1$  and  $a_2$  boundaries of the DSIII equation. The constant matrices  $r^{(1)}$  and  $r^{(2)}$  are subjected to the only condition to be hermitian. To these boundaries corresponds the most general discrete spectral transform of  $Q$ , which is given by

$$R_d(k, l) = -i\pi \begin{pmatrix} 0 & -\tau_2 \exp[-i(k^2 + l^2)t] \\ \tau_1 \exp[i(k^2 + l^2)t] & 0 \end{pmatrix} \quad (4.6)$$

for DSII [9] and by

$$R_d(k, l) = -i\pi \begin{pmatrix} 0 & \tau_2 \exp[i(k^2 - l^2)t] \\ \tau_1 \exp[i(k^2 - l^2)t] & 0 \end{pmatrix} \quad (4.7)$$

for DSIII with

$$\begin{aligned} \tau_1 &= \sum_{in} \rho_{in} \exp[i l u_o(l) + i k v_o(k)] \\ &\quad \times \left[ (l - \mu_n) \delta(l - \mu_n) + \sum_m C_{nm} \bar{\mu}_{mm} \delta(l - \bar{\mu}_m) \right] \delta(k - \lambda_i) \end{aligned} \quad (4.8)$$

$$\begin{aligned} \tau_2 &= \sum_{in} \eta_{in} \exp[-i l v_o(l) - i k u_o(k)] \\ &\quad \times \left[ (l - \lambda_n) \delta(l - \lambda_n) + \sum_m D_{nm} \bar{\lambda}_{mm} \delta(l - \bar{\lambda}_m) \right] \delta(k - \mu_i) \end{aligned} \quad (4.9)$$

where  $\rho$  and  $\eta$  are arbitrary complex constant matrices,  $v_o(\lambda_n) = v_o(\bar{\lambda}_n) = v_{on}$ ,  $u_o(\mu_n) = u_o(\bar{\mu}_n) = u_{on}$  fix the initial position of the  $(n, m)$  soliton and  $C_{mn}$ ,  $D_{mn}$  are related to  $r^{(1)}$

and  $r^{(2)}$  (see [9] for details and for the constraints obtained as a consequence of a possible reduction). The distributions  $\delta(l - l_o)$  and  $(l - l_o)\delta(l - l_o)$  are defined as follows

$$\iint dl \wedge d\bar{l} \delta(l - l_o) f(l) = f(l_o) \quad (4.10)$$

$$\iint dl \wedge d\bar{l} (l - l_o)\delta(l - l_o) f(l) = \text{Res}(f, l_o). \quad (4.11)$$

We can conclude that the  $N \times M$ -soliton solution of the DSI equation can be easily modified to obtain the  $N \times M$ -soliton solution of the DSIII equation. Also for this equation the boundaries fix the kinematics of the solitons while the dynamics of their mutual interaction is fixed by the chosen initial condition. Moreover, also for the DSIII equation the solution can simulate quantum effects as inelastic scattering, fusion and fission, creation and annihilation.

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# ON GASDYNAMIC-SOLITONIC CONNECTIONS

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**ABSTRACT.** A link between a class of infinitesimal Bäcklund transformations of the hodograph equations of isentropic gasdynamics and novel  $2 + 1$ -dimensional integrable systems is reviewed. In addition, it is noted that a special Monge-Ampère equation descriptive of  $1 + 1$ -dimensional anisentropic gas flows with an appropriate equation of state may be linked in a geometric manner to the Dodd-Bullough equation. The Monge-Ampère equation is shown to admit a reciprocal invariance of Haar-type related to a duality inherent in the integrable gasdynamic system.

## 1. Introduction

Loewner in 1950, in a study of approximation methods in plane, isentropic gasdynamics was led to introduce a generalization of the classical notion of a Bäcklund transformation. Thus, in [1], Loewner sought reductions of hodograph systems via finite matrix Bäcklund transformations to the Cauchy-Riemann, Tricomi or classical wave equations in subsonic, transonic and supersonic régimes in turn. Such reductions may be achieved for certain multi-parameter equations of state which may be used to approximate real gas behaviour. A detailed survey of that work along with other physical applications of the Loewner transformations to boundary value problems in nonlinear elasticity and dielectrics is given in [2]. Connections between these transformations and Moutard's theorem as used in the construction of dromion solutions of the Nizhnik-Novikov-Veselov equation [3] have recently been described by Athorne [4].

In subsequent work, Loewner [5] extended his analysis to infinitesimal Bäcklund transformations which asymptotically reduce the hodograph equations to appropriate canonical forms. Recently, Konopelchenko and Rogers [6,7] have shown that, remarkably, on suitable interpretation, this work of Loewner has a strong connection with soliton theory. Thus, if the continuous parameter in the infinitesimal Bäcklund transformations is regarded as a third independent variable, then a linear triad is evident in the Loewner scheme. The compatibility conditions for the latter generate a wide class of integrable  $2 + 1$ -dimensional nonlinear evolution equations wherein the spatial variables occur on an equal basis. A key feature of the  $2 + 1$ -dimensional nonlinear systems so constructed is that they admit

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a compact parametrization. The class contains, in particular, 2 + 1-dimensional integrable extensions of the principal chiral fields model and Toda lattice equations as well as a matrix generalization of an integrable system recently discussed by Dryuma [8] and Tsarëv [9, 10].

The Davey-Stewartson and Nizhnik-Novikov-Veselov equations represent, in turn, 2 + 1-dimensional integrable generalizations of the nonlinear Schrödinger and Korteweg-de Vries equations wherein the two spatial variables have the same status. Moreover, both of these equations admit coherent structure solutions of dromion-type. Konopelchenko and Rogers in [6] have derived a 2 + 1-dimensional integrable extension of the classical sine-Gordon equation in which the spatial variables likewise occur in a symmetric manner. This has been shown to be amenable to the  $\bar{\partial}$ - $\partial$  dressing method by Konopelchenko and Dubrovsky [11]. An auto-Bäcklund transformation has been constructed in [12] which extends the classical result for the sine-Gordon equation to 2+1-dimensions. Nimmo [13,14] has applied Darboux-type transformations to obtain solutions of the 2 + 1-dimensional sine-Gordon equation in terms of Pfaffians of solutions of the associated spectral problem. In a recent development, Schief [15] has obtained localized soliton-like solutions.

In anisentropic gasdynamics, a connection with soliton theory has been developed in a series of interesting papers by Gaffet [16–19]. Therein it has been shown ‘inter-alia’ that the Lagrangian equations of gasdynamics with a certain class of equations of state may be reduced to the integrable Dodd-Bullough equation. In [16] a spinorial connection was made with a special Monge-Ampère equation. Here, it is shown that the latter may be linked to the Dodd-Bullough equation in an interesting geometric manner implicit in a paper by Tzitzéica [20], published in 1910. It is also noted that a duality principle admitted by the integrable gasdynamic system is equivalent to a Haar-type invariance of the canonical Monge-Ampère equation. The auto-Bäcklund transformation for the Dodd-Bullough equation as derived in [18] turns out to be a direct consequence of Moutard’s theorem.

## 2. The Loewner transformations. Gasdynamic genesis

Loewner [1] undertook a systematic investigation of the reduction of the hodograph equations of gasdynamics to appropriate canonical forms in subsonic, transonic and supersonic régimes. Multi-parameter gas laws were sought for which such reductions are admitted. To this end, Loewner introduced a generalization of the notion of classical Bäcklund transformation in the form

$$\mathbf{B}_i(x^a, u, \partial u / \partial x^a, v, \partial v / \partial x^a; x'^a, u', \partial u' / \partial x'^a, v', \partial v' / \partial x'^a) = 0, \quad (2.1)$$

$i = 1, \dots, 6$ ,  $a = 1, 2$ , which relate surface elements  $\{x^a, u, \partial u / \partial x^a\}$ ,  $\{x^a, v, \partial v / \partial x^a\}$  and  $\{x'^a, u', \partial u' / \partial x'^a\}$ ,  $\{x'^a, v', \partial v' / \partial x'^a\}$  associated with pairs of surfaces  $u = u(x^a)$ ,  $v = v(x^a)$  and  $u' = u'(x'^a)$ ,  $v' = v'(x'^a)$ . In particular, in the gasdynamics context, Loewner was led to introduce linear Bäcklund transformations of the matrix type

$$\begin{aligned} \psi'_y &= \Lambda \psi_y + \mathbf{A} \psi + \mathbf{B} \psi', \\ \psi'_x &= \tilde{\Lambda} \psi_x + \tilde{\mathbf{A}} \psi + \tilde{\mathbf{B}} \psi', \\ x' &= x, \quad y' = y, \end{aligned} \quad \mathbf{L} \quad (2.2)$$

which take  $2 \times 2$  linear matrix systems

$$\psi_x = \mathbf{S} \psi_y \quad (2.3)$$

to associated matrix systems

$$\psi'_x = \mathbf{S}'\psi'_y \quad (2.4)$$

with a view to the reduction of the latter to specific canonical forms associated with the Cauchy-Riemann, Tricomi and classical wave equations. In Loewner's work (2.3) was taken as the Molenbroek-Chaplygin version of the hodograph equations.

It is noted that the Loewner transformations (2.2) can be written as a composition of Darboux transformations

$$\psi \xrightarrow{\mathcal{D}_x(\mathbf{A})} \psi' = (\mathbf{I}\partial_x + \mathbf{A})\psi,$$

and gauge transformations

$$\psi \xrightarrow{G(\mathbf{V})} \psi' = \mathbf{V}\psi,$$

according to

$$\begin{aligned} \psi' &= \mathcal{D}_y^{-1}(-\mathbf{B})G(\Lambda)\mathcal{D}_y(\Lambda^{-1}\mathbf{A})\psi, \\ &= \mathcal{D}_x^{-1}(-\tilde{\mathbf{B}})G(\tilde{\Lambda})\mathcal{D}_x(\tilde{\Lambda}^{-1}\tilde{\mathbf{A}})\psi. \end{aligned}$$

The Moutard transformation

$$\psi' = 1/\psi, \quad U' = U - 2(\ln \psi)_{xy},$$

which takes the linear hyperbolic equation

$$\psi_{xy} = U\psi,$$

to its primed counterpart

$$\psi'_{xy} = U'\psi'$$

is readily obtained as a scalar specialization of  $\mathbb{L}$ .

In  $1+1$ -dimensions, Darboux's theorem is retrieved by appropriate separation of variables. In  $2+1$ -dimensions, Moutard's theorem has recently been employed by Athorne and Nimmo [3] to construct dromion solutions of the Nizhnik-Novikov-Veselov equation

$$u_t = \left[ u_{xx} + 3u(\partial_x\partial_y^{-1}u) \right]_x + \left[ u_{yy} + 3u(\partial_y\partial_x^{-1}u) \right]_y. \quad (2.5)$$

The dromion solutions derived therein apply '*mutatis mutandis*' to the  $2+1$ -dimensional Dodd-Bullough equation [7]

$$\partial_t(e^\phi) = (\partial_x e^\phi \partial_y^{-1} e^{-2\phi} \partial_x + \partial_y e^\phi \partial_x^{-1} e^{-2\phi} \partial_y)(e^{2\phi} \phi_{xy} + e^{3\phi}) \quad (2.6)$$

obtained by setting  $u = e^\phi$  in (2.5). It may be shown that a version of Moutard's theorem can be used to construct an auto-Bäcklund transformation for the  $2+1$ -dimensional sine-Gordon equation [12]. Plane wave soliton and breather solutions of an integrable system in which the latter equation is embedded have been generated via Moutard's theorem by Nimmo [13].

It is concluded that the finite Bäcklund transformations of Loewner-type are, through Moutard's theorem, of relevance to the study of 2 + 1-dimensional integrable systems. In interesting recent work, Athorne [4] has established an extension of a theorem due to Goursat which allows the characterization of those Loewner transformations which may be regarded as generalized Moutard transformations. Here, it will be seen that an infinitesimal version of the Loewner transformations leads, on appropriate interpretation, to a novel class of 2 + 1-dimensional integrable systems.

In 1952 Loewner [5], again in a gasdynamic context, introduced the notion of an infinitesimal Bäcklund transformation. This adopted the form

$$\psi_{y\epsilon} = \mathbf{U}\psi_\epsilon + \mathbf{V}\psi_y + \mathbf{W}\psi, \quad (2.7)$$

$$\psi_{x\epsilon} = \tilde{\mathbf{U}}\psi_\epsilon + \tilde{\mathbf{V}}\psi_x + \tilde{\mathbf{W}}\psi, \quad (2.8)$$

and, on application to the matrix equation (2.3) may be shown to lead to the restrictions

$$\begin{aligned} \tilde{\mathbf{U}} &= \mathbf{S}\mathbf{U}, \quad \tilde{\mathbf{V}} = \mathbf{V}, \quad \tilde{\mathbf{W}} = \mathbf{S}\mathbf{W}, \\ \mathbf{U}_x - \tilde{\mathbf{U}}_y + [\mathbf{U}, \tilde{\mathbf{U}}] &= \mathbf{0}, \\ \mathbf{W}_x - \tilde{\mathbf{W}}_y + \mathbf{U}\tilde{\mathbf{W}} - \tilde{\mathbf{U}}\mathbf{W} &= \mathbf{0}, \\ \mathbf{V}_x - \tilde{\mathbf{W}} - \tilde{\mathbf{U}}\mathbf{V} - (\tilde{\mathbf{V}}_y - \mathbf{W} - \mathbf{U}\mathbf{V})\mathbf{S} &= \mathbf{0}, \\ \mathbf{S}_\epsilon &= [\mathbf{V}, \mathbf{S}]. \end{aligned} \quad (2.9)$$

It is observed that (2.3) augmented by (2.7), (2.8) may be viewed as a linear triad with compatibility conditions the nonlinear system (2.9). If the continuous parameter  $\epsilon$  in the infinitesimal Bäcklund transformations is reinterpreted as a new distinguished independent variable  $z$ , then the compatibility conditions may be shown to yield novel integrable 2 + 1-dimensional systems. These are described in the next section.

### 3. A novel class of integrable 2+1-dimensional equations

Here, we shall be concerned with the 2 + 1-dimensional nonlinear equations obtained as compatibility conditions for a linear triad of generalized Loewner-type, namely

$$\begin{aligned} \mathbf{L}_1\Psi &:= (\mathbf{I}\partial_x - \mathbf{S}\partial_y - \mathbf{P})\Psi = \mathbf{0}, \\ \mathbf{L}_2\Psi &:= (\mathbf{I}\partial_y\partial_z - \mathbf{U}\partial_z - \mathbf{V}\partial_y - \mathbf{W})\Psi = \mathbf{0}, \\ \mathbf{L}_3\Psi &:= (\mathbf{I}\partial_x\partial_z - \tilde{\mathbf{U}}\partial_z - \tilde{\mathbf{V}}\partial_x - \tilde{\mathbf{W}})\Psi = \mathbf{0}, \end{aligned} \quad (3.1)$$

where  $\mathbf{U}, \mathbf{V}, \mathbf{W}, \tilde{\mathbf{U}}, \tilde{\mathbf{V}}, \tilde{\mathbf{W}}, \mathbf{S}$  and  $\mathbf{P}$  are  $N \times N$  matrices dependent on the three variables  $x, y$  and  $z$ .

It is readily shown that the compatibility conditions for the linear system (3.1) comprise the nonlinear system

$$\begin{aligned} \mathbf{S}_z &= [\mathbf{V}, \mathbf{S}], \quad \mathbf{P}_z = \tilde{\mathbf{W}} - \mathbf{S}\mathbf{W} + \mathbf{V}\mathbf{P}, \\ \tilde{\mathbf{U}} &= \mathbf{S}\mathbf{U} + \mathbf{P}, \quad \tilde{\mathbf{V}} = \mathbf{V}, \quad \mathbf{U}_x - \tilde{\mathbf{U}}_y + [\mathbf{U}, \tilde{\mathbf{U}}] = \mathbf{0}, \\ \mathbf{V}_x - \tilde{\mathbf{U}}\mathbf{V} - \tilde{\mathbf{W}} + (\mathbf{U}\mathbf{V} - \mathbf{W} - \mathbf{V}_y)\mathbf{S} &= \mathbf{0}, \\ \mathbf{W}_x - \tilde{\mathbf{W}}_y + \mathbf{U}\tilde{\mathbf{W}} - \tilde{\mathbf{U}}\mathbf{W} + (\mathbf{U}\mathbf{V} - \mathbf{W} - \mathbf{V}_y)\mathbf{P} &= \mathbf{0}. \end{aligned} \quad (3.2)$$

The constraints (3.2) admit an alternative representation [7]

$$\begin{aligned} \mathbf{S}_z &= [\phi_1 + \phi_2(\partial_y^{-1}\tilde{\phi}_3), \mathbf{S}], \\ \mathbf{P}_z &= \phi_{1x} - \mathbf{S}\phi_{1y} - [\mathbf{P}, \phi_1] + \phi_2 \left[ \partial_y^{-1}\tilde{\phi}_{3x} - \tilde{\phi}_3\mathbf{S} + (\partial_y^{-1}\tilde{\phi}_3)\mathbf{P} \right], \\ \phi_{2x} - \mathbf{S}\phi_{2y} - \mathbf{P}\phi_2 &= \mathbf{0}, \\ \tilde{\phi}_{3x} - (\tilde{\phi}_3\mathbf{S})_y + \tilde{\phi}_3\mathbf{P} &= \mathbf{0}, \end{aligned} \quad (3.3)$$

where, in terms of the  $N \times N$  matrices  $\phi_1, \phi_2, \tilde{\phi}_3$  and  $\mathbf{S}$ ,

$$\begin{aligned} \mathbf{U} &= \phi_{2y}\phi_2^{-1}, \quad \tilde{\mathbf{U}} = \phi_{2x}\phi_2^{-1}, \quad \mathbf{V} = \tilde{\mathbf{V}} = \phi_1 + \phi_2(\partial_y^{-1}\tilde{\phi}_3), \\ \mathbf{W} &= \phi_2(\phi_2^{-1}\phi_1)_y, \quad \tilde{\mathbf{W}} = \phi_2(\phi_2^{-1}\phi_1)_x + \phi_2(\partial_y^{-1}\tilde{\phi}_{3x}) - \phi_2\tilde{\phi}_3\mathbf{S}. \end{aligned} \quad (3.4)$$

The nonlinear system (3.3) includes a number of interesting subcases. Thus, if  $\mathbf{S}$  is a constant matrix then the so-called 2 + 1-dimensional non-Abelian sine-Gordon system

$$\begin{aligned} \mathbf{P}_z &= [\mathbf{S}, \phi_2\tilde{\phi}_3], \\ \phi_{2x} - \mathbf{S}\phi_{2y} - \mathbf{P}\phi_2 &= \mathbf{0}, \\ \phi_{3x} - \phi_{3y}\mathbf{S} + \phi_3\mathbf{P} &= \mathbf{0}, \end{aligned} \quad (3.5)$$

results. If  $\mathbf{P} = \mathbf{0}$  then the linear triad system (3.1) reduces to

$$\begin{aligned} \mathbf{L}_1\Psi &:= [\mathbf{I}\partial_x - \mathbf{S}\partial_y]\Psi = \mathbf{0}, \\ \mathbf{L}_2\Psi &:= [\mathbf{I}\partial_y\partial_z - \phi_{2y}\phi_2^{-1}\partial_z - (\phi_1 + \phi_2\phi_3)\partial_y - \phi_2(\phi_2^{-1}\phi_1)_y]\Psi = \mathbf{0}, \\ \mathbf{L}_3\Psi &:= [\mathbf{I}\partial_x\partial_z - \phi_{2x}\phi_2^{-1}\partial_z - (\phi_1 + \phi_2\phi_3)\partial_x - \phi_2(\phi_2^{-1}\phi_1)_x]\Psi = \mathbf{0}, \end{aligned} \quad (3.6)$$

with compatibility conditions the nonlinear system

$$\mathbf{S}_z = [\phi_1 + \phi_2\phi_3, \mathbf{S}], \quad \phi_{1x} = \mathbf{S}\phi_{1y}, \quad \phi_{2x} = \mathbf{S}\phi_{2y}, \quad \phi_{3x} = \phi_{3y}\mathbf{S} \quad (3.7)$$

( $\phi_3 := \partial_y^{-1}\tilde{\phi}_3$ ). Two subcases of the system are of particular importance and are considered below:

CASE 1,  $\phi_1 = \mathbf{I}_N$ . In this case, the system (3.7) adopts the form

$$\mathbf{S}_z = [\phi_2\phi_3, \mathbf{S}], \quad \phi_{2x} = \mathbf{S}\phi_{2y}, \quad \phi_{3x} = \phi_{3y}\mathbf{S}, \quad (3.8)$$

or, equivalently, since  $\mathbf{W} = -\mathbf{U}$ ,

$$\begin{aligned} \mathbf{S}_z &= [\mathbf{V}, \mathbf{S}], \\ \mathbf{U}_x - (\mathbf{S}\mathbf{U})_y + [\mathbf{U}, \mathbf{S}\mathbf{U}] &= \mathbf{0}, \\ \mathbf{V}_x - \mathbf{V}_y\mathbf{S} - [\mathbf{S}, \mathbf{U}\mathbf{V}] + [\mathbf{S}, \mathbf{U}] &= \mathbf{0}. \end{aligned} \quad (3.9)$$

In the 1 + 1-dimensional limit  $\mathbf{S}_y = \mathbf{V}_y = \mathbf{0}$ , the specialization  $U = \mathbf{I}_N$  in (3.9) leads to the system

$$\mathbf{S}_z - [\mathbf{V}, \mathbf{S}] = \mathbf{0}, \quad \mathbf{V}_x + [\mathbf{V}, \mathbf{S}] = \mathbf{0}.$$

This is equivalent to the principal chiral fields model equation

$$(\Omega_x \Omega^{-1})_z + (\Omega_z \Omega^{-1})_x = 0 \quad (3.10)$$

with  $S = \frac{1}{2} \Omega_x \Omega^{-1}$ ,  $V = \frac{1}{2} \Omega_z \Omega^{-1}$ .

CASE 2,  $\phi_2 = I_N$ . In this case, the system (3.7) yields

$$S_z = [\phi_1 + \phi_3, S], \quad \phi_{1x} = S\phi_{1y}, \quad \phi_{3x} = \phi_{3y}S,$$

or, equivalently, since  $U = \tilde{U} = 0$ ,

$$\begin{aligned} S_z - [V, S] &= 0, \\ V_x - V_y S + [W, S] &= 0, \\ W_x - (SW)_y &= 0. \end{aligned} \quad (3.11)$$

In the  $1+1$ -dimensional limit  $S_y = V_y = W_y = 0$ , the system (3.11) reduces to the pair

$$S_z + [S, \partial_x^{-1}[S, W]] = 0, \quad W_x = 0 \quad (3.12)$$

as considered by Bruschi and Ragnisco [21]. Therein, the special case of the reduction  $S^N = I$  with

$$S = \begin{pmatrix} 0 & g_1 & 0 & \dots & 0 \\ 0 & 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ g_N & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (3.13)$$

where  $g_1, g_2, \dots, g_N$  are scalar functions subject to the condition

$$\det S = g_1 g_2 \dots g_N = 1, \quad (3.14)$$

was shown to lead to the two-dimensional Toda lattice scheme [22]. The latter contains the sine-Gordon and Dodd-Bullough equations as reductions with  $N = 2$  and  $N = 3$  respectively. The sharpline limit of the self-induced transparency (SIT) system was also retrieved in [21] as a reduction of (3.12) with  $S$  of the form (3.13). It is noted that this SIT system is gauge-equivalent to the Regge-Lund system and may also be transformed to the stimulated Raman scattering (SRS) system (Steudel [23]).

Insertion of the representation (3.13) into the nonlinear system (3.11) produces an integrable  $2+1$ -dimensional Toda lattice scheme [6]. In particular, as a special reduction with  $N = 2$  it yields the  $2+1$ -dimensional sine-Gordon system

$$\left( \frac{\theta_{xz}}{\sin \theta} \right)_x - \left( \frac{\theta_{zy}}{\sin \theta} \right)_y + \left( \frac{\theta_x \tilde{\theta}_y - \theta_y \tilde{\theta}_x}{\sin^2 \theta} \right) = 0, \quad (3.15a)$$

$$\left( \frac{\tilde{\theta}_x}{\sin \theta} \right)_x - \left( \frac{\tilde{\theta}_y}{\sin \theta} \right)_y + \left( \frac{\theta_x \theta_{zy} - \theta_y \theta_{xz}}{\sin^2 \theta} \right) = 0, \quad (3.15b)$$

corresponding to the linear triad representation (3.1) with

$$\begin{aligned} S &= - \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \\ P = U = \tilde{U} &= 0, \quad V = \tilde{V} = \frac{1}{2} \begin{pmatrix} 0 & -\theta_z \\ \theta_z & 0 \end{pmatrix}, \\ W &= \frac{1}{4} \partial_y \begin{pmatrix} \partial_x^{-1} K, & -\theta_z + \tilde{\theta} \\ \theta_z + \tilde{\theta}, & \partial_x^{-1} L \end{pmatrix}, \quad \tilde{W} = \frac{1}{4} \partial_x \begin{pmatrix} \partial_x^{-1} K, & -\theta_z + \tilde{\theta} \\ \theta_z + \tilde{\theta}, & \partial_x^{-1} L \end{pmatrix}, \end{aligned} \quad (3.16)$$

where

$$K = (\theta_z + \tilde{\theta})_x \cot \theta - (\theta_z + \tilde{\theta})_y \operatorname{cosec} \theta, \quad L = (\theta_z - \tilde{\theta})_x \cot \theta - (\theta_z - \tilde{\theta})_y \operatorname{cosec} \theta.$$

The specialization  $\tilde{\theta} = \theta_z$  in (3.15) produces the single 2 + 1-dimensional sine-Gordon equation (Konopelchenko and Rogers [6])

$$(\theta_{zx}/\sin \theta)_x - (\theta_{zy}/\sin \theta)_y + (\theta_x \theta_{zy} - \theta_y \theta_{zx})/\sin^2 \theta = 0. \quad (3.17)$$

In an interesting recent development [24] it has been shown that the Zakharov-Manakov-type linear triad [25]

$$\begin{aligned} \psi_{2uv} &= \alpha_v \alpha^{-1} \psi_{2u} + \beta_u \beta^{-1} \psi_{2v}, \\ \psi_{2uz} &= \beta_z \beta^{-1} \psi_{2v} + \gamma_v \gamma^{-1} \psi_{2z}, \\ \psi_{2zu} &= \gamma_u \gamma^{-1} \psi_{2z} + \alpha_z \alpha^{-1} \psi_{2u}, \end{aligned} \quad (3.18)$$

with integrability conditions the nonlinear matrix system

$$\begin{aligned} \alpha_{uz} &= \gamma_v \gamma^{-1} \alpha_z + \beta_z \beta^{-1} \alpha_v, \\ \beta_{zu} &= \alpha_z \alpha^{-1} \beta_u + \gamma_u \gamma^{-1} \beta_z, \\ \gamma_{uv} &= \beta_u \beta^{-1} \gamma_u + \alpha_v \alpha^{-1} \gamma_u, \end{aligned} \quad (3.19)$$

may be incorporated into the subcase  $\phi_2 = I_N$  of the Loewner system with  $S^2 = I$ . This reduction is achieved by setting  $\Psi = (\psi_1, \psi_2)^t$  where  $\psi_1 = \gamma^{-1} \psi_{2z}$  and introducing the coordinates  $x, y$  according to  $x = u + v, y = u - v$ .

The system (3.18), in the scalar case, appears in the context of the theory of orthogonal systems of curvilinear coordinates in a treatise by Darboux which appeared in 1910 [26]. It has recently been discussed by Dryuma [8] and Tsarëv [9, 10].

The Darboux system (3.18) is of current interest since, in particular, the scalar specialization

$$\alpha = \cos\left(\frac{1}{2}\theta\right), \quad \beta = \sin\left(\frac{1}{2}\theta\right), \quad \gamma = \left(\frac{1}{2}\theta_z\right), \quad (3.20)$$

produces, via the integrability conditions (3.19), the 2+1-dimensional sine-Gordon equation (3.17). Moreover, the SIT equations may be shown to be contained within the Darboux system as a 1 + 1-dimensional reduction [24].

#### 4. Geometry and Haar invariance of an integrable system in anisentropic gasdynamics

It has been shown in [17, 18] that, remarkably, for a special class of gas laws, the governing equations of 1 + 1-dimensional anisentropic gasdynamics may be reduced to the Dodd-Bullough equation via a Riemann invariant formalism. Thus, the resulting gasdynamic system is integrable and, in principle, is amenable to the inverse scattering method with a third-order scattering operator. Soliton solutions have been constructed in [18]. It was also demonstrated in [16] that this integrable system admits reduction to an interesting Monge-Ampère equation in spinor components. Here, it is shown that this Monge-Ampère equation has its origins in a geometric problem studied by Tzitzéica in 1910 [20]. Therein

is found an implicit link not only to the Dodd-Bullough equation but also to its linear triad representation. In another historical link, a gasdynamic duality principle set down in [17] is noted to be equivalent to a Haar-type invariance published in 1928 [27, 28]. Moreover, the auto-Bäcklund transformation for the Dodd-Bullough equation turns out to be a simple consequence of Moutard's theorem as applied to a linear triad representation. The latter can be obtained as a  $1 + 1$ -dimensional reduction of a Loewner system.

The Lagrangian equations of  $1 + 1$ -dimensional anisentropic gasdynamics are

$$v_X = e_t, \quad (4.1)$$

$$\rho_0 v_t = -p_X, \quad (4.2)$$

together with an appropriate equation of state

$$p = -\Sigma(e, X). \quad (4.3)$$

Here,  $X$  is a material coordinate and  $t$  denotes time. If  $x = x(X, t)$  denotes the Eulerian coordinate then

$$v = \frac{\partial x}{\partial t} \Big|_X, \quad e = \frac{\partial x}{\partial X} \Big|_t - 1 = \frac{\rho_0}{\rho} - 1 \quad (4.4), (4.5)$$

designate, in turn, the material velocity and stretch;  $p$  and  $\rho$  denote the gas pressure and density respectively.

The subsequent results will hold 'mutatis mutandis' for the uniaxial deformation  $X \rightarrow x(X, t)$  of certain nonlinear elastic materials with appropriate density distribution  $\rho_0(X)$  in the undeformed configuration. Therein, the governing equations consist of the compatibility condition (4.1) together with the equation of motion

$$\rho_0(X)v_t = T_X$$

and stress-strain law

$$T = T \left( \frac{\partial X}{\partial x} \right),$$

where  $T$  denotes the traction. The resultant nonlinear equation of motion is (Keller and Ting [29])

$$\partial_X((\Sigma'(e)/\rho_0(X))e_X) = e_{tt}.$$

Here, we proceed with the anisentropic gasdynamic system (4.1), (4.2) and restrict attention to polytropic gas laws

$$p = \rho^\gamma X^{-b}. \quad (4.6)$$

The continuity equation (4.1) is embodied in the relation

$$dx = \rho^{-1}dX + vdt \quad (4.7)$$

whence, on introduction of the new independent variable  $w = vt - x$  it is seen that

$$dX = \rho(tdv - dw). \quad (4.8)$$

Accordingly, the continuity equation in terms of  $v, w$  coordinates becomes

$$\rho_v + (\rho t)_w = 0. \quad (4.9)$$

The momentum equation (4.2) implies the existence of  $\Pi(X, t)$  such that

$$vdX - pdt = d\Pi, \quad (4.10)$$

or, in terms of  $v, w$  coordinates

$$dv(\rho vt - pt_v) + dw(-\rho v - pt_w) = d\Pi$$

whence

$$p_w t_v - p_v t_w = -\rho. \quad (4.11)$$

At this stage, it proves convenient to introduce a new material coordinate

$$\Lambda = \alpha X^\delta \quad (4.12)$$

whence (4.9) yields

$$t = -X_v/X_w = -\Lambda_v/\Lambda_w. \quad (4.13)$$

Substitution in the momentum equation (4.11) gives, on requiring that  $p = p(\Lambda_w)$ , the Monge-Ampère equation

$$\Lambda_{vv}\Lambda_{ww} - \Lambda_{vw}^2 = \rho\Lambda_w/p'(\Lambda_w). \quad (4.14)$$

In view of the gas law (4.6) and the relation

$$\rho = -X_w = -\Lambda_w/\alpha\delta(\Lambda/\alpha)^{(\delta-1)/\delta},$$

it is seen that

$$p = \left( \frac{-\Lambda_w}{\alpha\delta(\Lambda/\alpha)^{(\delta-1)/\delta}} \right)^\gamma \left( \frac{\Lambda}{\alpha} \right)^{-b/\delta}.$$

The condition  $p = p(\Lambda_w)$  requires that  $\delta = (\gamma - b)/\gamma$  in (4.12) whence

$$p = \left( \frac{-\Lambda_w}{\alpha\delta} \right)^\gamma. \quad (4.15)$$

If we set  $\gamma = 3$ ,  $\alpha\delta = \pm\sqrt{3}$  then the Monge-Ampère equation (4.14) reduces to

$$\Lambda_{vv}\Lambda_{ww} - \Lambda_{vw}^2 = \Lambda^{(1-\delta)/\delta}. \quad (4.16)$$

Of particular interest are the cases  $\delta = 1$  and  $\delta = \frac{1}{3}$  corresponding, in turn, to the isentropic case  $\gamma = 3$ ,  $b = 0$  and the anisentropic case  $\gamma = 3$ ,  $b = 4$  as considered in [16-19].

In the isentropic case, the original Lagrangian system may be linearized via a hodograph transformation so that, in particular, the Monge-Ampère equation

$$\Lambda_{vv}\Lambda_{ww} - \Lambda_{vw}^2 = 1 \quad (4.17)$$

corresponding to the isentropic case with  $\gamma = 3$  must be linearizable. Indeed, this may be achieved by the transformation  $R$

$$\Lambda^* = -\Lambda + \frac{1}{2}(\Lambda_v^2 + \Lambda_w^2), \quad v^* = v - \Lambda_v, \quad w^* = w - \Lambda_w \quad (4.18)$$

(note that  $R^2 = I$ ), which takes (4.17) to

$$\Lambda_{x^*x^*}^* + \Lambda_{t^*t^*}^* = 1. \quad (4.19)$$

A transformation of the type (4.18) was introduced by Hoskins and Bretherton [30] in connection with the linearization of a boundary value problem in atmospheric frontogenesis. The transformation  $R$  is sometimes known in that context as the geostrophic coordinate transformation (Blumen [31]). Extension of such transformations to  $3+1$ -dimensions have been recently introduced to linearize higher-dimensional Monge-Ampère equations which arise naturally in the study of the propagation of discontinuity waves [32].

In the anisentropic case  $\gamma = 3$ ,  $b = 4$  corresponding to the gas law

$$p = \rho^3 X^{-4} \quad (4.20)$$

the Monge-Ampère equation

$$\Lambda_{vv}\Lambda_{vw} - \Lambda_{vw}^2 = \Lambda^{-4} \quad (4.21)$$

is obtained [16]. Here, it is noted that (4.21) may be linked to the integrable Dodd-Bullough equation and its linear triad representation via a classical geometric problem considered by Tzitzéica [20] in 1910.

Thus, we consider surfaces  $\Sigma$  such that the total curvature  $K$  at a point  $P$  is proportional to  $d^4$  where  $d$  is the distance from the origin to the tangent plane to  $\Sigma$  at  $P$ . If the equation of  $\Sigma$  is taken in the Monge form  $z = f(x, y)$  then this condition translates to

$$f_{xx}f_{yy} - f_{xy}^2 = (f - xf_x - yf_y)^4 \quad (4.22)$$

where the constant of proportionality has been taken to be unity without loss of generality. Introduction of the Legendre transformation

$$\Phi = f - xf_x - yf_y, \quad \bar{x} = f_x, \quad \bar{y} = f_y \quad (4.23)$$

takes (4.22) to the Monge-Ampère equation

$$\Phi_{\bar{x}\bar{x}}\Phi_{\bar{y}\bar{y}} - \Phi_{\bar{x}\bar{y}}^2 = \Phi^{-4}. \quad (4.24)$$

Alternatively, in terms of its asymptotic lines  $(\xi, \eta)$ , the surface  $\Sigma : x = x(\xi, \eta), y = y(\xi, \eta), z = z(\xi, \eta)$  is given by three linearly independent integrals of the system [20]

$$\begin{aligned} \psi_{2\xi\eta} &= e^{-\theta}\psi_2, \\ \psi_{2\xi\xi} &= e^\theta\psi_{2\eta} - \theta_\xi\psi_{2\xi}, \\ \psi_{2\eta\eta} &= e^\theta\psi_{2\xi} - \theta_\eta\psi_{2\eta}, \end{aligned} \quad (4.25)$$

where  $\theta$  satisfies the Dodd-Bullough equation

$$\theta_{\xi\eta} = e^{2\theta} - e^{-\theta}. \quad (4.26)$$

Indeed composition of (4.25) with the invariance  $\xi \rightarrow \lambda^{-1}\xi, \eta \rightarrow \lambda\eta$  of (4.26) produces a linear representation. The latter may be also obtained via the 1 + 1-dimensional reduced Loewner triad representation

$$L_1\Psi := \left( I\partial_\xi - \lambda \begin{pmatrix} 0 & e^{-\theta} & 0 \\ 0 & 0 & e^{-\theta} \\ e^{2\theta} & 0 & 0 \end{pmatrix} \right) \Psi = \mathbf{0}, \quad (4.27)$$

$$L_2\Psi := \left( \lambda I\partial_\eta - \lambda V - \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \Psi = \mathbf{0}, \quad (4.27)$$

$$L_3\Psi := \left( I\partial_\xi\partial_\eta - V\partial_\xi - \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & e^{2\theta} \end{pmatrix} \right) \Psi = \mathbf{0}, \quad (4.27)$$

where

$$V = \partial_\xi^{-1} \begin{pmatrix} e^{-\theta} - e^{2\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{2\theta} - e^{-\theta} \end{pmatrix} \quad (4.28)$$

and  $\Psi = (\psi_1, \psi_2, \psi_3)^t$ .

An auto-Bäcklund transformation for (4.26) is readily generated via Moutard's theorem. Thus, (4.25) is seen to be invariant under

$$\psi'_2 = 1/\psi_2, \quad u' = u + 2(\ln \psi_2)_{\xi\eta}, \quad \xi' = -\xi, \quad \eta' = -\eta, \quad (4.29)$$

where  $u = -e^{-\theta}$ . The relation (4.29)<sub>2</sub> yields

$$e^{-\theta} + e^{-\theta'} = 2\psi_2\xi\psi_{2\eta}/\psi_2^2 \quad (4.30)$$

which, combined with (4.25)<sub>1</sub> provides the auto-Bäcklund transformation for the Dodd-Bullough equation as originally obtained in [17]. An auto-Bäcklund transformation for the 2 + 1-dimensional Dodd-Bullough equation may be generated in a similar manner via Moutard's theorem.

In conclusion, we note an elegant invariance of the Monge-Ampère equation (4.24) associated with the Dodd-Bullough equation. Thus, on introduction of the Haar-type transformation [27,28]

$$\Phi^* = \frac{1}{\Phi - \bar{x}\Phi_{\bar{x}} - \bar{y}\Phi_{\bar{y}}}, \quad x^* = \frac{\Phi_{\bar{x}}}{\Phi - \bar{x}\Phi_{\bar{x}} - \bar{y}\Phi_{\bar{y}}}, \quad y^* = \frac{\Phi_{\bar{y}}}{\Phi - \bar{x}\Phi_{\bar{x}} - \bar{y}\Phi_{\bar{y}}}, \quad (4.31)$$

it is readily demonstrated that

$$(\Phi_{\bar{x}\bar{x}}\Phi_{\bar{y}\bar{y}} - \Phi_{\bar{x}\bar{y}}^2)(\Phi_{x^*x^*}\Phi_{y^*y^*} - \Phi_{x^*y^*}^2) = 1/(\Phi^*\Phi)^4. \quad (4.32)$$

Accordingly, given a solution  $\Phi = \Phi(\bar{x}, \bar{y})$  of (4.24), a solution  $\Phi^* = \Phi^*(x^*, y^*)$  of its starred counterpart is given parametrically by (4.31). This invariance may be shown to correspond to the duality principle enunciated for the original integrable anisentropic gasdynamic system in [17]. It is noted that invariance of gasdynamic and magnetogasdynamics systems under reciprocal and adjoint-type transformations have been investigated in [33–37].

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# INTEGRABLE CELLULAR AUTOMATA AND INTEGRABLE ALGEBRAIC AND FUNCTIONAL EQUATIONS

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**ABSTRACT.** Integrable nonlinear equations can have different forms; they appear as differential, partial differential, discrete and integro-differential equations. In this lecture we show that also cellular automata and algebraic and functional equations can be included in this general picture.

## 1. Integrable Cellular Automata

Here we present a short summary of the papers [1]–[3], written in collaboration with M. Bruschi and O. Ragnisco.

1 + 1-dimensional cellular automata (CA) are dynamical systems in the discrete space variable  $n \in \mathbb{Z}$  and time variable  $t \in \mathbb{Z}$ , in which the field variable  $q = q(n, t)$  takes values on a finite field, say  $\{0, 1, \dots, m-1\}, m \in \mathbb{B}$ . CA exhibit a very large variety of dynamics and empirical attempts of classification have been made [4]. A new class of 1+1-dimensional CA, called parity-rule filter automata (PRFA) (see below for the definition), has been introduced in [5] several years ago. These PRFA exhibit a vast array of coherent structures, called particles, which in many cases do interact solitonically. The discovery of a “fast rule” [6], an effective method of solving the PRFA, has allowed one to investigate analytically the rich properties of the solitonic structures [7]; see also [8]. Generalizations to states taking values in finite groups were obtained in [9]; variants of the above PRFA were introduced in [10]. In a recent paper [11] a class of time reversible CA, which incorporate the most interesting features of the above PRFA, was proposed. In this work the issues of time reversibility and stability were fully addressed. These CA possess also conservation laws [12], therefore they share with soliton equations many of the relevant properties of integrability.

Here we illustrate a procedure to construct CA as integrability conditions for linear operators [1], namely through a Lax pair [13] compatibility, and we consider, as a prototype example, the discrete Schrodinger spectral problem [14]

$$\psi(n-1; z) + (1 + q(n))\psi(n+1; z) = (z + 1/z)\psi(n; z), \quad n \in \mathbb{Z}, \quad z \in \mathbb{C}. \quad (1)$$

Assuming that  $\psi$  and  $q$  depend on the discrete time variable  $t \in \mathbb{Z}$ , we seek a compatible

time evolution in the form

$$\psi(n, t+1; z) = \sum_{j=0}^M V^{(j)}(n, t)\psi(n+2j, t; z), \quad M \in \mathbb{B}. \quad (2)$$

If  $M = 1$ , compatibility between (1) and (2) yields

$$V^{(0)}(n-1, t) = V^{(0)}(n, t) = \text{const.} = 1 \text{ (without loss of generality)}, \quad (3a)$$

$$q(n, t+1) + V^{(1)}(n-1, t) = q(n, t) + V^{(1)}(n, t), \quad (3b)$$

$$(1 + q(n, t+1))V^{(1)}(n+1, t) = (1 + q(n+2, t))V^{(1)}(n, t). \quad (3c)$$

In order to view equations (3) as defining a CA, we have to interpret equalities as “modulo congruences”, indicated hereafter with the symbol “mod  $m$ ”, where the modulo  $m$  is a prime number. In other words we seek solutions of equations (3) in the finite field  $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} = \{0, 1, \dots, m-1\}$ . Equation (3c) then admits the basic solution

$$V^{(1)}(n, t) = b\delta(1 + q(n-1, t+1))\delta(1 + q(n+2, t)), \quad \text{mod } m, \quad (4)$$

where  $b$  is a constant in  $\mathbb{Z}_m$  and  $\delta(x) = 0 \pmod{m}$ , if and only if  $x \neq 0 \pmod{m}$ . In the following, we will use the representation

$$\delta(x) = \prod_{k=1}^{m-1} (k+x) \pmod{m}, \quad x \in \mathbb{Z}_m. \quad (5)$$

Substituting equation (4) into equation (3b) we obtain the following explicit law of the CA:

$$\begin{aligned} q(n, t+1) + b\delta(1 + q(n-2, t+1))\delta(1 + q(n+1, t)) &= q(n, t) \\ + b\delta(1 + q(n-1, t+1))\delta(1 + q(n+2, t)) &\pmod{m}, \quad b \in \mathbb{Z}_m, \end{aligned} \quad (6)$$

which reduces to the dynamics

$$q(n, t+1) = q(n, t) + q(n-2, t+1)q(n+1, t) + q(n-1, t+1)q(n+2, t) \pmod{2}, \quad (7)$$

for  $m = 2$  and  $b = 1$ .

Applying the same strategy we obtain, for  $M \geq 1$ , the following hierarchy of cellular automata [2]:

$$q(n, t+1) = q(n, t) + V(n, t) - V(n-1, t), \quad (8a)$$

$$V(n, t) = \delta(1 + q(n+M+1, t))\delta(1 + q(n-M, t+1))$$

$$\times \prod_{s=1}^{M-1} \delta(\rho_{M-s} + q(n+M+1-s, t)\delta(\rho_{M-s} + q(n-M+s, t+1))), \quad \rho_s \in \mathbb{Z}_m. \quad (8b)$$

The above CA possess the following properties.

- i) They are explicitly computable for any initial condition  $q(n, 0)$  on “compact support”, that is, if  $q(n, 0) = 0$  for  $|n| > N$ , some  $N \in \mathbb{B}$ .
- ii) They are filter CA [5], since the value of  $q(n_0, t+1)$  depends not only on  $q(n, t), n \in \{n_0, n_0+1, \dots, n_0+r\}$  but also on  $q(n, t+1), n \in \{n_0-r, n_0-r+1, \dots, n_0-1\}$ , where  $r$  is the so-called “interaction radius”.

- iii) Each member of the above hierarchy of basic CA has a different radius  $r$  of interaction. Precisely,  $r = M + 1$ .
- iv) Decomposing an arbitrary initial configuration on compact support in a finite number of regions separated by at least  $2r - 1$  consecutive zeroes, it is easy to show that these CA induce evolution only inside such regions, indeed strictly inside, since the boundary values are constant under the evolution. Therefore we call such regions “islands”. Note that this property implies trivially the stability of these CA; the number of nonzero sites stays finite for any finite time.
- v) These CA are “time-reversible”, namely they possess the symmetry  $q(n+j, t) \leftrightarrow q(n-j, t+1)$ .
- vi) From iv) and v) it follows that these CA are “completely periodic”, i.e. any initial condition is part of a periodic configuration.

We stress the novelty of our solution of equation (3) viewed as a modulo congruence with respect to the ordinary solution of the same equation

$$V^{(1)}(n, t) = V^{(1)}(-\infty, t) \prod_{j=0}^{\infty} \frac{1 + q(n+1-j, t)}{1 + q(n-1-j, t+1)}, \quad (9)$$

which yields the simplest Backlund transformation (BT) of the Kac-Moerbeke lattice [14]. It turns out that the discrete time evolution described by such BT is different from that of the corresponding CA. More precisely, if we let the state  $q(n, t)$  evolve according to both equations, we discover that the state  $q(n, t+1)$ , evolved according to the BT, is **not** modulo congruent to the state evolved according to the CA.

The Lax equations (1) and (2), viewed as modulo congruences, should provide in principle the linearization scheme for the above hierarchy of CA. Here we concentrate on those aspects of the theory which are sufficient to prove that the above CA possess a countable number of constants of motion and to construct them explicitly.

Since the spectral data are functions of the spectral parameter  $z$ , we first introduce the following definitions:

- i) Two entire functions  $h_1(z), h_2(z)$  of  $z \in \mathbb{C}$

$$h_i(z) = \sum_{k=0}^{\infty} c_k^{(i)} z^k, \quad i = 1, 2, \quad (10a)$$

are modulo congruent,  $h_1(z) = h_2(z) \pmod{m}$ , if and only if  $c_k^{(1)} = c_k^{(2)} \pmod{m}, \forall k$ .

- ii) Two meromorphic functions  $g_1(z), g_2(z)$  of  $z$ :

$$g_i(z) = \frac{h_i(z)}{\tilde{h}_i(z)}, \quad i = 1, 2, \quad (10b)$$

are modulo congruent,  $g_1(z) = g_2(z) \pmod{m}$ , if and only if  $h_1(z) = h_2(z) \pmod{m}$ , and  $\tilde{h}_1(z) = \tilde{h}_2(z) \pmod{m}$ , with  $\tilde{h}_j(z) \neq 0 \pmod{m}$ .

Now we consider an initial configuration on the compact support  $\{n_0, \dots, n_0 + N - 1\}$ , i.e.  $q(n) = 0$  if  $n < n_0$  or  $n > n_0 + N - 1$ , and we look for the solution  $\psi(n; z)$  of (1) behaving like  $z^n$  at  $n \rightarrow +\infty$ . Then the scattering data  $a(z)$  and  $b(z)$ , defined by

$$\psi \rightarrow a(z)z^n + b(z)z^{-n}, \quad n \rightarrow -\infty, \quad (11)$$

have the following form

$$a(z) = \frac{1}{z^2 - 1} \sum_{j=0}^{N-1} a_j z^{2j}, \quad a_0 = -1; \quad b(z) = \frac{z^{2(n_0+1)}}{z^2 - 1} \sum_{j=0}^{N-1} b_j z^{2j}, \quad (12)$$

where the coefficients  $a_j$ 's and  $b_j$ 's can be expressed in terms of  $q$ . For instance

$$a_1 = 1 + \sum_{l=n_0}^{n_0+N-1} q(l), \quad (13a)$$

$$a_2 = - \sum_{s=n_0}^{n_0+N-1} q(s) \sum_{l=s+2}^{n_0+N-1} q(l), \quad (13b)$$

$$a_3 = - \sum_{s=n_0}^{n_0+N-1} q(s) \sum_{l=s+3}^{n_0+N-1} q(l) + \sum_{r=n_0}^{n_0+N-1} q(r) \sum_{s=r+2}^{n_0+N-1} q(s) \sum_{l=s+2}^{n_0+N-1} q(l), \quad (13c)$$

⋮

$$a_{N-1} = -q(n_0)q(n_0 + N - 1), \quad (13d)$$

$$b_0 = -q(n_0), \quad (13e)$$

$$] \quad b_1 = -q(n_0 + 1) + q(n_0) \sum_{l=n_0+2}^{n_0+N-1} q(l), \quad (13f)$$

$$b_2 = -q(n_0 + 2) + q(n_0 + 1) \sum_{l=n_0+3}^{n_0+N-1} q(l) + q(n_0) \left[ \sum_{l=n_0+3}^{n_0+N-1} q(l) - \sum_{s=n_0+2}^{n_0+N-1} q(s) \sum_{l=s+2}^{n_0+N-1} q(l) \right], \quad (13g)$$

⋮

$$b_{N-1} = -q(n_0 + N - 1). \quad (13h)$$

Noting that  $V^{(j)} \rightarrow 0$  as  $|n| \rightarrow \infty$  for  $j \neq 0$ , equations (2) and (3a) show that  $\psi(n, t+1; z)$  and  $\psi(n, t; z)$  are asymptotically (in  $n$ ) congruent modulo  $m$ ; therefore

$$a(t+1; z) = a(t; z), \quad b(t+1; z) = b(t; z) \quad \text{mod } m, \quad (14)$$

and equations (12) imply that

$$a_j(t+1) = a_j(t), \quad b_j(t+1) = b_j(t) \quad \text{mod } m, \quad \forall j, \quad (15)$$

i.e. the  $a_j$ 's and the  $b_j$ 's are constants of motion for the above CA.

#### REMARKS

- i) In the usual case, evolutions according to the integrable nonlinear equations associated with (1) (which include the Kac-Moerbeke lattice [14]), imply the conservation of the transmission coefficient  $T(z) = 1/a(z)$  and then of the countable number of coefficients of any suitable asymptotic expansion of  $T$  in  $z$ . These constants of motion, being just combinations of the  $a_j$ 's, are shared also by our CA which, in addition, exhibits the constants of motion  $b_j$ 's. Note however that not all the constants of motion here derived are independent; for instance the conservation of (13e) and (13h) implies the conservation of (13d).

- ii) From the conservation of (13e) and (13h) it also follows that the values of  $q(n)$  at the boundaries of the support do not evolve, as mentioned above.
- iii) Contrary to the solitonic equations of KdV type, the linear problem (1) can apparently be solved uniquely for eigenfunctions  $\psi(n; z)$  normalized at  $+\infty$ , as we did above to generate the constants of motion, but not for eigenfunctions normalized at  $-\infty$  since, in this case, one should deal with the denominator  $1 + q(n)$  which could be zero in  $\mathbb{Z}_m$ . This novel feature and the possible limitations of the proposed integrability scheme (1) and (2) will be discussed in a forthcoming paper.

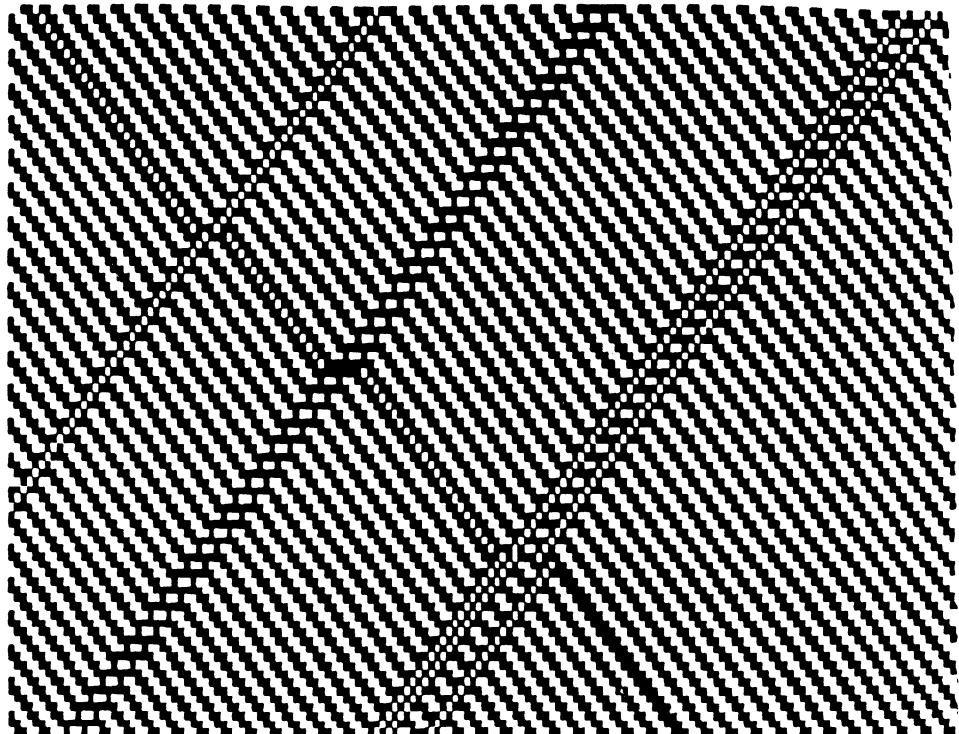


Figure 1a

#### QUALITATIVE FEATURES

Although completely periodic, the dynamics of our CA is far from trivial. Indeed a computerized investigation of the dynamics in sufficiently large islands shows the following phenomena:

- 1) existence of suitable ordered backgrounds in which perturbations can move ( see Figures

- 1a, 1b );
- 2) two perturbations interact solitonically, possibly with creation and annihilation phenomena (see Figures 1a,1b);
  - 3) perturbations are reflected by the boundaries of the island, interacting with them as in point 2) above, (see Figure 1b);
  - 4) the only velocities empirically observed are  $0, \pm 1, \infty$  (see Figures 1a,1b). An infinite velocity of propagation is possible in filter CA.

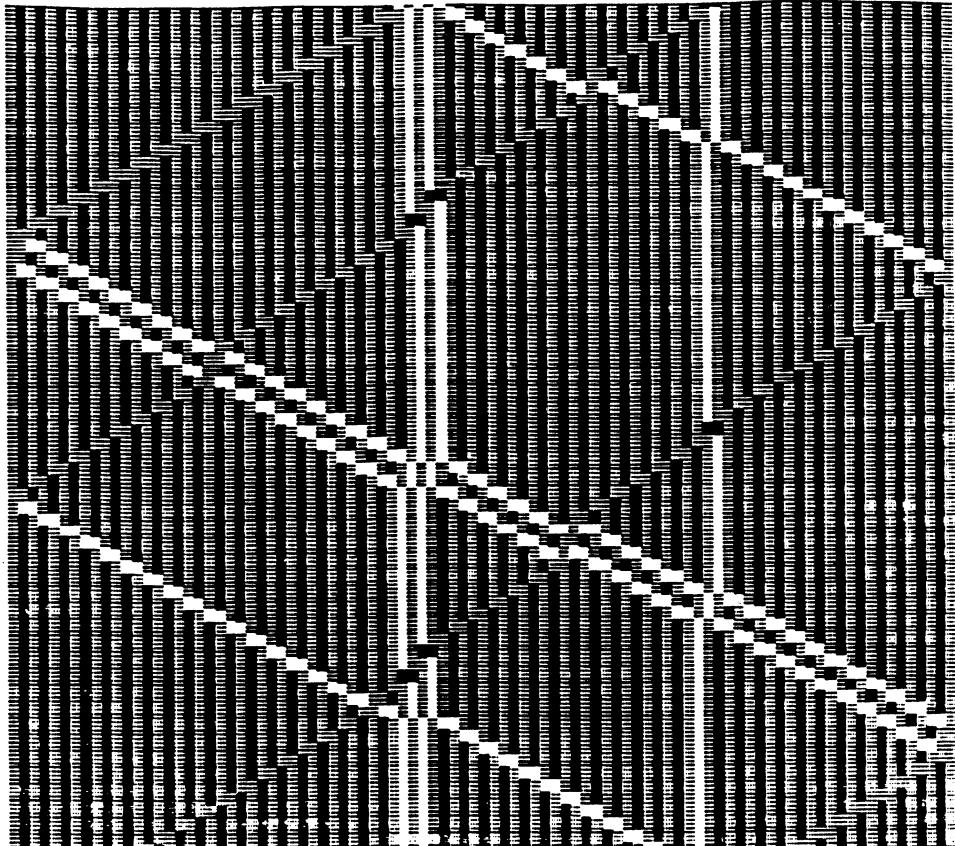


Figure 1b

The analogies with the motion of “dislocations” in Solid Physics and with the “quark confinement” in Particle Physics are suggestive and give the above localized perturbations a particle-like content. The configurations which we call islands are referred in the literature [4]-[11] as particles with zero velocity and internal dynamical structure. Therefore what we observe inside an island can be interpreted as confined particles moving inside a steady particle.

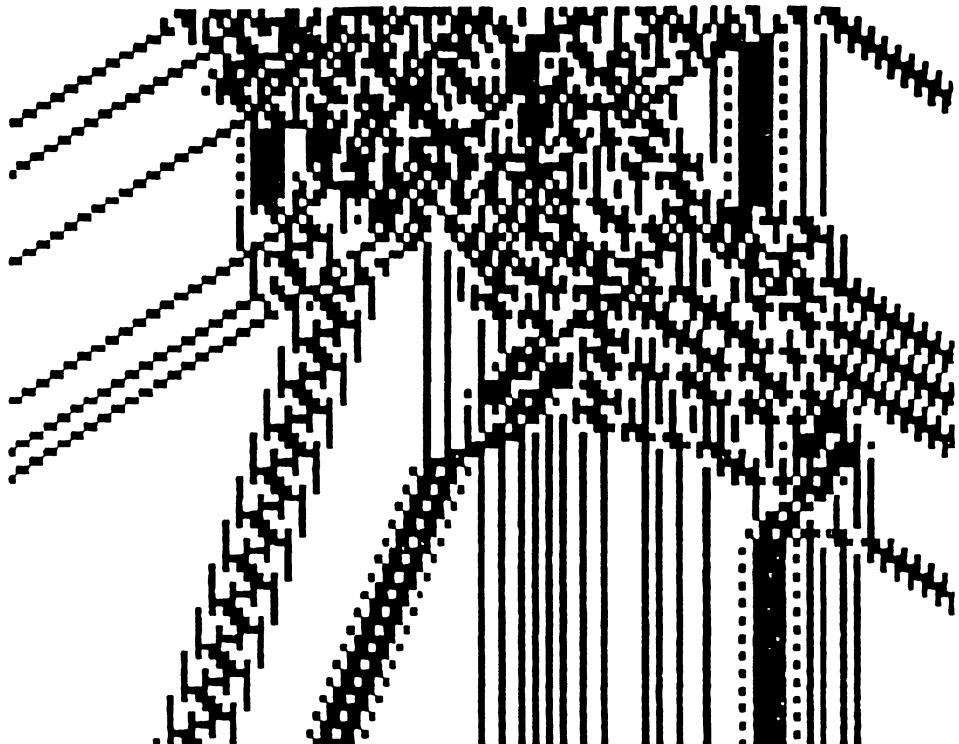


Figure 2a

## FINAL REMARKS

- i) There exists an interesting mechanism, based on a factorization problem, which allows one to "defilter" the hierarchy (8) and to construct the associated constants of motion, through an alternative approach [2].
- ii) The above theory has been extended to higher space dimensions [3].
- iii) Following a philosophy closer in spirit of [11], we have also introduced a different class of 1 + 1-dimensional CA, which are filter, quadratic, reversible and stable [1]. For instance we obtain

$$\begin{aligned} q(n, t+1) = & q(n, t) + q(n+1, t) + q(n-1, t+1) + q(n-2, t+1)q(n-1, t+1) \\ & + q(n+1, t)q(n+2, t) \quad \text{mod } 2 \end{aligned} \quad (16)$$

and

$$q(n, t+1) = q(n, t) + q(n-3, t+1)q(n-2, t+1) + q(n-2, t+1)q(n-1, t+1)$$

$$+ q(n+2,t)q(n+1,t) + q(n+3,t)q(n+2,t) \quad \text{mod } 2. \quad (17)$$

To the basic structures (16) and (17) one can always add, without altering the requested properties, the interaction term

$$\begin{aligned} & \sum_{j=1}^r \alpha_j q(n-j, t+1)q(n+j, t) \\ & + \sum_{j>k=1}^r \beta_{jk}(q(n-j, t+1)q(n+k, t) + q(n-k, t+1)q(n+j, t)), \end{aligned} \quad (18)$$

where  $\alpha_j, \beta_{jk} \in \{0, 1\}$ .

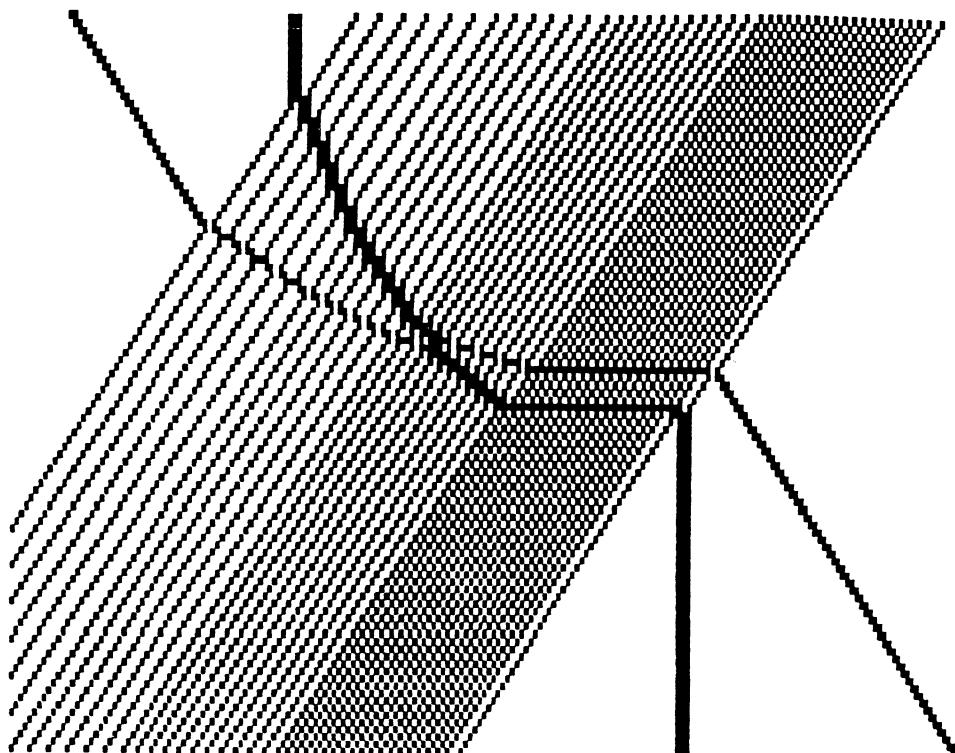


Figure 2b

The above CA exhibit a very rich phenomenology, investigated via computer, which we summarize in the following points.

- 1) There exist an enormous array of coherent structures, which we call particles, travelling with different velocities in both directions (see Figure 2a,2b).
- 2) Any finite initial condition splits, asymptotically in time, into a finite number of particles (see Figure 2a).
- 3) The interaction between two or more particles can be completely solitonic, that is, their interaction produces only a phase shift, preserving number, shape and velocity of each particle, or one can have production or gluing of particles (see Figures 2a,2b).
- 4) It is possible to construct various media, possibly nonhomogeneous. In these media two or more particles interact between each other as in iii) as well as with the medium, exhibiting a refraction phenomenon (see Figure 2b).

The problem of associating with the CA (16), (17) any integration scheme is presently under investigation. Here we note only the existence of some constants of motion:

$$c_1 = \sum_{j \in \mathbb{Z}} q(j)q(j+1), \quad c_2 = \sum_{j \in \mathbb{Z}} q(j)q(j+1)q(j+2) \pmod{2},$$

and

$$\tilde{c}_1 = \sum_{j \in \mathbb{Z}} q(j) \pmod{2}$$

for the CA (16) and (17) respectively.

- iv) A different approach to the construction of integrable CA, in some sense complementary to ours, has been presented at this Workshop [15].

## 2. Solvable Algebraic and Functional Equations

Here we present a short summary of some of the results of the paper [16], written in collaboration with B. Dubrovin and A.S. Fokas.

Our goal is to *construct* and *solve* classes of nonlinear algebraic and functional equations using spectral methods. In the literature, spectral techniques were used to investigate a famous example of functional equation, the so-called “main commutation relation” [17].

The basic example we consider is given by the following equations [18],[16]:

$$\frac{q_{ij}q_{jk}}{q_{ik}} - r_{ij} + r_{kj} = \sum_{\alpha=1}^{n-d} \gamma_j^{(\alpha)} p_{ik}^{(\alpha)}, \quad (19)$$

where  $d \leq n \in \mathbb{B}$ ,  $\gamma_j^{(\alpha)}$ ,  $\alpha = 1, \dots, n-d$ ;  $j = 1, \dots, n$  is a given rectangular matrix of maximal rank and  $\mathbf{Q} = (q_{ij})$ ,  $\mathbf{R} = (r_{ij})$  and  $\mathbf{P} = (p_{ij})$  are unknown  $n \times n$  off-diagonal matrices.

Such a system of equations is equivalent to the commutativity condition of a  $d$ -dimensional ( $d \leq n$ ) algebra of linear  $\lambda$  matrices:

$$[\mathbf{L}_i, \mathbf{L}_j] = 0, \quad i, j = 1, \dots, d, \quad (20a)$$

$$\mathbf{L}_i = \lambda \mathbf{A}_i + \mathbf{U}_i. \quad (20b)$$

The diagonal matrices  $\mathbf{A}_i = \text{diag}(a_1^{(i)}, \dots, a_n^{(i)})$  satisfy the constraint  $\sum_{s=1}^n \gamma_s^{(\alpha)} a_s^{(i)} = 0$ , where  $\alpha = 1, \dots, n-d$ ,  $i = 1, \dots, n$ , and the matrices  $\mathbf{U}_i$ ,  $i = 1, \dots, d$  are expressed in terms of two

off-diagonal matrices  $\mathbf{Q} = (q_{ij})$  and  $\mathbf{R} = (r_{ij})$  in the following way:

$$\begin{aligned}\mathbf{U}_i &= \mathbf{D}_i + [\mathbf{A}_i, \mathbf{Q}], \\ \mathbf{D}_i &= \text{diag} \left( d_1^{(i)}, \dots, d_n^{(i)} \right), \quad d_j^{(i)} = \sum_{s=1}^n (a_s^{(i)} - a_j^{(i)}) r_{js}. \end{aligned}\quad (21)$$

We have proven in [12] that if  $d = n$ , the spectral curve associated with (20) is a rational curve. The commutative algebra (20) can be parametrized in terms of rational functions and the inverse spectral transform

$$q_{ij} = \frac{\nu_i}{\nu_j} \frac{\lambda_j}{z_i - z_j}, \quad r_{ij} = \frac{\lambda_j}{z_i - z_j}, \quad i \neq j \quad (22a)$$

parametrizes the general solution of the algebraic-functional equation

$$\frac{q_{ij} q_{jk}}{q_{ik}} - r_{ij} + r_{kj} = 0 \quad (22b)$$

in terms of the spectral data  $\nu_i, \lambda_i, z_i \in \mathbb{C}, i = 1, \dots, n$ .

If  $d = n - 1$ , the spectral curve is elliptic, the commutative algebra (20) of  $\lambda$  matrices can be parametrized in terms of elliptic functions and the inverse spectral transform

$$q_{ij} = \lambda_j \frac{\nu_i}{\nu_j} \frac{\sigma(z_i - z_j + z_0)}{\sigma(z_0)\sigma(z_i - z_j)}, \quad r_{ij} = \lambda_j \zeta(z_i - z_j), \quad i \neq j \quad (23a)$$

parametrizes the general solution of the algebraic functional equation

$$\frac{q_{ij} q_{jk}}{q_{ik}} - r_{ij} + r_{kj} = \gamma_j p_{ik} \quad (23b)$$

in terms of Weierstrass  $\sigma$ -and  $\zeta$ -functions, where  $\nu_i, \lambda_i, z_i, i = 1, \dots, n$  are suitable spectral data and  $\lambda_i = \gamma_i$ . In addition we have that  $p_{ik} = \zeta(z_0) - \zeta(z_i - z_k + z_0)$ .

We remark that an approach similar to the one used to solve equation (19) could, in principle, be used to solve the continuous version of equations (19) [16]:

$$\frac{q(x, y)q(y, z)}{q(x, z)} - r(x, y) + r(z, y) = \int dl \gamma(y; l) p(x, z; l), \quad (24)$$

associated with the spectral operator

$$\begin{aligned}(\mathbf{L}f)(x, \lambda) &= \lambda a(x)f(x) + \int dy [a(x) - a(y)]q(x, y)f(y) \\ &\quad + f(x) \int dy [a(y) - a(x)]r(x, y). \end{aligned}\quad (25)$$

In particular, the functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} - r(x, y) + r(z, y) = 0 \quad (26a)$$

has solution

$$q(x, y) = \frac{\lambda(y)\nu(x)}{\nu(y)[z(x) - z(y)]}, \quad r(x, y) = \frac{\lambda(y)}{z(x) - z(y)}, \quad (26b)$$

while the functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} - r(x, y) + r(z, y) = \gamma(y)p(x, z) \quad (27a)$$

has solution

$$q(x, y) = \frac{\gamma(y)\nu(x)}{\nu(y)} \frac{\sigma(z(x) - z(y) + z_0)}{\sigma(z_0)\sigma(z(x) - z(y))}, \quad (27b)$$

$$r(x, y) = \gamma(y)\zeta(z(x) - z(y)), \quad (27c)$$

$$p(x, y) = \zeta(z_0) - \zeta(z(x) - z(y)). \quad (27d)$$

The above equations admit the following one-dimensional reductions  $f(x, y) = f(x - y) = f(X)$ :

$$\frac{q(X)q(Y)}{q(X + Y)} - r(X) + r(-Y) = 0, \quad (28a)$$

$$\frac{q(X)q(Y)}{q(X + Y)} - r(X) + r(-Y) = p(X + Y), \quad (28b)$$

whose solutions are obtained by specifying in a proper way the arbitrary functions  $\lambda(x)$ ,  $\nu(x)$  and  $z(x)$  in (26b) and in (27b).

The functional equation (28b) was introduced a few years ago in the context of integrable many-body problems, and was solved by direct means [19].

We finally remark that a possible application of the results of this section is the complete characterization of the stationary points of integrable dynamical systems of the form  $\mathbf{L}_t = [\mathbf{L}, \mathbf{M}]$ , where  $\mathbf{L}$  and  $\mathbf{M}$  are suitable linear  $\lambda$  matrices (Arnold and Manakov tops are in this class) [20].

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# CLASSICAL DIFFERENTIAL GEOMETRY AND INTEGRABILITY OF SYSTEMS OF HYDRODYNAMIC TYPE

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**ABSTRACT.** A remarkable parallelism between the theory of integrable systems of first-order quasilinear PDE and some old results in projective and affine differential geometry of conjugate nets, Laplace equations and their Bianchi-Bäcklund transformations is exposed. These results were recently applied by Krichever and Dubrovin to prove integrability of some models in topological field theories. Within a geometric framework we derive some new integrable (in a sense to be discussed) generalizations describing  $N$ -wave resonant interactions.

## 1. Introduction

Ten years ago [10] a natural Hamiltonian formalism was proposed for the class of homogeneous systems of the PDE

$$(1+1, h) \quad \begin{pmatrix} u_t^1 \\ \vdots \\ u_t^n \end{pmatrix} = \begin{pmatrix} v_1^1(\mathbf{u}) & \cdots & v_n^1(\mathbf{u}) \\ \vdots & \ddots & \vdots \\ v_1^n(\mathbf{u}) & \cdots & v_n^n(\mathbf{u}) \end{pmatrix} \begin{pmatrix} u_x^1 \\ \vdots \\ u_x^n \end{pmatrix},$$

$u^i = u^i(x, t)$ ,  $i = 1, \dots, n$ , called “one-dimensional systems of hydrodynamic type”. Later (see [11]), it was generalized to the class of multidimensional systems:

$$(N+1, h) \quad \begin{pmatrix} u_t^1 \\ \vdots \\ u_t^n \end{pmatrix} = \sum_{j=1}^N \begin{pmatrix} v_1^{(j)1}(\mathbf{u}) & \cdots & v_n^{(j)1}(\mathbf{u}) \\ \vdots & \ddots & \vdots \\ v_1^{(j)n}(\mathbf{u}) & \cdots & v_n^{(j)n}(\mathbf{u}) \end{pmatrix} \begin{pmatrix} u_{x_j}^1 \\ \vdots \\ u_{x_j}^n \end{pmatrix}$$

where  $u^i = u^i(t, x_1, \dots, x_N)$ ,  $i = 1, \dots, n$  and non-homogeneous systems:

$$(1+1, nh) \quad \begin{pmatrix} u_t^1 \\ \vdots \\ u_t^n \end{pmatrix} = \begin{pmatrix} v_1^1(\mathbf{u}) & \cdots & v_n^1(\mathbf{u}) \\ \vdots & \ddots & \vdots \\ v_1^n(\mathbf{u}) & \cdots & v_n^n(\mathbf{u}) \end{pmatrix} \begin{pmatrix} u_x^1 \\ \vdots \\ u_x^n \end{pmatrix} + \begin{pmatrix} f_1(\mathbf{u}) \\ \vdots \\ f_n(\mathbf{u}) \end{pmatrix},$$

where  $u^i = u^i(x, t)$ ,  $i = 1, \dots, n$ .

Some systems of type  $(1+1, h)$  which are of physical importance, such as the Whitham equations (the averaged 1-phase KdV equation) and the Benney equations, have the notable property of being diagonalizable: under a suitable choice of field variables  $u^i$  (namely the Riemann invariants) the equations become

$$u_t^i(x) = v_i(\mathbf{u}) u_x^i \quad (1)$$

(where there is no summation over  $i$ ). As we have proved in [32], the Hamiltonian property and diagonalizability imply integrability. Deeper insight into this type of integrability is given by the theory of orthogonal curvilinear coordinate systems. This chapter of classical differential geometry was intensively developed at the beginning of the twentieth century [6,8,20]. In fact this theory gives the geometric background for integrability of systems  $(1+1, h), (N+1, h), (1+1, nh)$ . These forgotten corners of differential geometry seem to be worth revisiting.

**EXAMPLE.** The well-known Bullough-Dodd-Jiber-Shabat equation

$$u_{xx} - u_{tt} = e^u - e^{-2u} \quad (2)$$

(in the form  $(\ln h)_{uv} = h - 1/h^2$ ) was introduced for the first time in [35] where the respective linear problem was given, as was a proper Bäcklund transformation for it. It is much simpler and more geometric than the Bäcklund transformations discussed recently [1,31] in the context of integrable systems.

In this paper we sketch some applications of methods originating from classical differential geometry to equations of types  $(1+1, h), (N+1, h)$  and  $(1+1, nh)$ .

## 2. Diagonal systems of hydrodynamic type and orthogonal curvilinear coordinate systems in $\mathbb{R}^n$

Let us recall briefly the main results of [10,32]. A (generally nondiagonal) system  $u_t^i = \sum_{j=1}^n v_j^i(\mathbf{u}) u_x^j$  is Hamiltonian if there exist a Hamiltonian  $H = \int h(\mathbf{u}) dx$  and a Hamiltonian operator

$$\hat{A}^{ij} = g^{ij}(\mathbf{u}) \frac{d}{dx} + b_k^{ij}(\mathbf{u}) u_x^k,$$

which define a skew-symmetric Poisson bracket on functionals

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \hat{A}_{ij} \frac{\delta J}{\delta u^j(x)} dx,$$

satisfying the Jacobi identity and which generate the system

$$u_t^i(x) = \{u^i(x), H\} = \hat{A}_{ij} \frac{\partial H}{\partial u^j(x)} = \left( g^{ij} \partial_k \partial_j h + b_k^{ij} \partial_j h \right) u_x^k = v_k^i(\mathbf{u}) u_x^k \quad (3)$$

where  $\partial_s = \partial/\partial u^s$ . B.A. Dubrovin and S.P. Novikov [10] proved that the necessary and sufficient conditions for  $\hat{A}_{ij}$  to be a Hamiltonian operator in the case of non-degeneracy of the matrix  $g^{ij}$  are:

- a)  $g^{ij} = g^{ji}$ , i.e. the inverse matrix  $\mathbf{g}^{-1}$  defines a Riemannian metric.
- b)  $b_k^{ij} = -g^{is} \Gamma_{sk}^j$  for the standard Christoffel symbols  $\Gamma_{sk}^j$  generated by  $g_{ij}$ .
- c) the metric  $g_{ij}$  has identically vanishing curvature tensor.

In such case we have  $v_j^i(\mathbf{u}) = \nabla^i \nabla_j h = g^{is} \nabla_s \nabla_j h$  with the covariant derivatives defined by  $g_{ij}$ .

LEMMA [32,33]. *In order that a matrix  $v_j^i(\mathbf{u})$  be a matrix of a Hamiltonian system  $(1+h)$  with a nondegenerate metric in  $\hat{A}^{ij}$  it is necessary and sufficient that there exists a nondegenerate zero curvature metric  $g_{ij}$  such that*

a)  $g_{ik} v_j^k = g_{jk} v_i^k$  and

b)  $\nabla_j v_k^i = \nabla_k v_j^i$ ,

where  $\nabla$  is the covariant differentiation generated by the metric  $g_{ij}$ .

For a diagonal matrix  $v_j^i(\mathbf{u}) = v_j(\mathbf{u}) \delta_j^i$  this implies (see [32,33]) that  $g_{ij}$  is also diagonal and

$$\frac{\partial_i v_k}{(v_i - v_k)} = \Gamma_{ki}^k = \frac{1}{2} \partial_i \ln g_{kk}, \quad \partial_i = \partial / \partial u^i \quad (4)$$

(hereafter summation on repeated indices is not implied). From (3) we deduce

$$\partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right), \quad i \neq j \neq k. \quad (5)$$

From a differential geometric point of view, a zero curvature nondegenerate diagonal metric is equivalent to an orthogonal curvilinear coordinate system on a flat (possibly pseudo-Euclidean) space (see [6]). Locally these coordinate systems are determined by  $\frac{1}{2}n(n-1)$  functions of two variables (Bianchi). A striking fact can be discovered: formula (3) can be found in [6] (p. 353)! This formula is crucial for the integrability property of diagonal Hamiltonian systems (1). If we interpret formula (3) as an overdetermined system (compatible in view of zero curvature property of  $g$ ) on  $n$  unknown functions  $v_j(\mathbf{u})$  ( $g_{ii}$  given) we can generate from every solution  $\bar{v}_j(\mathbf{u})$  a symmetry (commuting flow)

$$u_t^i = \bar{v}_i(\mathbf{u}) u_x^i, \quad i = 1, \dots, n,$$

of (1) and a solution of (1) (the generalized hodograph method, see [33] for the details). One can prove [33] the completeness property for this class of symmetries and solutions parametrized by  $n$  functions of 1 variable — the generic Cauchy data for our diagonal system (1).

The corresponding geometric notion used in the theory of orthogonal curvilinear coordinate systems corresponding to (3) is the so called Combescure transformation (see [6]).

**DEFINITION.** Two orthogonal curvilinear coordinate systems  $x^i = x^i(u^1, \dots, u^n)$  and  $\hat{x}^i = \hat{x}^i(u^1, \dots, u^n)$  in the same flat (pseudo)-Euclidean space  $\mathbb{R}^n = \{(x^1, \dots, x^n)\}$  are said to be related by a *Combescure transformation* (or are simply parallel) if and only if their tangent frames  $e_i = \partial x^i / \partial u^i$  and  $\hat{e}_i = \partial \hat{x}^i / \partial u^i$  are parallel in points corresponding to the same values of curvilinear coordinates  $u^i$ .

Let us take the quantities  $H_i(\mathbf{u}) = |e_i| = \sqrt{g_{ii}}$ ,  $\hat{H}_i(\mathbf{u}) = |\hat{e}_i|$  i.e. the Lamé coefficients.

**PROPOSITION.** The quantities  $\bar{v}_i(\mathbf{u}) = \hat{H}_i(\mathbf{u}) / H_i(\mathbf{u})$  satisfy (3) with  $\Gamma_{ki}^k = \partial_i H_k / H_k$ , the connection coefficients for the metric  $g_{ii} = H_i^2$ . Conversely, for any solution  $\bar{v}_i$  of (3),  $\hat{g}_{ii} = (\bar{v}_i H_i)^2$  will give an orthogonal curvilinear coordinate system related to the coordinate system with the metric  $g_{ii} = H_i^2$  by a Combescure transformation.

The theory of Combescure transformations coincides with the theory of integrable diagonal systems of hydrodynamic type.

Physical examples of such systems (Whitham equations, Benney equations) have Hamiltonian structures (2) with diagonal metrics  $g_{ii}$  possessing the so called Egorov property:  $\partial_i g_{kk} = \partial_k g_{ii}$ . As we have demonstrated earlier [34] this is a consequence of Galilei invariance of the original systems (see also [12] for the algebro-geometric background of this property for averaged integrable systems). Using this property and the homogeneity of the coefficients one can find explicit formulas for solutions of (3) for such systems [33,34].

The class of Egorov orthogonal curvilinear coordinate systems is interesting in itself and merits our special attention.

### 3. Egorov coordinate systems, the $N$ -wave problem and its generalizations

Introducing rotation coefficients of a given orthogonal curvilinear coordinate system with  $g_{ii} = H_i^2$ ,  $\beta_{ik}(\mathbf{u}) = \partial_i H_k / H_i$ ,  $i \neq k$ ,  $\beta_{ii}(\mathbf{u}) = 0$  ( see [6]) one can easily check the following:

- the vanishing of the curvature tensor is equivalent to

$$\partial_j \beta_{ik} = \beta_{ij} \beta_{jk}, \quad i \neq j \neq k, \quad (6)$$

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{s \neq i, k} \beta_{si} \beta_{sk} = 0, \quad i \neq k. \quad (7)$$

$$(8)$$

b) the Egorov property  $\partial_i g_{kk} = \partial_k g_{ii}$  reduces to

$$\beta_{ik} = \beta_{ki}. \quad (9)$$

In the Egorov case condition (6) is equivalent to  $\hat{\mathbf{T}}\beta_{ik} = 0$ ,  $\hat{\mathbf{T}} = \partial_1 + \dots + \partial_n$ . Consequently the problem of classification of Egorov coordinate systems is reduced to description of all off-diagonal symmetric matrices  $(\beta_{ik})$  satisfying (5) and  $\hat{\mathbf{T}}\beta_{ik} = 0$ .

Dubrovin [12] has recently observed that this problem coincides with the purely imaginary reduction of the well-known integrable system describing resonant  $N$ -wave interactions. Namely, restriction of  $\beta_{ik}$  on any  $(x, t)$  plane  $u^i = a^i x + b^i t$  gives (compare, for example, [28])

$$[\mathbf{A}, \Gamma_t] - [\mathbf{B}, \Gamma_x] = [[\mathbf{A}, \Gamma], [\mathbf{B}, \Gamma]],$$

where  $\mathbf{A} = \text{diag}(a^1, \dots, a^n)$ ,  $\mathbf{B} = \text{diag}(b^1, \dots, b^n)$  and  $\Gamma = (\beta_{ik})$  with the additional reduction  $\Im(\Gamma) = 0$ ,  $\Gamma^T = \Gamma$ . For the case  $N = 3$  this reduces to

$$\begin{cases} b_t^1 + c_1 b_x^1 = \kappa b^2 b^3, \\ b_t^2 + c_2 b_x^2 = \kappa b^1 b^3, \\ b_t^3 + c_3 b_x^3 = \kappa b^1 b^2. \end{cases} \quad (10)$$

This is a system of type  $(1+1, nh)$ , integrable by the IST method [28].

Now we can compare the progress achieved in the modern integrability theory for (8) and the results obtained more than 70 years ago in the theory of Egorov coordinate systems initiated by Darboux in 1866 and continued by Egorov in 1901 in his thesis (see [13]). It was Darboux [6] who proposed to call this special type of coordinate systems Egorov type systems. From the point of view of integrability properties remarkable progress was achieved by Bianchi in 1915 (see [2]). He found a Bäcklund transformation for this problem and established the permutability property as well as the superposition formula for it. We remark

here that the pioneering results on Bäcklund transformations and their permutability in the well-known theory of constant curvature surfaces in  $\mathbf{R}^3$  are also due to Bianchi.

Let us take an orthogonal curvilinear coordinate system (not necessary of Egorov type) with Lamé coefficients  $H_i(\mathbf{u})$  and rotation coefficients  $\beta_{ik}(\mathbf{u})$ . Bianchi applied to it a generalization of Ribaucour transformations known in the theory of transformations of surfaces. We recall that two surfaces  $\mathbf{x}(u, v)$  and  $\hat{\mathbf{x}}(u, v)$  in  $\mathbf{R}^3$  are related by a Ribaucour transformation if and only if there exists a two-parameter family of spheres  $S(u, v)$  such that each sphere  $S(u_0, v_0)$  is tangent to both surfaces in corresponding points  $\mathbf{x}(u_0, v_0), \hat{\mathbf{x}}(u_0, v_0)$  and this correspondence preserves the curvature lines on the surfaces. For the case of a pair of orthogonal curvilinear coordinate systems in  $\mathbf{R}^3$  we need a three-parameter family of spheres (or an  $n$ -parameter family for the  $n$ -dimensional case) tangent in the corresponding points to one of three families of coordinate surfaces as well as to a coordinate surface of the other curvilinear coordinate system. Since the classical Dupin theorem implies that coordinate lines in any orthogonal curvilinear coordinate system are curvature lines their correspondence is guaranteed. In terms of the rotation coefficients  $\beta_{ik}$  one finds a solution  $\gamma_i(\mathbf{u})$  of

$$\partial_i \gamma_k = \beta_{ki} \gamma_i, \quad i \neq k, \quad (11)$$

to define the corresponding Ribaucour transformation [2]

$$\hat{\beta}_{ik} = \beta_{ik} - \frac{2\gamma_i}{A} \left( \partial_k \gamma_k + \sum_{s \neq k} \beta_{sk} \gamma_s \right), \quad A = \sum_p (\gamma_p)^2. \quad (12)$$

For the case of Egorov systems we shall complete (9) and restrict  $\gamma_i$  to satisfy

$$\partial_i \gamma_i = c \gamma_i - \sum_{s \neq i} \beta_{si} \gamma_s, \quad (c = \text{constant}),$$

or

$$\hat{T} \gamma_i = c \gamma_i, \quad \hat{T} = \partial_1 + \dots + \partial_n \quad (13)$$

The Bäcklund transformation in question is then

$$\hat{\beta}_{ik} = \beta_{ik} - 2c \gamma_i \gamma_k / A. \quad (14)$$

The permutability property for any orthogonal coordinate system requires a quadrature, but for Egorov systems it may be found explicitly and provides the following formulas for the fourth Egorov system  $\bar{\beta}_{ik}$  related to  $\beta'_{ik}, \beta''_{ik}$  obtained from a given Egorov system  $\beta_{ik}$  with constants  $c', c''$  ( $c' + c'' \neq 0$ ) and potentials  $\gamma'_i, \gamma''_i$  in (10):

$$\bar{\gamma}'_i = \gamma''_i - \frac{2c' \gamma_i \sum_s (\gamma'_s \gamma''_s)}{(c' + c'') \sum_s (\gamma'_s)^2}, \quad \bar{\gamma}''_i = \gamma'_i - \frac{2c'' \gamma_i \sum_s (\gamma'_s \gamma''_s)}{(c' + c'') \sum_s (\gamma''_s)^2}.$$

One can enjoy reading [4,23] where these formulas were rediscovered in the context of 3-wave system. So the basic integrability results for (8) (with the exception of the IST transformation) were established long ago by Darboux, Egorov and Bianchi.

An unexpected result (hidden in [6]) consists in the existence of a *homogeneous* system  $(1+1, h)$  of three equations related to (8) by a nonlocal transformation. Geometrically this

is trivial. Given an orthogonal curvilinear coordinate system in  $\mathbf{R}^3$  we have in each its points  $P(x_0, y_0, z_0)$  the orthogonal 3-frame of tangent planes

$$z = p_k(x - x_0) + q_k(y - y_0) + z_0, \quad k = 1, 2, 3. \quad (15)$$

Let us parametrize this by 3 functions  $A(x, y, z), B(x, y, z), C(x, y, z)$ , with the coefficients  $p_k$  and  $q_k$  of the tangent planes being three solutions of

$$\begin{cases} pq + Ap + Bq = 0, \\ p^2 - q^2 + 2(Cp + Hq) = 0, \\ 2(BC - AH) + 1 = 0, \end{cases}$$

that are different from the trivial solution  $p = q = 0$  [3]. Then the Frobenius compatibility conditions for these three families of distributions (13) give a system of three homogeneous first-order equations of type  $(2+1, h)$ :

$$\begin{cases} 2(AC_z - CA_z) = 2C_y + B_y - A_x \\ 2(BH_z - HB_z) = 2H_x + B_y - A_x \\ AH_z - HA_z + BC_z - CB_z = A_y - B_x \end{cases} \quad (16)$$

where  $H = (2BC - 1)/2A$ .

For this system one can reformulate the Bäcklund-like transformation (10) given in terms of  $\beta_{ik}(\mathbf{u})$ . A number of different transformations producing (with quadratures) solutions of (14) parametrized by arbitrarily many functions of one variable may be found in [6]. Thus (14) is integrable in a sense to be discussed elsewhere.

If we search for solutions of (14) which do not depend on  $z$  then a remarkable integrable  $(1+1, h)$  system of three equation appears. Since one can easily prove the equivalence of  $z$ -independence in (14) and the Egorov property (7) we have a homogeneous system related to (8) by a nonlocal change of variables. In the Euler parametrization  $(\varphi, \psi, \theta)$  of orthogonal 3-frames it reads

$$\begin{pmatrix} \psi_t \\ \theta_t \\ \varphi_t \end{pmatrix} = \begin{pmatrix} -\cos^2 \varphi & -\sin \varphi \cos \varphi / \sin \theta & 0 \\ -\sin \theta \sin \varphi \cos \varphi & -\sin^2 \varphi & 0 \\ -\cos \theta(1 + \cos^2 \varphi) & -\sin \varphi \cos \varphi \cos \theta / \sin \theta & 1 \end{pmatrix} \begin{pmatrix} \psi_x \\ \theta_x \\ \varphi_x \end{pmatrix} \quad (17)$$

This nonlocal change does not affect the existence of higher order conserved densities. Recently Ferapontov [17] has proved the uniqueness result for such  $3 \times 3$  homogeneous systems possessing higher-order conserved densities: they may be transformed to (15) by reciprocal and point transformations. Another nonlocal transition from (8) to (15) is also given there.

The matrix of (15) has constant eigenvalues  $-1, 0$  and  $+1$  but its eigenvector fields (properly normalized) form an  $so(3)$  Lie algebra, and consequently (15) is a non-diagonalizable  $(1+1, h)$  integrable system.

The complete system (14) certainly may be called a  $(2+1)$ -dimensional generalization of the  $(1+1)$ -dimensional 3-wave system (8). Orthogonal curvilinear coordinate systems in  $\mathbf{R}^n$  provide also only a  $(2+1)$ -dimensional generalization of the  $(1+1)$ -dimensional  $N$ -wave system since they are parametrized by  $\frac{1}{2}n(n-1)$  functions of two variables (Bianchi).

#### 4. Semi-Hamiltonian diagonal systems and coordinate systems with conjugate lines

The class of integrable diagonal systems (1) is wider than the class of Hamiltonian systems of this type, since property (4) is a weaker consequence of the Hamiltonian property [33]. Let us call a diagonal system *semi-Hamiltonian* if  $n = 2$  or if  $n > 2$  and  $v_i(\mathbf{u})$  satisfy (4). As a physical example of a semi-Hamiltonian (but non-Hamiltonian for  $n > 3$ ) system one can mention the ideal Langmuir chromatography and electrophoresis systems [33].

To every semi-Hamiltonian system we can relate a diagonal metric  $g_{ii}$  via

$$\frac{1}{2} \partial_i \ln g_{kk} = \frac{\partial_i v_k}{v_i - v_k}.$$

This metric is not flat in general though some coefficients of the curvature tensor vanish as a consequence of (4). On introducing  $H_i = \sqrt{g_{ii}}$ ,  $\beta_{ik}(\mathbf{u}) = \partial_i H_k / H_i$ , we find that (4) is equivalent to the set (5) of equations for  $\beta_{ik}$ . Solutions of (5) may be parametrized by  $n(n-1)$  functions of 2 variables. This system coincides with the compatibility conditions for a linear system

$$\partial_i \psi_k = \beta_{ik} \psi_i, \quad i \neq k.$$

Restricting (5) to 3-dimensional planes  $u^i = a^i x + b^i y + c^i z$  in  $\mathbb{R}^n$  we obtain (for general nonvanishing constants  $a^i, b^i, c^i$ ) a  $(2+1, nh)$  system on  $n(n-1)$  quantities  $\beta_{ik}(x, y, z)$ .

As we have previously seen the theory of Hamiltonian diagonal systems (1) is closely related to the theory of orthogonal curvilinear coordinate systems in  $\mathbb{R}^n$ . The geometric background for the theory of semi-Hamiltonian systems is given by the theory of coordinate systems with conjugate coordinate lines (see [6] and [7], Vol. 4, ch. 12). A general (non-orthogonal) coordinate system  $\mathbf{x}(u_1, u_2, u_3)$  in  $\mathbb{R}^3$  is called a system with conjugate coordinate lines (or simply a conjugate coordinate system) if on every coordinate surface  $S_{i_0} = \{u_0^i = \text{const}\}$  in every point  $P(x_0, y_0, z_0)$  the lines of intersection of this surface with two other coordinate surfaces belonging to other one-parametric families of coordinate surfaces and containing  $P(x_0, y_0, z_0)$  are conjugate on  $S_{i_0}$  (with respect to its second fundamental form). Every orthogonal curvilinear coordinate system is conjugate due to the Dupin theorem mentioned above. The theory of conjugate coordinate systems was developed by Darboux and others and borrowed many results from the classical theory of conjugate coordinate nets on surfaces in  $\mathbb{R}^3$  (known as “nets” or “réseaux”, see [14,21,22,36]). A number of Bäcklund-like transformations for these coordinate systems was given with permutability properties (though the superposition formulas therein require quadratures).

Every conjugate coordinate system  $x^i = x^i(u^1, \dots, u^n)$  in  $\mathbb{R}^n = \{(x^1, \dots, x^n)\}$  is characterized by the conditions of conjugacy of coordinate lines:

$$\partial_i \partial_k \mathbf{x} = \Gamma_{ki}^k(\mathbf{u}) \partial_k \mathbf{x} + \Gamma_{ik}^i(\mathbf{u}) \partial_i \mathbf{x}, \quad i \neq k. \quad (18)$$

This system of equations coincides with the system describing hydrodynamic type conserved quantities of a semi-Hamiltonian system [33]. Quantities  $\Gamma_{ki}^k$  in (16) satisfy the compatibility conditions

$$\partial_j \Gamma_{ki}^k = \partial_i \Gamma_{kj}^k, \quad \partial_j \Gamma_{ki}^k = \Gamma_{kj}^k \Gamma_{ji}^j + \Gamma_{ki}^k \Gamma_{ij}^i - \Gamma_{ki}^k \Gamma_{kj}^k, \quad i \neq j \neq k.$$

which are equivalent to the semi-Hamiltonian property (4). Introducing  $H_i(\mathbf{u})$  as solutions of  $\partial_i H_k(\mathbf{u}) = \Gamma_{ki}^k H_k(\mathbf{u})$  and  $\beta_{ik} = \partial_i H_k / H_i$ ,  $i \neq k$ , we receive a set of  $\beta_{ik}$  satisfying (5). The

converse is also true: given a solution  $\beta_{ik}$  of (5) one can find a number of semi-Hamiltonian systems related to it. Any semi-Hamiltonian system also may be related to a Combescure transformation of conjugate coordinate systems [6].

This geometric interpretation provides another example of an integrable  $(2+1, h)$  system. Namely, given a conjugate coordinate system in  $\mathbb{R}^3$  one can take the field of its (non-orthogonal) tangent 3-frames  $(e_1, e_2, e_3)$  and parametrize it by 6 independent functions  $e_k^i(x, y, z), i = 1, 2, k = 1, 2, 3$  (the coefficients  $e_k^3$  may be set to 1 due to normalization). Then the Frobenius compatibility conditions give three homogeneous first-order PDE on  $e_k^i(x, y, z), i = 1, 2, k = 1, 2, 3$ . Another three equations are given by the conjugacy condition  $\det((e_i \cdot \nabla)e_k, e_i, e_k) = 0, i < k$ . This system of six equations is the homogeneous  $(2+1, h)$  system in question. Its  $z$ -independent solutions satisfy a  $(1+1, h)$  system enjoying properties analogous to those of (15): it has constant eigenvalues  $-1, 0, +1$  (all doubly degenerate) and six linearly independent fields of eigenvectors forming (if properly normalized) a nontrivial Lie algebra. This remarkable system will be studied in subsequent publications.

## 5. Additional topics

Recently Mokhov and Ferapontov [27,16] found a nonlocal generalization of the Hamiltonian formalism of hydrodynamic type (2). Ferapontov communicated to the author about a further generalization resulting in the following beautiful theorem: any semi-Hamiltonian system (1) has a nonlocal Hamiltonian structure with a hydrodynamic Hamiltonian and a Hamiltonian operator with (possibly infinitely many) nonlocal terms similar to those in [16]. Grinevich [19] derived a series of nonlocal symmetries for the Whitham equations as well as for the original KdV equations.

Weakly nonlinear semi-Hamiltonian systems (i.e. systems (6) with  $\partial_i v_i = 0$  without summation on  $i$ ; such systems are also called “linearly degenerate”) were studied in [15,30]. The theory of such systems is connected to the theory of  $n$ -webs on Euclidean plane, Dupin cyclids and Stäckel metrics (Ferapontov). Among the results are the quasi-periodic behavior of their solutions [30] and the complete description of such systems and complete sets of their hydrodynamic symmetries [15]. Ferapontov communicated to the author the following fact: any  $n$ -phase ( $n$ -zone) quasi-periodic solution (or an  $n$ -soliton) of the KdV equation can be represented by a solution of a weakly nonlinear semi-Hamiltonian system  $R_t^i = (\sum_{k \neq i} R^k) R_x^i, i = 1, \dots, n$ . These results compare with Curro and Fusco’s results [5] in the soliton-like interactions of Riemann simple waves for some  $2 \times 2$  systems.

IST-like methods were developed in [18,24,25] for some diagonal Hamiltonian systems of physical importance. Certainly this approach shall be related to our geometric methods.

In a series of preprints (see [9],[26] and references therein) Krichever and Dubrovin exposed a remarkable link between the theory of Egorov coordinate systems and Witten-Dijgraagh-Verlinder-Verlinder equations for the correlation functions of topological conformal field theories, proving their integrability.

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# ON THE ELLIPTIC 2 + 1 TODA EQUATION

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**ABSTRACT.** The elliptic version of the Toda equation in two continuous and one discrete variable is considered. Conditions on the scattering data are found which allow a unique solution to be constructed. In one case a radiation condition, analogous to the well known Sommerfeld radiation condition used in certain solutions of linear problems, is required.

## 1. Introduction

In this article we shall consider the elliptic 2 + 1 Toda “molecule” equation

$$\frac{\partial^2 \theta_n}{\partial x^2} - \varepsilon^2 \frac{\partial^2 \theta_n}{\partial y^2} = 2\sigma^2 [\exp \{2(\theta_{n+1} - \theta_n)\} - \exp \{2(\theta_n - \theta_{n-1})\}] \quad (1)$$

where  $\theta = \theta(x, n, y) \equiv \theta_n(x, y)$ ;  $\varepsilon^2 = -1$ ,  $x, y$  are continuous variables,  $n$  is discrete (e.g.,  $n \in \mathbb{Z}$ ) and  $\sigma^2 = \pm 1$ . This equation is a natural reduction of the self dual Yang-Mills system [1-3], reduces to the Toda lattice in 1 + 1 dimensions and has asymptotic reductions to the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations [3]. It also has the property of being a coupled system of Lorentz invariant nonlinear fields. The elliptic problem is a three dimensional prototype of some problems studied in two dimensions such as the unstable sine-Gordon and SIT systems [4].

Even though the initial value problem is ill posed nevertheless we are still able to discuss the solution of a boundary value problem in the domain  $-\infty < x < \infty, -\infty < n < \infty, y > 0$  given  $\theta(x, n, 0)$ . The two cases:  $\sigma^2 = \pm 1, \varepsilon^2 = -1$  are discussed here (see also [5,6]). When  $\sigma^2 = -1$  we find constraints on the scattering data which yield a (implicit) functional relationship that determines  $\partial\theta(x, n, 0)/\partial y$  as a functional of  $\theta(x, n, 0)$ . In the case when  $\sigma^2 = +1$  in addition to constraints such as those discussed above, in order to obtain a unique solution, it is also necessary to impose a boundary condition, which is analogous to the Sommerfeld radiation condition used in linear problems.

The method of solution for rapidly decaying (in  $x$  and  $n$ ) data requires implementation of the DBAR method [7] in which regions of smooth nonholomorphic variation and jump discontinuities in the underlying eigenfunctions are both present. The solution therefore contains both elements which were present individually in the well known cases, such as

the KP and DS problems [8,9]. Such a situation was previously encountered in some other initial value problems [10,11].

The Toda molecule arises from the compatibility of the following spectral problems [12]

$$L\mu \equiv \mu_{n,x} + (\varepsilon\theta_{n,y} - \theta_{n,x})\mu_n + i\sigma\{k\mu_{n+1} - (k + k^{-1})\mu_n + \frac{C_{n-1}^2}{k}\mu_{n-1}\} = 0 \quad (2a)$$

$$M\mu \equiv \mu_{n,t} + (\varepsilon^{-1}\theta_{n,x} - \theta_{n,y})\mu_n + \frac{i\sigma}{\varepsilon}\{(k - k^{-1})\mu_n + \frac{C_{n-1}^2}{k}\mu_{n-1} - k\mu_{n+1}\} = 0 \quad (2b)$$

where  $k \equiv R \exp(i\phi)$  is the spectral parameter and we have defined  $C_n \equiv \exp\{\theta_{n+1} - \theta_n\}$ .

As mentioned above, we are interested in the elliptic version of (1), i.e. the cases corresponding to  $\varepsilon^2 = -1$ . The initial value problem is therefore ill posed. Natural problems to consider for these equations are boundary value problems. We note that the Inverse Scattering Transform (IST) has primarily been used to solve initial value problems.

As we shall see, consideration of boundary problems involves a non-trivial extension of the IST method, posing new difficulties to face. The first of these is, of course, determination of the natural boundary value problem to consider for the equation.

Another difficulty is that now the method involves solving, besides the standard linear problems, an integral relationship which serves as a constraint for  $\theta_{ny}$  as a functional of  $\theta_n$ . As we shall see, this relationship can be understood by means of a series expansion. However a rigorous investigation regarding its exact solvability still must be undertaken. For some other work on elliptic boundary problems we refer the reader to [13,14].

## 2. Case I: $\sigma = -i$

A well posed boundary problem is the following. Determine  $\theta_n(x, y)$  satisfying (1) where  $\theta_n(x, 0)$  is given for  $|x| < \infty$ ,  $n \in \mathbb{Z}$ ,  $0 \leq y < \infty$  and  $\theta_n(x, y)$  decays sufficiently rapidly as  $x^2 + n^2 \rightarrow \infty$  and  $y \rightarrow \infty$ . To solve the above problems we need the spectral properties of a suitable eigenfunction  $\mu(x, n, y, k)$  where  $\mu \rightarrow 1$  as  $|k| \rightarrow \infty$ . It turns out that this function is nonholomorphic and its departure from holomorphicity is given by (frequently we simply write  $\mu(n)$  to represent  $\mu(x, n, y, k)$ )

$$\frac{\partial \mu}{\partial k} = \frac{\text{sign}(k_I)}{2\pi i k} F(k, y) \exp\left\{2i\left[(R - R^{-1})x \sin \phi - n\phi\right]\right\} \mu(\bar{k}) \quad (3)$$

where

$$F(k, y) \equiv \int_{-\infty}^{\infty} dx \sum_{-\infty}^{\infty} \exp\left\{-2i\left[(R - R^{-1})x \sin \phi - n\phi\right]\right\} (V\mu)(x, n, y, k), \quad (4)$$

$$(V\mu)(x, n, k) \equiv (\theta_{n,x} + \theta_{n,y})\mu_n + \left(\frac{1 - C_{n-1}^2}{k}\right) \mu_{n-1}. \quad (5)$$

The data  $F(k, y)$  satisfies:

$$F(k, y) = F(k, 0) \exp\left\{2(R + R^{-1}) \sin \phi y\right\} \quad (6)$$

since  $\mu \rightarrow 1$  as  $|k| \rightarrow \infty$ . The equation of the inverse problem is found from the generalized Cauchy formula:

$$\mu(k) = 1 + \frac{1}{2ki} \int_{\mathbb{C}} \frac{\partial \mu}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - k}. \quad (7)$$

Finally, the solution is recovered from

$$\theta_n(x, y) = \frac{1}{2} \ln \{\mu(x, n, y, k=0)\}. \quad (8)$$

We now claim:

(i) boundedness requires

$$F(k, 0) = 0 \quad \text{for } k_I > 0; \quad (9)$$

- (ii) the constraint (9) allows one to eliminate, in principle, the dependence in the spectral problem upon  $\theta_{n,y}(x, 0)$ ; and  
 (iii) the solution is unique.

To justify the above, one solves the equations of the inverse problem in the asymptotic domain  $r^2 \equiv x^2 + y^2 \rightarrow \infty$  and obtains that

$$\mu = 1 + \frac{A(k)e^{-4r} + B(k)e^{4r}}{r^{1/2}} + o(1/r^{1/2}) \quad (10)$$

where  $A, B$  are proportional to  $F(k, y)$  in the regions  $k_I < 0, k_I > 0$  respectively. Boundedness requires  $B = 0$  and (i) follows.

Note that (2a) and (9) at  $y = 0$  define a system of linear equations whereupon in principle it is possible to obtain  $\theta_{ny}(x, 0)$  as a functional of  $\theta_n(x, 0)$ . In practice this is far from being trivial. We show how to do it, informally, by series expansion. Let  $\lambda \ll 1$  and

$$\begin{aligned} \theta_n(x, y = 0) &= \lambda \theta_n^{(1)}(x, y = 0) + \lambda^2 \theta_n^{(2)}(x, y = 0) + \dots \\ \mu(x, n, y = 0, k) &= 1 + \lambda \mu^{(1)}(x, n, y = 0, k) + \dots \end{aligned} \quad (11)$$

For convenience we introduce the coordinates  $(p, \psi)$  where

$$k = \frac{\left(p \pm \sqrt{p^2 + 16 \sin^2\left(\frac{1}{2}\psi\right)}\right)}{4 \sin\left(\frac{1}{2}\psi\right)} e^{i\psi} \quad (12)$$

according to whether  $\pm \sin \phi > 0$ . Let us denote:  $F_{\pm}(k) = F(k)$  for  $\pm \sin \phi > 0$  (recall  $k = Re^{i\phi}$ ). We solve the equation  $F_+(k) = 0$  iteratively. At first order it reduces to

$$\int dx \sum_n e^{-ipx+in\psi} \left\{ \theta_n - \frac{\theta_{n,y}}{\sqrt{p^2 + 16 \sin^2\left(\frac{1}{2}\psi\right)}} \right\} = 0 \quad (13)$$

which can be trivially inverted to obtain,

$$\theta_{ny}(x, y = 0) = \int dx' \sum_{n'} \mathcal{G}(x - x', n - n') \theta(x', n', y = 0) \quad (14)$$

where

$$\mathcal{G}(x, n) = \int_{-\infty}^{\infty} dp \int_0^{2x} d\psi e^{i(px-n\psi)} \sqrt{p^2 + 16 \sin^2\left(\frac{1}{2}\psi\right)}. \quad (15)$$

The first term in the expansion of the scattering data  $F_-^{(1)}(k)$  is given by

$$F_-^{(1)}(p, \psi) = 2\sqrt{p^2 + 16 \sin^2\left(\frac{1}{2}\psi\right)} (\mathcal{F}\theta^{(1)})(p, \psi) \quad (16)$$

where  $\mathcal{F}$  is the usual Fourier transform:

$$(\mathcal{F}\theta^{(1)})(p, \psi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_n e^{-ipx+in\psi} \theta_n^{(1)}(x, y=0) dx. \quad (17)$$

The  $y$  dependence follows from (6) and reads

$$F_-(p, \psi, y) = \exp \left\{ -y\sqrt{p^2 + 16 \sin^2\left(\frac{1}{2}\psi\right)} \right\} F_-(p, \psi, 0). \quad (18)$$

These expansions also determine  $\mu(x, n, y, k)$  from which we may also determine the solutions  $\theta_n(x, y)$  in series form, and the leading term is given by:

$$\begin{aligned} \theta_n(x, y) \sim \theta_n^{(1)}(x, y) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp \int_0^{2x} d\psi (\mathcal{F}\theta^{(1)})(y=0) \\ &\times \exp \left\{ i(px - n\psi) - y\sqrt{p^2 + 16 \sin^2\left(\frac{1}{2}\psi\right)} \right\}. \end{aligned} \quad (19)$$

Hence at this order our procedure and results reduce to those of the linear problem.

It is straightforward to show that at any order the above expansion yields  $\theta_{ny}^{(2)}(x, y=0)$  in terms of  $\mu^{(n)}(x, n, y=0, k)$  and  $\theta_n^{(m)}(x, y=0), 1 \leq m \leq j$  and therefore fixes  $\theta_{ny}(x, y=0)$  as a functional of  $\theta_n(x, y=0)$ .

We now turn to the issue of uniqueness. Assume two solutions  $\theta_n^{(a)}$  and  $\theta_n^{(b)}$  to (1) exist, satisfying the same initial conditions. Call  $w_n \equiv \theta_n^{(a)} - \theta_n^{(b)}$  and let  $D$  be the half-plane

$$D = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}.$$

Manipulating equation (1) for  $\theta^{(a)}$  and  $\theta^{(b)}$  we have the identity

$$\begin{aligned} &\int_{\partial D} \sum_{n=-\infty}^{\infty} (w_n w_{n,x} dy - w_n w_{n,y} dx) \\ &= \int_D dx dy \sum_{n=-\infty}^{\infty} \left\{ w_{n,x}^2 + w_{n,y}^2 + 2 \exp \left\{ 2(\theta_{n+1}^{(b)} - \theta_n^{(b)}) \right\} (w_{n+1} - w_n) \right\} \\ &\quad \times \left\{ \exp \{2(w_{n+1} - w_n)\} - 1 \right\}. \end{aligned} \quad (20)$$

The left hand side vanishes in view of the boundary conditions, while the right hand side is positive definite. It follows that  $w_n = 0$ ; i.e.  $\theta_n^{(a)} = \theta_n^{(b)}$ , and hence uniqueness. These facts establish the well posedness of the problem.

### 3. Case II: $\sigma = 1$

The nature of this case is different. It is neither well posed as an initial value problem nor as a pure boundary problem. Actually it is a radiation problem, i.e. a correctly posed problem is the following: solve equation (1) with given  $\theta_n(x, 0)$  decaying sufficiently rapidly as  $x^2 + n^2 + y^2 \rightarrow \infty$ ,  $0 \leq y < \infty$ ,  $|x| < \infty$  and satisfying the following analogue of the Sommerfeld radiation condition used in linear problems,

$$\lim_{r \rightarrow \infty} \left\{ \frac{\partial \theta_n}{\partial r}(x, y) - 4i[(1 - c^2)^{1/2} - \arccos c]\theta_n(x, y) \right\} = 0, \quad (21)$$

where  $-1 < c < 1$ , with  $c \equiv n/(2r)$ . In scattering space this extra complexity translates into a richer structure. The corresponding eigenfunction  $\mu(x, n, y, k)$ , possesses both smooth regions of nonholomorphicity and a discontinuity across the unit circle. We find see [5,6]

$$\frac{\partial \mu}{\partial k} = \exp \left\{ 2i \left[ (R - R^{-1})x \cos \phi - 2n\phi \right] \right\} (-1)^n F(k) \mu(x, n, -\bar{k}) \quad |k| \neq 1 \quad (22)$$

and

$$(\mu_+ - \mu_-)(x, n, k) = \int_0^{2\pi} d\alpha (z/k)^n \mu_-(x, n, z) \quad (23)$$

where in (23)  $k = e^{i\phi}$  ( $|k| = 1$ ) and  $z = e^{i\alpha}$ .  $\mu_+$  and  $\mu_-$  are the boundary values of  $\mu(k)$  evaluated by taking limits from the outside ( $\mu_+$ ) and inside ( $\mu_-$ ) of the unit circle.  $F(k)$  and  $D(z, k)$  are the scattering data of the problem which can be expressed in terms of  $\theta_n(x, 0), \theta_{ny}(x, 0)$  and initial eigenfunctions  $\mu(x, n, y = 0, k)$ . To prove that the aforementioned radiation problem is correctly posed one needs to establish the following:

1. boundedness requires,

$$F(k, 0) = 0 \quad (24)$$

for  $\{k_R > 0, |k| > 1\}$  and  $\{k_R < 0, |k| < 1\}$ .

2. The radiation condition implies

$$D(z, k, 0) = 0 \quad \text{for } \sin \alpha > \sin \phi \quad (z \equiv e^{i\alpha}, \quad k \equiv e^{i\phi}). \quad (25)$$

3. The solution  $\theta_n(x, y)$  behaves as

$$\begin{aligned} \theta(x, n, y) &= \frac{\sin \omega}{2\pi r} \exp \left\{ i \{ 4r((1 - c^2)^{1/2} - \arccos c) - \pi/2 \} \right\} \\ &\times D(\exp \{ i\Omega_+ \}, \exp \{ i\Omega_- \}) + O(1/r^2) \end{aligned} \quad (26)$$

where  $x \equiv r \cos w$ ,  $y \equiv r \sin w$ ,  $\Omega_{\pm} = \pm \arccos c + w - \pi/2$  and  $c \equiv n/(2r)$  is fixed.

4. The solution to the boundary problem is unique if we require (24) and (25)
5. Equations (24-25) allow us to eliminate, in principle, the dependence in the spectral problem upon  $\theta_{ny}(x, 0)$ . The proof of the above facts is similar, in spirit, to those of case I, although the details are considerably more tedious [6].

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# CANONICAL BILINEAR SYSTEMS AND SOLITON RESONANCES

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**ABSTRACT.** A direct method, based on the use of Hirota's binary operators, is developed for the search of families of soliton equations. A crucial step is the explicit construction of canonical bilinear equations from which recursion-operators and consequently canonical Bäcklund-transformations and (or) Lax-pairs can be deduced. This construction is illustrated in the case of the KdV-, Sawada-Kotera (SK)- and mKdV-hierarchies.

One space dimensional soliton resonances are investigated for a family of integrable equations which are related by translation to the SK-hierarchy. These resonances are expected to induce a structural instability responsible for the decay of solitons belonging to a definite amplitude interval.

## 1. Introduction

The bilinear method [1] is usually presented as a convenient way to construct special solutions to NLPDE's. Finding an appropriate dependent variable transformation which bilinearizes the equation is the first but crucial step in this procedure.

The best known example is perhaps that of the KdV-equation:

$$u_t + u_{3x} + 6uu_x = 0 \quad (1.1)$$

which can be bilinearized through the dependent variable transformation  $u(x, t) = 2\partial_x^2(\ln f)$ :

$$F_3(D_x, D_t)f \bullet f \equiv D_x(D_t + D_x^3)f \bullet f = 0. \quad (1.2)$$

In [1] it is shown how to find the KdV-two-soliton solution  $u_2 = 2\partial_x^2(\ln f_2)$  from this bilinear form:

$$\begin{aligned} f_2 &= 1 + \exp(\theta_1) + \exp(\theta_2) + A_{12}^{\text{KdV}} \exp(\theta_1 + \theta_2) \\ A_{12}^{\text{KdV}} &= \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad \theta_i = -k_i x + k_i^3 t + \tau_i, \quad i = 1, 2. \end{aligned} \quad (1.3)$$

In our approach, the bilinear operators are used in a direct way [2] in order to construct bilinear equations which will have two-soliton solutions of a specific type.

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If, for example, we want to find the second member of the KdV-hierarchy [3], we look for a bilinear equation which reduces to a fifth-order NLPDE with two-soliton solutions of the KdV-type (i.e. with the KdV coupling-factor  $A_{12}^{\text{KdV}}$ ).

Introducing an extra time-variable  $t_3$  (which corresponds to the KdV equation itself) and taking  $\tau_i = k_i^3 t_3 + \tau'_i$ :

$$\begin{aligned} f_2 &= 1 + \exp(\theta_1) + \exp(\theta_2) + A_{12}^{\text{KdV}} \exp(\theta_1 + \theta_2) \\ \theta_i &= -k_i x + k_i^5 t_5 + k_i^3 t_3 + \tau'_i, \quad i = 1, 2. \end{aligned} \quad (1.4)$$

one finds:

$$D_x(D_{t_5} + D_x^5)f_2 \bullet f_2 = 20k_1^2 k_2^2 (k_1 - k_2)^2 \exp(\theta_1 + \theta_2) \quad (1.5)$$

and:

$$D_x^3(D_{t_3} + D_x^3)f_2 \bullet f_2 = 24k_1^2 k_2^2 (k_1 - k_2)^2 \exp(\theta_1 + \theta_2) \quad (1.6)$$

$$D_{t_3}(D_{t_3} + D_x^3)f_2 \bullet f_2 = 12k_1^2 k_2^2 (k_1 - k_2)^2 \exp(\theta_1 + \theta_2). \quad (1.7)$$

such that the following one-parameter family has  $f_2$  as a solution:

$$\left[ D_x(D_{t_5} + D_x^5) + \alpha D_x^3(D_{t_3} + D_x^3) - \left(\frac{5}{3} + 2\alpha\right) D_{t_3}(D_{t_3} + D_x^3) \right] f \bullet f = 0. \quad (1.8)$$

Setting:  $u = 2\partial_x^2(\ln f)$ , differentiating once with respect to  $x$  and eliminating the  $t_3$  dependencies with the KdV-equation (1.1), we find the well known Lax-equation [4]:

$$u_{t_5} + u_{5x} + 10uu_{3x} + 20u_xu_{2x} + 30u^2u_x = 0 \quad (1.9)$$

In this paper we show that this explicit construction of multidimensional bilinear equations can be developed into a direct method for the search of integrable hierarchies. Our main point is that crucial features of integrability such as recursion operators, Lax-pairs and Bäcklund transformations can be derived from canonical bilinear systems which are obtained at the two-soliton level.

Canonical bilinear systems are also used to show that soliton resonances and instabilities can be generic for an integrable hierarchy in  $1+1$  dimensions.

## 2. Canonical bilinear forms for the KdV- and SK-hierarchies

It is interesting to see whether it is possible to repeat the above procedure at higher orders and, if so, whether one can find a “canonical” bilinear equation for an entire family of NLPDE’s, all sharing the same two-soliton solutions. The first question we have to address is that of the structure of the hierarchy.

Let us start with the KdV-case. A perturbational analysis [5] (which will not be discussed here) shows that NLPDE’s with a linear part  $L_m(u) = u_t + u_{mx}$ , with a polynomial quadratic part and with two-soliton solutions  $u_2(\theta_1, \theta_2, A_{12}^{\text{KdV}})$ ,  $\theta_i = -k_i x + k_i^m t + \tau_i$ , can only exist at odd  $m$  levels.

We require the canonical bilinear form  $F(D)f \bullet f$  we are looking for to satisfy the minimal degree hypothesis: it has to be of *constant and minimal degree* in the D-operators. A first

attempt therefore is to look for a polynomial expression (of homogeneous weight  $m+3$ ) in the D-operators

$$F_{m+2}(D_x, D_{t_{m+2}}, D_{t_m}, D_{t_3})f \bullet f = 0, \quad \text{odd } m \geq 3 \quad (2.1)$$

which has two-soliton solutions

$$\begin{aligned} f_2 &= 1 + \exp(\theta_1) + \exp(\theta_2) + A_{12}^{\text{KdV}} \exp(\theta_1 + \theta_2) \\ \theta_i &= -k_i x + k_i^{m+2} t_{m+2} + k_i^m t_m + k_i^3 t_3, \quad i = 1, 2. \end{aligned} \quad (2.2)$$

and which depends on only two auxiliary time-variables:  $t_3$  since it certainly appears in the expression for  $F_5$  and  $t_m$ , since the corresponding D-operator will produce the lowest possible degree terms.

At  $m = 3$  there is a unique bilinear form (corresponding to  $\alpha = -1$  in the one-parameter family (1.8)) which satisfies the minimal degree hypothesis:

$$F_5(D_x, D_{t_5}, D_{t_3})f \bullet f \equiv [D_x D_{t_5} - \frac{2}{3} D_x^3 D_{t_3} + \frac{1}{3} D_{t_3}^2] f \bullet f = 0. \quad (2.3)$$

(Notice that the  $D_x^6$  term has disappeared).

At  $m = 5$  we also find a unique bilinear form satisfying the minimal degree hypothesis:

$$F_7(D_x, D_{t_7}, D_{t_5}, D_{t_3})f \bullet f \equiv [D_x D_{t_7} - \frac{2}{3} D_x^3 D_{t_5} + \frac{1}{3} D_{t_5} D_{t_3}] f \bullet f = 0. \quad (2.4)$$

Considering these results, we propose:

$$F_{m+2}(D_x, D_{t_{m+2}}, D_{t_m}, D_{t_3})f \bullet f = 0, \quad \text{odd } m \geq 3 \quad (2.5)$$

with

$$F_{m+2}(D_x, D_{t_{m+2}}, D_{t_m}, D_{t_3}) \equiv D_x D_{t_{m+2}} - \frac{2}{3} D_x^3 D_{t_m} + \frac{1}{3} D_{t_3} D_{t_m}, \quad (2.6)$$

as a canonical bilinear form for the KdV-hierarchy.

It is easy to check that:

$$F_{m+2}(D_x, D_{t_{m+2}}, D_{t_m}, D_{t_3})f_2 \bullet f_2 = 0 \quad (2.7)$$

with  $f_2$  defined by (2.2). (Notice that at  $m = 1$  one finds the KdV-equation if and only if  $D_{t_1} = -D_x$ .)

Taking now  $u = 2\partial_x^2(\ln f)$ , differentiating with respect to  $x$  and eliminating the  $t_3$  dependencies with the KdV-equation, this canonical bilinear equation becomes:

$$u_{t_{m+2}} - u_{2x,t_m} - 4u u_{t_m} - 2u_x \int_{-\infty}^x u_{t_m} dx' = 0 \quad (2.8)$$

defining an operator which relates two successive orders in the KdV-hierarchy:

$$u_{t_{m+2}} = \Delta(u) \bullet u_{t_m}, \quad \text{odd } m \geq 1, \quad u_{t_1} = -u_x \quad (2.9)$$

with

$$\Delta(u) = (\partial_x^2 + 4u + 2u_x \partial_x^{-1}), \quad (2.10)$$

where  $\partial_x^{-1}(u) = \int_{-\infty}^x u(x')dx'$  ( $u$  vanishes with its derivatives as  $x \rightarrow \pm\infty$ ).

It is obvious that (2.8) has two-soliton solutions:

$$u_2(\theta_1, \theta_2, A_{12}^{\text{KdV}}) = 2\partial_x^2[\ln f_2(\theta_1, \theta_2, A_{12}^{\text{KdV}})],$$

with  $\theta_i$  defined by (2.2). The operator  $\Delta(u)$  is the well-known Lenard recursion operator for the KdV equation [6].

Let us now consider the SK-equation:

$$u_t + u_{5x} + 15uu_{3x} + 15u_xu_{2x} + 45u^2u_x = 0 \quad (2.11)$$

which, by setting  $u = 2\partial_x^2(\ln f)$ , can be bilinearized as:

$$G_5(D_x, D_{t_5})f \bullet f \equiv D_x(D_{t_5} + D_x^5)f \bullet f = 0. \quad (2.12)$$

The corresponding two-soliton solution is  $u_2(\theta_1, \theta_2, A_{12}^{\text{SK}}) = 2\partial_x^2[\ln f_2(\theta_1, \theta_2, A_{12}^{\text{SK}})]$  with:

$$\begin{aligned} f_2 &= 1 + \exp(\theta_1) + \exp(\theta_2) + A_{12}^{\text{SK}} \exp(\theta_1 + \theta_2) \\ A_{12}^{\text{SK}} &= \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)}, \quad \theta_i = -k_i x + k_i^5 t_5 + \tau_i, \quad i = 1, 2. \end{aligned} \quad (2.13)$$

Again, in order to find a canonical bilinear form for the entire SK-hierarchy [3], one must have an idea of the structure of this hierarchy. In this case one can see from the perturbational analysis [5] that NLPDE's with a linear part  $L_m(u) = u_t + u_{mx}$ , with a polynomial quadratic part and with two-soliton solutions  $u_2(\theta_1, \theta_2, A_{12}^{\text{SK}})$ ,  $\theta_i = -k_i x + k_i^m t + \tau_i$ , can only exist when  $m$  is odd and not a multiple of 3. Therefore, the next equation of this hierarchy is to be found at  $m = 7$ :

$$\begin{aligned} G_7(D_x, D_{t_7}, D_{t_5})f \bullet f &\equiv \left[ D_x(D_{t_7} + D_x^7) - \frac{7}{10}D_x^3(D_{t_5} + D_x^5) \right] f \bullet f \\ &\equiv \left[ D_x D_{t_7} + \frac{3}{10}D_x^8 - \frac{7}{10}D_x^3 D_{t_5} \right] f \bullet f = 0 \end{aligned} \quad (2.14)$$

which is the unique bilinear equation with solution  $f_2(\theta_1, \theta_2, A_{12}^{\text{SK}})$  (where the phase-functions  $\theta_i = -k_i x + k_i^7 t_7 + k_i^5 t_5 + \tau_i$  depend on only one auxiliary variable: the SK-time  $t_5$ ). Introducing  $u = 2\partial_x^2(\ln f)$ , differentiating with respect to  $x$  and eliminating the  $t_5$  dependencies with the SK-equation (2.11), one finds the second member of the SK-hierarchy [7]:

$$u_{t_7} + u_{7x} + 21(uu_{4x} + u_xu_{3x} + u_{2x}^2 + 6u^2u_{2x} + 3uu_x^2 + 3u^4)_x = 0 \quad (2.15)$$

It is clear that a canonical bilinear form for this hierarchy should:

1. obey the minimal degree hypothesis;
2. possess solutions  $f_2(\theta_1, \theta_2, A_{12}^{\text{SK}})$ ; that is for the structure of the SK-hierarchy

$$G_{m+6}(D)f \bullet f = 0 \quad m \in \{6p + 1, 6p + 5 | p \in \mathbb{B}\}. \quad (2.16)$$

(Notice that we do not ask for this bilinear form to describe the SK-equation itself, since the  $G_5$  polynomial contains too few terms to be described by a canonical form).

We will therefore look for a bilinear form (of homogeneous weight  $m+7$ ):

$$G_{m+6}(D_x, D_{t_{m+6}}, D_{t_m}, D_{t_7}, D_{t_5})f \bullet f = 0 \quad (2.17)$$

which depends on three auxiliary time-variables:  $t_3$  since it is certainly present in the  $G_7$  bilinear form and  $t_m$  and  $t_7$  because their corresponding D-operators will produce the lowest possible degree terms. The phase-functions  $\theta_i$  in the two-soliton solutions will then be of th form:

$$\theta_i = -k_i x + k_i^{m+6} t_{m+6} + k_i^m t_m + k_i^7 t_7 + k_i^5 t_5 + \tau_i \quad i = 1, 2. \quad (2.18)$$

At  $m = 5$  there is only one bilinear form which satisfies the above requirements:

$$\begin{aligned} G_{11}(D_x, D_{t_{11}}, D_{t_7}, D_{t_5}) f \bullet f &\equiv \left[ D_x D_{t_{11}} - \frac{9}{35} D_x^7 D_{t_5} \right. \\ &+ \left. \frac{1}{7} D_{t_7} D_{t_5} + \frac{21}{35} D_x^2 D_{t_5}^2 \right] f \bullet f = 0. \end{aligned} \quad (2.19)$$

Since all the possible combinations of D-operators which are allowed by the minimal degree hypothesis, can be identified with terms in  $G_{11}$ , we propose the following canonical form for the SK-hierarchy:

$$\begin{aligned} G_{m+6}(D_x, D_{t_{m+6}}, D_{t_m}, D_{t_7}, D_{t_5}) f \bullet f &\equiv \left[ D_x D_{t_{m+6}} - \frac{9}{35} D_x^7 D_{t_m} + \frac{1}{7} D_{t_7} D_{t_m} \right. \\ &+ \left. \frac{21}{35} D_x^2 D_{t_5} D_{t_m} \right] f \bullet f = 0. \end{aligned} \quad (2.20)$$

It is a straightforward matter to check that:

$$G_{m+6}(D_x, D_{t_{m+6}}, D_{t_m}, D_{t_7}, D_{t_5}) f_2 \bullet f_2 = 0, \quad m \in \{6p + 1, 6p + 5 | p \in \mathbb{B}\} \quad (2.21)$$

for  $f_2(\theta_1, \theta_2, A_{12}^{\text{SK}})$  with phase-functions defined by (2.18). Notice also that at  $m = 1$ , (2.20) reduces to (2.14) if we set  $D_{t_1} = -D_x$ . Setting  $u = 2\partial_x^2(\ln f)$  in (2.20) gives, after differentiation with respect to  $x$  and elimination of the  $t_5$  and  $t_7$  dependencies (with the use of eqns(2.11) and (2.15)):

$$u_{t_{m+6}} = \Delta'(u) \bullet u_{t_m}, \quad m \in \{6p + 1, 6p + 5 | p \in \mathbb{B}\}, \quad u_{t_1} = -u_x \quad (2.22)$$

$$\begin{aligned} \Delta'(u) = \partial_x^6 + 18u\partial_x^4 + 27u_x\partial_x^3 + (33u_{2x} + 81u^2)\partial_x^2 + (30u_{3x} + 189uu_x)\partial_x \\ + (15u_{4x} + 144uu_{2x} + 54u_x^2 + 108u^3) \\ + (3u_{5x} + 45uu_{3x} + 45u_xu_{2x} + 135u^2u_x)\partial_x^{-1} + 9u_x\partial_x^{-1}(2u_{2x} + 3u^2). \end{aligned} \quad (2.23)$$

The operator  $\Delta'(u)$  linking the members of the SK-hierarchy, turns out to be the recursion operator for the SK equation [8]. The bilinear equation (2.20) has previously been reported in [9].

### 3. The translated SK-hierarchy

In their study of one space dimensional soliton resonances, Hirota and Ito [10] investigated the SK-equation (2.11) with non-vanishing boundary conditions:

$$u \xrightarrow[x \rightarrow \pm\infty]{} C, \quad C = \text{constant} \neq 0. \quad (3.1)$$

The map

$$u \longrightarrow \tilde{u} = u - C \quad (3.2)$$

transforms (2.11) into a translated SK-equation:

$$\widetilde{SK}_5 \equiv \tilde{u}_t + \tilde{u}_{5x} + 15C\tilde{u}_{3x} + 45C^2\tilde{u}_x + 15(\tilde{u}\tilde{u}_{2x} + 3C\tilde{u}^2 + \tilde{u}^3)_x = 0. \quad (3.3)$$

This equation admits N-soliton solutions satisfying zero boundary conditions (in addition to the translated N-soliton solutions of the SK-equation) among which two-soliton solutions  $u_2 = 2\partial_x^2(\ln f_2)$  with:

$$f_2 = 1 + \exp(\theta_1) + \exp(\theta_2) + A_{12}^{\widetilde{SK}} \exp(\theta_1 + \theta_2),$$

$$\theta_i = -k_i x + \tilde{\omega}_i^{(5)} t + \tau_i, \quad \tilde{\omega}_i^{(5)} = \tilde{\omega}^{(5)}(k_i) \equiv k_i^5 + 15Ck_i^3 + 45C^2k_i, \quad i = 1, 2. \quad (3.4)$$

$$A_{12}^{\widetilde{SK}} = \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2 + 9C)}{(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2 + 9C)}. \quad (3.5)$$

These solutions comprise resonant soliton triads when  $C < 0$  and when  $(k_1, k_2)$  is located on the elliptic boundary of the regularity domain:

$$k_1^2 + k_1k_2 + k_2^2 - 9|C| = 0, \quad (3.6)$$

where  $A_{12}^{\widetilde{SK}}$  is unbounded, and where the resonance condition is satisfied:

$$\tilde{\omega}^{(5)}(k_1 + k_2) = \tilde{\omega}^{(5)}(k_1) + \tilde{\omega}^{(5)}(k_2). \quad (3.7)$$

By choosing  $\tau_1 = -\ln A_{12}^{\widetilde{SK}}$ ,  $\tau_2 = 0$  and by considering points  $(k_1, k_2)$  close enough to the ellipse (3.6), with  $0 < k_1 < \sqrt{3|C|}$ , it is found that (3.3) has regular two-soliton solutions which describe a particle-like resonance process in which two incoming solitons (with parameters  $k_1$  and  $k_2$ ) collide and fuse to form a larger “intermediate solitary pulse” which eventually decays into the same two outgoing solitons. The amplitude of the intermediate sech-squared pulse is determined by  $k_R = k_1 + k_2$  with  $\sqrt{3|C|} < k_R < 2\sqrt{3|C|}$ . Its shape remains practically unchanged during its lifetime (which is proportional to  $\ln A_{12}^{\widetilde{SK}}$ ) except for continuous but unperceivable changes in the details of its profile. When  $(k_1, k_2)$  reaches the boundary (3.6) the solution degenerates into a resonant soliton triad describing the merger of two incoming solitons (with parameters  $k_1$  and  $k_2$ ) into one outgoing soliton (with parameter  $k_1 + k_2$ ), or the reversed decay-process. The existence of “intermediate solitary waves”, as  $A_{12}^{\widetilde{SK}} \rightarrow \infty$ , implies a structural instability for a particular set of solitons, with amplitude ranging from  $\frac{9}{2}|C|$  to  $6|C|$ . This instability is due to the fact that any initial soliton profile within this range can be approximated with arbitrary accuracy by intermediate solitary pulses which are bound to decay after a finite period of time. A similar instability due to similar resonances in the two-soliton solutions has been reported for the “good Boussinesq” ( $gB$ ) equation [11,12]. The structural instability of large  $gB$ -solitons (with amplitude above a critical value) was confirmed by the observed instability of supercritical lattice excitations of a nonlinear atomic chain [13], modelled by the  $gB$ -equation in the continuum limit. The instability of particular sets of sech-squared solitons, though first regarded as an exceptional phenomenon due to a peculiarity of the  $gB$ -dispersion (small  $gB$ -solitons move faster than larger ones) [11], is a general feature of the dynamics described by the two-soliton solutions  $u_2 = 2\partial_x^2(\ln f_2)$  which depends less on the dispersion law (linear terms in the equation) than on the particular form of the two-soliton coupling factor  $A_{12}$  (linear plus quadratic terms).

We shall now see that the above resonant triads for (3.3), and the related soliton instabilities, are generic for an infinite family of  $\widetilde{SK}$ -equations which is related to the SK-hierarchy by the translation (3.2). The next equation of this  $\widetilde{SK}$ -family should be of seventh order and should correspond to (2.15). It is quite easy to verify that both equations can be derived from similar three-dimensional bilinear systems. Indeed, the only bilinear candidate for  $\widetilde{SK}_7$  is the translated version of the system (2.12), (2.14):

$$Bil\widetilde{SK}_7(\tilde{f}) \equiv \begin{cases} \tilde{f}^{-2} \tilde{G}_7(D_x, D_{t_7}, D_{t_5}) \tilde{f} \bullet \tilde{f} = 0 \\ \tilde{f}^{-2} \tilde{G}_5(D_x, D_{t_5}) \tilde{f} \bullet \tilde{f} = 0 \end{cases} \quad (3.8)$$

with

$$\begin{aligned} \tilde{G}_7(D_x, D_{t_7}, D_{t_5}) &= D_x(D_{t_7} + D_x^7 + 21CD_x^5 + 126C^2D_x^3 + 252C^3D_x) \\ &\quad - \frac{7}{10}D_x^3(D_{t_5} + D_x^5 + 15CD_x^3 + 45C^2D_x) \end{aligned} \quad (3.9)$$

and

$$\tilde{G}_5(D_x, D_{t_5}) = D_x(D_{t_5} + D_x^5 + 15CD_x^3 + 45C^2D_x). \quad (3.10)$$

The first term on the right side of (3.9) accounts for the correct  $\widetilde{SK}_7$ -dispersion as determined by the linear part of the translated version of (2.15):

$$\tilde{L}_7(\tilde{u}) = \tilde{u}_{t_7} + \tilde{u}_{7x} + 21C\tilde{u}_{5x} + 126C^2\tilde{u}_{3x} + 252C^3\tilde{u}_x. \quad (3.11)$$

The second term is determined by relation (2.14) and by expression (3.10) which is the translated version of formula (2.12), according to the linear part of  $\widetilde{SK}_5(\tilde{u})$ . It is a straightforward matter to check that the system (3.8) admits two-soliton generating solutions  $f_2$  with  $A_{12} = A_{12}^{\widetilde{SK}}$  and with:

$$\begin{aligned} \theta_i &= -k_i x + \tilde{\omega}_i^{(7)} t_7 + \tilde{\omega}_i^{(5)} t_5 \tau_i \\ \tilde{\omega}_i^{(7)} &= \tilde{\omega}^{(7)}(k_i) \equiv k_i^7 + 21CK_i^5 + 126C^2k_i^3 + 252C^3k_i. \end{aligned} \quad (3.12)$$

Eliminating the auxiliary  $\widetilde{SK}_5$ -time  $t_5$  by means of (3.3) one finds that the system (3.8) produces a seventh-order equation for  $\tilde{u}(x, t_7)$  which corresponds to (2.15) through the map (3.2). This equation admits resonant soliton triads which are bound to occur on the ellipse (3.6) where the resonance condition is satisfied:

$$\begin{aligned} \tilde{\omega}^{(7)}(k_1 + k_2) - \tilde{\omega}^{(7)}(k_1) - \tilde{\omega}^{(7)}(k_2) &= 7k_1 k_2 (k_1 + k_2)[k_1^2 + k_1 k_2 + k_2^2 + 9C] \\ &\times [k_1^2 + k_1 k_2 + k_2^2 + 6C] = 0 \end{aligned} \quad (3.13)$$

It should be noticed that the resonance condition (3.13) is also satisfied on the ellipse  $k_1^2 + k_1 k_2 + k_2^2 - 6|C| = 0$  but that this curve does not correspond to actual resonances ( $A_{12}^{\widetilde{SK}}$  remain finite!).

Let us now derive a canonical bilinear system for the  $\widetilde{SK}$ -family, from the above system (2.20) for the SK-hierarchy, by applying a map:

$$f \longrightarrow \tilde{f} = f \exp(r/2) \quad (3.14)$$

with an appropriate choice of the function  $r$ , subject to the condition:

$$r_{2x} = -C \quad (3.15)$$

At  $m = 1$  one finds that the simplest possible map transforming the system:

$$BilSK_7(f) \equiv \begin{cases} f^{-2} G_7(D_x, D_{t_7}, D_{t_5}) f \bullet f = 0 \\ f^{-2} G_5(D_x, D_{t_5}) f \bullet f = 0 \end{cases} \quad (3.16)$$

into the system (3.8), is obtained through formula (3.14) with:

$$r = r_7(x, t_7, t_5) \equiv -\frac{1}{2}Cx^2 + 15C^3xt_5 + 63C^4xt_7 \quad (3.17)$$

This particular quadratic form is determined by the condition (3.15) and by demanding that the transformed expressions:

$$f^{-2} G_7(D_x, D_{t_7}, D_{t_5}) f \bullet f|_{f=\tilde{f}\exp(-r/2)}, \quad f^{-2} G_5(D_x, D_{t_5}) f \bullet f|_{f=\tilde{f}\exp(-r/2)}, \quad (3.18)$$

be free of any term independent of  $f$ .

In order to transform the bilinear expressions under the map (3.14) one may use the following property of the D-operators. Let  $p_1, \dots, p_n$  be a set of integers with  $p_1 + \dots + p_n = \text{even}$ . Then:

$$\begin{aligned} D_{x_1}^{p_1} \dots D_{x_n}^{p_n} fg \bullet fg &= \sum_{\ell_1, \ell_2, \dots, \ell_n} \left[ \prod_{i=1}^n \binom{p_i}{p_i - \ell_i, \ell_i} \right] \\ &\times [D_{x_1}^{\ell_1} \dots D_{x_n}^{\ell_n} f \bullet f] [D_{x_1}^{\ell_1} \dots D_{x_n}^{\ell_n} g \bullet g] \end{aligned} \quad (3.19)$$

with  $0 \leq \ell_i \leq p_i$  and the restriction:  $\sum_{i=1}^n \ell_i = \text{even or zero}$ .

The next equation in the  $\widetilde{SK}$ -family, of order 11, can similarly be derived at  $m = 3$  from a bilinear system of the form:

$$Bil\widetilde{SK}_{11}(\tilde{f}) \equiv \begin{cases} \tilde{f}^{-2} \tilde{G}_{11}(D_x, D_{t_{11}}, D_{t_7}, D_{t_5}) \tilde{f} \bullet \tilde{f} = 0 \\ \tilde{f}^{-2} \tilde{G}_7(D_x, D_{t_7}, D_{t_5}) \tilde{f} \bullet \tilde{f} = 0 \\ \tilde{f}^{-2} \tilde{G}_5(D_x, D_{t_5}) \tilde{f} \bullet \tilde{f} = 0 \end{cases} \quad (3.20)$$

in which  $\tilde{G}_{11}(D_x, D_{t_{11}}, D_{t_7}, D_{t_5}) \tilde{f} \bullet \tilde{f}$  corresponds to the expression (2.19) through the map (3.14) with:

$$r = r_{11}(x, t_{11}, t_7, t_5) \equiv r_7(x, t_7, t_5) + 1188C^6xt_{11} - \frac{405}{2}C^5t_5^2 - 1890C^6t_5t_7. \quad (3.21)$$

This quadratic expression is determined by conditions similar to the above conditions (3.15) and (3.18), and by the additional condition that the elimination of the auxiliary times ( $t_5$  and  $t_7$ ) from the system (3.20) should lead to a two-dimensional NLPDE for the “primary” field variable  $\tilde{q}(x, t_{11}) = 2\ln[\tilde{f}(x, t_{11})]$  in which the term proportional to  $\tilde{q}_{2x}$  coincides with the corresponding term in the binomial expansion of  $1188(\tilde{q}_{2x} + C)^6$ . It may be noticed that the term  $1188\tilde{q}_{2x}^6$  is the highest degree nonlinear term in the “primary” SK-equation of order 11. It also follows from the canonical bilinear form (2.20) for the SK-hierarchy that

the highest degree nonlinear term in the “primary” SK-equation of order  $m + 6$  ( $m$  odd,  $(m + 6)/3 \notin \mathbb{B}$ ) is:

$$c_{m+6} q_{2x}^{(m+7)/2} \quad (3.22)$$

with

$$c_{m+6} = \frac{108(m+3)(m+6)}{(m+5)(m+7)} c_m, \quad c_1 = 1, \quad c_5 = 5. \quad (3.23)$$

This enables us to determine a quadratic form  $r_{m+6}(x, t_5, \dots, t_{m+6}, \dots)$  to be used in the map (3.14) in order to obtain a translated expression  $Bil\widetilde{SK}_{m+6}(\tilde{f})$  out of the canonical bilinear system for the SK-hierarchy. It suffices to impose conditions similar to those used at  $m = 3$  to obtain the following general expression for  $r$ :

$$\begin{aligned} r(x, t_5, \dots) &= -\frac{1}{2}Cx^2 + \sum_{m \text{ odd}, (m+6)/3 \notin \mathbb{B}} c_{m+4} C^{(m+5)/2} t_{m+4} R(x, t_5, t_7) \\ R(x, t_5, t_7) &= x - \frac{45(m+5)}{(m+9)(1+\delta_{m,1})} C^2 t_5 - \frac{252(m+5)}{(m+11)(1+\delta_{m,3})} C^3 t_7 \end{aligned} \quad (3.24)$$

where  $\delta_{m,n}$  stands for the Kronecker symbol.

With this quadratic form for  $r$  (auxiliary  $t$ -variables with index exceeding  $m+6$  have no bearing at any fixed value of  $m$ ) it is found that the map (3.14) transforms the canonical form (2.20) into:

$$\begin{aligned} \tilde{G}_{m+6}(D_x, D_{t_{m+6}}, D_{t_m}, D_{t_7}, D_{t_5}) &= D_x D_{t_{m+6}} - \frac{9}{35} D_x^7 D_{t_m} + \frac{1}{7} D_{t_7} D_{t_m} \\ &+ \frac{3}{5} D_x^2 D_{t_5} D_{t_m} - \frac{27}{5} C D_x^5 D_{t_m} + \frac{3}{5} C D_{t_5} D_{t_m} - 27 C^2 D_x^3 D_{t_m} \\ &- 45 C^3 D_x D_{t_m} + \frac{9}{5} C^{(m+1)/2} c_m D_x^6 - \frac{6}{5} C^{(m+1)/2} c_m D_x D_{t_5} \\ &+ 27 C^{(m+3)/2} c_m D_x^4 + \frac{108(m+4)}{(m+5)} C^{(m+5)/2} c_m D_x^2 \end{aligned} \quad (3.25)$$

It is straightforward, but tedious, to check that the corresponding  $\widetilde{SK}$ -equation of order  $m + 6$  has two-soliton solutions with the  $\widetilde{SK}$ -coupling (3.5), exhibiting the same resonances and soliton instabilities as  $\widetilde{SK}_5, \widetilde{SK}_7$ .

#### 4. Canonical Lax-pairs for the KdV- and SK-hierarchies

In §2 we have shown that the equations generated by (2.9) and (2.22) all possess two-soliton solutions of KdV- and SK-type respectively. Although these equations can be recognized as those generated by the known recursion operators for the KdV- and SK-hierarchies we will show their integrability by the explicit construction of canonical Lax-pairs. Let us first consider the case of the KdV-hierarchy. It is well known that the following system of bilinear equations forms a Bäcklund transformation for the KdV-equation [1]:

$$D_x^2 f \bullet g = \lambda f g \quad (4.1)$$

$$[D_{t_3} + D_x^3 + 3\lambda D_x] f \bullet g = 0. \quad (4.2)$$

This system can be transformed into a Lax-pair for the KdV-equation by setting  $\psi = f/g$  and  $u = 2\partial_x^2(\ln g)$ :

$$\psi_{2x} + u\psi = \lambda\psi \quad (4.3)$$

$$\psi_{t_3} + (4\lambda + 2u)\psi_x - u_x\psi = 0 \quad (4.4)$$

The technique which is used to derive these results can now provide us with a canonical Bäcklund transformation in bilinear form for the entire KdV-hierarchy. Using the following “exchange” formulas:

$$\begin{aligned} (D_{x_1} D_{x_2} f \bullet f)g^2 - f^2(D_{x_1} D_{x_2} g \bullet g) &= 2D_{x_1} [(D_{x_2} f \bullet g) \bullet fg] \\ &= 2D_{x_2} [(D_{x_1} f \bullet g) \bullet fg] \end{aligned} \quad (4.5)$$

$$\begin{aligned} (D_{x_1}^3 D_{x_2} f \bullet f)g^2 - f^2(D_{x_1}^3 D_{x_2} g \bullet g) &= -D_{x_2} [(D_{x_1}^3 f \bullet g) \bullet fg] \\ &- 3D_{x_2} [(D_{x_1}^2 f \bullet g) \bullet (D_{x_1} f \bullet g)] + 3D_{x_1} [(D_{x_1}^2 D_{x_2} f \bullet g) \bullet fg] \\ &- 3D_{x_1} [(D_{x_1}^2 f \bullet g) \bullet (D_{x_2} f \bullet g)], \end{aligned} \quad (4.6)$$

one can show, on account of (4.1), that:

$$\begin{aligned} (F_{m+2} f \bullet f)g^2 - f^2(F_{m+2} g \bullet g) &= 2D_x \left[ ((D_{t_{m+2}} - D_x^2 D_{t_m})f \bullet g) \bullet fg \right] + 2D_x [\lambda f g \bullet (D_{t_m} f \bullet g)] \\ &+ \frac{2}{3}D_{t_m} \left[ ((D_{t_3} + D_x^3)f \bullet g) \bullet fg \right] + 2D_{t_m} [\lambda f g \bullet (D_x f \bullet g)] \\ &= 2D_x \left[ ((D_{t_{m+2}} - D_x^2 D_{t_m} - 3\lambda D_{t_m})f \bullet g) \bullet fg \right] \\ &+ \frac{2}{3}D_{t_m} \left[ ((D_{t_3} + D_x^3 + 3\lambda D_x)f \bullet g) \bullet fg \right]. \end{aligned} \quad (4.7)$$

Due to this last equality, it is clear that the following system is a Bäcklund transformation for the canonical bilinear form of the KdV-hierarchy:

$$\begin{cases} D_x^2 f \bullet g = \lambda f g \\ \left[ D_{t_{m+2}} - D_x^2 D_{t_m} - 3\lambda D_{t_m} \right] f \bullet g = 0, \end{cases} \quad (4.8)$$

$$(m \text{ odd}, D_{t_1} = -D_x). \quad (4.9)$$

Notice that at  $m=1$  we find the above mentioned Bäcklund transformation for the KdV-equation. Introducing now  $\psi = f/g$  and  $q = 2\ln g$  and eliminating the auxiliary  $t_3$  variable through the use of (4.4), the system (4.8), (4.9) is transformed into a canonical Lax-pair for the KdV-hierarchy:

$$\begin{cases} L_1\psi \equiv \psi_{2x} + (q_{2x} - \lambda)\psi = 0 \\ L_2\psi \equiv \psi_{t_{m+2}} - 4\lambda\psi_{t_m} - 2q_{xt_m}\psi_x + q_{2x,t_m}\psi = 0 \end{cases} \quad (4.10)$$

$$L_2\psi \equiv \psi_{t_{m+2}} - 4\lambda\psi_{t_m} - 2q_{xt_m}\psi_x + q_{2x,t_m}\psi = 0 \quad (4.11)$$

The compatibility condition of the Lax-operators:

$$[L_1, L_2]\psi = 0 \quad (4.12)$$

is equivalent to

$$u_{t_{m+2}} = \Delta(u) \bullet u(t_m), \quad u = \partial_x^2 q \quad (4.13)$$

thus settling the Lax-integrability of the equations obtained from the canonical bilinear form (2.6).

In the same way as described above, one can derive a Bäcklund transformation in bilinear form for the SK-equation [14]:

$$\left\{ \begin{array}{l} D_x^3 f \bullet g = \lambda f g \\ \left[ D_{t_5} - \frac{15}{2} \lambda D_x^2 - \frac{3}{2} D_x^5 \right] f \bullet g = 0, \end{array} \right. \quad (4.14)$$

$$(4.15)$$

Taking  $\psi = f/g$  and  $u = 2\partial_x^2(\ln g)$  we find

$$\left\{ \begin{array}{l} \psi_{3x} + 3u\psi_x + \lambda\psi = 0 \\ \psi_{t_5} + 9(\lambda + u_x)\psi_{2x} + 3(3u^2 - u_{2x})\psi_x + 18\lambda u\psi = 0, \end{array} \right. \quad (4.16)$$

$$(4.17)$$

as a Lax-pair for the SK-equation.

At this point, we encounter the following problem: due to the complexity of the canonical bilinear form (2.20) we found for the SK-hierarchy, the above procedure can not be easily extended to the general case. In order to derive a canonical Lax-pair one has to proceed in a different manner.

Since the members of the SK-hierarchy all share the same spatial part of their Lax-pairs (they have the same coupling factor in their two-soliton solutions), we propose as a first Lax-operator:

$$L'_1 \psi \equiv \psi_{3x} + 3u\psi_x + \lambda\psi = 0 \quad (4.18)$$

or

$$D_x^3 f \bullet g = \lambda f g \quad (4.19)$$

(in bilinear notation). As a second bilinear representative of a Lax-operator we choose:

$$\left[ D_{t_{m+6}} + a D_x^6 D_{t_m} + b D_x D_{t_5} D_{t_m} + c \lambda D_x^3 D_{t_m} + d \lambda^2 D_{t_m} \right] f \bullet g = 0$$

$$m \in \{6p + 1, 6p + 5 | p \in \mathbb{B}\}, \quad D_{t_1} = -D_x \quad (4.20)$$

which is the most general bilinear equation which could possibly be derived through the use of exchange formulas. Taking again  $\psi = f/g$  and  $q = 2\ln g$  and eliminating  $t_5$  and higher  $x$  derivatives of  $\psi$  with eqns.(4.17) and (4.18) respectively, we get (modulo (2.11)):

$$\begin{aligned} L'_2 \psi \equiv & \psi_{t_{m+6}} + A_1 \psi_{2x,t_m} + A_2 \psi_{x,t_m} + A_3 \psi_{t_m} + A_4 \psi_{2x} \\ & + A_5 \psi_x + A_6 \psi = 0 \end{aligned} \quad (4.21)$$

where the  $A_i$  are functionals of  $u$  and its derivatives (the expressions of which are too involved to be printed here). Forcing now the compatibility of the two Lax-operators:

$$[L'_1, L'_2] \psi = 0 \quad (4.22)$$

on the solution space of

$$u_{t_{m+6}} = \Delta'(u) \bullet u_{t_m}, \quad u = \partial_x^2 q \quad (4.23)$$

gives the following values for the parameters:

$$\{a = \frac{3}{5}; b = \frac{3}{5}; c = 6; d = 15\}. \quad (4.24)$$

and therefore:

$$\begin{aligned} L'_2 \psi \equiv & \psi_{t_m+6} + 27\lambda^2 \psi_{t_m} - 9[q_{4x,t_m} + 3q_{xt_m}q_{3x} + 3q_{2x}q_{2x,t_m} + 3\lambda q_{xt_m}] \psi_{2x} \\ & + \frac{3}{5}\psi_x [q_{t_5,t_m} + 6q_{5x,t_m} + 15q_{xt_m}q_{4x} + 45q_{2x,t_m}q_{3x} - 45q_{xt_m}q_{2x}^2 \\ & + 45\lambda q_{2x,t_m}] - 18\lambda\psi[3q_{xt_m}q_{2x} + q_{3x,t_m}] = 0 \end{aligned} \quad (4.25)$$

which, together with (4.18), forms a canonical Lax-pair for the SK-hierarchy.

## 5. Modified KdV: canonical bilinear form and Lax-pair

The mKdV-equation:

$$v_t + v_{xxx} - 6v^2v_x = 0 \quad (5.1)$$

has a well known bilinear representation as a system of bilinear equations [3,7]:

$$\left\{ \begin{array}{l} D_x^2 f \bullet g = 0 \\ (D_{t_3} + D_x^3) f \bullet g = 0 \end{array} \right. \quad (5.2)$$

with

$$v = \partial_x[\ln(f/g)]. \quad (5.4)$$

This is easily seen in the following way. Since the Miura-transformation:

$$u = -v_x - v^2 \quad (5.5)$$

linearizes through  $v = \partial_x(\ln \psi)$  to the spatial part of the linear problem for the KdV-equation (with the spectral parameter  $\lambda$  set to zero):

$$\psi_{2x} + u\psi = 0 \quad (5.6)$$

which in its turn (as was previously shown) can be bilinearized as  $D_x^2 f \bullet g = 0$  through the transformations:  $\psi = f/g$  and  $u = 2\partial_x^2(\ln g)$  ( $u$  a solution of the KdV-equation). It is clear that (5.4) expresses  $v$  in terms of the solutions  $f, g$  of the bilinear system (4.1), (4.2) which forms a Bäcklund transformation for KdV. A similar argument should hold for the entire mKdV-hierarchy. Hence we propose the canonical Bäcklund transformation of the KdV-hierarchy, with  $\lambda = 0$ , as a canonical bilinear system, representing all the members of the mKdV-hierarchy:

$$\left\{ \begin{array}{l} D_x^2 f \bullet g = 0 \\ (D_{t_{m+2}} - D_x^2 D_{t_m}) f \bullet g = 0, \quad \text{odd } m \geq 1, \quad D_{t_1} = -D_x \end{array} \right. \quad (5.7)$$

$$(5.8)$$

Notice that at  $m = 1$  ( $D_{t_1} = -D_x$ ) we find the bilinear system for mKdV.

Introducing  $v = \partial_x[\ln(f/g)]$  into the canonical system, we find (after differentiation with respect to  $x$ ):

$$v_{t_{m+2}} = R(v) \bullet v_{t_m}, \quad \text{odd } m \quad (u_{t_1} = -u_x) \quad (5.9)$$

where the operator

$$R(v) \equiv \partial_x^2 - 4v^2 - 4v_x\partial_x^{-1}v, \quad (5.10)$$

can be recognized as the recursion operator for the MKdV-equation [6]. In this case it proves to be possible to derive a canonical Bäcklund transformation by performing the same type of exchange formalism as was used for the KdV-hierarchy. We were able to prove that:

$$\begin{cases} D_x f \bullet g' = \lambda f' \bullet g \\ D_x f' \bullet g = -\lambda f \bullet g' \end{cases} \quad (5.11)$$

$$(D_{t_{m+2}} - 3\lambda^2 D_{t_m} - D_x^2 D_{t_m}) f \bullet f' = 0 \quad (5.12)$$

$$(D_{t_{m+2}} - 3\lambda^2 D_{t_m} - D_x^2 D_{t_m}) g \bullet g' = 0, \quad \text{odd } m \geq 1, \quad D_{t_1} = -D_x \quad (5.14)$$

forms a Bäcklund transformation for the canonical form (5.7), (5.8) of the mKdV-hierarchy. At  $m = 1$  ( $D_{t_1} = -D_x$ ) this system reduces to the Bäcklund transformation for the mKdV-equation [7,15].

If we take now  $\psi = g'/g$ ,  $\tilde{\psi} = f'/f$  and  $v = \partial_x[\ln(f/g)]$ , we get the following canonical linear problem for the mKdV-hierarchy:

$$\begin{pmatrix} \psi_x \\ \tilde{\psi}_x \end{pmatrix} = \begin{pmatrix} v & -\lambda \\ -\lambda & -v \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \quad (5.15)$$

$$\begin{pmatrix} \psi_{t_{m+2}} \\ \tilde{\psi}_{t_{m+2}} \end{pmatrix} = 4\lambda^2 \begin{pmatrix} \psi_{t_m} \\ \tilde{\psi}_{t_m} \end{pmatrix} + \begin{pmatrix} v_{xt_m} - 4v\partial_x^{-1}vv_{t_m} & 2\lambda(v_{t_m} + 2\partial_x^{-1}vv_{t_m}) \\ -2\lambda(v_{t_m} - 2\partial_x^{-1}vv_{t_m}) & -v_{xt_m} + 4v\partial_x^{-1}vv_{t_m} \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \quad (5.16)$$

This system of differential equations is found to be compatible if and only if:

$$\begin{pmatrix} v_{t_{m+2}} - R(v) \bullet v_{t_m} & 0 \\ 0 & -[v_{t_{m+2}} - R(v) \bullet v_{t_m}] \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.17)$$

where the operator  $R(v) \equiv \partial_x^2 - 4v^2 - 4v_x\partial_x^{-1}v$ , is the recursion operator for the mKdV-equation.

We remark that when one performs the following transformations  $v = q$ ,  $\lambda = i\xi$  and

$$\begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.18)$$

the system (5.15), (5.16) is transformed into:

$$\begin{pmatrix} v_{1x} \\ v_{2x} \end{pmatrix} = \begin{pmatrix} -i\xi & q \\ q & i\xi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.19)$$

and

$$\begin{pmatrix} v_{1t_{m+2}} \\ v_{2t_{m+2}} \end{pmatrix} = -4\xi^2 \begin{pmatrix} v_{1t_m} \\ v_{2t_m} \end{pmatrix} + \begin{pmatrix} 4i\xi\partial_x^{-1}qq_{t_m} & q_{xt_m} - 2i\xi q_{t_m} - 4q\partial_x^{-1}qq_{t_m} \\ q_{xt_m} + 2i\xi q_{t_m} - 4q\partial_x^{-1}qq_{t_m} & -4i\xi\partial_x^{-1}qq_{t_m} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.20)$$

which at  $m = 1$ ,  $q_{t_1} = -q_x$  reduces to the usual linear problem for the mKdV-equation [16].

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# A PERTURBATIVE EXTENSION OF THE PAINLEVÉ TEST

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**ABSTRACT.** We further improve the Painlevé test so that negative indices (“resonances”) can be treated: we demand single-valuedness not only for any pole-like expansions as in the usual Painlevé test, but also for every solution close to it, represented as a perturbation series in a small parameter  $\epsilon$ . Order zero is the usual test. Order one, already treated in a preliminary paper, reduces to a (linear) Fuchs analysis near a regular singularity and allows the introduction of all missing arbitrary coefficients. Higher orders lead to the analysis of a linear, Fuchsian type inhomogeneous equation.

We obtain an infinite sequence of necessary conditions for the absence of movable logarithmic branch points, arising at every integer index, whether positive or negative, and at every order; those arising at negative indices, including  $-1$ , are new, while some conditions may not arise before some high perturbation order.

We discuss the understanding of negative indices, and conclude that they are indistinguishable from positive indices, just as in the Fuchs theory. In particular, negative indices give rise to doubly infinite Laurent series.

## 1. Introduction

In the history of “Painlevé analysis” there have been two main strands of thought. Fuchs, Poincaré, the Painlevé School [19], Chazy [8] and Bureau [4] have been interested in *defining new classes of transcendental functions* as single-valued general solutions of ordinary differential equations (ODEs) in the complex plane, a problem mathematically defined in [20] (page 59):

“Déterminer toutes les équations différentielles algébriques du premier ordre, puis du second ordre, puis du troisième ordre, etc., dont l’intégrale générale est uniforme.”

On the other hand, Kowalevskaya [15,16] was interested in the relationship between complete integrability in classical mechanical systems (in the Liouville sense) and the meromorphy of solutions.

The current interest in the Painlevé method stems from the connection, observed by Ablowitz and Segur [1], between partial differential equations (PDEs) of soliton type and

Painlevé transcedents: they showed that ODEs arising as reductions of PDEs soluble by IST can be solved *via* a linear integral equation whose solutions could be proven meromorphic. This led them to their famous conjecture [2,3]: “All reductions of completely integrable PDEs are of Painlevé type, perhaps only after a change of variables”; by “Painlevé type”, they meant that every solution should be meromorphic. This focussed attention on Kowalevskaya’s *pole* type Laurent expansion technique. They then presented an algorithm (essentially equivalent to that of Kowalevskaya [15,16] and Gambier [12,13]) known as the “Painlevé test”, to give *necessary* conditions for an equation to be of Painlevé type. Weiss, Tabor and Carnevale (WTC) [23] later extended this algorithm to directly handle NLPDEs. Two modifications can be used to simplify the laborious computations involved: the first [14] if one is only interested in *testing* the equation, the second [9] if one is also interested in extracting all possible information from the whole WTC test (including the truncation procedure).

**REMARK 1.** To Ablowitz, Ramani and Segur (ARS) [2,3] and most subsequent workers in this field, the existence of an essential singularity is a *bad* thing, violating their meromorphy requirement. On the other hand, the Painlevé school did not forbid essential singularities provided they were not multi-valued.

As remarked above, ODE reductions of PDEs solvable by standard IST have meromorphic solutions and hence such ODEs and PDEs ought to pass the ARS-WTC test. However, even here there are some questions. For instance, the fifth order member of the KdV hierarchy possesses two expansion families. The first one, the so-called “principal”, has indices (“resonances”)  $(-1, 2, 5, 6, 8)$  and the usual Painlevé expansion gives a local representation of the general solution. The second expansion family has indices  $(-3, -1, 6, 8, 10)$ , and the negative index  $-3$  hides part of the solution; that is, the expansion contains only four, rather than the required five, arbitrary functions, so is not a representation of the general solution. The problem is that this is a particular solution, whereas the general solution (with arbitrary data corresponding to every index) may still be multivalued. The ARS-WTC algorithm gives no way of testing the potential “compatibility condition” of this negative index. Of course, in this example, IST tells us that no such compatibility problems arise.

However, there do exist equations for which the latter argument is not available and yet do possess negative indices (*untestable* by the ARS-WTC algorithm), thus rendering the equation’s integrability *indeterminate*. The best known example of such an equation is that of Chazy [7]

$$u_{xxx} - 2uu_{xx} + 3u_x^2 = 0. \quad (1.1)$$

This equation is known to have as a movable natural boundary a circle (the centre and radius of which depend upon the initial conditions) of essential singularities, beyond which the solution cannot be analytically continued, so is not within the usual ARS Painlevé type (movable poles only). Chazy solved his equation by giving a parametric representation of  $u$  and  $x$  in terms of two solutions of the hypergeometric equation. The solution is holomorphic within its domain of definition.

Fordy and Pickering [11] introduced the Fuchs-Painlevé test, which simultaneously examines the Painlevé property (PP) of a nonlinear equation *and* its linearisation. Applied to the linearised equation, the Painlevé analysis reduces to a Fuchsian analysis about a regular (but movable) singularity. Their examples illustrate that vital information can be extracted from negative resonances, whether or not there exists a “generic” family in the sense of ARS.

In the present paper we present the details of an extension of the Fuchs-Painlevé test, which we call the “perturbative Painlevé test”. We consider a perturbation expansion (in a formal “small” parameter  $\varepsilon$ ) for the given nonlinear equation, in which the ARS-WTC expansion is the zeroth order term in  $\varepsilon$ . Truncation at first order recovers the Fuchs-Painlevé test. At each index and each perturbation order, *new* compatibility conditions can arise, thus giving the possibility of further *necessary* conditions for an equation to have the PP.

Furthermore, whereas a standard Painlevé expansion can only contain arbitrary coefficients at *nonnegative integer indices* (together with index  $-1$ ), our perturbation introduces an arbitrary coefficient for *every* index. Unless all the indices satisfy stringent conditions, our perturbed solution *exhibits branching* and thus the equation *fails* our test.

Like all our predecessors, we construct a *local* representation for the general solution. The ARS test does this by constructing a pole-like Laurent series with arbitrary constants arising at each of the *nonnegative* integer indices (“resonances”). In the perturbative-Painlevé approach of this paper we require a ‘full set’ of *integer* indices, but these can now be negative. At each order of perturbation we also construct a pole-like Laurent series, but the order of the pole grows with the order of the perturbation, so the final result is a full Laurent series, which may even represent an essential singularity.

In this paper we just present the basic method and an example for which we need a perturbation of order two. In [10,22] many more examples are presented, together with a broader historical context.

## 2. The Painlevé method: basic definitions and conventions

Let us consider a nonlinear differential equation (DE) with any number of equations  $\mathbf{E}$ , dependent variables  $\mathbf{u}$  and independent variables  $\mathbf{x}$  (boldface characters mean multicomponent)

$$\mathbf{E} \equiv \mathbf{K}[\mathbf{u}, \mathbf{x}] = \mathbf{0}, \quad (2.1)$$

polynomial in  $\mathbf{u}$  and its derivatives, analytic in  $\mathbf{x}$ . This includes ordinary and partial differential systems of equations. We want to build necessary conditions (sometimes referred to as stability conditions for brevity) for the absence of movable critical points in the general solution  $\mathbf{u}$ .

First, we perform a usual Painlevé analysis of this nonlinear DE, using the framework of the unified invariant Painlevé analysis [9], which simplifies many of the expressions and is valid for any DE. In this analysis, the gradient of the expansion variable  $\chi$  is a second degree polynomial in  $\chi$  (see the Appendix).

About any family of movable pole-like singularities of  $\mathbf{u}$  and its transform under (A1), the standard Painlevé expansion takes the form:

$$\mathbf{u} = \sum_{j=0}^{+\infty} \mathbf{u}_j \chi^{j+p}, \quad \mathbf{E} = \sum_{j=0}^{+\infty} \mathbf{E}_j \chi^{j+q}, \quad (2.2)$$

in which negative integers  $p$  and  $q$  are the respective singularity orders of  $\mathbf{u}$  and  $\mathbf{E}$ .

**REMARK 2.** We are using compact notation here, in order to suppress co-ordinate indices. If  $\mathbf{u} = (u_1, \dots, u_N)$ ,  $\mathbf{p} = (p_1, \dots, p_N)$ , then:

$$u_j \sim \chi^{p_j} \text{ and } \mathbf{u}_j \chi^{j+p} = (u_{1j} \chi^{j+p_1}, \dots, u_{Nj} \chi^{j+p_N}). \quad (2.3)$$

The singularity order analysis is a purely *algebraic* process, which provides different sets of dominant terms  $\hat{\mathbf{K}}[\mathbf{u}, \mathbf{x}]$ , and each set of dominant terms provides different sets of leading behaviours, solutions of the algebraic equation

$$\hat{\mathbf{K}}[\mathbf{u}_0 \chi^{\mathbf{P}}, \mathbf{x}]|_{\nabla \mathbf{u}_0=0, \nabla \chi=\mathbf{A}_0} \equiv \mathbf{P}(\mathbf{u}_0, \mathbf{x}) \chi^{\mathbf{q}} = 0, \quad (2.4)$$

in which  $\mathbf{A}_0$  is assumed constant, and  $\mathbf{P}$  is polynomial in  $\mathbf{u}_0$ . Each such pair  $(\mathbf{p}, \mathbf{u}_0)$  defines an *expansion family*.

**REMARK 3.** An *expansion family* is usually called a “branch” in the literature, but we prefer to avoid the confusion with the “branching” of a solution.

For each expansion family the coefficients  $\mathbf{u}_j$  are determined recursively by

$$(\hat{\mathbf{K}}'[\mathbf{u}_0 \chi^{\mathbf{P}}, \mathbf{x}] \chi^{j+\mathbf{p}}|_{\nabla \mathbf{u}_0=0, \nabla \chi=\mathbf{A}_0}) \mathbf{u}_j = \mathbf{F}_j(\mathbf{u}_0, \dots, \mathbf{u}_{j-1}, \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2). \quad (2.5)$$

The determinant of the linear operator acting on  $\mathbf{u}_j$  in the lhs of the above equation vanishes for certain values of  $j$  (not necessarily positive integer) called “indices” by classical authors like Chazy and Bureau, and “resonances” by ARS-WTC. The number  $N$  of these indices is usually equal to the order of the system equivalent to the dominant expression  $\hat{\mathbf{K}}[\mathbf{u}, \mathbf{x}]$ , itself lower than or equal to the order of  $\mathbf{K}[\mathbf{u}, \mathbf{x}]$ , but counterexamples do exist [22]. In order to ensure the existence of  $\mathbf{u}_j$  without introducing movable logarithmic terms whenever  $j$  reaches a *positive* integer index, one must enforce some consistency conditions, namely: the rhs of (2.5) must also be orthogonal to the kernel of the adjoint of the lhs linear operator. No conditions arise at indices which are *not positive integers*.

Arbitrary coefficients enter the series at non-negative integer indices (a further one corresponding to the arbitrariness of  $\chi$ ).

A *maximal family* denotes any family with a number of indices equal to the order of the equation being considered.

A *generic expansion* is any expansion in which the number of arbitrary constants is equal to the order of  $\mathbf{K}$ , and is therefore a local representation of the general solution.

A *principal family* is any maximal family with, apart from  $-1$ , all indices nonnegative integers.

**REMARK 4.** A necessary condition for a pole-like expansion to be generic in the case of one DE, whether ordinary or partial, is that indices be  $-1$  and distinct nonnegative integers.

In the case of there being not enough arbitrary coefficients, then the Painlevé expansion (2.2) only represents a particular (or even singular) solution. This may happen for a number of reasons: for example, a nonmaximal family, or a negative integer index apart from  $-1$ . The method we present next guarantees an arbitrary coefficient at *each* index.

### 3. The perturbative Painlevé method

In this section we seek a Laurent expansion for any solution which is *near to* the solution obtained by the standard Painlevé method of section 2. We do this by considering a perturbation expansion. For a nonprincipal but maximal Painlevé family our perturbation extends the particular solution into a representation of the general solution.

Let us denote the Painlevé expansion (2.2) as  $(\mathbf{u}^{(0)}, \mathbf{E}^{(0)})$ , and look for a nearby solution formally represented by an infinite perturbation series in a small parameter  $\varepsilon$

$$\mathbf{u} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{u}^{(n)}, \quad \mathbf{K}[\mathbf{u}, \mathbf{x}] = 0, \quad (3.1a)$$

$$\mathbf{E} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{E}^{(n)} = \mathbf{0}. \quad (3.1b)$$

The condition that this expansion still be a solution generates the infinite sequence of successive differential equations

$$n = 0 : \mathbf{E}^{(0)} \equiv \mathbf{K}[\mathbf{u}^{(0)}, \mathbf{x}] = \mathbf{0}, \quad (3.2a)$$

$$n \geq 1 : \mathbf{E}^{(n)} \equiv \hat{\mathbf{K}}'[\mathbf{u}^{(0)}, \mathbf{x}] \mathbf{u}^{(n)} - \mathbf{R}^{(n)}(\mathbf{u}^{(0)}, \dots, \mathbf{u}^{(n-1)}) = \mathbf{0}, \quad (3.2b)$$

in which the expression  $\mathbf{R}^{(n)}$  represents the contribution of previous terms of the expansion;  $\mathbf{R}^{(1)}$  is identically zero. Each higher order equation (3.2b) is a linear inhomogeneous differential equation whose left hand side is the Fréchet operator  $\hat{\mathbf{K}}'[\mathbf{u}^{(0)}, \mathbf{x}]$  (independent of  $n$ ) acting on  $\mathbf{u}^{(n)}$ .

As already explained [11], when the Painlevé series  $\mathbf{u}^{(0)}$ , a Laurent series bounded from below, is substituted into the linearised equation:

$$\hat{\mathbf{K}}'[\mathbf{u}^{(0)}, \mathbf{x}] \mathbf{u}^{(1)} = \mathbf{0}, \quad (3.3)$$

the resulting equation for  $\mathbf{u}^{(1)}$  is of Fuchsian type, the movable singularity  $\chi = 0$  of equation (2.1) is a regular singularity for (3.3), and its Fuchs indices are  $i + p$ , where  $i$  runs over the Painlevé “resonances”.

At first order, an arbitrary coefficient is introduced at each index, regardless of type. Not all of these are *new* since we already have a coefficient in  $\mathbf{u}^{(0)}$ , corresponding to each positive integer index (regardless of whether or not it satisfied the stability condition). The coefficients introduced into  $\mathbf{u}^{(1)}$  at the corresponding indices ( $i + p$ ,  $i$  a positive integer) just perturb the already arbitrary coefficients, so add nothing new. However, *all* other indices give rise to *new* arbitrary coefficients. Thus the expression  $\mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)}$  already contains as many arbitrary coefficients as there are indices in the family. If the family is *maximal* then the perturbed solution is a local representation of the *general solution*. Furthermore, since the equation for  $\mathbf{u}^{(1)}$  is *linear* the usual Fuchs-Frobenius theory holds. For each solution  $\sigma$  (not necessarily integer) of the indicial equation, there is a solution  $\mathbf{u}^{(1)}$  with leading order behaviour  $\chi^\sigma$ . If any  $\sigma$  is noninteger, then the solution exhibits branching. When (for a single equation) any of the roots are repeated, then logarithms enter the solution. When roots differ by an integer, then a condition must be satisfied in order to avoid logarithms.

After first order, the function  $\mathbf{u}^{(n)}$  satisfies an inhomogeneous, *linear* differential equation. The indicial equation is the same for all  $n \geq 1$ , but for  $n \geq 2$  the leading order behaviour of  $\mathbf{u}^{(n)}$  is determined by the singularity order of the rhs, not by the kernel of  $\hat{\mathbf{K}}'[\mathbf{u}^{(0)}, \mathbf{x}]$

$$\mathbf{u}^{(n)} = \sum_{j=n\rho}^{+\infty} \mathbf{u}_j^{(n)} \chi^{j+p}, \quad (3.4a)$$

$$\mathbf{E}^{(n)} = \sum_{j=n\rho}^{+\infty} \mathbf{E}_j^{(n)} \chi^{j+q}, \quad (3.4b)$$

$$\mathbf{u} = \sum_{n=0}^{+\infty} \varepsilon^n \left[ \sum_{j=n\rho}^{+\infty} \mathbf{u}_j^{(n)} \chi^{j+p} \right] = \sum_{j=-\infty}^{+\infty} \mathbf{u}_j \chi^{j+p}, \quad (3.5)$$

where  $\rho$  denotes the smallest index of nonlinear equation (2.1), an integer lower or equal to  $-1$ .

Whilst there is no need to introduce arbitrary coefficients for  $n \geq 2$ , stability conditions do arise at each index

$$Q_i^{(n)} \equiv E_i^{(n)} = 0, \quad i \in \text{indices}. \quad (3.6)$$

These must be satisfied at *each* level of perturbation if we are to avoid logarithms.

We can summarise the above in the following theorem.

**THEOREM 1.** *The perturbed Painlevé expansion (3.1a) exhibits branching if:*

1. *any index is noninteger,*
2. *any integer index is degenerate (for a single equation this means repeated),*
3. *any nondegenerate, integer index (positive, zero or negative) generates an incompatible stability condition.*

Whilst the above statements have been *believed* to be true by most people working in the field [17], our method gives a simple *proof*.

**REMARK 5.** It is sometimes believed that negative indices cannot have *any* meaning in Painlevé analysis since they are said to contradict the leading balance, but this is evidently false.

Thus, if all indices are non-degenerate and integer (positive, zero or negative) and compatible at each level of perturbation, then our maximal family gives a single valued, local representation of the general solution to (2.1). However, since we have no *a priori* limit on the perturbation order required to give *all* possible stability conditions, the process of finding *all* these necessary conditions is not finite. If it were finite, we would have obtained necessary *and sufficient* conditions for the absence of movable critical points of logarithmic type. The main aim of the perturbative Painlevé test is, however, not to give *all* possible stability conditions, but to give *more* conditions, if they exist. Each new stability condition (if it does not ‘kill’ the equation) leads to further restrictions, which makes the next step (that of attempting to *prove* the global PP) easier.

**REMARK 6.** However, it is possible to prove [22] that, if the expansion  $\mathbf{u}^{(0)}$  is generic, then the stability conditions at *all* perturbation levels are automatically satisfied, so give no new restrictions.

We are aware of only two situations where order  $n = 0$  is sufficient to build all necessary conditions for the absence of movable logarithmic branch points:

1. for any family having only positive indices, apart from  $-1$ , with the adequate multiplicity,
2. for any family of a scaled ODE having only negative indices with the adequate multiplicity; the general solution can then be represented by a Laurent series ranging from the negative singularity order to  $-\infty$ .

In all other cases, we do not know in advance at which  $n$  to stop and, for instance, a simple fourth order ODE (example 6.4 of [10]) requires an order seven perturbation to conclude.

**REMARK 7.** The above theorem is slightly stronger than the one in [4], which involves only all indices except one, and is indecisive about some examples (see 5.2 of [10]).

**REMARK 8.** In the case of a single equation, the condition  $Q_\rho^{(1)} = 0$  is identically satisfied, since this just corresponds to the Fuchs index  $i = \rho + p$ . However, even at first order perturbation the condition corresponding to resonance  $-1$  need not be satisfied (see equation

(4.7b)). The frequently encountered statement that resonance  $-1$  should be assumed compatible is clearly erroneous. Numerous nontrivial stability conditions  $Q_{-1}^{(n)} = 0$  at index  $-1$  can be found in our examples [10].

#### 4. Example: Chazy's class V

This equation has two expansion families which are equivalent, but this can now only be seen by a *second* order perturbation. Chazy's class V appears in Bureau [6] as (20.5):

$$\begin{aligned} E \equiv & -u''' + a u u'' + 2 a u'^2 - \frac{1}{2} a^2 u^2 u' \\ & + a_1 u'' + c_1 u u' + c_0 u' + d_3 u^3 + d_2 u^2 + d_1 u + d_0 = 0, \end{aligned} \quad (4.1)$$

( $a, a_i, c_i, d_i$  arbitrary analytic functions of  $x$ , and  $a \neq 0$ ).

Let us first set  $a = -2$  by a transformation (A1). The equation has two families

$$p = -1, u_0^{(0)} = 1, \text{ indices } (-1, 1, 4), \hat{K} = -u''' - 2 u u'' - 4 u'^2 - 2 u^2 u', \quad (4.2a)$$

$$p = -1, u_0^{(0)} = 3, \text{ indices } (-3, -1, 4), \hat{K} = -u''' - 2 u u'' - 4 u'^2 - 2 u^2 u'. \quad (4.2b)$$

The length of the calculation is sensitive to the choice of gauge. The most convenient choice for the *first family* is  $a = -2, c_1 = 0, d_3 = 0$ , requiring only two quadratures in the definition of transformation (A1):

$$\frac{\alpha'}{\alpha} = \frac{2d_3}{a^2}, \quad \frac{\xi'}{\alpha} = -\frac{a}{2}, \quad \beta = \frac{14d_3}{a^3} + \frac{c_1 + a'}{a^2}. \quad (4.3)$$

The necessary stability conditions are, at order  $n = 0$  (with  $S = 0$ ):

$$Q_1^{(0)} \equiv 2a_1 = 0, \quad (4.4a)$$

$$\begin{aligned} Q_4^{(0)} \equiv & 2d_2 \left[ u_1^{(0)} \right]^2 + (2d_1 - \frac{1}{2}a_1 d_2 - c'_0 + 3d'_2) u_1^{(0)} + d_0 \\ & + \frac{1}{4} [2d_2(c_0 - d_2) - a_1 d_1 + 4d'_1 + a_1(c_0 - d_2)' - 2(c_0 - d_2)''] = 0, \end{aligned} \quad (4.4b)$$

and they provide four conditions, equivalent to:

$$a_1 = 0, \quad d_2 = 0, \quad d_1 = \frac{1}{2}c'_0, \quad d_0 = 0. \quad (4.5)$$

Since the *second family* is nonprincipal, it is necessary to consider at least order  $n = 1$ . The most convenient choice of gauge is

$$a = -2, \quad a_1 = \frac{9}{5}d_3, \quad c_1 = \frac{21}{5}d_3. \quad (4.6)$$

With  $S = 0$  ( $\chi' = 1$ ), we find at  $\varepsilon^0$  that  $u_4^{(0)}$  is arbitrary and:

$$\begin{aligned} Q_4^{(0)} \equiv & d_0 - \frac{27}{5}d_1 d_3 + 3d'_1 \\ & + \frac{1}{50} [45d_2 - 243d_3^2 + 135d'_3 + 270d_3 \partial_x - 75\partial_x^2] (c_0 - 3d_2) = 0. \end{aligned} \quad (4.7a)$$

At  $\varepsilon^1$  we have arbitrary coefficients  $u_{-3}^{(1)}, u_{-1}^{(1)}$  and stability conditions:

$$Q_{-3}^{(1)} \equiv 0, Q_{-1}^{(1)} \equiv -6d_2 u_{-3}^{(1)} = 0, \quad (4.7b)$$

$$\begin{aligned} Q_4^{(1)} &\equiv \left( -5d_1 - 5c_0 d_3 + \frac{827}{40} d_2 d_3 - \frac{27}{100} d_3^3 \right. \\ &\quad \left. + \frac{5}{2}(c_0 - 7d_2)' + \frac{51}{20} d_3 d_3' - 3d_3'' \right) u_4^{(0)} u_{-3}^{(1)} + F(d_3, d_2, d_1, c_0) u_{-3}^{(1)} = 0, \end{aligned} \quad (4.7c)$$

where  $F$  is an expression with 112 terms. At  $\varepsilon^2$  we have no new arbitrary coefficients, but do have the following two stability conditions:

$$\begin{aligned} Q_{-3}^{(2)} &\equiv \left[ -6d_1 + 3c_0' - 13d_2' \right. \\ &\quad \left. + \left( \frac{913}{50} d_2 - \frac{143}{25} c_0 + \frac{27}{500} d_3^2 - \frac{51}{100} d_3 \partial_x + \frac{3}{5} \partial_x^2 \right) d_3 \right] (u_{-3}^{(1)})^2 = 0, \end{aligned} \quad (4.7d)$$

$$Q_{-1}^{(2)} \equiv [Z(d_3, d_2, d_1, d_0, c_0) - 25d_3 u_4^{(0)}] (u_{-3}^{(1)})^2 + (3d_2 d_3 - 10d_2') u_{-3}^{(1)} u_{-1}^{(1)} = 0, \quad (4.7e)$$

in which  $Z$  is a 31-term expression vanishing when  $d_2, d_1, d_0$  are replaced by their value computed respectively from  $Q_{-1}^{(1)}$ , the coefficient of  $u_4^{(0)} u_{-3}^{(1)}$  in  $Q_4^{(1)}, Q_4^{(0)}$ . The higher condition  $Q_4^{(2)} = 0$  brings no additional information.

The final stability conditions are identical to that for the first family and, since they are also sufficient, this proves the equivalence of the two families.

## 5. Conclusions

The present method, which is based on perturbation theory, provides necessary conditions for the stability of differential equations. It proves the identity of Painlevé “resonances” with the Fuchs indices of a *linear* equation. In case some Fuchs indices are negative integers (apart from the ever present  $-1$ ), these necessary conditions are *new* and the information they contain can be decisive.

If one takes account of the necessary conditions generated by the very first stages of the  $\alpha$ -method [21], the present method provides infinitely many *necessary and sufficient* conditions for the absence of movable critical points of algebraic and logarithmic types. We do not yet know an upper bound, if one exists, on the perturbation order  $n$ , required to find *all* possible compatibility conditions, so our process is not finite, as would be required to define an algorithm. As explained by Painlevé there are unfortunately no methods to determine when an essential singular point is not critical.

In [10] we present many examples which illustrate important features of our method. In one particular example, we need a *seventh order* perturbation in order to expose logarithmic branching.

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## Appendix 1. Homographic Transformations

It is a classical result of complex function theory that the only bijections of the analytic plane (i.e. the complex plane and the point at infinity) are the homographic transformations, or Möbius group, or  $\text{PSL}(2, \mathbb{C})$ . Such transformations do not alter the two classifications of singularities: critical points and noncritical points, fixed and movable singularities. Taking account of uniformizing transformations leaving invariant the structure of movable singularities, the PP is invariant under an arbitrary homographic transformation of the dependent variable  $u$  and an arbitrary change of the independent variable  $x$ :

$$(u, x) \rightarrow (U, X) : u = \frac{\alpha(x)U + \beta(x)}{\gamma(x)U + \delta(x)}, \quad X = \xi(x), \quad \alpha\delta - \beta\gamma \neq 0, \quad (\text{A.1})$$

where  $\alpha, \beta, \gamma, \delta, \xi$  denote arbitrary analytic functions. Such a transformation is sometimes denoted  $T(\alpha, \beta, \gamma, \delta; \xi)$  in the text. This provides four gauges to reduce the complexity of the study. Details can be found in [5].

## Appendix 2. Invariant analysis

In this appendix we state some formulae from the invariant analysis [9].

The expansion function  $\chi$ , whose explicit expression is *not* required, is defined by its gradient

$$\nabla\chi = \mathbf{A}_0 + \mathbf{A}_1\chi + \mathbf{A}_2\chi^2, \quad (\text{A.2})$$

where the  $N$  components of three vectors  $\mathbf{A}_k$  depend on  $N$  elementary functions ( $N$  denoting the number of independent variables) linked by some Schwarz cross-derivative conditions. For two independent variables  $(x, t)$ , these Riccati equations are

$$\chi_x = 1 + \frac{1}{2}S\chi^2, \quad \chi_t = -C + C_x\chi - \frac{1}{2}(C_{xx} + CS)\chi^2, \quad (\text{A.3})$$

with integrability condition:

$$S_t + C_{xxx} + 2C_xS + CS_x = 0. \quad (\text{A.4})$$

Although the above relations are sufficient to perform the whole analysis, we give, for information only, the explicit expressions for  $(\chi, S, C)$ :

$$\chi = \left[ \frac{\varphi_x}{\varphi} - \frac{\varphi_{xx}}{2\varphi_x} \right]^{-1}, \quad S = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2} \left[ \frac{\varphi_{xx}}{\varphi_x} \right]^2, \quad C = -\frac{\varphi_t}{\varphi_x}. \quad (\text{A.5})$$

Function  $S$  (Schwarzian of  $\varphi$ ) is the unique [18] differential invariant of the Möbius group. The practical advantage of using this approach is to reduce the size of many expressions and thus obtain directly the invariant form of the results (e.g. the WTC truncation equations).

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# EXACT SOLUTIONS TO THE COMPLEX GINZBURG-LANDAU EQUATION FROM A LINEAR SYSTEM

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**ABSTRACT.** The one-dimensional complex Ginzburg-Landau equation is highly nonintegrable, and only a few exact solutions are known. We address the problem of finding exact solutions by singularity analysis only. By introducing a linear partial differential system with constant coefficients, we reduce the problem to the finding of a finite number of constant coefficients. We thus recover easily the five solutions of Nozaki and Bekki, each represented in our method by a set of four constants.

## 1. Introduction

The one-dimensional Ginzburg-Landau equation

$$iA_t + pA_{xx} + q|A|^2A - i\gamma A = 0, \quad pq \neq 0, \quad A, p, q \in \mathbb{C}, \quad \gamma \in \mathbb{R}, \quad (1)$$

in which  $p, q, \gamma$  are constants, is quite important in physics where it occurs for instance in turbulence and superconductivity (for a recent review, see van Saarloos and Hohenberg 1992). It is called complex in the case  $\text{Im}(p/q) \neq 0$ , real otherwise, and the real case reduces for  $\gamma = 0$  to the nonlinear Schrödinger equation (NLS).

The complex equation is nonintegrable: it fails the Painlevé test (Keeffe 1986, Cariello and Tabor 1989) and no bilinear form is known for it. To our knowledge, apart from the plane wave solution

$$A = A_0 e^{i(Kx - \Omega t + \delta)}, \quad \Omega - pK^2 - i\gamma + q|A_0|^2 = 0, \quad (2)$$

only five particular solutions have been found (Pereira and Stenflo 1977; Nozaki and Bekki 1984; Bekki and Nozaki 1985). This was done by a clever but *ad hoc* modification (Nozaki and Bekki 1984) of the Hirota bilinear method, by looking for solutions in the class

$$A = G(x, t)e^{i(Kx - \Omega t)} F(x, t)^{-1-i\alpha}, \quad (3)$$

in which  $\alpha, K, \Omega$  are real constants, and  $G$  and  $F$  are sums of two exponentials linear in  $x$  and  $t$ .

The problem we address is to retrieve these solutions by singularity analysis, for which there exist well established methods, and possibly to find new ones. Achieving this goal is not so easy and for instance a Painlevé analysis of (1) (Cariello and Tabor 1989) could only retrieve some of the four solutions mentioned in Nozaki and Bekki (1984).

## 2. Linear systems for solitary waves

Nonlinear partial differential equations (NLPDEs)  $E(u) = 0$  are either integrable (whatever this means) or nonintegrable. In the first case, they admit a Lax pair, i.e. a system of two linear differential operators  $L_1(u), L_2(u)$  whose commutator vanishes as a consequence of  $E(u) = 0$ , and there exists a well known method for building solitary wave solutions from the solutions of the linear system

$$L_1(u_0)\psi = 0, \quad L_2(u_0)\psi = 0, \quad (4)$$

in which  $u_0$  is a particular constant solution called *vacuum*.

In the nonintegrable case, there is no Lax pair any more, but nevertheless some solitary waves or other simple solutions have been found for many such equations. Let us remark that many, if not all, physical NLPDEs admit at least a vacuum solution: (2) with  $\Omega = K = 0$  for (1);  $u$  = arbitrary constant for the Kuramoto-Sivashinsky equation, etc. Then, since this vacuum solution is a constant, it is physically (and mathematically) reasonable to assume the existence of a remnant of a Lax pair in the form of a set of two linear PDEs with constant coefficients; we are then back to the previous situation and a solitary wave can usually be deduced from the consideration of this linear system.

The simplest such linear system is the second order one

$$\psi_{xx} - k^2\psi = 0, \quad (5a)$$

$$\psi_t + c\psi_x + g\psi = 0, \quad (5b)$$

where the constant  $g$  can be set to zero without loss of generality. From the two-parameter general solution

$$\psi = c_1 \cosh(k\xi + c_2), \quad c_1, c_2 \text{ arbitrary}, \quad \xi = x - ct, \quad (6)$$

one builds the elementary variable  $Y = k^{-1} \log(\psi)_x$  and, from  $Y$  and  $Y^{-1}$ , the two elementary functions

$$\sigma(2\theta) = \operatorname{sech}(2\theta), \quad \tau(2\theta) = \tanh(2\theta), \quad \theta = k\xi + c_2, \quad \xi = x - ct, \quad (7)$$

proportional respectively to  $Y - Y^{-1}$  and  $Y + Y^{-1}$ .

The large majority of one-soliton solutions of integrable equations (KdV, Boussinesq, modified KdV, NLS, Zhiber-Shabat, etc) as well as solitary wave solutions of nonintegrable equations (Kuramoto-Sivashinsky, KPP, etc) happen to be polynomial in these two elementary solitary waves  $\sigma$  and  $\tau$ . Such a class (polynomials in  $(\sigma, \tau)$ ) of solitary wave solutions

can be found by a method (Conte and Musette 1992) needing only the determination of a *finite* number of coefficients. This method is equivalent to the generalization of the Weiss truncation procedure developed recently (Pickering 1993).

However, some physically interesting solutions lay outside the previous class, e.g.: the one-soliton solution

$$u = k^2 \frac{1 + 2 \cosh k\xi}{(\cosh k\xi + 2)^2}, \quad c = k^4, \quad (8)$$

to the integrable Kaup-Kupershmidt (KK) equation (Kaup 1980; Fordy and Gibbons 1980)

$$u_t + \left( u_{xxxx} + 30uu_{xx} + \frac{45}{2}u_x^2 + 60u^3 \right)_x = 0, \quad (9)$$

and the “collision of two shocks” solution (Nozaki and Bekki 1984) to the nonintegrable complex Ginzburg-Landau equation (1)

$$|A| = \text{const} \frac{\sinh k(x - x_0)}{\cosh k(x - x_0) + \frac{1}{2} \exp(-kct)}. \quad (10)$$

These last two solutions can be captured easily if one extends the order of the linear system and considers instead

$$\psi_{xxx} - a_1\psi_x - a_2\psi = 0, \quad (11a)$$

$$\psi_t - b_1\psi_{xx} - b_2\psi_x - b_3\psi = 0, \quad (11b)$$

( $a_i, b_i$  constants) whose general solution depends on three arbitrary constants  $C_n$

$$\psi = \exp(b_3 t) \sum_{n=1}^3 C_n \exp \left\{ k_n(x + b_2 t) + k_n^2 b_1 t \right\}, \quad k_n^3 - a_1 k_n - a_2 = 0, \quad (12)$$

with  $k_1 + k_2 + k_3 = 0$ , at least when the  $k'_n$ s are distinct. Without loss of generality, one can set  $b_3 = 0$ . Then *all* solutions described above are polynomial in the two elementary variables

$$Y_1 = \frac{1}{k} \frac{\psi_x}{\psi}, \quad Y_2 = \frac{1}{k^2} \frac{\psi_{xx}}{\psi}, \quad (13)$$

where  $k$  is some homogeneity constant equal to the inverse of a length, the polynomial coefficients  $c_{j,l}$  being *constant* for a PDE with constant coefficients.

Hence we have an extension of our previous method for finding particular exact solutions of nonlinear equations  $E(u) = 0$  polynomial in  $u$  and its derivatives. One expands  $u$  and, correspondingly, the LHS  $E(u)$  on the basis  $(Y_1, Y_2)$  with a *finite* number of coefficients. The coefficients of  $E(u)$  equated to zero define the *determining equations* for the unknowns  $(a_i, b_i)$  and the coefficients of  $u$ .

Details about this method are explained elsewhere (Musette and Conte 1993).

### 3. Application to the complex Ginzburg-Landau equation

Equation (1) admits the invariance transformation

$$\forall \alpha \in \mathbb{R} : A \rightarrow Ae^{i\alpha}, \quad (17)$$

which implies that  $\arg A$  only contributes by its gradient.

Let us perform the Painlevé analysis (Weiss, Tabor and Carnevale 1983) of equation (1). This technique is equivalent (Conte 1989, 1992) to looking, in the neighborhood of a movable singular manifold  $\Psi(x, t) = 0$ , for representations of  $A$  as Laurent series in the expansion function  $\chi = \Psi/\Psi_x$ , where  $\Psi$  satisfies the second order linear system

$$\Psi_{xx} + \frac{1}{2}S\Psi = 0, \quad (18a)$$

$$\Psi_t + C\Psi_x - \frac{1}{2}C_x\Psi = 0, \quad (18b)$$

depending on two adjustable functions  $(S, C)$  constrained by the Schwarz cross-derivative condition

$$S_t + C_{xxx} + 2C_xS + CS_x = 0. \quad (19)$$

As found by Cariello and Tabor (1989), the field  $A$  behaves like a noninteger, complex power

$$A \sim A_0 \left[ \frac{\Psi}{\Psi_x} \right]^{-1+i\Theta_{01}} \quad (20a)$$

$$|A_0|^2 = \frac{9|p|^2}{2D_i^2}(D_r + \Delta), \quad \Theta_{01} = \frac{3}{2D_i}(D_r + \Delta) \quad (20b)$$

$$\Delta^2 = D_r^2 + \frac{8}{9}D_i^2, \quad D_r = p_r q_r + p_i q_i, \quad D_i = p_r q_i - p_i q_r, \quad (20c)$$

with both signs allowed for  $\Delta$ .

Step one of our algorithm therefore fails, for no *integer* polynomial degree can be found for  $A$  and  $\bar{A}$ , except when  $\Theta_{01}$  vanishes, i.e.  $\text{Im}(p/q) = 0$ , a case which includes real Ginzburg-Landau (Keefe 1986), NLS (Chudnovsky, Chudnovsky and Tabor 1983) and KPP (Kolmogorov, Petrovskii and Piskunov 1937; Newell and Whitehead 1969) equations. However, this transcendental behaviour for  $A$  and  $\bar{A}$  is transformed into a more usual *pole-like* behaviour if one takes for  $A$  the representation  $A = |A|e^{i\arg A}$ ; indeed, the real variables  $|A|$  and  $\text{grad } \arg A$  have a simple pole-like behaviour

$$|A| \sim |A_0| \left[ \frac{\Psi}{\Psi_x} \right]^{-1}, \quad \arg A \sim \Theta_{01} \log \Psi. \quad (21)$$

Consequently, the complex Ginzburg-Landau equation now fits the requirement for the existence of integer polynomial degrees.

The result of our algorithm is that the four solutions called respectively solitary wave, hole, shock, collision of two shocks by Nozaki and Bekki (1984) have the same remarkably simple expression for the argument

$$\arg A = \Theta_{01} \log \psi + d_{00}x + e_{00}t, \quad (22)$$

where  $d_{00}$  and  $e_{00}$  are two real constants depending on  $(p, q, \gamma)$ , and their modulus is given respectively by

$$\frac{|A|}{|A_0|} = \sigma, \tau, \tau \pm 1, Y_1. \quad (23)$$

As to the five constants of the linear system (11), they are  $b_3 = 0$  (gauge),  $a_1 = k^2, a_2 = 0$  and, respectively,  $(b_1, b_2) = (0, 0), (0, 0), (0, -c), (c/k, 0)$ , where the wavevector  $k$  and the speed  $c$  depend only on  $(p, q, \gamma)$ . Care must be taken that the elementary functions  $(\sigma, \tau, Y_1)$  have different expressions in  $(x, t)$  in the four cases, namely

$$\psi = \cosh(\tfrac{1}{2}kx), \cosh(\tfrac{1}{2}kx), \cosh[\tfrac{1}{2}k(x - ct)], e^{kct} \cosh(kx) + \tfrac{1}{2}. \quad (24)$$

None of these four solutions depends on any arbitrary constant, except the two arbitrary origins of  $x$  and  $t$ . As to the fifth solution (propagating hole), found later (Bekki and Nozaki 1985), it depends on one arbitrary constant (the speed  $c$ ) but, at first glance, it seems to escape our method. Its expression is indeed

$$A = |A_0|(\tanh(k(x - ct)) + c_{00}) \exp\{i(\Theta_{01} \log \cosh k(x - ct) + d_{00}x + e_{00}t)\}, \quad (25)$$

where  $d_{00}$  and  $e_{00}$  are two real constants but where  $c_{00}$  is a *complex* constant

$$c_{00} = \frac{2p_i + 3i(p_i\Delta + p_r D_i - |p|^2 q_i)}{12D_i |p|^2} c. \quad (26)$$

There are two ways to incorporate this fifth solution into our scheme. The first one is to notice that equation (1) is still polynomial in the variables  $(A^2, \bar{A}^2)$  and to consider the real variables  $|A|^2$  and  $\text{grad arg } A$ :  $|\tanh(k(x - ct)) + c_{00}|^2$  is then polynomial in  $\tau$ .

The second one is dictated by a subtle feature of the WTC truncation procedure (Musette and Conte 1993) and leads to represent the complex field  $A$  not with the couple  $(|A|, \arg A)$ , but with the mixed variables  $(Z, \Theta)$  uniquely defined by

$$A = Ze^{i\Theta}, Z \in \mathbb{C}, \Theta \in \mathbb{R} \quad (27a)$$

$$\Theta = \Theta_{01} \log \Psi + \Theta_0, \Theta_0 \text{ arbitrary}, \quad (27b)$$

in which  $\Theta_0$  accounts for the arbitrary function which must be introduced in the expansion of  $\arg A$  at this level.

The regular part of the Painlevé expansion of the argument, i.e. the one which follows the  $\Theta_0$  term, has been incorporated into the modulus to yield a “complexified modulus”  $Z$ . Consequently, variables  $(Z, \text{grad } \Theta)$  now fit the assumptions of our algorithm and the fifth solution is easily retrieved in this way.

#### 4. Conclusion

The complex field  $A$  of the one-dimensional complex Ginzburg-Landau equation must be represented in a way adapted to the singularity structure of the equation. The main result is that its modulus must be “complexified”. Then, the five solutions of Nozaki and Bekki can all be simply represented by polynomials in at most two elementary variables.

Conversely, it is natural to systematically look for all possible solutions in the class  $(Z, \text{grad } \Theta)$  polynomials of degree one in  $(\sigma, \tau)$  or  $(Y_1, Y_2)$ . Such an investigation is still in progress and its results will be presented later.

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## GENERALISED SOLUTIONS OF THE PERTURBED KdV EQUATION FOR CONVECTING FLUIDS

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ABSTRACT. New and generalised solutions of the perturbed KdV equation

$$u_t + \lambda_1 uu_x + \lambda_2 u_{xxx} + \lambda_3 u_{xxxx} + \lambda_4 (uu_x)_x + \lambda_5 u_{xx} = 0,$$

which describes the evolution of long shallow waves in a convecting fluid, are presented using a full Painlevé Analysis.

Recently some exact solitary waves for a kind of perturbed Korteweg-de Vries equation have been found [1] with the aim of applying them to describe the nonlinear behaviour of a convecting fluid near the transition point. The purpose here is to report on a generalization of these solutions by fully using both the classical and more recent tools provided by Painlevé Analysis. Due to the lack of space we cannot give all the technical details of the full formalisation that it has been used in the search for these results but an excellent account can be found in [2,3]. Indeed, this formalism has already been used by some authors in the search for solutions in particular case of the equation under consideration [4]. Also we have incorporated recent developments in regard to the extremely fruitful Truncation Method [5] as well as the recent work relating Painlevé Analysis with the problem of the symmetries in the corresponding partial differential equation [6].

The equation under consideration is [1]

$$u_t + \lambda_1 uu_x + \lambda_2 u_{xxx} + \lambda_3 u_{xxxx} + \lambda_4 (uu_x)_x + \lambda_5 u_{xx} = 0, \quad (1)$$

where  $\lambda_1, \lambda_2, \dots, \lambda_5$  are arbitrary constants. Trivial rescalings of  $u$ ,  $x$  and  $t$  allow us to rewrite it in the form

$$u_t + (5\beta + 6\lambda)uu_x + \lambda u_{xxx} + [u_{xxx} + 6uu_x]_x + \alpha u_{xx} = 0, \quad (2)$$

where  $\alpha, \beta, \lambda$  are arbitrary constants. This rescaling forces us to consider only non zero values for  $\lambda_3$  and  $\lambda_4$ . After applying the aforementioned formalism we have found the following cases containing exact solutions of (2):

(i),  $\beta = 0$ . The form of the solutions is

$$u_1(x, t) = -\frac{1}{6}(\alpha + 4K_0^2) + 2K_0^2[1 - \tanh^2\{K_0(x + vt + x_0)\}], \quad (3)$$

where  $K_0$  and  $x_0$  are arbitrary constants and  $v(\alpha, \lambda) = \alpha\lambda$ . This solution corresponds to solutions (21-24) of [1].

(ii),  $\beta = -\lambda$ . Now the exact solution takes the form

$$\begin{aligned} u_2(x, t) = & \left( \frac{4}{3}K_0^2 + \frac{1}{36}\lambda^2 - \frac{1}{6}\alpha \right) + \frac{1}{3}\lambda K_0 \tanh\{K_0(x + vt + x_0)\} \\ & -(2K_0^2) \tanh^2\{K_0(x + vt + x_0)\} \end{aligned} \quad (4)$$

with  $K_0$  and  $x_0$  arbitrary constants and

$$v(\alpha, \lambda, K_0) = \frac{2}{3}\lambda K_0^2 + \frac{1}{36}\lambda(6\alpha - \lambda^2). \quad (5)$$

This solution has no analogue in [1].

(iii),  $\beta \neq 0$ . This last case yields the expression

$$u_3(x, t) = -[(36\alpha + 19\beta^2 + 30\beta\lambda)/216] - 2\phi_x^2\wp(\phi), \quad (6a)$$

$$\phi(x, t) = M_0 + N_0 \exp\left\{\frac{1}{6}\beta(x + vt)\right\}, \quad (6b)$$

$$v(\alpha, \beta, \lambda) = \alpha(\lambda + \frac{5}{6}\beta) + \frac{5}{6}\beta(\lambda + \frac{5}{6}\beta)^2, \quad (6c)$$

where  $\wp(\phi)$  is the Weierstrass Elliptic Function

$$\wp(\phi) = \wp(\phi(x, t); 0, a_0), \quad (7)$$

and  $M_0, N_0, a_0$  are arbitrary constants. If  $a_0$  tends to zero, the limit yields  $\wp(\phi) \sim \phi^{-2}$ . Then, we recover the solutions (14-18) of [1]. Thus, this case represents a generalization of the above solutions to which is reduces in a special case.

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# MODIFIED SINGULAR MANIFOLD EXPANSION: APPLICATION TO THE BOUSSINESQ AND MIKHAILOV-SHABAT SYSTEMS

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**ABSTRACT.** We present a modified treatment of the singular manifold method as an improved variant of the Painlevé analysis for partial differential equations with two branches in the Painlevé expansion. We illustrate the method by fully applying it to the Classical Boussinesq system and to the Mikhailov-Shabat system.

## 1. Introduction

Integrability seems to be a concept with elusive meaning; many efforts have been dedicated in the last decade to the conquest of a more definite and precise concept of integrability. We shall try to make some progress on the relationship between integrability and the Painlevé property [1], which is closely related to the classic paper by Weiss, Tabor and Carnevale [2] on the Painlevé test for PDEs.

Our first observation is that the Painlevé test for PDEs has paved the way for establishing (as it has been shown in a variety of examples [3,4]) the relation between the singular manifold method [5] based in the truncation of the Painlevé series, and Hirota's bilinear formalism based in the definition and use of the  $\tau$ -function [6,20] which has been proved to be extremely successful for the explicit construction of  $N$ -soliton solutions. We will try to give here strong support to the conjecture that the number of  $\tau$ -functions used in Hirota's method equals the number of branch expansions in Painlevé's singular manifold approach.

For this to be true we shall need to generalize the latter to more than one singular manifold expansion. We shall do this in two cases: the Classical Boussinesq system (CB) [7]

$$u_t + \omega_x + uu_x = 0, \quad (1.1a)$$

$$\omega_t + u_{xxx} + (u\omega)_x = 0, \quad (1.1b)$$

and the Mikhailov-Shabat system (MS) that appears in the classification of integrable systems given by those authors [8,9]:

$$p_t = p_{xx} + (p+q)q_x - \frac{1}{6}(p+q)^3, \quad (1.2a)$$

$$- q_t = q_{xx} - (p+q)p_x - \frac{1}{6}(p+q)^3. \quad (1.2b)$$

Hirota and Satsuma have constructed  $N$ -soliton solutions of both systems using the bilinear formalism [10]. Sachs has analysed the Painlevé property for CB [11] and Flaschka, Newell and Tabor for MS [12]. Conte [13] has studied the Painlevé property and found a Lax pair for a version of MS in PDE form that is described below.

In §2 we shall briefly review the Painlevé test in the WTC version [2] for the CB system and the problems arising in the traditional singular manifold method will be discussed. In the rest of the §2 we will motivate our conjecture. This section constitutes the core of the paper; we introduce a new expansion using two singular manifolds and the two different expansion branches will be presented and discussed. Using our new procedure we shall be able to construct both the auto-Bäcklund transformations and the Lax pair for the CB system in §3. The analysis will be applied again to the MS system in §4 to provide further support for our conjecture. As a bonus a method for generating solutions in an iterative manner using only linear equations shall be briefly described. The relationship between Hirota's method and ours will also be described.

## 2. The Boussinesq System and the Singular Manifold Method

The Classical Boussinesq system (CB)

$$u_t + \omega_x + uu_x = 0, \quad (2.1a)$$

$$\omega_t + u_{xxx} + (u\omega)_x = 0, \quad (2.1b)$$

is known to be an appropriate model for describing the behaviour of water waves in shallow channels [7]. From the theoretical point of view it is also known to be the compatibility conditions of a Lax pair [15] and to be solvable through the inverse scattering method [14,15]. On the other hand Hirota has shown [16] that the bilinear formulation of (2.1) is a reduction of the Modified KP Equation. This allows us to obtain multi-solitonic solutions for CB through the reduction of those obtained by Jimbo and Miwa [17] for the KP hierarchy. Sachs has shown [11] that (2.1) has the Painlevé property. We shall briefly sketch here his main results. The Painlevé test in the WTC version [2] requires the functions  $u$  and  $\omega$  to have expansions in the form

$$u(x, t) = \sum_{j=0}^{\infty} u_j(x, t)\phi^{j-r}(x, t), \quad (2.2a)$$

$$\omega(x, t) = \sum_{j=0}^{\infty} \omega_j(x, t)\phi^{j-s}(x, t), \quad (2.2b)$$

where  $\phi(x, t)$ , an arbitrary function, is the celebrated movable singularity manifold. The exponents  $r$  and  $s$  and the functions  $u_0$  and  $\omega_0$  in (2.2) are determined through the usual balance of dominant terms. If there exist several solutions of these balance equations one usually speaks of different expansion branches. For instance, in the case of the system (2.1) one has the following solutions of the balance equations

$$r = 1, \quad s = 2, \quad u_0 = 2a\phi_x, \quad \omega_0 = -2(\phi_x)^2, \quad (2.3)$$

but  $a$  can take the values  $\pm 1$  so that (2.1) has two different expansion branches. Sachs has shown [11] that (2.1) possesses the Painlevé property in both expansion branches. He has applied the singular manifold method [5] to this system: by assuming a truncated expansion a function  $\phi$  is sought such that

$$u(x, t) = \sum_{j=0}^r u_j(x, t)\phi^{j-r}(x, t), \quad (2.4a)$$

$$\omega(x, t) = \sum_{j=0}^s \omega_j(x, t)\phi^{j-s}(x, t). \quad (2.4b)$$

The usual way to proceed after inserting (2.4) into (2.1) is to assume that all coefficients for each power of  $\phi$  vanish independently. In so doing one obtains

$$u_0 = 2a\phi_x, \quad u_1 = -[\phi_t + a\phi_{xx}]/\phi_x, \quad (2.5b)$$

$$\omega_0 = -2(\phi_x)^2, \quad \omega_1 = 2\phi_{xx}, \quad \omega_2 = a(u_1)_x, \quad (2.5b)$$

where  $a = \pm 1$  and  $u_1$  and  $\omega_2$  must be solutions of (2.1). This is why the truncated expansion (2.4) is actually an auto-Bäcklund transformation between solutions of (2.1).

It was pointed out by Sachs [11] that the equations (2.5) restrict to solutions of (2.1), satisfying in addition the constraint  $\omega = au_x$ , so that the system reduces to Burgers' equation:

$$u_t + uu_x + au_{xx} = 0. \quad (2.6)$$

As it has been described, the singular manifold method is inadequate to deal with several expansion branches and consequently a connection with Hirota's bilinear method is not possible. We propose now an important modification of the method based upon the following considerations:

1. Several criticisms have already appeared in the literature concerning the singular manifold method as it has been described ([2] and [3]). The assumption that the coefficients have to vanish independently for each power of  $\phi$  has been seen as too restrictive since all that we need is that the truncation ansatz (2.4) be satisfied. This requirement could be fulfilled by setting to zero the sum of all contributions for each different power of  $\phi$ . Such modifications have already been discussed in the framework of some particular evolution equations ([2] and [4]).
2. The connection between the singular manifold method and the Hirota formalism [4] lies in the interpretation of (2.4) as an auto-Bäcklund transformation where  $(u_r, \omega_s)$  and  $(u, \omega)$  are solutions of the system (2.1). As is the case for any Bäcklund transformation one has an iterative procedure for deriving increasingly complicated solutions starting with the simplest ones. The problem with having several expansion branches is that we must make a choice of a seed solution one definite branch. In doing so we will always remain on that branch missing solutions from the other branches.

From these considerations we propose to modify the singular manifold expansion in the following way. First we identify the number of independent expansion branches. Then we modify the expansion (2.4) by using as many singular manifolds as expansion branches. The generalized form of (2.4) for two expansion branches takes the form

$$u(x, t) = \sum_{j=0}^{r-1} u_j(x, t) \phi^{j-r}(x, t) + \sum_{j=0}^{s-1} u'_j(x, t) \sigma^{j-r}(x, t) + u_r, \quad (2.7a)$$

$$\omega(x, t) = \sum_{j=0}^{r-1} \omega_j(x, t) \phi^{j-r}(x, t) + \sum_{j=0}^{s-1} \omega'_j(x, t) \sigma^{j-r}(x, t) + \omega_s(x, t), \quad (2.7b)$$

where we have taken  $(u_0, \omega_0)$  to be the leading order terms for one of the expansion branch and  $(u'_0, \omega'_0)$  to be the leading order terms for the other.

If we attempt to express the solutions using expansions such as (2.7) we have to deal with products of the coefficients of  $\phi$  and  $\sigma$  when inserting (2.7) into (2.1). We remind the reader of the remarks made previously in regard to setting coefficients to zero. In addition, we have to find a way of rewriting auto-Bäcklund transformations of the kind described above if we want to be successful in relating Painlevé analysis to the Hirota formalism. These ideas will be developed by applying them specifically to the CB system. We will do this in the next Section.

### 3. The Method of two singular manifolds for the Boussinesq system

We construct an auto-Bäcklund Transformation of the form (2.7) for the CB system (2.1). Restricting ourselves for the moment to the  $a = 1$  branch, we look for a truncated expansion of the form

$$u = 2(\phi_x/\phi) + u_1, \quad (3.1a)$$

$$\omega = -2(\phi_x/\phi)^2 + 2(\phi_{xx}/\phi) + \omega_2, \quad (3.1b)$$

which after substitution into (2.1) yields

$$(u_1)_t + u_1(u_1)_x + (\omega_2)_x + 2\{(\phi_{xx}/\phi) + (\phi_t/\phi) + u_1(\phi_x/\phi)\}_x = 0, \quad (3.2a)$$

$$(\omega_2)_t + (u_1)_{xxx} + (\omega_2 u_1)_x + 2\{(\phi_x/\phi)_t + (\phi_{xx}/\phi)_x + \omega_2(\phi_x/\phi) + u_1(\phi_x/\phi)_x\}_x = 0. \quad (3.2b)$$

Dropping for a moment the requirement that different powers of  $\phi$  must have independent vanishing coefficients, then  $(u_1, \omega_2)$  do not have to be a set of solutions of CB and we can reinterpret (3.2) as a new system for which we look again for a truncated expansion. Since the dominant terms of (3.2) are of the same nature as those of (2.1) we now take the other branch  $a = -1$  in the form

$$u_1 = -2(\sigma_x/\sigma) + \alpha, \quad (3.3a)$$

$$\omega_2 = -2(\sigma_x/\sigma)^2 + 2(\sigma_{xx}/\sigma) + \beta. \quad (3.3b)$$

Combining this last expression with (3.1) we find

$$u = 2\{(\ln \phi)_x - (\ln \sigma)_x\} + \alpha, \quad (3.4a)$$

$$\omega = 2\{(\ln \phi)_x + (\ln \sigma)_x\} + \beta. \quad (3.4b)$$

We now demand that  $(\alpha, \beta)$  be a solution of (2.1). Automatically (3.4) will be the auto-Bäcklund transformation we are seeking. Substituting (3.3) into (3.2) we obtain the expressions

$$(\ln \phi)_x[\alpha + v_1 + w_1] - (\ln \sigma)_x[\alpha - v_2 + w_2] = 2(\ln \sigma)_x(\ln \phi)_x, \quad (3.5a)$$

$$\begin{aligned} & (\ln \phi)_x[\beta + v_{1x} + w_{1x} + \{v_1 - (\ln \phi)_x\}\{\alpha + v_1 + w_1 - 2(\ln \sigma)_x\}] \\ & - (\ln \sigma)_x[\beta + v_{2x} - w_{2x} - \{v_2 - (\ln \sigma)_x\}\{\alpha - v_2 + w_2 + 2(\ln \phi)_x\}] = 0, \end{aligned} \quad (3.5b)$$

where  $v_i$  and  $w_i$  are defined as usual ([18,19]):

$$v_1 = (\phi_{xx}/\phi_x), \quad w_1 = (\phi_t/\phi_x), \quad (3.6a)$$

$$v_2 = (\sigma_{xx}/\sigma_x), \quad w_2 = (\sigma_t/\sigma_x). \quad (3.6b)$$

We see that (3.4) is an auto-Bäcklund transformation that allows us to generate a solution  $(u, \omega)$  starting from another known solution  $(\alpha, \beta)$  using the two singular manifolds  $\phi$  and  $\sigma$  satisfying (3.5).

Furthermore, the pair of equations (3.5) can be simplified greatly by substituting (3.5a) into (3.5b) and into the derivative of the former. After this step we impose the requirement that the coefficients of  $\phi_x$  and  $\sigma_x$  vanish. Then we obtain

$$\alpha = -v_1 - w_1 + v_2 - w_2, \quad (3.7a)$$

$$\beta = \frac{1}{2}[w_1^2 + w_2^2 - v_1^2 - v_2^2], \quad (3.7b)$$

$$v_{1x} + w_{1x} + \frac{1}{2}(v_1 + w_1)(v_2 - w_2 - v_1 + w_1) = 0, \quad (3.7c)$$

$$v_{2x} - w_{2x} + \frac{1}{2}(v_2 - w_2)(-v_2 - w_2 + v_1 + w_1) = 0. \quad (3.7d)$$

We recall that  $(\alpha, \beta)$  must be solutions of the CB system. On imposing that (3.7a,b) satisfy (2.1) we are led to the following relation:

$$(v_1 + w_1 + v_2 - w_2)_t = [w_1(v_1 + w_1) + w_2(v_2 - w_2)]_x. \quad (3.8)$$

A crucial observation at this point is that (3.8) can be written as the compatibility condition of a linear system in the form

$$\Psi_x = b_0[v_1 + w_1 + v_2 - w_2]\Psi, \quad (3.9a)$$

$$\Psi_t = b_0[w_1(v_1 + w_1) + w_2(v_2 - w_2)]\Psi, \quad (3.9b)$$

Taking  $b_0 = \frac{1}{4}$  we are able to express (3.9) as a function of the solutions  $(\alpha, \beta)$  in the form

$$\Psi_{xx} = [\frac{1}{16}\alpha^2 - \frac{1}{4}\beta]\Psi, \quad (3.10a)$$

$$\Psi_t = \frac{1}{4}\alpha_x\Psi - \frac{1}{2}\alpha\Psi_x. \quad (3.10b)$$

Using (3.7a) and (3.9a) and taking into account (3.6) we obtain for  $\phi$  and  $\sigma$  the following linear equations:

$$\phi_{xx} + \phi_t + [\frac{1}{2}\alpha - 2(\ln \Psi)_x]\phi_x = 0, \quad (3.11a)$$

$$\sigma_{xx} - \sigma_t - [\frac{1}{2}\alpha + 2(\ln \Psi)_x]\sigma_x = 0. \quad (3.11b)$$

Let us briefly summarize the procedure we have used to generate solutions of the CB system:

1. We begin with a known solution  $(\alpha, \beta)$  of (2.1). Inserting this in (3.10) we obtain  $\Psi(x, t)$ .
2. With  $\Psi(x, t)$  we solve the linear equations (3.11) obtaining  $\phi(x, t)$  and  $\sigma(x, t)$ . However, we should recall that  $\phi$  and  $\sigma$  must also satisfy (3.5a) since so far only the derivative of this equation had been used.
3. Finally we can generate a new solution  $(u, \omega)$  using the auto-Bäcklund transformation (3.4).

Let us apply this procedure to the CB system (2.1) using the solution

$$\alpha = \alpha_0 = \text{constant}, \quad \beta = \beta_0 = \text{constant}. \quad (3.12)$$

A solution of (3.10) in this case looks like

$$\Psi(x, t) = \exp\{k(x - vt)\} \quad (3.13)$$

where

$$v = \frac{1}{2}\alpha_0, \quad \beta_0 = v^2 - 4k^2. \quad (3.14)$$

Now the solutions of (3.11) are

$$\phi(x, t) = A_0 + \sum A_n \exp\{2k_n(x - v_n t)\}, \quad (3.15a)$$

$$\sigma(x, t) = B_0 + \sum B_n \exp\{2k'_n(x - v'_n t)\}, \quad (3.15b)$$

where  $A_0, B_0, A_n, B_n$ , are arbitrary constants while  $v_n, k_n, v'_n$  and  $k'_n$  satisfy

$$2k_n - v_n = 2k - v, \quad 2k'_n + v'_n = 2k + v. \quad (3.16)$$

However  $\phi$  and  $\sigma$  must also satisfy (3.5a). Inserting (3.15) in (3.5a) gives a different set of solutions depending on whether  $\beta_0 = 0$  or  $\beta_0 \neq 0$ . We shall deal with these two cases separately.

*Case 1.*  $\beta_0 = 0$ . According to (3.14b)  $\beta_0 = 0$  implies  $v = \pm 2k$ .

We have two cases:

*Subcase (a).*  $v = -2k$ . In this case (3.5a) leads to the following solutions for  $\phi$  and  $\sigma$

$$\phi(x, t) = A \exp\{2k(x + 2kt)\}, \quad (3.17a)$$

$$\sigma(x, t) = B_0 + \sum B_n \exp\{2k'_n(x + 2k'_n t)\}, \quad (3.17b)$$

and then the solution (3.4) takes the form

$$u(x, t) = -2(\ln \sigma)_x, \quad (3.18a)$$

$$\omega(x, t) = -u_x. \quad (3.18b)$$

Using (3.18b) in (2.1) we can easily reduce the CB system to Burgers' equation. Also (3.18a) corresponds to a particular solution of this equation describing the confluence of shock waves [7].

*Subcase (b).*  $v = 2k$ . Using the same procedure as before we obtain

$$\Phi(x, t) = A_0 + \sum A_n \exp\{2k_n(x - 2k_n t)\}, \quad (3.19a)$$

$$\sigma(x, t) = B \exp\{2k(x - 2kt)\}, \quad (3.19b)$$

$$u(x, t) = 2(\phi_x/\phi), \quad (3.19c)$$

$$\omega(x, t) = u_x, \quad (3.19d)$$

corresponding via (3.19d) to another reduction of Burgers' equation.

Case 2.  $\beta_0 \neq 0$ . In this case equation (3.5a) leads to

$$k_n = k'_n = k, \quad v_n = v'_n = v, \quad (3.20)$$

so that we obtain

$$\phi(x, t) = A_0[1 + q \exp\{2k(x - vt) + \psi_0\}], \quad (3.21a)$$

$$\sigma(x, t) = B_0[1 + p \exp\{2k(x - vt) + \psi_0\}], \quad (3.21b)$$

where  $p$  and  $q$  are given by

$$q = v + 2k, \quad p = v - 2k. \quad (3.22)$$

Finally the solution is:

$$u(x, t) = 2v + 2[(\ln \phi)_x - (\ln \sigma)_x], \quad (3.23a)$$

$$\omega(x, t) = v^2 - 4k^2 + 2[(\ln \phi)_x + (\ln \sigma)_x]_x, \quad (3.23b)$$

which is the soliton found by Kaup [15] using the inverse scattering method, and by Hirota [16] using his bilinear formalism.

At this point, one can easily establish the relationship of our method with Hirota's bilinear formalism. From the initial solution  $(\alpha_0, \beta_0)$  we apply the transformation (3.4) an arbitrary number of times following the procedure described for each one of the three steps. After  $n$  iterations we obtain

$$u^{(n)} = 2\{(\ln \phi_n)_x - (\ln \sigma_n)_x\} + \dots + 2\{(\ln \phi_1)_x - (\ln \sigma_1)_x\} + \alpha_0, \quad (3.24a)$$

$$\omega^{(n)} = 2\{(\ln \phi_n)_x + (\ln \sigma_n)_x\} + \dots + 2\{(\ln \phi_1)_x + (\ln \sigma_1)_x\}_x + \beta_0. \quad (3.24b)$$

To find both sets of  $\tau$ -functions we just define

$$\tau = \phi_1 \phi_2 \dots \phi_n, \quad (3.25a)$$

$$\tau' = \sigma_1 \sigma_2 \dots \sigma_n, \quad (3.25b)$$

and the solutions of the CB system can be now expressed in the form

$$u = \alpha_0 + 2[\ln(\tau/\tau')]_x, \quad (3.26a)$$

$$\omega = \beta_0 + 2[\ln(\tau\tau')]_{xx}, \quad (3.26b)$$

as required by Hirota's bilinear formalism [16].

A final observation concerns the Galilean invariance of the CB system. Under the transformation

$$x' = x - \lambda t, \quad u' = u - \lambda, \quad (3.27)$$

the equations (2.1) remain invariant. This symmetry can be used to introduce a spectral parameter in the linear system (3.10). To see this, apply the transformation (3.27) to (3.10) obtaining

$$\Psi_{xx} = \{[\frac{1}{4}(\alpha + \lambda)]^2 - \frac{1}{4}\beta\}\Psi, \quad (3.28a)$$

$$\Psi_t = \frac{1}{4}\alpha_x\Psi - [\frac{1}{2}(\alpha - \lambda)]\Psi_x, \quad (3.28b)$$

This is exactly the Lax pair found by Jaulent and Miodek [14].

#### 4. The Method of the two singular manifolds for the Mikhailov-Shabat system

As mentioned in §1, the system (1.2) was first classified as an integrable system by Mikhailov and Shabat ([8] and [9]). We shall refer to it as the MS system. We introduce the transformation

$$u = p + q, \quad \omega = q_x - p_x. \quad (4.1)$$

using which the MS system (1.2) becomes

$$u_t + \omega_x - uu_x = 0, \quad (4.2a)$$

$$\omega_t + u_{xxx} + uu_x + u\omega_x + u_x\omega - u^2u_x = 0. \quad (4.2b)$$

After substituting  $u = v_x$ , (4.2) transforms into the single PDE

$$v_{tt} - v_{xxxx} + v_tv_{xx} - \frac{1}{2}v_x^2v_{xx} = 0. \quad (4.3)$$

This equation corresponds to one of the modified Boussinesq equations found by Hirota and Satsuma [10]. The singular manifold analysis has been carried out by Conte [13]. We shall apply to this equation our method, first taking into account the number of expansion branches. A simple look to the dominant terms of the expansion (2.4) for (4.3) leads to the choices:

$$r = 1, \quad s = 2, \quad u_0 = a\gamma\phi_x, \quad \omega_0 = -6\phi_x^2, \quad (4.4)$$

where  $\gamma^2 = -12$  and again  $a$  can be  $+1$  or  $-1$ . We have once again a two expansion branch problem. As in the previous case we shall assume the existence of two different singular manifolds  $\phi$  and  $\sigma$  that give a truncation solution of (4.2) of the form

$$u = \gamma[(\ln \phi)_x - (\ln \sigma)_x] + \alpha, \quad (4.5a)$$

$$\omega = 6[(\ln \phi)_x + (\ln \sigma)_x] + \beta, \quad (4.5b)$$

where  $(\alpha, \beta)$  is a set of solutions of the MS system (4.2). Inserting (4.5) into (4.2) gives the following equations for  $\phi$  and  $\sigma$ :

$$(\ln \phi)_x[\gamma w_1 + 6v_1 - \gamma\alpha] + (\ln \sigma)_x[-\gamma w_2 + 6v_2 + \gamma\alpha] = 12(\ln \sigma)_x(\ln \phi)_x, \quad (4.6a)$$

$$\begin{aligned} & (\ln \phi)_x\{6(w_{1x} + w_1v_1) + \gamma(v_{1x} + v_1^2) + \gamma\beta + 6v_1\alpha - \gamma\alpha^2\} \\ & + (\ln \sigma)_x\{6(w_{2x} + w_2v_2) - \gamma(v_{2x} + v_2^2) - \gamma\beta + 6v_2\alpha + \gamma\alpha^2\} \\ & + [(\ln \phi)_x]^2\{3\gamma v_1 + 6\alpha - 6w_1\} + [(\ln \sigma)_x]^2\{-3\gamma v_2 + 6\alpha - 6w_2\} \\ & + (\ln \sigma)_x(\ln \phi)_x\{6\gamma[v_2 - v_1 + (\ln \sigma)_x - (\ln \phi)_x] - 24\alpha\} = 0. \end{aligned} \quad (4.6b)$$

As in §3,  $v_1$  and  $w_1$  are defined by (3.6). We now combine the previous equations in the following form. Substitute (4.6a) into (4.6b) and also in the derivative of (4.6a). After that we set to zero all coefficients in  $\sigma_x$  and  $\phi_x$ . A rather tedious and cumbersome calculation yields

$$\alpha = -w_1 - w_2 - \frac{1}{2}\gamma v_1 + \frac{1}{2}\gamma v_2, \quad (4.7a)$$

$$2\beta = -4v_{1x} - 4v_{2x} - v_1^2 + w_1^2 - v_2^2 + w_2^2, \quad (4.7b)$$

$$\gamma w_{1x} + \gamma w_{2x} - 2v_{1x} + 2v_{2x} + v_1^2 - v_2^2 - w_1^2 + w_2^2 = 0, \quad (4.7c)$$

$$\begin{aligned} & \gamma w_{1x} - \gamma w_{2x} + 6v_{1x} + 6v_{2x} - 3(v_1 - v_2)^2 - (w_1 - w_2)^2 \\ & + \gamma(w_1v_1 - w_2v_2) + \gamma(w_1 + w_2)(v_1 - v_2) = 0. \end{aligned} \quad (4.7d)$$

Since  $(\alpha, \beta)$  must also be a solution of (4.2) we require that equation (4.2) must be satisfied by (4.7). Then we arrive at the condition

$$\begin{aligned} & [w_1 - w_2 + \frac{1}{2}\gamma v_1 + \frac{1}{2}\gamma v_2]_t \\ &= [2v_{1x} - 2v_{2x} + \frac{1}{2}v_1(\gamma w_1 - 2v_1) + \frac{1}{2}v_2(\gamma w_2 + 2v_2)]_x, \end{aligned} \quad (4.8)$$

Again, as in §3, (4.8) is the compatibility condition of the linear system

$$\Psi_x = b_0[w_1 - w_2 + \frac{1}{2}\gamma v_1 + \frac{1}{2}\gamma v_2], \quad (4.9a)$$

$$\Psi_t = b_0[2v_{1x} - 2v_{2x} + \frac{1}{2}v_1(\gamma w_1 - 2v_1) + \frac{1}{2}v_2(\gamma w_2 + 2v_2)]. \quad (4.9b)$$

Setting  $b_0 = (2\gamma)^{-1/2}$ , using (4.7) allows one to re-express (4.9) in terms of  $(\alpha, \beta)$  only in the form

$$\Psi_{xx} = [\frac{1}{16}\alpha^2 - \frac{1}{4}\beta]\Psi, \quad (4.10a)$$

$$\Psi_t = \frac{1}{4}\alpha_x\Psi - \frac{1}{2}\alpha\Psi_x. \quad (4.10b)$$

We call the attention of the reader to the curious fact that (4.10) represents exactly the same Lax pair as for the CB system. [See (3.10).] There is only one but nevertheless important difference. The spectral parameter has disappeared from (4.10). While one can use Galilean invariance to reinstate the spectral parameter in the CB system, the MS system lacks such an invariance. The question of the spectral parameter is certainly an intriguing one but we can generate the solutions from (4.10) all the same. Starting from some known solution  $(\alpha, \beta)$  we solve (4.10). Then using (4.7a) and (4.9a) we find  $\phi$  and  $\sigma$  from the pair of LINEAR equations

$$\gamma\phi_t - 6\phi_{xx} + [\frac{1}{2}\gamma\alpha + 12(\ln\Psi)_x]\phi_x = 0, \quad (4.11a)$$

$$\gamma\sigma_t + 6\sigma_{xx} + [\frac{1}{2}\gamma\alpha - 12(\ln\Psi)_x]\sigma_x = 0. \quad (4.11b)$$

We also have to check whether  $\phi$  and  $\sigma$  satisfy (4.6a). Finally, the auto-Bäcklund transformation (4.5) provides us with a new solution  $(u, \omega)$ .

As in the previous section we choose  $(\alpha, \beta) = (\alpha_0, \beta_0) = (\text{constants})$ . Then we obtain

$$\phi(x, t) = 1 + q \exp\{2k(x - vt + \psi_0)\}, \quad (4.12a)$$

$$\sigma(x, t) = 1 + p \exp\{2k(x - vt + \psi_0)\}, \quad (4.12b)$$

where

$$v = \frac{1}{2}\alpha_0, \quad \beta_0 = v^2 - 4k^2, \quad q = \gamma v + 4k, \quad p = \gamma v - 4k. \quad (4.13)$$

Finally from (4.5) we obtain the solution  $(u, \omega)$  in the form

$$u(x, t) = 2v + (8k^2 A_0/v)\{A_0 + \cosh[2k(x - vt + \theta_0)]\}^{-1}, \quad (4.15a)$$

$$\omega(x, t) = v^2 - 4k^2 + 24k^2 \left\{ \frac{1 + A_0 \cosh[2k(x - vt + \theta_0)]}{\{A_0 + \cosh[2k(x - vt + \theta_0)]\}^2} \right\}, \quad (4.15b)$$

where

$$A_0 = [3v^2/(3v^2 + 4k^2)], \quad \exp(2k\theta_0) = (pq)^{1/2} \exp(2k\psi_0). \quad (4.16)$$

Notice that in spite of  $\gamma$  being an imaginary constant the obtained solitonic solution is indeed real.

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# DISCRETE PAINLEVÉ EQUATIONS: DERIVATION AND PROPERTIES

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**ABSTRACT.** We present a review of the various methods that can be used for the derivation of discrete Painlevé equations. Particular emphasis is put on the singularity confinement method that plays the role of the “Painlevé” singularity analysis for discrete systems. The various properties of the discrete Painlevé equations are also presented and this allows a close parallel between the continuous and the discrete case to be established.

## 1. Introduction

Discrete Painlevé equations have made their appearance recently in connection with field theoretical models. While computing a partition function for closed strings in a model of two-dimensional gravity, by the method of orthogonal polynomials, Brézin and Kazakov [1] obtained a recursion relation that reads:

$$x_{n+1} + x_n + x_{n-1} = \frac{an + b}{x_n} + c. \quad (1.1)$$

By taking to the continuous limit they have shown that this relation is a discretization of the continuous Painlevé equation P<sub>I</sub>. Fokas and Its [2] presented a detailed study of this integrable mapping using isomonodromy techniques. Shortly afterwards another Painlevé equation appeared in a similar model. Periwal and Shevitz [3] studied a model similar to that in [1], but with a different interaction and obtained the recursion relation:

$$x_{n+1} + x_{n-1} = \frac{x_n(an + b) + c}{1 - x_n^2}. \quad (1.2)$$

They proceeded to identify (1.2) as the discrete  $P_{II}$  ( $d-P_{II}$ ), based on its continuous limit. The same  $d-P_{II}$  appeared also in a context more familiar to specialists of nonlinear phenomena. In [4], Nijhoff and Papageorgiou studied the similarity reductions of the discrete mKdV equation. They derived the similarity constraint in the form of a nonlinear nonautonomous discrete lattice equation. In order to obtain the one-dimensional reduction they considered a semicontinuous limit in which one lattice variable tends to infinity (while the corresponding lattice constant goes to zero), the other variable remaining discrete. The result was again (1.2), which they identified (correctly) as  $d-P_{II}$ .

The time was thus ripe for a more systematic approach like the one used by Painlevé and Gambier for the derivation of the continuous Painlevé equations [5]. Curiously, an approach akin to the singularity analysis adapted to discrete systems has been long in the finding. Early unpublished investigations [6] have realized the importance of singularities for integrable discrete systems. The reason why these works remained unpublished was that they, somehow, failed to formulate a working conjecture allowing an algorithmic prediction of integrability. Joshi [7] came closest to the target with her concept of “orbits with pole-like behaviour”. However, the first truly powerful method for the study of the integrability of discrete systems was the “singularity confinement” approach we introduced in [8]. It is based on the simple observation that (movable) singularities in *integrable* discrete systems do not propagate *ad infinitum* but are confined to a few iteration steps. With a tool similar to the continuous singularity analysis at our disposal, the derivation of the discrete Painlevé equations became a straightforward calculation. Discrete  $P_I$  and  $P_{II}$  were thus rederived and the forms of  $d-P_{III}$ ,  $d-P_{IV}$  and  $d-P_V$  were obtained for the first time [9]. Only  $d-P_{VI}$  is not explicitly obtained as yet, but this is due to the cumbersome computations that its derivation involves and not to any fundamental difficulty.

Once these nonautonomous mappings have been obtained one must ask the fundamental question: are they *indeed* “Painlevé” equations? Clearly, the fact that they reduce to the latter at their continuous limit is not (and by far) reason enough. The Painlevé equations possess a host of other properties that single them out among all nonlinear second order ODEs. The aim of this paper is to show that the equations we call discrete Painlevé are called so justifiably. Not only a) are they derived using the analogue of the Painlevé method and b) have the Painlevé equations as continuous limits but, just like the latter: they c) have coalescence properties that allows one to obtain the simpler ones from the more complex, d) can be obtained from a compatibility condition for a linear system (Lax pair), e) possess elementary solutions for particular values of their parameters and f) are related by Bäcklund and Miura transformations. Of course, all the above points have not yet been fully implemented for every known discrete Painlevé equation: after all, their implementation in the continuous case took almost a century. Still all these points will be illustrated in the following sections.

Finally, the last section will be devoted to a recent development that bridges the gap between continuous and discrete Painlevé equations. The combination of singularity confinement with the Painlevé method has furnished an integrability test for differential-difference systems. An important offshoot of this new algorithm is the derivation of the delay-Painlevé equations. The latter are differential-difference systems that possess the singularity confinement property and tend to the usual Painlevé equations at the continuous limit. From the emerging results it appears that these delay equations also possess properties similar to the ones of the continuous Painlevé equation.

## 2. Derivation of the discrete Painlevé equations

Several methods have appeared in the literature for the derivation of the discrete Painlevé equations. Our presentation will not respect a strict historical order; instead we will try to give the essence of each method.

### 2.1. THE ORTHOGONAL POLYNOMIAL METHOD

In this approach [10,11], one starts by considering a family of polynomials  $P_n(z)$ , ( $P_n = z^n + \dots$ ), orthogonal under the measure  $d\mu = \exp(-U(z)) dz$ , where  $U(z)$  is a given function of  $z$  depending on parameters  $t$  with:

$$\int P_n(z)P_m(z) d\mu(z) = \delta_{nm} h_n. \quad (2.1)$$

It can be shown that the  $P_n$ 's satisfy a linear recursion relation:

$$zP_n = P_{n+1} + w_n P_{n-1}, \quad w_n = h_n/h_{n+1} \quad (2.2)$$

and moreover  $h_n$  satisfies the discrete equation:

$$(1 + 2n)h_n = \int zP_n^2(z)U'(z) d\mu. \quad (2.3)$$

Using the recursion relation (2.2) one can, for a given  $U(z)$ , eliminate the  $z$  variable from the right-hand side of (2.3) and obtain a nonlinear difference equation for  $h_n$ . The discrete  $P_1$  (1.1) was obtained with  $U(z) = tz^2 + z^4$ .

### 2.2. THE DISCRETE AKNS METHOD

Here one starts with a given eigenvalue problem and tries to obtain all the nonlinear equations associated with it [12]. For the problem at hand, a discrete eigenvalue problem [13] is the starting point:

$$\Phi_{n+1} = \mathbf{U}_n(\zeta)\Phi_n \quad (2.4)$$

where  $\mathbf{U}_n(\zeta)$  is a given matrix and  $\zeta$  is the spectral parameter. In order to find discrete Painlevé equations, we look for a compatible equation of the form:

$$\Phi_{n,\zeta} = \mathbf{A}_n(\zeta)\Phi_n. \quad (2.5)$$

The compatibility condition between the two equations leads to:

$$\mathbf{U}_{n,\zeta} = \mathbf{A}_{n+1}\mathbf{U}_n - \mathbf{U}_n\mathbf{A}_n. \quad (2.6)$$

Given  $\mathbf{U}_n$  and assuming an appropriate expansion in  $\zeta$  for  $\mathbf{A}_n(\zeta)$ , equation (2.6) yields the explicit form of  $\mathbf{A}_n$  and the compatibility condition is just a discrete Painlevé equation. Equations (2.4) and (2.5) define its Lax pair. Further details on this method as well as results from its application can be found in [14].

### 2.3. THE USE OF BÄCKLUND AND SCHLESINGER TRANSFORMS

It is known that there exist auto-Bäcklund transformations that map a solution of a given Painlevé equation (other than the parameter-free P<sub>I</sub>) to a solution of the same equation with a different value of parameters [15]. (The Schlesinger transforms [16,17] are just a special type of auto-Bäcklund.) These transforms can be used in order to derive discrete Painlevé equations. The general form of an auto-Bäcklund is:

$$w(z, \bar{\alpha}) = F(w'(z, \alpha), w(z, \alpha), z) \quad (2.7)$$

where  $w(z, \alpha)$  is a solution of a Painlevé equation corresponding to the parameter  $\alpha$ . In order to obtain a discrete equation it suffices to find another value of the parameter  $\underline{\alpha}$  for which a relation of the form:

$$w(z, \underline{\alpha}) = G(w'(z, \alpha), w(z, \alpha), z) \quad (2.8)$$

exists. Once  $w'$  is eliminated, one obtains a discrete equation relating  $w(\bar{\alpha})$ ,  $w(\underline{\alpha})$  and  $w(\alpha)$ , where  $z$  enters as a simple parameter with  $\alpha$  being the independent variable of the mapping.

Let us illustrate this approach in the case of P<sub>II</sub>. In fact, this is one of the oldest examples of discrete Painlevé equations appearing in the literature [16]. If  $w(z, \alpha)$  is a solution of P<sub>II</sub> ( $w'' = 2w^3 + zw + \alpha$ ), then:

$$\begin{aligned} w(z, -\alpha) &= -w(z, \alpha) \\ w(z, \alpha + 1) &= -w(z, \alpha) - \frac{1 + 2\alpha}{2w^2(z, \alpha) + 2w'(z, \alpha) + z} \end{aligned} \quad (2.9)$$

are also solutions of the same equation with the appropriate  $\alpha$ . Combining these two transforms we construct  $w(z, \alpha - 1)$  and eliminate  $w'$ . We obtain thus:

$$\frac{\alpha + \frac{1}{2}}{w(\alpha + 1) + w(\alpha)} + \frac{\alpha - \frac{1}{2}}{w(\alpha - 1) + w(\alpha)} = -[2w^2(\alpha) + z]. \quad (2.10)$$

This equation is an alternate form of d-P<sub>I</sub>.

Since the discrete Painlevé equations also possess auto-Bäcklund transforms one can use the latter in order to construct further discrete Painlevé equations. In fact, starting from the auto-Bäcklund of d-P<sub>II</sub> [18], we can construct another alternate d-P<sub>I</sub>.

### 2.4. THE USE OF DISCRETE ANALOGUES OF MIURA TRANSFORMS

In the Gambier classification [19,20] of the continuous Painlevé equations, out of the 50 equations with the Painlevé property, only 24 are considered as fundamental, while the remaining 26 are obtained from the former through Bäcklund or Miura transforms. Similarly, starting from a given discrete equation one can use a discrete transform to obtain a new discrete Painlevé equation. Indeed, starting from d-P<sub>II</sub> rewritten as:

$$x_{n+1} + x_{n-1} = \frac{x_n(2(\alpha n + \beta) - \alpha) + \delta + \alpha}{1 - x_n^2} \quad (2.11)$$

we introduce the Miura transformation [18]:

$$y_n = (x_n - 1)(x_{n+1} + 1) + (\alpha n + \beta) \quad (2.12)$$

and we obtain:

$$(y_n + y_{n+1})(y_n + y_{n-1}) = \frac{-4y_n^2 + \delta^2}{y_n - (\alpha n + \beta)}. \quad (2.13)$$

Equation (2.13) is d-P<sub>34</sub>, i.e. the discrete form of equation 34 in the Gambier classification, in perfect analogy to what happens in the continuous case.

## 2.5. SIMILARITY REDUCTIONS OF INTEGRABLE LATTICES

We will limit ourselves to only one example (more can be found in [21]) of an integrable lattice and its similarity reduction: the discrete modified Korteweg de Vries. In [4] Nijhoff and Papageorgiou studied this problem in detail. They have shown that the lattice potential mKdV:

$$\begin{aligned} &(\epsilon - \delta)[(u(n, m)u(n+1, m) - u(n, m+1)u(n+1, m+1)] \\ &+ (\epsilon + \delta)[(u(n, m)u(n, m+1) - u(n+1, m)u(n+1, m+1)] = 0 \end{aligned} \quad (2.14)$$

can be associated to the similarity constraint:

$$n \frac{u(n+1, m) - u(n-1, m)}{u(n+1, m) + u(n-1, m)} + m \frac{u(n, m+1) - u(n, m-1)}{u(n, m+1) + u(n, m-1)} = 0. \quad (2.15)$$

The situation is reminiscent of the continuous case where one supplements the original equation with a second *integrable* one: the similarity constraint. Unlike the continuous case one cannot solve here for a similarity variable. It turns out that the dimensional reduction of the constrained system (2.14-2.15) requires a limiting procedure. Let us assume that  $m$  and  $n$  go to infinity in such a way that  $k = m - n$  remains finite. Correspondingly, we assume that the lattice parameter  $\delta$  in the direction of  $l = m + n$  goes to zero so that  $\tau = l\delta$  is finite. Let us now define  $v(l, k) = u(\frac{l-k}{2}, \frac{l+k}{2})$ . Expanding  $v$  in powers of  $\delta$  we obtain:

$$\begin{aligned} k(v(k+1) - v(k-1)) &= 2\tau \frac{v(k-1)v'(k+1) + v'(k-1)v(k+1)}{v(k-1) + v(k+1)} \\ \frac{\epsilon v'(k)}{v(k)} &= \frac{v(k+1) - v(k-1)}{v(k+1) + v(k-1)} \end{aligned} \quad (2.16)$$

where ' stands for the derivative in the direction of  $l$  and where, since everything is now computed for the same value of the index  $l$ , we have dropped this index. We can now eliminate  $v'(k)$  from these last two equations to get:

$$\begin{aligned} k[v(k+1) - v(k-1)][v(k+1) + v(k-1)] &= \\ \frac{2\tau v(k+1)v(k-1)}{\epsilon} \left[ \frac{v(k+2) - v(k)}{v(k+2) + v(k)} + \frac{v(k) - v(k-2)}{v(k) + v(k-2)} \right]. \end{aligned} \quad (2.17)$$

On defining

$$w = \frac{v(k+1) - v(k-1)}{v(k+1) + v(k-1)} \quad (2.18)$$

one can rewrite this equation as:

$$w(k+1) + w(k-1) = \frac{2k\epsilon}{\tau} \frac{w}{1-w^2} \quad (2.19)$$

which is just d-P<sub>II</sub> with the constant taken equal to zero.

## 2.6. THE DIRECT METHOD THROUGH SINGULARITY CONFINEMENT

This method is based on a recent conjecture [8] that relates the behaviour of the singularities that appear in discrete systems to their integrability. This conjecture can be formulated in the following (intuitive) way: the movable singularities of integrable mappings are confined i.e. they are cancelled out after a finite number of steps. Moreover the memory of the initial condition is not lost whenever a singularity is crossed. One must be careful as to whether singularities are possible, within a given discrete-time system, without assuming divergent initial conditions. In the latter case the singularity is not a *movable* one.

The implementation of the singularity confinement method is quite simple. Given a mapping one must first find all possible ways a singularity can emerge (this step follows closely the first step of the algorithm for ODEs where one looks for all possible leading singular behaviours). The system is said to have passed the test (and is thus a candidate for integrability) if this divergence does not propagate in (discrete) time, i.e. that it remains confined. The second step is therefore to find how far it has to propagate before it has a chance to leave room for a regular behaviour (this is somewhat reminiscent of the “search for resonances” in the ARS [22] algorithm), and finally one has to verify that indeed the singularity does not propagate beyond that (this last step is the equivalent of the “resonance condition”).

As an illustration of the singularity confinement algorithm used as integrability detector we will examine a mapping of the Quispel [23] type:

$$x_{n+1} + x_{n-1} = -\frac{\delta x_n^2 + \epsilon x_n + \zeta}{\alpha x_n^2 + \beta x_n + \gamma}. \quad (2.20)$$

Applying our criterion we will deduce the values of the parameters  $\alpha, \beta, \dots, \zeta$  for the mapping to be integrable. Let us assume that at a certain time step the denominator vanishes:  $\alpha x_n^2 + \beta x_n + \gamma = 0$ . We obtain then  $x_{n+1} = \infty$  and  $x_{n+2} = -x_n - \delta/\alpha$ . The problems arise at the next step:  $x_{n+3}$  will diverge because of the presence of  $x_{n+1}$  unless the right-hand side diverges also so as to cancel the divergence of  $x_{n+1}$ . This can happen only if  $\alpha x_{n+2}^2 + \beta x_{n+2} + \gamma = 0$ . Substituting  $x_{n+2}$  we obtain an equation for  $x_n$  and demand that it be proportional to  $\alpha x_n^2 + \beta x_n + \gamma = 0$ . We thus obtain (the only) constraint for the confinement of the singularity  $\delta = \beta$ . This is precisely the only relation between the parameters  $\alpha, \beta, \dots, \zeta$  that makes the mapping a member of the integrable Quispel family.

In order to obtain the discrete Painlevé equations through the application of the singularity confinement method we start from the general Quispel mapping:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}. \quad (2.21)$$

The reason for this choice is that this mapping is integrable in terms of elliptic functions. Since the autonomous limits of the continuous Painlevé equations are integrable in terms of elliptic functions one would expect to find the discrete forms by “de-autonomizing” the Quispel map. In order to gain some insight into the choice of the  $f_i$ 's we rewrite the Quispel map as:

$$f_3(x_n)\Pi - f_2(x_n)\Sigma + f_1(x_n) = 0 \quad (2.22)$$

where  $\Sigma = x_{n+1} + x_{n-1}$ ,  $\Pi = x_{n+1}x_{n-1}$ , the  $f_i$  are quartic polynomials and we ask that this equation go over to the continuous Painlevé under consideration at the continuous limit.

We introduce a lattice parameter  $\delta$  and obtain:

$$\begin{aligned}\Sigma &= 2x + \delta^2 x'' + \mathcal{O}(\delta^4) \\ \Pi &= x^2 + \delta^2(x x'' - x'^2) + \mathcal{O}(\delta^4)\end{aligned}\quad (2.23)$$

and when we extract from (2.23) the part involving derivatives, we obtain a continuous limit (as  $\delta$  goes to zero) of the form:

$$x'' = \frac{f_3(x)}{x f_3(x) - f_2(x)} (x')^2 + g(x). \quad (2.24)$$

If we are looking for a specific Painlevé equation we must first choose  $f_2$ ,  $f_3$  so as to get  $f_3(x)/[x f_3(x) - f_2(x)]$  to coincide with the factor multiplying  $(x')^2$  in that equation.

For  $P_I$  and  $P_{II}$  we have, clearly,  $f_3 = 0$ . The form we choose for d- $P_I$  is:

$$x_{n+1} + x_{n-1} = -x_n + B(n) + C(n)/x_n. \quad (2.25)$$

A first condition for singularity confinement is  $B(n+1) - B(n) = 0$ . Thus  $B$  is constant. Once this is implemented we find a second (and sufficient) condition:  $C(n+3) - C(n+2) - C(n+1) + C(n) = 0$ . The general solution of this equation is  $C(n) = \alpha n + \beta + \gamma(-1)^n$ . We remark here that the last term of  $C(n)$  will disappear in the continuous limit. Here again (for  $\gamma = 0$  and  $B = b$ ) we obtain:

$$x_{n+1} + x_{n-1} + x_n = b + (\alpha n + \beta)/x_n \quad (2.26)$$

which is precisely the d- $P_I$ . For d- $P_{II}$  we start from the form (2.20) after rescaling:

$$x_{n+1} + x_{n-1} = -\frac{x_n^2 + B(n)x_n + C(n)}{x_n(x_n + 1)} \quad (2.27)$$

As in the autonomous case we assume that  $x_n = 0$ , which leads to the following condition for singularity confinement:  $C(n+1) - C(n-1) - B(n+1) + B(n) = 0$ . Similarly starting from the second root of the denominator  $x_n = -1$ , we find:  $C(n+1) - C(n-1) + B(n-1) - B(n) = 0$ . Combining the two equations we obtain:  $B(n+1) - 2B(n) + B(n-1) = 0$  and  $B(n) = \lambda n + \mu$ . Substituting back we obtain for  $C$ :  $C(n+1) - C(n-1) = \lambda$  and thus  $C(n) = \frac{1}{2}\lambda n + \nu$ , neglecting the even-odd dependence. With these expressions for  $B$  and  $C$  and with  $z_n = 2x_n + 1$  we find indeed a d- $P_{II}$  of the form:

$$z_{n+1} + z_{n-1} = \frac{z_n(\alpha n + \beta) + \gamma}{1 - z_n^2}. \quad (2.28)$$

In the case of  $P_{III}$  we have  $x'' = x'^2/x + g(x)$ . Instead of deriving the discrete form of the “usual”  $P_{III}$  we will work with the more convenient form:

$$w'' = \frac{(w')^2}{w} + e^z(aw^2 + b) + e^{2z} \left( cw^3 + \frac{d}{w} \right) \quad (2.29)$$

obtained from the usual one through the transformation  $z \rightarrow e^z$  that absorbs the  $w'/z$  term. This form agrees with (2.24) if we simply take  $f_2 = 0$ . In that case, the mapping takes the form:

$$x_{n+1}x_{n-1} = \frac{\kappa(n)x_n^2 + \zeta(n)x_n + \mu(n)}{x_n^2 + \beta(n)x_n + \gamma(n)}. \quad (2.30)$$

In order to fix the  $n$ -dependent coefficients we will study the singularity behaviour as described before. When one solves for  $x_{n+1}$  there are two possible sources of singularity for this mapping. Either  $x_n$  is a zero of the denominator or  $x_{n-1}$  becomes zero. In the first case, the singularity sequence is the following:  $x_{n+1}$  diverges,  $x_{n+2}$  has a finite value  $\kappa(n+1)/x_n$  and  $x_{n+3}$  would in principle be proportional to  $1/x_{n+1}$  and thus zero. This would lead to a new divergence. The only way out is to ask that  $x_{n+2}$  also be a zero of the appropriate denominator, so that  $x_{n+3}$  does not vanish. Expressing  $x_{n+2}$  in terms of  $x_n$  and taking into account that this must be true for both zeros of  $x_n$ , we obtain

$$\beta(n) = \frac{\beta(n+2)\kappa(n+1)}{\gamma(n+2)}, \quad \gamma(n) = \frac{\kappa^2(n+1)}{\gamma(n+2)}.$$

Multiplying  $x_n$  by an arbitrary function of  $n$  does not change the form of (2.30) but only affects the coefficients. This scaling freedom allows us to take a constant value  $\beta$  for  $\beta(n)$ , resulting to  $\kappa(n+1) = \gamma(n+2)$ ,  $\gamma(n) = \gamma(n+2)$ . Thus the  $\gamma$ 's and  $\kappa$ 's must be constants within a given parity:  $\gamma(\text{even})=\kappa(\text{odd})=\gamma_+$ ,  $\gamma(\text{odd})=\kappa(\text{even})=\gamma_-$ . In order to study the second kind of singularity, we start with  $x_n$  such that  $x_{n+1}$  vanishes (i.e.  $\kappa(n)x_n^2 + \zeta(n)x_n + \mu(n) = 0$ ). We find then that  $x_{n+2}$  has a finite value  $\mu(n+1)/[\gamma(n+1)x_n]$  and this would lead to a divergent  $x_{n+3}$  unless the numerator also vanishes. Substituting the expression for  $x_{n+2}$  and using the fact that again this must be true for both zeros of  $\kappa(n)x_n^2 + \zeta(n)x_n + \mu(n)$ , we obtain

$$\mu(n) = \frac{\zeta(n)\mu(n+1)}{\zeta(n+2)} = \frac{\mu^2(n+1)}{\mu(n+2)}.$$

The solution to these equations is straightforward:  $\mu(n) = \mu_0\lambda^{2n}$  and  $\zeta(n) = \zeta_{0,\pm}\lambda^n$ , where  $\mu_0$ ,  $\zeta_{0,\pm}$ , are constants, the  $\pm$  sign being related to the parity of  $n$ . Note that, in that case, there is no second kind of singularity at all! Indeed  $x_{n+3}$  is not allowed to diverge even though  $x_{n+1} = 0$ . (This is reminiscent of the case of continuous equations where, if a denominator appears, one must consider the values of the dependent variable that makes this denominator vanish to ascertain that this does not generate a singularity). Since at the continuous limit the distinction between even and odd must disappear, we neglect the even-odd dependence and write d-P<sub>III</sub>, after a change of the mapping variable  $z_n = \lambda^{n/2}x_n$  as:

$$z_{n+1}z_{n-1} = \frac{\gamma z_n^2 + \zeta_0\lambda^{n/2}z_n + \mu_0\lambda^n}{\lambda^n z_n^2 + \beta\lambda^{n/2}z_n + \gamma}. \quad (2.31)$$

No derivation will be presented here for d-P<sub>IV</sub> and d-P<sub>V</sub>. We will only give their forms, for completeness reasons. We have thus for d-P<sub>IV</sub>:

$$x_{n+1}x_{n-1} + x_n(x_{n+1} + x_{n-1}) = \frac{-(an+b)x_n^3 + (d - \frac{1}{4}(an+b)^2)x_n^2 + m}{x_n^2 + (an+b)x_n + c + \frac{1}{4}(an+b)^2} \quad (2.32)$$

and for d-P<sub>V</sub>:

$$(2x_n - 1)x_{n+1}x_{n-1} - x_n(x_{n+1} + x_{n-1}) = \frac{\frac{1}{2}(c - a\lambda^{2n})x_n^3 + (d + \frac{1}{4}(a\lambda^{2n} - 2b\lambda^n + c))x_n^2 - 2mx_n + m}{a\lambda^{2n}x_n^2 + (b\lambda^n - a\lambda^{2n})x_n + \frac{1}{4}(a\lambda^{2n} - 2b\lambda^n + c)} \quad (2.33)$$

The forms presented above are far from being unique. In the previous sections we have exhibited forms of d-P<sub>I</sub> different from the above. More details can be found in [14]. Here, it suffices to say that there exist alternate forms for almost all the discrete Painlevé equations. One interesting result is that some of these discrete Painlevé equations do not belong to the Quispel family at the autonomous limit: examples are the symmetric (2.21) and the asymmetric:

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}, \quad y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}. \quad (2.34)$$

In fact, for some of these equations the relation between  $z_{n+1}$  and  $z_n$  is not homographic any more and the right-hand side of the equivalent of (2.34) contains higher nonlinearities. This immediately raises the question of the inverse evolution of the mapping i.e. from  $n+1$  to  $n$ . Clearly, if nonlinearities are involved, the solution of the algebraic relations will entail multideterminacies and thus a given point  $z_n$  will have an exponentially growing number of preimages. This is contrary to the intuitive notion of how an integrable mapping should behave. It turns out that, for all the cases examined (see also examples in [24]), the inverse of the mapping is uniquely defined. This led us into introducing a further conjecture for integrable mappings: the preimages of an integrable mapping do not proliferate. This necessary condition for integrability is not particularly strong and cannot be used easily in order to construct integrable candidates, since it leads to cumbersome problems of divisibility of polynomials. Still, for a given mapping the nonproliferation condition can be checked in a most straightforward way (see also [25]).

One last remark is at order at this point. From our presentation of the singularity confinement method one might have concluded that infinities play a particular role. This is not the case! In order to see that, consider the Quispel map (2.21) under the assumption that all the  $f_i$ 's have the same degree. In this case an infinite  $x_n$  will lead (generically) to a finite  $x_{n+1}$  and no further singularity will appear. However this does not mean that there is no danger of unconfined singularities. It may turn out that for a certain  $n$  the mapping (apparently) loses one degree of freedom. This occurs when  $x_{n+1}$  is defined independently of  $x_{n-1}$ , and this happens whenever:  $f_1(x_n)f_3(x_n) - f_2^2(x_n) = 0$ . Once  $x_n$  is obtained from the latter, one can compute  $x_{n+1}$  simply as:

$$x_{n+1} = \frac{f_1(x_n)}{f_2(x_n)} = \frac{f_2(x_n)}{f_3(x_n)},$$

unless  $x_{n-1}$  is such that both the numerator and the denominator of the fraction containing  $x_{n+1}$  vanished i.e.

$$x_{n-1} = \frac{f_1(x_n)}{f_2(x_n)} = \frac{f_2(x_n)}{f_3(x_n)}.$$

If the latter is not satisfied, one degree of freedom will be lost at this stage and  $x_{n+2}$  will be determined in terms of  $x_n$  alone. The only way out is to demand that both numerator and the denominator of the fraction that defines it vanish i.e.

$$x_n = \frac{f_1(x_{n+1})}{f_2(x_{n+1})} = \frac{f_2(x_{n+1})}{f_3(x_{n+1})}$$

(see [25]). Thus the lost degree of freedom reappears at this step provided the conditions on the  $f_i$  are satisfied (we could, of course, imagine an arbitrary number of intermediate steps before the reemergence of the lost degree of freedom).

### 3. Properties of the discrete Painlevé equations

In the Introduction we have presented a list of properties that distinguish the continuous Painlevé equations. In what follows we shall show how these properties find their analogues in the discrete case.

#### 3.1. CONTINUOUS LIMITS

This is of course something that characterizes the discrete systems. It allows also to give a name to each of the discrete Painlevé equations. Going to the continuous limit is a delicate procedure and one must be particularly careful not to lose any free constants in the process. For d-P<sub>I</sub> (1.1), for example, one must take:  $x = -\frac{1}{2} + \epsilon^2 w$ ,  $c = -3$ ,  $an + b = -\frac{1}{4}(3 + 2\epsilon^4 t)$ . At the limit  $\epsilon \rightarrow 0$  equation (1.1) reduces to the continuous P<sub>I</sub> equation,  $w'' = 6w^2 + t$ . Similarly for the alternate d-P<sub>I</sub>, (2.10), one simply takes:  $w = \rho(1 + \epsilon^2 u)$ ,  $z = -6\rho^2$ ,  $\alpha + \frac{1}{2} = \rho^3(4 + \epsilon^4 t)$ , with  $\rho^3\epsilon^5 = 1$  and finds P<sub>I</sub> for  $u(t)$ .

For d-P<sub>II</sub> it suffices to take  $x = \epsilon w$ ,  $an + b = 2 + \epsilon^2 t$ ,  $c = \epsilon^3 \mu$  in order to find:  $w'' = 2w^3 + tw + \mu$ , i.e. P<sub>II</sub>, when  $\epsilon$  goes to zero. For P<sub>III</sub>, in order to recover (2.29) from (2.31) it suffices to take:  $\gamma = -1/(c\epsilon^2)$ ,  $\beta = a/c$ ,  $\mu_0 = -d/c$ ,  $\zeta_0 = -b/c$  and simultaneously  $\lambda = 1 + 2\epsilon$  [26]. Similar results can be obtained for all the other known discrete Painlevé equations.

#### 3.2. COALESCENCE CASCADE

The continuous Painlevé equations form a coalescence cascade i.e. the “lower” (in order) equations can be obtained from the “higher” ones through adequate limiting processes involving the dependent variable and the free constants entering the equation. The reduction scheme follows the pattern: P<sub>VI</sub> → P<sub>V</sub> → {P<sub>IV</sub>, P<sub>III</sub>} → P<sub>II</sub> → P<sub>I</sub>. We have shown in [9] that the discrete Painlevé equations follow exactly the same cascade. In order to illustrate this procedure let us start with d-P<sub>II</sub> (1.2). Putting  $x = 1 + \nu X$ ,  $a = -2A\nu^2$ ,  $b = -4 - 2C\nu$  and  $c = 4 + 2C\nu - 2B\nu^2$  we recover exactly d-P<sub>I</sub> (with obvious notations) at the limit  $\nu \rightarrow 0$ . The mapping (1.1) is not the only coalescence limit of d-P<sub>II</sub>. Putting  $x = X/\nu$ ,  $an + b = -\alpha n + \beta/\nu^2$ , and  $c = -\gamma/\nu^3$  we recover an alternate d-P<sub>I</sub> at the limit  $\nu \rightarrow 0$ :

$$X_{n+1} + X_{n-1} = \frac{\gamma}{X_n^2} + \frac{\alpha n + \beta}{X_n} \quad (3.1)$$

On the other hand the alternate d-P<sub>I</sub> (2.10) does not belong to the same cascade but comes from an alternate d-P<sub>II</sub>:

$$\frac{a(n+1)+b}{x_n x_{n+1} - 1} + \frac{an+b}{x_n x_{n-1} - 1} = c \left( x_n + \frac{1}{x_n} \right) - an + m. \quad (3.2)$$

Similar results hold for the other known discrete Painlevé equations.

#### 3.3. LAX PAIRS

The ultimate proof of the integrability of the discrete Painlevé equations is their effective linearization i.e. their transcription as the compatibility condition for a linear isospectral

deformation problem. In most cases of known Lax pairs the linear system assumes the form:

$$\begin{aligned}\zeta \Phi_{n,\zeta} &= \mathbf{L}_n(\zeta) \Phi_n \\ \Phi_{n+1} &= \mathbf{M}_n(\zeta) \Phi_n\end{aligned}\quad (3.3)$$

leading to the compatibility condition:

$$\zeta \mathbf{M}_{n,\zeta} = \mathbf{L}_{n+1} \mathbf{M}_n - \mathbf{M}_n \mathbf{L}_n. \quad (3.4)$$

The latter gives the discrete Painlevé equation. Thus for d-P<sub>I</sub> we have found [27]:

$$\mathbf{L}(\zeta) = \begin{pmatrix} \kappa & v_2 & 1 \\ \zeta & \lambda & v_3 \\ \zeta v_1 & \zeta & \mu \end{pmatrix}, \quad \mathbf{M}(\zeta) = \begin{pmatrix} d_1 & 1 & 0 \\ 0 & d_2 & 1 \\ \zeta & 0 & 0 \end{pmatrix}. \quad (3.5)$$

Taking  $d_2 = 0$  leads to the mapping:

$$\bar{v}_1 = v_2, \quad \bar{v}_2 = v_3 + d_1, \quad \bar{v}_3 = v_1 - d_1 \quad (3.6a)$$

together with

$$d_1 = (\kappa - \lambda)/v_2 \quad \text{and} \quad \kappa = \text{const.}, \quad \lambda = \lambda(n) = \text{linear in } n, \quad \mu = \bar{\lambda} \quad (3.6b)$$

where the overline indicates the next iteration in the mapping i.e.  $\bar{z} = z(n+1)$ . Using the Casimir  $C = v_1 + v_2 + v_3$  and taking  $x_n \equiv v_2$  we obtain d-P<sub>I</sub>:

$$x_{n+1} + x_n + x_{n-1} = C + [\kappa - \lambda(n)]/x_n \quad (3.7)$$

In [2,27] a  $2 \times 2$  isomonodromy problem was considered for the same discrete Painlevé equation. An interesting result is obtained when one does not assume  $d_2 = 0$ . Then one obtains:

$$d_1 = \frac{\kappa - \mu}{\bar{v}_1} \quad \text{and} \quad \kappa = \text{const.}, \quad \lambda = \text{const.}, \quad \mu = \mu(n) = \text{linear in } n \quad (3.6c)$$

Putting  $x_n \equiv v_1$  and  $y_n \equiv v_3$  we find:

$$v_1 + \bar{v}_1 = C - v_3 + (\mu - \lambda)/v_3, \quad v_3 + \bar{v}_3 = C - \bar{v}_1 + (\mu - \kappa)/\bar{v}_1. \quad (3.8)$$

This is precisely the most general d-P<sub>I</sub> as it has been derived in the paragraph preceding equation (2.26) and includes the even-odd dependence, a further proof of the consistence of the singularity confinement approach.

Lax pairs have been obtained in [27] for the alternate d-P<sub>I</sub> (2.10) and d-P<sub>II</sub> (while in [28] a  $2 \times 2$  pair was given). This shows that the linearization of the discrete Painlevé equations is not unique. A most interesting result is the isospectral problem associated to the d-P<sub>III</sub>. Here a  $q$ -difference scheme is necessary instead of a differential one:

$$\begin{aligned}\Phi_n(q\zeta) &= \mathbf{L}_n(\zeta) \Phi_n(\zeta) \\ \Phi_{n+1}(\zeta) &= \mathbf{M}_n(\zeta) \Phi_n(\zeta)\end{aligned}\quad (3.9)$$

leading to:

$$\mathbf{M}_n(q\zeta) \mathbf{L}_n(\zeta) = \mathbf{L}_{n+1}(\zeta) \mathbf{M}_n(\zeta) \quad (3.10)$$

The resulting Lax pair is written in terms of  $4 \times 4$  matrices.

Although the list of Lax pairs for the known discrete Painlevé equations is far from being complete there is reasonable hope that eventually all of them will be derived. From the results of [29] we can surmise that more general isospectral problems, being simultaneously both  $q$ -difference and differential, will be needed.

### 3.4. MIURA AND BÄCKLUND RELATIONS

Just as in the continuous case [15] the discrete Painlevé equations possess (auto-) Bäcklund and Miura transformations that allow us to establish a dense net of relationships among them. The domain is barely touched upon and much work is needed in order to derive all the relations that surely exist. We will illustrate our point in the case of d-P<sub>II</sub>. In fact, in section 2.4 we have shown how one can derive d-P<sub>34</sub> as a Miura of d-P<sub>II</sub> making one equation the modified of the other.

As in the case of the continuous Painlevé equations the existence of a Miura relation is rich in consequences. Since d-P<sub>34</sub> (2.13) contains only  $\delta^2$  choosing the plus or minus sign for  $\delta$  in the Miura transform

$$y_n = \frac{\delta + x_{n-1} - x_n}{x_{n-1} + x_n} \quad (3.11)$$

(that complements (2.12)), would lead to solutions of d-P<sub>II</sub> with  $\pm\delta$  respectively. Defining  $z_n \equiv -y_n(-\delta)$  we find that, while  $y_n$  is a solution of d-P<sub>II</sub> with the constant term ( $c$  in equation 1.2) equal to  $\delta + \alpha$ ,  $z_n$  corresponds to  $\delta - \alpha$ . We have thus (eliminating  $x_n$ ):

$$z_n = -y_n + \frac{2\delta}{y_n(y_{n+1} + y_{n-1}) - y_{n+1} + y_{n-1} + 2(\alpha n + \beta - 1) - \alpha} \quad (3.12a)$$

$$y_n = -z_n + \frac{2\delta}{z_n(z_{n+1} + z_{n-1}) + z_{n+1} - z_{n-1} + 2(\alpha n + \beta - 1) - \alpha}. \quad (3.12b)$$

From equation (3.12) we can easily construct the auto-Bäcklund transformation for d-P<sub>II</sub>. Using d-P<sub>II</sub> to eliminate  $y_{n-1}$  ( $z_{n-1}$ ) from the right-hand side of (3.12) we obtain:

$$y_n + z_n = \frac{\delta(1 - y_n)}{(y_{n+1} + 1)(y_n - 1) + (\alpha n + \beta) - \frac{1}{2}\delta} \quad (3.13a)$$

$$y_n + z_n = \frac{\delta(1 + z_n)}{(z_{n+1} - 1)(z_n + 1) + (\alpha n + \beta) + \frac{1}{2}\delta}. \quad (3.13b)$$

System (3.13) is indeed auto-Bäcklund since eliminating either of the two variables gives d-P<sub>II</sub> for the other. As we have explained in section 2.3 these auto-Bäcklund relations can be used in order to generate new discrete Painlevé equations.

### 3.5. PARTICULAR SOLUTIONS

It is well known [30] that the continuous Painlevé equations P<sub>II</sub> to P<sub>VI</sub> possess elementary solutions for specific values of their parameters. Some of them are special functions (Airy, Bessel, Weber-Hermite, Whittaker and Gauss hypergeometric) while the others are just rational ones. Quite remarkably the discrete Painlevé equations have the same property and, in fact, their “special function”-type solutions are solutions of linear difference equations that are discretizations of the corresponding equations for the continuous special functions [31]. We will illustrate this point with the example of d-P<sub>II</sub>. We start from the discrete Miura for d-P<sub>II</sub> that we rewrite as:

$$\lambda x_n = (y_n - 1)(y_{n+1} + 1) - (\alpha n + \beta) \quad (3.14a)$$

$$y_n = \frac{\delta + x_{n-1} - x_n}{x_{n-1} + x_n} \quad (3.14b)$$

When  $\lambda = 0$  the second equation decouples but  $y_n$  still satisfies d-P<sub>II</sub> for this value of the parameter. Rewriting (3.14a) as  $y_{n+1} = -1 + (\alpha n + \beta)/(y_n - 1)$  and putting  $y_n = B_n/A_n$  we linearize (3.14a) to:

$$A_{n+1} = -B_n + A_n, \quad B_{n+1} = B_n - (\alpha n + \beta + 1)A_n \quad (3.15)$$

or equivalently:

$$A_{n+2} - 2A_{n+1} + (\alpha n + \beta)A_n = 0. \quad (3.16)$$

The latter is the discrete analogue of the Airy equation and once its solution is obtained we can construct  $y_n$  through  $y_n = 1 - A_{n+1}/A_n$ . Higher “Airy” type solutions can be constructed using the auto-Bäcklund (3.13) for  $\delta = \pm 2k\alpha$ . Rational solutions can be obtained along the same lines. We remark that when  $\delta = -\alpha$  in (2.11),  $y_n = 0$  is a solution. Injecting this “seed” solution in (3.13a) we obtain  $z_n$ , i.e. the solution for  $\delta = -3\alpha$ , as

$$z_n = \frac{\alpha}{\alpha n + \beta + 1 - \alpha/2},$$

and, similarly, solutions for all  $\delta = \pm(2k + 1)\alpha$ . Results for the higher Painlevé equations can be obtained in a similar way [31,32].

The list of properties of the discrete Painlevé equations presented above is not exhaustive. Since this work was presented at the Exeter workshop a new result in yet another direction has been obtained: we have shown [33], in fact, that the discrete Painlevé equations can be cast in bilinear form reminiscent of the ones for the continuous Painlevé equations.

#### 4. Summary and outlook: introducing the delay-Painlevé equations

The discovery of the discrete Painlevé equations has coincided with a revival in the interest in the discrete systems in particular from the point of view of integrability. Painlevé transcendental equations are the most sophisticated integrable (low order) ODEs and the fact that they possess discrete versions indicates that the other, simpler, forms of integrability should also possess discrete analogues. This is indeed the case. We have shown, for example, that Cole-Hopf type linearizability exists in discrete systems [25] leading to second order (three-point) mappings that discretize the linearizable equations classified by Painlevé and Gambier [5]. This is just another manifestation of something that we hold to be true: there exists a close parallel between continuous and discrete systems and their dynamic behaviours may well be indistinguishable. (This statement is intentionally vague; given time, we may return to it some day.)

A most important breakthrough in the domain of discrete integrability was the introduction of the singularity confinement method that plays the same role as the Painlevé analysis of continuous systems. Thanks to this method we are not burdened by the difficulties of the constructive methods in which integrability is proven by the explicit computation of the integrals, derivation of the Lax pairs, linearization of the system and so on. Thanks to this new method, easily implemented predictions of integrability can be made, identifying integrability candidates, which can be subsequently confirmed at leisure. The important thing is that, with the singularity confinement tool at hand, one does not have to grope in the dark for new integrable cases.

Our outlook will focus on a most recent direction of investigation. It has been known for years that the Painlevé analysis becomes cumbersome when one tries to apply it to

differential-difference systems. Once the workings of the singularity confinement method were understood it was straightforward to blend the two methods and obtain an integrability detector [35] for systems that are both discrete and continuous. Here we will not go into any details concerning this new method but limit ourselves to one all-important result of this approach: delay-Painlevé equations. In fact delay equations (that have appeared recently as reductions of integrable lattices [36,37]) can be treated with our method just like differential-difference systems. We have performed a detailed study of first order two-point such systems [38] and identified three such equations that have P<sub>I</sub>, P<sub>II</sub> and a special form of P<sub>III</sub> as continuous limits. With  $u = u(t)$  and  $\bar{u} = u(t + \tau)$  (where  $\tau$  is the delay) we write:

$$u' + \bar{u}' = (u - \bar{u})^2 + k(u + \bar{u}) + \alpha e^{\omega t} + \beta \quad (4.1)$$

$$(u\bar{u})' = e^{\omega t}(\alpha u^2 + \beta \bar{u}^2) \quad (4.2)$$

$$u\bar{u}' - \bar{u}u' = e^{\omega t}(\alpha(u\bar{u})^2 + \beta). \quad (4.3)$$

Note that D-P<sub>II</sub> can be written in  $q$ -difference form as:

$$(u(t)u(qt))' = \alpha u^2(t) + \beta u^2(qt).$$

Moreover it can coalesce to a particular form of D-P<sub>I</sub>:

$$u' + \bar{u}' = (u - \bar{u})^2 + k(u + \bar{u}) + \lambda t$$

and it can be shown to possess rational solutions. The fact that we do not obtain exactly (4.1) may be an indication that the delay Painlevé equations are objects that go beyond the Painlevé transcedents. Our classification [38] assumes a certain canonical form. Alternate forms of these equations can be found if the constraint of the form is relaxed somewhat. For instance, Shabat's [39] equation:

$$u' + \bar{u}' = u^2 - \bar{u}^2 + k$$

is another discrete form of D-P<sub>I</sub>. First-order three-point delay Painlevé equations obtained as extensions of the Quispel family can also be shown to exist, and are derived from:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_4(x_n) - x_{n-1}f_3(x_n)} \quad (4.4)$$

where the  $f_i$  are now Riccati-like quantities:  $\alpha x' + \beta x^2 + \gamma x + \delta$ . This investigation is in its initial phase but already some new discrete Painlevé equations are identified:  $\frac{u'}{u} = \frac{\bar{u}}{u} + \lambda t$  as D-P<sub>I</sub> (the delay Kac-Moerbeke

$$\alpha + \beta \frac{u'}{u} = \bar{u} - u$$

is another form of D-P<sub>I</sub>) and

$$\frac{u'}{u} = e^{\omega t} \frac{\bar{u} - u}{\bar{u} + u} + k$$

as D-P<sub>II</sub>.

It is clear from our review that the domain of integrability enters a new phase with a large field of investigation, efficient tools and interesting preliminary findings that will certainly spur further explorations.

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# A STUDY OF THE FOURTH PAINLEVÉ EQUATION

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ABSTRACT. We investigate numerically solutions of a special case of the fourth Painlevé equation given by

$$\frac{d^2\eta}{d\xi^2} = 3\eta^5 + 2\xi\eta^3 + \left(\frac{1}{4}\xi^2 - \nu - \frac{1}{2}\right)\eta, \quad (1a)$$

with  $\nu$  a parameter, satisfying the boundary condition

$$\eta(\xi) \rightarrow 0, \quad \text{as} \quad \xi \rightarrow +\infty. \quad (1b)$$

Equation (1a) arises as a symmetry reduction of the Derivative Nonlinear Schrödinger (DNLS) equation, which is a completely integrable soliton equation solvable by inverse scattering techniques. Previous results concerned with solutions of (1) are largely restricted to the case when  $\nu$  is an integer and very little has been proved when  $\nu$  is a non-integer. Here we use both analytical and numerical techniques to describe solutions of (1) for  $\nu$  non-integer. In addition we obtain information characterising both the amplitude and phase connection formulae associated with (1) which describe how the asymptotic behaviours of solutions as  $\xi \rightarrow +\infty$  relate to those as  $\xi \rightarrow -\infty$ .

## 1. Introduction

In a previous paper (Bassom *et al.* 1992, hereafter referred to as BCHM), we discussed solutions of the equation

$$\frac{d^2\eta}{d\xi^2} = 3\eta^5 + 2\xi\eta^3 + \left(\frac{1}{4}\xi^2 - \nu - \frac{1}{2}\right)\eta, \quad (1.1a)$$

where  $\nu = n$  is a positive integer, satisfying the boundary condition

$$\eta(\xi) \rightarrow 0, \quad \text{as} \quad \xi \rightarrow +\infty. \quad (1.1b)$$

Equation (1.1a), which may be thought of as a nonlinear harmonic oscillator (see §2), is a special case of the fourth Painlevé equation (PIV)

$$ww'' = \frac{1}{2}(w')^2 + \frac{3}{2}w^4 + 4zw^3 + 2(z^2 - \alpha)w^2 + \beta, \quad (1.2)$$

with  $' \equiv d/dz$ , and  $\alpha, \beta$  arbitrary constants, since if in (1.1) we make the transformation

$$\eta(\xi) = 2^{-3/4} \sqrt{w(z)}, \quad z = \xi/\sqrt{2}, \quad (1.3)$$

then  $w(z)$  satisfies PIV with  $\alpha = 2\nu + 1$  and  $\beta = 0$ . Equation (1.1a) arises as a symmetry reduction of the Derivative Nonlinear Schrödinger (DNLS) equation

$$iq_t + q_{xx} \pm i(|q|^2 q)_x = 0,$$

which is solvable by inverse scattering techniques (Kaup and Newell, 1978). Consequently the solution of (1.1a) satisfying (1.1b) can be expressed in terms of the solutions of linear integral equations (see equations (2.3) below) at least for  $\xi$  sufficiently large. Using these integral equations, several properties of solutions of (1.1) have been derived including, global existence and uniqueness of solutions, bound state solutions which decay exponentially as  $\xi \rightarrow \pm\infty$  and connection formulae relating the asymptotic behaviour of solutions as  $\xi \rightarrow +\infty$  to the asymptotic behaviour as  $\xi \rightarrow -\infty$  (Ablowitz *et al.*, 1980; BCHM; Clarkson and McLeod, 1992).

We remark here that PIV has appeared in several physical applications including quantum gravity, nonlinear optics and fluid mechanics. In addition, PIV arises as a symmetry reduction of a great many physically interesting nonlinear PDE's. (References may be found in Ablowitz and Clarkson (1991), Kruskal and Clarkson (1992) and Clarkson (1992).) In this paper we continue our study of (1.1) for general  $\nu$  mainly through extensive numerical investigations. In §2 we briefly review the existing analytical results concerned with (1.1). Most of these are restricted to the case when  $\nu$  is an integer and are described in detail in BCHM. In §3 and §4 we describe the numerical techniques implemented in order to examine solution properties of (1.1). Finally we conclude with a discussion.

## 2. Properties of solutions of (1.1)

When  $\eta$  is small we linearise (1.1a) and the solutions satisfying  $\eta \rightarrow 0$  as  $\xi \rightarrow +\infty$  are multiples of the *parabolic cylinder function*  $D_\nu(z)$ , which is defined to be the solution of

$$D_\nu''(z) = (\frac{1}{4}z^2 - \nu - \frac{1}{2})D_\nu(z),$$

satisfying

$$D_\nu(z) \sim z^\nu \exp(-\frac{1}{4}z^2), \quad \text{as } z \rightarrow +\infty, \quad (2.1a)$$

$$D_\nu(z) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} \exp(\frac{1}{4}z^2), \quad \text{as } z \rightarrow -\infty, \quad (2.1b)$$

provided that  $\nu$  is not an integer. When  $\nu$  is a positive integer

$$D_n(z) = \text{He}_n(z) \exp(-\frac{1}{4}z^2),$$

where  $\text{He}_n(z)$  is the *Hermite polynomial* of degree  $n$  whose properties are given in Abramowitz and Stegun (1965). In BCHM we proved the following:

**THEOREM 2.1.** *Any solution of (1.1a) satisfying (1.1b), for any real  $\nu$  (either integer or non-integer), is asymptotic to  $kD_\nu(\xi)$  for some  $k$  and conversely, for any  $k$ , there is a unique solution of (1.2a) asymptotic to  $kD_\nu(\xi)$ . If we denote this solution by  $\eta_k(\xi; \nu)$ , then as  $\xi \rightarrow \infty$*

$$\eta_k(\xi; \nu) \sim kD_\nu(\xi) \sim k\xi^\nu \exp(-\frac{1}{4}\xi^2). \quad (2.2)$$

We note here that since (1.1) is left unchanged by the transformation  $\eta \rightarrow -\eta$ , henceforth we shall assume without loss of generality that  $k > 0$ . BCHM showed that for  $\xi$  sufficiently large (dependent upon  $k$ ), say  $\xi > \xi_1$ , the (unique) solution of (1.1) is given by

$$\eta_k(\xi; \nu) = [K_\nu(\xi, \xi) K_\nu^*(\xi, \xi)]^{1/2},$$

where  $K_\nu(\xi, \zeta)$  and  $K_\nu^*(\xi, \zeta)$  are the solutions of the integral equations

$$K_\nu(\xi, \zeta) = kG_\nu(\xi + \zeta) + k^2 \int_\xi^\infty \int_\xi^\infty K_\nu(\xi, s) F_\nu'(s + t) G_\nu(t + \zeta) ds dt, \quad (2.3a)$$

$$K_\nu^*(\xi, \zeta) = kF_\nu(\xi + \zeta) - k^2 \int_\xi^\infty \int_\xi^\infty K_\nu^*(\xi, s) G_\nu'(s + t) F_\nu(t + \zeta) ds dt, \quad (2.3b)$$

with

$$F_\nu(\xi) = D_\nu(\frac{1}{2}\xi) \exp(\xi^2/16), \quad G_\nu(\xi) = D_\nu(\frac{1}{2}\xi) \exp(-\xi^2/16).$$

The following conjecture is discussed by Clarkson and McLeod (1992):

**CONJECTURE 2.2.** *If  $|k| < k_\nu^*$ , for some  $k_\nu^*$ , then  $\eta_k(\xi; \nu)$  exists for all  $\xi$  and as  $\xi \rightarrow -\infty$*

$$\begin{aligned} \eta_k(\xi; \nu) &\sim \kappa_n D_n(\xi), & \text{if } \nu = n \in \mathbb{Z}^+, \\ \eta_k(\xi; \nu) &= (-1)^\mu \left(-\frac{1}{6}\xi\right)^{1/2} + d(-\xi)^{-1/2} \sin \phi(\xi) + O((- \xi)^{-3/2}), & \text{if } \nu \notin \mathbb{Z}^+, \end{aligned} \quad (2.4a)$$

where  $\mu = [\nu + 1]$ , the integer part of  $\nu + 1$ ,

$$\phi(\xi) = \frac{\xi^2}{2\sqrt{3}} - \frac{4d^2}{\sqrt{3}} \ln(-\xi) + c + O(\xi^{-2}), \quad (2.4b)$$

and the constants  $\kappa_n$ ,  $d$ ,  $c$  are dependent on  $k$ . If  $|k| = k_\nu^*$ , then as  $\xi \rightarrow -\infty$

$$\eta_k(\xi; \nu) \sim \operatorname{sgn}(k) \left(-\frac{1}{2}\xi\right)^{1/2},$$

and if  $|k| > k_\nu^*$ , then  $\eta_k(\xi; \nu)$  has a pole at a finite value of  $\xi$ , say  $\xi_0$ , dependent on  $k$ , so

$$\eta_k(\xi; \nu) \sim \operatorname{sgn}(k)(\xi - \xi_0)^{-1/2}, \quad \text{as } \xi \downarrow \xi_0.$$

(We remark that both Clarkson and McLeod (1992) and BCHM made errors in their statements of (2.4).)

Clearly proof of the existence of  $k_\nu^*$  as defined in Conjecture 2.2 above and the determination of its value are important. For the case  $\nu = n$ , with  $n$  an integer, BCHM proved the following Theorem (henceforth whenever we refer to  $\nu = n$  an integer, we mean a positive integer which will be assumed without further comment):

**THEOREM 2.3.** *If  $n$  is an integer then there exists a  $k_n^*$  such that whenever  $k < k_n^*$ ,  $\eta_k(\xi; n)$  exists for all  $\xi$ , whilst for  $k > k_n^*$ ,  $\eta_k(\xi; n)$  blows up at a finite  $\xi$ . Furthermore*

$$(k_n^*)^2 = \frac{1}{2\sqrt{2\pi} n!}. \quad (2.5)$$

This result suggests the following for general  $\nu$ :

**CONJECTURE 2.4.** *There exists  $k_\nu^*$  such that whenever  $k < k_\nu^*$ ,  $\eta_k(\xi; \nu)$  exists for all  $\xi$  and*

$$(k_\nu^*)^2 = \frac{1}{2\sqrt{2\pi} \Gamma(1 + \nu)}. \quad (2.6)$$

Clarkson and McLeod (1992) derived (2.6) heuristically, and in §3 we verify this result numerically for several non-integer values of  $\nu$ . BCHM also proved

**THEOREM 2.5.** *Suppose that  $k_\nu^*$  as defined in Conjecture 2.2 exists. Then, for  $|k| < k_\nu^*$ ,  $\eta_k(\xi; \nu)$  has the same number of zeros as  $D_\nu(\xi)$ .*

In the special case when  $\nu = n$ , with  $n$  an integer, then  $F_\nu(\xi)$  and  $G_\nu(\xi)$  take values which enable integral equations (2.5) to be reduced to simplified forms. These equations may be used in order to derive exact “bound state” solutions of (1.1) for integer  $\nu$ . Clarkson and McLeod (1992) deduced solutions for  $n = 0, 1$  and BCHM reduced the problem of solving the integral equations for general  $n$  to solving two sets of linear algebraic equations. In particular, the following “bound state” solutions were obtained:

$$\begin{aligned} \eta_k(\xi; 0) &= \frac{k \exp(-\frac{1}{4}\xi^2)}{[1 - k^2 I(\xi)]^{1/2}}, \\ \eta_k(\xi; 1) &= \frac{k \exp(-\frac{1}{4}\xi^2) \{ \xi [1 - k^2 I(\xi)] + 2k^2 \exp(-\frac{1}{2}\xi^2) \}}{\phi_1(\xi) [1 - k^2 I(\xi)]^{1/2}}, \\ \eta_k(\xi; 2) &= \frac{k \exp(-\frac{1}{4}\xi^2)}{\phi_2(\xi)} \\ &\quad \times \left\{ \frac{J^2(\xi)(\xi^2 - 1) + 12k^2 \xi J(\xi) \exp(-\frac{1}{2}\xi^2) + 32k^4 \exp(-\xi^2)}{[J^2(\xi) - 4\xi k^2 J(\xi) \exp(-\frac{1}{2}\xi^2) - 16k^4 \exp(-\xi^2)]^{1/2}} \right\}, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} I(\xi) &= 2 \int_{\xi}^{\infty} \exp\left(-\frac{1}{2}s^2\right) ds, \\ J(\xi) &\equiv 1 - 2k^2 I(\xi), \\ \phi_1^2(\xi) &\equiv [1 - k^2 I(\xi)]^2 - 2\xi k^2 [1 - k^2 I(\xi)] \exp(-\frac{1}{2}\xi^2) - 4k^4 \exp(-\xi^2), \\ \phi_2^2(\xi) &\equiv J^3(\xi) - 2k^2 \xi (3 + \xi^2) J^2(\xi) \exp(-\frac{1}{2}\xi^2) \\ &\quad - 8k^4 (2\xi^2 + 3) J(\xi) \exp(-\xi^2) - 32k^6 \xi \exp(-\frac{3}{2}\xi^2). \end{aligned}$$

Also if  $\eta(\xi; \mu)$  is a solution of (1.1) with  $\nu = \mu$  then

$$\eta_\kappa(\xi; \mu + 1) = \frac{\xi \eta_k(\xi; \mu) + 2\eta_k^3(\xi; \mu) - 2\eta'_k(\xi; \mu)}{2[\mu + 1 + 2\eta_k(\xi; \mu)\eta'_k(\xi; \mu) - \xi\eta_k^2(\xi; \mu) - 2\eta_k^4(\xi; \mu)]^{1/2}}, \quad (2.8)$$

is also a solution with  $\nu = \mu + 1$  and  $\kappa = k/\sqrt{\mu + 1}$ . (We remark that Murata (1985) has obtained an analogous Bäcklund transformation to (2.8) for the PIV equation (1.2).) The Bäcklund transformation (2.8) provides a procedure for generating all the bound state solutions of the nonlinear harmonic oscillator (1.1) without the need of solving either the integral equations (2.3) or the associated linear algebraic systems.

### 3. Numerical Investigations for General $\nu$

The principal aim of the current study is to use numerical methods so as to describe characteristics of solutions of (1.1) for several values of the parameter  $\nu$ . Existing theory for this problem has already been summarised in the preceding section where we saw that there is a distinct difference between the behaviours of the solutions depending on whether  $\nu$  is an integer or not (recall conjecture 2.2).

#### 3.1. INTEGER $\nu$

In figures 1a and 1b we have plotted the functions  $\eta_k(\xi; 2)$  defined by solution (2.7). It was postulated in Conjecture 2.2 that if  $\nu = n$  then  $\eta_k(\xi; n) \rightarrow \kappa_n D_n(\xi)$  as  $\xi \rightarrow -\infty$  whenever  $|k| < k_n^*$ , where  $\kappa_n$  is a function of  $k$ . Moreover,  $(k_n^*)^2 = 1/(2\sqrt{2\pi} n!)$  from Theorem 2.3. Given bound state solutions for  $n = 0, 1$  and  $2$  and the Bäcklund transformation (2.8) an elementary inductive proof yields

$$\kappa_n^2 = \frac{k^2}{1 - 2\sqrt{2\pi} n! k^2} \equiv \frac{k^2 (k_n^*)^2}{(k_n^*)^2 - k^2}, \quad (3.1)$$

a result which may also be derived from equations (4.4) of BCHM. This formula breaks down as  $k$  approaches  $k_n^*$  and in this limit  $\eta_k(\xi; n)$  asymptotes to the parabola  $(-\frac{1}{2}\xi)^{1/2}$ . Numerical simulations have verified all the properties proved for  $\nu = n$ . In particular, estimates of  $k_n^*$  were obtained and these were found to be accurate to within  $10^{-8}$  of the analytical values.

#### 3.2. NON-INTEGER $\nu$

Compared with the situation for  $\nu = n$ , relatively few results have been rigorously proved for non-integer  $\nu$  although several conjectures have been made. In particular, we refer to Conjecture 2.4 which asserts that for such  $\nu$  the critical value  $k_\nu^*$  is defined by

$$(k_\nu^*)^2 = \frac{1}{2\sqrt{2\pi} \Gamma(1 + \nu)}. \quad (2.6)$$

Clarkson & McLeod (1992) showed numerically that this formula holds when  $\nu$  takes the half-integer values  $\frac{3}{2}$  and  $\frac{5}{2}$ . In order to provide a more extensive verification of (2.6) we integrated (1.1) for a wide range of non-integer  $\nu$  and for each value of  $\nu$  we found that the value of  $k_\nu^*$  determined numerically differs from the conjectured values (2.6) by less than  $10^{-8}$  (see cases (iii) and (iv) below). Heuristically there appears to be four options for the behaviour of  $\eta_k(\xi; \nu)$  as  $\xi \rightarrow -\infty$  and we now review each of these possibilities.

CASE (i)  $\eta_k(\xi; \nu) \rightarrow 0$  as  $\xi \rightarrow -\infty$ . This occurs only for integer  $\nu$  and  $k < k_\nu^*$  as exemplified in figures 1a and 1b.

CASE (ii)  $\eta_k(\xi; \nu)$  blows up at some finite  $\xi$ . This occurs for  $k > k_\nu^*$  and for all  $\nu > -1$ .

CASE (iii)  $\eta_k(\xi; \nu) \sim \pm(-\frac{1}{2}\xi)^{1/2}$  as  $\xi \rightarrow -\infty$ . This possibility for solution behaviour as  $\xi \rightarrow -\infty$  occurs only in the special situation when  $k = k_\nu^*$ . The bound state solutions (2.12–14) and the Bäcklund transformation (2.17) show that  $\eta_k(\xi; n) \sim \pm(-\frac{1}{2}\xi)^{1/2}$  as  $\xi \rightarrow -\infty$  when  $k$  takes exactly the critical value  $k_n^*$ . A similar outcome is likely for non-integer  $\nu$  and on inspection of figure 2 (which has  $\nu = 1.4$ ) we observe that as  $k$  increases then the closer it becomes to  $k_\nu^*$  the further the solution  $\eta_k(\xi; \nu)$  follows the outer of the two dotted parabolas

(which has the equation  $y = (-\frac{1}{2}\xi)^{1/2}$ ) before diverging from this curve. It then eventually settles down to a stable oscillatory behaviour around the inner parabola which is given by  $y = (-\frac{1}{6}\xi)^{1/2}$ . Analogously to the integer  $\nu$  case we would therefore expect that as  $k \rightarrow k_\nu^*$  so  $\eta_k(\xi; \nu)$  would follow the upper parabola to larger and larger values of  $-\xi$ . Ultimately, when  $k = k_\nu^*$  precisely we would expect that then  $\eta_k(\xi; \nu) \sim (-\frac{1}{2}\xi)^{1/2}$  as  $\xi \rightarrow -\infty$  though a numerical verification of this fact would be almost impossible due to the fact that solutions near this parabola are unstable. We remark here that the parabola  $y = (-\frac{1}{6}\xi)^{1/2}$  is just the first term in an infinite algebraic series which describes the *algebraic* behaviour of the solutions (there is also an infinite series of oscillatory terms in the general expansion of  $\eta_k(\xi; \nu)$  as  $\xi \rightarrow -\infty$ ). We would also point out that there is a similar infinite series of algebraically decaying terms in front our approximation to the parabola  $y = (-\frac{1}{2}\xi)^{1/2}$ .

CASE (iv)  $\eta_k(\xi; \nu) \sim \pm(-\frac{1}{6}\xi)^{1/2}$  as  $\xi \rightarrow -\infty$ . We observe that for all  $0 < k < k_\nu^*$  when  $\nu$  is non-integer and  $\nu > -1$  then  $\eta_k(\xi; \nu)$  approaches this parabola as  $\xi \rightarrow -\infty$ . This is illustrated in figures 3a and 3b which show the solution for values  $k_1, k_2$  and  $k_3$  with  $0 < k_1 < k_2 < k_\nu^* < k_3$  and for  $\nu = -\frac{1}{2}$  and  $\nu = \frac{1}{2}$ . It is seen that  $\eta_k(\xi; \nu)$  has  $[1 + \nu]$  zeros when  $0 < k < k_\nu^*$  but when  $k > k_\nu^*$  this solution blows up at a finite value of  $\xi$  and has no zeros at all between this value and  $\xi = \infty$ . We note that when  $\nu = -\frac{1}{2}$  the right side of (1.1a) factors exactly as  $3\eta(\eta^2 + \frac{1}{6}\xi)(\eta^2 + \frac{1}{2}\xi)$  and so the solution for the critical value  $k = k_\nu^*$  follows this parabola very closely indeed.

We compare the solutions to the linearised equation (dashed line) in figures 4a and 4b, for half integer  $\nu$ . Note the similarity in the solutions before the position is reached beyond which the linearised solution grows without bound (as indicated by properties (2.1)). We remark on the general similarity here with the behaviour in the integer case where, as demonstrated in figure 1, we saw that for values of  $k$  which are a sizable proportion of  $k_n^*$  then nonlinear effects are relatively unimportant and  $\eta_k(\xi; n)$  differs only slightly from  $kD_n(\xi)$ . The critical value  $k_\nu^*$  (given by Conjecture 2.4) appears to hold for all  $\nu$  greater than  $-1$ . When  $\nu < -1$  real valued  $k_\nu^*$  may occur for  $\nu$  in the range  $-(2m+1) < \nu < -2m$ ,  $m \in \mathbb{Z}^+$  (see figure 5). All these solutions for  $k_\nu > 0$  appear to blow up at finite  $\xi$ , as illustrated in figure 6, which shows the solutions for many different values of  $k_\nu$ , including  $k_\nu \approx |k_\nu^*|$ . For  $\nu$  in the range  $-2m < \nu < -2m+1$ ,  $\Gamma(1+\nu) < 0$  and so  $k_\nu^*$  is not real valued. Again the solutions blow up at finite  $\xi$  and exhibit the same type of behaviour as those shown in figure 6.

#### 4. Asymptotic properties of $\eta_k(\xi; \nu)$ as $\xi \rightarrow -\infty$

One property of (1.1) not investigated thus far is the dependencies of the connection formulae  $c(k; \nu)$  and  $d(k; \nu)$  in the asymptotic formula (2.4). These parameters are of importance and we initiated some numerical investigations of this aspect of the problem, since we know of no conjecture or other evidence describing the properties of these functions. We found it to be convenient to regard the parameters  $c$  and  $d$  as functions not of  $k$  and  $\nu$  but, equivalently, of  $\hat{k}_\nu := k/k_\nu^*$  and  $\nu$ . Substitution of the asymptotic behaviours (2.4) as  $\xi \rightarrow -\infty$  into the Bäcklund transformation (2.17) yields, provided  $\nu > -1$ ,

$$d(\hat{k}_\nu; \nu + 1) = d(\hat{k}_\nu; \nu), \quad c(\hat{k}_\nu; \nu + 1) = c(\hat{k}_\nu; \nu) + \frac{1}{3}\pi. \quad (4.1a,b)$$

It follows that  $d(\hat{k}_\nu; \nu)$  is a periodic function of  $\nu$  of period 1. Without loss of generality we may restrict  $c(\hat{k}_\nu; \nu)$  to lie in the range  $0 \leq c < 2\pi$ , so that  $c$  can be regarded as a

periodic function in  $\nu$  of period 6. Thus knowledge of  $d(\hat{k}_\nu; \nu)$  and  $c(\hat{k}_\nu; \nu)$  for  $0 < \nu < 1$  effectively gives us these parameters for all  $\nu > -1$ . Estimates of the connection parameters  $c(\hat{k}_\nu; \nu)$  and  $d(\hat{k}_\nu; \nu)$  were obtained using a numerical integration technique based upon a fourth order Runge-Kutta scheme with matching. Details of the scheme may be found in Bassom, Clarkson and Hicks (1993). Routine numerical consistency checks have led us to believe that our results for  $d(\hat{k}_\nu; \nu)$  are accurate to 8 or 9 decimal places. This function is given in figure 7 for the choices  $\nu = (i) 0.01, (ii) 0.97, (iii) 0.2, (iv) 0.7$  and  $(v) 0.5, 1.5$ . Of course, since  $d(\hat{k}_\nu; \frac{1}{2}) = d(\hat{k}_\nu; \frac{3}{2})$  we are unable to distinguish their graphs. (We remark that the differences in the numerically determined values of  $d$  for  $\nu = \frac{1}{2}$  and  $\nu = \frac{3}{2}$  were less than  $10^{-9}$ ). It is noted that as  $\hat{k}_\nu \rightarrow 1$  (i.e.  $k \rightarrow k_\nu^*$ ) or  $\hat{k}_\nu \rightarrow 0$  then  $d \rightarrow \infty$  and that as  $\nu$  increases towards an integer value so the graphs of  $d(\hat{k}_\nu; \nu)$  appear to become flatter across the majority of  $0 < \hat{k}_\nu < 1$ . In addition, for a fixed value of  $\hat{k}_\nu$ , as  $\nu$  increases from zero so  $d$  decreases until  $\nu$  is approximately  $\frac{1}{2}$  and then  $d$  grows again as  $\nu$  increases further. At this stage one might ask if a solution of  $d(\hat{k}_\nu; \nu)$  for  $0 < \nu < \frac{1}{2}$  will meet a solution of  $d(\hat{k}; \nu)$  for  $\frac{1}{2} < \nu < 1$ . We believe this to be the case. We find that the values of  $d(\hat{k}_\nu; \nu)$  for  $\nu$  and  $1 - \nu$  agree to 9 decimal places for the cases  $\nu = 0.3, \nu = 0.7$  and  $\nu = 0.2, \nu = 0.8$ . Thus we conjecture that

$$d(\hat{k}_\nu; \nu) = d(\hat{k}_\nu; 1 - \nu)$$

for all  $\nu > -1$  and consequently in any subsequent analytic determination of the amplitude  $d(\hat{k}_\nu; \nu)$  attention may be restricted to  $\nu$  in the range  $0 < \nu \leq \frac{1}{2}$ . In addition, on figure 7 it appears that for  $\nu = \frac{1}{2}$  there exists a critical value of  $\hat{k}_\nu$ ,  $\hat{k}_{\nu*}$  at which  $d$  vanishes. We have determined that  $\hat{k}_{\nu*} = 1/\sqrt{2}$  (to 10 decimal places) and at this value of  $\hat{k}_\nu$  we believe  $d(\hat{k}_\nu; \nu)$  to be zero. Thus for  $\nu = \frac{1}{2} + m$  ( $m \in \mathbb{Z}^+ \cup \{-1\}$ ) and  $\hat{k}_{\nu*} = 1/\sqrt{2}$ , we would expect that the solution  $\eta_{k_\nu}(\xi; \nu)$  does not oscillate about the parabola  $(-\xi/6)^{1/2}$  and the solution as  $\xi \rightarrow -\infty$  can then be given in terms of an infinite algebraic series of the form

$$\eta_{k_{\nu*}}(\xi; m + \frac{1}{2}) = (-1)^{1+m} \sum_{j=0}^{\infty} a_j (-\xi)^{1/2-2j},$$

where

$$a_0 = 1/\sqrt{6}, \quad a_1 = -\frac{1}{2}\sqrt{6}(j+1) \quad \text{and} \quad a_2 = \frac{\sqrt{6}}{16} \left[ 5 + 12(j + \frac{1}{2})(j + \frac{3}{2}) \right],$$

and the remaining coefficients  $a_j$  can be determined by a non-trivial nonlinear recurrence relation. We have plotted  $\eta_{k_\nu}(\xi; \frac{1}{2})$  in figure 8 for  $\hat{k}_\nu = 1/\sqrt{2} \pm \epsilon$  where  $\epsilon$  is taken to be zero, or typically, 10 percent of  $\hat{k}_\nu^*$ . For the non-zero values of  $\epsilon$  we see oscillations in the solution (the dashed lines) but none for  $\epsilon = 0$ . (We remark here that we have determined the asymptotic expansion of  $\eta_{k_\nu}(\xi; \nu)$  for general  $\nu$  for terms up to and including  $O((-\xi)^{-7/2})$  and we also have found the  $O((-\xi)^2)$  term in the  $\phi$  expansion (2.4b).)

Examination of the numerical solutions of  $d$  show that for  $0 < \nu < 1$ ,  $d$  attains its (positive) minimum value at  $\hat{k}_\nu = 1/\sqrt{2}$ . Thus when  $d$  attains its minimum value we would expect that the oscillations of the solution  $\eta_{k_\nu}(\xi; \nu)$  to have a minimum amplitude. This particular feature is illustrated in figures 9 and 10 with  $\hat{k}_\nu = 1/\sqrt{2} \pm \epsilon$  for the cases  $\nu = 0.7, \epsilon = 0.1$  and  $\nu = 0.97, \epsilon = 0.2$ .

We have also determined the function  $c(\hat{k}_\nu; \nu)$  and we believe that these solutions ( $\text{mod}(2\pi)$ ) are accurate to within *at least*  $5 \times 10^{-5}$ . In figure 11 we have plotted  $c$  for the cases  $\nu = (i) 0.01, (ii) 0.97, (iii) 0.2, (iv) 0.7, (v) 0.5$  and  $(vi) 1.5$ . Unlike the case for the amplitude  $d(\hat{k}_\nu; \nu)$ , we can see no similarity between the solutions  $c(\hat{k}_\nu; \nu)$  and  $c(\hat{k}_\nu; 1 - \nu)$ . Plots (v) and (vi) show that the recurrence formula (4.1b) holds and the numerically determined values of  $c$  for  $\nu = \frac{1}{2}$  and  $\nu = \frac{3}{2}$  differed by  $\pi/3 + \delta$  with  $|\delta| < 10^{-7}$ . We also observe that in these cases the function is discontinuous at one point. This discontinuity in  $c(\hat{k}_\nu; \nu)$  occurs for half integer  $\nu > -1$  and for values of  $\hat{k}_\nu = 1/\sqrt{2} = \hat{k}_{\nu*}$  precisely when we believe  $d(\hat{k}_\nu; \nu)$  to be zero. At this value of  $\hat{k}_{\nu*}$ ,  $c(\hat{k}_\nu; \nu)$  is not defined. Extensive numerical checks have confirmed the existence of this jump across the critical value of  $\hat{k}_{\nu*}$ , and tests indicate that  $c(\hat{k}_\nu; \nu)$  jumps by the value  $\pi$ . We believe that this shift is a direct result of our imposing that the connection parameter  $d$  in (2.4) be non-negative. It is clear that we can adjust the phase  $c$  if necessary in order to ensure that  $d \geq 0$  but this has the consequence that if anywhere in parameter space we predict  $d = 0$  then in the vicinity of this point jumps in  $c$  can be anticipated. It is most likely that this behaviour will be confirmed or refuted as a by-product of a rigorous analytical derivation of the dependences  $c(\hat{k}_\nu; \nu)$  and  $d(\hat{k}_\nu; \nu)$ ; work which is currently in progress.

## 5. Discussion

In this study we have performed a careful numerical investigation of (1.1) which is a special case of PIV. In particular, we have contrasted the solution properties when the parameter  $\nu$  is an integer with those when  $\nu$  is non-integer. Little has been rigorously proved when  $\nu \notin \mathbb{Z}^+$  and the known results are summarised by Conjectures 2.2 and 2.4 which speculate upon the existence of the critical value  $k_\nu^*$  and on its precise value (as given by (2.6)). The numerical procedures of §3 have verified that  $(k_\nu^*)^2 = 1/[2\sqrt{2\pi}\Gamma(1+\nu)]$  and the solution characteristics for the separate cases  $k < k_\nu^*$  and  $k > k_\nu^*$  are typified by the results presented in figures 2–4. Furthermore, figures 5 and 6 demonstrate that  $\eta_k(\xi; \nu)$  blows up for all non-zero  $k$  whenever  $\nu < -1$ .

The results summarised by figures 7, 8–10 and 11 together with the numerical and analytical results in §4 describe properties of the connection parameters  $d(\hat{k}_\nu; \nu)$  and  $c(\hat{k}_\nu; \nu)$ . These are useful additions to the existing knowledge relating to  $\eta_k(\xi; \nu)$  and should prove valuable in the verification of any subsequent analytical predictions for  $c$  and  $d$ .

## Acknowledgements

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Figure (1a). Functions  $\eta_k(\xi; 2)$  (as given by (2.7)) and  $kD_2(\xi)$  (dashed line) for  $k = 0.2$ .

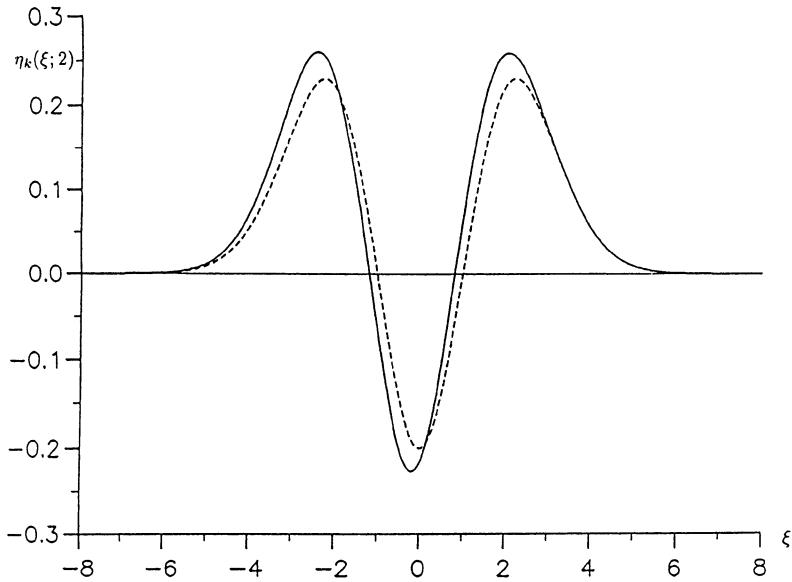


Figure (1b). Function  $\eta_k(\xi; 2)$  as defined in equation (2.7) in the cases (i),  $k = 0.3156$ , (ii),  $k = 0.3158$  and (iii),  $k = 0.31581$ . [ $k_\nu^* = 0.315809388873\dots$  ].

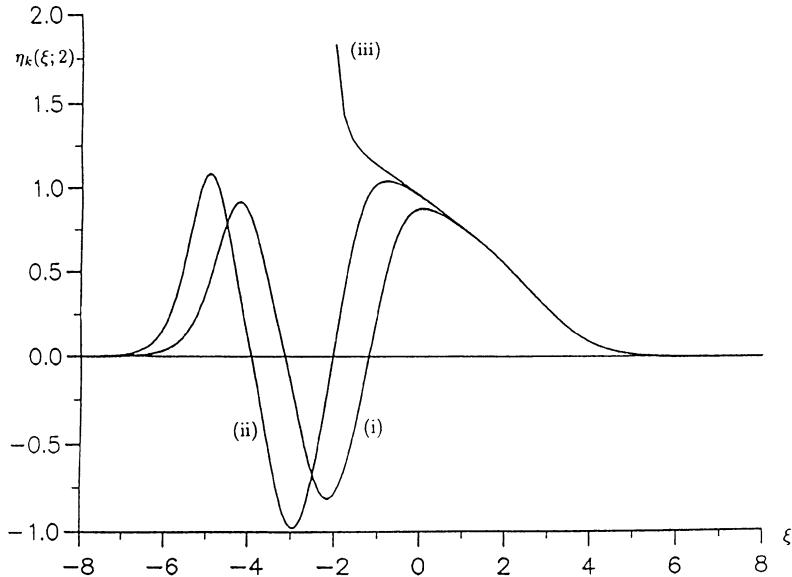


Figure (2). Plots of  $\eta_k(\xi; \nu)$  for  $\nu = 1.4$  with (i)  $k_1 = 0.40072$ , (ii)  $k_2 = 0.40072794$ , (iii)  $k_3 = 0.40072795$ . [  $k_\nu^* = 0.400727947088\dots$  ]

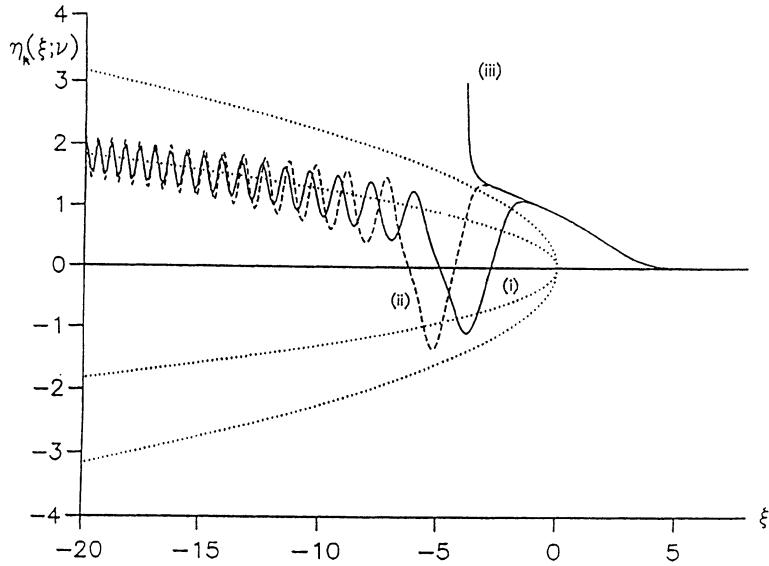


Figure (3a). Plots of  $\eta_k(\xi; \nu)$  for  $\nu = -0.5$  with (i)  $k_1 = 0.33546$ , (ii)  $k_2 = 0.33546913$ , (iii)  $k_3 = 0.33546914$ . [  $k_\nu^* = 0.335469133482\dots$  ]

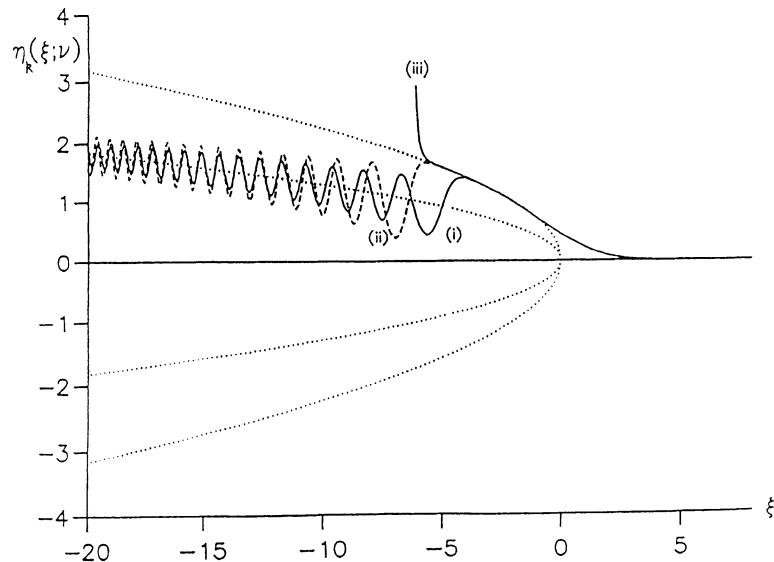


Figure (3b). Plots of  $\eta_k(\xi; \nu)$  for  $\nu = 0.5$  with (i)  $k_1 = 0.47442$ , (ii)  $k_2 = 0.47442499$ ,  
 (iii)  $k_3 = 0.474425$ . [  $k_\nu^* = 0.474424998328\dots$  ].

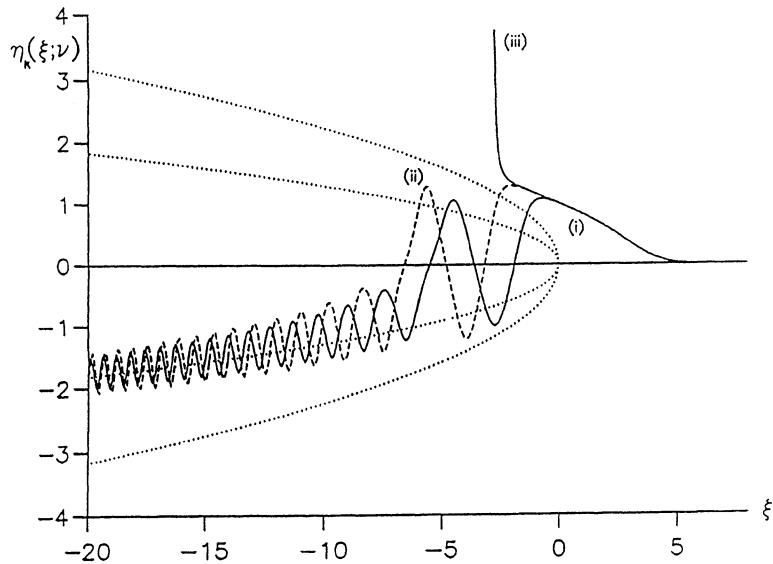


Figure (4a).  $\nu = 0.5$  : Plots of (i)  $kD_\nu(\xi)$  with  $k = 0.4$ , (ii)  $\eta_k(\xi; \nu)$  with  $k = 0.4$  and  
 (iii)  $\eta_k(\xi; \nu)$  with  $k = 0.4745$ . [  $k_\nu^* = 0.474424998328\dots$  ].

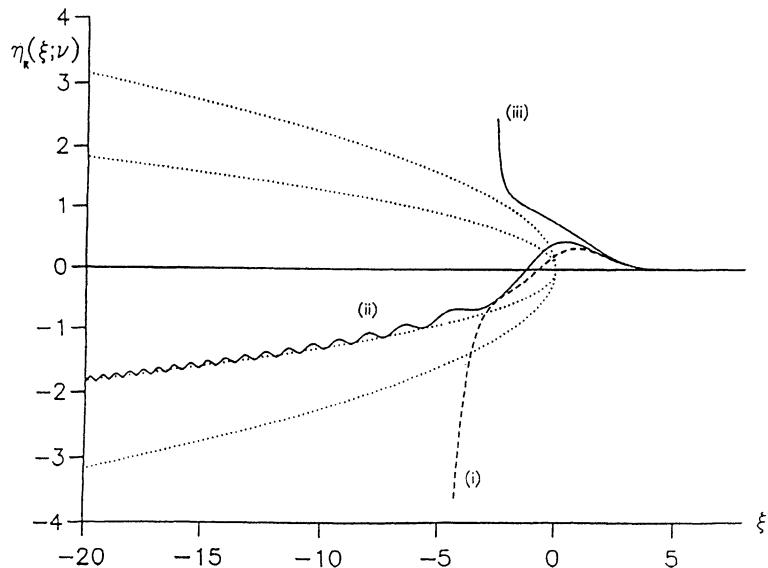


Figure (4b).  $\nu = 1.5$  : Plots of (i)  $kD_\nu(\xi)$  with  $k = 0.37$ , (ii)  $\eta_k(\xi; \nu)$  with  $k = 0.37$  and (iii)  $\eta_k(\xi; \nu)$  with  $k = 0.38736639$ . [  $k_\nu^* = 0.387366389042\dots$  ].

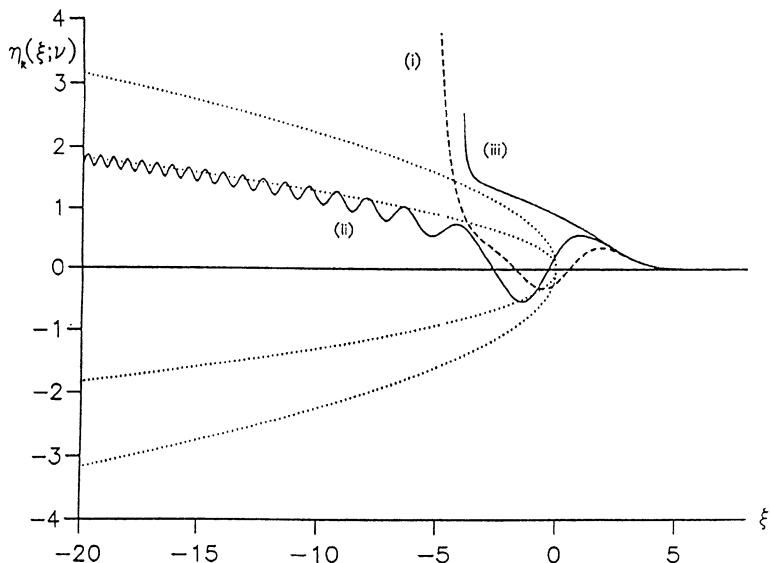


Figure (5). Function  $k_\nu^*$  as given by Conjecture 2.4.

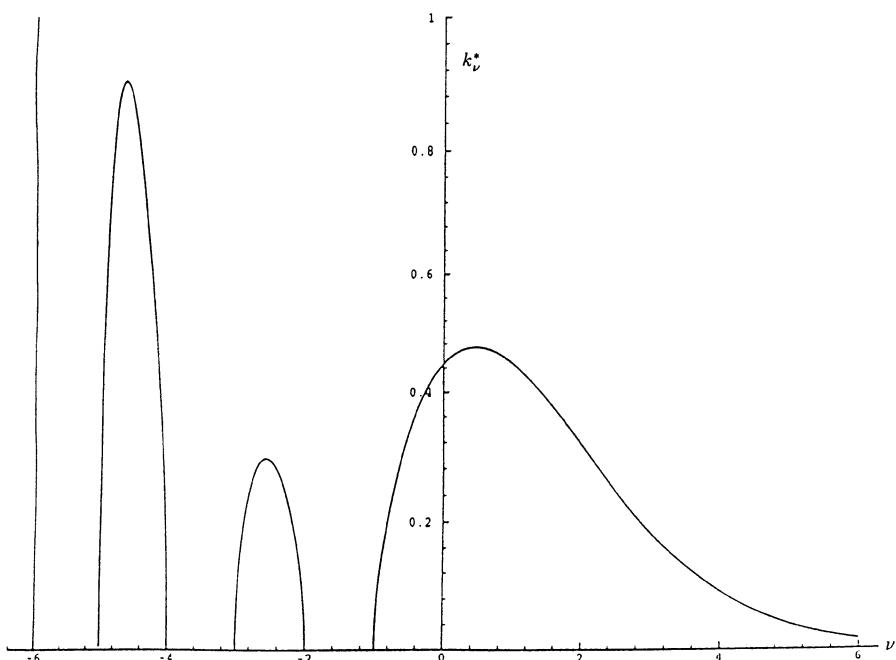


Figure (6).  $\nu = -2.5$  : Functions (i)  $kD_\nu(\xi)$  with  $k = 0.2$  and  $\eta_k(\xi; \nu)$  with (ii)  $k = 0.290524 \approx k^*$ , (iii)  $k = 0.2$ , (iv)  $k = 0.1$ , (v)  $k = 0.01$  and (vi)  $k = 0.001$ .

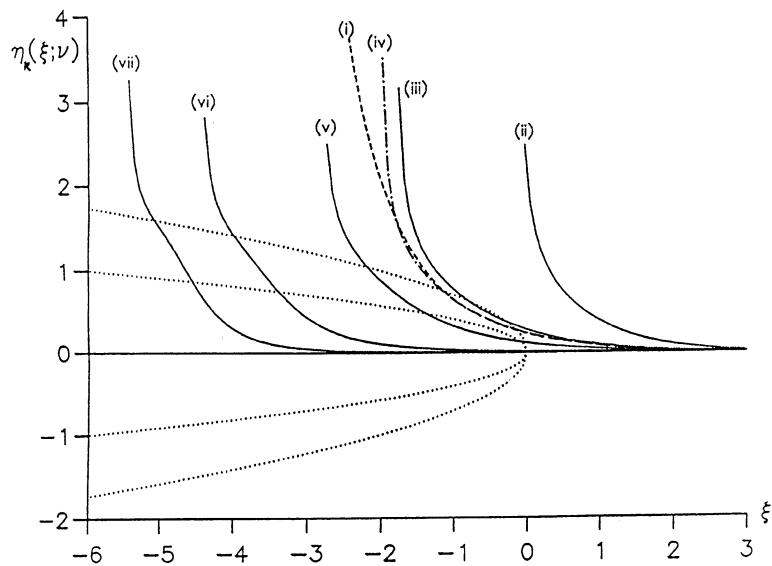


Figure (7). Dependence of  $d(k_\nu; \nu)$  upon  $k_\nu/k_\nu^*$  for (i)  $\nu = 0.01$ , (ii)  $\nu = 0.97$ , (iii)  $\nu = 0.2$ , (iv)  $\nu = 0.7$  and (v)  $\nu = 0.5, 1.5$ .

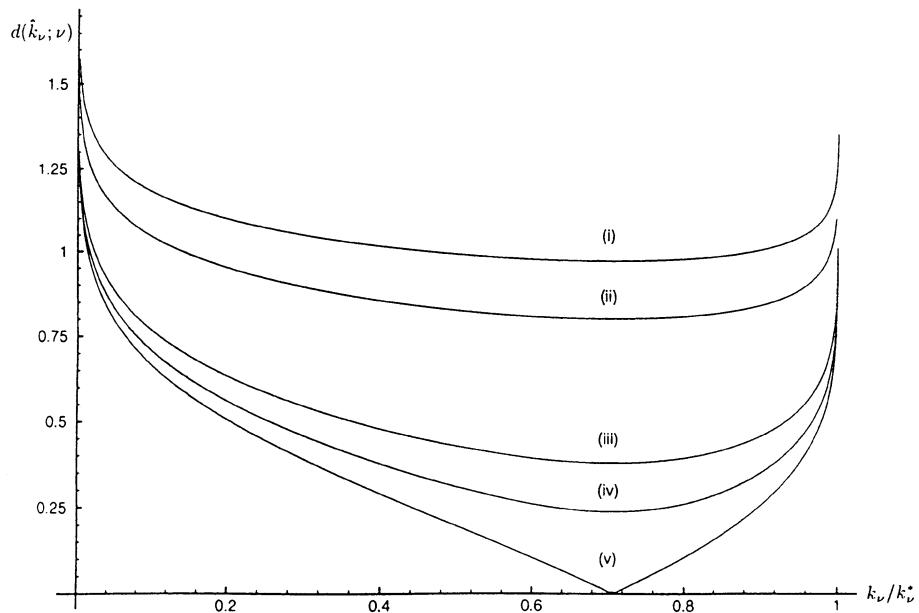


Figure (8). Functions  $\eta_{k_\nu}(\xi; \frac{1}{2})$  with  $\hat{k}_\nu = 1/\sqrt{2} \pm \epsilon$  for  $\epsilon = 0$  (no oscillation) and  $\epsilon = 0.1$  (dashed lines).

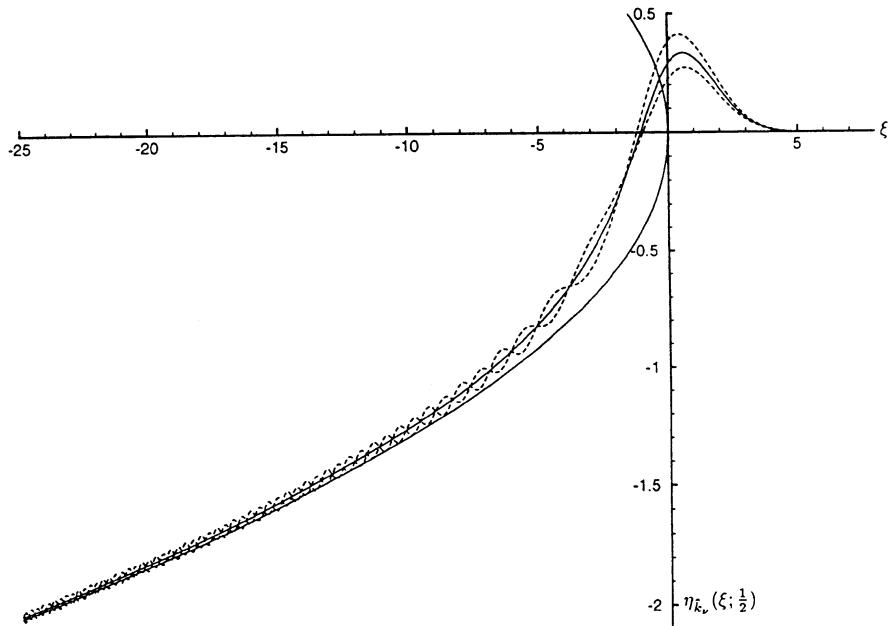


Figure (9). Functions  $\eta_{k_\nu}(\xi; 0.7)$  with  $\hat{k}_\nu = 1/\sqrt{2} \pm \epsilon$  for  $\epsilon = 0$  (minimum amplitude) and  $\epsilon = 0.1$  (dashed lines).

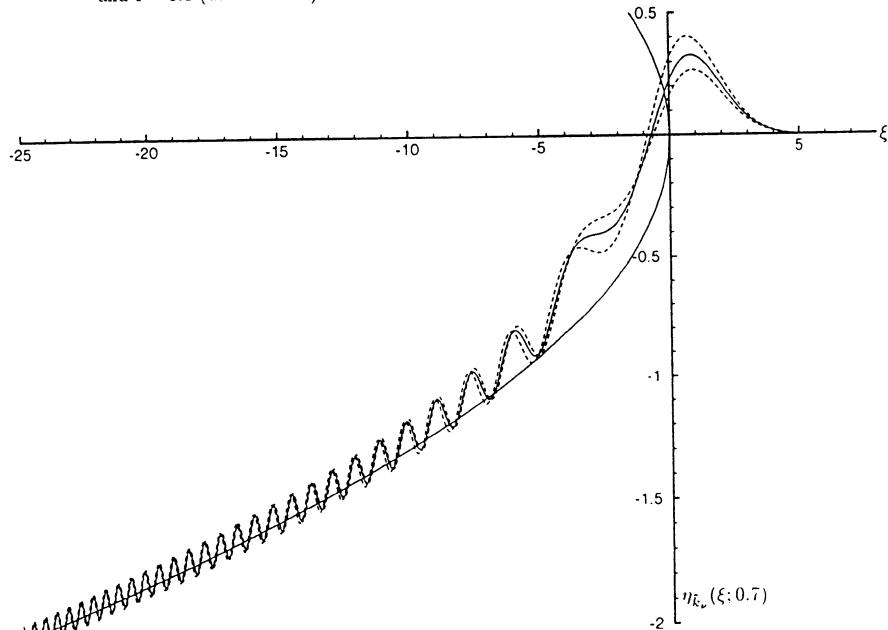


Figure (10). Functions  $\eta_{\hat{k}_\nu}(\xi; 0.97)$  with  $\hat{k}_\nu = 1/\sqrt{2} \pm \epsilon$  for  $\epsilon = 0$  (minimum amplitude) and  $\epsilon = 0.2$  (dashed lines).

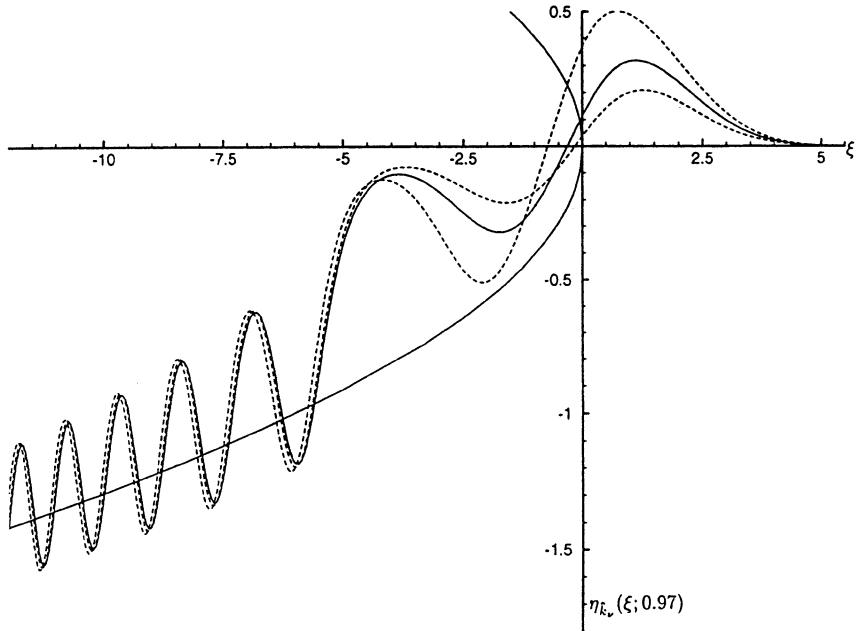
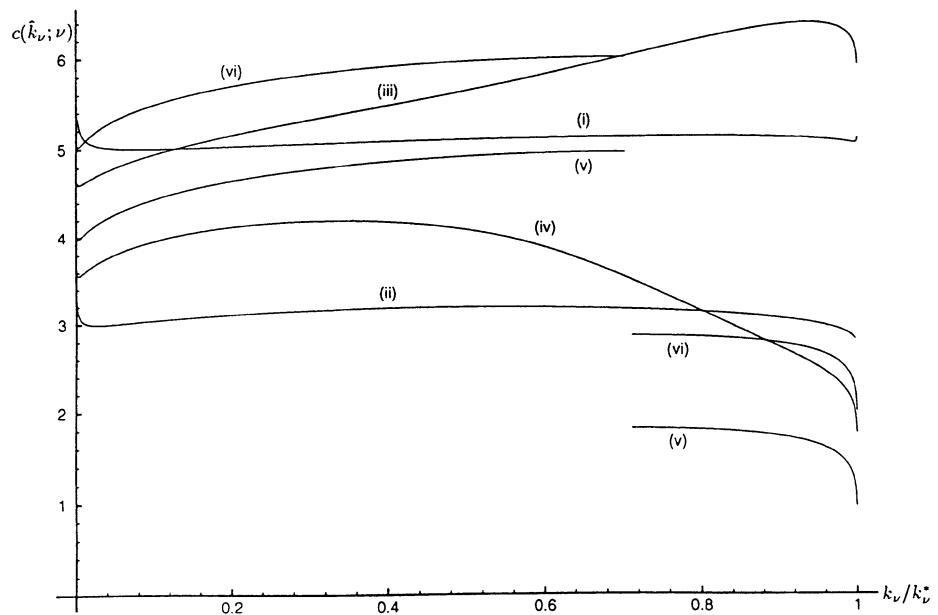


Figure (11). Dependence of  $c(\hat{k}_\nu; \nu)$  upon  $k_\nu/k_\nu^*$  for (i)  $\nu = 0.01$ , (ii)  $\nu = 0.97$ , (iii)  $\nu = 0.2$ , (iv)  $\nu = 0.7$ , (v)  $\nu = 0.5$  and (vi)  $\nu = 1.5$ .



# A LOCAL ASYMPTOTIC METHOD OF SEEING THE NATURAL BARRIER OF THE SOLUTIONS OF THE CHAZY EQUATION

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**ABSTRACT.** The Chazy equation is a third-order nonlinear ordinary differential equation whose solutions are known to have a movable natural barrier, i.e. a closed curve on the complex sphere whose location depends on initial conditions and through which a solution (defined in the interior or exterior of the region it encloses) cannot be analytically continued. This is usually shown by invoking the Chazy equation's well known relationship to the Hypergeometric equation. Here we describe a new (local asymptotic) method which exhibits the natural barrier directly without appealing to the latter. Furthermore, we show that the Chazy equation has a three-dimensional nonsolvable Lie group of symmetries which may be used to reduce its order by two.

## 1. Introduction

The Chazy equation

$$y''' = 2yy'' - 3(y')^2, \quad (1)$$

where  $y$  is a function of  $z$  and the primes denote differentiation with respect to  $z$ , has been the focus of much recent interest due to its derivation as a reduction (an equation governing some subset of solutions) of the self-dual Yang-Mills equations [1]. The latter are completely integrable partial differential equations (PDEs), that is, they can be exactly solved through an associated system of linear equations (see Ward [2]). According to the conjecture of Ablowitz, Ramani, and Segur (ARS) [3] the Chazy equation must therefore possess the Painlevé property, that is, all movable singularities of all solutions must be poles (perhaps after a change of variables). However, the general (three-parameter) solution of the Chazy equation is known to have a movable natural barrier with no other singularities

whatsoever; there is also a (two-parameter) movable double-pole solution with no barrier. The only method known previously for exhibiting the barrier is to use the transformation [4]

$$z = \frac{u_1(t)}{u(t)}, \quad (2a)$$

$$y = \frac{6}{u(t)} \frac{du}{dt}, \quad (2b)$$

where  $u_1, u$  are two solutions of the Hypergeometric equation

$$t(t-1) \frac{d^2u}{dt^2} + \left(\frac{1}{2} - \frac{7}{6}t\right) \frac{du}{dt} - \frac{u}{144} = 0. \quad (3)$$

Obviously, transformation to a linear equation is not a route applicable to all nonlinear ODEs of interest. Indeed the power of the ARS conjecture, and its popular use as a test for integrability, lie in the algorithmic procedures known for finding movable poles or algebraic (or logarithmic) branched singularities of solutions directly from a differential equation, without first having to solve (or linearize) it. Thus the natural question arises: what is a direct method of seeing the natural barrier of a given ODE, if it admits one? The purpose of this paper is to give an asymptotic method for seeing the natural barrier of the general solution of the Chazy equation directly from the equation. Our underlying motivation is to provide a direct method for testing other ODEs.

Chazy [4] studied his eponymous Equation (1) as part of his attempt to generalize the work of Painlevé [5], Gambier [6], and R. Fuchs [7] on the classification of ODEs with the Painlevé property to third-order equations. He pointed out that the general solution of Equation (1) is analytic wherever it is defined, that is, on one side of a barrier which is movable. He also showed that it has a special solution

$$y(z) = \frac{A}{(z - z_0)^2} - \frac{6}{(z - z_0)}, \quad (4)$$

where  $A$  and  $z_0$  are arbitrary. By virtue of its finite (convergent) expression it is clear that the double pole solution (4) is defined everywhere in the complex  $z$ -plane punctured at  $z_0$ , and therefore has no barrier. The general solution, in contrast, is defined through the composition of the inverse function  $t(z)$  of (2a) with the transformation (2b). Following the work of Halphen, Chazy pointed out that the function  $z(t)$  is a Schwarz triangle function which maps the upper-half  $t$ -plane with punctures at 0, 1, and  $\infty$ , to a circular triangle with angles  $\pi/2$ ,  $\pi/3$ , and 0 in the  $z$ -plane. The continuation of the solutions  $u, u_1$  in the  $t$ -plane through one of the intervals  $(0, 1), (1, \infty)$ , or  $(-\infty, 0)$ , leads to successively smaller reflections through each side of the circular triangle in the  $z$ -plane which “tessellate” the interior of a circle or one side of a straight line. As a result, the circle or the line forms a curve of essential singularities which is a barrier for  $y(z)$ . He also pointed out (see p. 345 of [4]) that (1) (actually a more general version of (1)) is unchanged by the transformation

$$z \mapsto \frac{az + b}{cz + d}, \quad y \mapsto \frac{ad - bc}{(cz + d)^2} y - \frac{6c}{cz + d}, \quad (5a)$$

where  $a, b, c$ , and  $d$  are arbitrary constants. Hence from a particular solution  $y = f(z)$ , we can construct a more general solution

$$y = \frac{ad - bc}{(cz + d)^2} f\left(\frac{az + b}{cz + d}\right) - \frac{6c}{cz + d}. \quad (5b)$$

More recently, Chakravarty, Ablowitz, and Clarkson [1] showed that (1) arises as a similarity reduction of the self-dual Yang-Mills equations. Knowing its relationship to the Hypergeometric equation, Ablowitz and Clarkson [8] have derived various properties of the general solution. In particular, they rediscovered the transformation (5b) and termed it a Bäcklund transformation. We show in Section 3 that this arises naturally from a Lie-point analysis of the Chazy equation as a third symmetry in addition to the two well known symmetries (invariant changes of variables) given by translation of  $z$  and scaling of  $y$  and  $z$ . Although Chazy gave the transformation (5) in 1910 it is not widely known that it is in fact a third symmetry. Even so, the third-order Chazy equation cannot be reduced to quadratures because the generators of these symmetries do not form a solvable Lie group (see [9] or [10]). At best, it can be reduced to a first-order (nonautonomous, nonlinear) ODE. We believe that this reduction process must lead inevitably to the Hypergeometric equation (3). Full results will be communicated in a forthcoming paper.

The commonly used technique for exhibiting the movable singularities of solutions is to find all formal series expansions around an arbitrary point  $z_0$  (excepting fixed singular points of the equation) in the form of a series of powers of  $(z - z_0)$  starting with a leading term, logarithmic factors also being allowed. Usually one substitutes a Laurent-type series with leading term of the form

$$y(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^{n+\rho}, \quad (6)$$

or a more general series with powers of  $\log(z - z_0)$  and/or iterated logarithms, into the equation in the limit as  $y$ , or its derivatives, approach a value for which the equation is singular. The leading order power  $\rho$  is determined by a dominant balance of the equation (a limiting form of the equation in which the terms of largest size remain); often there is more than one balance, and therefore more than one value of  $\rho$ , possible. Applying this procedure to the Chazy equation, the maximal (three-term) dominant balance yields  $\rho = -1$ . The result is easily shown to be just the solution (4) with  $A = 0$ . To find more free parameters, we may search for powers with free coefficients in an expansion of the type (6) by substituting

$$y(z) = \frac{-6}{(z - z_0)} + \beta(z - z_0)^{-1+r}, \quad (7)$$

into the ODE and demanding that  $\beta$  be free. Values of  $r$  for which this holds are called resonances. For the Chazy equation substitution of (7) gives

$$\beta(r+1)(r+2)(r+3) = 0.$$

For  $\beta$  to be free, we must therefore have  $r = -1$ ,  $r = -2$ , or  $r = -3$ . The case  $r = -1$  gives precisely the solution (4) with  $A = \beta$ . The case  $r = -2$  can then be explained as the freedom of  $z_0$ . However, the case  $r = -3$  has no explanation, as its presence generates an infinite series of negative powers of  $(z - z_0)$  which ordinarily only converges in a neighborhood of  $\infty$ . Such negative resonances have not been completely explained in the literature although Fordy and Pickering [11] have given criteria (based on analogy with Fröbenius analysis) under which negative resonances are compatible with possession of the Painlevé property.

Since the presence of the negative resonances above indicates that the algebraic (and/or logarithmic) series (6) fails to describe all possible solutions near the singular point  $z_0$ , we

regard it as (the nonlinear analogue of) an irregular singular point. There is a universal ansatz used for the representation of solutions near such a point (see Bender and Orszag [12]), i.e. the exponential (or WKB) representation  $\exp(S)$ . We therefore take

$$y(z) = \frac{A}{(z - z_0)^2} - \frac{6}{(z - z_0)} + \exp(S(z)) \quad (8)$$

where  $\exp S$  is to be regarded as small (perhaps in a restricted region) near  $z_0$ . That is, we perturb the (exact) double-pole solution in the space of all solutions of the Chazy equation (in a restricted region) by an exponential. We show in Section 2 that this perturbation generates an asymptotic series involving powers of  $\exp(-2A/z)$  valid in a half-plane. In fact, this region of asymptotic validity can be extended. We show that the boundary of the extended region is a circle of small radius whose circumference goes through the point  $z_0$ . Hence, the effect of the perturbation (7) is to open out the isolated pole of the unperturbed solution into a small circular barrier through which it cannot be continued.

A similar series involving exponentials was derived by Ablowitz and Clarkson [8] from the transformation (2) of the Hypergeometric equation. However, the advantage of our approach is that no information about this transformation is required. Hence it is a general method that may be extended to other equations with more than one negative resonance.

The plan of the paper is as follows. We exhibit the method of carrying out the exponential expansion in Section 2. The boundary of its region of validity is also derived here. In Section 3, we find the three-parameter Lie group of symmetries of the Chazy equation, and show how to reduce it to an Abel equation.

## 2. The Exponential Expansion

In this section, we develop an asymptotic representation of a three-parameter solution of the Chazy equation in a restricted region near a nonisolated essential singularity  $z_0$ . By deducing the boundary of validity of this expansion, we show how the natural barrier of this solution may be seen directly.

Consider the perturbation (8). Since the Chazy equation is autonomous, we take (without loss of generality)  $z_0 = 0$ . For simplicity, we also take  $A = \frac{1}{2}$ . Then substitution of (8) into the Chazy equation gives

$$\begin{aligned} S'^3 + 3S'S'' + S''' &= \left(\frac{1}{z^2} - \frac{12}{z}\right) (S'^2 + S'') + 6\left(\frac{1}{z^3} - \frac{6}{z^2}\right) S' \\ &\quad + 6\left(\frac{1}{z^4} - \frac{4}{z^3}\right) + 2[(S')^2 + S''] e^S - 3S'^2 e^S. \end{aligned} \quad (9)$$

As is usual in irregular singular point theory, we assume that  $S'' \ll (S')^2$ ,  $S''' \ll (S')^3$ . Also it should be kept in mind that  $\exp(S) \ll 1$  with  $S' \gg 1$ , while  $z \ll 1$ . To the first few orders, (9) gives

$$(S')^3 \approx \frac{(S')^2}{z^2} - 3S'S'' - \frac{12}{z}(S')^2 + \frac{S''}{z^2} + \frac{6}{z^3} S',$$

which implies that

$$S' \approx \frac{1}{z^2} - \frac{2}{z},$$

and hence, after integrating,

$$S \approx -\frac{1}{z} - 2 \log z + K, \quad (10)$$

where the constant  $K$  is the third free parameter we sought to fully describe the general solution of the Chazy equation. The perturbation (8) then becomes

$$y(z) \approx \frac{1}{2z^2} - \frac{6}{z} + \frac{k}{z^2} \exp(-1/z) \quad (11)$$

where  $k = \exp(K)$ . Because of the terms involving  $\exp(S)$  on the right side of (9), the corrections to the leading order perturbation (11) involve a double series consisting of powers of  $\exp(-1/z)$  and algebraic terms in  $z$ . Carrying out the asymptotic analysis of (9) to the next exponential order, i.e.  $\exp(2S)$ , we get

$$\begin{aligned} y(z) &= \frac{1}{2z^2} - \frac{6}{z} + \frac{k}{z^2} \exp(-1/z) [1 + O(z)] \\ &\quad + \frac{k^2}{8z^2} \exp(-2/z) [1 + O(z)] + O\left(\frac{\exp(-3/z)}{z^2}\right). \end{aligned} \quad (12)$$

Note that each successive power of  $\exp(-1/z)$  appears to be multiplied by the same power in  $z$ , i.e.  $1/z^2$ . To confirm this to all orders we make the transformation

$$y(z) = \frac{Q(s)}{z^2} - \frac{6}{z}, \quad (13)$$

where  $s = \exp(-1/z)$ . Then the Chazy equation becomes

$$s^3 Q_{sss} = s^2(2Q - 3)Q_{ss} + s(2Q - 1)Q_s - 3s^2 Q_s^2, \quad (14)$$

where subscripts denote differentiation with respect to  $s$ . Since  $Q$  satisfies an equation that depends purely on  $s$ , we have shown that every exponential term in the perturbation series (12) is indeed multiplied only by  $1/z^2$ . Note that  $Q = A$  is a solution of (14) which is consistent with the special solution (4).

As an aside, we note here that (14) is transformed exactly to the Chazy equation under the substitution  $s = \exp(t)$ . That is (putting the parameters  $A$  and  $z_0$  back in), the mapping

$$y(z) = \frac{2A}{(z - z_0)^2} Q(t) - \frac{6}{z - z_0}, \quad t = -\frac{2A}{z - z_0}, \quad (15)$$

leaves the Chazy equation invariant. This is just a special case of the map (5b) and can be obtained through the symmetry analysis that we present in Section 3.

Now consider the region of validity of the perturbation series given to the first few orders by (12). Clearly, the exponentials remain small, as required, in a neighbourhood of the origin in (the interior of) the half-plane  $\Re(z) > 0$ . We assume that  $|k| \ll 1$  in order to extend this neighbourhood towards infinity. Then the asymptotic series (12) remains valid wherever

$$|k \exp(-1/z)| \ll 1, \quad (16)$$

To investigate where it breaks down asymptotically, we write  $z = -\xi + i\eta$  where  $\xi > 0$ . Then we have

$$-\frac{1}{z} = \frac{\xi}{\xi^2 + \eta^2} + i\frac{\eta}{\xi^2 + \eta^2}, \quad (17)$$

and so the requirement (16) becomes in terms of  $\xi, \eta$

$$\frac{\xi}{\xi^2 + \eta^2} \ll \log\left(\frac{1}{|k|}\right). \quad (18)$$

If we let

$$\delta = -1/(2\log|k|), \quad (19)$$

( $\delta > 0$  because  $|k| \ll 1$ ) and complete the square in (18) (after multiplication by  $\xi^2 + \eta^2$ ) we get

$$(\xi - \delta)^2 + \eta^2 \gg \delta^2. \quad (20)$$

That is, the region of validity of the series (12) lies outside the circle of radius  $\delta$  centred at  $-\delta$  (on the negative  $z$ -axis). In other words, the asymptotic validity of the perturbation series (12) breaks down on the circle.

This (small) circle is nothing other than the natural barrier of the general solution that we have asymptotically represented by the series (12).

### 3. The Symmetries of the Chazy Equation

The purpose of this section is to show that the Chazy equation has a three-dimensional Lie group of symmetries (changes of variables under which it is invariant), which is however, not solvable. We deduce here the first-order reduction of the Chazy equation under a two-dimensional subgroup and show that it can be transformed to an Abel equation [13].

To find the symmetries and the corresponding reductions of the Chazy equation, we first convert it to a system of three first-order equations:

$$u' = v, \quad (21a)$$

$$v' = w, \quad (21b)$$

$$w' = 2uw - 3v^2, \quad (21c)$$

where  $u = y$ ,  $v = y'$ , and  $w = y''$ . According to the standard theory, (see [9], [10]), a one-parameter ( $\epsilon$ ) group of (point) symmetries can be found by expanding near the identity ( $\epsilon = 0$ ). Hence we assume that an invariant transformation of (21) is given by

$$Z = z + \epsilon\zeta(z, u), \quad (22a)$$

$$U = u + \epsilon v(z, u), \quad (22b)$$

$$V = v + \epsilon\phi(z, u, v), \quad (22c)$$

$$W = w + \epsilon\psi(z, u, v, w). \quad (22d)$$

The assumption of the functional dependence of  $\zeta$  and  $v$  on  $z$ ,  $u$  is standard for a so called point symmetry. We have also assumed that  $\phi$  and  $\psi$  depend respectively on the first

and the first and second derivatives of  $u$  as an alternative to calculating the prolongations (the transformations of the derivatives) after deriving the symmetry. The application of the transformations (22) to (21) and the assumption of invariance lead to the *determining equations*

$$\phi = v_z + vv_u - v(\zeta_x + v\zeta_u), \quad (23a)$$

$$\psi = \phi_z + v\phi_u + w\phi_v - w(\zeta_x + v\zeta_u), \quad (23b)$$

$$2\psi u + 2vw - 6\phi v = \psi_x + v\psi_u + w\psi_v + (2uw - 3v^2)(\psi_w - \zeta_x - v\zeta_u). \quad (23c)$$

These (overdetermined) equations can be solved in the usual way (by assuming that  $u$ ,  $v$ ,  $w$ ,  $z$  are independent variables in a space extended from the solution space of (21)). The solutions are

$$\zeta = k_1 z^2 + k_2 z + k_3, \quad (24a)$$

$$v = -(2k_1 z + k_2)u - 6k_1, \quad (24b)$$

$$\phi = -2k_1 u - 2(2k_1 z + k_2)v, \quad (24c)$$

$$\psi = -6k_1 v - 3(2k_1 z + k_2)w, \quad (24d)$$

where  $k_1$ ,  $k_2$ ,  $k_3$  are arbitrary constants.

Thus, there are three independent symmetries which are obtained by taking pairs of  $k_1$ ,  $k_2$ ,  $k_3$ , to be zero in turn. Normalizing the remaining nonzero constant in each case to be unity, we get for  $k_1 = 0$ ,  $k_2 = 0$ :

$$Z = z + \epsilon + O(\epsilon^2), \quad (25a)$$

$$U = u + O(\epsilon^2), \quad (25b)$$

$$V = v + O(\epsilon^2), \quad (25c)$$

$$W = w + O(\epsilon^2), \quad (25d)$$

for  $k_1 = 0$ ,  $k_3 = 0$ :

$$Z = z + \epsilon z + O(\epsilon^2), \quad (26a)$$

$$U = u - \epsilon u + O(\epsilon^2), \quad (26b)$$

$$V = v - 2\epsilon v + O(\epsilon^2), \quad (26c)$$

$$W = w - 3\epsilon w + O(\epsilon^2), \quad (26d)$$

and for  $k_2 = 0$ ,  $k_3 = 0$ :

$$Z = z + \epsilon z^2 + O(\epsilon^2), \quad (27a)$$

$$U = u - 2\epsilon(3 + zu) + O(\epsilon^2), \quad (27b)$$

$$V = v - 2\epsilon(u + 2zv) + O(\epsilon^2), \quad (27c)$$

$$W = w - 6\epsilon(v + zw) + O(\epsilon^2). \quad (27d)$$

An alternative description of the symmetries is given by the vector fields that generate them. For the symmetries above these are respectively

$$T = \frac{\partial}{\partial z}, \quad (28a)$$

$$S = z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v} - 3w \frac{\partial}{\partial w}, \quad (28b)$$

$$R = z^2 \frac{\partial}{\partial z} - 2(3 + zu) \frac{\partial}{\partial u} - 2(u + 2zv) \frac{\partial}{\partial v} - 6(v + zw) \frac{\partial}{\partial w}. \quad (28c)$$

Here  $T$  and  $S$  represent respectively the translation and scaling invariances of the Chazy equation. The invariance represented by  $R$  is not as transparent. Integration of the characteristic equations of  $R$  shows, in fact, that it represents the invariance given by (5b).

These vector fields satisfy the commutation relations

$$[T, S] = T, \quad [T, R] = 2S, \quad [S, R] = R. \quad (29)$$

Let  $\mathcal{L}$  be the Lie group generated by  $T$ ,  $S$ , and  $R$ . To be a solvable group, there must be a basis  $v_1, v_2, v_3$  of its Lie algebra such that

$$[v_i, v_j] = \sum_{k=1}^{j-1} c_{ij}^k v_k,$$

for some constants  $c_{ij}^k$ , whenever  $i < j$ . Since  $T$ ,  $S$ ,  $R$  (and their linear combinations) do not satisfy this property,  $\mathcal{L}$  is nonsolvable. An important consequence (see [9] or [10]) is that they cannot be used to reduce the order of the Chazy equation by three.

However, we can reduce it to a first-order ODE by using the two-dimensional (solvable) subgroup generated by  $S$ ,  $T$ . To find the reductions, it is useful to know the (differential) invariants of each vector field. In Table 1, we give the invariants of  $T$ ,  $S$ , and  $R$ , labelled by their respective lower case letter.

Table 1. Differential Invariants of  $T$ ,  $S$ , and  $R$

$T$	$S$	$R$
$t_1 = u$	$s_1 = zu$	$r_1 = 6z + z^2u$
$t_2 = v$	$s_2 = z^2v$	$r_2 = 6z^2 + 2z^3u + z^4v$
$t_3 = w$	$s_3 = z^3w$	$r_3 = 12z^3 + 6z^4u + 6z^5v + z^6w$

To reduce the Chazy equation under the symmetry generated by  $T$ , we recast it (by using the chain rule) in terms of the invariants of  $T$ . Then we get

$$\frac{dv}{du} = \frac{w}{v}, \quad (30a)$$

$$\frac{dw}{du} = 2u\frac{w}{v} - 3v, \quad (30b)$$

where  $v, w$ , are to be thought of as functions of  $u$ . In terms of the new variables,  $S$  becomes

$$S = -u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 3w\frac{\partial}{\partial w}. \quad (31)$$

(This can be obtained by using the first commutation relation in (29).) The invariants of  $S$  are now

$$p = v/u^2 \quad q = w/u^3.$$

Reduction of (30) in terms of  $S$  then gives

$$\frac{dq}{dp} = \frac{2q - 3p^2 - 3pq}{q - 2p^2}. \quad (32)$$

The change of variables  $1/Q = q(p) - 2p^2$  transforms this to

$$\frac{dQ}{dp} = (7p - 2)Q^2 + p^2(6p - 1)Q^3, \quad (33)$$

which is an Abel equation of the first kind [13].

We believe that this equation can be made to yield further information about the natural barrier. Full details will be given in a later paper.

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# RATIONAL SOLUTIONS AND BÄCKLUND TRANSFORMATIONS FOR THE THIRD PAINLEVÉ EQUATION

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**ABSTRACT.** In this paper we discuss rational solutions, one-parameter families of solutions expressible in terms of Bessel functions for the third Painlevé equation (PIII)

$$\frac{d^2y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \quad \text{PIII}$$

where  $\alpha, \beta, \gamma$  and  $\delta$  are arbitrary constants. PIII is interesting since it arises in many physical applications and also as a similarity reduction of several soliton equations. Using the Bäcklund transformations for PIII we construct hierarchies of exact solutions.

## 1. Introduction

The Painlevé equations were discovered by Painlevé and his colleagues around the turn of the century in an investigation into which second order equations of the form

$$\frac{d^2y}{dx^2} = F \left( x, y, \frac{dy}{dx} \right), \quad (1.1)$$

where  $F$  is rational in  $y$  and  $dy/dx$  and analytic in  $x$ , have the property that the singularities other than poles of any of the solutions are independent of the particular solution and so dependent only on the equation. Painlevé et al. showed that there are fifty canonical equations of the form (1.1) with this property (now known as the *Painlevé property*). Distinguished amongst these fifty equations are the six *Painlevé equations* PI–PVI; any of the other forty-four equations are solvable either in terms of known functions (e.g. elliptic functions) or one of the Painlevé equations — for a survey see [20,21].

Recently there has been considerable interest in the Painlevé equations, primarily due to the fact that they arise as symmetry reductions of soliton equations solvable by inverse scattering, which, in effect, expresses the solution of a nonlinear partial differential equation in terms of the solution of a linear integral equation [1,6]. This relationship was first observed by Ablowitz and Segur [5] and led to the formulation of the so-called Painlevé Conjecture [2,3,19,38].

In this paper we discuss rational solutions, one-parameter families of solutions expressible in terms of Bessel functions and Bäcklund transformations for the third Painlevé equation

$$\frac{d^2y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \quad \text{PIII}$$

where  $\alpha, \beta, \gamma$  and  $\delta$  are arbitrary constants.

PIII arises in the following physical applications: general relativity [9,27,31,52,57,58,59]; scattering of electromagnetic radiation [44]; statistical mechanics (Ising model [8,37], XY-model [34,35]); stimulated Raman Scattering [39]; study of polyelectrolytes in excess salt solution [33].

PIII arises as a similarity reduction of the following partial differential equations: Sine-Gordon equation [5]; Bullough-Dodd-Mikhailov equation [12]; Pohlmeyer-Regge-Lund equation [24]; SU(2) Self-Dual Yang-Mills equations [1,53]; Bianchi-IX equations [56]; Ernst equation [9,27,31,32,52,57,58,59]; Toda Molecule equations [16]; Three-Wave interaction equations [26]; two-dimensional dispersive long wave equations [51].

We remark that PIII has been studied through the isomonodromy deformation method [22, 23], which expresses PIII as the compatibility condition of two linear systems of equations. This method, which can be thought of as a nonlinear analogue of Laplace's method for solving linear ordinary differential equations, is an extension of the inverse scattering method and can be used to investigate the asymptotic behaviour of solutions and obtain connection formulae for PIII [15,25,36,41,42,47,48]. Other studies of properties of PIII include [11,30,37,49,50,54,55].

PIII possesses transformations which map solutions of the equation into new solutions with different values of the parameters [7,13,17,18,29,43,50]. These are commonly known as *Bäcklund transformations* and are given as follows: if  $Y(x; \alpha, \beta, \gamma, \delta)$  is a solution of PIII, then so are

$$\tilde{Y}(x; \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = \frac{xY' - [1 + \beta(-\delta)^{-1/2}] Y + x(-\delta)^{1/2}}{Y^2}, \quad (1.2a)$$

$$\tilde{\alpha} = (-\delta)^{1/2}, \quad \tilde{\beta} = \alpha [2 + \beta(-\delta)^{-1/2}], \quad \tilde{\gamma} = 0, \quad \tilde{\delta} = -\alpha^2; \quad (1.2b)$$

$$\check{Y}(x; \check{\alpha}, \check{\beta}, \check{\gamma}, \check{\delta}) = \left( \frac{\gamma}{\check{\gamma}} \right)^{1/2} Y \left\{ 1 + \frac{[2 + \beta(-\delta)^{-1/2} + \alpha\gamma^{-1/2}] Y}{x [Y' + \gamma^{1/2} Y^2 + (-\delta)^{1/2}] - [1 + \beta(-\delta)^{-1/2}] Y} \right\}, \quad (1.3a)$$

$$\check{\alpha} = -[2 + \beta(-\delta)^{-1/2}] \gamma^{1/2}, \quad \check{\beta} = -[2 + \alpha\gamma^{-1/2}] (-\delta)^{1/2} (\gamma/\check{\gamma})^{1/2}, \quad (1.3b)$$

$$(-\check{\delta})^{1/2} = (-\delta)^{1/2} (\gamma/\check{\gamma})^{1/2} \quad (1.3c)$$

with  $\check{\gamma}$  arbitrary, assuming  $\gamma \neq 0$  and  $2 + \alpha\gamma^{-1/2} + \beta(-\delta)^{1/2} \neq 0$ ;

$$\hat{Y}(x; \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) = \frac{2xR(R-2)}{2xR' + 2(\alpha e - 2)R + 2(\beta - \alpha e + 2)}, \quad (1.4a)$$

$$\hat{\alpha} = 4e - \alpha, \quad \hat{\beta} = -\beta, \quad \hat{\gamma} = 1, \quad \hat{\delta} = -1, \quad e^2 = 1, \quad (1.4b)$$

where

$$R = Y' - eY^2 - \frac{(\alpha e - 1)Y}{x} + 1, \quad (1.4c)$$

provided that  $R(R - 2) \neq 0$ ; and

$$\bar{Y}(x; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) = -Y(x; \alpha, \beta, \gamma, \delta) \quad (1.5a)$$

$$\bar{\alpha} = -\alpha, \quad \bar{\beta} = -\beta, \quad \bar{\gamma} = \gamma, \quad \bar{\delta} = \delta. \quad (1.5b)$$

It is well known that PIII possesses rational solutions and one-parameter families of solutions expressible in terms of Bessel functions [7,13,18,28,50]. Starting with these known rational and one-parameter family solutions, hierarchies of solutions of PIII can be generated by means of the above Bäcklund transformations.

## 2. Rational solutions

By successively applying the Bäcklund transformation (1.2) to the simple rational solution

$$Y(x; f, -\frac{27}{4}f^2k^3, 0, -\frac{729}{64}f^4k^6) = \frac{k(9fkx^{2/3} + 4)}{4x^{1/3}}, \quad (2.1)$$

we obtain the following solutions,

$$Y_0(x; \frac{27}{8}f^2k^3, 0, 0, -f^2) = \frac{2x^{1/3}}{3k}, \quad (2.2a)$$

$$Y_1(x; f, \frac{27}{4}f^2k^3, 0, -\frac{729}{64}f^4k^6) = \frac{k(9fkx^{2/3} - 4)}{4x^{1/3}}, \quad (2.2b)$$

$$Y_2(x; \frac{27}{8}f^2k^3, 4f, 0, -f^2) = \frac{2x^{1/3}(81f^2k^2x^{4/3} - 144fkx^{2/3} + 80)}{3k(9fkx^{2/3} - 4)^2}, \quad (2.2c)$$

$$Y_3(x; f, \frac{81}{4}f^2k^3, 0, -\frac{729}{64}f^4k^6) = \frac{k(9fkx^{2/3} - 4)}{4x^{1/3}} \\ \times \frac{(6561f^4k^4x^{8/3} - 29160f^3k^3x^2 + 51840f^2k^2x^{4/3} - 40320fkx^{2/3} + 8960)}{(81f^2k^2x^{4/3} - 144fkx^{2/3} + 80)^2}, \quad (2.2d)$$

Plots of these solutions with  $f = 1$  and  $k = 2$  are given in Figure 2.1 (for simplicity, we plot  $Y_n$  against  $x^3$ , for  $n = 0, 1, 2, 3$ .)

It can be shown by induction the parameters take the general form:

$$\begin{aligned} \alpha_{2n} &= \frac{27}{8}f^2k^3, & \beta_{2n} &= 4nf, & \gamma_{2n} &= 0, & \delta_{2n} &= -f^2, \\ \alpha_{2n+1} &= f, & \beta_{2n+1} &= \frac{27}{4}(2n+1)f^2k^3, & \gamma_{2n+1} &= 0, & \delta_{2n+1} &= -\frac{729}{64}f^4k^6. \end{aligned}$$

PIII is invariant under the rotation  $y(x) = -iY(ix)$  with  $\beta$  changing sign. Applying this rotation to Bäcklund transformation (1.2) gives

$$\tilde{Y} = \frac{-xY' + [1 - \beta(-\delta)^{-1/2}]Y + x(-\delta)^{1/2}}{Y^2}, \quad (2.3a)$$

$$\tilde{\alpha} = (-\delta)^{1/2}, \quad \tilde{\beta} = \alpha [\beta(-\delta)^{-1/2} - 2], \quad \tilde{\gamma} = 0, \quad \tilde{\delta} = -\alpha^2, \quad (2.3b)$$

which is in fact the inverse of (1.2).

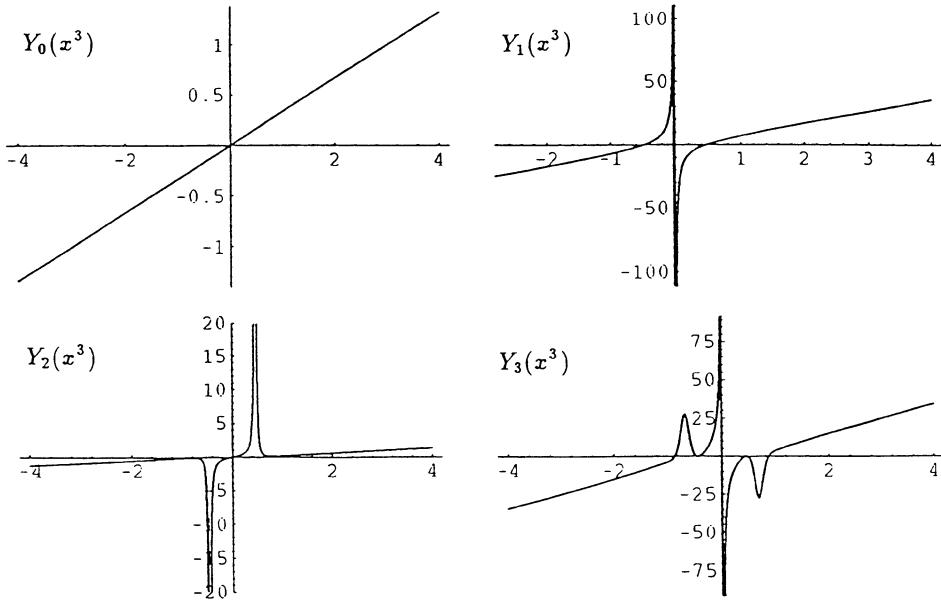


Figure 2.1

The Bäcklund transformation (1.3) is in fact its own inverse. To generate a pattern of solutions, different combinations of signs of  $\gamma^{1/2}$  and  $(-\delta)^{1/2}$  must be taken at each application of the Bäcklund transformation. In this example the signs of the square roots of  $\gamma$  and  $-\delta$  simultaneously alternate between positive and negative. Starting with the solution

$$Y(x; \alpha, \beta, \gamma, \delta) = \frac{x + g}{kx + f}, \quad (2.4a)$$

$$\alpha = \frac{k^2(3f - gk)}{(gk - f)^2}, \quad \beta = \frac{f - 3gk}{(gk - f)^2}, \quad \gamma = \frac{k^4}{(gk - f)^2}, \quad \delta = \frac{1}{(gk - f)^2}, \quad (2.4b)$$

the following set of solutions are generated,

$$Y_0(x; \alpha_0, \beta_0, \gamma_0, \delta_0) = \frac{1}{k}, \quad (2.5a)$$

$$\alpha_0 = \frac{k^2(gk + f)}{(gk - f)^2}, \quad \beta_0 = \frac{-gk + f}{(gk - f)^2}, \quad \gamma_0 = \frac{k^4}{(gk - f)^2}, \quad \delta_0 = \frac{1}{(gk - f)^2}, \quad (2.5b)$$

$$Y_1(x; \alpha_1, \beta_1, \gamma_1, \delta_1) = \frac{kx + f}{k^2(x + g)}, \quad (2.6a)$$

$$\alpha_1 = \frac{k^2(3gk - f)}{(gk - f)^2}, \quad \beta_1 = \frac{gk - 3f}{(gk - f)^2}, \quad \gamma_1 = \frac{k^4}{(gk - f)^2}, \quad \delta_1 = \frac{1}{(gk - f)^2}, \quad (2.6b)$$

$$Y_2(x; \alpha_2, \beta_2, \gamma_2, \delta_2) = \frac{(kx + f)(k^2x^3 + 3gk^2x^2 + 3g^2k^2x + 2fg^2k - f^2g)}{k^2(x + g)(k^2x^3 + 3fkx^2 + 3f^2x - fg^2k + 2f^2g)}, \quad (2.7a)$$

$$\alpha_2 = \frac{k^2(3gk - f)}{(gk - f)^2}, \quad \beta_2 = \frac{gk - 3f}{(gk - f)^2}, \quad \gamma_2 = \frac{k^4}{(gk - f)^2}, \quad \delta_2 = \frac{1}{(gk - f)^2}. \quad (2.7b)$$

Plots of these solutions with  $k = -1$ ,  $f = 1$  and  $g = -2$  are given in Figure 2.2.

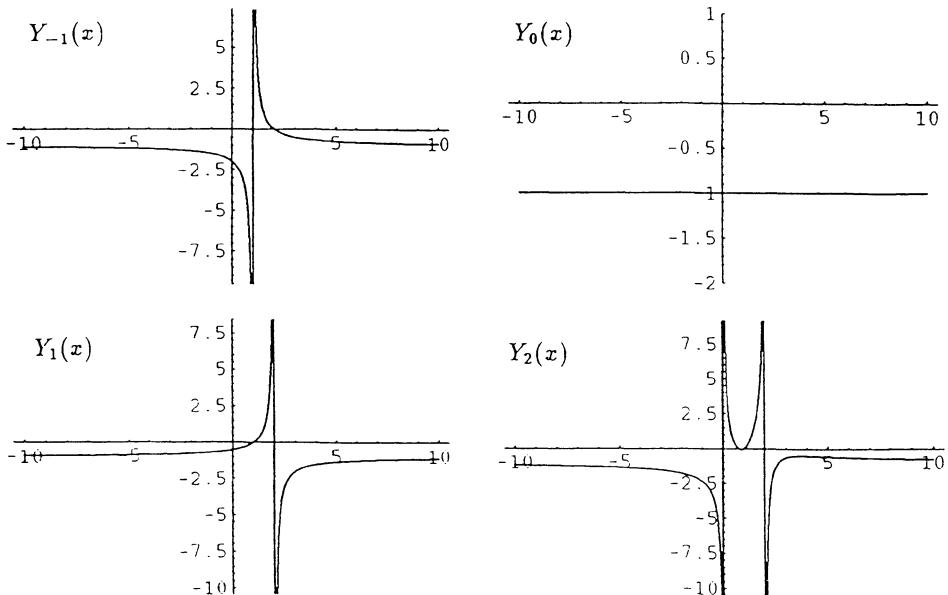


Figure 2.2

It can be shown by induction that following the above scheme of alternating parameters, the parameters take the general form

$$\begin{aligned} \alpha_{2n} &= [4n + \alpha_0\gamma^{-1/2}]\gamma^{1/2}, & \beta_{2n} &= [4n + \beta_0(-\delta)^{-1/2}](-\delta)^{1/2}, & \gamma_n &= \gamma, & \delta_n &= \delta, \\ \alpha_{2n+1} &= -[4n + 2 + \beta_0(-\delta)^{-1/2}]\gamma^{1/2}, & \beta_{2n+1} &= -[4n + 2 + \alpha_0\gamma^{-1/2}](-\delta)^{1/2}. \end{aligned}$$

The Bäcklund transformation (1.4) is also its own inverse.  $e$  must assume alternative values of +1 and -1 at each step in the series to generate a pattern of solutions. By repeatedly applying this Bäcklund transformation to the rational solution

$$Y(x; 4 - k, k, 1, -1) = -\frac{2x + k - 1}{2x + k - 3}, \quad (2.8)$$

we obtain the following solutions

$$Y_0(x; k, -k, 1, -1) = 1, \quad (2.9a)$$

$$Y_1(x; -(k+4), k, 1, -1) = -\frac{2x+k+1}{2x+k+3}, \quad (2.9b)$$

$$Y_2(x; k+8, -k, 1, -1) = \frac{2x+k+5}{2x+k+3} \times \frac{8x^3 + 12(k+3)x^2 + 6(k+3)^2x + (k+1)(k+3)(k+5)}{8x^3 + 12(k+5)x^2 + 6(k+5)^2x + (k+3)(k+5)(k+7)}, \quad (2.9c)$$

Plots of these solutions with  $k = 1$  are given in Figure 2.3.

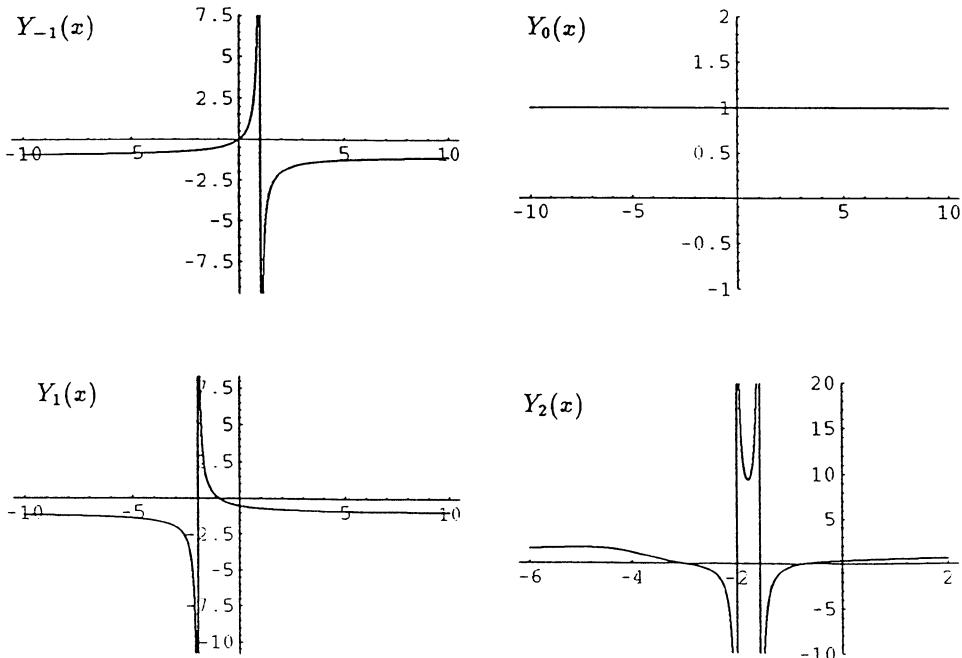


Figure 2.3

The general form of the parameters is

$$\begin{aligned} \alpha_{2n} &= -8n + \alpha_0 & \beta_{2n} &= b_0, & \gamma_n &= 1, & \delta_n &= -1, \\ \alpha_{2n+1} &= 8n + 4 - \alpha_0 & \beta_{2n+1} &= -b_0. \end{aligned}$$

### 3. One-Parameter Families of Bessel Function Solutions

The one-parameter family of solutions for PIII is characterised by the Riccati equation

$$Y' = -\gamma^{1/2}Y^2 - \left(1 + \frac{\alpha}{\gamma^{1/2}}\right)\frac{Y}{x} - (-\delta)^{1/2} \quad (3.1)$$

where  $2 + \alpha\gamma^{-1/2} + \beta(-\delta)^{-1/2} = 0$ . This equation is linearized by setting  $Y = \phi' / (\gamma^{1/2}\phi)$ , yielding

$$\phi'' + \left(1 + \frac{\alpha}{\gamma^{1/2}}\right) \frac{\phi'}{x} + \gamma^{1/2}(-\delta)^{1/2}\phi = 0 \quad (3.2)$$

which has the solution

$$\phi(x) = Ax^{-\nu}J_\nu(\mu x) + Bx^{-\nu}Y_\nu(\mu x) \quad (3.3)$$

where  $\nu = \alpha/(2\gamma^{1/2})$ ,  $\mu = \gamma^{1/4}(-\delta)^{1/4}$  and  $J_\nu(z)$  and  $Y_\nu(z)$  are Bessel functions of the 1st and 2nd kind.

The solution,  $\psi = \phi' / (\gamma^{1/2}\phi)$ , of (3.1) can now be used with an appropriate Bäcklund transformation to generate a hierarchy of solutions of PIII.  $\psi$  can not be used directly with the Bäcklund transformation (1.3) to generate new solutions since the denominator of (1.3) vanishes and so the Bäcklund transformation breaks down. To overcome this difficulty we need to first use the Bäcklund transformation (1.5), then apply the Bäcklund transformation (1.3) to  $-\psi$  once and then again use the Bäcklund transformation (1.5) to get the solution  $Y_1$  of the hierarchy before continuing to apply (1.3). This is illustrated below:

$$Y_0 = \psi \xrightarrow{(1.5)} \bar{Y}_0 = -\psi \xrightarrow{(1.3)} \bar{Y}_1 \xrightarrow{(1.5)} Y_1 \xrightarrow{(1.3)} \dots$$

For certain choices of the parameters,  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ ,  $\phi$  and thus  $\psi$ , the solution to PIII, can be expressed in terms of spherical Bessel functions. For example if  $\alpha = 1$ ,  $\beta = -3$ ,  $\gamma^{1/2} = 1$ ,  $(-\delta)^{1/2} = 1$ , then (3.3) becomes

$$\phi(x) = ax^{-1/2}J_{1/2}(x) + bx^{-1/2}J_{-1/2}(x) = (a \sin x + b \cos x)/x. \quad (3.4)$$

Applying the Bäcklund transformations (1.5) and (1.3), to the solution  $\psi = \phi'/\phi$ , we obtain the following set of solutions,

$$Y_0(x; 1, -3, 1, -1) = \psi = \frac{\phi'}{\phi} = \frac{(ax - b)\cos x - (bx + a)\sin x}{x(a \sin x + b \cos x)}, \quad (3.5a)$$

$$\bar{Y}_0(x; -1, 3, 1, -1) = -\psi = -\frac{\phi'}{\phi}, \quad (3.5b)$$

$$\bar{Y}_1(x; -5, -1, 1, -1) = -\frac{\psi[(\psi^2 + 1)x + \psi]}{(\psi^2 + 1)x + 3\psi}, \quad (3.5c)$$

$$Y_1(x; 5, 1, 1, -1) = \frac{\psi[(\psi^2 + 1)x + \psi]}{(\psi^2 + 1)x + 3\psi}, \quad (3.5d)$$

$$Y_2(-3, 7, 1, -1) = -\frac{[(\psi^2 + 1)x + \psi][2(\psi^2 + 1)x^2 + (3\psi^3 + 7)x + 3\psi^2]}{[(\psi^2 + 1)x + 3\psi][2(\psi^3 + \psi)x^2 + (5\psi^2 + 1)x + 3\psi]}. \quad (3.5e)$$

Plots of these solutions are given in Figure 3.1.

Similarly, by other choices of the parameters  $\psi = \phi' / (\gamma^{1/2}\phi)$  can be expressed in terms of modified spherical Bessel functions. For example, if  $\alpha = 1$ ,  $\beta = -3$ ,  $\gamma^{1/2} = 1$  and  $(-\delta)^{1/2} = -1$ , then

$$\phi(x) = ax^{-1/2}I_{1/2}(x) + bx^{-1/2}I_{-1/2}(x) = (a \sinh x + b \cosh x)/x. \quad (3.6)$$

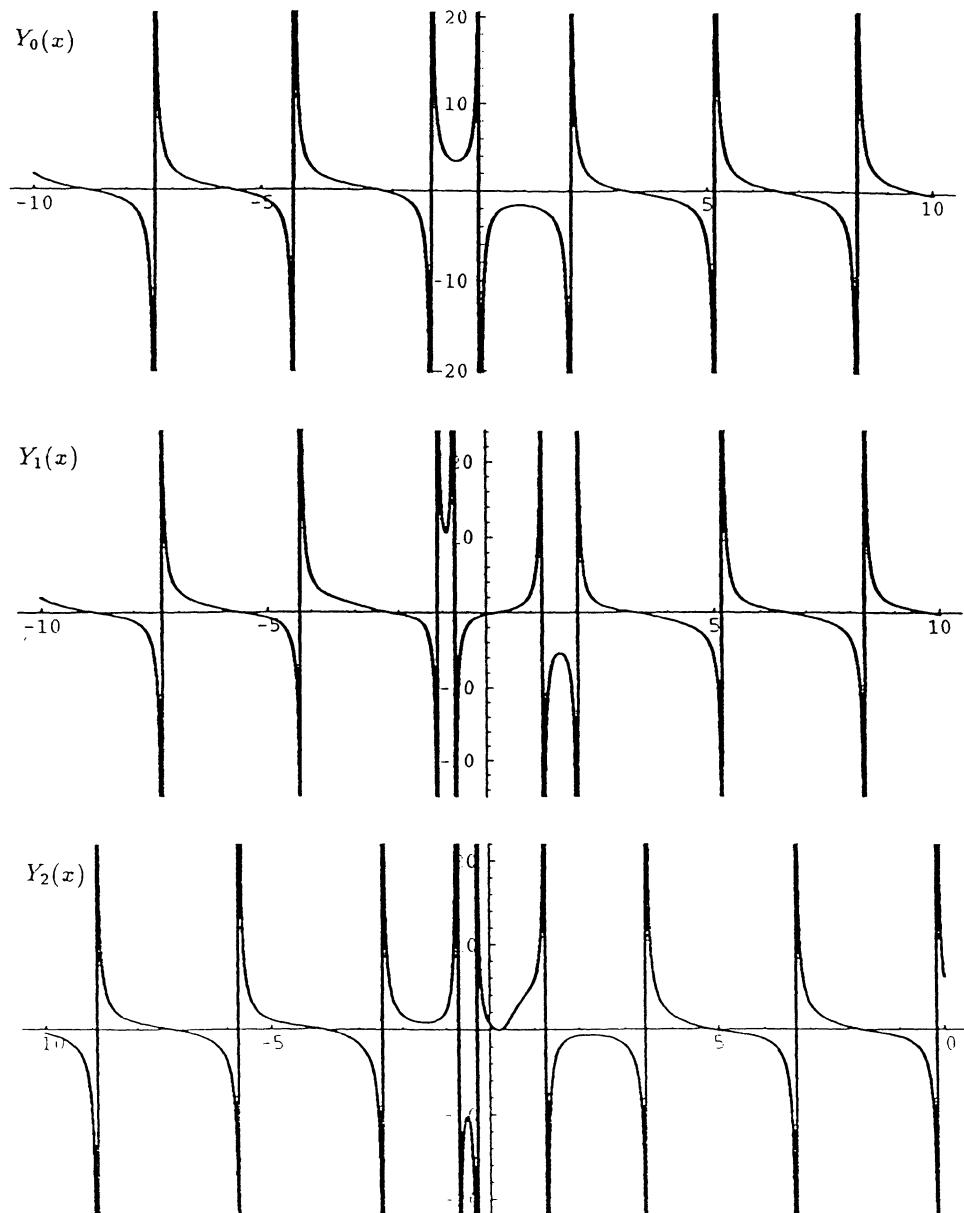


Figure 3.1

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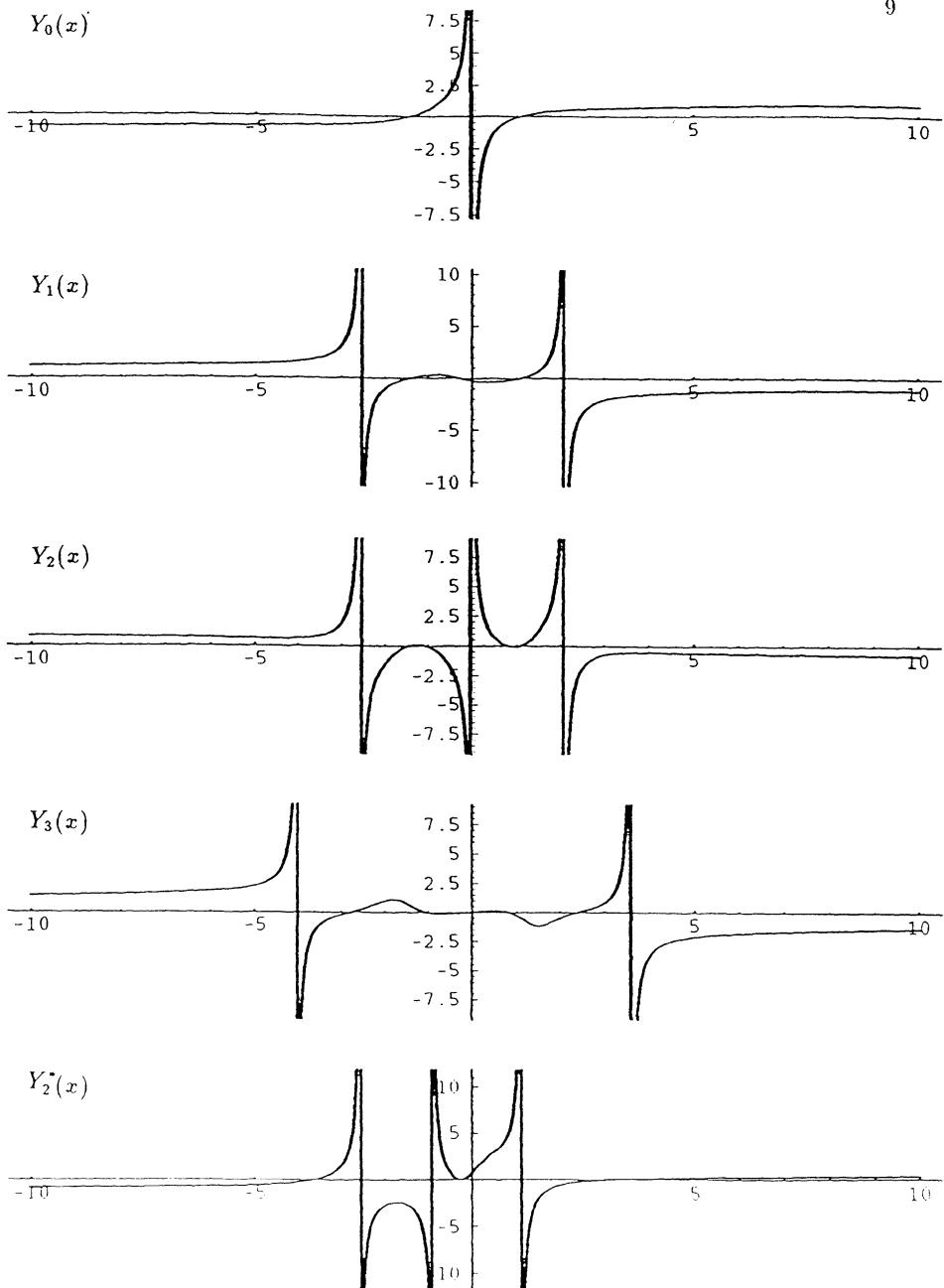


Figure 3.2

By applying the Bäcklund transformations (1.5) and (1.3), the following solutions are generated,

$$Y_0(x; 1, -3, 1, -1) = \psi = \frac{\phi'}{\phi} = \frac{(ax - b) \cosh x + (bx - a) \sinh x}{x(a \sinh x + b \cosh x)}, \quad (3.7a)$$

$$\bar{Y}_0(x; -1, 3, 1, -1) = -\psi = -\frac{\phi'}{\phi}, \quad (3.7b)$$

$$\bar{Y}_1(x; -5, 1, 1, -1) = -\frac{\psi[(\psi^2 - 1)x + \psi]}{(\psi^2 - 1)x + 3\psi}, \quad (3.7c)$$

$$Y_1(x; 5, -1, 1, -1) = \frac{\psi[(\psi^2 - 1)x + \psi]}{(\psi^2 - 1)x + 3\psi}, \quad (3.7d)$$

$$Y_2(-1, -7, 1, -1) = \frac{\psi[(1 - \psi^2)x^2 - \psi x - 6\psi^2]}{(3\psi - x)[(\psi^2 - 1)x + 3\psi]}, \quad (3.7e)$$

$$Y_3(x; 9, 3, 1, -1) = \frac{x[(1 - \psi^2)x^2 - \psi x - 6\psi^2](\psi x^2 - 5x + 15\psi)}{(3\psi - x)[(\psi^2 - 1)x^4 + 3\psi x^3 - 10x^2 + 60\psi x - 90\psi^2]}. \quad (3.7f)$$

$Y_2^*$  below is an illustration of how by taking the square root of  $-\delta$  to be positive as opposed to negative at the step  $Y_1 \rightarrow Y_2$ , a different solution is generated.

$$Y_2^*(-3, 7, 1, -1) = -\frac{[(\psi^2 - 1)x + \psi][2(1 - \psi^2)x^2 + (3\psi^3 - 7\psi)x + 3\psi]}{[(\psi^2 - 1)x + 3\psi][2(\psi^3 - \psi)x^2 + (5\psi^2 - 1)x + 3\psi]}. \quad (3.8)$$

Plots of these solutions are given in Figure 3.2.

By considering the spherical modified Bessel function solutions, four special cases arise.

*Case (i)*  $a = 0$ . In this case, (3.7a) reduces to  $\psi(x) = \tanh x - 1/x$ . This solution is plotted in Figure 3.3.

*Case (ii)*  $b = 0$ . In this case, (3.7a) reduces to  $\psi(x) = \coth x - 1/x$ . This solution is plotted in Figure 3.3. (Note that there is no singularity at  $x = 0$ .)

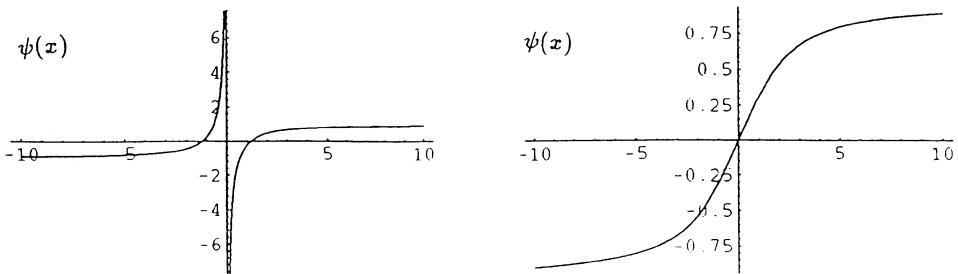


Figure 3.3

*Case (iii)*  $a = -b$ . In this case the solutions reduce to rational solutions. For example, (3.7a) yields  $\psi(x) = -1 - 1/x$ . Applying the Bäcklund transformation (1.3) to this solution generates the same set of solutions as given in §2 before, with  $f$ ,  $k$  and  $g$  suitably chosen. These solutions are asymptotically the same as those generated by applying the Bäcklund transformation (1.3) to the initial solution  $\psi = \phi' / (\gamma^{1/2} \phi)$ , where  $\phi$  is expressed in terms of Bessel functions of the second kind only, i.e.  $A = 0$  in (3.3).

*Case (iv)*  $a = b$ . Again the solutions reduce to rational solutions. For example, (3.7a) yields  $\psi(x) = 1 - 1/x$ . This time the solutions generated when the Bäcklund transformation (1.3) is applied are asymptotically the same as the solutions produced by applying to  $\psi = \phi' / (\gamma^{1/2} \phi)$ , where  $\phi$  is expressed in terms of Bessel functions of the first kind only, i.e.  $B = 0$  in (3.3).

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# SOME EXTENSIONS OF THE TRUNCATION PROCESS IN PAINLEVÉ ANALYSIS

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**ABSTRACT.** We consider two extensions of the truncation process in Painlevé analysis. The first of these arises from the recently developed perturbative Painlevé test. A simple example is presented, for which we show that it is possible to obtain the recursion operator from the truncation of the expansion for the linearisation. In addition, we find that the truncation of the perturbation series is induced by that for the original PDE, together with its singular manifold equation. Our second extension addresses the question of at what level a truncation can be made. We demonstrate that it is possible to truncate at levels other than constant level.

## 1. Introduction

The truncation of Painlevé expansions [1], and in particular the singular manifold method [2], is well-known as a means of recovering Lax pairs for completely integrable PDEs. The simplest formalism in which to perform such a truncation is the homographic invariant analysis introduced by Conte [3] (see Appendix A). Recently there have been attempts [4] to use this formalism to make the path from truncation to Lax pair more algorithmic.

Of course, substitution of a truncated expansion into a differential equation does not necessarily yield a singular manifold equation (SME) leading to a Lax pair. However we may still be able to find some special solutions of our equation (see for example [5,6,7]); this can be done for both integrable and non-integrable equations. Such solutions are understood to be a subset of the solutions which may be obtained from the full expansion [6].

Recently we have introduced a perturbative extension to the Painlevé test [8,9] (see also in these proceedings). Given a differential equation, say

$$U_t = K[U], \quad (1)$$

we seek a solution as a perturbation expansion in some parameter  $\varepsilon$ ;

$$U = \sum_{i=0}^{\infty} \bar{U}_i \varepsilon^i. \quad (2)$$

Substitution in (1) gives a hierarchy of differential equations; we successively seek solutions of each member of this hierarchy in the form of Painlevé expansions,

$$\bar{U}_i = \chi^{-\alpha} \sum_{j=i\rho}^{\infty} U_{i,j} \chi^j. \quad (3)$$

Here  $\rho = r_1$  is the lowest of the resonances  $\mathcal{R} = \{r_1, \dots, r_n\}$  of the branch with leading order behaviour

$$U \sim U_{0,0}\chi^{-\alpha}. \quad (4)$$

The case  $i = 0$  is of course the standard Painlevé test, in which case we may write our expansion coefficients  $U_{0,j}$  simply as  $U_j$ .

This test allows the construction of compatibility conditions for negative resonances. In order for (2) to provide a local representation of the general solution we only require that  $n$  is equal to the order of the system (1), irrespective of the signs of resonances. Having introduced such a test it is natural to ask if truncation in  $\chi$  of the perturbation expansion is possible. We see in Section 2 that this is indeed the case. In fact, such a truncation is found to follow from that for the original nonlinear equation.

It is usually assumed that a truncation can only be made at constant level. This certainly seems to be the case when the expansion is made in  $\phi$ ;

$$U_T = \phi^{-\alpha} \sum_{j=0}^{\alpha} U'_j \phi^j \quad (5)$$

(we use ' to denote WTC expansion coefficients). In rewriting (5) in the invariant form,

$$U_T = \chi^{-\alpha} \sum_{j=0}^{\alpha} U_j \chi^j, \quad (6)$$

it is still assumed that this is the only truncation which can be made. This ignores a new freedom afforded by the change in expansion variable. It is this we exploit in Section 3.

## 2. The truncation of perturbation expansions

We discuss the truncation of perturbation expansions within the context of a simple example, the KdV equation. We consider just the first two coefficients of (2), for ease of notation writing  $\bar{U}_0 = U$  and  $\bar{U}_1 = V$ . This obtains

$$U_t = (U_{xx} + 3U^2)_x, \quad (7)$$

$$V_t = \partial(\partial^2 + 6U)V, \quad (8)$$

where (8) is of course just the linearised equation. We recall that (7) has the single branch

$$\alpha = 2, \quad U_0 = -2, \quad \mathcal{R} = \{-1, 4, 6\}. \quad (9)$$

Taking account of the singularity orders (3) we seek truncated expansions;

$$U_T = U_0 \chi^{-2} + U_1 \chi^{-1} + U_2, \quad (10)$$

$$V_T = V_{-1} \chi^{-3} + V_0 \chi^{-2} + V_1 \chi^{-1} + V_2. \quad (11)$$

Substitution of (10,11) into (7,8) gives

$$\sum_{j=0}^5 Q_j(U_i, S, C) \chi^{5-j} = 0, \quad (12)$$

$$\sum_{j=-1}^5 R_j(U_i, V_k, S, C) \chi^{5-j} = 0, \quad (13)$$

for some functions  $Q_j, R_j$ , which we label as  $U_j, V_j$  in (10,11).

We already know [2] that the truncation for the KdV equation is

$$U_T = -2\chi^{-2} - \frac{1}{6}(C + 4S), \quad (14)$$

with SME,

$$\text{SME} \equiv C + S + \lambda = 0. \quad (15)$$

We now consider the truncation of  $V$ . Just as with the full perturbation expansion, our approach is to successively construct solutions to equations appearing at each order of  $\varepsilon$ . So we seek a solution  $V_T$  of (13), assuming  $U$  to be given by  $U_T$  in (14).

The resonance at  $-1$  of the linearisation is compatible, of course, and  $V_0, V_1$  and  $V_2$  are all determined in terms of  $V_{-1}, S$  and  $C$ ;

$$V_0 = -(V_{-1})_x, \quad (16)$$

$$V_1 = \frac{1}{2}\{(V_{-1})_{xx} + SV_{-1}\}, \quad (17)$$

$$V_2 = \frac{1}{24}\{(V_{-1})_t - 4(V_{-1})_{xxx} + (C - 8S)(V_{-1})_x - (2C_x + 5S_x)V_{-1}\}. \quad (18)$$

Then the coefficients  $R_3, R_4$  and  $R_5$  give three Painlevé-Darboux equations. The first two of these are (up to some factors);

$$\begin{aligned} R_3 &\equiv \{(V_{-1})_t - (V_{-1})_{xxx} - (2S - C)(V_{-1})_x\}_x \\ &\quad - (C + S)_{xx}V_{-1} - 2(C + S)_x(V_{-1})_x = 0, \end{aligned} \quad (19)$$

$$R_4 \equiv (R_3)_x + \{S_t + C_{xxx} + 2SC_x + S_xC\}V_{-1} = 0, \quad (20)$$

and we see that the second of these, corresponding to the resonance at 4, is satisfied as a consequence of the first and the cross-derivative condition (73). Taking (15), (19) and (73) into account, our final Painlevé-Darboux equation,  $R_5 = 0$ , tells us;

$$\{(V_{-1})_t - (V_{-1})_{xxx} - (2S - C)(V_{-1})_x\}_t = 0. \quad (21)$$

We thus obtain the SME for the linearisation;

$$(V_{-1})_t - (V_{-1})_{xxx} - (2S - C)(V_{-1})_x = \bar{\lambda}, \quad (22)$$

where  $\bar{\lambda}$  is some arbitrary constant.

We use (15), (22) to eliminate  $C$  and  $(V_{-1})_t$ , and rewrite our truncation as,

$$U_T = -2\chi^{-2} - \frac{1}{2}S + \frac{1}{6}\lambda, \quad (23)$$

$$\begin{aligned} V_T &= V_{-1}\chi^{-3} - (V_{-1})_x\chi^{-2} + \frac{1}{2}\{(V_{-1})_{xx} + SV_{-1}\}\chi^{-1} \\ &\quad - \frac{1}{8}\{(V_{-1})_{xxx} + 2S(V_{-1})_x + S_xV_{-1}\} + \frac{1}{24}\bar{\lambda}, \end{aligned} \quad (24)$$

and our SMEs as,

$$C + S + \lambda = 0, \quad (25)$$

$$(V_{-1})_t = (V_{-1})_{xxx} + (3S + \lambda)(V_{-1})_x + \bar{\lambda}. \quad (26)$$

We might expect  $Q_5$  to determine an evolution of  $U_2$ , and thus of  $S$ . Substituting for  $U_j$  and  $C$  from (23,25) into  $Q_5$  gives

$$S_t = S_{xxx} + (3S + \lambda)S_x, \quad (27)$$

which tells us that

$$U_L = \frac{1}{6}\lambda + \frac{1}{2}S \quad (28)$$

is also a solution of the KdV equation;

$$(U_L)_t = \left( (U_L)_{xx} + 3U_L^2 \right)_x. \quad (29)$$

Equation (28) is just the classical Darboux transformation (CDT), as observed by Musette and Conte [4], with  $U_L$  being the solution of the KdV equation appearing in the Lax pair obtained from the SME.

Similarly, we might expect  $R_5$  to give an evolution of  $V_2$ . To determine this evolution we substitute for the coefficients of  $U_T$  from (23), for  $V_0$  and  $V_1$  from (24), and eliminate  $C$  and any  $t$ -derivatives of  $V_{-1}$  using (25,26). We then use the definition of  $V_2$  to solve for third and higher order  $x$ -derivatives of  $V_{-1}$ ;

$$(V_{-1})_{xxx} = \left( \frac{1}{3}\bar{\lambda} - 8V_2 \right) - 2S(V_{-1})_x - S_x V_{-1}. \quad (30)$$

When we do this we find in fact that *all* terms in  $V_{-1}$  vanish from  $R_5$ , which then gives;

$$(V_2)_t = \partial\{\partial^2 + (3S + \lambda)\}V_2 - \frac{1}{4}\bar{\lambda}S_x. \quad (31)$$

Using the CDT (28) to replace  $S$  with  $U_L$  gives

$$(V_2)_t = \partial\{\partial^2 + 6U_L\}V_2 - \frac{1}{2}\bar{\lambda}(U_L)_x. \quad (32)$$

Doing the same with (26) and (30) gives

$$(V_{-1})_t = \{\partial^2 + 6U_L\}(V_{-1})_x + \bar{\lambda}, \quad (33)$$

$$V_2 = -\frac{1}{8}\{\partial^3 + 2(U_L\partial + \partial U_L)\}V_{-1} + \frac{1}{12}\lambda(V_{-1})_x + \frac{1}{24}\bar{\lambda}. \quad (34)$$

Remember that in (32–34)  $U_L$  is a solution of the KdV equation (29).

The equations (29,32–34) are consistent; when (29) and (33) are satisfied,  $V_2$  defined by (34) is a solution of (32). It is a simple matter to recover the recursion operator for the KdV hierarchy from (29,32–34); setting  $\bar{\lambda}$  to be zero and differentiating (33) with respect to  $x$ , we get from (34) the map

$$V_2 = R(V_{-1})_x, \quad R = \{\partial^3 + 2(U_L\partial + \partial U_L)\}\partial^{-1}, \quad (35)$$

between two generalised symmetries,  $(V_{-1})_x$  and  $V_2$ , of the KdV equation (29). We remark that the equations (29,32,33,34) not only define the recursion operator (35), but also its function.

So we see that for the KdV equation, important information can be recovered from the truncation for the linearisation, just as it can from the truncation for the nonlinear equation. Similar results hold for the fifth-order KdV equation [8].

We now demonstrate how the truncation for the perturbation expansion may be derived from that for the original nonlinear equation. A feature of our perturbative analysis is that when the  $\varepsilon^0$  pole expansion corresponds to the general solution (of the nonlinear equation),

the perturbation expansion (2) may be recovered from  $\bar{U}_0$  by making a suitable perturbation of the arbitrary data. The same holds for truncations. Sending

$$\phi \rightarrow \phi - (\alpha U'_0)^{-1} \sum_{i=1}^{\infty} E_i(x, t) \varepsilon^i, \quad \lambda \rightarrow \sum_{i=0}^{\infty} \lambda_i \varepsilon^i, \quad (36)$$

in  $U_T$  (23), and SME (15), gives

$$U_T \rightarrow U_T + \varepsilon V_T + \varepsilon^2 W_T + \dots, \quad \text{SME} \rightarrow \text{SME}_0 + \varepsilon \text{SME}_1 + \varepsilon^2 \text{SME}_2 + \dots \quad (37)$$

Each  $E_i$  gives rise to an arbitrary function for the resonance  $-1$ . In particular  $E_1 = V'_{-1}$  is the arbitrary coefficient at  $\phi^{-3}$  in  $V_T$ ;

$$\begin{aligned} V_T = & V'_{-1} \phi^{-3} - \left[ \frac{(V'_{-1})_x}{\phi_x} - \frac{3V'_{-1}\phi_{xx}}{2\phi_x^2} \right] \phi^{-2} \\ & + \left[ \frac{(V'_{-1})_{xx}}{2\phi_x^2} - \frac{V'_{-1}\phi_{xxx} + 2(V'_{-1})_x\phi_{xx}}{\phi_x^3} + \frac{3V'_{-1}\phi_{xx}^2}{\phi_x^4} \right] \phi^{-1} \\ & + \left[ \frac{\lambda_1}{6} - \frac{(V'_{-1})_{xxx}}{8\phi_x^3} + \frac{7(V'_{-1})_x\phi_{xxx} + 7(V'_{-1})_{xx}\phi_{xx} + 2V'_{-1}\phi_{xxxx}}{8\phi_x^4} \right. \\ & \left. - \frac{23(V'_{-1})_x\phi_{xx}^2 + 22V'_{-1}\phi_{xx}\phi_{xxx}}{8\phi_x^5} + \frac{4V'_{-1}\phi_{xx}^3}{\phi_x^6} \right]. \end{aligned} \quad (38)$$

We then find that  $\text{SME}_1$  defines an evolution of  $V'_{-1}$ ;

$$\begin{aligned} (V'_{-1})_t = & (V'_{-1})_{xxx} + \lambda_0 (V'_{-1})_x + 4\lambda_1 \phi_x^3 - \frac{9(V'_{-1})_{xx}\phi_{xx} + 6(V'_{-1})_x\phi_{xxx}}{\phi_x} \\ & + \frac{63(V'_{-1})_x\phi_{xx}^2 + 36V'_{-1}\phi_{xx}\phi_{xxx}}{2\phi_x^2} - \frac{42V'_{-1}\phi_{xx}^3}{\phi_x^3}. \end{aligned} \quad (39)$$

Rewriting (38,39) in invariant form, with  $V'_{-1} = V_{-1}\phi_x^3$ , gives precisely the expressions (24,26) with  $\lambda = \lambda_0$  and  $\bar{\lambda} = 4\lambda_1$ . In [8] we find the truncation to order  $\varepsilon^2$  from that at order  $\varepsilon^0$  and the SME.

So we see that the truncation of coefficients of higher powers of  $\varepsilon$  is induced by the truncation at  $\varepsilon^0$ , together with its SME.

### 3. Higher-order truncations

As we remarked earlier, most authors seem content to use the invariant analysis to rewrite truncations of WTC expansions, which are necessarily at constant level. However the change of expansion variable allows the possibility of truncation at higher levels.

A leading order analysis of equation (1) consists of a choice of  $\alpha$ ,  $U_0$ , and dominant terms  $\hat{K}[U]$ , say of weight  $\beta$ . This means;

$$U = \chi^{-\alpha} \sum_{j=0}^{\infty} U_j \chi^j, \quad K[U] - U_t = \chi^{-\beta} \sum_{j=0}^{\infty} Q_j \chi^j. \quad (40)$$

If we consider the Riccati equations (72) we see that this leading order analysis is mirrored at positive powers; the terms  $\hat{K}[U]$  are again dominant. This means that we can consider the truncation:

$$U_T = \chi^{-\alpha} \sum_{j=0}^{2\alpha} U_j \chi^j, \quad K[U] - U_t = \chi^{-\beta} \sum_{j=0}^{2\beta} Q_j \chi^j. \quad (41)$$

We find that each choice of  $U_0$  leads to a choice of  $U_{2\alpha}$ ,

$$U_{2\alpha} = \left(-\frac{1}{2}S\right)^\alpha U_0, \quad (42)$$

and so the first and last coefficients of our truncated expansion are determined as

$$U_T = U_0 \chi^{-\alpha} + \dots + \left(-\frac{1}{2}S\right)^\alpha \tilde{U}_0 \chi^\alpha, \quad (43)$$

where  $U_0$  and  $\tilde{U}_0$  correspond to two branches, not necessarily the same, of (1). From (43) we see that if our truncation for (1) is to be at  $\chi^\alpha$ , we must insist that  $S \neq 0$ .

### 3.1. KDV EQUATION

The only solution of the KdV equation (7) of the form

$$U_T = \chi^{-2} \sum_{j=0}^4 U_j \chi^j, \quad U_0 U_4 \neq 0, \quad (44)$$

is

$$U_T = -2\chi^{-2} - \frac{1}{6}(C + 4S) - \frac{1}{2}S^2\chi^2, \quad (45)$$

where both  $S$  and  $C$  are required to be constant. Accordingly, we write

$$S = -\frac{1}{2}k^2, \quad C = c, \quad (46)$$

so that we may use the results of Appendix B. Writing (45) in terms of  $q(x, t)$  (see (78,79)) gives

$$U_T = -\frac{1}{2}k^2\{q^{-2} + q^2\} + \frac{1}{6}\{2k^2 - c\}. \quad (47)$$

When we truncate at constant level we are able to identify the finite invariant and non-invariant expansions. We can of course still do this; the difference here is that the WTC expansion is still an infinite series. The truncation (45), or more generally (41), therefore corresponds to the summation of such a series.

Taking into account the transformation between invariant and non-invariant analyses [3], we see that our truncation (45) corresponds to the WTC expansion,

$$U = \phi^{-2} \sum_{j=0}^{\infty} U'_j \phi^j, \quad (48)$$

with the choice of arbitrary data

$$U'_4 = -\frac{k^4}{8\phi_x^2}, \quad U'_6 = -\frac{3k^4\phi_{xx}^2}{16\phi_x^6}, \quad (49)$$

$\phi$  being subject to the constraints

$$C = -\frac{\phi_t}{\phi_x} = c, \quad S = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2} \left(\frac{\phi_{xx}}{\phi_x}\right)^2 = -\frac{1}{2}k^2. \quad (50)$$

A simple identity enables us to rewrite the solution (47) as

$$U_T = -2k^2 \tanh^2\{k(x - ct + F)\} + \frac{1}{6}(8k^2 - c), \quad F = E + \frac{i\pi}{2k}, \quad (51)$$

and so our truncation yields a different representation of a well-known solution.

### 3.2. GENERALISED KURAMOTO-SIVASHINSKY EQUATION

We now consider the equation [7,10,11]

$$U_t + K[U] = U_t + UU_x + aU_{xx} + bU_{xxx} + dU_{xxxx} = 0. \quad (52)$$

Since our analysis assumes  $d \neq 0$ , we may rescale to set  $d = 1$ ;

$$U_t + K[U] = U_t + UU_x + aU_{xx} + bU_{xxx} + U_{xxxx} = 0. \quad (53)$$

We then have the branch,

$$\alpha = 3, \quad U_0 = 120, \quad \hat{K}[U] = UU_x + U_{xxxx}, \quad \beta = 7, \quad \mathcal{R} = \{-1, 6, r, s\}, \quad (54)$$

where  $r$  and  $s$  are the roots of

$$r^2 - 13r + 60 = 0. \quad (55)$$

For the Kuramoto-Sivashinsky equation ( $b = 0$ ), the resonance at 6 is compatible [5]; it remains so for the equation (53).

For reasons of simplicity, we seek solutions in the form

$$U_T = \chi^{-3} \sum_{j=0}^6 U_j \chi^j, \quad S, C \text{ constant.} \quad (56)$$

We find that such solutions exist,

$$\begin{aligned} U_T = & 120\chi^{-3} - 15b\chi^{-2} + \frac{15}{76}(304S - b^2 + 16a)\chi^{-1} \\ & + \frac{1}{608}(608C - 3040Sb - 13b^3 + 56ba) - \frac{15}{152}S(304S - b^2 + 16a)\chi \\ & - \frac{15}{4}bS^2\chi^2 - 15S^3\chi^3, \end{aligned} \quad (57)$$

provided that

$$b(6080S - 13b^2 + 56a)(b^2 - 16a) = 0, \quad (58)$$

$$1478656S^2 - 6080Sb^2 + 97280Sa - 131b^4 + 696ab^2 - 704a^2 = 0. \quad (59)$$

Equations (58,59) have five solutions;

$$\frac{b^2}{a} = 16, \quad S^2 = \frac{b^4}{16384}, \quad (60)$$

$$\frac{b^2}{a} = \frac{256}{73}, \quad S = -\frac{b^2}{2048}, \quad (61)$$

$$\frac{b^2}{a} = \frac{144}{47}, \quad S = -\frac{b^2}{1152}, \quad (62)$$

$$b = 0, \quad S = -\frac{11a}{152}, \quad (63)$$

$$b = 0, \quad S = \frac{a}{152}. \quad (64)$$

The parameter ratios

$$\frac{b^2}{a} = 16, \quad \frac{256}{73}, \quad \frac{144}{47}, \quad 0, \quad (65)$$

are exactly those that Kudryashov [7] finds are required in order to allow solitary wave solutions of (53). Cases (63) and (64), for which we must have  $b = 0$ , correspond to solutions of the original Kuramoto-Sivashinsky equation.

We now write (57) in terms of  $q(x, t)$  for each of the above cases:

CASE 1. Choose

$$b = 8k, \quad S = -\frac{b^2}{128} = -\frac{1}{2}k^2, \quad a = \frac{b^2}{16} = 4k^2;$$

$$U_T = 15k^3\{q^{-3} + q^3\} - 30k^3\{q^{-2} + q^2\} - 15k^3\{q^{-1} + q\} + \{c + 12k^3\}. \quad (66)$$

CASE 2. Choose

$$b = 32k, \quad S = -\frac{b^2}{2048} = -\frac{1}{2}k^2, \quad a = \frac{73b^2}{256} = 292k^2;$$

$$U_T = 15k^3\{q^{-3} + q^3\} - 120k^3\{q^{-2} + q^2\} + 345k^3\{q^{-1} + q\} + \{c + 240k^3\}. \quad (67)$$

CASE 3. Choose

$$b = 24k, \quad S = -\frac{b^2}{1152} = -\frac{1}{2}k^2, \quad a = \frac{47b^2}{144} = 188k^2;$$

$$U_T = 15k^3\{q^{-3} + q^3\} - 90k^3\{q^{-2} + q^2\} + 225k^3\{q^{-1} + q\} + \{c + 180k^3\}. \quad (68)$$

CASE 4. Choose

$$b = 0, \quad a = \frac{76k^2}{11}, \quad S = -\frac{11a}{152} = -\frac{1}{2}k^2;$$

$$U_T = 15k^3\{q^{-3} + q^3\} - \frac{45}{11}k^3\{q^{-1} + q\} + c. \quad (69)$$

CASE 5. Choose

$$b = 0, \quad a = -76k^2, \quad S = \frac{a}{152} = -\frac{1}{2}k^2;$$

$$U_T = 15k^3\{q^{-3} + q^3\} - 135k^3\{q^{-1} + q\} + c. \quad (70)$$

Once again we find that the solutions we obtain are different representations of known solutions. However our point here is to demonstrate the *possibility* of truncation at levels other than constant level. This allows the summation of infinite WTC expansions for certain choices of arbitrary data. The question of whether new solutions can be obtained via such truncations is addressed elsewhere [8].

### Appendix A: The Invariant Analysis

In this appendix we state some formulae from Conte's invariant analysis [3]. The functions  $\chi$  and the usual WTC  $\phi$  are related by;

$$\chi(\phi) = \left( \frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x} \right)^{-1}. \quad (71)$$

This  $\chi$  satisfies the Riccati equations,

$$\chi_x = 1 + \frac{1}{2}S\chi^2, \quad \chi_t = -C + C_x\chi - \frac{1}{2}(C_{xx} + CS)\chi^2, \quad (72)$$

with integrability condition,

$$S_t + C_{xxx} + 2C_xS + CS_x = 0, \quad (73)$$

$S$  and  $C$  being the differential invariants of the Möbius group given by;

$$S = \left( \frac{\phi_{xx}}{\phi_x} \right)_x - \frac{1}{2} \left( \frac{\phi_{xx}}{\phi_x} \right)^2, \quad C = -\frac{\phi_t}{\phi_x}. \quad (74)$$

In addition to its theoretical significance, this approach has the practical advantage of greatly simplifying many calculations.

### Appendix B: General Solution of Invariant Equations

The general solution of

$$C = c, \quad S = -\frac{1}{2}k^2, \quad (75)$$

where  $c$  and  $k$  are arbitrary constants, is [5,12];

$$\phi = \frac{\alpha + \beta e^{k(x-ct)}}{\gamma + \delta e^{k(x-ct)}}, \quad \text{arbitrary } \alpha, \beta, \gamma, \delta, \quad \alpha\beta\gamma\delta \neq 0. \quad (76)$$

Alternatively we may write the general solution of (75) as

$$\phi = \frac{2B}{k} \tanh \left( \frac{1}{2}k(x - ct + A) \right) + D, \quad \text{arbitrary } A, B, D. \quad (77)$$

For  $\phi$  given by (77), we find

$$\chi = \frac{2}{k} \tanh^{-1} \left( \frac{1}{2}k(x - ct + E) \right), \quad E = A + \frac{2}{k} \tanh^{-1} \left( \frac{2B}{kD} \right), \quad (78)$$

and it is then convenient to define  $q(x, t)$  by

$$\chi = \frac{2}{kq(x, t)}. \quad (79)$$

This helps to simplify the expressions for truncated expansions.

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## POTENTIAL SYMMETRIES AND LINEARIZATION

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**ABSTRACT.** A nonlinear scalar PDE (system of PDES) is linearizable by an invertible contact (point) transformation if and only if it admits an infinite-parameter Lie group of contact (point) transformations satisfying specific criteria. Algorithms exist to find such symmetry groups and to construct such linearizations when they exist.

Most of the interesting linearizations involve non-invertible transformations. It will be shown how such linearizations can be found by embedding given nonlinear PDES in auxiliary systems of PDES. These are discovered by replacing equations of given PDES by equivalent conserved forms (conservation laws), introducing  $n - 1$  auxiliary potential variables ( $n$  represents the number of independent variables), and determining point symmetries of the resulting auxiliary systems of PDES. Such point symmetries correspond to nonlocal (potential) symmetries of the given nonlinear PDES.

A new algorithm is presented which significantly simplifies the process of determining whether or not a given nonlinear system of PDES can be linearized. This algorithm depends on a recently proved result concerning the invariance of conserved forms under arbitrary contact transformations.

Several examples are given.

### 1. Introduction

For a given nonlinear system of PDES, one can usually determine algorithmically whether or not there exists a change of variables which linearizes it, i.e. leads to the system being "C-integrable" in the sense of [1]. In particular, by determining the (Lie) point symmetries of a system of PDES one can determine algorithmically whether or not it can be linearized by an invertible point transformation (see [2-4]). By embedding a given system of PDES in an auxiliary system of PDES through the use of conserved forms (conservation laws) one can determine whether or not the given system admits potential symmetries leading to linearization by a change of variables involving a non-invertible mapping. It will be shown that the algorithm (see [4,5]) to discover such linearizations by non-invertible mappings can be simplified through searching for all conserved forms of a particular type for a given system of PDES.

Consider a nonlinear system of  $m$  PDES  $R\{x, u\}$  given by the relations

$$G^\sigma(x, u, u_1, u_2, \dots, u_k) = 0, \quad \sigma = 1, 2, \dots, m, \quad (1.1)$$

with independent variables  $x = (x_1, x_2, \dots, x_n)$ , dependent variables  $u = (u^1, u^2, \dots, u^m)$  and where  $u_j$  denotes the set of coordinates corresponding to all  $j$ th order partial derivatives

of  $u$  with respect to  $x$  (a coordinate in  $u$  is denoted by  $u_{i_1 i_2 \dots i_j}^\gamma \equiv \frac{\partial^j u^\gamma}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}}$  with  $\gamma = 1, 2, \dots, m$ ;  $i_j = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, k$ ).

**DEFINITION 1.1.** A *symmetry* admitted by  $R\{x, u\}$  is a transformation which maps any solution of (1.1) into another solution of (1.1).

**DEFINITION 1.2.** A *Lie point symmetry* admitted by  $R\{x, u\}$  is a symmetry characterized by an infinitesimal generator of the form

$$\mathbf{X} = \xi_i(x, u) \frac{\partial}{\partial x_i} + \alpha^\mu(x, u) \frac{\partial}{\partial u^\mu}. \quad (1.2)$$

(Summation over a repeated index is assumed throughout this paper.)

A point symmetry (1.2) corresponds to a one-parameter ( $\varepsilon$ ) Lie group of point transformations

$$\begin{aligned} x_i^* &= x_i + \varepsilon \xi_i(x, u) + O(\varepsilon^2), \quad i = 1, 2, \dots, n, \\ (u^\mu)^* &= u^\mu + \varepsilon \alpha^\mu(x, u) + O(\varepsilon^2), \quad \mu = 1, 2, \dots, m. \end{aligned}$$

In its action on mapping solutions of (1.1) into other solutions of (1.1),  $\mathbf{X}$ , given by (1.2), is equivalent to the infinitesimal generator

$$\tilde{\mathbf{X}} = \eta^\mu(x, u, u) \frac{\partial}{\partial u^\mu} \quad (1.3)$$

with  $\eta^\mu(x, u, u) = \alpha^\mu(x, u) - \xi_i(x, u) u_i^\mu$ , corresponding to the transformations (see [4])

$$\begin{aligned} x_i^* &= x_i, \quad i = 1, 2, \dots, n, \\ (u^\mu)^* &= u^\mu + \varepsilon \eta^\mu(x, u, u) + O(\varepsilon^2), \quad \mu = 1, 2, \dots, m. \end{aligned}$$

If one knows all infinitesimal generators  $\mathbf{X}$  admitted by  $R\{x, u\}$  ( $m \geq 2$ ) one can determine whether or not  $R\{x, u\}$  can be linearized by an *invertible mapping*, acting on a finite-dimensional manifold involving  $(x, u, u, u, \dots, u)$  for some finite  $p$  and construct such

a mapping when it exists. This follows from two theorems (for proofs and extensions to the scalar case  $m = 1$  see [2–4]):

**THEOREM 1.3** (necessary conditions). *If there is an invertible mapping which transforms  $R\{x, u\}$  to a linear system of PDES then (I) the mapping is a point transformation*

$$\begin{aligned} z_j &= \phi_j(x, u), \quad j = 1, 2, \dots, n, \\ w^\gamma &= \psi^\gamma(x, u), \quad \gamma = 1, 2, \dots, m. \end{aligned}$$

(II)  $R\{x, u\}$  must admit an infinitesimal generator, corresponding to an infinite-parameter group, of the form

$$\mathbf{X} = \xi_i(x, u) \frac{\partial}{\partial x_i} + \alpha^\mu(x, u) \frac{\partial}{\partial u^\mu}, \quad (1.4a)$$

with

$$\xi_i(x, u) = a_i^\rho(x, u)F^\rho(x, u), \quad \alpha^\mu(x, u) = b_\mu^\rho(x, u)F^\rho(x, u), \quad (1.4b)$$

where  $a_i^\rho$ ,  $b_\mu^\rho$  are specific functions of  $(x, u)$  and  $F = (F^1, F^2, \dots, F^m)$  is an arbitrary solution of some linear system

$$L[X]F = 0;$$

$L[X]$  is a linear operator depending on independent variables

$$X = (X_1(x, u), X_2(x, u), \dots, X_n(x, u)). \quad (1.5)$$

**THEOREM 1.4** (sufficient conditions). *If the linear system of  $m$  first order PDES*

$$a_i^\sigma \frac{\partial \Phi}{\partial x_i} + b_\mu^\sigma \frac{\partial \Phi}{\partial u^\mu} = 0, \quad \sigma = 1, 2, \dots, m, \quad (1.6)$$

whose coefficients  $a_i^\sigma$ ,  $b_\mu^\sigma$  are given by (1.4a,b), has  $X_1(x, u), X_2(x, u), \dots, X_n(x, u)$  (given by (1.5)) as  $n$  functionally independent solutions, and the linear system of  $m^2$  first order PDES ( $\delta^{\gamma\sigma}$  is the Kronecker symbol)

$$a_i^\sigma \frac{\partial \psi^\gamma}{\partial x_i} + b_\mu^\sigma \frac{\partial \psi^\gamma}{\partial u^\mu} = \delta^{\gamma\sigma}, \quad \gamma, \sigma = 1, 2, \dots, m, \quad (1.7)$$

has a solution

$$\psi(x, u) = (\psi^1(x, u), \psi^2(x, u), \dots, \psi^m(x, u)),$$

then the invertible mapping

$$z_j = \phi_j(x, u) = X_j(x, u), \quad j = 1, 2, \dots, n,$$

$$w^\gamma = \psi^\gamma(x, u), \quad \gamma = 1, 2, \dots, m,$$

transforms  $R\{x, u\}$  to a linear system  $\hat{R}\{z, w\}$  given by

$$L[z]w = g(z) \quad (1.8)$$

for some specific function  $g(z)$ .

Consequently if  $R\{x, u\}$  admits a generator of the form (1.4a,b) one knows the coordinates  $z = \phi(x, u) = X(x, u)$  for the independent variables and the linear operator  $L[z]$  of the linearizing PDE; dependent variables  $w = \psi(x, u)$  of the linearizing transformation follow from finding a particular solution of system (1.7). [The solution of (1.7) is not unique. Different solutions of (1.7) lead to different nonhomogeneous terms  $g(z)$  in (1.8). There is always a solution of (1.7) for which  $g(z)$  is zero. There is no known example for which Theorem 1.3 holds and Theorem 1.4 does not hold.]

## 2. Determination of point symmetries

A point symmetry  $\mathbf{X}$  acting on  $(x, u)$ -space “naturally” extends (prolongs) to its  $j$ -th extension ( $j$ -th prolongation) acting on  $(x, u, u_1, u_2, \dots, u_j)$ -space by requiring invariance of contact conditions

$$\begin{aligned} du^\gamma &= u_i^\gamma dx_i, \\ &\vdots \\ du_{i_1 i_2 \dots i_{j-1}}^\gamma &= u_{i_1 i_2 \dots i_j}^\gamma dx_{i_j}, \quad \gamma = 1, 2, \dots, m, \end{aligned} \tag{2.1}$$

$i_q = 1, 2, \dots, n$ , for  $q = 1, 2, \dots, j - 1$ .

Let

$$\mathbf{X}^{(j)} = \mathbf{X} + \alpha_i^{(1)\mu}(x, u, u_1) \frac{\partial}{\partial u_i^\mu} + \dots + \alpha_{i_1 i_2 \dots i_j}^{(j)\mu}(x, u, u_1, u_2, \dots, u_j) \frac{\partial}{\partial u_{i_1 i_2 \dots i_j}^\mu}$$

where

$$\begin{aligned} \alpha_i^{(1)\mu} &= D_i \alpha^\mu - (D_i \xi_\ell) u_\ell^\mu, \\ &\vdots \\ \alpha_{i_1 i_2 \dots i_j}^{(j)\mu} &= D_{i_j} \alpha_{i_1 i_2 \dots i_{j-1}}^{(j-1)\mu} - (D_{i_j} \xi_\ell) u_{i_1 i_2 \dots i_{j-1}}^\mu, \end{aligned}$$

and  $D_i$  is the total derivative operator

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_i^\gamma \frac{\partial}{\partial u^\gamma} + \dots + u_{i_1 i_2 \dots i_\ell}^\gamma \frac{\partial}{\partial u_{i_1 i_2 \dots i_\ell}^\gamma} + \dots$$

The algorithm to determine all infinitesimal generators  $\mathbf{X}$  admitted by  $R\{x, u\}$  involves finding all solutions  $\{\xi_i, \alpha^\mu\}$  of the system of PDES

$$\mathbf{X}^{(k)} G^\sigma(x, u, u_1, u_2, \dots, u_k) = 0, \quad \sigma = 1, 2, \dots, m \tag{2.2}$$

where  $u(x)$  is any solution of  $R\{x, u\}$ . This leads to an overdetermined system of linear PDES (“determining equations”) satisfied by the  $m + n$  components  $\{\xi_i, \alpha^\mu\}$  of  $\mathbf{X}$ . For details see [4], [6–8]. Various symbolic manipulation algorithms [9–15] exist in REDUCE, MUMATH, MACSYMA, MAPLE, SCRATCHPAD, MATHEMATICA, etc., which set up the determining equations, simplify them, sometimes solve them and/or find the dimension of their solution space automatically. Problems can arise if the dimension of the solution space is infinite, especially if the system of PDES admits a group leading to linearization.

## 3. Potential symmetries; linearizations by noninvertible mappings

Often linearizations obtained by a change of variables do not arise as invertible point transformations. Our aim is to find such non-invertible transformations by algorithmic procedures. We show that the use of conserved forms is a natural way to discover such linearizations.

Suppose one PDE of  $R\{x, u\}$ , without loss of generality  $G^m = 0$ , is a conserved form

$$\frac{\partial}{\partial x_i} f^i(x, u, u_1, u_2, \dots, u_{k-1}) = 0.$$

Then  $R\{x, u\}$  is the system

$$G^\sigma(x, u, u_1, u_2, \dots, u_k) = 0, \quad \sigma = 1, 2, \dots, m-1, \quad (3.1a)$$

$$\frac{\partial}{\partial x_i} f^i(x, u, u_1, u_2, \dots, u_{k-1}) = 0. \quad (3.1b)$$

Through (3.1b), we can introduce  $n - 1$  auxiliary variables (*potentials*)  $V = (V^1, V^2, \dots, V^{n-1})$  and form an *auxiliary system* of  $m + n - 1$  PDES  $S\{x, u, V\}$ :

$$\begin{aligned} f^1(x, u, u_1, u_2, \dots, u_{k-1}) &= \frac{\partial V^1}{\partial x_2}, \\ f^j(x, u, u_1, u_2, \dots, u_{k-1}) &= (-1)^{j-1} \left[ \frac{\partial V^j}{\partial x_{j+1}} + \frac{\partial V^{j-1}}{\partial x_{j-1}} \right], \quad 2 < j < n, \\ f^n(x, u, u_1, u_2, \dots, u_{k-1}) &= (-1)^{n-1} \frac{\partial V^{n-1}}{\partial x_{n-1}}, \\ G^\sigma(x, u, u_1, u_2, \dots, u_k) &= 0, \quad \sigma = 1, 2, \dots, m-1. \end{aligned} \quad (3.2)$$

If  $(u(x), V(x))$  solves  $S\{x, u, V\}$  then  $u(x)$  solves  $R\{x, u\}$  from the integrability conditions of (3.2). From these integrability conditions it also follows that if  $u(x)$  solves  $R\{x, u\}$  then there is some  $V(x)$  such that  $(u(x), V(x))$  solves  $S\{x, u, V\}$ . But even though all solutions of  $R\{x, u\}$  are embedded in the solutions of  $S\{x, u, V\}$  and, conversely, all solutions of  $S\{x, u, V\}$  can be found from the solutions of  $R\{x, u\}$ , the relationship between  $R\{x, u\}$  and  $S\{x, u, V\}$  is not invertible: if  $(u(x), V_1(x), V_2(x), \dots, V_{n-1}(x))$  solves  $S\{x, u, V\}$  then so does  $(u(x), V_1(x) + c_1, V_2(x) + c_2, \dots, V_{n-1}(x) + c_{n-1})$  for any constants  $(c_1, c_2, \dots, c_{n-1})$ .

From Definition 1.1 and the relationship between  $R\{x, u\}$  and  $S\{x, u, V\}$  it follows that a symmetry of  $S\{x, u, V\}$  defines (induces) a symmetry of  $R\{x, u\}$ . (The converse is also true.) However, a point symmetry of  $S\{x, u, V\}$  could yield a nonlocal symmetry (*potential symmetry*) of  $R\{x, u\}$  since from (3.2) we can see that each component of  $V(x)$  cannot be expressed in terms of the components of  $(x, u, u_1, u_2, \dots, u_p)$  for any finite  $p$ .

**DEFINITION 3.1.** A *potential symmetry* of  $R\{x, u\}$  is derived from a point symmetry

$$\mathbf{X}_s = \xi_{si}(x, u, V) \frac{\partial}{\partial x_i} + \alpha_s^\mu(x, u, V) \frac{\partial}{\partial u^\mu} + \beta_s^v(x, u, V) \frac{\partial}{\partial V^v},$$

admitted by  $S\{x, u, V\}$ , when the components of  $\{\xi_{si}, \alpha_s^\mu\}$  of  $\mathbf{X}$  depend essentially on  $V$ .

Consequently if  $S\{x, u, V\}$  admits a point symmetry which satisfies the criteria of Theorems 1.3 and 1.4 and this point symmetry yields a potential symmetry of  $R\{x, u\}$ , it follows that the linearization of  $S\{x, u, V\}$  by an invertible point transformation yields the linearization of  $R\{x, u\}$  by a non-invertible change of variables.

It could turn out that neither  $R\{x, u\}$  nor  $S\{x, u, V\}$  admits point symmetries yielding linearizations through Theorems 1.3 and 1.4 but a PDE of  $S\{x, u, V\}$  is a conserved form yielding another auxiliary system  $T\{x, u, V, W\}$  of  $m + 2(n - 1)$  PDES which admits point symmetries satisfying the criteria of Theorems 1.3 and 1.4. In this case one can explicitly linearize  $T\{x, u, V, W\}$  by an invertible point transformation which in turn leads to the linearizations of  $S\{x, u, V\}$  and  $R\{x, u\}$  by non-invertible transformations. (One could also move in the reverse order: it could turn out that a given system  $R\{x, u\}$  admits no linearizing point symmetries but that a subsystem of PDES, satisfied by  $M < m$  components of  $u$  admits linearizing point symmetries and hence yields the linearization of  $R\{x, u\}$  by a non-invertible transformation.)

#### 4. Examples of linearizations

##### 4.1. BURGERS' EQUATION

Suppose  $R\{x, u\}$  is Burgers' equation ( $x = (x, t)$ )

$$\frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0. \quad (4.1)$$

It is easy to show that (4.1) admits only five infinitesimal generators  $\mathbf{X}$ , corresponding to its invariance under a five-parameter Lie group, and hence cannot be linearized by an invertible point transformation. The extension of Theorem 1.3 to the scalar case involves the computation of contact symmetries of (4.1). One can show that the contact symmetries of (4.1) are its point symmetries and hence linearization by an invertible contact transformation is impossible. However (4.1) can be written in the conserved form

$$\frac{D}{Dx} \left( 2 \frac{\partial u}{\partial x} - u^2 \right) - \frac{D}{Dt}(2u) = 0.$$

The corresponding system  $S\{x, u, V\}$  given by

$$\frac{\partial V}{\partial x} = 2u, \quad \frac{\partial V}{\partial t} = 2 \frac{\partial u}{\partial x} - u \quad (4.2)$$

admits

$$\mathbf{X}_s = e^{V/4} \left\{ \left[ 2F^2(x, t) + uF^1(x, t) \right] \frac{\partial}{\partial u} + 4F^1(x, t) \frac{\partial}{\partial V} \right\}$$

where  $F = (F^1, F^2)$  is an arbitrary solution of the linear system  $L[X]F = 0$  given by ( $X = (x, t)$ )

$$\frac{\partial F^1}{\partial x} = F^2, \quad \frac{\partial F^2}{\partial x} = \frac{\partial F^1}{\partial t}. \quad (4.3)$$

Hence Theorem 1.3 is satisfied ( $u^1 = u, u^2 = V$ ) with coefficients  $a_1^1 = a_1^2 = a_2^1 = a_2^2 = b_2^1 = 0, b_1^1 = ue^{V/4}, b_1^2 = 2e^{V/4}, b_2^2 = 4e^{V/4}$ . Then Theorem (1.4) yields  $(X_1, X_2) = (x, t)$  and  $(\psi^1, \psi^2) = e^{-V/4}(-1, \frac{1}{2}u)$ . Thus the invertible mapping  $z_1 = x, z_2 = t, w^1 = -e^{-V/4}, w^2 = \frac{1}{2}ue^{-V/4}$  transforms (4.4) to

$$\frac{\partial w^1}{\partial z_1} = w^2, \quad \frac{\partial w^2}{\partial z_1} = \frac{\partial w^1}{\partial z_2},$$

and hence yields the well-known Hopf-Cole transformation  $u = -\frac{2w^2}{w^1} = -\frac{2}{w^1} \frac{\partial w^1}{\partial z_1}$  which embeds  $R\{x, u\}$  in the linear heat equation

$$\frac{\partial^2 w^1}{\partial z_1^2} = \frac{\partial w^1}{\partial z_2}.$$

#### 4.2. NONLINEAR DIFFUSION EQUATION

The nonlinear diffusion equation  $R\{x, u\}$  given by

$$\frac{D}{Dx} \left( \frac{1}{u^2} \frac{\partial u}{\partial x} \right) - \frac{Du}{Dt} = 0 \quad (4.4)$$

admits no linearizing point or contact symmetries. However  $S\{x, u, V\}$  given by

$$\frac{\partial V}{\partial x} = u, \quad \frac{\partial V}{\partial t} = \frac{1}{u^2} \frac{\partial u}{\partial x} \quad (4.5)$$

admits

$$\mathbf{X}_s = F^1(t, V) \frac{\partial}{\partial x} - u^2 F^2(t, V) \frac{\partial}{\partial u}$$

where  $F = (F^1, F^2)$  is an arbitrary solution of the linear system ( $X = (V, t)$ )

$$F^2 = \frac{\partial F^1}{\partial V}, \quad \frac{\partial F^2}{\partial V} = \frac{\partial F^1}{\partial t}.$$

Application of Theorems 1.3 and 1.4 leads to the invertible mapping  $z_1 = V$ ,  $z_2 = t$ ,  $w^1 = x$ ,  $w^2 = u^{-1}$ , which transforms (4.7) to the linear system

$$\frac{\partial w^1}{\partial z_1} = w^2, \quad \frac{\partial w^1}{\partial z_2} = \frac{\partial w^2}{\partial z_1}$$

and hence to the linearization of (4.4) by a non-invertible mapping to the linear heat equation

$$\frac{\partial w^1}{\partial z_2} = \frac{\partial^2 w^1}{\partial z_1^2}.$$

#### 4.3. NONLINEAR TELEGRAPH EQUATION

The nonlinear telegraph equation  $R\{x, u\}$  given by

$$\frac{D}{Dx} \frac{\partial u}{\partial x} = \frac{D}{Dt} \left( \frac{1}{u^2} \frac{\partial u}{\partial t} + 1 - \frac{1}{u} \right) \quad (4.6)$$

admits no linearizing point or contact symmetries. But  $S\{x, u, V\}$  given by

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial t}, \quad \frac{\partial u}{\partial t} = u^2 \frac{\partial V}{\partial x} + u(1-u) \quad (4.7)$$

admits

$$\mathbf{X}_s = F^1 \frac{\partial}{\partial x} + e^{-t} F^2 \frac{\partial}{\partial t} + e^{-t} u F^2 \frac{\partial}{\partial u} + F^1 \frac{\partial}{\partial V} \quad (4.8)$$

where  $F = (F^1, F^2)$  is an arbitrary solution of the linear system  $L[X]F = 0$  given by  $(X = (X_1, X_2) = (x - V, t - \ln u))$

$$\frac{\partial F^1}{\partial X_2} = e^{-X_2} \frac{\partial F^2}{\partial X_1}, \quad \frac{\partial F^1}{\partial X_1} = e^{-X_2} \frac{\partial F^2}{\partial X_2}.$$

Then Theorems 1.3 and 1.4 yield the mapping  $z_1 = x - V$ ,  $z_2 = t - \ln u$ ,  $w^1 = x, w^2 = e^t$  which transforms (4.7) to the linear system

$$\frac{\partial w^1}{\partial z_2} = e^{-z_2} \frac{\partial w^2}{\partial z_1}, \quad \frac{\partial w^1}{\partial z_1} = e^{-z_2} \frac{\partial w^2}{\partial z_2}.$$

Consequently one obtains a non-invertible mapping which linearizes (4.6).

In this example it turns out to be difficult to discover the linearizing point symmetry (4.8) of system (4.7). In particular, in solving the determining equations it is tricky to find the appropriate coordinates  $(X_1, X_2) = (x - V, t - \ln u)$ .

## 5. A new approach to discover linearizations

In using Theorems 1.3 and 1.4 to discover the existence or non-existence of linearization of a given system of PDES by an invertible mapping the main difficulty is in solving the determining equations. The number of unknowns is  $m + n$ . We now give a new algorithm useful for discovering linearizations, which also involves the determination of the nature of the solution set of a system of linear PDES. Here the number of unknowns usually turns out to be  $m$ . The linear system to be solved is the system of determining equations for point symmetries if and only if the determining equations are self-adjoint.

**DEFINITION 5.1.** Factors (multipliers, characteristics)  $\{\lambda^\sigma(x, u, u, u, \dots, u)\}$ ,  $\sigma = 1, 2, \dots, m$ , yield a *conserved form* for  $R\{x, u\}$  if and only if

$$\lambda^\sigma G^\sigma(x, u, u, u, \dots, u) = \frac{D}{Dx_i} f^i(x, u, u, u, \dots, u) \quad (5.1)$$

for some  $\{f^i(x, u, u, u, \dots, u)\}$ , for any solution of  $R\{x, u\}$ .

**DEFINITION 5.2.** A conserved form (5.1) for  $R\{x, u\}$  is an *equivalent conserved form* for  $R\{x, u\}$  if and only if for some  $\rho = 1, 2, \dots, m$ , any solution of  $\lambda^\rho(x, u, u, u, \dots, u) = 0$  is also a solution of  $R\{x, u\}$ .

By taking the Fréchet derivative of  $R\{x, u\}$  (let  $u \rightarrow u + \epsilon U$ ) one obtains the “linearized” system of PDES

$$\mathcal{L}_\kappa^\sigma[u]U^\kappa = 0, \quad \sigma = 1, 2, \dots, m, \quad (5.2)$$

which must hold for any solution of  $R\{x, u\}$  (see [7, p. 317]), denoted by  $\mathcal{L}[u]U = 0$ ;  $\mathcal{L}[u]$  is a linear operator whose coefficients depend on components of  $(x, u, u, u, \dots, u)$ .

Let  $\mathcal{L}^*[u]$  be the adjoint operator of  $\mathcal{L}[u]$ . Then the following theorem holds:

**THEOREM 5.3.** *Factors  $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^m)$  leading to equivalent conserved forms for  $R\{x, u\}$  must satisfy the linear system*

$$\mathcal{L}^*[u]\lambda = 0, \quad (5.3)$$

for any solution  $u(x)$  of  $R\{x, u\}$ .

[It is easy to see that when  $\mathcal{L}[u]$  is self-adjoint, i.e.  $\mathcal{L}[u] = \mathcal{L}^*[u]$ , then a factor  $\lambda = \eta$  yields an infinitesimal generator of the form (1.3) admitted by  $R\{x, u\}$  if  $\lambda$  depends at most linearly on the components of  $u$ . However the converse often does not hold, i.e., a symmetry (1.3)<sub>1</sub> admitted by  $R\{x, u\}$  does not necessarily yield an equivalent conserved form of  $R\{x, u\}$ .]

If  $R\{x, u\}$  is a linear system of PDES, denoted by  $L[x]u = 0$ , i.e.

$$G^\sigma = L_\kappa^\sigma[x]u^\kappa = 0, \quad \sigma = 1, 2, \dots, m, \quad (5.4)$$

for some linear operator  $L[x]$ , then the following theorem is easily proved:

**THEOREM 5.4.** *Factors  $\lambda(x) = (\lambda^1(x), \lambda^2(x), \dots, \lambda^m(x))$  yield equivalent conserved forms for a linear system (5.4) if and only if*

$$L^*[x]\lambda = 0, \quad (5.5)$$

where  $L^*[x]$  is the adjoint operator of  $L[x]$ .

Suppose  $R\{x, u\}$  is a quasilinear system of first order PDES which linearizes by an invertible mapping

$$\begin{aligned} z_j &= X_j = \phi_j(x, u), \quad j = 1, 2, \dots, n, \\ w^\gamma &= \psi^\gamma(x, u), \quad \gamma = 1, 2, \dots, m. \end{aligned}$$

Then there exist specific functions  $\{\mu_\sigma^\rho(x, u, u)\}$  such that

$$\mu_\sigma^\rho G^\sigma = L_\gamma^\rho[X]w^\gamma = 0, \quad \rho = 1, 2, \dots, m,$$

corresponding to  $L[X]w = 0$ , for some linear operator  $L[X]$ . From Theorem 5.4 it follows that there exist factors  $\lambda(x) = (\lambda^1(x), \lambda^2(x), \dots, \lambda^m(x))$  such that

$$\lambda^\rho \mu_\sigma^\rho G^\sigma = \frac{D\mathcal{F}^i}{DX_j} \quad \text{for some } \{\mathcal{F}^i\},$$

with  $L^*[X]\lambda = 0$ , for any solution of  $R\{x, u\}$ .

One can show that conserved forms are invariant under any change of coordinates defined by an invertible point transformation [16]. In particular

$$J \frac{D\mathcal{F}^j}{DX_j} = \frac{D\Phi^k}{Dx_k} \quad \text{for some } \{\Phi^k\}$$

where the Jacobian  $J = \frac{\partial(X_1, X_2, \dots, X_n)}{\partial(x_1, x_2, \dots, x_n)}$ . Consequently the following theorem has been proved:

**THEOREM 5.5.** *If  $R\{x, u\}$  is a quasilinear system of first order PDES which is linearized by an invertible mapping to some linear system  $L[X]w = 0$ , then there exist factors  $\Lambda = (\Lambda^1(x, u, u_1), \Lambda^2(x, u, u_1), \dots, \Lambda^m(x, u, u_1))$ , satisfying  $L^*[u]\Lambda = 0$  for any solution of  $R\{x, u\}$ , such that*

$$\Lambda^\sigma G^\sigma = \frac{D\Phi^k}{Dx_k} \quad \text{for some } \{\Phi^k\}$$

with  $\Lambda^\sigma = \Psi_\rho^\sigma(x, u, u_1) \lambda^\rho(X)$  for some specific functions  $\{\Psi_\rho^\sigma(x, u, u_1) = J\mu_\rho^\sigma\}$  with  $\{\lambda^\rho(X)\}$  satisfying

$$L^*[X]\lambda = 0.$$

Thus in determining all factors  $\Lambda$  which yield an equivalent conserved form for  $R\{x, u\}$  one essentially discovers:

1. whether or not  $R\{x, u\}$  can be linearized by an invertible point transformation;
2. independent variables  $X = (X_1, X_2, \dots, X_n)$  of a linearization;
3. the adjoint of the linearizing system of PDES.

[It turns out that for all known examples the specific functions  $\{\Psi_\rho^\sigma\}$  are independent of  $u_1$  and hence  $\Lambda = (\Lambda^1(x, u), \Lambda^2(x, u), \dots, \Lambda^m(x, u))$ .]

Now we re-examine the linearization examples of §4.

### 5.1. BURGERS' EQUATION

One can straightforwardly show that system (4.2) admits as factors  $\Lambda = (\Lambda^1, \Lambda^2)$  where

$$\Lambda^1 = \frac{e^{-V}}{V} \left[ \left( 1 + \frac{u}{V} \right) \lambda^1 + \lambda^2 \right], \quad \Lambda^2 = \frac{e^{-V}}{V} \lambda^1$$

with  $X = (x, t)$ , and  $\lambda = (\lambda^1(x), \lambda^2(x))$  satisfies  $L^*[X]\lambda = 0$  given by

$$\lambda^2 = \frac{\partial \lambda^1}{\partial x}, \quad \frac{\partial \lambda^2}{\partial x} = -\frac{\partial \lambda^1}{\partial t}.$$

### 5.2. NONLINEAR DIFFUSION EQUATION

It is easy to derive that system (4.5) admits as factors

$$\Lambda = (\Lambda^1, \Lambda^2) = \left( \frac{1}{u} \lambda^1, \lambda^2 \right)$$

with  $X = (V, t)$  and  $\lambda(X) = (\lambda^1(X), \lambda^2(X))$  an arbitrary solution of  $L^*[X]\lambda = 0$  given by

$$\frac{\partial \lambda^2}{\partial V} = \lambda^1, \quad \frac{\partial \lambda^1}{\partial V} + \frac{\partial \lambda^2}{\partial t} + \frac{2}{V} \lambda^1 = 0.$$

### 5.3. NONLINEAR TELEGRAPH EQUATION

It is much less effort in computation to discover that system (4.7) admits as factors

$$\Lambda = (\Lambda^1, \Lambda^2) = \frac{1}{V^2 - 1} \left( V\lambda^1 - \lambda^2, \frac{1}{u}(\lambda^1 - V\lambda^2) \right)$$

with  $X = (X^1, X^2) = (x - V, t - \ln u)$  and  $\lambda(X) = (\lambda^1(X), \lambda^2(X))$  an arbitrary solution of  $L^*[X] = 0$  given by

$$\frac{\partial \lambda^1}{\partial X_2} = \frac{\partial \lambda^2}{\partial X_1}, \quad \frac{\partial \lambda^1}{\partial X_1} = \frac{\partial \lambda^2}{\partial X_2} - \lambda^2,$$

than it is to discover that system (4.7) admits the infinitesimal generator (4.8).

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## NONCLASSICAL SYMMETRY REDUCTIONS AND EXACT SOLUTIONS OF NONLINEAR REACTION-DIFFUSION EQUATIONS

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**ABSTRACT.** In this paper we present some nonclassical symmetry reductions and exact solutions for several physically significant nonlinear reaction-diffusion equations that are not obtainable using the classical Lie method for finding symmetry reductions. These are obtained using the *nonclassical method* due to Bluman and Cole and the *direct method* originally developed by Clarkson and Kruskal to study symmetry reductions of the Boussinesq equation.

### 1. Introduction

The classical method for finding symmetry reductions of PDES is the Lie group method of infinitesimal transformations [3,7,8,44,67,72,80,86], for which symbolic manipulation programmes have been developed, e.g. in MACSYMA [12], MATHEMATICA [5], MAPLE [11,77–79], MUMATH [42] and REDUCE [49,64,83–85]. A survey of the different packages presently available and a discussion of their strengths and applications is given in [12].

To apply the classical method to the second order reaction-diffusion equation

$$u_t = \Omega(x, t, u, u_x, u_{xx}), \quad (1.1)$$

where  $x$  and  $t$  are the independent variables and subscripts denote partial derivatives, we consider the one-parameter Lie group of infinitesimal transformations in  $(x, t, u)$  given by

$$\tilde{x} = x + \varepsilon\xi(x, t, u) + O(\varepsilon^2), \quad (1.2a)$$

$$\tilde{t} = t + \varepsilon\tau(x, t, u) + O(\varepsilon^2), \quad (1.2b)$$

$$\tilde{u} = u + \varepsilon\eta(x, t, u) + O(\varepsilon^2), \quad (1.2c)$$

where  $\varepsilon$  is the group parameter. Requiring that (1.1) is invariant under this transformation yields an overdetermined, linear system of equations for the infinitesimals  $\xi(x, t, u)$ ,  $\tau(x, t, u)$  and  $\eta(x, t, u)$ . The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \quad (1.3)$$

There have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole [6], in their study of symmetry reductions of the linear heat equation, proposed the so-called nonclassical method of group-invariant solutions, also known as the “method of conditional symmetries” [34–36] (see also [37, 70, 73, 74, 90]) and the “method of partial symmetries of the first type” [87]. In this technique, the original system of PDES (1.1) is augmented with the invariant surface condition

$$\psi \equiv \xi(x, t, u)u_x + \tau(x, t, u)u_t - \eta(x, t, u) = 0, \quad (1.4)$$

which is associated with the vector field (1.3). By requiring that both (1.1) and (1.4) are invariant under the transformation (1.2) one obtains an overdetermined, nonlinear system of equations for the infinitesimals  $\xi(x, t, u)$ ,  $\tau(x, t, u)$  and  $\eta(x, t, u)$ , which appear in both the transformation (1.2) and the supplementary condition (1.4). The set of solutions is larger than for the classical method since the number of determining equations is smaller. However, it should be emphasized that the associated vector fields do not form a vector space, still less a Lie algebra, since the invariant surface condition (1.4) depends upon the particular reduction. Subsequently, these methods were further generalized by Olver and Rosenau [69, 70] to include “weak symmetries” and, even more generally, “side conditions” or “differential constraints”, and they concluded that “the unifying theme behind finding special solutions of PDES is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of PDES”.

Solutions of nonlinear overdetermined systems of PDES can be found systematically if a differential Gröbner basis (**DGB**) for the system can be calculated, since a **DGB** provides a “triangulation” of the system [60]. Programs designed to calculate a **DGB** for a system have recently become available [59, 77–79]. The triangulations calculated in this paper were calculated using the **MAPLE** package **diffgrob2** [59].

It is known that there do exist PDES which possess symmetry reductions that are not obtained using the classical Lie group method [69–71, 75–76, 81]. Motivated by this, Clarkson and Kruskal [24] developed a direct, algorithmic method for finding symmetry reductions (in the sequel referred to as the *direct method*), which they used to obtain previously unknown reductions of the Boussinesq equation. Subsequently, Levi and Winternitz [51] showed that all the new reductions of the Boussinesq equation could be obtained using the nonclassical method of Bluman and Cole [6]. The novel characteristic about the direct method, in comparison to the others mentioned above, is that it involves no use of group theory. For many PDES, the direct method appears simpler to implement than either the classical or nonclassical methods and has been successfully applied to obtain new symmetry reductions and exact solutions for several physically significant PDES [15–26, 33, 46, 52–58, 66]. We remark that the direct method has certain resemblances to the so-called “method of free parameter analysis” [41] (though in the latter method, the boundary conditions are crucially used in the determination of the reduction whereas they are not used in the direct method).

Recently, Galaktionov [38–40] has developed a method of “nonlinear separation” in an extensive study of solutions of nonlinear heat equations with source terms. As for the direct method, Galaktionov’s method uses a specific ansatz though it is more general since it reduces a single PDE to a coupled system of ODES.

## 2. The Fitzhugh-Nagumo Equation

In this section we discuss symmetry reductions of the Fitzhugh-Nagumo (FN) equation

$$u_t = u_{xx} + u(1-u)(u-a), \quad (2.1)$$

which arises in population genetics [4] and models the transmission of nerve impulses [32,61].

Applying the classical Lie method to (2.1) yields the infinitesimals

$$\xi = \lambda, \quad \tau = \mu, \quad \eta = 0,$$

where  $\lambda$  and  $\mu$  are arbitrary constants. Hence the only symmetry reduction obtained by this method is the travelling wave solution

$$u(x, t) = w(z), \quad z = \mu x - \lambda t. \quad (2.2)$$

To apply the direct method to (2.1) it suffices to seek a solution in the form

$$u(x, t) = \beta(x, t)w(z) + \alpha(x, t),$$

where  $\alpha(x, t)$ ,  $\beta(x, t)$  and  $z(x, t)$  are functions to be determined. Substituting this into (2.1) yields

$$\begin{aligned} & \beta z_x^2 w'' + (2\beta_x z_x + \beta z_{xx} - \beta z_t)w' - \beta^3 w^3 + \beta^2(a+1-3\alpha)w^2 \\ & + [2(a+1)\alpha\beta - 3\beta\alpha^2 - a\beta + \beta_{xx} - \beta_t]w + [\alpha_{xx} + \alpha_t - \alpha(\alpha-a)(\alpha-1)] = 0. \end{aligned} \quad (2.3)$$

We require that this is an ODE for  $w(z)$  and so the ratios of the coefficients of the powers and derivatives of  $w$  have to be functions of  $z$  only. This generates a system of equations for  $\beta(x, t)$ ,  $\alpha(x, t)$  and  $z(x, t)$ . In the generic case when  $z_x \neq 0$ , it can be shown that  $\beta = z_x$  and  $\alpha = \frac{1}{3}(1+a)$ . Hence we obtain the reduction

$$u(x, t) = z_x w(z) + \frac{1}{3}(1+a),$$

where  $z(x, t)$  satisfies the overdetermined system of equations

$$3z_{xx} - z_t = z_x^2 \Gamma_1(z), \quad (2.4a)$$

$$z_{xxx} - z_{xt} + \frac{1}{3}(a^2 - a + 1)z_x = z_x^3 \Gamma_2(z), \quad (2.4b)$$

$$(a+1)(2a-1)(a-2) = z_x^3 \Gamma_3(z), \quad (2.4c)$$

with  $\Gamma_1(z)$ ,  $\Gamma_2(z)$  and  $\Gamma_3(z)$  functions to be determined, and  $w(z)$  satisfies

$$w'' = w^3 - \Gamma_1(z)w' - \Gamma_2(z)w - \frac{1}{27}\Gamma_3(z) \quad (2.5)$$

(see Nucci and Clarkson [66] for further details). Therefore, to determine symmetry reductions we need to find solutions of (2.4). There are two cases to consider.

CASE 2.2.1.  $(a+1)(2a-1)(a-2) \neq 0$ . In this generic case, the only symmetry reduction obtained by the direct method is travelling wave solution (2.2).

CASE 2.2.2.  $(a+1)(2a-1)(a-2) = 0$ . In this case, from (2.4c) we see that  $\Gamma_3(z) \equiv 0$ . In the special case when  $\Gamma_1(z) = 0$  and  $\Gamma_2(z) = 0$ , the compatibility of (2.4a,b) yields

$$z_{xxx} - \mu^2 z_x = 0, \quad (2.6)$$

where  $\mu^2 = \frac{1}{6}(a^2 - a + 1)$  (i.e.,  $\mu^2 = \frac{1}{2}$ , if  $a = -1$  or  $a = 2$ , and  $\mu^2 = \frac{1}{8}$ , if  $a = \frac{1}{2}$ ). Solving (2.6) and ensuring that the solution also satisfies (2.4a) with  $\Gamma_1(z) = 0$  yields

$$z(x, t) = c_1 \exp(\mu x + 3\mu^2 t) + c_2 \exp(-\mu x + 3\mu^2 t) + c_3,$$

where  $c_1$ ,  $c_2$  and  $c_3$  are arbitrary constants. If  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$ , then (2.5) is solvable in terms of Jacobi elliptic functions [1,29]. Thus we obtain the following exact solutions of the FN equation (2.1), for  $a = -1$

$$u(x, t) = \frac{1}{2}\sqrt{2} \left[ c_1 \exp\left(\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) - c_2 \exp\left(-\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) \right] \operatorname{ds}\left(z; \frac{1}{2}\sqrt{2}\right), \quad (2.7a)$$

$$z(x, t) = c_1 \exp\left(\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) + c_2 \exp\left(-\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) + c_3, \quad (2.7b)$$

for  $a = 2$

$$u(x, t) = \frac{1}{2}\sqrt{2} \left[ c_1 \exp\left(\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) - c_2 \exp\left(-\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) \right] \operatorname{ds}\left(z; \frac{1}{2}\sqrt{2}\right) + 1, \quad (2.8a)$$

$$z(x, t) = c_1 \exp\left(\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) + c_2 \exp\left(-\frac{1}{2}\sqrt{2}x + \frac{3}{2}t\right) + c_3, \quad (2.8b)$$

and for  $a = \frac{1}{2}$

$$u(x, t) = \frac{1}{4}\sqrt{2} \left[ c_1 \exp\left(\frac{1}{4}\sqrt{2}x + \frac{3}{8}t\right) - c_2 \exp\left(-\frac{1}{4}\sqrt{2}x + \frac{3}{8}t\right) \right] \operatorname{ds}\left(z; \frac{1}{2}\sqrt{2}\right) + \frac{1}{2}, \quad (2.9a)$$

$$z(x, t) = c_1 \exp\left(\frac{1}{4}\sqrt{2}x + \frac{3}{8}t\right) + c_2 \exp\left(-\frac{1}{4}\sqrt{2}x + \frac{3}{8}t\right) + c_3, \quad (2.9b)$$

where  $\operatorname{ds}(z; k)$  is the Jacobi elliptic function satisfying

$$(\eta')^2 = \eta^4 + (2k^2 - 1)\eta^2 + k^2(k^2 - 1). \quad (2.10)$$

(The special case where  $k = \frac{1}{2}\sqrt{2}$  is known as the lemniscate case [1,29].) Cosgrove [28] has solved the overdetermined system (2.4a,b) in the case when  $\Gamma_1(z)$  and  $\Gamma_2(z)$  are not both identically zero and shown that all consistent solutions of this system yields solutions of the FN equation (2.1) which are equivalent under a gauge transformation to either the travelling wave solution (2.2) or the special solutions (2.7–2.9) given above.

Applying the nonclassical method to the FN equation (2.1) yields the following determining equations for the infinitesimals

$$\xi_{uu} = 0,$$

$$\eta_{uu} - 2(\xi_{xu} - \xi\xi_u) = 0,$$

$$2\eta_{xu} - 2\xi_u\eta - 3u(u-a)(u-1)\xi_u - \xi_{xx} + 2\xi\xi_x + \xi_t = 0,$$

$$\eta_t - \eta_{xx} + 2\eta\xi_x + (2\xi_x - \eta_u)u(u-a)(u-1) + [3u^2 - 2(a+1)u + a]\eta = 0.$$

A preliminary triangulation of this system leads to the condition

$$\xi_u(2\xi_u^2 - 9) = 0.$$

Setting  $\xi_u = k$ , where  $k \neq 0$ , the triangulation of the system is

$$-2k\xi - 3(a+1) + 9u = 0,$$

$$2\eta + 3u(u-a)(u-1) = 0.$$

Setting  $\xi_u = 0$  one obtains the triangulation

$$3\eta + (3u - a - 1)\xi_x = 0,$$

$$\xi_t - 3\xi_{xx} + 2\xi\xi_x = 0,$$

$$(1 - a + a^2)\xi_x + 3\xi_{xxx} - 3\xi\xi_{xx} = 0,$$

$$\xi_{xx}(3\xi_{xx} - 2\xi\xi_x) = 0,$$

$$\xi_x^3\xi^2(1 + a^2 - a)[3(a^2 - a + 1) - 2\xi^2 + 6\xi_x] = 0,$$

$$\xi_x(a - 2)(2a - 1)(a + 1) = 0.$$

CASE 2.3.1.  $a = -1$ . In this case, (2.1) reduces to the real Newell-Whitehead (RNW) equation [63] (or Kolmogoroff-Petrovsky-Piscounov equation [50])

$$u_t = u_{xx} + u - u^3. \quad (2.11)$$

The infinitesimals are  $\tau = 1$ , and

$$\xi = -\frac{3}{\sqrt{2}} \frac{c_1 \exp(\sqrt{2}x) + c_2}{c_1 \exp(\sqrt{2}x) - c_2}, \quad \eta = -\frac{6uc_1c_2 \exp(\sqrt{2}x)}{\left[c_1 \exp(\sqrt{2}x) - c_2\right]^2}, \quad (2.12)$$

where  $c_1$  and  $c_2$  are arbitrary constants. Consequently, solving the invariant surface condition (1.4) gives the exact solution (2.7) of the RNW equation (2.11). We remark that Cariello and Tabor [9,10] found the exact solution (2.7) of the RNW equation (2.11) using a truncated Painlevé expansion and verified that it derives from a nonclassical symmetry reduction (see also [27]).

CASE 2.3.2.  $a = \frac{1}{2}$ . The infinitesimals are  $\tau = 1$ , and

$$\xi = -\frac{3}{2\sqrt{2}} \frac{c_1 \exp\left(\frac{1}{2}\sqrt{2}x\right) + c_2}{c_1 \exp\left(\frac{1}{2}\sqrt{2}x\right) - c_2}, \quad \eta = -\frac{3(2u-1)c_1c_2 \exp\left(\frac{1}{2}\sqrt{2}x\right)}{4 \left[c_1 \exp\left(\frac{1}{2}\sqrt{2}x\right) - c_2\right]^2}, \quad (2.13)$$

where  $c_1$  and  $c_2$  are arbitrary constants. Solving the invariant surface condition (1.4) yields the exact solution (2.9).

CASE 2.3.3.  $a = 2$ . The infinitesimals are  $\tau = 1$ ,  $\xi$  is the same as for Case 2.3.1, and  $\eta = -\xi_x(u-1)$ . An analogous procedure to that used in Case 2.3.1 yields the exact solution (2.8).

CASE 2.3.4.  $a$  arbitrary. The infinitesimals are  $\tau = 1$ ,  $\xi = \pm\frac{1}{2}\sqrt{2}(3u - a - 1)$  and  $\eta = -\frac{3}{2}u(u - a)(u - 1)$ , and the associated invariant surface condition is

$$u_t \pm \frac{1}{2}\sqrt{2}(3u - a - 1)u_x + \frac{3}{2}u(u - a)(u - 1) = 0. \quad (2.14)$$

Eliminating  $u_t$  in the FN equation (2.1) yields a member of the Riccati-chain [2,65], i.e.

$$u_{xx} = \pm \frac{1}{2} \sqrt{2} (a + 1 - 3u) u_x - \frac{1}{2} u(u - a)(u - 1),$$

which can be linearized by the transformation  $u = \pm \sqrt{2} (\ln \phi)_x$ , yielding

$$2\phi_{xxx} \mp \sqrt{2} (1 + a) \phi_{xx} + a\phi_x = 0.$$

By solving this equation and requiring that  $u$  also satisfies (2.14), we obtain the following exact solutions of the FN equation (2.1): (i), for  $a \neq 0, a \neq 1$

$$u(x, t) = \frac{ac_1 \exp \left\{ \frac{1}{2} (\pm \sqrt{2} ax + a^2 t) \right\} + c_2 \exp \left\{ \frac{1}{2} (\pm \sqrt{2} x + t) \right\}}{c_1 \exp \left\{ \frac{1}{2} (\pm \sqrt{2} ax + a^2 t) \right\} + c_2 \exp \left\{ \frac{1}{2} (\pm \sqrt{2} x + t) \right\} + c_3 \exp (at)}; \quad (2.15)$$

(ii), for  $a = 0$

$$u(x, t) = \frac{c_1 \exp \left\{ \frac{1}{2} (\sqrt{2} x + t) \right\} + \sqrt{2} c_2}{c_1 \exp \left\{ \frac{1}{2} (\sqrt{2} x + t) \right\} + c_2 (x - \sqrt{2} t) + c_3}, \quad (2.16)$$

and (iii), for  $a = 1$

$$u(x, t) = \frac{c_2 [x - \sqrt{2}(t + 1)] + c_3}{c_1 \exp \left\{ \frac{1}{2} (\sqrt{2} x + t) \right\} + c_2 (x - \sqrt{2} t) + c_3}, \quad (2.17)$$

where  $c_1, c_2$  and  $c_3$  are arbitrary constants. The solution (2.15) was obtained by Vorob'ev [87] (who calls the associated symmetry a “partial symmetry of the first type”), Kawahara and Tanaka [48] (using Hirota’s bi-linear method [45]) and Hereman [43] (using the truncated Painlevé expansion method [62,88,89]). Plots of (2.15) for (i),  $a = -0.5$ , (ii),  $a = 0.4$ , (iii),  $a = 0.7$ , and (iv),  $a = 1.5$  are given in Figure 1. The solution (2.16) was obtained by Chen and Guo [13] using a truncated Painlevé expansion (see also [14,31]). However, these solutions (2.15,2.16) are *not* obtainable using the direct method unless  $a = -1$ ,  $a = 2$  or  $a = \frac{1}{2}$  when (2.15) is a special case of (2.7–2.9), respectively.

Estevez [30] has recently shown that the exact solution (2.15) can be obtained using a generalization of the direct method. If  $\beta = z_x$  and  $z(x, t)$  and  $\alpha(x, t)$  satisfy

$$3z_{xx} - z_t = \pm \sqrt{2} (a + 1 - 3\alpha) z_x, \quad (2.18a)$$

$$z_{xxx} - z_{xt} = [3\alpha^2 - 2(a + 1)\alpha + a] z_x, \quad (2.18b)$$

$$\alpha_{xx} - \alpha_t + \alpha(\alpha - 1)(a - \alpha) = 0, \quad (2.18c)$$

then (2.3) reduces to

$$z_x (w'' - w^3) + (a + 1 - 3\alpha) (\pm \sqrt{2} w' + w^2) = 0.$$

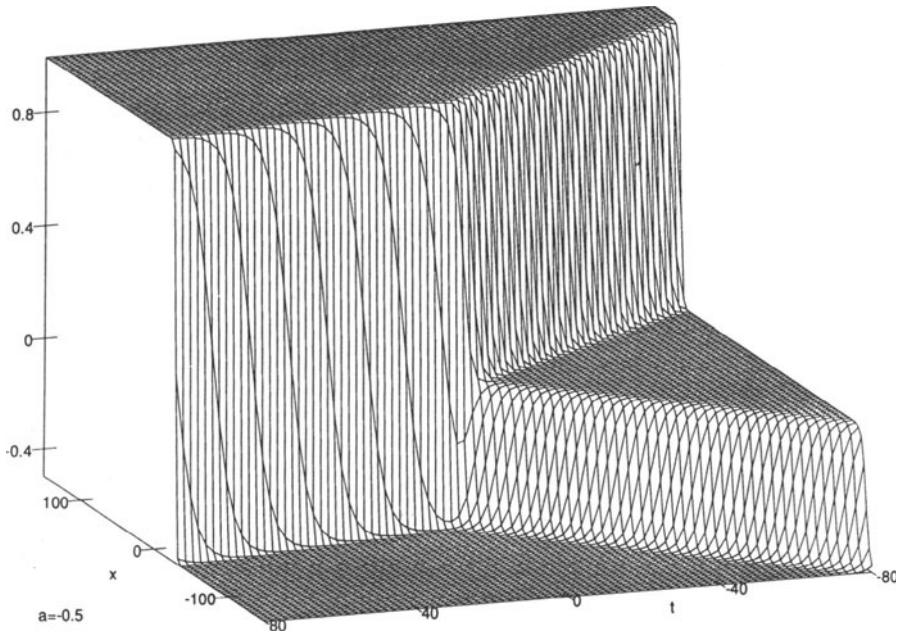


Figure 1(i): Plot of (2.15) for  $a = -0.5$

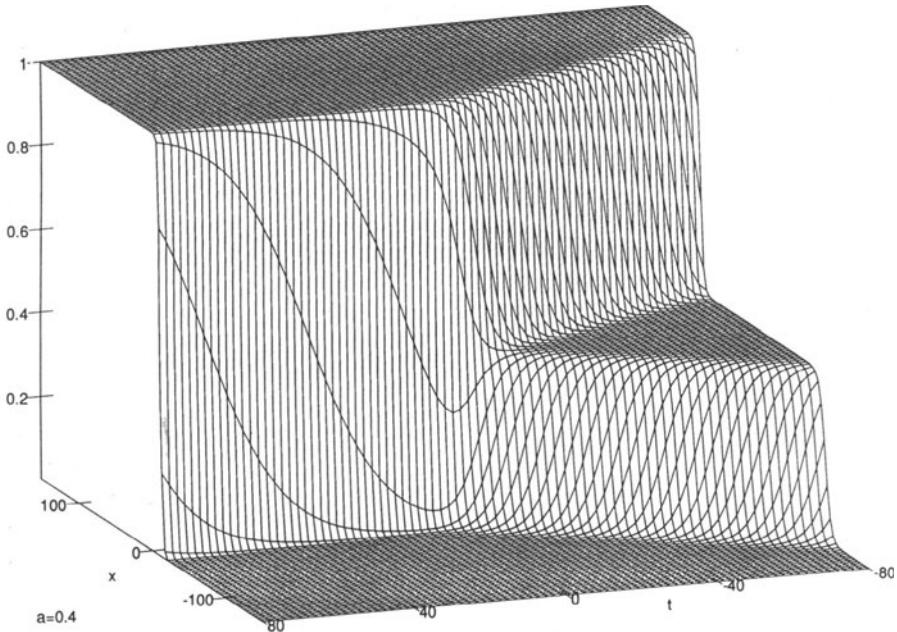
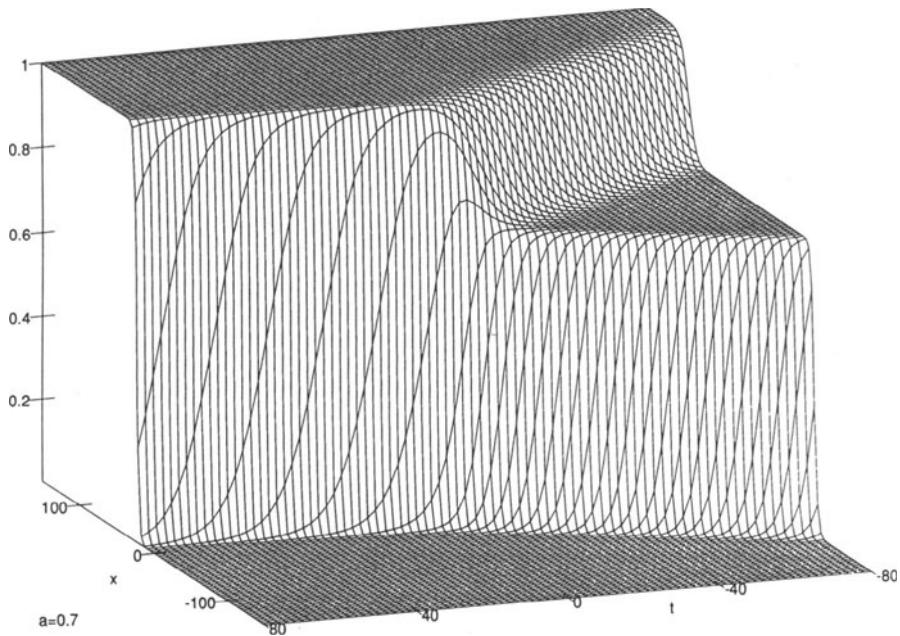
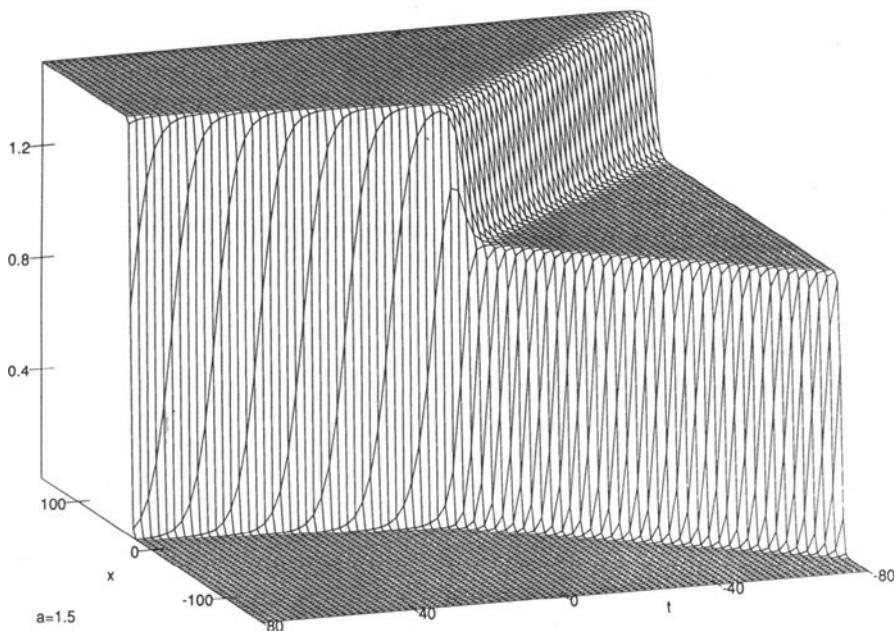


Figure 1(ii): Plot of (2.15) for  $a = 0.4$

Figure 1(iii): Plot of (2.15) for  $a = 0.7$ Figure 1(iv): Plot of (2.15) for  $a = 1.5$

Therefore if  $w(z)$  satisfies the overdetermined system of equations

$$w'' - w^3 = 0, \quad \pm\sqrt{2}w' + w^2 = 0, \quad (2.19)$$

which have common solution

$$w(z) = \pm\sqrt{2}/(z - z_0), \quad (2.20)$$

with  $z_0$  an arbitrary constant, then we obtain an exact solution of the FN equation (2.1). Equation (2.18c) has three constant solutions,  $\alpha = 0$ ,  $\alpha = 1$  and  $\alpha = a$ , then solving (2.18a,b), given  $\alpha$ , and substituting in (2.20) yields (2.15) for all three values of  $\alpha$ . We remark that (2.18) are the same equations as those which arise in the truncated Painlevé expansion method [43].

### 3. The Nonlinear Heat Equation

In this section we discuss symmetry reductions of the nonlinear heat equation

$$u_t = u_{xx} + u^3. \quad (3.1)$$

This diffusion equation with absorption arises in several applications including: the spatial diffusion of biological populations, in several chemical diffusion processes where the physical structure changes with concentration, and materials undergoing heating by microwave radiation which exhibit thermal conductivities and body heating that are strongly dependent upon temperature.

Applying the classical Lie method to (3.1) yields the infinitesimals

$$\xi = \kappa x + \lambda, \quad \tau = 2\kappa t + \mu, \quad \eta = -\kappa u,$$

where  $\kappa$ ,  $\lambda$  and  $\mu$  are arbitrary constants. Hence the symmetry reduction obtained by this method are: (i), the travelling wave solution (for  $\kappa = 0$ )

$$u(x, t) = w(z), \quad z = \mu x - \lambda t; \quad (3.2)$$

and (ii), the scaling reduction (for  $\kappa \neq 0$ , set  $\lambda = \mu = 0$ , without loss of generality)

$$u(x, t) = t^{-1/2}w(z), \quad z = x/t^{1/2}. \quad (3.3)$$

Applying the direct method to (3.1) yields one additional reduction

$$u(x, t) = xw(z), \quad z = \frac{1}{2}x^2 + 3t, \quad (3.4)$$

where  $w(z)$  satisfies

$$w'' + w^3 = 0,$$

which is solvable in terms of Jacobi elliptic functions [1,29]. Hence we obtain the exact solution

$$u(x, t) = \frac{1}{2}\sqrt{2}x \operatorname{sd}\left(\frac{1}{2}x^2 + 3t; \frac{1}{2}\sqrt{2}\right), \quad (3.5)$$

where  $\text{sd}(z; k)$  is the Jacobi elliptic function satisfying

$$(\eta')^2 = 1 + (2k^2 - 1)\eta^2 + k^2(k^2 - 1)\eta^4. \quad (3.6)$$

Applying the nonclassical method to (3.1) yields the following equations for the infinitesimals

$$\begin{aligned} \xi_{uu} &= 0, \\ \eta_{uu} - 2(\xi_{xu} - \xi\xi_u) &= 0, \\ 2\eta_{xu} - 2\xi_u\eta + 3u^3\xi_u - \xi_{xx} + 2\xi\xi_x + \xi_t &= 0, \\ \eta_t - \eta_{xx} + 2\eta\xi_x + 3u^2\eta + (\eta_u - \xi_x)u^3 &= 0. \end{aligned}$$

A preliminary triangulation yields the condition

$$u^3\xi_u^2(2\xi^2 + 9u^2) = 0.$$

In the case  $\xi_u = 0$  we obtain the triangulation

$$\begin{aligned} \eta + u\xi_x &= 0, \\ \xi_t - 3\xi_{xx} + 2\xi\xi_x &= 0, \\ \xi_{xxx} - \xi\xi_{xx} &= 0, \\ \xi_{xx}(-3\xi_{xx} + 2\xi\xi_x) &= 0, \\ \xi_{xx}\xi_x\xi(3\xi_x - \xi^2) &= 0. \end{aligned}$$

In the case  $\xi = \pm \frac{3}{2}i\sqrt{2}u$ , we obtain the additional equation

$$\eta = \frac{3}{2}u^3.$$

There are thus four canonical solutions of these equations:

1.  $\xi = c$ , a constant, and  $\eta = 0$  which are associated infinitesimals for the travelling wave solution (3.2);
2.  $\xi = x/(2t)$  and  $\eta = -u/(2t)$  which are associated infinitesimals for the scaling reduction (3.3);
3.  $\xi = -3/x$  and  $\eta = -3u/t^2$ , which are associated infinitesimals for the reduction (3.4); and
4.  $\xi = ku$  and  $\eta = \frac{3}{2}u^3$ , where  $k^2 = -\frac{9}{2}$ .

In case (4) we solve (3.1) together with the associated invariant surface condition

$$kuu_x + u_t - \frac{3}{2}u^3 = 0. \quad (3.7)$$

Eliminating  $u_t$  between (3.1) and (3.7) yields

$$u_{xx} + kuu_x - \frac{1}{2}u^3 = 0,$$

which has general solution

$$u(x, t) = \frac{3}{k} \left\{ \frac{1}{x + \lambda(t)} + \frac{1}{x + \mu(t)} \right\}.$$

Requiring that this also satisfies (3.1) yields the exact solution

$$u(x, t) = \frac{2\sqrt{2}i(x + x_0)}{(x + x_0)^2 + 6(t + t_0)}, \quad (3.8)$$

where  $x_0$  and  $t_0$  are arbitrary constants. This is a special case of the exact solution (3.5) obtained using the direct method. It is interesting to note that whereas (3.8) is a special case of a symmetry reduction obtained using the direct method, the analogous infinitesimals for the FN equation (2.1), i.e. Case 2.3.4 above, yield the exact solutions (2.15, 2.16) which are not special cases of reductions obtained using the direct method, unless either  $a = -1$ ,  $a = \frac{1}{2}$  or  $a = 2$ .

#### 4. The Real Landau-Ginzburg Equation

In this section we discuss symmetry reductions of the real Landau-Ginzburg (RLG) equation

$$u_t + u_{xx} + 2u^2v = \gamma u, \quad (4.1a)$$

$$v_t + v_{xx} + 2uv^2 = \gamma v, \quad (4.1b)$$

where  $\gamma$  is a real constant, which arise in several physical applications including Bénard and Taylor problems, chemical reactions and semi-conductors.

If  $\gamma \neq 0$ , then the RLG equation (4.1) possesses the following two symmetry reductions: (i), the travelling wave reduction

$$u(x, t) = \exp(\mu_2 t) U(z), \quad v(x, t) = \exp(-\mu_2 t) V(z), \quad z(x, t) = x - \mu_1 t, \quad (4.2)$$

where  $\mu_1$  and  $\mu_2$  are arbitrary constants; and (ii),

$$u(x, t) = \mu [c_1 \cos(\mu x) - c_2 \sin(\mu x)] \exp\left(\frac{3}{2}\gamma t\right) U(z), \quad (4.3a)$$

$$v(x, t) = \mu [c_1 \cos(\mu x) - c_2 \sin(\mu x)] \exp\left(\frac{3}{2}\gamma t\right) V(z), \quad (4.3b)$$

$$z(x, t) = [c_1 \sin(\mu x) + c_2 \cos(\mu x)] \exp\left(\frac{3}{2}\gamma t\right), \quad (4.3c)$$

where  $c_1$  and  $c_2$  are arbitrary constants and  $\mu^2 = \frac{1}{2}\gamma$ . The reduction (4.2) is obtainable using the classical Lie group method whereas (4.3) is not. The reduction (4.3), which is the analogue of (2.7) for the FN equation, is obtainable using both the direct and nonclassical methods.

If  $\gamma = 0$ , then the RLG equation (4.1) possesses three symmetry reductions: (i), the travelling wave reduction (4.2); (ii), the scaling reduction

$$u(x, t) = (2t)^\mu U(z), \quad v(x, t) = (2t)^{-1-\mu} V(z), \quad z(x, t) = x/(2t)^{1/2}, \quad (4.4)$$

where  $\mu$  is an arbitrary constant; and (iii),

$$u(x, t) = xU(z), \quad v(x, t) = xV(z), \quad z(x, t) = \frac{1}{2}x^2 - 3t. \quad (4.5)$$

The reductions (4.2,4.4) are obtainable using the classical Lie group method whereas (4.5) is not. The reduction (4.5), which is the analogue of (3.4) for the nonlinear heat equation (3.1), is obtainable using both the direct and nonclassical methods.

We remark that for reductions (4.3) and (4.5),  $U(z)$  and  $V(z)$  satisfy

$$U'' + 2U^2V = 0, \quad V'' + 2UV^2 = 0. \quad (4.6)$$

These equations are solved by setting  $W = -UV$ , which yields the Weierstrass elliptic function equation

$$W'' = 6W^2 - g_2, \quad (4.7)$$

where  $g_2$  is an arbitrary constant. Thus  $W(z) = \wp(z; g_2, g_3)$ , where  $g_3$  is an arbitrary constant, and so  $U(z)$  and  $V(z)$  satisfy the Lamé equations

$$U'' - 2\wp(z; g_2, g_3)U = 0, \quad V'' - 2\wp(z; g_2, g_3)V = 0. \quad (4.8)$$

The general solution of the Lamé equation

$$w'' - 2\wp(z)w = 0,$$

is

$$w(z) = c_1 \exp\{-z\zeta(a)\} \frac{\sigma(z+a)}{\sigma(z)} + c_2 \exp\{z\zeta(a)\} \frac{\sigma(z-a)}{\sigma(z)},$$

where  $c_1, c_2$  are arbitrary constants,  $\zeta(z)$  and  $\sigma(z)$  are the Weierstrass zeta and sigma functions and  $a$  is any solution of the transcendental equation  $\wp(a) = 0$  [47, p379].

## 5. Discussion

In this paper we have reviewed the direct method developed by Clarkson and Kruskal [24] as a technique for determining symmetry reductions of nonlinear PDES. As illustrated here, this method has been successfully applied to obtain many new symmetry reductions and exact solutions for several physically significant PDES.

Now we make some general remarks about symmetry reductions of PDES. Generally, given a PDE with a symmetry (i.e., a transformation of the dependent and/or independent variables that leaves the equation invariant), the action of the symmetry group takes one solution of the equation into another solution of the equation. Starting with a fixed solution that corresponds to the identity element of the group, then every element of the group corresponds to some solution of the (same) equation — the starting solution can be any solution of the equation. This mapping can be used to define a symmetry, and the group carries the set of all solutions of the PDE into itself. Given such a symmetry of a PDE, one can seek solutions which are mapped into themselves under the action of the group. These are symmetry reductions corresponding to the group. For a PDE with two independent and one dependent variables, these solutions typically are solutions of an ODE.

Alternatively, the ODE can be taken as a means of generating symmetry reductions (or special solutions) of the PDE, without regard to what maps into what. Then the ODE appears to be an example of the side condition introduced by Olver and Rosenau [69,70]. This seems to be the way that symmetry reductions are generally used. The symmetry reductions obtained here are defined through an ODE that is “compatible” with the PDE (in the sense that they have common solutions). Again, the ODE is a side condition on the PDE, and the surprise is that there exist common solutions. The issue of mapping solutions of the

PDE does not arise in the direct method used in this paper and so there is no connection with remarks in the previous paragraph (in fact, as shown above, the transformation groups associated with the additional symmetry reductions do not map solutions of the PDE into itself). The direct method appears to be an effective procedure of generating symmetry reductions of given PDES without using group theory.

In their generalization of the nonclassical method due to Bluman and Cole [6], Olver and Rosenau [69,70] show that in order to determine a group-invariant solution of a given PDE, one can use any group of infinitesimal transformations. However, in general, given any group of infinitesimal transformations and any PDE, there will be no solutions invariant under the group and so the question becomes how does one determine *a priori* whether a given group will give a meaningful symmetry reduction? One possibility is that by seeking a solution in a certain form (as we have done in this paper), one is naturally lead to the appropriate group (i.e., the requirement that the symmetry reduction reduces the PDE to an ODE is equivalent to the “side conditions” in the terminology of Olver and Rosenau [69,70]). The results obtained using the direct method provides further evidence to support the remark by Olver and Rosenau quoted in the introduction. Nevertheless, group theory clearly remains important in the determination of explicit, physically significant, special solutions of PDES (as also demonstrated by Olver and Rosenau [70]).

In §2 it was demonstrated that there exist symmetry reductions of the FN equation (2.1) which are obtainable using the nonclassical method due to Bluman and Cole [6], but not using the direct method due to Clarkson and Kruskal [24]. This answers the question posed by Clarkson and Kruskal on the relationship between these two methods; the nonclassical method is more general than the direct method, at least as it was originally formulated. Clarkson and Hood [22], following Olver and Rosenau [69,70], have shown that every reduction obtained by the direct method is necessarily obtainable using the nonclassical method. A recent extension of the direct method by Estevez [30] does find all the symmetry reduction solutions of the FN equation found by the nonclassical method. Since the original development of the direct method, it has been known that there exist exact solutions of PDES which are not obtainable using the direct method; for example, the two-soliton solution (and more generally the  $N$ -soliton solution), of the Korteweg-de Vries equation. We remark that the solutions (2.15,2.16) have certain similarities to a “two-soliton” solution (further they have also been obtained using Hirota’s bi-linear method [45]). These results pose the following important open question: “for which PDES does the nonclassical method yield more symmetry reductions than the direct method?” Furthermore, it remains an open question to determine *a priori* which PDES possess symmetry reductions that are not obtainable using the classical Lie group approach.

The ansatz  $u(x,t) = F(x,t,w(z))$  with  $z = z(x,t)$  used in the direct method assumes that the symmetry variable  $z$  does not depend upon  $u$ . Consequently it is implicitly assumed that one of the ratios of infinitesimals  $\xi/\tau$ ,  $\xi/\eta$  or  $\tau/\eta$  are independent on  $u$ . In Case 2.3.4 above, all three ratios of infinitesimals are dependent upon  $u$ . However even if all these ratios are dependent upon  $u$ , then this does not guarantee that the associated symmetry solution is not obtainable using the direct method. For example, consider Case 4 for equation (3.1) above.

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## BOUNDARY CONDITIONS ON SIMILARITY SOLUTIONS

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### 1. Introduction

Group-invariant solutions of a differential equation are obtained by expressing the equation in terms of the invariants of a subgroup of its symmetry group, and solving this reduced form of the equation. Boundary conditions may be applied to the reduced form if they are invariant under the subgroup. Sometimes this may be effected by a suitable choice of the subgroup used for the reduction, but often rather few boundary conditions are invariant under any subgroup. One may then consider modelling the conditions using functions that are invariant, and this requires a list of such allowed conditions. It is thus convenient to have a classification of all conditions invariant under some subgroup of a given symmetry group. This is analogous to the classification problem for equations, identifying special values of constants allowing a larger symmetry group.

For the simplest case of an equation for  $u(x, y)$ , a boundary is a curve in the domain, and a condition that specifies  $u$  on this boundary is a curve in space. The condition is invariant under a subgroup if both these curves are invariant. Acceptable boundary conditions correspond to invariant curves.

### 2. Invariant Conditions

The boundary condition

$$u = M(x, y) \quad \text{on} \quad B(x, y) = 0 \quad (1)$$

is invariant under the transformations generated by

$$\xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \zeta(x, y, u) \frac{\partial}{\partial u} \quad (2)$$

if, when (1) holds,

$$\xi \frac{\partial M}{\partial x} + \eta \frac{\partial M}{\partial y} - \zeta = 0 \quad (3)$$

$$\xi \frac{\partial B}{\partial x} + \eta \frac{\partial B}{\partial y} = 0. \quad (4)$$

The generator (2) is assumed known from the symmetry algebra of a given differential equation, and will depend linearly on parameters which will be determined or related when (3) and (4) are solved. The solution proceeds using the associated characteristic equations. This is just a standard calculation of invariants.

The characteristic equations may also be given in a parametric form:  $u = f(\theta)$  on the boundary  $x = g(\theta)$ ,  $y = h(\theta)$  is invariant under (2) if

$$\frac{g'(\theta)}{\xi[g(\theta), h(\theta)]} = \frac{h'(\theta)}{\eta[g(\theta), h(\theta)]} = \frac{f'(\theta)}{\zeta[g(\theta), h(\theta), f(\theta)]} \quad (5)$$

### 3. A Diffusion-Convection Equation

The equation

$$\frac{\partial u}{\partial y} = a \frac{\partial}{\partial x} \left[ u^p \frac{\partial u}{\partial x} \right] + c u^b \frac{\partial u}{\partial x} \quad (6)$$

admits a 3-parameter group generated by

$$[\alpha + \gamma(p - b)x] \frac{\partial}{\partial x} + [\beta + \gamma(p - 2b)y] \frac{\partial}{\partial y} + \gamma u \frac{\partial}{\partial u}. \quad (7)$$

Assume that  $\gamma$ ,  $(p - b)$  and  $(p - 2b)$  are nonzero, and take  $\gamma = 1$ ; the parameters  $\alpha$  and  $\beta$  will then be fixed by the boundary condition invariance. The general solution of (5) is

$$x = g(\theta) = x_0 + c_1 \theta^s, \quad y = h(\theta) = y_0 + c_2 \theta^r \quad (8)$$

$$u = f(\theta) = k_1 \theta^{sr} \quad (9)$$

where  $r = 1/(p - b)$  and  $s = 1/(p - 2b)$ . Equations (8) describe all possible invariant boundaries, and (9) the invariant boundary conditions on these boundaries. They are easily put into Cartesian form. The subgroup under which the curves are invariant will depend on  $x_0$  and  $y_0$ , the parameters in the generator (7) taking the values  $\alpha = -x_0/r$ ,  $\beta = -y_0/s$ .

### 4. Conditions on Derivatives

To consider invariance of ( $u_x = \partial u / \partial x$ )

$$u_x = M(x, y) \quad \text{on} \quad B(x, y) = 0 \quad (10)$$

requires prolongation of the generator. If  $\xi_u = \eta_u = \eta_x = 0$ , add to (2)  $\chi \partial / \partial u_x$  where

$$\chi = \zeta_x + \zeta_u u_x - \xi_x u_x. \quad (11)$$

Then (3) is changed by replacing  $\zeta$  by  $\chi$ , and the corresponding change made in (5). The possible invariant boundaries are the same; the solution of (3) or (5) now determines certain invariant curves in the prolongation space.

For the example given by (6) and (7),  $\chi = (1 + b - p)u_x$ , and the invariant condition, on the same boundaries (8), is

$$u_x = k_2 \theta^{sr-s}. \quad (12)$$

Because the subgroup is determined by the boundaries only, conditions (9) and (12) are simultaneously invariant.

Physical restrictions are often conditions on the directional derivatives either normal or tangential to the boundary, say  $u_N$  and  $u_T$ . For the boundary  $B(x, y) = 0$ , satisfying (4),

$$u_T = \frac{B_y u_x - B_x u_y}{[B_x^2 + B_y^2]^{1/2}}, \quad u_N = \frac{B_x u_x + B_y u_y}{[B_x^2 + B_y^2]^{1/2}}. \quad (13)$$

If  $V$  is the complete first prolongation of (2), including a  $\partial/\partial u_y$  term, then  $u_T = M(x, y)$  is invariant if

$$V[u_T - M(x, y)] = 0 \quad \text{when} \quad B(x, y) = 0. \quad (14)$$

(Alternatively  $u_T$  may be replaced by  $u_N$ .) These conditions are complicated by the denominators in (13), which have to be differentiated when  $V$  is applied in (14). The calculation is simple if  $M(x, y) = 0$ . For the example (7),  $u_T = 0$  is invariant, but  $u_N = 0$  is invariant only for the special form of (6) with  $b = 0$ .

## 5. Conclusion

When obtaining similarity solutions to differential equations, it is useful to know the boundary conditions invariant under different subgroups, so that an appropriate reduction may be determined. In three dimensions, these conditions correspond to invariant curves, either in the physical space or the prolongation space. Complete results for various diffusion type equations will be given elsewhere. Further work is in progress on conditions that specify a derivative which is tangential or normal to the boundary curve.

# SYMMETRY REDUCTIONS AND EXACT SOLUTIONS OF THE DAVEY-STEWARTSON SYSTEM

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**ABSTRACT.** In this paper, symmetry reductions and exact solutions are presented for the Davey-Stewartson (DS) system which has the completely integrable DS I and DS II systems as special cases. These symmetry reductions are obtained using the Direct Method originally developed by Clarkson & Kruskal to study symmetry reductions of the Boussinesq equation; the method involves no group theoretic techniques. Using these reductions we obtain exact solutions expressible in terms of the second and fourth Painlevé equations and special cases thereof, and also in terms of elementary functions. In special cases of the DS system the solutions contain arbitrary spatial functions in addition to the usual temporal ones.

## 1. Introduction

In this paper we study symmetry reductions and exact solutions of the Davey-Stewartson (DS) system first derived by Davey and Stewartson (1974),

$$ip_t + \epsilon_1 p_{xx} + p_{yy} - \epsilon_2(|p|^2 + v)p = 0, \quad (1.1a)$$

$$\delta_1 v_{xx} + v_{yy} - \delta_2 \left( |p|^2 \right)_{xx} = 0, \quad (1.1b)$$

where  $\delta_1$  and  $\delta_2$  take any real values, and  $\epsilon_1$  and  $\epsilon_2$  take the values 1 or  $-1$ .

In the case of purely one-dimensional propagation (along the  $y$  axis)  $p_x \equiv 0$ , so we can consider solutions with  $v \equiv 0$ . In this case (1.1) reduces to the ubiquitous cubic nonlinear Schrödinger equation,

$$ip_t + \epsilon_1 p_{xx} = \epsilon_2 |p|^2 p. \quad (1.2)$$

The DS system therefore provides a 2-dimensional generalisation in which the basic direction of propagation remains a privileged one.

As a consequence of the non-analytic term in (1.1a) we introduce  $q(x, r, t)$ , which is the (formal) complex conjugate of  $p(x, y, t)$  for  $x, y$  and  $t$  real, but otherwise is independent. Hence we obtain

$$ip_t + \epsilon_1 p_{xx} + p_{yy} - \epsilon_2(v - pq)p = 0, \quad (1.3a)$$

$$-iq_t + \epsilon_1 q_{xx} + q_{yy} - \epsilon_2(v - pq)q = 0, \quad (1.3b)$$

$$\delta_1 v_{xx} + v_{yy} - \delta_2(pq)_{xx} = 0. \quad (1.3c)$$

There are two special cases of these equations to consider: (i), if  $\delta_1 = \delta_2 = -1$  and  $\epsilon_1 = \epsilon_2 = 1$ , then we obtain the Davey-Stewartson I (DSI) equations; and (ii), if  $\delta_1 = \delta_2 = \epsilon_2 = 1$  and  $\epsilon_1 = -1$ , we obtain the Davey-Stewartson II (DSII) equations.

### 1.1. PHYSICAL BACKGROUND

The Davey-Stewartson system was first derived in order to describe the evolutions of progressive waves with slowly varying amplitude under gravity in water of finite depth (Davey and Stewartson, 1974), using the method of *multiple scales*. They considered the velocity potential,  $\phi(x, y, z, t)$ , such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (1.4)$$

with appropriate boundary conditions. By writing

$$\phi(x, y, z, t) = \sum_{n=-\infty}^{\infty} \phi_n \exp[in(kx - \omega t)], \quad \phi_n = \sum_{j=n}^{\infty} \epsilon^j \phi_{nj}, \quad (1.5)$$

they derived the system

$$i \frac{\partial A}{\partial \tau} + \lambda \frac{\partial^2 A}{\partial \xi^2} + \mu \frac{\partial^2 A}{\partial \eta^2} = \nu |A|^2 A + \nu_1 A Q, \quad (1.6a)$$

$$\lambda_1 \frac{\partial^2 Q}{\partial \xi^2} + \mu_1 \frac{\partial^2 Q}{\partial \mu^2} = \kappa_1 \frac{\partial^2 |A|^2}{\partial \eta^2}, \quad (1.6b)$$

i.e., essentially (1.1), where

$$\phi_{11} = A(\xi, \eta, \tau) \frac{\cosh k(z+h)}{\cosh kh}, \quad (1.6c)$$

$z = \zeta$  is the free surface elevation,  $\xi = \epsilon(x - c_g t)$ ,  $\eta = \epsilon y$ ,  $\tau = \epsilon^2 t$  ( $\epsilon$  being a small parameter),  $c_g$  is the group velocity, and  $\lambda$ ,  $\lambda_1$ ,  $\mu$ ,  $\mu_1$ ,  $\nu$ ,  $\nu_1$  and  $\kappa_1$  depend upon two dimensionless parameters,  $kh$  and  $k^2 T/g$ , with  $k$  the wave-number in the  $x$ -direction of the underlying wave train,  $h$  the depth,  $T$  the surface tension and  $g$  the acceleration due to gravity. In the shallow water limit as  $kh \rightarrow 0$  (and after scaling  $x$  and  $y$ ) (1.6) simplify to (1.3).

### 1.2. MATHEMATICAL BACKGROUND

We may write the DSI and DSII systems as

$$ip_t + \frac{1}{2}(\sigma^2 p_{xx} + p_{yy}) = (v - |p|^2)p, \quad (1.7a)$$

$$v_{xx} - \sigma^2 v_{yy} = 2(|p|)_{xx}, \quad (1.7b)$$

where  $\sigma^2 = 1$  for DSI and  $\sigma^2 = -1$  for DSII. The equations DSI and DSII are two of the few systems in more than  $(1+1)$ -dimensions for which the Cauchy initial value problem is solvable by the inverse scattering method (Fokas and Ablowitz, 1984), and are therefore said to be *completely integrable*.

The Lax Pair for DS<sub>I</sub> and DS<sub>II</sub> is

$$\begin{aligned}\mathbf{v}_x &= -\sigma \mathbf{J} \mathbf{v}_y + \mathbf{Q} \mathbf{v}, \\ \mathbf{v}_t &= \mathbf{A} \mathbf{v} - i\sigma^{-1} \mathbf{Q} \mathbf{v}_y + i \mathbf{J} \mathbf{v}_{yy},\end{aligned}$$

where

$$\mathbf{J} = \text{diag}(1, -1), \quad \mathbf{Q} = \begin{pmatrix} 0 & p(x, y) \\ q(x, y) & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} A_{11}(x, y) & A_{12}(x, y) \\ A_{21}(x, y) & A_{22}(x, y) \end{pmatrix},$$

with

$$\begin{aligned}A_{12} &= \frac{1}{2}i\sigma^2(\partial_x - \sigma\partial_y)p, \\ A_{21} &= -\frac{1}{2}i\sigma^2(\partial_x + \sigma\partial_y)q, \\ (\partial_x + \sigma\partial_y)A_{11} &= -\frac{1}{2}i\sigma^2(\partial_x - \sigma\partial_y)(pq), \\ (\partial_x - \sigma\partial_y)A_{22} &= \frac{1}{2}i\sigma^2(\partial_x + \sigma\partial_y)(pq).\end{aligned}$$

The inverse scattering scheme in the case  $\sigma = -1$  (DS<sub>I</sub>) is formulated in terms of a Riemann-Hilbert boundary value problem and in the case  $\sigma = i$  (DS<sub>II</sub>) it is formulated in terms of a  $\bar{\partial}$  problem.

According to the *Generalised Painlevé Conjecture*, any PDE solvable by inverse scattering will pass the Painlevé PDE Test, as devised by Weiss, Tabor and Canevale (1983). This is indeed the case for DS<sub>I</sub> and DS<sub>II</sub> (Ganesan and Lakshmanan, 1987; Leo *et al.*, 1988). It turns out that the Hirota bi-linear representation for DS<sub>I</sub> and DS<sub>II</sub> may be obtained in a straight forward way from the Painlevé PDE analysis (Ganesan and Lakshmanan, 1987); the method yields the  $N$ -soliton solution (Leo *et al.*, 1988).

DS<sub>I</sub> possesses “dromion” solutions which are localised coherent structures generated by the boundary conditions (Boiti *et al.*, 1988; Fokas and Santini, 1989, 1990). A method developed by Gilson and Nimmo (1991), related to the Hirota bilinear representation, has been applied to DS<sub>I</sub> and three classes of localised special solutions were found: solitons (plane waves); “solitoffs” (which are localised in every direction bar one and resemble solitons, except that they tend to a non-zero value in only one direction); and “dromions” (which might be described as exponential lumps).

The mathematical structure of the nonintegrable cases of the DS system appears to be much less rich.

### 1.3. THE DIRECT METHOD

The *Direct Method* is a method of finding symmetry reductions of a partial differential equation (PDE). The novel characteristic of this method is that unlike other methods for finding symmetry reductions of PDES, it does not make use of group theory, though a group theoretic explanation of these results was subsequently given by Levi and Winternitz (1989). The basic idea is to seek a reduction of a given system of PDES

$$\Delta_j(x, y, t, u_1(x, y, t), \dots, u_n(x, y, t)) = 0, \quad j = 1, 2, 3, \quad (1.8)$$

in the form

$$u_j(x, y, t) = F_j(x, y, t, w_j(z(x, y, t))), \quad j = 1, 2, 3. \quad (1.9)$$

Substituting this into (1.8) and demanding that the result be a system of ODES for  $w_j(z)$  imposes conditions on  $F_j(x, y, t)$  and  $z(x, y, t)$  and their derivatives in the form of an over-determined system of nonlinear PDES. This in principle allows one to solve for  $F_j$  and  $z$ . We remark that it is often sufficient to take  $F_j$  in a simpler form (e.g., linear in  $w_j(z)$ ).

There are freedoms in the determination of  $F_j$  and  $z$ , that we may exploit without loss of generality, which are valuable in keeping the method manageable: translation and rescaling of  $w_j(z)$  and redefinition of  $z$ . For example, if  $F_j$  is linear in  $w_j$ , i.e.,  $F_j = g_j(x, y, t)w_j(z) + h_j(x, y, t)$ , then

1. if  $h_j$  is of the form  $h_j(x, y, t) = h_0(x, y, t) + g_j(x, y, t)\Gamma_1(z)$ , where  $h_0(x, y, t)$  is specified, then we may assume  $\Gamma_1(z) \equiv 0$ , by translating  $w_j(z) \rightarrow w_j(z) - \Gamma_1(z)$ ,
2. and if  $g_j(x, y, t) = g_0(x, y, t)\Gamma_2(z)$ , where  $g_0(x, y, t)$  is specified, then we may take  $\Gamma_2(z) \equiv 1$  by rescaling  $w_j(z) \rightarrow w_j(z)/\Gamma_2(z)$ .
3. Further, if  $\Gamma_3(z) = z_0(x, y, t)$ , where  $z_0(x, y, t)$  is specified, we may redefine  $z \rightarrow \Gamma_3^{-1}(z)$  to obtain  $z = z_0(x, y, t)$ .

Each freedom may be used at most once. We remark that it is sometimes more difficult to use the general ansatz (1.9) and so we use a simplified one, as we do in the following calculations.

#### 1.4. NOTATION

Unless otherwise stated, the following notation is used:  $\Gamma_i(z)$ ,  $i = 1, 2, 3, \dots$ , are functions to be determined arising when balancing coefficients of powers and derivatives of  $w_j$ ,  $j = 1, 2, 3$ ;  $\gamma_i$ ,  $i = 1, 2, 3, \dots$ , are constants to be determined which are generated from the  $\Gamma_i(z)$ ; and  $\lambda_i$ ,  $i = 1, 2, 3, \dots$ , are constants of integration.

## 2. Application of the Direct Method to the DS System.

We assume reductions of the form

$$p(x, y, t) = R(t)\xi(z) \exp[i\Phi(x, y, t)], \quad (2.1a)$$

$$q(x, y, t) = R(t)\zeta(z) \exp[-i\Phi(x, y, t)], \quad (2.1b)$$

$$v(x, y, t) = S(x, y, t)\chi(z) + T(x, y, t), \quad (2.1c)$$

$$z(x, y, t) = \theta(t)(x + \kappa y) + \psi(t). \quad (2.1d)$$

Substituting this ansatz into the DS equations (1.3) and rescaling  $\chi(z)$  (recall freedom (ii)) it is easy to show that  $S$  is a function of  $t$  only and that we may take

$$R^2(t) = S(t). \quad (2.2)$$

We are then left with the determining equations,

$$R^2\Gamma_1(z) = -\Phi_t - \epsilon_1(\Phi_x)^2 - (\Phi_y)^2 - T, \quad (I)$$

$$R^2\Gamma_2(z) = \frac{R'}{R} - \epsilon_1\Phi_{xx} + \Phi_{yy}, \quad (II)$$

$$R^2\Gamma_3(z) = (x + \kappa y)\theta' + \psi' + 2\theta(\epsilon_1\Phi_x + \kappa\Phi_y), \quad (III)$$

$$R^2\Gamma_4(z) = \theta^2(\epsilon_1 + \kappa^2), \quad (IV)$$

$$R^4\Gamma_5(z) = \delta_1 T_{xx} + T_{yy}, \quad (V)$$

where  $\Gamma_i(z)$  are unknown functions of  $z$ , to be determined, and then  $\xi(z)$ ,  $\zeta(z)$  and  $\chi(z)$  satisfy the system of ODES

$$\Gamma_4(z)\xi'' - \epsilon_2\xi^2\zeta - \epsilon_2\xi\chi + i\Gamma_3(z)\xi' + [\Gamma_1(z) + i\Gamma_2(z)]\xi = 0, \quad (2.3a)$$

$$\Gamma_4(z)\zeta'' - \epsilon_2\xi\zeta^2 - \epsilon_2\zeta\chi - i\Gamma_3(z)\zeta' + [\Gamma_1(z) - i\Gamma_2(z)]\zeta = 0, \quad (2.3b)$$

$$(\delta_1 + \kappa^2)\chi'' + \Gamma_5(z) - \delta_2(\xi\zeta)'' = 0. \quad (2.3c)$$

## REMARKS

1. Both  $\epsilon_2$  and  $\delta_2$  appear only as multiplicative constants to terms in (2.3). These terms are trivially ‘‘balanced’’ with the normalising term and so neither  $\epsilon_2$  nor  $\delta_2$  plays any role in the system of determining equations, or the reductions found.
2. By contrast, the values of the other constants,  $\epsilon_1$  and  $\delta_1$ , are crucial in determining how the system of determining equations (I) through (V) behaves. Firstly, consider (IV); with  $\epsilon_1 + \kappa^2 \neq 0$ , then we take  $\Gamma_4(z) \equiv \gamma_4 = \epsilon_1 + \kappa^2$ , and so after rescaling one of  $\theta(t)$  and  $R(t)$  if necessary, we may take  $\theta(t) \equiv R(t)$ . But if  $\epsilon_1 + \kappa^2 = 0$ , then we set  $\Gamma_4(z) \equiv 0$  and  $\theta(t)$  and  $R(t)$  are no longer linked in this way. We consider the former case first.

### 2.1. THE GENERAL CASE: $\theta \equiv R$

Equation (III) is a first order linear PDE and so may be integrated by the method of characteristics to yield

$$\Phi(x, y, t) = \frac{\tilde{\Gamma}_3(z)}{2} - \frac{\theta'(t)}{2\epsilon_1\theta(t)} \left( \frac{x^2}{2}(1 - \kappa^2/\epsilon_1) + \kappa xy \right) - \frac{\psi'(t)x}{2\epsilon_1\theta(t)} + F(\kappa x/\epsilon_1 - y; t). \quad (2.4a)$$

Rescaling  $p(z)$  and  $q(z)$  (recall freedom (ii)) allows us to set  $\Gamma_3(z) \equiv 0$ . It is easy to show that we may assume

$$F(\kappa x/\epsilon_1 - y; t) = f_2(t)(\kappa x/\epsilon_1 - y)^2 + f_1(t)(\kappa x/\epsilon_1 - y) + f_0(t), \quad (2.4b)$$

then, equation (II) gives

$$f_2(t) = \tilde{\gamma}_2\theta^2(t) - \frac{\theta'}{4\theta(t)}, \quad \tilde{\gamma}_2 = \frac{\gamma_2}{2(1 + \kappa^2/\epsilon_1)}. \quad (2.5)$$

Freedom (i) allows us to set  $\Gamma_1(z) \equiv 0$ , and so (I) gives us  $T(x, y, t)$  in terms of  $\Phi(x, y, t)$ . Finally, equation (V) yields

$$\begin{aligned} \gamma_5\theta^6 &= (\delta_1 + \epsilon_1)\theta\theta'' + \kappa^2[2(1 + \delta_1\epsilon_1)(\epsilon_1 - 1) - 2(\delta_1 + \epsilon_1)](\theta')^2 \\ &\quad + 8\tilde{\gamma}_2\kappa^2[1 - \delta_1 + \epsilon_1(\delta_1 - 1)]\theta^3\theta' - 16(\tilde{\gamma}_2)^2[\delta_1\kappa^4 + \epsilon_1(\delta_1 + \epsilon_1)\kappa^2 + \epsilon_1]\theta^6. \end{aligned} \quad (2.6)$$

## REMARKS

1. The functions  $\psi(t)$ ,  $f_1(t)$  and  $f_0(t)$  all remain arbitrary. We write  $\psi(t) = f_3(t)$  for consistency below.
2. There are two distinct cases of (2.6) to consider: first, the integrable case  $\delta_1 + \epsilon_1 = 0$ ; and second, the nonintegrable case  $\delta_1 + \epsilon_1 \neq 0$ .

**2.1.1. Integrable Case:**  $\delta_1 + \epsilon_1 = 0$ . For the integrable cases of the DS system ( $\delta_1 = -1$ ,  $\epsilon_1 = 1$ , for DSI and  $\delta_1 = 1$ ,  $\epsilon_1 = -1$  for DSII) (2.6) becomes

$$\theta^6(t)[\gamma_5 \pm 8(\tilde{\gamma}_2)^2(\kappa^4 - 1)] = 0. \quad (2.7)$$

Since  $\theta(t) \neq 0$ , then  $\gamma_5 \pm 8(\tilde{\gamma}_2)^2(\kappa^4 - 1) = 0$ , and  $\theta(t)$  is an arbitrary function.

**REDUCTION 1.** We set  $\theta(t) = f_4(t)$  and obtain the reduction for DSI and DSII given by,

$$\begin{aligned} p(x, y, t) &= f_4(t)\xi(t) \left[ i \left( -\frac{f'_4(t)}{2\epsilon_1 f_4(t)} \left( \frac{x^2}{2} \left( 1 - \frac{\kappa^2}{\epsilon_1} \right) + \kappa x y \right) - \frac{f'_3(t)x}{2\epsilon_1 f_4(t)} \right. \right. \\ &\quad \left. \left. + \left( \tilde{\gamma}_2 f_4^2(t) - \frac{f'_4(t)}{4f_4(t)} \right) (\kappa x - y)^2 + f_1(t) \left( \frac{\kappa x}{\epsilon_1} - y \right) + f_0(t) \right) \right], \end{aligned} \quad (2.8a)$$

$$v(x, y, t) = f_4^2(t)\chi(z) + T(x, y, t), \quad (2.8b)$$

$$z(x, y, t) = f_4(t)(x + \kappa y) + f_3(t), \quad (2.8c)$$

where

$$\begin{aligned} T(x, y, t) &= \left( \frac{x^2}{\epsilon_1} + y^2 \right) \left[ \frac{\theta''}{4\theta} - \frac{(\theta')^2}{2\theta^2} \right] - \frac{4\tilde{\gamma}_2^2(\kappa^2 + \epsilon_1)}{\epsilon_1} \left( \frac{\kappa x}{\epsilon_1} - y \right)^2 \theta^4 \\ &\quad + x \left[ \frac{\kappa\theta'f_1}{\epsilon_1\theta} - \frac{\psi'\theta'}{\epsilon_1\theta^2} - \frac{\kappa f'_1}{\epsilon_1} + \frac{\psi''}{2\epsilon_1\theta} - 4\kappa\tilde{\gamma}_2 \left( \kappa^2 + \frac{1}{\epsilon_1} \right) \theta^2 f_1 + 2\tilde{\gamma}_2 \kappa^2 \psi'\theta \right] \\ &\quad + y \left[ 4\tilde{\gamma}_2 \left( 1 + \frac{\kappa^2}{\epsilon_1} \right) \theta^2 f_1 - \frac{\theta' f_1}{\theta} - \frac{2\tilde{\gamma}_2 \kappa \psi' \theta}{\epsilon_1} \theta + f'_1 \right] \\ &\quad + \frac{\kappa \psi' f_1}{\epsilon_1 \theta} - \frac{(\psi')^2}{4\epsilon_1 \theta^2} - \frac{\kappa^2 f_1^2}{\epsilon_1} - f_1^2 - f'_0, \end{aligned} \quad (2.8d)$$

and  $\xi(z)$ ,  $\zeta(z)$  and  $\chi(z)$  satisfy the system,

$$(\epsilon_1 + \kappa^2)\xi'' - \epsilon_2 \xi^2 \zeta - \xi \chi + i\gamma_2 \xi = 0, \quad (2.8e)$$

$$(\epsilon_1 + \kappa^2)\zeta'' - \epsilon_2 \xi \zeta^2 - \zeta \chi - i\gamma_2 \zeta = 0, \quad (2.8f)$$

$$(\delta_1 + \kappa^2)\chi'' - \delta_2(\xi \zeta)'' + \gamma_5 = 0, \quad (2.8g)$$

provided that  $\gamma_5 \pm 8\tilde{\gamma}_2^2(\kappa^4 - 1) = 0$ . These ODES may be integrated in terms of the second Painlevé equation (if  $\gamma_2 = 0$ ) or the fourth Painlevé equation (cf., Gagnon *et al.*, 1989, Gagnon and Winternitz 1989); which are

$$\frac{d^2u}{dz^2} = 2u^3 + zu + \alpha, \quad P_{II}$$

$$\frac{d^2u}{dz^2} = \frac{1}{2u} \left( \frac{du}{dz} \right)^2 + \frac{3u^3}{2} + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u}. \quad P_{IV}$$

**2.1.2. Nonintegrable Case:**  $\delta_1 + \epsilon_1 \neq 0$ . Here we consider the nonintegrable case of the DS system (1.3). i.e., we no longer assume specific values of the parameters  $\delta_1$ ,  $\delta_2$ ,  $\epsilon_1$  and  $\epsilon_2$  (other than that  $\epsilon_1$  and  $\epsilon_2$  may take only the values  $-1$ , and  $1$ ). We may write (2.6) as

$$A\theta\theta'' + B(\theta')^2 + C\theta^3\theta' + D\theta^6 = 0, \quad (2.9)$$

where  $A, B, C$  and  $D$  are defined in the obvious way. [We note that  $A = 0$  if and only if  $\delta_1 + \epsilon_1 = 0$ , which is exactly the integrable case considered above.] We therefore take  $A = 1$

Solutions of (2.9) are given by:

1. if  $B = C = 0$  or  $B = 1$  and  $C = 0$  then (2.9) is solvable in terms of Jacobian elliptic functions;
2. if  $C = D = 0$  then  $\theta^{B+1}(t) = (B+1)(\lambda_1 t + \lambda_2)$ ;
3. if  $B = D = 0$  then  $\int(\lambda_1 - C\theta^3/3)^{-1}d\theta = t + \lambda_2$ ; and
4. if  $B = C = D = 0$  then  $\theta(t) = \lambda_1 t + \lambda_2$ ;
5. if  $B = 1, C \neq 0$  and  $D = C^2/18$  then

$$\theta^2(t) = \frac{3}{C} \left( \frac{1}{z + \lambda_1} + \frac{1}{z + \lambda_2} \right);$$

6. if  $B = 1, C \neq 0$  and  $D = -\frac{1}{2}C^2$  then (2.9) is solvable in terms of Weierstrass elliptic functions;
7. if  $B = 1, C \neq 0$  and  $D = 0$ , then  $\theta^2(t) = \lambda_1 \tanh(C\lambda_1 t + \lambda_2)$ ;
8. if  $D = (1+B)C^2/[4(2+B)^2], B \neq -1, B \neq -2$  and  $C \neq 0$  then

$$\theta^2(t) = \frac{(m+2)(\lambda_1 t + \lambda_2)^m}{mC \int^t (\lambda_1 s + \lambda_2)^m ds},$$

where  $m = 2/(1+B)$ .

REDUCTION 2. In this non-integrable case we obtain the reduction

$$\begin{aligned} p(x, y, t) &= \theta(t)\xi(t) \exp \left[ i \left( -\frac{\theta'(t)}{2\epsilon_1\theta(t)} \left( \frac{x^2}{2} \left( 1 - \frac{\kappa^2}{\epsilon_1} \right) + \kappa xy \right) - \frac{f'_3(t)x}{2\epsilon_1\theta(t)} \right. \right. \\ &\quad \left. \left. + \left( \tilde{\gamma}_2\theta^2(t) - \frac{\theta'(t)}{4\theta(t)} \right) (\kappa x - y)^2 + f_1(t) \left( \frac{\kappa x}{\epsilon_1} - y \right) + f_0(t) \right) \right], \end{aligned} \quad (2.10a)$$

$$v(x, y, t) = \theta^2(t)\chi(z) + T(x, y, t), \quad (2.10b)$$

$$z(x, y, t) = \theta(t)(x + \kappa y) + f_3(t), \quad (2.10c)$$

where  $T(x, y, t)$  is the same as for Reduction 1,  $\theta(t)$  takes one of the forms given above and  $\xi(z), \zeta(z)$  and  $\chi(z)$  satisfy the same system of ODES as given in Reduction 1, and so we obtain solutions in terms of the second and fourth Painlevé equations (and special cases thereof).

## 2.2. THE SPECIAL CASE $\kappa = \pm 1$ .

We now consider the special case where determining equation (V) is trivially satisfied. i.e.,  $\Gamma_4(z) \equiv 0$  and  $\kappa^2 + \epsilon_1 = 0$ . Since we assume that  $\kappa$  is real and  $\epsilon_1$  is restricted to the values  $\pm 1$ , then the special case corresponds to  $\kappa = \pm 1$  with  $\epsilon_1 = -1$ . (If  $\delta_1 = 1$  also then we have the integrable case DSII.)

Equation (III) has the (symmetric) general solution

$$\Phi(x, y, t) = [\psi' - R^2\Gamma_3(z)] \frac{(x \mp y)}{2\theta} + \frac{\theta'}{4\theta}(x^2 - y^2) + F(y \pm x; t). \quad (2.11)$$

REMARK. One might choose to absorb the arbitrary function  $\Gamma_3(z)$  in  $F$  (which is also arbitrary at this point). However, this would mean that all derivatives in the system of ODES

generated by the reduction would be zero. i.e this is a totally degenerate case, yielding only the trivial solution  $\xi(z) = \zeta(z) = 0$ . Instead we set  $\Gamma_3(z) \equiv 1$ , the other canonical form and proceed.

Equation (II) gives the Bernoulli equation

$$\gamma_2 R^3(t) = R' - \frac{\theta'}{2\theta} R, \quad (2.12)$$

for  $R(t)$  given  $\theta(t)$ . We may assume that  $\Gamma_1(z) \equiv 0$  [recall freedom (i), section 1] and so (I) gives

$$\begin{aligned} T(x, y, t) &= \left[ \frac{\theta''}{4\theta} - \frac{(\theta')^2}{2\theta^2} \right] (y^2 - x^2) + \left[ \frac{\theta'}{\theta} (y \pm x) \pm \left( \frac{\psi' - R^2}{\theta} \right) \right] F_X \\ &\quad + \left[ \frac{\theta' \psi'}{\theta^2} - \frac{\psi''}{\theta} - \frac{\theta' R^2}{2\theta^2} + \frac{R R'}{\theta} \right] (x \mp y) - F_t. \end{aligned} \quad (2.13)$$

Finally, (V) gives a linear PDE for  $F$ ,

$$\begin{aligned} \theta^2 R^2 \Gamma_5(z) &= 2(\delta_1 + 1) \frac{\theta'}{\theta} F_{XX} + (\delta_1 + 1) \left[ \frac{\theta' X}{\theta} \pm \left( \frac{\psi' - R^2}{\theta} \right) \right] F_{XXX} \\ &\quad - (\delta_1 + 1) F_{XXt} - (\delta_1 - 1) \left[ \frac{\theta''}{2\theta} - \frac{(\theta')^2}{\theta^2} \right], \end{aligned} \quad (2.14)$$

where  $X = y \pm x$ , the argument of  $F$ . There are two distinct cases. With  $\delta_1 + 1 \neq 0$  then  $F$  is indeed given by (2.14) and we obtain Reductions 3 and 4. If however,  $\delta_1 = -1$ , then  $F$  remains arbitrary and (2.14) becomes

$$\gamma_5 \theta^4 R^2 = -2(\theta')^2 + \theta \theta''. \quad (2.15)$$

We therefore have a nonlinear system consisting of the Bernoulli equation, (2.12) and (2.15) for  $\theta(t)$  and  $R(t)$ . This case yields Reduction 5 (see below).

**REDUCTION 3.** In the case when  $\gamma_2 = 0$ ,  $\theta(t)$  remains arbitrary (we write  $\theta(t) = f_4(t)$ ) and we obtain the reduction

$$\begin{aligned} p(x, y, t) &= \lambda_2 f_4^{1/2}(z) \xi(z) \exp \left[ i \left( \frac{\psi'(t) - \lambda_2^2 f_4(t)}{2f_4(t)} (x \mp y) \right. \right. \\ &\quad \left. \left. + \frac{f'_4(t)}{4f_4(t)} (x^2 - y^2) + F(y \pm x; t) \right) \right], \end{aligned} \quad (2.16a)$$

$$v(x, y, t) = \lambda_2^2 f_4(t) \chi(z) + T(x, y, t), \quad (2.16b)$$

$$z(x, y, t) = f_4(t)(x \pm y) + f_3(t), \quad (2.16c)$$

where  $T(x, y, t)$  is given by (2.13) and where  $F(X; t)$  is found from the quadrature

$$\begin{aligned} F_{XX}(X; t) &= \frac{\theta^2}{1 + \delta_1} \int^t \left\{ R^2(s) + (\delta_1 - 1) \left[ \frac{\theta''(s)}{2\theta^3(s)} - \frac{(\theta'(s))^2}{\theta^4(s)} \right] \right\} dt \\ &\quad + G(\theta X \mp \int^t (\psi'(s) - R^2(s)) ds), \end{aligned} \quad (2.16d)$$

and  $\xi(z)$ ,  $\zeta(z)$  and  $\chi(z)$  satisfy the system,

$$\epsilon_2 \xi^2 \zeta + \xi \chi - i \xi' = 0, \quad (2.16e)$$

$$\epsilon_2 \xi \zeta^2 + \zeta \chi + i \zeta' = 0, \quad (2.16f)$$

$$(\delta_1 + 1) \chi'' - \delta_2 (\xi \zeta)'' + 1 = 0, \quad (2.16g)$$

These ODES are easily solved in terms of elementary functions.

REDUCTION 4. With  $\gamma_2 \neq 0$  again  $\theta(t)$  is arbitrary and we have the reduction

$$p(x, y, t) = R(t) \xi(z) \exp \left[ i \left( \frac{(\psi'(t) - R^2(t))}{2f_4(t)} (x \mp y) + \frac{f'_4(t)}{4f_4(t)} (x^2 - y^2) + F(y \pm x; t) \right) \right], \quad (2.17a)$$

$$v(x, y, t) = R^2(t) \chi(z) + T(x, y, t), \quad (2.17b)$$

$$z(x, y, t) = f_4(t)(x \pm y) + f_3(t), \quad (2.17c)$$

where  $T(x, y, t)$  is given by (2.13),  $F(X; t)$  is the same as for Reduction 3,

$$\frac{1}{R^2(t)} = \frac{\lambda_1}{f_4(t)} - \frac{2\lambda_1}{f_4(t)} \int f_4(t) dt, \quad (2.17d)$$

and  $\xi(z)$ ,  $\zeta(z)$  and  $\chi(z)$  satisfy the system,

$$\epsilon_2 \xi^2 \zeta + \xi \chi - i \xi' - i \gamma_2 \xi = 0, \quad (2.17e)$$

$$\epsilon_2 \xi \zeta^2 + \zeta \chi + i \zeta' + i \gamma_2 \zeta = 0, \quad (2.17f)$$

$$(\delta_1 + 1) \chi'' - \delta_2 (\xi \zeta)'' + 1 = 0. \quad (2.17g)$$

Again these ODES are easily integrated in terms of elementary functions.

REDUCTION 5.

$$p(x, y, t) = R(t) \xi(z) \exp \left[ i \left( (\psi'(t) - R^2(t)) \frac{x}{2\theta(t)} + \frac{x\theta'}{2\theta} (x \pm y) + F(y \pm x; t) \right) \right], \quad (2.18a)$$

$$v(x, y, t) = R^2(t) \chi(z) + T(x, y, t), \quad (2.18b)$$

$$z(x, y, t) = \theta(t)(x + \kappa y) + \psi(t), \quad (2.18c)$$

where  $T(x, y, t)$  is given by (2.13) and  $\xi(z)$ ,  $\zeta(z)$  and  $\chi(z)$  satisfy the system,

$$\epsilon_2 \xi^2 \zeta + \xi \chi - i \xi' - i \gamma_2 \xi = 0, \quad (2.18d)$$

$$\epsilon_2 \xi \zeta^2 + \zeta \chi + i \zeta' + i \gamma_2 \zeta = 0, \quad (2.18e)$$

$$\delta_2 (\xi \zeta)'' - \gamma_5 = 0, \quad (2.18f)$$

In this case the system of ODES has degenerated into an overdetermined system in which  $\chi(z)$  plays only a parametric role. It is easy to show that the only (non-trivial) solution of this system requires  $\gamma_2 = \gamma_5 = 0$ , and so we have

$$\theta(t) = \lambda_2/(t + \lambda_3), \quad R^2(t) = \lambda_1/(t + \lambda_3). \quad (2.18g)$$

### 2.3. DISCUSSION

To our knowledge, previous studies of reductions of the DS system to ODES have been done by first reducing to a system in  $(1+1)$ -dimensions, so the reductions and exact solutions found here represent new results. In accordance with the Painlevé conjecture, all ODES generated by reductions found for the integrable cases of DS<sub>I</sub> and DS<sub>II</sub> are of Painlevé type and these were integrated in terms of the second and fourth Painlevé equations, or, in the special case, in terms of elementary functions.

Of particular note is the solution found in reduction 5. This comes about because  $z$  has the same spatial structure as the characteristics of the DS<sub>II</sub> equations.

It is of interest to compare the symmetry reductions found for the DS systems to those found for the  $(2+1)$ -dimensional Schrödinger equation. For the latter, taking the same ansatz yields four possible forms for  $\theta(t)$ , one of which is constant, with corresponding forms for  $\psi(t)$ . This contrasts with the arbitrariness of both  $\theta(t)$  and  $\psi(t)$  for the DS system. This increase in freedom may be interpreted (in terms of the Direct Method) as a consequence of the freedom of translation in  $\chi(z)$ , c.f. equation (I). Recall also that the NLS equation has “radial” reductions with  $z = \theta(t)\sqrt{x^2 + y^2}$ . No such reductions exist for the DS system; this is to be expected as (unlike the NLS equation) there is a preferential direction of travel for the DS.

We remark that in the above we have not determined the most general form for  $z(x, y, t)$ . In order to do this we must find the general solution of the system

$$R^2(t)\Gamma_1(z) = (z_x)^2 + (z_y)^2, \quad (3.1a)$$

$$R^2(t)\Gamma_2(z) = z_{xx} + z_{yy}, \quad (3.1b)$$

where  $\Gamma_1(z)$  and  $\Gamma_2(z)$  are functions of  $z$  to be determined. Recent work on Differential Gröbner Bases as applied to nonlinear overdetermined systems of PDES (cf., Mansfield, 1992) should provide a straightforward method for solving such equations.

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## DIMENSIONAL REDUCTION FOR EQUATIONS INVOLVING DISCRETE AND CONTINUOUS VARIABLES

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**ABSTRACT.** We show how ordinary and conditional Lie point symmetries can be found for differential-difference equations (DDES). Once found, these symmetries can be combined with discrete ones to reduce the DDES to various types of equations with fewer independent variables.

### 1. Introduction

Lie group theory provides a powerful tool for obtaining analytic solutions of a very large class of partial differential equations (PDES) [1,2,3]. The most standard method, provided by Lie group theory, is that of symmetry reduction. The method consists of several steps. First, one must find the symmetry group  $G$  of the equation, i.e. the Lie group of local point transformations, transforming solutions into solutions. Second, we classify subgroups  $G_i \subset G$  of the symmetry group. Third, we find the invariants of each subgroup  $G_i$  and rewrite the equation in terms of invariants. This provides a dimensional reduction: solutions invariant under  $G_i$  depend on fewer variables, than the general solutions.

A further method for obtaining particular solutions of PDES yields so called partially invariant solutions [2,4,5]. Recently, Clarkson and Kruskal [6] have proposed a direct reduction method, that in many cases provides more reductions (and solutions) than the method of invariant solutions. It was subsequently shown [7] that Lie group theory also provides a framework for this larger class of solutions. They are obtained from "conditional symmetries": transformations that leave a subclass of solutions invariant, rather than all solutions. Conditional symmetries are in turn related to the nonclassical method of Bluman and Cole [8].

In this presentation we shall point out how the symmetry techniques can be extended to differential-difference equations (DDES). Further details can be found in Ref. 9 and 10.

## 2. Lie Point Symmetries of DDES

For notational simplicity, let us restrict ourselves to a “scalar” DDE for one real function  $u(n, \mathbf{x})$  of one discrete variable  $n$  and  $N$  continuous real variables  $\mathbf{x} \in \mathbf{R}^N$ . Moreover, let us consider DDES involving only first and second derivatives. We write such a DDE as

$$\Delta_n^{(2)} \equiv \Delta(\mathbf{x}, n, u(n+k)|_{k=-a}^b, u_{x_i}(n+k)|_{k=-a_i}^{b_i}, u_{x_i x_j}(n+k)|_{k=-a_{ij}}^{b_{ij}}) = 0 \quad (1)$$

$$a, b, a_i, b_i, a_{ij}, b_{ij} \in \mathbf{Z} \geq 0.$$

An example of such a system is the two-dimensional Toda system (TDTS) [11,12]

$$\Delta_n = u_{xt}(n) - \exp\{u(n-1) - u(n)\} + \exp\{u(n) - u(n+1)\} = 0 \quad (2)$$

with  $u(n) = u(n, \mathbf{x}, t)$ .

Lie point transformations that we are looking for will leave the solution set of (1) invariant and will have the form

$$\tilde{x} = \Lambda_g(x, u(n, x)), \quad \tilde{u}(\tilde{n}, \tilde{x}) = \Omega_g(x, n, u(n, x)), \quad \tilde{n} = \nu_g(n), \quad (3)$$

where  $g$  represents continuous, or discrete group parameters.

Continuous transformations of the form (3) are generated by a Lie algebra of vector fields of the form

$$\mathbf{v} = \sum_{i=1}^p \xi_i(\mathbf{x}, u(n)) \partial_{x_i} + \phi(n, \mathbf{x}, u(n)) \partial_{u(n)}, \quad (4)$$

where  $n$  is treated as a discrete variable and we have set  $\tilde{n} = n$ .

The condition that  $\tilde{u}(n, \tilde{x})$  is a solution whenever  $u(n, x)$  is one is assured by requiring that the second prolongation  $\text{pr}^{(2)}\mathbf{v}$  of  $\mathbf{v}$  should annihilate  $\Delta_n^{(2)}$  on the solution set of (1):

$$\text{pr}^{(2)}\mathbf{v} \Delta_n^{(2)} \Big|_{\Delta_n^{(2)}=0} = 0, \quad (5a)$$

$$\begin{aligned} \text{pr}^{(2)}\mathbf{v} &= \sum_{i=1}^p \xi_i \partial_{x_i} + \sum_{k=-a}^b \phi(k) \partial_{u(k)} \\ &+ \sum_{i=1}^p \sum_{k=-a_i}^{b_i} \phi^{x_i}(k) \partial_{u_{x_i}(k)} + \sum_{i,j=1}^p \sum_{k=-a_{ij}}^{b_{ij}} \phi^{x_i x_j}(k) \partial_{u_{x_i x_j}(k)}, \end{aligned} \quad (5b)$$

$$\phi^{x_i}(k) = D_{x_i} \phi(k) - \sum_{j=1}^p (D_{x_i} \xi_j) u_{x_j}(k), \quad (5c)$$

$$\phi^{x_i x_j}(k) = D_{x_j} \phi^{x_i}(k) - \sum_{\ell=1}^p (D_{x_j} \xi_\ell) u_{x_i x_\ell}. \quad (5d)$$

The only difference between (5a) for differential equations and for DDES lies in the summations over  $k$  in (5b) (that would be absent for differential equations).

We call three points to attention:

1. All sums involved in (5) are finite, if the ranges of  $k$  in (1) are finite.
2. Equation (5a) is to be viewed as just one equation ( $n$  is a variable).
3. The coefficient  $\phi(k, \mathbf{x}, u(k))$  in (4) (and hence also  $\phi^{x_i}(k), \phi^{x_i x_j}(k)$  in (5)) depends on  $u(k)$ , but not on, say  $u(k+1)$ , or  $u(k-1)$ .

We hence have a finite algorithm for obtaining the “determining equations”: a usually overdetermined system of linear PDES for  $\xi = \xi(\mathbf{x}, u(n))$  and  $\phi(n) = \phi(n, \mathbf{x}, u(n))$ .

If we apply the above algorithm to the two-dimensional Toda systems (2), we obtain an infinite dimensional Lie algebra [9] with a Kac-Moody-Virasoro structure [3], namely

$$\mathrm{T}(f) = f(t) \partial_t + f'(t)n \partial_{u(n)}, \quad \mathrm{U}(k) = k(t) \partial_{u(n)}, \quad (6a)$$

$$\mathrm{X}(g) = g(x) \partial_x + g'(x)n \partial_{u(n)}, \quad \mathrm{W}(h) = h(x) \partial_{u(n)}, \quad (6b)$$

where  $f(t)$ ,  $k(t)$ ,  $g(x)$  and  $h(x)$  are arbitrary functions. A typical group transformation, leaving (2) invariant is hence

$$\begin{aligned} t &= F_{\lambda_1}(\tilde{t}), \quad x = G_{\lambda_2}(\tilde{x}), \\ \tilde{u}(n, \tilde{x}, \tilde{t}) &= u(n, x, t) + n \ln \left[ F'_{\lambda_1}(\tilde{t}) G'_{\lambda_2}(\tilde{x}) \right] + \lambda_3 h(\tilde{x}) + \lambda_4 k(\tilde{t}), \end{aligned} \quad (7)$$

where the functions  $F$  and  $G$  in the group are related to those in the algebra by

$$F_{\lambda}(\tilde{t}) = \phi^{-1} \left[ -\lambda + \phi(\tilde{t}) \right], \quad \phi(t) = \int \frac{dt}{f(t)}, \quad (8)$$

and similarly for  $G_{\lambda}(\tilde{x})$ . The primes in (7) are derivatives with respect to the argument.

To the transformations (7) we add some discrete ones, found by inspection to leave (2), invariant namely:

$$n' = n + N, \quad N \in \mathbb{Z}, \quad (9)$$

and also

$$(x, t, u(n)) \rightarrow (-x, -t, u(n)), \quad (10a)$$

$$(x, t, u(n)) \rightarrow (t, x, u(n)), \quad (10b)$$

$$(x, t, u(n)) \rightarrow (x, t, -u(-n)). \quad (10c)$$

### 3. Differential Equations Approach and Higher Symmetries of DDE

The system (1) could also be viewed as a system of differential equations for the functions  $u_n(x, t) \equiv u(n, x, t)$ . In general, we have infinitely many equations for infinitely many functions. In particular, if the equations do not depend explicitly on  $n$ , i.e. the symmetry (9) pertains, we can impose periodicity in  $n$

$$u(n + N, \mathbf{x}) = u(n, \mathbf{x}), \quad (11)$$

and reduce the system to a finite one.

The vector fields realizing the symmetry algebra in this case (equation (1) treated as a system of differential equations) would have the form

$$\mathbf{v} = \sum_{i=1}^p \xi_i(\mathbf{x}, \mathbf{u}) \partial_{x_i} + \sum_k \phi_k(\mathbf{x}, \mathbf{u}) \partial_{u(k)}. \quad (12)$$

The vector field (12) differs from (4) in two aspects. First, we have a summation over  $k$  in (12); its range is finite in the periodic case, infinite otherwise. Secondly,  $\xi_i$  and  $\phi_k$  depend on all functions  $u_j$ , rather than just on one. Calculating the second prolongation  $\text{pr}^{(2)}\mathbf{v}$  in a standard manner [1,2,3] and imposing

$$\left. \text{pr}^{(2)}\mathbf{v}\Delta_n^{(2)} \right|_{\Delta_j^{(2)}=0} = 0, \quad \forall n, j, \quad (13)$$

we obtain, in general an infinite system of determining equations for an infinite number of functions  $\{\xi_i, \phi_k, i = 1, \dots, p, k = \dots, 2, -1, 0, 1, 2, \dots\}$ .

Conceptually speaking, the “differential equation method” of this section may give a larger symmetry group than the “intrinsic” method of Section 2. The reason for this is that the intrinsic method yields purely point transformations so that finally we have  $\xi = \xi(\mathbf{x}, u(n)), \phi = \phi(n, \mathbf{x}, u(n))$ . The differential equation method can yield transformations that are “higher symmetries” with respect to the discrete variables. Indeed,  $\xi_i$  and  $\phi_n$  may depend not only on  $u(n)$ , but also on  $u(n+k), k \in \mathbb{Z}$ , or, in other words, on differences such as  $u(n+k) - u(n)$ . The coefficients  $\xi_i$  and  $\phi_n$  will not depend on derivatives of  $u(n)$  and are thus point symmetries as far as the continuous variables are concerned.

In practice, it quite often turns out that no higher symmetries, (nor contact symmetries, involving only nearest neighbours  $\phi(n, u(n-1), u(n), u(n+1), \mathbf{x})$ ) exist. The two methods then give the same result.

For example, consider the equation

$$u_{xt}(n) - F_n(x, t, \dots, u(n-1), u(n), u(n+1), \dots) = 0, \quad (14)$$

and assume that the interaction is nonlinear:

$$\frac{\partial^2 F_n}{\partial u(j) \partial u(k)} \neq 0, \quad (15a)$$

for at least one pair  $(j, k)$  (e.g.  $j = k = n$ ), and has finite range:

$$\frac{\partial F_n}{\partial u(k)} = 0, \quad n + b < k \quad \text{and} \quad k < n - a. \quad (15b)$$

Equation (13) then provides determining equations from which we can deduce that the vector field (12) must have the form

$$\mathbf{v} = \xi(x) \partial_x + \tau(t) \partial_t + \sum_n [A_n u(n) + B_n(x, t)] \partial_{u(n)}, \quad (16)$$

(with  $A_n = \text{const}$ ). The two methods thus give the same result.

In particular, the differential equation method for the nonlinear Toda system (2) gives the Lie algebra (6), already obtained above.

#### 4. Symmetry Reduction for the Two-Dimensional Toda System

We write the symmetry group of (1) as

$$G = G_D \triangleright G_C, \quad (17)$$

where  $G_D$  are the discrete transformations (9) and (10) and the invariant subgroup  $G_C$  in (17) corresponds to the Lie algebra (6).

A complete classification of the low-dimensional subgroups of  $G_C$  has been performed [10]. We shall now look for solutions of (2) invariant under various subgroups.

#### 4.1. REDUCTIONS BY CONTINUOUS SUBGROUPS $G_0 \subset G_C$

4.1.1.  $T(1)$ . Invariance implies  $u = u(n, x)$ . This function satisfies a difference equation obtained from (2). The solution is

$$u(n, x, t) = \alpha(x) n + \beta(x), \quad (18)$$

where  $\alpha(x)$  and  $\beta(x)$  are arbitrary functions.

The subalgebra  $X(1) = \partial_x$  leads to a similar solution

$$u(n, x, t) = \alpha(t) n + \beta(t) \quad (19)$$

#### 4.1.2. $T(1) - \epsilon X(1)$ . We have

$$u(n, x, t) = u(n, \xi), \quad \zeta = x + \epsilon t, \quad \epsilon^2 = 1 \quad (20)$$

and (2) reduces to the usual Toda equation

$$\epsilon u_{\zeta\zeta} = \exp\{u(n-1) - u(n)\} - \exp\{u(n) - u(n+1)\}. \quad (21)$$

Any solution of (21) will provide a family of solutions of (2), depending on 4 arbitrary functions, once use is made of the transformations (7).

For instance, we have

$$\begin{aligned} u(n, x, t) &= 2(n+a) \ln(F(t) + G(x)) + n F_t G_x \\ &+ H(x) + K(t) - \sum_{j=1}^n \ln[C + 2(a-j-1)\epsilon j + \epsilon j^2], \quad n \geq 0, \end{aligned} \quad (22)$$

#### 4.1.3. $T(1) + \epsilon W(x)$ , $\epsilon^2 = 1$ . The invariant solutions are

$$u(n, x, t) = axt - \sum_{j=1}^n \ln(-aj + b) + c, \quad n \geq 0. \quad (23)$$

#### 4.2. REDUCTION BY DISCRETE SUBGROUPS $G_0 \in G_D$

The invariance under discrete translations of  $n$  (9) makes it possible to impose periodicity as in (11). For instance, setting  $N = 2$  and putting  $u(2k+1) = u(1)$ ,  $u(2k) = u(2)$ ,  $2u(1) = -2u(2) = \phi(x, t)$  we obtain the sinh-Gordon equation

$$\phi_{xt} = -4 \sinh \phi. \quad (24)$$

For  $N = 3$  we obtain the Dodd-Bullough equation [11,12,13].

#### 4.3. REDUCTION BY A COMBINATION OF DISCRETE AND CONTINUOUS SYMMETRIES

Let us now consider a subgroup  $G_0 \subset G$  that is not contained in  $G_D$ , nor  $G_C$ . A typical transformation will have the form

$$n' = n + N, \quad t' = T(t, x, u, N, \lambda), \quad (25a)$$

$$x' = X(t, x, u, N, \lambda), \quad \frac{\partial W}{\partial u} \neq 0, \quad \frac{\partial(T, X)}{\partial(t, x)} \neq 0, \quad (25b)$$

where  $N$  and  $\lambda$  are group parameters ( $N \in \mathbb{Z}, \lambda \in \mathbb{R}$ ). The transformation (25) will have two invariants  $I_\alpha(x, t, n, u)$ ,  $\alpha = 1, 2$ . If these invariants satisfy

$$I_1 = \xi(x, t, u), \quad \frac{\partial I_2}{\partial u} \neq 0 \quad (26)$$

we can put  $I_2 = F(\xi)$  on the solution set of (2), solve for  $u$  and obtain

$$u(u, x, t) = U(n, x, t, F(\xi)). \quad (27)$$

Substituting  $u$  back into the original DDE, we obtain a reduced equation for  $F(\xi)$ , just as in the case of continuous symmetries.

As a matter of fact, the usual Lie algebraic techniques can be applied, if we complement the symmetry algebra by formally introducing a vector field

$$Z = \frac{\partial}{\partial n} \quad (28)$$

and requiring, at the end of the calculations, that the corresponding group parameter be an integer.

Let us now consider the one-dimensional Toda lattice (21). Its symmetry algebra is given by

$$D = \xi \partial_\xi + 2n \partial_{u(n)}, \quad T = \partial_\xi, \quad W = \xi \partial_{u(n)}, \quad U = \partial_{u(n)}, \quad (29)$$

to which we add  $Z$  of (28). Let us consider reductions involving  $Z$ .

*4.3.1.  $Z + aD + bU$ ,  $a, b \in \mathbb{R}$ .* The group transformation is

$$n' = n + N, \quad \xi' = e^{aN} \xi, \quad u' = u + (2an + b)N + aN^2. \quad (30)$$

The invariants are

$$I_1 \equiv \eta = \xi e^{-an}, \quad I_2 = u - n^2 a - bn. \quad (31)$$

They imply that the invariant solution will have the form

$$u(n, \xi) = n^2 a + bn + F(\eta), \quad (32)$$

with  $F(\eta)$  satisfying

$$F''(\eta) = \epsilon e^{-b} [\exp\{F(\eta e^a) - F(\eta) + a\} - \exp\{F(\eta) - F(\eta e^{-a}) - a\}]. \quad (33)$$

Equation (33) can be called a “dilation-delay” type equation. It involves one independent variable  $\eta$ , but the function  $F$  and its derivatives are evaluated at point  $\eta$  and at “dilated” points:  $\eta e^a, \eta e^{-a}$ .

4.3.2.  $Z + aT + kW$ ,  $a, k \in \mathbf{R}$ . The group transformation is

$$n' = n + N, \quad \xi' = \xi + aN, \quad u' = u + k(\frac{1}{2}aN^2 + \xi N). \quad (34)$$

The invariants are

$$I_1 \equiv \eta = \xi - an, \quad I_2 = u - \frac{k}{2a}\xi^2, \quad (35)$$

and we have

$$u(n, \xi) = \frac{k}{2a}\xi^2 + F(\eta), \quad \eta = \xi - an, \quad (36)$$

where  $F(\eta)$  satisfies the reduced equation

$$\epsilon \left( \frac{k}{a} + F'' \right) = \exp\{F(\eta + a) - F(\eta)\} - \exp\{F(\eta) - F(\eta - a)\}. \quad (37)$$

This is a differential-delay equation:  $F$  and its derivatives are evaluated at  $\eta, \eta + a$  and  $\eta - a$ . Among the interesting known solutions of the Toda lattice equations, the soliton

$$u(n, x, t) = \ln \left\{ \frac{1 + \exp[2\eta \sinh \alpha - \alpha]}{1 + \exp[2\eta \sinh \alpha]} \right\}, \quad a = -\frac{\alpha}{\sinh \alpha}, \quad (38)$$

and the periodic theta function solution

$$u(n, x, t) = \ln \{ \theta_4[\nu(\eta - a)] \} - \ln \{ \theta_4(\nu\eta) \}, \quad (39)$$

satisfy the delay equation (37).

## 5. Conditional Symmetries of DDES

Conditional symmetries can be introduced for DDES, just as they were for PDES [7].

As in the case of ordinary symmetries, two approaches are possible: the intrinsic approach and the differential equations one. In both cases we complement (1) that we are studying by a further equation: a “side condition” in the Olver and Rosenau terminology [16]. We choose this condition in a manner that is adapted to the form of the vector field (4), or (12), that we are looking for. Thus, we write the condition as

$$\Delta_n^{(1)} = \xi_i u_{x_i}(n) - \phi(n) = 0, \quad (40)$$

where the functions  $\xi_i$  and  $\phi(n)$  are the same as in the vector field.

We shall now construct simultaneous solutions of (40) and (1) and we shall do this by first finding vector fields  $\mathbf{v}$ , the prolongation of which annihilates both equations simultaneously, on the solution set of both:

$$\text{pr}^2 \mathbf{v} \Delta_n^{(2)} \Big|_{\Delta_n^{(2)}=0, \Delta_n^{(1)}=0} = 0, \quad (41)$$

$$\text{pr}^{(1)} \mathbf{v} \Delta_n^{(1)} \Big|_{\Delta_n^{(1)}=0} = 0. \quad (42)$$

The point is that (42) is satisfied automatically and imposes no constraints on the vector field  $\mathbf{v}$ .

The essential properties that the conditional symmetries for DDES share with those of differential equations are:

1. When they exist, they provide reductions of DDES, just like ordinary symmetries do.
2. They provide all results of the Clarkson-Kruskal method [6], sometimes more [17,18].
3. The vector fields corresponding to conditional symmetries do not form a vector space, nor a Lie algebra. The reason is that each vector field corresponds to a different condition (40).
4. The determining equations are nonlinear.

5. Conditional symmetries, distinct from ordinary ones, exist whenever (1) and the side condition (40) are compatible (and (40) does not correspond to an ordinary symmetry).

In the “intrinsic method”  $\xi_i$  and  $\phi(n)$  depend only on one function  $u(n)$ , and (40) is just one equation. In the differential equations method  $\xi_i$  and  $\phi(n)$  depends on the entire vector  $\mathbf{u}$  (with components  $u(n)$ ).

Let us now turn to conditional symmetries of the TDTs of (2). The results can be summed up quite simply.

1. The intrinsic method for the TDTs provides no conditional symmetries that are not ordinary ones.
2. For the differential equations method we write the vector field as

$$\mathbf{v} = \tau(x, t, \mathbf{u}) \partial_t + \xi(x, t, \mathbf{u}) \partial_x + \phi_n(x, t, \mathbf{u}) \partial_{u(n)}. \quad (43)$$

For  $\tau \neq 0, \xi \neq 0$  the differential equations method reduces to the intrinsic one and provides no conditional symmetries.

3. For  $\tau \neq 0, \xi = 0$  the differential equations method provides a rather complicated system of nonlinear determining equations [10] for  $\phi_n(x, t, \dots, u_{n-1}, u_n, u_{n+1}, \dots)$ . The Bäcklund transformations for the TDTs are obtained as special solutions of this system:

$$\phi_n = f_{n,t} + a [\exp\{u(n) - f_{n+1}\} - \exp\{u(n-1) - f_n\}], \quad (44)$$

where  $f_n(x, t) \equiv \tilde{u}(n)$  is a solution of the TDTs. Since in this case the vector field is  $\mathbf{v} = \partial_t + \phi_n \partial_u(n)$ , we obtain the reductions formula  $du(n)/\phi_n = dt$ , i.e.

$$u_t(n) - \tilde{u}_t(n) = a [\exp\{u(n) - \tilde{u}(n+1)\} - \exp\{u(n-1) - \tilde{u}(n)\}]. \quad (45a)$$

This is half of the (known) Bäcklund transformations [11,12] for the TDTs. The other half is obtained by combining (45) with (2):

$$u_x(n-1) - \tilde{u}_x(n) = \frac{1}{a} [\exp\{\tilde{u}(n-1) - u(n-1)\} - \exp\{\tilde{u}(n) - u(n)\}]. \quad (45b)$$

## 6. Conclusions

The main conclusions that we can draw are:

1. Symmetry methods work quite well for DDES and should be further developed. In particular, systematic methods for finding nonobvious discrete symmetries are needed. For work along these see e.g. the talks by G.R.W. Quispel, H.W. Capel, F.W. Nijhoff and others at this meeting.
2. The intrinsic method and the differential equation method sometimes give different results. In other words, contact symmetries and higher symmetries are of interest for DDES.

3. Conditional symmetries make sense for DDES.
4. As in the case of PDES, similarity techniques can be successfully combined with singularity analysis of PDES. In particular differential delay equations like (33), or (37) have the Painlevé property, in the sense of singularity confinement, that has received much attention recently [19–26].

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# SYMMETRY REDUCTIONS AND EXACT SOLUTIONS FOR A GENERALISED BOUSSINESQ EQUATION

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**ABSTRACT.** In this paper, new nonclassical symmetry reductions and exact solutions are presented for a Generalised Boussinesq equation  $u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + u_{tt} = 0$ , which has the modified Boussinesq equation ( $q = 0, r = -\frac{1}{2}p^2$ ) and dispersive water wave equation, or classical Boussinesq equations ( $q = 2p, r = \frac{3}{2}p^2$ ) as special cases. These symmetry reductions are obtained using the Direct Method, originally developed by Clarkson & Kruskal to study symmetry reductions of the Boussinesq equation, which involves no group theoretic techniques and using these reductions, we obtain exact solutions expressible in terms of solutions of the second and fourth Painlevé equations, Jacobi and Weierstrass elliptic functions, and elementary functions, for certain values of the parameters  $p, q$  and  $r$ . Furthermore, in the case when  $q = p$  and  $r = \frac{1}{2}p^2$ , symmetry reductions are obtained which are reminiscent of reductions of the 2+1-dimensional cubic nonlinear Schrödinger equation arising from the Talanov lens transformation.

## 1. Introduction

In this paper we consider symmetry reductions of the Generalised Boussinesq (GBQ) equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + u_{tt} = 0, \quad (1.1)$$

where  $p, q$  and  $r$  are constants such that  $r \neq 0$  and subscripts denote partial derivatives.

A special case of the Generalised Boussinesq equation (1.1) is the modified Boussinesq (MBQ) equation

$$u_{tt} + pu_t u_{xx} - \frac{1}{2}p^2 u_x^2 u_{xx} + u_{xxxx} = 0, \quad (1.2)$$

which is the special case of (1.1) where  $q = 0$  and  $r = -\frac{1}{2}p^2$ . This is a soliton equation solvable by inverse scattering (Quispel *et al.*, 1982) and there is a “Miura type” transformation relating solutions of (1.2) to solutions of the Boussinesq equation (Boussinesq, 1871, 1872)

$$u_{tt} + au_{xx} + b(u^2)_{xx} + cu_{xxxx} = 0, \quad (1.3)$$

where  $a, b, c$  are constants, which arises in several physical applications and also is a soliton equation solvable by inverse scattering (Zakharov, 1974; Caudrey, 1982; Deift *et al.*, 1982).

Specifically, if  $u(x, t)$  satisfies (1.2) with  $p = -1$  (which we can assume without loss of generality), then using the Bäcklund transformation (Hirota and Satsuma, 1977)

$$v_x = -u_t + \sqrt{3}u_{xx} - \frac{1}{2}u_x^2, \quad (1.4a)$$

$$v_t = \sqrt{3}u_{xt} + u_{xxx} - u_xu_t - \frac{1}{6}u_x^3 + \delta, \quad (1.4b)$$

where  $\delta$  is a constant, it is easily shown that  $v(x, t)$  is a solution of the potential Boussinesq equation

$$v_{tt} + v_xv_{xx} + v_{xxxx} = 0,$$

and so  $u(x, t) = v_x(x, t)$  is a solution of the Boussinesq (1.3) with  $a = 0, b = \frac{1}{2}, c = 1$ , which we can assume without loss of generality (see also Quispel et al., 1982; Gromak, 1987). We remark that the MBQ equation (1.2) may be derived from the Mikhailov-Shabat system

$$\eta_t = \eta_{xx} + (\eta + \phi)\phi_x - \frac{1}{6}(\eta + \phi)^3, \quad (1.5a)$$

$$-\phi_t = \phi_{xx} - (\eta + \phi)\eta_x - \frac{1}{6}(\eta + \phi)^3 \quad (1.5b)$$

(Mikhailov et al., 1987, 1991). Setting  $u_x = \eta + \phi, v = \eta - \phi$ , and eliminating  $v$  in (1.5) yields (1.2), after suitably rescaling the variables. Studies of the MBQ equation (1.2) include Clarkson (1986), Conte (1992), Estévez et al. (1992), Flaschka et al. (1991), Fordy and Gibbons (1981), Gromak (1987) Hirota and Satsuma (1977) and Quispel et al. (1982).

A second special case of (1.1) which has attracted considerable attention is the dispersive water-wave (DWW) equation

$$u_{xxxx} + pu_tu_{xx} + 2pu_xu_{xt} + \frac{3}{2}p^2u_x^2u_{xx} + u_{tt} = 0, \quad (1.6)$$

which is the special case of (1.1) where  $q = 2p$  and  $r = \frac{3}{2}p^2$ , and also is a soliton equation solvable by inverse scattering (Kaup, 1975).

Equation (1.6) can easily be derived from the so-called classical Boussinesq equations

$$\eta_t + \{(1 + a\eta)v\}_x - v_{xxx} = 0, \quad (1.7a)$$

$$v_t + \eta_x + avv_x = 0, \quad (1.7b)$$

with  $a$  an arbitrary constant, which arise in the description of surface waves propagating in shallow water (Whitham, 1974; Broer, 1976). Further the Hirota bi-linear representation of (1.7) is a reduction of equations of the Kadomtsev-Petviashvili and modified Kadomtsev-Petviashvili hierarchies (Hirota, 1985, 1986) and so multi-soliton solutions of (1.7) can be obtained (see also Freeman et al. 1990). Setting  $v = w_x$  in (1.7b), solving for  $\eta$ , and eliminating in (1.7a) yields

$$w_{xxxx} + \{aw_t - [1 + af(t)]\}w_{xx} + 2aw_xw_{xt} + \frac{3}{2}a^2w_x^2w_{xx} + w_{tt} = f'(t), \quad (1.8)$$

with  $f(t)$  an arbitrary function, which is equivalent to (1.6). We remark that (1.6) can be derived in an analogous manner from the systems of equations studied by Jaulent and Miodek (1976)

$$\eta_t + \eta v_x + \frac{1}{2}\eta_xv - \frac{1}{4}v_{xxx} = 0, \quad (1.9a)$$

$$v_t + \eta_x + \frac{3}{2}vv_x = 0, \quad (1.9b)$$

and by Kuperschmidt (1986)

$$\eta_t - (\eta v)_x - av_{xxx} + b\eta_{xx} = 0, \quad (1.10a)$$

$$v_t - \eta_x - vv_x - bv_{xx} = 0, \quad (1.10b)$$

where  $a$  and  $b$  are arbitrary constants. Studies of the DWW equation (1.6) include Estévez et al. (1992), Hirota (1985), Kuperschmidt (1985), Roy & Chowdhury (1988) and Sachs (1988).

The classical method for finding symmetry reductions of partial differential equations (PDES) is the Lie group method of infinitesimal transformations (cf., Bluman & Cole, 1974; Bluman & Kumei, 1989; Hill, 1992; Olver, 1986). There have been several generalizations of the classical Lie method for symmetry reductions. Bluman & Cole (1969), in their study of symmetry reductions of the linear heat equation, proposed the nonclassical method of group-invariant solutions (in the sequel referred to as the *nonclassical method*). Subsequently, this method was further generalized by Olver & Rosenau (1986, 1987) to include “weak symmetries” and, even more generally, “side conditions” or “differential constraints”.

Motivated by the fact that there exist PDES which possess reductions that are not obtained using the classical Lie group method, Clarkson & Kruskal (1989), hereafter referred to as CK, developed a direct and algorithmic method for finding symmetry reductions (in the sequel referred to as the *Direct Method*) which was used to obtain previously unknown reductions of the Boussinesq equation (1.2). Levi & Winternitz (1989) subsequently gave a group theoretical explanation of these results by showing that all the new reductions of (1.2) can also be obtained using the nonclassical method of Bluman & Cole (1969).

There is much current interest in the determination of symmetry reductions of PDES which reduce the equations to ordinary differential equations (ODEs). One then checks if the resulting ODE is of *Painlevé type* (i.e., its solutions have no movable singularities other than poles). It appears to be the case that whenever the ODE is of Painlevé type then it can be solved explicitly, leading to exact solutions to the original equation. Conversely, if the resulting ODE is not of Painlevé type, then one is often unable to solve it explicitly.

In §2 we derive symmetry reductions of the Generalized Boussinesq equation (1.1) using the Direct Method. Using these symmetry reductions we obtain exact solutions of (1.1) expressible in terms of solutions of the second and fourth Painlevé equations, Jacobi and Weierstrass elliptic functions, and elementary functions, for certain values of the parameters  $p$ ,  $q$  and  $r$ . In §3 we compare the symmetry reductions obtained using the Direct Method with those derived using the classical Lie group method. In §4 we apply the Painlevé PDE test due to Weiss et al. (1983) to the GBQ equation (1.1) and show that this test suggests the MBQ equation (1.2) and the DWW equation (1.6) are the only special case of the GBQ equation (1.1) which are completely integrable.

## 2. Symmetry Reductions obtained from the Clarkson-Kruskal Direct Method

To apply the Direct Method to the Generalised Boussinesq equation (1.1), we seek a solution in the form

$$u(x, t) = F(x, t, w(z)), \quad (2.1)$$

where  $z = z(x, t)$ . Substituting this into (1.1) and requiring that the result be an ODE for  $w(z)$  imposes conditions upon  $F$ ,  $z$  and their derivatives in the form of a nonlinear, over-determined system of equations, whose solution yields the desired reductions. The novel characteristic about the Direct Method, in comparison to the others mentioned above, is that

it involves no use of group theory. It has been employed to obtain new symmetry reductions and exact solutions for several physically significant PDES, which represent significant progress. Since solutions of PDES asymptotically tend to symmetry reductions, some of these special solutions will illustrate important physical phenomena and they can also be used to test computer coding. We remark that the Direct Method has certain resemblances to the so-called “method of free parameter analysis” (cf., Hansen, 1964), though in the latter method, the boundary conditions are crucially used in the determination of the reduction whereas they are not used in the Direct Method. Following the development of the Direct Method, there has been renewed interest in the nonclassical method, and the relationship between these two methods is discussed by Clarkson & Hood (1992), Nucci & Clarkson (1992) and Pucci (1992).

The application of the direct method is split into two cases,  $z_x \neq 0$  and  $z_x \equiv 0$ .

### 2.1. $z_x \neq 0$ .

In the generic case when  $z_x \neq 0$  it is readily found that, as in the application of the Direct Method to the MBQ equation (1.2) by Clarkson (1989a), it suffices to seek solutions in the linear form

$$u(x, t) = \alpha(x, t) + w(z), \quad z = x\theta(t) + \phi(t),$$

where  $\alpha(x, t)$ ,  $\theta(t)$  and  $\phi(t)$  are sufficiently well-behaved functions, with  $\theta(t) \neq 0$ , rather than (2.1). Substituting this into (1.1) and requiring that the result be an ODE for  $w(z)$  imposes conditions upon  $\alpha(x, t)$ ,  $\theta(t)$  and  $\phi(t)$ , and their derivatives in the form of a nonlinear, overdetermined system of equations, whose solution yields the desired symmetry reductions. There are three cases to consider: (i),  $\frac{d\theta}{dt} \neq 0$ , with  $q \neq p$  and  $r \neq \frac{1}{2}p^2$ ; (ii),  $\frac{d\theta}{dt} \neq 0$ , with  $q = p$  and  $r = \frac{1}{2}p^2$ ; and (iii),  $\frac{d\theta}{dt} \equiv 0$ .

Case 2.1.1.  $\frac{d\theta}{dt} \neq 0$ , with  $q \neq p$  and  $r \neq \frac{1}{2}p^2$ . In this case it can be shown that

$$\alpha = -\frac{(p+q)}{4r\theta} \frac{d\theta}{dt} x^2 - \frac{(p+q)}{2r\theta} \frac{d\phi}{dt} x - \psi(t), \quad (2.2)$$

where  $\theta(t)$  satisfies

$$\frac{d\theta}{dt} = -\frac{1}{2}\theta^3, \quad (2.3)$$

which has the canonical solution  $\theta(t) = t^{-1/2}$ , and  $\psi(t)$  and  $\phi(t)$  satisfy

$$\begin{aligned} 4p(p+q)t^2 \frac{d^2\phi}{dt^2} + 2[4r + (p+q)(2p-q)]t \frac{d\phi}{dt} + [4r - (p+q)^2]\phi &= c_1, \\ 8p(p+q)t^2\phi \frac{d^2\phi}{dt^2} + 4[4r + p^2 - q^2]t^2 \left(\frac{d\phi}{dt}\right)^2 + 4[4r + (p+q)(2p-q)]t\phi \frac{d\phi}{dt} \\ + [4r - (p+q)^2]\phi^2 - 16prt \frac{d\psi}{dt} &= c_2, \\ 2[2r - q(p+q)]t^2 \frac{d^2\phi}{dt^2} - q(p+q)t \frac{d\phi}{dt} - [3r - q(p+q)]\phi &= c_3 \end{aligned}$$

$$(p+q)\{16rt^3\frac{d^3\phi}{dt^3} + 4[4r + (p+q)^2]t^2\frac{d^2\phi}{dt^2} - 4[r-p(p+q)]t\frac{d\phi}{dt} + [16r - (p+q)(p+3q)]\phi\} = c_4,$$

$$16(p+q)t^3\left\{2r\phi\frac{d^3\phi}{dt^3} + q(p+q)\frac{d\phi}{dt}\frac{d^2\phi}{dt^2}\right\} + 4(p+q)t^2\left\{2[4r + (p+q)^2]\phi\frac{d^2\phi}{dt^2} + (p+q)(p+3q)\left(\frac{d\phi}{dt}\right)^2\right\} - 8(p+q)[r-p(p+q)]t\phi\frac{d\phi}{dt} + (p+q)[16r - (p+q)(p+3q)]\phi^2 - 64r^2t^2\frac{d^2\psi}{dt^2} - 16pr(p+q)t\frac{d\psi}{dt} = c_5,$$

where  $c_1, c_2, c_3, c_4$  and  $c_5$  are arbitrary constants. By solving this overdetermined system we obtain the following five reductions:

**REDUCTION 2.1.** *Arbitrary  $p, q$  and  $r$ .*

$$u(x, t) = w(z) + \lambda_1 \log t, \quad z = xt^{-1/2}, \quad (2.4)$$

where  $\lambda_1$  is an arbitrary constant and  $W(z) := w'(z)$  satisfies,

$$W''' - \frac{1}{2}(p+q)zWW' + rW^2W' + \left(\frac{1}{4}z^2 + p\lambda_1\right)W' - \frac{1}{2}qW^2 + \frac{3}{4}zW - \lambda_1 = 0. \quad (2.5)$$

This equation is of Painlevé type if either (i),  $q = 0$  and  $r = -\frac{1}{2}p^2$ , or (ii),  $q = 2p$  and  $r = \frac{3}{2}p^2$ . If  $q = 0$  and  $r = -\frac{1}{2}p^2$  and we make the transformation

$$W(z) = -[3^{3/4}y(x) - z]/p, \quad x = -\frac{1}{2}3^{1/4}z, \quad (2.6)$$

then  $y(x)$  satisfies the fourth Painlevé equation (PIV)

$$\frac{d^2y}{dx^2} = \frac{1}{2y}\left(\frac{dy}{dx}\right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - A)y + \frac{B}{y}, \quad (2.7)$$

with  $A = p\lambda_1/\sqrt{3}$  and  $B$  a constant of integration. If  $q = 2p$  and  $r = \frac{3}{2}p^2$  and we make the transformation

$$W(z) = [e^{3i\pi/4}y(x) + z]/p, \quad x = -\frac{1}{2}e^{-i\pi/4}z, \quad (2.8)$$

then  $y(x)$  satisfies PIV with  $A = ip\lambda_1$  and  $B$  a constant of integration.

**REDUCTION 2.2.**  *$p = 0, q$  and  $r$  arbitrary.*

$$u(x, t) = w(z) + \lambda_1 t + \lambda_2 \log t, \quad z = xt^{-1/2}, \quad (2.9)$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W''' - \frac{1}{2}qzWW' + rW^2W' + \frac{1}{4}z^2W' - \frac{1}{2}qW^2 + \frac{3}{4}zW - \lambda_2 = 0, \quad (2.10)$$

which is not of Painlevé type.

REDUCTION 2.3.  $q = 0, r = -\frac{1}{2}p^2 \neq 0$ .

$$u(x, t) = w(z) + 2\lambda_1 xt + \lambda_2 \log t, \quad z = xt^{-1/2} + \lambda_1 pt^{3/2}, \quad (2.11)$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W''' - \frac{1}{2}p^2 W^2 W' - \frac{1}{2}pzWW' + \left(\frac{1}{4}z^2 + p\lambda_2\right)W' + \frac{3}{4}zW - \lambda_2 = 0. \quad (2.12)$$

If in this equation we make the transformation (2.6) then  $y(x)$  satisfies PIV with  $A = p\lambda_2/\sqrt{3}$  and  $B$  a constant of integration.

REDUCTION 2.4.  $r = \frac{1}{4}q(p+q) \neq 0$ .

$$u(x, t) = w(z) - 2\lambda_1 x - \lambda_1^2 qt + \lambda_2 \log t, \quad z = xt^{-1/2} + \lambda_1 qt^{1/2}, \quad (2.13)$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W''' + \frac{1}{4}q(p+q)W^2 W' - \frac{1}{2}(p+q)zWW' + \left(\frac{1}{4}z^2 + p\lambda_2\right)W' - \frac{1}{2}qW^2 + \frac{3}{4}zW - \lambda_2 = 0. \quad (2.14)$$

This equation is of Painlevé type if and only if  $q = 2p$  and  $r = \frac{3}{2}p^2$ . If in this case we make the transformation (2.8), then  $y(x)$  satisfies PIV with  $A = ip\lambda_2$  and  $B$  a constant of integration.

REDUCTION 2.5.  $r = \frac{1}{4}q^2 \neq 0, p = 0$ .

$$u(x, t) = w(z) - 2\lambda_1 x + \lambda_2 \log t + \lambda_3 t, \quad z = xt^{-1/2} + \lambda_1 qt^{1/2}, \quad (2.15)$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W''' + \frac{1}{4}q^2 W^2 W' - \frac{1}{2}qzWW' + \frac{1}{4}z^2 W' - \frac{1}{2}qW^2 + \frac{3}{4}zW - \lambda_2 = 0, \quad (2.16)$$

which is not of Painlevé type.

Case 2.1.2.  $\frac{d\theta}{dt} \neq 0, q = p$  and  $r = \frac{1}{2}p^2$ . In this case it can be shown that:  $\alpha(x, t)$  satisfies (2.2);  $\theta(t)$  satisfies

$$\theta \frac{d^2\theta}{dt^2} - 2 \left( \frac{d\theta}{dt} \right)^2 = c\theta^6, \quad (2.17)$$

where  $c$  is an arbitrary constant, which has the canonical solution  $\theta(t) = (t^2 - c)^{-1/2}$ , and  $\psi(t)$  and  $\phi(t)$  satisfy

$$\begin{aligned} & (t^2 - c)^2 \frac{d^2\phi}{dt^2} + 2t(t^2 - c) \frac{d\phi}{dt} - c\phi = c_1, \\ & 2(t^2 - c)^2 \phi \frac{d^2\phi}{dt^2} + (t^2 - c)^2 \left( \frac{d\phi}{dt} \right)^2 + 4t(t^2 - c)\phi \frac{d\phi}{dt} - c\phi^2 - p(t^2 - c) \frac{d\psi}{dt} = c_2, \\ & (t^2 - c) \left\{ (t^2 - c)^2 \frac{d^3\phi}{dt^3} + 6t(t^2 - c) \frac{d^2\phi}{dt^2} + (6t^2 - 3c) \frac{d\phi}{dt} \right\} = c_3, \\ & (t^2 - c) \left\{ 2(t^2 - c)^2 \phi \frac{d^3\phi}{dt^3} + 4(t^2 - c)^2 \frac{d\phi}{dt} \frac{d^2\phi}{dt^2} + 12t(t^2 - c)\phi \frac{d^2\phi}{dt^2} \right. \\ & \left. + 8t(t^2 - c) \left( \frac{d\phi}{dt} \right)^2 + 6(2t^2 - c)\phi \frac{d\phi}{dt} - p(t^2 - c) \frac{d^2\psi}{dt^2} - 2pt \frac{d\psi}{dt} \right\} = c_4, \end{aligned}$$

where  $c_1, c_2, c_3$  and  $c_4$  are arbitrary constants. By solving this overdetermined system we obtain the following two reductions:

REDUCTION 2.6.  $p = q, r = \frac{1}{2}p^2 \neq 0$ .

$$u(x, t) = w(z) + \frac{x^2 t - 2\lambda_1 \lambda_2^2 x + \lambda_1^2 \lambda_2^2 t}{p(t^2 - \lambda_2^2)} + \lambda_3 \log \left( \frac{t - \lambda_2}{t + \lambda_2} \right), \quad z = \frac{x - \lambda_1 t}{(t^2 - \lambda_2^2)^{1/2}}, \quad (2.18)$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$WW'' - \frac{1}{2}(W')^2 + \frac{1}{8}p^2 W^4 - \lambda_2(\frac{1}{2}\lambda_2 z^2 - \lambda_3 p)W^2 = \lambda_4, \quad (2.19)$$

with  $\lambda_4$  a constant of integration, which is not of Painlevé type unless  $\lambda_2 = 0$ , when it is solvable in terms of Jacobian elliptic functions.

REDUCTION 2.7.  $p = q, r = \frac{1}{2}p^2 \neq 0$ .

$$u(x, t) = w(z) + \frac{x^2}{pt} - \frac{2\lambda_1 x}{pt^2} + \frac{2\lambda_1^2}{3pt^3} + \frac{\lambda_2}{pt}, \quad z = \frac{x}{t} - \frac{\lambda_1}{2t^2}, \quad (2.20)$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$WW'' - \frac{1}{2}(W')^2 + \frac{1}{8}p^2 W^4 + (\lambda_1 z - \frac{1}{2}\lambda_2)W^2 = \lambda_3, \quad (2.21)$$

with  $\lambda_3$  a constant of integration. This equation is not of Painlevé type unless  $\lambda_1 = 0$ , when it is solvable in terms of Jacobian elliptic functions.

Case 2.1.3.  $\frac{d\theta}{dt} \equiv 0$  (*set  $\theta(t) \equiv 1$ , without loss of generality*). In this case it can be shown that

$$\alpha = -\frac{(p+q)}{2r} \frac{d\phi}{dt} x - \psi(t) \quad (2.22)$$

and  $\psi(t)$  and  $\phi(t)$  satisfy

$$\begin{aligned} \frac{d^2\phi}{dt^2} &= c_1, \\ 2p(p+q)\phi \frac{d^2\phi}{dt^2} + [4r + p^2 - q^2] \left( \frac{d\phi}{dt} \right)^2 - 4pr \frac{d\psi}{dt} &= c_2, \\ q(p+q)^2 \frac{d\phi}{dt} \frac{d^2\phi}{dt^2} - 4r^2 \frac{d^2\psi}{dt^2} &= c_3, \end{aligned}$$

where  $c_1, c_2$  and  $c_3$  are arbitrary constants. By solving this overdetermined system we obtain the following five reductions:

REDUCTION 2.8.  $p, q$  and  $r$  arbitrary.

$$u(x, t) = w(z) + \lambda_2 t, \quad z = x + \lambda_1 t, \quad (2.23)$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$(W')^2 + \frac{1}{6}rW^4 + \frac{1}{3}\lambda_1(p+q)W^3 + (p\lambda_2 + \lambda_1^2)W^2 + \lambda_3W + \lambda_4 = 0, \quad (2.24)$$

which is solvable in terms of Jacobian elliptic functions.

REDUCTION 2.9.  $p = 0, q$  and  $r$  arbitrary.

$$u(x, t) = w(z) + \lambda_2 t^2 + \lambda_3 t, \quad z = x + \lambda_1 t, \quad (2.25)$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W'' + \frac{1}{3}rW^3 + \frac{1}{2}\lambda_1 qW^2 + \lambda_1^2 W + 2\lambda_2 z = \lambda_4, \quad (2.26)$$

with  $\lambda_4$  a constant of integration. This equation is not of Painlevé type unless  $\lambda_2 = 0$ , when it is solvable in terms of Jacobian elliptic functions.

REDUCTION 2.10.  $r = \frac{1}{4}q(p+q) \neq 0$ .

$$u(x, t) = w(z) - 4\lambda_1 xt + \lambda_2 t - \frac{8}{3}\lambda_1^2 qt^3, \quad z = x + \lambda_1 qt^2, \quad (2.27)$$

where  $\lambda_1 \neq 0$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W''' + \frac{1}{4}q(p+q)W^2W' + (-4\lambda_1 pz + \lambda_2 p)W' - 2\lambda_1 qW = 0, \quad (2.28)$$

This equation is of Painlevé type if and only if  $q = 2p$  and  $r = \frac{3}{2}p^2$ , when after one integration we obtain

$$W'' + \frac{1}{2}p^2W^3 + (-4\lambda_1 pz + \lambda_2 p)W = \lambda_3, \quad (2.29)$$

with  $\lambda_3$  a constant of integration, which is equivalent to the second Painlevé equation (PII)

$$\frac{d^2y}{dx^2} = 2y^3 + xy + A, \quad (2.30)$$

where  $A$  is a constant.

REDUCTION 2.11.  $r = -\frac{1}{2}p(p+q) \neq 0$ .

$$u(x, t) = w(z) + 2\lambda_1 xt + \lambda_2 t - \frac{2}{3}q\lambda_1^2 t^3, \quad z = x + \lambda_1 pt^2, \quad (2.31)$$

where  $\lambda_1 \neq 0$  and  $\lambda_2$  are arbitrary constants and  $W(z) := w'(z)$  satisfies,

$$W''' - \frac{1}{2}p(p+q)W^2W' + (2p\lambda_1 z + p\lambda_2)W' + 2\lambda_1(p+q)W = 0. \quad (2.32)$$

This equation is of Painlevé type if and only if  $q = 0$  and  $r = -\frac{1}{2}p^2$ , when after one integration we obtain

$$W'' - \frac{1}{2}p^2W^3 + (2p\lambda_1 z + p\lambda_2)W = \lambda_3, \quad (2.33)$$

with  $\lambda_3$  a constant of integration, which is equivalent to PII.

REDUCTION 2.12.  $p = 0, r = \frac{1}{4}q^2 \neq 0$ .

$$u(x, t) = w(z) - 4\lambda_1 xt - \frac{8}{3}\lambda_1^2 qt^3 + \lambda_2 t^2 + \lambda_3 t, \quad z = x + \lambda_1 qt^2, \quad (2.34)$$

where  $\lambda_1 \neq 0$ ,  $\lambda_2$  and  $\lambda_3$  are arbitrary constants and  $w(z)$  satisfies,

$$w''' + \frac{1}{12}q^2(w')^3 - 2\lambda_1 qw + 2\lambda_2 z + \lambda_4 = 0, \quad (2.35)$$

with  $\lambda_4$  a constant of integration, which is not of Painlevé type.

Case 2.2.  $z_x \equiv 0$

In this case it can be shown that all similarity reductions are of the form

$$u(x, t) = x^2\phi_2(t) + x\phi_1(t) + \phi_0(t), \quad (2.36)$$

where  $\phi_2(t)$ ,  $\phi_1(t)$  and  $\phi_0(t)$  satisfy

$$\frac{d^2\phi_2}{dt^2} + 2(p+2q)\phi_2 \frac{d\phi_2}{dt} + 8r\phi_2^3 = 0, \quad (2.37a)$$

$$\frac{d^2\phi_1}{dt^2} + 2(p+q)\phi_2 \frac{d\phi_1}{dt} + 2\left(q \frac{d\phi_2}{dt} + 4r\phi_2^2\right)\phi_1 = 0, \quad (2.37b)$$

$$\frac{d^2\phi_0}{dt^2} + 2p\phi_2 \frac{d\phi_0}{dt} + q\phi_1 \frac{d\phi_1}{dt} + 2r\phi_2\phi_1^2 = 0. \quad (2.37c)$$

Equation (2.37a) possesses the solution  $\phi_2 = \gamma/(t+\lambda)$ , where  $\lambda$  is an arbitrary constant and  $\gamma$  is a solution of

$$4r\gamma^2 - 3(p+2q)\gamma + 1 = 0. \quad (2.38)$$

Further, (2.37a) is of Painlevé-type if: (i),  $r = (p+2q)/18$ ; (ii),  $r = -\frac{1}{2}(p+2q)^2$ ; (iii),  $p = -2q$ ; or (iv),  $r = 0$  (a case which we excluded).

Case (i).  $r = (p+2q)/18$ . In this case the general solution of (2.37a) is

$$\phi_2(t) = \frac{3}{2(p+2q)} \left\{ \frac{1}{t+\lambda_1} + \frac{1}{t+\lambda_2} \right\},$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary constants.

Case (ii).  $r = -\frac{1}{2}(p+2q)^2$ . In this case, setting

$$\phi_2 = \frac{1}{2(p+2q)\psi} \frac{d\psi}{dt},$$

in (2.37a) and integrating twice yields

$$\left( \frac{d\psi}{dt} \right)^2 = \lambda_3 \psi^3 + \lambda_4, \quad (2.39)$$

where  $\lambda_3$  and  $\lambda_4$  are arbitrary constants. If  $\lambda_3\lambda_4 \neq 0$ , then the solution of (2.39) is

$$\psi(t) = 4\lambda_3^{-1}\wp(t+t_0; 0, -\lambda_3^2\lambda_4/16),$$

where  $t_0$  is an arbitrary constant and  $\wp(t; g_2, g_3)$  is the Weierstrass elliptic function (cf., Whittaker & Watson, 1927). If  $\lambda_4 = 0$ , then the solution of (2.39) is  $\psi(t) = 4/[\lambda_3(t + t_0)^2]$ , whilst if  $\lambda_3 = 0$ , then the solution of (2.39) is  $\psi(t) = \lambda_4^{1/2}(t + t_0)$ .

Case (iii).  $p = -2q$ . In this case multiplying (2.37a) by  $d\phi_2/dt$  and integrating yields

$$\left(\frac{d\phi_2}{dt}\right)^2 + 4r\phi_2^4 = \lambda_5, \quad (2.40)$$

where  $\lambda_5$  is an arbitrary constant. If  $\lambda_5 \neq 0$ , then the solution of (2.40) is

$$\phi_2(t) = [\lambda_5/(4r)]^{1/4} \operatorname{sn}(z; i), \quad z = (4r\lambda_5)^{1/4}(t + t_0),$$

where  $t_0$  is an arbitrary constant and  $\operatorname{sn}(z; k)$  is the Jacobi elliptic function (cf., Whittaker & Watson, 1927), whilst if  $\lambda_5 = 0$ , then

$$\phi_2(t) = i/\left[(4r)^{1/2}(t + t_0)\right].$$

Having solved (2.37a) for  $\phi_2(t)$ , then (2.37b) is a linear equation for  $\phi_1(t)$  with  $\phi_2(t)$  as one solution and so is solvable by quadratures. Finally, given  $\phi_1(t)$  and  $\phi_2(t)$ , then (2.37c) is a linear equation for  $\phi_0(t)$  which also is solvable by quadratures.

### 3. Symmetry Reductions obtained from the Classical Lie Group Method

To apply the classical Lie group method to the gBQ equation (1.1), consider the one-parameter Lie group of infinitesimal transformations in  $(x, t, u)$  given by

$$\tilde{x} = x + \varepsilon\xi(x, t, u) + O(\varepsilon^2), \quad (3.1a)$$

$$\tilde{t} = t + \varepsilon\tau(x, t, u) + O(\varepsilon^2), \quad (3.1b)$$

$$\tilde{u} = u + \varepsilon\eta(x, t, u) + O(\varepsilon^2), \quad (3.1c)$$

where  $\varepsilon$  is the group parameter. Requiring that (1.1) is invariant under the transformation (3.1) yields a linear, overdetermined system of equations for the infinitesimals  $\xi(x, t, u)$ ,  $\tau(x, t, u)$  and  $\eta(x, t, u)$ . The associated Lie algebra is the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \quad (3.2)$$

Solving the determining equations yields the following infinitesimals:

$$p, q, r \text{ arbitrary} \quad \begin{aligned} \xi &= c_1x + c_2, \\ \tau &= 2c_1t + c_3, \end{aligned}$$

$$\eta = c_4,$$

$$p = 0, q, r \text{ arbitrary} \quad \begin{aligned} \xi &= c_1x + c_2, \\ \tau &= 2c_1t + c_3, \end{aligned}$$

$$\eta = c_4t + c_5,$$

$$r = \frac{1}{4}q(p + q), p \neq 0 \quad \begin{aligned} \xi &= c_1x + c_2qt + c_3, \\ \tau &= 2c_1t + c_4, \end{aligned}$$

$$\eta = 2c_2x + c_5,$$

$$r = \frac{1}{4}q^2, p = 0 \quad \begin{aligned} \xi &= c_1x + c_2qt + c_3, \\ \tau &= 2c_1t + c_4, \end{aligned}$$

$$\eta = 2c_2x + c_5t + c_6,$$

$$q = p, r = \frac{1}{2}p^2 \quad \begin{aligned} \xi &= c_7xt + c_1x + c_2pt + c_3, \\ \tau &= c_7pt^2 + 2c_1t + c_4, \end{aligned}$$

$$\eta = 2c_2x + c_7x^2 + c_6,$$

where  $c_1, c_2, \dots, c_7$  are arbitrary constants. The symmetry variables are then found by solving the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \quad (3.3)$$

and then substituting the resulting expressions into (1.1), one obtains the reduced equation. It is easily shown that using the infinitesimals above, one obtains Reductions 2.1, 2.2, 2.4–2.10 and 2.12, i.e., all the reductions in §2.1 above except Reductions 2.3 and 2.11.

In Reductions 2.10 and 2.11, the following symmetry reduction of the GBQ equation (1.1) is given

$$u(x, t) = w(z) - (p + q)\mu xt - \frac{1}{6}q(p + q)^2\mu^2t^3, \quad z = x + \mu rt^2, \quad (3.4)$$

where  $\mu \neq 0$ , provided that either  $r = \frac{1}{4}p(p + q)$  [Reduction 2.10] or  $r = -\frac{1}{2}q(p + q)$  [Reduction 2.11]. An associated one-parameter group of transformations for this reduction is given by

$$\tilde{x} = x + \varepsilon \mu r(2t - \varepsilon), \quad (3.5a)$$

$$\tilde{t} = t - \varepsilon, \quad (3.5b)$$

$$\tilde{u} = u - \varepsilon \mu(p + q) \left[ \mu r(\varepsilon^2 - 3\varepsilon t + 2t^2) - x \right] + \frac{1}{6}\varepsilon \mu^2 q(p + q)^2(\varepsilon^2 - 3\varepsilon t + 3t^2). \quad (3.5c)$$

This transformation maps (1.1) to

$$\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + p\tilde{u}_{\tilde{t}}\tilde{u}_{\tilde{x}\tilde{x}} + q\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{t}} + r\tilde{u}_{\tilde{x}}^2\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}_{\tilde{t}\tilde{t}} = -\varepsilon \mu [4r - q(p + q)]\Phi(\tilde{x}, \tilde{t}, \tilde{u}), \quad (3.6)$$

where

$$\Phi(\tilde{x}, \tilde{t}, \tilde{u}) = \tilde{u}_{\tilde{x}\tilde{t}} + \mu(p + q) + \mu \left\{ p(p + q)\tilde{t} + \frac{1}{2}\varepsilon [2r + p(p + q)] \right\} \tilde{u}_{\tilde{x}\tilde{x}}. \quad (3.7)$$

The infinitesimals associated with the transformation are

$$\xi = 2\mu rt, \quad \tau = -1, \quad \eta = \mu(p + q)(x - 2\mu rt^2) + \frac{1}{2}\mu^2 q(p + q)^2 t^2, \quad (3.8)$$

and the invariant surface condition is given by

$$\psi(x, t, u) \equiv 2\mu rtu_x - u_t - \mu(p + q)(x - 2\mu rt^2) - \frac{1}{2}\mu^2 q(p + q)^2 t^2 = 0. \quad (3.9)$$

It is easily shown that

$$\Phi(x, t, u) = -\psi_x + [2r + p(p + q)]\mu(t + \frac{1}{2}\varepsilon)u_{xx}. \quad (3.10)$$

Hence we conclude that whereas Reduction 2.10 is classical reduction (since the transformation (3.5) maps (1.1) into itself), Reduction 2.11 is not. The group (3.5) is a conditional symmetry of the equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} - \frac{1}{2}q(p + q)u_x^2 u_{xx} + u_{tt} = 0, \quad (3.11)$$

since it does not map the equation into itself, yet nevertheless provides a reduction to an ODE. Furthermore, it is a classical symmetry of the system of equations (3.9,3.11). (In the terminology of Olver and Rosenau (1986, 1987), equation (3.9) is a side condition.)

We remark that Reduction 2.11 will also be obtainable using the nonclassical method due to Bluman and Cole (1969). In this method, the **GBQ** equation (1.1) is augmented with the invariant surface condition

$$\psi \equiv \xi(x, t, u)u_x + \tau(x, t, u)u_t - \eta(x, t, u) = 0, \quad (3.12)$$

which is associated with the vector field (3.2). By requiring that both (1.1) and the invariant surface condition (3.12) are invariant under the transformation (3.1), one obtains a nonlinear, overdetermined system of equations for the infinitesimals  $\xi$ ,  $\tau$  and  $\eta$  which appear both in the transformation (3.1) and the supplementary condition (3.12). The set of solutions is larger than for the classical method since the number of determining equations is smaller. However, it should be emphasized that the associated vector fields do not form a vector space, still less a Lie algebra, since the invariant surface condition (3.12) depends upon the particular reduction. Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equations (3.3) and then substituting the resulting expressions into (1.1); one obtains then the reduced equation.

Reduction 2.3 is nonclassical in the case when  $\lambda_1 \neq 0$ , see Clarkson (1989a) for further details.

We conclude this section with some observations about the symmetry reduction of the **GBQ** equation (1.1) in the special case when  $q = p$  and  $r = \frac{1}{2}p^2$  given by

$$u(x, t) = w(z) - x^2/(pt), \quad z = x/t, \quad (3.13)$$

which is a special case of Reductions 2.5 and 2.6 in §2. An associated one-parameter transformation group for (3.13) is given by

$$\tilde{x} = e^\varepsilon x, \quad \tilde{t} = e^\varepsilon t, \quad \tilde{u} = u + (e^\varepsilon - 1)x^2/(pt). \quad (3.14)$$

It is easily shown that this transformation does *not* map (1.1) with  $q = p$  and  $r = \frac{1}{2}p^2$  into itself. Instead it maps the equation to

$$u_{xxxx} + pu_t u_{xx} + pu_x u_{xt} + \frac{1}{2}p^2 u_x^2 u_{xx} + u_{tt} = \Phi(x, t, u)/t^2, \quad (3.15a)$$

where

$$\Phi(x, t, u) = pt(1 - e^{-\varepsilon})(u_{xx}\psi + u_x\psi_x) + t(1 - e^{-2\varepsilon})\psi_t - (1 - e^{-\varepsilon})^2(\psi + x\psi_x), \quad (3.15b)$$

and

$$\psi := xu_x + tu_t - x^2/(pt), \quad (3.15c)$$

which is the invariant surface condition, with associated infinitesimals  $\xi = x$ ,  $\tau = t$  and  $\eta = x^2/(pt)$ , which are not obtainable using the classical Lie group method.

However, a second associated one-parameter transformation group for (3.13) is given by

$$\tilde{x} = x/(1 - \varepsilon t), \quad \tilde{t} = t/(1 - \varepsilon t), \quad \tilde{u} = u + \varepsilon x^2/[p(1 - \varepsilon t)], \quad (3.16)$$

which does map (1.1) with  $q = p$  and  $r = \frac{1}{2}p^2$  into itself. Here the invariant surface condition is

$$\psi := xt u_x + t^2 u_t - x^2/p,$$

with associated infinitesimals  $\xi = xt$ ,  $\tau = t^2$  and  $\eta = x^2/p$ , which are obtained using the classical Lie group method.

More generally, it is easily shown that the one-parameter transformation group

$$\tilde{x} = xf(t; \varepsilon), \quad \tilde{t} = tf(t; \varepsilon), \quad \tilde{u} = u + [f(t; \varepsilon) - 1]x^2/(pt), \quad (3.17)$$

where  $f(t; \varepsilon)$  forms a group in  $\varepsilon$  with  $f(t; 0) = 1$ , maps (1.1) with  $q = p$  and  $r = \frac{1}{2}p^2$  into itself if and only if  $f(t; \varepsilon) = 1/(1 - \varepsilon t)$ .

The reduction (3.13) is similar to the reductions of the generalized nonlinear Schrödinger equation

$$iu_t + u_{xx} + (b_1 + ib_2)(|u|^2 u)_x + c|u^4|^2 u = 0 \quad (3.18)$$

(Clarkson, 1992; Florjańczyk & Gagnon, 1992) which arise from the so-called *Talanov lens* transformation (Talanov, 1970), or conformal point symmetry, which is similar to the transformation group (3.15). The Talanov lens transformation also arises in connection with reductions of the 2 + 1-dimensional, cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + u_{yy} + \kappa|u|^2 u = 0 \quad (3.19)$$

(Gagnon, 1990; Gagnon & Paré, 1991).

#### 4. The Painlevé Tests

In this section we apply the Painlevé PDE test due to Weiss *et al.* (1983) to the GBQ equation (1.1). We seek a solution to (1.1) in the form

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) \phi^{k+s}(x, t), \quad \phi = x + \psi(t), \quad (4.1)$$

where  $\psi(t)$  is an arbitrary analytic function and  $u_k(t)$ ,  $k = 0, 1, 2, \dots$ , analytic functions such that  $u_0 \not\equiv 0$ , in the neighbourhood of an arbitrary, non-characteristic movable singularity manifold defined by  $\phi(x, t) = 0$ , and  $s$  is an arbitrary constant to be determined. By leading order analysis we find that  $s = 0$ . This suggests we need to modify (4.1) and look for a solution of the form

$$u(x, t) = v(t) \ln \phi(x, t) + \sum_{k=0}^{\infty} u_k(t) \phi^k(x, t) \quad (4.2)$$

(cf., Clarkson, 1986), where the same notation applies as in (4.1) and  $v(t)$  is an analytic function to be determined. By leading order analysis we find that

$$v^2(t) = -6/r. \quad (4.3)$$

Using this, we now substitute (4.2) into (1.1) to obtain from the coefficient of  $\phi^{k-4}$ ,

$$k(k+1)(k-3)(k-4)u_k = H_k(u_{k-1}, u_{k-2}, \dots, u_0; p, q, r), \quad (4.4a)$$

where

$$\begin{aligned}
H_k(u_{k-1}, u_{k-2}, \dots, u_0; p, q, r) = & -rv \sum_{l=1}^{k-1} l(k-l)(2k-2l-3)u_l u_{k-l} \\
& - r \sum_{j=2}^{k-2} \sum_{l=1}^{j-1} l(j-l)(k-j)(k-j-1)u_l u_{j-l} u_{k-j} \\
& - (p+q) \frac{d\psi}{dt} \sum_{l=2}^{k-2} (k-l-1)l(l-1)u_{k-l-1} u_l - p \sum_{l=0}^{k-2} l(l-1) \frac{du_{k-l-2}}{dt} u_l \\
& - q \sum_{l=1}^{k-3} l(k-l-2)u_{k-l-2} \frac{du_l}{dt} + (p+q)v^2 \frac{d\psi}{dt} \delta_{k,1} - (p+q)(k-1)(k-3)v \frac{d\psi}{dt} u_{k-1} \\
& + pv \frac{du_{k-2}}{dt} - qv(k-2) \frac{du_{k-2}}{dt} + v \left( \frac{d\psi}{dt} \right)^2 \delta_{k,2} - (k-2)(k-3)u_{k-2} \left( \frac{d\psi}{dt} \right)^2 \\
& - v \delta_{k,3} \frac{d^2\psi}{dt^2} - 2(k-3) \frac{du_{k-3}}{dt} \frac{d\psi}{dt} - (k-3)u_{k-3} \frac{d^2\psi}{dt^2} - \frac{d^2u_{k-4}}{dt^2}, \tag{4.4b}
\end{aligned}$$

for  $k = 0, 1, 2, \dots$ , where  $\delta_{k,j}$  is the kronekar delta function (define  $u_k = 0$  for all  $k < 0$ ). This defines  $u_k$  unless  $k = 0$ ,  $k = 3$  or  $k = 4$  which are the resonances. At each positive resonance there is a compatibility condition which must be identically satisfied for the expansion (4.4) to be valid, i.e., we require that  $H_0 \equiv 0$ ,  $H_3 \equiv 0$  and  $H_4 \equiv 0$  for (1.1) to have a solution of the form (4.4). Equations (4.4) with  $k = 0$  is identically satisfied which implies that  $u_0(t)$  is arbitrary. Equations (4.4) with  $k = 1$  and  $k = 2$  yield

$$\begin{aligned}
u_1 &= -\frac{(p+q)}{2r} \frac{d\psi}{dt}, \\
u_2 &= \frac{pv}{12} \frac{du_0}{dt} + \frac{v[4r-(p+q)^2]}{48r} \left( \frac{d\psi}{dt} \right)^2,
\end{aligned}$$

respectively. The compatibility condition  $H_3 \equiv 0$  yields

$$(q^2 - p^2 - 2r)v \frac{d^2\psi}{dt^2} = 0. \tag{4.5}$$

Since  $\psi$  is an arbitrary function then this implies that

$$r = \frac{1}{2}(q^2 - p^2), \tag{4.6}$$

is a necessary condition for (1.1) to have a solution of the form (4.4). The compatibility condition  $H_4 \equiv 0$  yields, after using (4.6),

$$q(q-2p) \left\{ \frac{1}{2}r \frac{d^2u_0}{dt^2} + (p+q) \frac{d\psi}{dt} \frac{d^2\psi}{dt^2} \right\} = 0. \tag{4.7}$$

and so we require  $q = 0$  or  $q = 2p$  for this to be satisfied. Therefore we conclude that (1.1) has a solution in the form (4.2) for two sets of parameters:

$$(i), \quad q = 0, \quad r = -\frac{1}{2}p^2; \quad (ii), \quad q = 2p, \quad r = \frac{3}{2}p^2. \tag{4.8}$$

In these two cases (1.1) reduces to the MBQ equation (1.2) and the DWW equation (1.6), respectively, both of which are known to be completely integrable. If otherwise, then it is necessary to introduce a  $\ln \phi(x, t)$  term into the expansion (4.2) at this order and at higher orders of  $\phi(x, t)$ , higher and higher powers of  $\ln \phi(x, t)$  are required, a strong indication of non-Painlevé behaviour. Hence the Painlevé PDE test suggests that there are no other integrable cases of the GBQ equation (1.1).

We conclude this section with the following remarks:

1. The form (4.2) is not strictly a Painlevé Laurent series, though the initial logarithmic term is of little consequence as it is easily removed by either differentiation or the transformation  $u = \ln U$ . Since  $v(t) = (-6/r)^{1/2}$ , a constant, the partial derivatives  $u_x(x, t)$  and  $u_t(x, t)$ , are expressible in terms of Laurent series, and only the partial derivatives of  $u$  arise in (1.1).
2. The compatibility conditions (4.5) and (4.7) are identically satisfied if  $d^2\psi/dt^2 \equiv 0$ , i.e., if  $\psi = \kappa_1 t + \kappa_0$ , with  $\kappa_1$  and  $\kappa_0$  arbitrary constants, and  $d^2u_0/dt^2 \equiv 0$  (which we can assume if  $\psi = \kappa_1 t + \kappa_0$ ). Consequently this shows why the ODES arising from Reduction 2.6 (in the special case when  $\lambda_2 = 0$ ), Reduction 2.7 (in the special case when  $\lambda_1 = 0$ ), and Reductions 2.8 and 2.9 (in the special case when  $\lambda_2 = 0$ ) are of Painlevé type even when the parameters  $p$ ,  $q$  and  $r$  do not satisfy the integrability condition (4.8).

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# CONTINUOUS SYMMETRIES AND PAINLEVÉ REDUCTION OF THE KAC-VAN MOERBEKE EQUATION

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**ABSTRACT.** A method is given to derive the point symmetries of partial differential-difference equations. Applying the method to the Kac-van Moerbeke equation we find its symmetries form a Kac-Moody-Virasoro algebra. Using the symmetries, a similarity reduction of the Kac-van Moerbeke equation to an ordinary differential-difference equation is obtained. This reduced equation possesses a Lax pair, reduces to the first Painlevé equation in the continuum limit, and satisfies a recently proposed discrete version of the Painlevé property.

## 1. Introduction

In recent years much effort has been invested in taking concepts we know and love in the theory of differential equations, and finding the analogous concepts (and methods and theorems) for difference equations [1, 2]. For example, integrable partial differential equations were generalized to integrable partial difference equations [3] (see also [4]), and integrable ordinary differential equations were generalized to integrable mappings [5-8]. Attempts were also made to find discrete analogues of the Painlevé equations and of the Painlevé property [9-14].

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In this paper we show how the method of Lie-symmetries which is so useful for finding similarity reductions of partial differential equations (PDEs) [15-21] can also be applied to partial differential-difference equations (PD $\Delta$ E)s.<sup>1</sup> In both cases the number of independent variables can be reduced. The difference is that, for example, a PDE in two independent variables reduces to an ordinary differential equation, whereas a PD $\Delta$ E in two independent variables reduces to an ordinary differential-difference equation (our convention is to call an equation in one independent variable “ordinary”, and an equation in more than one independent variable “partial”). One of the reasons similarity reductions have received a lot of attention is the Ablowitz-Ramani-Segur conjecture [22], which says that similarity reductions of integrable PDEs have the Painlevé property. Here we show that the similarity reduction of the Kac-van Moerbeke PD $\Delta$ E possesses the discrete Painlevé property [10]. An alternative approach is given in [23].

## 2. Symmetries

An equation for a scalar function  $u$  ( $\mathbb{R}^n \rightarrow \mathbb{R}^n$ ) is said to possess a Lie point symmetry if it is invariant under the infinitesimal transformation

$$\mathbf{x}^* = \mathbf{x} + \varepsilon \xi(\mathbf{x}, u(\mathbf{x})) + O(\varepsilon^2), \quad (1)$$

$$u^*(\mathbf{x}^*) = u(\mathbf{x}) + \varepsilon v(\mathbf{x}, u(\mathbf{x})) + O(\varepsilon^2). \quad (2)$$

For (differential-)difference equations the question now arises what the expression for  $u^*(\mathbf{x}^* + \omega)$  is, where  $\omega \in \mathbb{R}^n$  is some given fixed span appearing in the equation. To answer this question, we first express the right-hand-side of (2) in terms of  $\mathbf{x}^*$ , using (1):

$$u^*(\mathbf{x}^*) = u(\mathbf{x}^* - \varepsilon \xi(\mathbf{x}^*, u^*(\mathbf{x}^*))) + \varepsilon v(\mathbf{x}^*, u^*(\mathbf{x}^*)) + O(\varepsilon^2). \quad (3)$$

We now shift  $\mathbf{x}^* \rightarrow \mathbf{x}^* + \omega$ :

$$u^*(\mathbf{x}^* + \omega) = u(\mathbf{x}^* + \omega - \varepsilon \xi(\mathbf{x}^* + \omega, u^*(\mathbf{x}^* + \omega))) + \varepsilon v(\mathbf{x}^* + \omega, u^*(\mathbf{x}^* + \omega)) + O(\varepsilon^2). \quad (4)$$

Having done this, we express the right-hand-side in terms of  $\mathbf{x}$  again (using (1)):

$$\begin{aligned} u^*(\mathbf{x}^* + \omega) &= u(\mathbf{x} + \omega + \varepsilon \xi(\mathbf{x}, u(\mathbf{x})) - \varepsilon \xi(\mathbf{x} + \omega, u(\mathbf{x} + \omega))) \\ &\quad + \varepsilon v(\mathbf{x} + \omega, u(\mathbf{x} + \omega)) + O(\varepsilon^2). \end{aligned} \quad (5)$$

Expanding the right-hand-side in a Taylor series, we obtain the general result

$$\begin{aligned} u^*(\mathbf{x}^* + \omega) &= u(\mathbf{x} + \omega) + \varepsilon v(\mathbf{x} + \omega, u(\mathbf{x} + \omega)) \\ &\quad + \varepsilon \sum_{i=1}^n [\xi_i(\mathbf{x}, u(\mathbf{x})) - \xi_i(\mathbf{x} + \omega, u(\mathbf{x} + \omega))] \frac{du}{dx_i}(\mathbf{x} + \omega) + O(\varepsilon^2). \end{aligned} \quad (6)$$

We now apply this theory to a specific example. As our example we have chosen the Kac-van Moerbeke equation, however the same approach holds for numerous other PD $\Delta$ E's and partial difference equations [24].

The Kac-van Moerbeke equation [25,26], also known as the discrete Korteweg-de Vries equation, is given by

$$\frac{d}{dt} u(x, t) = u(x, t)[u(x+1, t) - u(x-1, t)], \quad (7)$$

<sup>1</sup> J.M. Hill has kindly pointed out to us that an application to ordinary differential-difference equations is given in [21].

where  $u(x, t)$  is a function  $\mathbf{R}^2 \rightarrow \mathbf{R}$ . (The reader should realize that although usually the independent variable  $x$  in this equation is taken to be integer, here we allow  $x$  to be any real number.) Equation (7) occurs in the study of the spectra of Langmuir oscillations in a plasma [27].

To derive the symmetries of the Kac-van Moerbeke equation (7), we assume that (7) is invariant under the infinitesimal point transformation (1)-(2), with  $\mathbf{x} = (x, t)$  and  $\boldsymbol{\xi} = (\nu, \tau)$ . The transformed equation is

$$\frac{d}{dt^*} u^*(x^*, t^*) = u^*(x^*, t^*)[u^*(x^* + 1, t^*) - u^*(x^* - 1, t^*)]. \quad (8)$$

The left-hand-side of (8) is known from the usual theory [15,17]. In the right-hand-side we insert (6), with  $\omega = (1, 0)$  and  $(-1, 0)$ , respectively.

It then follows immediately that

$$\nu(x + 1, t, u(x + 1, t)) = \nu(x, t, u(x, t)), \quad (9)$$

$$\tau(x + 1, t, u(x + 1, t)) = \tau(x, t, u(x, t)). \quad (10)$$

This, in turn, can only be true if  $\nu$  and  $\tau$  do not depend on  $u$ . We thus obtain

$$\begin{aligned} \frac{\partial v(x)}{\partial t} + \left[ \frac{\partial v(x)}{\partial u(x)} - \frac{\partial \tau(x)}{\partial t} \right] \frac{du(x)}{dx} - \frac{\partial \nu(x)}{\partial t} \frac{du(x)}{dx} \\ = v(x)[u(x + 1) - u(x - 1)] + u(x)[v(x + 1) - v(x - 1)], \end{aligned} \quad (11)$$

where we have not indicated the  $t$  and  $u$  dependence of  $v$  and  $\tau$  explicitly.

Substituting (7) in (11) and solving, we obtain the symmetries of the Kac-van Moerbeke equation:

$$u^*(x^*, t^*) = u(x, t) + \varepsilon \alpha(x)u(x, t), \quad (12)$$

$$t^* = t + \varepsilon[\beta(x) - \alpha(x)t], \quad (13)$$

$$x^* = x + \varepsilon\gamma(x), \quad (14)$$

where  $\alpha(x)$ ,  $\beta(x)$  and  $\gamma(x)$  are arbitrary unit periodic functions [28]

$$\alpha(x) = \alpha(x + 1), \quad \beta(x) = \beta(x + 1), \quad \gamma(x) = \gamma(x + 1). \quad (15)$$

Expanding the functions  $\alpha(x)$ ,  $\beta(x)$  and  $\gamma(x)$  in a Fourier series we obtain the generators of the symmetry algebra

$$G_{1,k} = -\frac{e^{2\pi i k x}}{2\pi i} \frac{\partial}{\partial x}, \quad G_{2,k} = e^{2\pi i k x} \frac{\partial}{\partial t}, \quad G_{3,k} = e^{2\pi i k x} \left[ t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} \right]. \quad (16)$$

These generators form a centreless Kac-Moody-Virasoro algebra [16] given by the non-zero commutation relations

$$\begin{aligned} [G_{1,k}, G_{1,\ell}] &= (k - \ell)G_{1,k+\ell}, \quad [G_{1,k}, G_{2,\ell}] = -\ell G_{2,k+\ell}, \\ [G_{1,k}, G_{3,\ell}] &= -\ell G_{3,k+\ell}, \quad [G_{2,k}, G_{3,\ell}] = G_{2,k+\ell}. \end{aligned} \quad (17)$$

### 3. Invariant reduction

The characteristic equation associated with the symmetries (12-14) is

$$\frac{du}{\alpha(x)u} = \frac{dt}{\beta(x) - \alpha(x)t} = \frac{dx}{\gamma(x)}. \quad (18)$$

Equation (18) represents a linear constraint leading to the similarity solution

$$u(x, t) = v(\eta)/[b(x) - at] \quad (19)$$

with

$$\eta = x + d(x) + \frac{c}{a} \log[b(x) - at], \quad (20)$$

where  $a$  and  $c$  are constants, and

$$b(x) = b(x+1), \quad d(x) = d(x+1). \quad (21)$$

The explicit expressions which we have derived for  $a$ ,  $c$ ,  $b(x)$  and  $d(x)$  in terms of integrals over  $x$ , involving the functions  $\alpha(x)$ ,  $\beta(x)$  and  $\gamma(x)$  are somewhat complicated and are given in the appendix. Note that the linear constraint (19) has been obtained directly from the Lie symmetries, without invoking the integrability of the Kac-van Moerbeke equation. Inserting (19-21) in (7), the Kac-van Moerbeke PD $\Delta$ E reduces to the following ordinary differential-difference equation

$$av(\eta) - c \frac{dv}{d\eta} = v(\eta)[v(\eta+1) - v(\eta-1)]. \quad (22)$$

In the special case that  $\alpha(x)$ ,  $\beta(x)$  and  $\gamma(x)$  are constants, we obtain

$$\frac{\alpha}{\gamma} = \frac{a}{c}, \quad \frac{\beta}{\gamma} = \frac{b}{c}, \quad d(x) = 0 \quad (23)$$

with  $b$  being a constant as well. This corresponds to the special similarity reduction

$$u(x, t) = v(\eta)/(b - at) \quad (24)$$

with

$$\eta = x + \frac{c}{a} \log(b - at), \quad (25)$$

where  $a$ ,  $b$  and  $c$  are constants. The function  $v(\eta)$  would again satisfy (22).

Equation (22) is unusual, in the sense that it is both “advanced” and “retarded”. This makes its integration difficult. Here we restrict ourselves to exhibiting two particular solutions for the case  $a = 0$ , cf. also [29]:

(i) Soliton solution

$$v(\eta) = \frac{c\kappa[1 + e^{\kappa(\eta+1)+\delta}][1 + e^{\kappa(\eta-2)+\delta}]}{(e^{-\kappa} - e^\kappa)[1 + e^{\kappa\eta+\delta}][1 + e^{\kappa(\eta-1)+\delta}]}, \quad (26)$$

where  $\kappa$  and  $\delta$  are arbitrary parameters.

(ii) Rational solution

$$v(\eta) = -\frac{c}{2} \frac{(\eta+1+\delta)(\eta-2+\delta)}{(\eta+\delta)(\eta-1+\delta)}, \quad (27)$$

where  $\delta$  is an arbitrary parameter.

A continuum limit of equation (22) is obtained by putting

$$v(\eta) = 1 + \varepsilon^2 w(\eta), \quad v(\eta + 1) - v(\eta - 1) = 2\varepsilon^2 \sinh\left(\varepsilon \frac{d}{d\eta}\right) w(\eta), \quad (28)$$

where  $a\varepsilon^5 A$  and  $c = -2\varepsilon$ . Then  $w(\eta)$  satisfies

$$\frac{1}{3} \frac{d^2 w}{d\eta^2} + w^2 - A\eta = 0. \quad (29)$$

Equation (29) is the first Painlevé equation [30], apart from a trivial transformation.

Up to this point, the fact that the Kac-van Moerbeke equation (7) is integrable has not played a special role. Now, however, we will use the fact that (7) is integrable, and has the following Lax representation [14]:

$$\begin{aligned} \frac{d}{dt} Q(x, t, z) &= -\frac{1}{2}[u(x, t) + u(x+1, t)]Q(x, t, z) - u^{1/2}(x, t)u^{1/2}(x-1, t)Q(x-2, t, z), \\ zQ(x, t, z) &= \frac{1}{2}u^{1/2}(x+1, t)Q(x+1, t, z) + \frac{1}{2}u^{1/2}(x, t)Q(x-1, t, z). \end{aligned} \quad (30)$$

This leads to the following Lax representation for the reduced equation (22):

$$\begin{aligned} \left[ c \frac{d}{d\eta} + \frac{1}{2}a\zeta \frac{d}{d\zeta} \right] R(\eta, \zeta) &= \frac{1}{2}[v(\eta) + v(\eta+1)]R(\eta, \zeta) + v^{1/2}(\eta)v^{1/2}(\eta-1)R(\eta-2, \zeta), \\ \zeta R(\eta, \zeta) &= \frac{1}{2}v^{1/2}(\eta+1)R(\eta+1, \zeta) + \frac{1}{2}v^{1/2}(\eta)R(\eta-1, \zeta). \end{aligned} \quad (31)$$

(This equation is obtained from (30) putting  $z = (b - at)^{-1/2}\zeta$  and  $Q(x, t, z) = R(\eta, \zeta)$ .) Indeed, after some algebra, the compatibility condition of (31) is just the reduced equation (22).

#### 4. Singularity confinement

Recently, a discrete version of the Painlevé property has been proposed [10].<sup>2</sup> Stated simply it says that difference equations have the Painlevé property if their singularities are confined. We assume the same definition for differential-difference equations. Rewriting (22) as

$$v(\eta+1) = -c \frac{v'(\eta)}{v(\eta)} + a + v(\eta-1), \quad v'(\eta) := \frac{dv}{d\eta}, \quad (32)$$

we see that equation (32) has a singularity if  $v(\eta_0) = 0$  and  $v'(\eta_0) \neq 0$ . We have expanded (shifted) equations for  $v(\eta_0+2)$ ,  $v(\eta_0+3)$  and  $v(\eta_0+4)$  in a Laurent series in  $v(\eta_0)$ . It turns out to be necessary to keep the leading three orders. Then, if  $v(\eta_0-1)$ ,  $v'(\eta_0-1)$ ,  $v'(\eta_0)$ ,  $v''(\eta_0)$  and  $v'''(\eta_0)$  are in general position, we obtain

$$\begin{aligned} v(\eta_0+1) &= -c \frac{v'(\eta_0)}{v(\eta_0)} + a + v(\eta_0-1) = O(1/v(\eta_0)) \\ v(\eta_0+2) &= c \frac{v'(\eta_0)}{v(\eta_0)} + v(\eta_0-1) + 2a - c \frac{v''(\eta_0)}{v'(\eta_0)} + v(\eta_0) \\ &\quad - \left\{ (a + v(\eta_0-1)) \frac{v''(\eta_0)}{v'(\eta_0)} - \frac{1}{c}(a + v(\eta_0-1))^2 - v'(\eta_0-1) \right\} \frac{v(\eta_0)}{v'(\eta_0)} \end{aligned}$$

<sup>2</sup> For an alternative discrete version of the Painlevé property see [9].

$$\begin{aligned}
& + O(v^2(\eta_0)) = O(1/v(\eta_0)) \\
v(\eta_0 + 3) &= \left\{ c \frac{v'''(\eta_0)}{v'(\eta_0)} - c \left( \frac{v''(\eta_0)}{v'(\eta_0)} \right)^2 - 3v'(\eta_0 - 1) + v(\eta_0 - 1) \frac{v''(\eta_0)}{v'(\eta_0)} \right. \\
&\quad \left. + \frac{a^2}{c} - \frac{1}{c} (v(\eta_0 - 1))^2 \right\} \frac{v(\eta_0)}{v'(\eta_0)} + O(v^2(\eta_0)) = O(v(\eta_0)) \\
v(\eta_0 + 4) &= \left( v(\eta_0 + 2) - c \frac{v'(\eta_0 + 3)}{v(\eta_0 + 3)} + a \right) = O(1)
\end{aligned} \tag{33}$$

Hence, the singularity is confined and (22) or (32) satisfies the singularity confinement criterion in a remarkably nontrivial way.

## 5. Final remarks

As we have indicated in §2, the above method applies to other PDΔEs as well, whether integrable or not. We here briefly mention the results of applying our method to the Reduced Two-Dimensional Toda Equation [31-33]

$$\frac{\partial^2 u(x, t)}{\partial x \partial t} = \exp \{u(x, t) - u(x - 1, t - 1)\} - \exp \{u(x + 1, t + 1) - u(x, t)\}. \tag{34}$$

The symmetries of this equation are

$$\begin{aligned}
x^* &= x + \varepsilon \alpha(x) \\
t^* &= t + \varepsilon \beta(t) \\
u^* &= u + \varepsilon[qx - qt - x\alpha'(x) + \gamma(x) - t\beta'(t) + \delta(t)].
\end{aligned} \tag{35}$$

In (35)  $\alpha, \beta, \gamma$  and  $\delta$  are arbitrary unit periodic functions:

$$\alpha(x) = \alpha(x + 1), \gamma(x) = \gamma(x + 1), \beta(t) = \beta(t + 1), \delta(t) = \delta(t + 1), \tag{36}$$

the primes denote derivatives, and  $q$  is an arbitrary constant. Expanding the functions  $\alpha(x), \beta(x), \gamma(x)$  and  $\delta(x)$  in a Fourier series we obtain the generators of the symmetry algebra

$$\begin{aligned}
G_{1,k} &= e^{2\pi i k x} \left[ kx \frac{\partial}{\partial u} - \frac{1}{2\pi i} \frac{\partial}{\partial x} \right], \quad G_{2,k} = \frac{e^{2\pi i k x}}{2\pi i} \frac{\partial}{\partial u}, \\
G_{3,k} &= e^{2\pi i k t} \left[ kt \frac{\partial}{\partial u} - \frac{1}{2\pi i} \frac{\partial}{\partial t} \right], \quad G_{4,k} = \frac{e^{2\pi i k x}}{2\pi i} \frac{\partial}{\partial u}, \quad G_5 = (x - t) \frac{\partial}{\partial u}.
\end{aligned} \tag{37}$$

These generators form an infinite-dimensional algebra given by the non-zero commutation relations

$$\begin{aligned}
[G_{1,k}, G_{1,l}] &= (k - l)G_{1,k+l} + (l - k)G_{1,k+l}, \\
[G_{1,k}, G_{2,l}] &= -lG_{2,k+l}, \\
[G_{1,k}, G_5] &= -G_{2,k}, \\
[G_{3,k}, G_{3,l}] &= (k - l)G_{3,k+l} + (l - k)G_{4,k+l}, \\
[G_{3,k}, G_{4,l}] &= -lG_{4,k+l}, \\
[G_{3,k}, G_5] &= -G_{4,k}.
\end{aligned} \tag{38}$$

Because  $G_5$  cannot be written as the commutator of two elements in the algebra, it follows that the algebra described by (38) is not a Kac-Moody-Virasoro algebra.

More results about similarity solutions and symmetry algebra of (34), and its derivation from the Two-Dimensional Toda Equation are given in [24]. A shorter version of this paper is published in [34].

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### Appendix A.

As a first step we integrate the second and third member involving  $dx$  and  $dt$  of (18). This gives

$$t \exp \left\{ \int_X^x \frac{\alpha(x')}{\gamma(x')} dx' \right\} = \int_X^x dx \frac{\beta(x')}{\gamma(x')} \exp \left\{ \int_X^{x'} dx'' \frac{\alpha(x'')}{\gamma(x'')} \right\} + \text{cst.} \quad (\text{A.1})$$

Equation (A.1) can be expressed as

$$\log f(x) + \log \left( \frac{g(x)}{f(x)} - t \right) = \text{cst.} \quad (\text{A.2})$$

in which

$$f(x) = \exp \left\{ \int_X^x \frac{\alpha(x')}{\gamma(x')} dx' \right\}, \quad g(x) = \int_X^x \frac{\beta(x')}{\gamma(x')} f(x') dx' \quad (\text{A.3})$$

For  $\alpha(x)/\gamma(x)$  we use the decomposition

$$\frac{\alpha(x)}{\gamma(x)} = \frac{a}{c} + h(x) \quad (\text{A.4})$$

with

$$h(x) = h(x+1), \quad \int_x^{x+1} h(x') dx' = 0, \quad \int_x^{x+1} \frac{\alpha(x')}{\gamma(x')} dx' = \frac{a}{c} \quad (\text{A.5})$$

This yields

$$f(x) = \exp \left\{ \frac{a[x + d(x)]}{c} \right\} \quad (\text{A.6})$$

with

$$d(x) = \int_X^x \left( \frac{c}{a} \frac{\alpha(x')}{\gamma(x')} - 1 \right) dx' = d(x+1) \quad (\text{A.7})$$

For the function  $g(x)/f(x)$  we have

$$\frac{g(x+1)}{f(x+1)} = \frac{g(x) + \Gamma}{f(x)}, \quad \Gamma = \int_{X-1}^X \frac{\beta(x')}{\gamma(x')} \exp \left\{ \frac{a[x' + d(x')]}{c} \right\} dx' \quad (\text{A.8})$$

From (A.2) we introduce a similarity variable  $\rho$  by identifying the constant with  $ac^{-1}\rho + \log a$ . Then  $\rho$  is given by

$$\rho = x + d(x) + \frac{c}{a} \log \left[ a \frac{g(x)}{f(x)} - at \right] \quad (\text{A.9})$$

Equation (A.9) differs from (20) by the fact that  $g(x)/f(x)$  is not periodic. This can be repaired by considering the periodic function

$$\frac{g(x+1)}{f(x+1)} + \frac{\Gamma}{(1 - e^{a/c}) f(x+1)} = \frac{g(x)}{f(x)} + \frac{\Gamma}{(1 - e^{a/c}) f(x)} \quad (\text{A.10})$$

and introducing the new variable

$$\eta = \rho + \frac{c}{a} \log \left[ 1 + a\Gamma \frac{e^{-a\rho/c}}{1 - e^{-a/c}} \right] = x + d(x) + \frac{c}{a} \log [b(x) - at] \quad (\text{A.11})$$

with

$$\begin{aligned} \frac{b(x)}{a} &= \frac{g(x)}{f(x)} + \frac{\Gamma}{(1 - e^{-a/c}) f(x)} \\ &= \exp \left\{ \frac{-a[x + d(x)]}{c} \right\} \int_X^x \frac{\beta(x')}{\gamma(x')} \exp \left\{ \frac{a[x' + d(x')]}{c} \right\} dx' \left( 1 + \frac{\Gamma}{1 - e^{-a/c}} \right) \end{aligned} \quad (\text{A.12})$$

in which the constant  $\Gamma$  is given by (A.8).

Finally, from the first and second member of (18) we obtain

$$\log u = \log f(x) + \text{cst.} \quad (\text{A.13})$$

Identifying the integration constant with  $\log(v(\eta) \exp\{-a\eta/c\})$  and using (A.6) we show that

$$u = \frac{v(\eta)}{b(x) - at}, \quad (\text{A.14})$$

which is (19).

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# A NEW APPROACH TO THE SEARCH FOR ANALYTICAL SOLUTIONS OF SECOND ORDER AUTONOMOUS NONLINEAR DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In this paper we present a method of finding analytical solutions to the class of nonlinear ordinary differential equations given by

$$f'' + A(f)f' + B(f)(f')^2 + C(f) = 0.$$

These types of equations appear frequently in mathematical physics and its applications to the physics of many-body problems. Except for very special cases, little is known about their analytical solutions.

## 1. Introduction

The nonlinear differential equations that are often found in the descriptions of complex physical systems (for example in the kinetics of critical systems) possess many interesting properties (e.g., solitons, limit cycles, instabilities, chaotic behaviour, etc.) but are also much more poorly understood than their linear counterparts. For example, unless an ordinary differential equation (ODE) satisfies a so-called Painlevé property [1], the hope of finding analytical solutions to it is virtually nonexistent [2–5]. Even when we fix our attention on equations with a single dependent variable (for single component critical systems, for example), the situation looks rather grim since most of the multidimensional (in terms of independent variables, i.e. space-time) symmetry reduction analyses for physically relevant partial differential equations (PDES) result in non-Painlevé ODES. However, many of those reductions lead to autonomous ODES which quite often take the form

$$\frac{d^2f}{dt^2} + A(f)\frac{df}{dt} + B(f)\left(\frac{df}{dt}\right)^2 + C(f) = 0. \quad (1.1)$$

This could be broadly classified as an anharmonic oscillator equation with both linear and quadratic friction (or dissipation) terms. Thus, it could also apply to a number of physically interesting cases of multistable systems with velocity-dependent friction terms. In particular, kink and nucleation center dynamics [6] falls in this category. It is interesting to note

that reductions of particular PDES to this form can be found in the multidimensional time-dependent Landau-Ginzburg equation [7,8] (for spatial-only, travelling-wave and spiral-type solutions), the nonlinear Schrodinger equation [9] (for travelling waves), and the nonlinear Klein-Gordon equation [10] (for a host of symmetry variables, including degenerate ones). It has only been recently demonstrated [11] that the celebrated Emden equation (for spherical and cylindrical patterns in critical dynamics) can be transformed to

$$f'' + f'(b_0 + b_1 f) + b_3 f(f - f_1)(f - f_2) = 0, \quad (1.2)$$

which is also a special case of (1.1). A very interesting result has been published [12] not long ago in connection with a transition from discrete lattice equations to continuum. This procedure has been shown to generate higher order friction terms yielding an equation of the form

$$f'' + \lambda_1 f' + \lambda_2 (f')^2 + \mu(f) = 0, \quad (1.3)$$

which is also a special case of (1.1).

Our motivation in the present paper is to study (1.1) and seek analytical ways of finding its solutions even if they turn out to be special ones only. The approach that will be presented is a systematic one and is a significant generalization of the one proposed in a recent paper by Otwinowski *et al.* [13] who has developed an ansatz approach to solving a generalized anharmonic oscillator equation with linear dissipation, i.e.

$$f'' + \lambda f' + C(f) = 0, \quad (1.4)$$

where  $C(f)$  is a polynomial in  $f$ . Their basic assumption was to seek solutions to (1.4) which also satisfy simultaneously the first order ODE.

$$f' = D(f), \quad (1.5)$$

where  $D(f)$  is *a priori* unknown. Our method goes a significant step beyond the one given by Otwinowski *et al.* [13] in both the scope of applications and the form of solutions. Theirs succeeds in finding only localized kink and bump solutions while ours will not be limited this way.

In what is to follow, we shall present a technique by which analytical solutions to (1.1) can be obtained. First, we shall look at some special cases when two of the functions in (1.1)  $A(f)$ ,  $B(f)$ , and  $C(f)$  are arbitrary functions and the third one is a specific function of  $f$  and give the solution to this problem. Second, we will look at the case when  $A(f)$ ,  $B(f)$ , and  $C(f)$  are polynomials in  $f$  with constant coefficients. In §2 we define a new function  $D(f)$  at the indefinite integral of  $B(f)$  (i.e.  $D(f) = \int df B(f)$ ). Throughout the following discussion, we shall consider  $A(f) \neq 0$  in (1.1) except for Case 1 in §2 which gives the general solution to (1.1) in principle, when  $A(f) = 0$ .

## 2. The Method and its Special Cases

In order to find special analytical solutions of (1.1) we make the following transformation

$$f = R(z)/S(z). \quad (2.1)$$

Also, we define the new independent variable  $z$  through

$$z' = T(z)/U(z), \quad (2.2)$$

so that

$$f' = TV/(US^2), \quad (2.3)$$

and

$$f'' = \frac{T}{U^3 S^3} \{T' U S V + T[U S V' - (S U' + 2 S' U) V]\}, \quad (2.4)$$

where  $V = R' S - R S'$ . Since the independent variable is transformed through (2.2) and the dependent one is assumed quite general in (2.3) our ansatz is much more general than that of (1.5) as postulated by Otwinowski *et al.* [11]. Substitution into (1.1) leads to

$$\begin{aligned} & \frac{T}{U^3 S^3} \{T' U S V + T[U S V' - (S U' + 2 S' U) V]\} A \left(\frac{R}{S}\right) \frac{TV}{U S^2} \\ & + B \left(\frac{R}{S}\right) \frac{T^2 V^2}{U^2 S^4} + C \left(\frac{R}{S}\right) = 0. \end{aligned} \quad (2.5)$$

The method we propose to consider relies on analyzing and solving (2.5) instead of (1.1). This can be done quite readily for some special cases listed below. In these *special cases*, we let

$$T = \frac{dz}{df} \exp \left\{ - \int^f dx B(x) \right\} = \frac{dz}{df} \exp(-D), \quad (2.6)$$

where

$$D = \int^f dx B(x), \quad (2.7)$$

and so (2.5) and (2.2) become respectively

$$\frac{dU}{df} = \{A(f) + C(f)e^D U\} e^D U^2, \quad (2.8)$$

and

$$t - t_0 = \int^f dx \exp\{D(x)\} U(x), \quad (2.9)$$

where the variable  $t$  is our original independent variable. Clearly, solving (2.8) and (2.9) is equivalent to solving (1.1). There are six special cases that solve (2.8) and (2.9), which are as follows:

### 2.1. SPECIAL CASE I

Here  $A(f) = 0$ . In this case, (1.1) takes the form

$$f'' + B(f)(f')^2 + C(f) = 0. \quad (2.10)$$

Clearly, the corresponding solution given in Table 1 is only formal since the integrals in general cannot be done. In fact the elliptic integrals satisfy an equation which is a special subcase of (2.10).

Interestingly, there is some information that can be gained without specifying what  $B(f)$  and  $C(f)$  are. To do this we consider the classical Hamiltonian  $H = (df/dt)^2 + V(f)$  where in this special case the corresponding potential is given by

$$V(f) = \exp\{-2D(f)\} \int^f dx C(x) \exp\{2D(x)\}. \quad (2.11)$$

For periodic solutions to exist we need bounded solutions which means that  $V(f)$  has to have at least one local minimum. Also, for the system to have a separatrix,  $V(f)$  needs to have a local maximum along with the local minimum. Thus,

$$\frac{dV}{df} \Big|_{f=f_1} = 0 = C(f_1) - 2B(f_1) \exp\{-2D(f_1)\} \int^{f_1} dx C(x) \exp\{2D(x)\}, \quad (2.12)$$

and

$$\frac{d^2V}{df^2} \Big|_{f=f_1} = \frac{dC}{df} \Big|_{f=f_1} - 2 \frac{dB}{df} \Big|_{f=f_1} \exp\{-2D(f_1)\} \int^{f_1} dx C(x) \exp\{2D(x)\}. \quad (2.13)$$

Clearly, the first derivative of the coefficient  $B$  plays a very important role in shifting a local maximum to a local minimum and vice versa. Also,  $B$  itself causes a shifting in the location of the local extrema. We shall now consider a simple example by noting that when  $B = 0$  and  $C = -2f$ , then we have the harmonic oscillator equation which has only periodic solutions so that when  $B = 0$  and  $C = -2f$ , then there are no periodic solutions. If we now consider the case  $B = b/f$  and  $C = -2f$ , then we do obtain periodic solutions when  $b < -3/2$  and no periodic solutions for  $b > -3/2$ . Since Special Case 1 covers all cases when  $A(f) = 0$ , we will consider  $A(f) \neq 0$  in the rest of the paper.

### 2.2. SPECIAL CASE II

Here,  $C(f) = 0$  and consequently, (1.1) takes the form

$$F'' + A(f)f' + B(f)(f')^2 = 0. \quad (2.14)$$

Also, the corresponding potential here is

$$V(f) = -\exp\{-2D(f)\} \left[ \int^f dx A(x) \exp\{2D(x)\} \right]^2. \quad (2.15)$$

### 2.3. SPECIAL CASE III

In this case we assume that  $A(f) = C(f) \exp\{D(f)\}$ , and thus (1.1) becomes

$$f'' + A(f)f' + B(f)(f')^2 + A(f) \exp\{-D(f)\} = 0. \quad (2.16)$$

In this case, the formula for its solution shown in Table 1 needs to be inverted before we can substitute for  $U$  in (2.9). However, depending on what the integral the left hand side of this equation is, it may be easier to take  $f$  as a function of  $U$  and then replace the integrand in (2.9) with the appropriate integral in  $U$ . Once this integral is solved, one may be able to invert (2.9) so as to obtain  $U$  as a function of  $t$  and thus obtain  $f$  as a function of  $t$ .

A simple approximation is to assume  $1/U \approx 0$  so that we can expand the logarithm to obtain

$$t - t_0 \approx \int^f dx \frac{\exp\{D(x)\}}{\sqrt{-1 - \int^x dy A(y) \exp\{2D(y)\}}}, \quad (2.17)$$

which has the potential

$$V(f) = -\exp\{-2D(f)\} \left[ \int^f dy A(y) \exp\{2D(y)\} \right]. \quad (2.18)$$

### 2.4. SPECIAL CASE IV

In this case, (1.1) becomes an integro-differential equation of the form

$$f'' + \frac{f'}{\sqrt{-\int^f dy [1 + C(y) \exp\{D(y)\}] \exp\{D(y)\}}} + B(f)(f')^2 + C(f) = 0. \quad (2.19)$$

Also, the corresponding potential is

$$V(f) = \exp\{-2D(f)\} \int^f dy [1 + C(y) \exp\{D(y)\}] \exp\{D(y)\}. \quad (2.20)$$

### 2.5. SPECIAL CASE V

In this case, (1.1) is

$$f'' + A(f)f' + B(f)(f')^2 - \int^f dy [A(y) + \exp\{D(y)\}] \exp\{D(y)\} = 0 \quad (2.21)$$

The potential in this case is

$$V(f) = -\exp\{-2D(f)\} \left[ \int^f dy [A(y) + \exp\{D(y)\}] \exp\{D(y)\} \right]^2. \quad (2.22)$$

Table 1: Special cases of Equation (1.1)

Case	Equation	Solution
I	$f'' + B(f)(f')^2 + C(f) = 0$	$t - t_0 = \int^f dx \frac{\exp\{D(x)\}}{\sqrt{- \int^x dx C(y) \exp\{2D(y)\}}}$
II	$f'' + A(f) f' + B(f) (f')^2 = 0$	$t - t_0 = - \int^f dx \frac{\exp\{D(x)\}}{\int^x dx A(y) \exp\{2D(y)\}}$
III	$f'' + A(f) f' + B(f)(f')^2 + A(f) \exp\{-D(f)\} = 0$	$t - t_0 = \int^f dx \frac{\exp\{D(x)\} u(x)}{\int^x dx \exp\{D(x)\} A(x) = \ln \left  1 + \frac{1}{U} \right  - \left( 1 + \frac{1}{U} \right)}$ , where
IV	$f'' + \frac{f'}{\sqrt{- \int^f dx [1 + C(x) \exp\{D(x)\}] \exp\{D(x)\}}}$	$t - t_0 = \int^f dx \frac{\exp\{D(x)\}}{\sqrt{- \int^x dy [1 + C(y) \exp\{D(y)\}] \exp\{D(y)\}}}$
V	$f'' + A(f)f' + B(f)(f')^2 + C(f) = 0$ $- \int^f dx [A(x) + \exp\{D(x)\}] \exp\{D(x)\} = 0$	$t - t_0 = - \int^f dx \frac{\exp\{D(x)\}}{\int^x dy [A(y) + \exp\{D(y)\}] \exp\{D(y)\}}$
VI	$f'' + A(f)f' + B(f)(f')^2 - c_0 A(f) - c_0^2 B(f) = 0$	$f = c_0 t + f_0$
VII	$f'' + af' + B(f)(f')^2 + \exp\{-D(f)\} = 0$	$t - t_0 = \int^f dx \exp\{D(x)\} U(x)$ , where $\int^f dx \exp\{D(x)\} = \frac{1}{a} \left\{ \frac{1}{a} \ln \left  1 + \frac{a}{U} \right  - \frac{1}{U} \right\}$

## 2.6. SPECIAL CASE VI

Here, we assume that

$$\exp\{2D(f)\} = \frac{1}{U(f)} \quad \text{or} \quad B(f) = \frac{-1}{U} \frac{dU}{df} = -A(f) - C(f),$$

and, as a result, we obtain

$$f = c_0 t + f_0. \quad (2.23)$$

In this case, (1.1) becomes

$$f'' + A(f)f' + B(f)(f')^2 - c_0 A(f) - c_0^2 B(f) = 0. \quad (2.24)$$

## 2.7. SPECIAL CASE VII

Here,  $A(f) = \text{constant}$  and  $C(f) = \exp\{-D(f)\}$ . Also, (1.1) becomes

$$f'' + Af' + B(f)(f')^2 + \exp\{-D(f)\} = 0. \quad (2.25)$$

In this case, the formula for its solution in Table 1 needs to be inverted before we can substitute for  $U$  in (2.9). However, depending on what the integral on the left hand side of (2.25) is, it may be easier to take  $f$  as a function of  $U$  and then replace the integrand in (2.9) with the appropriate integral in  $U$ . Once this integral is solved, one may be able to invert (2.9), so as to obtain  $U$  as a function of  $t$  and thus obtain  $f$  as a function of  $t$ .

The results of this section are summarized in Table 1 which gives the form of the equation and its corresponding solution. The solutions are written as indefinite integrals and thus the constants of integration must not be forgotten since this significantly changes the form of the solution. These integrals in general cannot be solved analytically as can be seen in case I which contains the elliptic equations as a special subcase. Cases II through VII are very particular cases but may be of interest in special applications in physics. In case VII, we have only one general function  $B(f)$ , while  $C(f)$  is related to  $B(f)$  and  $A(f)$  is a constant. In the next section we develop a general approach to analyzing and solving (1.1).

## 3. A General Analysis

This section deals with cases that do not fall in §2 and where  $A(f)$ ,  $B(f)$  and  $C(f)$  are ratios of polynomials in  $f$  with no explicit dependence on the independent variable  $t$ . We shall denote  $P(t; k)$  as representing a polynomial in the variable  $t$  to order  $k$  (i.e.  $p_0 + p_1 t + p_2 t^2 + \dots + p_k t^k$ ). Then,  $A(f)$ ,  $B(f)$  and  $C(f)$  can be represented as

$$A(f) = \frac{A_1(f)}{A_2(f)} = \frac{P(f; \alpha_1)}{P(f; \alpha_2)}, \quad (3.1)$$

$$B(f) = \frac{B_1(f)}{B_2(f)} = \frac{P(f; \beta_1)}{P(f; \beta_2)}, \quad (3.2)$$

$$C(f) = \frac{C_1(f)}{C_2(f)} = \frac{P(f; \chi_1)}{P(f; \chi_2)}. \quad (3.3)$$

Table 1: Special cases of Equation (1.1)

Case	Equation	Solution
I	$f'' + B(f)(f')^2 + C(f) = 0$	$t - t_0 = \int^f dx \frac{\exp\{D(x)\}}{\sqrt{- \int^x dy C(y) \exp\{2D(y)\}}}$
II	$f'' + A(f) f' + B(f) (f')^2 = 0$	$t - t_0 = - \int^f dx \frac{\exp\{D(x)\}}{\int^x dy A(y) \exp\{2D(y)\}}$
III	$f'' + A(f)f' + B(f)(f')^2 + A(f)\exp\{-D(f)\} = 0$	$t - t_0 = \int^f dx \frac{\exp\{D(x)\} u(x), \text{ where}}{\int^x dy \exp\{D(x)\} A(x) = \ln \left  1 + \frac{1}{U} \right  - \left( 1 + \frac{1}{U} \right)}$
IV	$f'' + \frac{f'}{\sqrt{- \int^f dx [1 + C(x) \exp\{D(x)\}] \exp\{D(x)\}}} + B(f)(f')^2 + C(f) = 0$	$t - t_0 = \int^f dx \frac{\exp\{D(x)\}}{\sqrt{- \int^x dy [1 + C(y) \exp\{D(y)\}] \exp\{D(y)\}}}$
V	$f'' + A(f)f' + B(f)(f')^2$ $- \int^f dx [A(x) + \exp\{D(x)\}] \exp\{D(x)\} = 0$	$t - t_0 = - \int^f dx \frac{\exp\{D(x)\}}{\int^x dy [A(y) + \exp\{D(y)\}] \exp\{D(y)\}}$
VI	$f'' + A(f)f' + B(f)(f')^2 - c_0 A(f) - c_0^2 B(f) = 0$	$f = c_0 t + f_0$
VII	$f'' + af' + B(f)(f')^2 + \exp\{-D(f)\} = 0$	$t - t_0 = \int^f dx \exp\{D(x)\} U(x), \text{ where}$ $\int^f dx \exp\{D(x)\} = \frac{1}{a} \left\{ \frac{1}{a} \ln \left  1 + \frac{a}{U} \right  - \frac{1}{U} \right\}$

Thus (1.1) can be written as follows

$$E_0(f) \frac{d^2 f}{dt^2} + A_0(f) \frac{df}{dt} + B_0(f) \left( \frac{df}{dt} \right)^2 + C_0(f) = 0, \quad (3.4)$$

where  $E_0(f) = P(f; \epsilon_0)$ ,  $A_0(f) = P(f; \alpha_0)$ ,  $B_0(f) = P(f; \beta_0)$  and  $C_0(f) = P(f; \chi_0)$ .

The purpose of this section is to reduce the problem of solving the nonlinear ordinary differential equation into an equivalent form of solving a known differential equation and a system of nonlinear algebraic equations. This is more desirable since the algebraic system is easier to solve using symbolic solver programs that are available on computers. We can accomplish this objective by introducing a new dependent variable and transforming (1.1) into a polynomial in the new dependent variable. Thus, for the polynomial to be zero, we must demand that each coefficient of each power of the new dependent variable is zero. This will produce a certain number of parameters and a certain number of coefficients which become our algebraic equations. Now, if we assume that

$$f = \frac{R(z)}{S(z)} = \frac{P(z; \kappa)}{P(z; \kappa)} \quad \text{and} \quad \frac{dz}{dt} = T(z) = P(z; n),$$

hence (3.4) becomes

$$\frac{F_1(z) + F_2(z)}{S^{\epsilon_0+3}} + \frac{F_3(z)}{S^{\alpha_0+2}} + \frac{F_4(z)}{S^{\beta_0+4}} + \frac{F_5(z)}{S^{\chi_0}} = 0, \quad (3.5)$$

where

$$F_1(z) = E_0 \left( \frac{R}{S} \right) S^{\epsilon_0} T^2 (SV' - S'V) = P[z; (3 + \epsilon_0)\kappa + 2n - 4], \quad (3.6)$$

$$F_2(z) = E_0 \left( \frac{R}{S} \right) S^{\epsilon_0} TV (T'S - TS') = P[z; (3 + \epsilon_0)\kappa + 2n - 4], \quad (3.7)$$

$$F_3(z) = A_0 \left( \frac{R}{S} \right) S^{\alpha_0} TV = P[z; (2 + \alpha_0)\kappa + n - 2], \quad (3.8)$$

$$F_4(z) = B_0 \left( \frac{R}{S} \right) S^{\beta_0} T^2 V^2 = P[z; (4 + \beta_0)\kappa + 2n - 4], \quad (3.9)$$

$$F_5(z) = C_0 \left( \frac{R}{S} \right) S^{\chi_0} = P[z; \chi_0\kappa], \quad (3.10)$$

$$V(z) = \frac{dR}{dz} S - R \frac{dS}{dz}. \quad (3.11)$$

The primes in (3.6) and (3.7) represent differentiation with respect of  $z$ . The  $F_1(z)$  and  $F_2(z)$  terms in (3.5) arise from  $f''$  in (3.4), the  $F_3(z)$  term comes from  $A_0(f)f'$ , the  $F_4(z)$  term comes from  $B_0(f)(f')^2$ , and the  $F_5(z)$  term comes from  $C_0(f)$ . As can be seen, the power  $n$  in each term of (3.5) appears as  $n - 2$  or  $2n - 4$  and thus can be eliminated by choosing  $n = 2$ . This corresponds to  $T(z)$  being a quadratic which means that along with solving the system of algebraic equations, we have to solve the following first order ordinary differential equation on the independent variable

$$\frac{dz}{dt} = \tau_0 + \tau_1 z + \tau_2 z^2, \quad (3.12)$$

and since the coefficients are complex constants in general, this leads to hyperbolic and/or trigonometric solutions. It should be noted that the definitions of  $F_1(z)$  and  $F_2(z)$  are not completely true and that they are polynomials  $P[z; (3 + \epsilon_0)\kappa + 2n - 3]$ . However, the coefficient of  $z^{(3+\epsilon_0)\kappa+2n-3}$  contains  $n - 2$  as a factor after  $F_1(z)$  and  $F_2(z)$  have been added together and thus cancel when  $n = 2$ . The  $n = 2$  requirement can also be interpreted as the square root of a quartic polynomial which changed (3.12) into an elliptic differential equation. However,  $F_3(z)$  gives rise to an extra square root, which would have to be eliminated. To eliminate the square root, we have two choices, one is to have  $A(f) = 0$  or else to place  $F_3(z)$  on the right hand side of (3.5) and square both sides. For  $A(f) = 0$ , all we need to do is use case I in Table 1 and thus this procedure is not necessary. The alternative of squaring both sides to eliminate the square root shall be considered later on and at present we will consider  $T(z)$  as given in (3.12). It should be noted that one could extend this argument to a polynomial of order  $2n$  taken to the  $(1/n)$ th power. However, this is impractical at this stage since the reduction is intended to obtain solvable first order differential equations. This extension also includes the  $z^{(3+\epsilon_0)\kappa+2n-3}$  term which in this case has the coefficient  $n/\ell - 2$  where  $n$  is the order of the polynomial and  $1/\ell$  is the power of the polynomial which is again eliminated by choosing  $n = 2\ell$ . Multiplying (3.5) by  $S(z)$  to the highest power in the denominator will make each of the five terms a polynomial to the same order as the other four. Of course, we want to have at least as many parameters as there are algebraic equations. This is obtained by restricting what  $\kappa$  is since from  $R$  we have  $\kappa + 1$  parameters and the same from  $S$ . We also obtain 3 parameters from  $T(z)$  for a total of  $2\kappa + 5$  parameters.

TABLE 2: Reduction to algebraic equations for exponential solutions

Case	$\kappa$	$a_0$	$\beta_0$	$\chi_0$	$\epsilon_0$	# of algebraic equations	$B(f)$
I	$\leq 4$	$\leq 1$	$= 0$	$\leq 3$	$= 0$	$3\kappa + 1$	$= 0$
II	$\leq 2$	$\leq 2$	$= 0$	$\leq 4$	$= 1$	$4\kappa + 1$	$= 0$
III	$\leq 1$	$\leq 4$	$= 0$	$\leq 6$	$= 3$	$6\kappa + 1$	$= 0$
IV	$\leq 4$	$= 1$	$= 0$	$\leq 3$	$= 0$	$3\kappa + 1$	$= 0$
V	$\leq 2$	$= 2$	$= 0$	$\leq 4$	$\leq 1$	$4\kappa + 1$	$= 0$
VI	$\leq 1$	$= 4$	$= 0$	$\leq 6$	$\leq 3$	$6\kappa + 1$	$= 0$
VII	$\leq 4$	$\leq 1$	$= 0$	$= 3$	$= 0$	$3\kappa + 1$	$= 0$
VIII	$\leq 2$	$\leq 2$	$= 0$	$= 4$	$\leq 1$	$4\kappa + 1$	$= 0$
IX	$\leq 1$	$\leq 4$	$= 0$	$= 6$	$\leq 3$	$6\kappa + 1$	$= 0$
X	$\leq 2$	$\leq 2$	$= 0$	$\leq 4$	$= 1$	$4\kappa + 1$	$\neq 0$
XI	$\leq 1$	$\leq 4$	$\leq 2$	$\leq 5$	$= 3$	$6\kappa + 1$	$\neq 0$
XII	$\leq 2$	$\leq 2$	$= 0$	$\leq 4$	$\leq 1$	$4\kappa + 1$	$\neq 0$
XIII	$\leq 1$	$\leq 4$	$\leq 2$	$\leq 6$	$\leq 3$	$6\kappa + 1$	$\neq 0$
XIV	$\leq 2$	$\leq 2$	$= 0$	$\leq 4$	$\leq 1$	$4\kappa + 1$	$\neq 0$
XV	$\leq 1$	$\leq 4$	$\leq 2$	$\leq 6$	$\leq 3$	$6\kappa + 1$	$\neq 0$
XVI	$\leq 2$	$\leq 2$	$= 0$	$\leq 4$	$\leq 1$	$4\kappa + 1$	$\neq 0$
XVII	$\leq 1$	$\leq 4$	$\leq 2$	$\leq 6$	$\leq 3$	$6\kappa + 1$	$\neq 0$

Table 2 gives for each case the maximum order for each of the polynomials of  $A_0$ ,  $B_0$ ,  $C_0$ , and  $E_0$ . This table also gives  $\kappa$  which is the maximum order that  $R$  and  $S$  can assume. Also, Table 2 contains two distinct classes, those when  $B(f) = 0$  (cases 1 through 9) and the cases when  $B(f) \neq 0$ . It is clear from the conditions on  $\kappa$  that there is always a solution

even if it is just the trivial solution  $\kappa = 0$ . This means that by equating parameters to zero, the number of equations reduces faster than the number of parameters.

As mentioned above, we shall now discuss the case when  $n = 4$ . The reduction leads to elliptic differential equations of the form

$$\frac{dz}{dt} = \sqrt{\tau_0 + \tau_1 z + \tau_2 z^2 + \tau_3 z^3 + \tau_4 z^4} = T(z), \quad (3.13)$$

instead of (3.12). The cases that are possible appear in Table 3

TABLE 3: Reduction to algebraic equations for elliptic solutions

Case	$\kappa$	$a_0$	$\beta_0$	$\kappa_0$	$\epsilon_0$	# of algebraic equations	$B(f)$
I	$\leq 1$	$\leq 2$	$= 0$	$\leq 4$	$= 1$	$8\kappa + 1$	$= 0$
II	$\leq 1$	$= 2$	$= 0$	$\leq 4$	$\leq 1$	$8\kappa + 1$	$= 0$
III	$\leq 1$	$\leq 2$	$= 0$	$= 4$	$\leq 1$	$8\kappa + 1$	$= 0$
IV	$\leq 1$	$\leq 2$	$= 0$	$\leq 4$	$= 1$	$8\kappa + 1$	$\neq 0$
V	$\leq 1$	$= 2$	$= 0$	$\leq 4$	$\leq 1$	$8\kappa + 1$	$\neq 0$
VI	$\leq 1$	$\leq 2$	$= 0$	$= 4$	$\leq 1$	$8\kappa + 1$	$\neq 0$
VII	$\leq 1$	$\leq 2$	$= 0$	$\leq 4$	$\leq 1$	$8\kappa + 1$	$\neq 0$

Clearly, all seven cases reduce to the same original differential equation, which is

$$\begin{aligned} \frac{d^2f}{dt^2} + \frac{a_0 + a_1 f + a_2 f^2}{e_0 + e_1 f} \left( \frac{df}{dt} \right) + \frac{b_0}{e_0 + e_1 f} \left( \frac{df}{dt} \right)^2 \\ + \frac{c_0 + c_1 f + c_2 f^2 + c_3 f^3 + c_4 f^4}{e_0 + e_1 f} = 0. \end{aligned} \quad (3.14)$$

Even though we have not investigated this equation, it would be interesting to see what type of solutions arise.

The above analysis covers a large class of nonlinear ordinary differential equations, however, in most cases when transforming to the type of differential equation as given in (1.1) there is usually some explicit dependence on the independent variable still in the equation. Thus, if we were to assume that the coefficient  $A_0(f, t) = P(f; \alpha_0)$  and similarly for  $B_0(f, t)$ ,  $C_0(f, t)$ , and  $E_0(f, t)$ , i.e. the equation would become nonautonomous, then we can use the method above with the parameters now functions of  $t$ . This, however, will lead to a system of coupled second order differential equations which are probably not solvable for general functions of  $T$  but may be solvable when the coefficients have a simple dependence on  $t$ . Also, (3.12) now becomes the generalized Riccati equation.

As a simple example, consider an *anharmonic oscillator equation with quadratic dissipation* given by

$$f'' + B(f')^2 + Cf + f^3 = 0,$$

where we have neglected a constant forcing term for simplicity. This is clearly in the form discussed as Case I in §1. We then readily find that

$$t - t_0 \int_{t_0}^f dx \exp\{D(x)\} \left\{ - \int_{x_0}^x dy C(y) \exp\{2D(y)\} \right\}^{-1/2}, \quad (3.15)$$

where

$$C(y) = Cy + y^3, \quad (3.16)$$

and

$$D(y) = By - By_0. \quad (3.17)$$

We therefore obtain an implicitly given solution in the form

$$t - t_0 = \exp \{-B(x_0 - y_0)\} \int_{f_0}^f dx \\ \times \left\{ C_0 e^{-2Bx} - \left[ \frac{Cx}{2B} - \frac{C}{4B^2} + \frac{x^3}{2B} - \frac{3x^2}{4B^2} - \frac{6x}{8B^3} - \frac{6}{16B^4} \right] \right\}^{-1/2} \quad (3.18)$$

with  $C_0$  denoting an integration constant.

This can be readily extended to a more general case where

$$f'' + B(f')^2 + C(f) = 0, \quad (3.19)$$

with  $C(f)$  being an arbitrary function of  $f$ . Examples could include  $C$  being a trigonometric function ( $\sin f$  or  $a \sin f + b \sin 2f$  for sine-Gordon and double-sine Gordon applications, respectively) or an exponential function for studying a damped Liouville equation. An implicit form of the solution is then found to be

$$t - t_0 = \exp \{B(y_0 - x_0)\} \int_{f_0}^f df \left[ -e^{-2Bx} \int_{x_0}^x dy e^{2By} C(y) \right]^{-1/2}. \quad (3.20)$$

It is interesting to note that the equation for separatrix trajectories in the  $(f', f)$ -space is

$$(f')^2 = -\frac{1}{2B} C(f), \quad (3.21)$$

while in the nondissipative case ( $B = 0$ ) one obtains

$$\frac{1}{2}(f')^2 = \int C(f) df + C_0, \quad (3.22)$$

with  $C_0$  corresponding to a local minimum of the potential function  $\int C(f) df$ . Since the right-hand side of (3.21) is proportional to  $B^{-1}$ , the transition from a dissipative to a nondissipative case ( $B \rightarrow 0$ ) is a *nonanalytic* one.

#### 4. Conclusions

In this paper we have focussed on a class of autonomous nonlinear differential equations which can be described as generalizations of the anharmonic oscillator equation with linear and quadratic dissipation. These types of equations appear frequently in the descriptions of dynamics of multistable dissipative systems and they also occur as reduced ODES for multidimensional PDES in conservative multistable systems. Physical examples of relevant

systems were given. The method that was proposed in the present paper relies on representing the solution as a ratio of polynomials. However, the independent variable is simultaneously transformed as well as, so that it can take a linear, exponential, trigonometric or hyperbolic form, depending on the case. Several special types of the investigated class of equations were solved analytically to either explicit or implicit forms but a general approach has been presented as an algorithm that results in a system of coupled algebraic equations on the polynomial coefficients in the ansatz.

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