Comments and Addenda

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Static axially symmetric solutions of self-dual SU(2) gauge fields in Euclidean four-dimensional space

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All solutions of the stationary axially symmetric Einstein equations are shown to correspond to static axially symmetric solutions of self-dual SU(2) gauge fields. Some simple examples are given.

Yang[1] has reduced the problem of finding self-dual SU(2) gauge fields on Euclidean four-dimensional flat space to solving a set of three Laplace-type equations for one real and one complex variable. In the R gauge, Yang's field equations for the variables ϕ, ρ, and $\bar{\rho}$ are

$$\begin{aligned} &\phi(\phi_{y\bar{y}}+\phi_{z\bar{z}})-\phi_y\phi_{\bar{y}}-\phi_z\phi_{\bar{z}}+\rho_y\bar{\rho}_{\bar{y}}+\rho_z\bar{\rho}_{\bar{z}}=0\,,\\ &\phi(\rho_{y\bar{y}}+\rho_{z\bar{z}})-2\rho_y\phi_{\bar{y}}-2\rho_z\phi_{\bar{z}}=0\,,\\ &\phi(\bar{\rho}_{y\bar{y}}+\bar{\rho}_{z\bar{z}})-2\bar{\rho}_{\bar{y}}\phi_y-2\bar{\rho}_{\bar{z}}\phi_z=0\,. \end{aligned} \tag{1}$$

The subscript denotes partial differentiation and

$$\begin{aligned} &\sqrt{2}\,y\equiv x_1+ix_2\,,\quad \sqrt{2}\,\bar{y}=x_1-ix_2\,,\\ &\sqrt{2}\,z\equiv x_3-ix_4\,,\quad \sqrt{2}\,\bar{z}\equiv x_3+ix_4 \end{aligned} \tag{2}$$

for the complexified Cartesian coordinates x_μ ($\mu=1,2,3,4$). For real values of x_μ (which is all we henceforth consider) $\bar{\rho}=\rho^*$ and ϕ is real. The coordinates of the self-dual potentials b_μ^i are given by

$$\begin{aligned} &\phi\vec{b}_y=(i\rho_y,\rho_y,-i\phi_{\bar{y}}),\quad \phi\vec{b}_{\bar{y}}=(-i\bar{\rho}_{\bar{y}},\bar{\rho}_{\bar{y}},i\phi_{\bar{y}})\,,\\ &\phi\vec{b}_z=(i\rho_z,\rho_z,-i\phi_{\bar{z}}),\quad \phi\vec{b}_{\bar{z}}=(-i\bar{\rho}_{\bar{z}},\bar{\rho}_{\bar{z}},i\phi_{\bar{z}})\,. \end{aligned} \tag{3}$$

Look for solutions of Eqs. (1) of the form $\rho=\sigma e^{i\alpha}$ where σ is a real function and α is a real constant; transform to the space coordinates x_μ and consider static solutions $(\partial/\partial x_4)=0$. Equations (1) become

$$\begin{aligned} &\phi(\nabla^2\phi)=\vec{\nabla}\phi\cdot\vec{\nabla}\phi-\vec{\nabla}\sigma\cdot\vec{\nabla}\sigma\,,\\ &\phi(\nabla^2\sigma)=2\vec{\nabla}\sigma\cdot\vec{\nabla}\sigma\,,\\ &\sigma_{x_1}\phi_{x_2}-\phi_{x_1}\sigma_{x_2}=0\,,\\ &\vec{\nabla}\equiv(\partial/\partial_{x_1},\partial/\partial_{x_2},\partial/\partial_{x_3})\,. \end{aligned} \tag{4}$$

Assuming axial symmetry about x_3, the third equation of the set (4) vanishes. With $\epsilon=\phi+i\sigma$, the first two equations become

$$\mathrm{Re}\,\epsilon(\nabla^2\epsilon)=\vec{\nabla}\epsilon\cdot\vec{\nabla}\epsilon\,. \tag{5}$$

This is the equation deduced by Ernst[2] for the axially symmetric gravitational field problem; ϵ is often called the Ernst potential. ϕ is the norm of the timelike Killing vector of the stationary space-time and σ is the twist potential.

Define

$$\epsilon\equiv\frac{(E-1)}{(E+1)}\,. \tag{6}$$

Then the equation for E is

$$(EE^*-1)\nabla^2E=2E^*\vec{\nabla}E\cdot\vec{\nabla}E\,, \tag{7}$$

and Eq. (7) is the Ernst equation. We have shown that any stationary axisymmetric gravitational field yields through ϕ and ρ ($=\sigma e^{i\alpha}$) a self-dual gauge field. A simple class of solutions is

$$\begin{aligned} &E=e^{-i\beta}\coth\psi\,,\\ &\nabla^2\psi=0\,, \end{aligned} \tag{8}$$

where β is a real constant and ψ any function that satisfies the Laplace equation. In gravitation these solutions are of interest only if $\beta=0$ (modπ); otherwise space-time is not asymptotically flat.

Another class of solutions of interest is the Tomimatsu-Sato[3] series of solutions; we state only the first two:

$$E=p\xi-iq\eta\,, \tag{9}$$

$$E=\frac{p^2\xi^4+q^2\eta^4-1-2ipq\xi\eta(\xi^2-\eta^2)}{2p\xi(\xi^2-1)-2iq\eta(1-\eta^2)}\,. \tag{10}$$

p, q are parameters such that $p^2+q^2=1$. ξ,η are prolate spheroidal coordinates

$$\begin{aligned}&x_3=c\xi\eta,\quad x_1=c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\sin\theta\,,\\&x_2=c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\cos\theta\,,\\&\xi\geq 1,\quad -1\leq\eta\leq 1,\quad 0\leq\theta\leq 2\pi\,.\end{aligned}\tag{11}$$

Equation (9) is the Kerr solution of general relativity with the special case of the Schwarzschild solution when $p=1$, $q=0$. As specific examples we calculate the gauge potentials for two special cases of the Kerr solution (9). One is $p=1$, $q=0$ (Schwarzschild), the other is $p=0$, $q=1$. First we write the gauge potentials in the x_μ coordinate system, taking for simplicity $\alpha=0$ (it can be shown that $\alpha\neq 0$ differs from $\alpha=0$ by a gauge transformation):

$$\begin{aligned}&\phi\vec{b}_{x_1}=(\sigma_{x_2},\sigma_{x_1},-\phi_{x_2})\,,\\&\phi\vec{b}_{x_2}=(-\sigma_{x_1},\sigma_{x_2},\phi_{x_1})\,,\\&\phi\vec{b}_{x_3}=(0,\sigma_{x_3},0)\,,\\&\phi\vec{b}_{x_4}=(\sigma_{x_3},0,-\phi_{x_3})\,.\end{aligned}\tag{12}$$

Example I: $p=1$, $q=0$ [in Eq. (9)].

$$\begin{aligned}&\vec{b}_{x_1}=\left(0,0,\frac{-x_2}{c(\xi^2-1)}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right),\\&\vec{b}_{x_2}=\left(0,0,\frac{x_1}{c(\xi^2-1)}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right),\\&\vec{b}_{x_3}=(0,0,0)\,,\\&\vec{b}_{x_4}=\left(0,0,-\frac{1}{c(\xi^2-1)}\left(\frac{x_3+c}{r_2}+\frac{x_3-c}{r_1}\right)\right).\end{aligned}\tag{13}$$

Example II: $p=0$, $q=1$ [in Eq. (9)].

$$\begin{aligned}&\vec{b}_{x_1}=\frac{1}{c(\eta^2+1)}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(-x_2,-x_1,+\frac{2\eta}{(\eta^2-1)}x_2\right),\\&\vec{b}_{x_2}=\frac{1}{c(\eta^2+1)}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(x_1,-x_2,-\frac{2\eta}{\eta^2-1}x_1\right),\\&\vec{b}_{x_3}=\frac{1}{c(\eta^2+1)}\left(0,\frac{x_3+c}{r_2}-\frac{x_3-c}{r_1},0\right),\\&\vec{b}_{x_4}=\frac{1}{c(\eta^2+1)}\left(\frac{x_3+c}{r_2}-\frac{x_3-c}{r_1},0,\frac{2\eta}{\eta^2-1}\left(\frac{x_3-c}{r_1}-\frac{x_3+c}{r_2}\right)\right),\\&r_1{}^2\equiv(x_3-c)^2+x_1{}^2+x_2{}^2,\quad r_2{}^2\equiv(x_3+c)^2+x_1{}^2+x_2{}^2\,,\\&\eta=(r_2-r_1)/2c\,,\quad \xi=(r_2+r_1)/2c\,.\end{aligned}\tag{14}$$

In example I, the potentials are singular at $r_1+r_2=2c$, which corresponds to the Schwarzschild horizon. Example II is singular at $\eta=\pm1$ which corresponds to $|x_3|\geq c$. The nature of these singularities is to be determined.

A whole industry has grown up in general relativity theory which generates solutions of the stationary axisymmetric field equations from other solutions. (I cite only some papers.[4]) Basically this industry arises from elementary considerations.

The first is that if ϵ is a solution of Eq. (5), so is ϵ' where

$$\epsilon'=\frac{a\epsilon+ib}{1+ic\epsilon}\,. \tag{15}$$

a, b, c, are real constants and Eq. (15) represents a three-parameter group of transformations, G, which transform solutions into solutions. G does not commute with coordinate transformations C, so that by alternating operations C with G one can usually get an endless chain of solutions depending eventually on an infinite number of parameters. Perhaps the best catalog of these transformations has been made by Kinnersley.[4] The most interesting solutions in general relativity are those that are asymptotically flat; these do not necessarily map into the most interesting self-dual SU(2) gauge fields.

Ward[5] has shown how to construct self-dual gauge fields. Atiyah and Ward[6] and Corrigan, Fairlie, Yates, and Goddard[6] have applied the Ward construction to finding finite-action solutions in the R gauge. It does not seem difficult to apply Ward's construction to find the static axisymmetric SU(2) self-dual fields in the R gauge and to select the fields for which $\rho=\bar{\rho}$ (up to a constant phase factor). Thus Ward's construction and the isomorphism described in this paper should give a constructive method of finding the stationary axisymmetric solutions of Einstein's equations.

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