JOURNAL OF MATHEMATICAL PHYSICS VOLUME 44, NUMBER 8 AUGUST 2003

Integrable systems and reductions of the self-dual Yang–Mills equations

M. J. Ablowitz
Department of Applied Mathematics, University of Colorado, Boulder, Colorado 80309

S. Chakravarty[a]
Department of Mathematics, University of Colorado, Colorado Springs, Colorado 80933-7150

R. G. Halburd
Department of Mathematical Sciences, Loughborough University, Loughborough, Leics, LE11 3TU, United Kingdom

(Received 5 April 2003; accepted 9 April 2003)

Many integrable equations are known to be reductions of the self-dual Yang–Mills equations. This article discusses some of the well known reductions including the standard soliton equations, the classical Painlevé equations and integrable generalizations of the Darboux–Halphen system and Chazy equations. The Chazy equation, first derived in 1909, is shown to correspond to the equations studied independently by Ramanujan in 1916. © *2003 American Institute of Physics.*
[DOI: 10.1063/1.1586967]

I. INTRODUCTION

The self-dual Yang–Mills (SDYM) equations (a system of equations for Lie algebra-valued functions of $\mathbb{C}^4$) play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics.

From the perspective of integrable systems, the study of the SDYM equations became particularly intriguing when, in 1985, R. S. Ward[59] conjectured that

> *... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.*

That the SDYM equations are a rich source of integrable systems is suggested by the fact that they are the compatibility condition of an associated linear problem which admits enormous freedom if one allows the associated Lie algebra (the so-called gauge algebra) to be arbitrary. In light of this and other results, the SDYM equations are often referred to as the master integrable system. The SDYM equations provide us with a means of generating and classifying many integrable systems and they also give a unified geometrical framework in which to analyze them. Moreover, in the context of the inverse scattering transform, an integrable equation admits well-behaved solutions obtained via the related linear problems.

The SDYM equations are of great importance in their own right and have found a remarkable number of applications in both physics and mathematics. These equations arise in the context of gauge theory,[49] in classical general relativity,[63,39] and can be used as a powerful tool in the analysis of four-manifolds.[29]

For finite-dimensional gauge groups the integrability of the SDYM equations can be understood from both the inverse scattering transform and geometric points of view.[58,13,26] An excellent reference related to the geometric aspects is Mason and Woodhouse.[41] Our point of view deals

[a]Author to whom correspondence should be addressed. Electronic mail: chuck@math.uccs.edu

with the algebraic and analytic aspects and novel reductions via infinite-dimensional gauge algebras.

The use of certain infinite-dimensional gauge algebras in the self-dual Yang–Mills equations is an important development in the theory. Using these Lie algebras, reductions of the SDYM equations to many important equations including the Kadomtsev–Petviashvili, Davey–Stewartson, $2+1$-dimensional n-wave, and Chazy equations have been found. Recently a generalized Darboux–Halphen system has been obtained as a reduction of the SDYM equations with an infinite-dimensional gauge algebra.[7] These equations are solvable via an associated linear problem, yet their solutions do not possess the Painlevé property—the characteristic singularity structure often thought to be the hallmark of integrability. A special case of the Darboux–Halphen system is equivalent to the generalized Chazy equation. Much work remains to be done in order to identify the class of infinite-dimensional gauge algebras for which the SDYM equations are integrable. These investigations force us to take a much closer look at the idea of integrability itself.

Throughout we will present geometrical interpretations and reasoning whenever we believe that they provide a deeper insight into the reduction process and the properties of the resulting equations. However, this article has been written with the nongeometer in mind. We hope that our survey will be accessible to a wide variety of researchers from many different branches of mathematics and physics.

In Sec. II we introduce the self-dual Yang–Mills equations and their underlying linear problem. In Sec. III we discuss reductions of the SDYM equations to integrable PDEs. Many of the reductions in this section can be found in Ablowitz and Clarkson.[1] In this section we also consider reductions to PDEs when the underlying Lie algebra is infinite dimensional. Such reductions include the Kadomtsev–Petviashvili and Davey–Stewartson equations. In Sec. IV we describe reductions of the SDYM equations with finite-dimensional Lie algebras to ODEs. In particular, the integrable cases of the equations of motion of a spinning top are recovered, together with some generalizations. We also describe the reductions of SDYM to the Painlevé equations due to Mason and Woodhouse.[41] Finally, Sec. V considers the reduction of the SDYM with an infinite-dimensional Lie algebra to a generalized Darboux–Halphen system whose general solution is densely branched about movable singularities and can contain movable natural barriers. This equation in turn has reductions to the Chazy equation and integrable generalizations of the Chazy equation.

II. THE SDYM EQUATIONS

In this section we motivate the SDYM equations from the points of view of both integrable systems and gauge theory.

A. Linear problems and integrable systems

Recall that many $1+1$-dimensional integrable systems are solved via related linear problems of the form

$$\mathbf{\Psi}_x = X\mathbf{\Psi}, \tag{1}$$

$$\mathbf{\Psi}_t = T\mathbf{\Psi}, \tag{2}$$

where X and T are square matrices of the same dimension which are functions of x, t, and the spectral parameter ζ. The compatibility of these equations (i.e., $\mathbf{\Psi}_{xt} = \mathbf{\Psi}_{tx}$) is equivalent to

$$X_t - T_x + [X,T] = 0. \tag{3}$$

In 1973, Ablowitz *et al.*[10,11] solved the inverse scattering problem in the case

$$X = \begin{pmatrix} -i\zeta & q(x,t) \\ r(x,t) & i\zeta \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{4}$$

where A,B,C are (Laurent) polynomials in ζ.

Below we list several choices of T that yield a number of integrable equations which will be discussed in later sections.

(i) The Korteweg–de Vries equation:

$$q_t+6qq_x+q_{xxx}=0,$$

$$T=\begin{pmatrix} -4i\zeta^3+2iq\zeta-q_x & 4q\zeta^2+2iq_x\zeta-(2q^2+q_{xx}) \\ -4\zeta^2+2q & 4i\zeta^3-2iq\zeta+q_x \end{pmatrix}$$

$$(r=-1).$$

(ii) The nonlinear Schrödinger equation:

$$iq_t=q_{xx}\pm 2q^2q^*,$$

$$T=\begin{pmatrix} 2i\zeta^2+iqr & -2q\zeta-iq_x \\ 2r\zeta+ir_x & -2i\zeta^2-iqr \end{pmatrix}$$

$$(r=\mp q^*).$$

(iii) The sine-Gordon equation:

$$u_{xt}=\sin u$$

$$T=\zeta^{-1}\begin{pmatrix} \frac{1}{4}i\cos u & \frac{1}{4}i\sin u \\ \frac{1}{4}i\sin u & -\frac{1}{4}i\cos u \end{pmatrix} \tag{5}$$

$$(q=-r=-\tfrac{1}{2}u_x).$$

It should be noted that the dependence on the spectral parameter and the restriction to 2×2 systems given by (4) is not the only choice for which the inverse scattering problem can be solved for Eq. (3) (see, for example, Refs. 3 and 57).

A simple, natural, and highly symmetric generalization of the linear problem (1) and (2) to four variables; x_1, x_2, t_1, t_2 is given by

$$\left(\frac{\partial}{\partial x_1}+\zeta\frac{\partial}{\partial x_2}\right)\mathbf{\Psi}=(X_1+\zeta X_2)\mathbf{\Psi}, \tag{6}$$

$$\left(\frac{\partial}{\partial t_1}+\zeta\frac{\partial}{\partial t_2}\right)\mathbf{\Psi}=(T_1+\zeta T_2)\mathbf{\Psi}, \tag{7}$$

where X_1, X_2; T_1, T_2 are functions from $\mathbb{C}^4$ to $\mathfrak{sl}(n;\mathbb{C})$—the Lie algebra of $n\times n$ trace-free matrices with complex-valued entries. The compatibility of this system is equivalent to the self-dual Yang–Mills equations with gauge algebra $\mathfrak{sl}(n;\mathbb{C})$ (see Sec. II C below). From the general form of this linear problem it is clear that many integrable equations are reductions of the self-dual Yang–Mills equations because their associated linear problems arise as reductions of the linear problem for the SDYM equations. Notice, however, that the right sides of Eqs. (6) and (7) are linear in ζ whereas in the AKNS scheme, the right side of Eq. (2) can be a Laurent polynomial in ζ. However, it turns out that reductions of (6) and (7) can have much more general dependence on the spectral parameter.

B. The Yang–Mills equations

Non-Abelian gauge theories first appeared in the seminal work of Yang and Mills[65] as a non-Abelian generalization of Maxwell's equations. Let G be a Lie group (referred to as the gauge group) with Lie algebra LG and let $\{x^\mu\}_{\mu=0,\dots,3}$ be coordinates on a four-dimensional manifold M which can be $\mathbb{R}^4$, $\mathbb{R}^{1,3}$ or $\mathbb{R}^{2,2}$. Given $A_\mu(\mathbf{x})\in \mathrm{LG}$, we introduce the covariant derivatives

$$D_\mu=\partial_\mu-A_\mu, \tag{8}$$

and their commutators

$$F_{\mu\nu}=-[D_\mu,D_\nu]=\partial_\mu A_\nu-\partial_\nu A_\mu-[A_\mu,A_\nu]. \tag{9}$$

The fact that the Yang–Mills equations have a natural geometric interpretation was recognized early on in the history of gauge theory. The covariant derivatives (8) can be used to obtain a local representation of a connection on a principal fiber bundle over M. The one-form $A:=A_\mu dx^\mu$ is called the *connection one-form* and $F:=\frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ the *curvature two-form* of the connection. F can also be expressed as the exterior covariant derivative of A given by

$$F=DA:=dA-A\wedge A.$$

Recall that the Hodge dual operator on the four-dimensional manifold M takes any two-form $T=\frac{1}{2}T_{\mu\nu}dx^\mu\wedge dx^\nu$ to the dual two-form $*T=\frac{1}{2}\varepsilon_{\mu\nu}{}^{\gamma\delta}T_{\gamma\delta}dx^\mu\wedge dx^\nu$ where $\varepsilon_{\mu\nu\gamma\delta}$ is totally antisymmetric with $\varepsilon_{0123}=1$ and the standard metric on M is used to raise and lower indices. The Yang–Mills equations then have the simple form

$$D{*F}=0$$

together with the Bianchi identity

$$DF=0,$$

which follows from the definitions of exterior covariant derivative D and F. Note that under the gauge transformation

$$A_\mu\mapsto g^{-1}A_\mu g+g^{-1}\partial_\mu g,\qquad g\in \mathrm{G}, \tag{10}$$

the components of the curvature two-form transform as

$$F_{\mu\nu}\mapsto g^{-1}F_{\mu\nu}g, \tag{11}$$

which corresponds to the transformation of the fibers by the right action of the structure group G on the principal bundle.

C. The self-dual Yang–Mills equations

The Yang–Mills equations are a set of coupled, second-order PDEs in four dimensions for the LG-valued gauge potential functions A_μ's, and are extremely difficult to solve in general. It is, however, possible to obtain a special class of first-order reductions of the full Yang–Mills equations by noting that any F that satisfies

$$*F=\lambda F \tag{12}$$

for some constant λ, also satisfies the Yang–Mills equations by virtue of the Bianchi identity: $DF=0$. From (12) we must have

$$**F=\lambda *F=\lambda^2 F. \tag{13}$$

However, $**F=gF$ where $g=\det[g_{\mu\nu}]$ is the determinant of the metric on M. Hence

$$\lambda=\begin{cases}\pm 1 & \text{on } \mathbb{R}^4,\ \mathbb{R}^{2,2};\\ \pm i & \text{on } \mathbb{R}^{3,1}.\end{cases}$$

All real solutions of the equations $*F=\pm iF$ are trivial. On $\mathbb{R}^4$ and $\mathbb{R}^{2,2}$, the equations $*F=(-)F$ are called the (anti-)self-dual Yang–Mills equations. We will work in $\mathbb{R}^4$ with the standard metric

$$ds^2=(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2.$$

The SDYM equations now take the form

$$F_{01}=F_{23},\quad F_{02}=F_{31},\quad F_{03}=F_{12}. \tag{14}$$

We introduce the null coordinates

$$\sigma=\frac{1}{\sqrt{2}}(x^1+ix^2),\quad \tau=\frac{1}{\sqrt{2}}(x^0-ix^3),$$

$$\tilde{\sigma}=\frac{1}{\sqrt{2}}(x^1-ix^2),\quad \tilde{\tau}=\frac{1}{\sqrt{2}}(x^0+ix^3).$$

It then follows from $A=A_\mu dx^\mu=A_\sigma d\sigma+A_{\tilde{\sigma}}d\tilde{\sigma}+A_\tau d\tau+A_{\tilde{\tau}}d\tilde{\tau}$ that

$$A_0=\frac{1}{\sqrt{2}}(A_\tau+A_{\tilde{\tau}}),\quad A_1=\frac{1}{\sqrt{2}}(A_\sigma+A_{\tilde{\sigma}}),$$

$$A_2=\frac{i}{\sqrt{2}}(A_\sigma-A_{\tilde{\sigma}}),\quad A_3=-\frac{i}{\sqrt{2}}(A_\tau-A_{\tilde{\tau}}),$$

and the SDYM equations become

$$F_{\sigma\tau}=0,\quad F_{\tilde{\sigma}\tilde{\tau}}=0,\quad F_{\sigma\tilde{\sigma}}+F_{\tau\tilde{\tau}}=0. \tag{15}$$

Equations (15) are the compatibility condition of the isospectral linear problem

$$(\partial_\sigma+\zeta\partial_{\tilde{\tau}})\mathbf{\Psi}=(A_\sigma+\zeta A_{\tilde{\tau}})\mathbf{\Psi}, \tag{16}$$

$$(\partial_\tau-\zeta\partial_{\tilde{\sigma}})\mathbf{\Psi}=(A_\tau-\zeta A_{\tilde{\sigma}})\mathbf{\Psi}, \tag{17}$$

where ζ is the spectral parameter and $\mathbf{\Psi}$ is a local section of the Yang–Mills fiber bundle. The compatibility condition is simply $(\partial_\tau-\zeta\partial_{\tilde{\sigma}})(\partial_\sigma+\zeta\partial_{\tilde{\tau}})\mathbf{\Psi}=(\partial_\sigma+\zeta\partial_{\tilde{\tau}})(\partial_\tau-\zeta\partial_{\tilde{\sigma}})\mathbf{\Psi}$. On using Eqs. (16) and (17), this gives

$$[F_{\sigma\tau}-\zeta(F_{\sigma\tilde{\sigma}}+F_{\tau\tilde{\tau}})+\zeta^2F_{\tilde{\sigma}\tilde{\tau}}]\mathbf{\Psi}=0.$$

The gauge transformation (10) can be understood by setting $\mathbf{\Psi}=g\tilde{\mathbf{\Psi}}$ in (16) and (17) and demanding that the A_μ's transform so as to preserve the form of these equations.

A very compact way of writing the SDYM equations was introduced by Pohlmeyer.[48] Following Yang,[64] and working with the Lie algebra su(2), Pohlmeyer noted that the vanishing of $F_{\sigma\tau}$ and $F_{\tilde{\sigma}\tilde{\tau}}$ allows us to write (locally)

$$A_\sigma = (\partial_\sigma C) C^{-1}, \quad A_\tau = (\partial_\tau C) C^{-1},$$

$$A_{\tilde\sigma} = (\partial_{\tilde\sigma} D) D^{-1}, \quad A_{\tilde\tau} = (\partial_{\tilde\tau} D) D^{-1},$$

for some C and D in the Lie group G. Letting $J = C^{-1} D \in \mathrm{G}$ we see that the last equation in (15) becomes

$$\partial_{\tilde\sigma}(J^{-1}\partial_\sigma J) + \partial_{\tilde\tau}(J^{-1}\partial_\tau J) = 0. \tag{18}$$

III. EXAMPLES OF REDUCTIONS

Perhaps the simplest reductions of the SDYM equations are those in which the A_μ's are taken to be independent of certain coordinates. With the exception of the reduction to the Ernst equation, all reductions in this section will be with respect to translational symmetries. That is, the A_μ's will be taken to depend only on two linear combinations of the variables x^0, x^1, x^2, and x^3 (or equivalently σ, $\tilde\sigma$, τ, and $\tilde\tau$).

A. The nonlinear Schrödinger equation

Following Mason and Sparling,[40] let us consider the case in which the Lie algebra is $\mathfrak{sl}(2;\mathbb{C})$ (trace-free 2×2 matrices over the field of complex numbers) and the A_μ's are functions of $x = \sigma + \tilde\sigma$ and $t = \tilde\tau$ only. We use the gauge freedom and take $A_\sigma = 0$. (Note that this involves solving a linear equation for g.)

In terms of the matrix-valued functions $P := A_\tau$, $Q := A_{\tilde\sigma}$ and $R := A_{\tilde\tau}$, the self-dual Yang–Mills equations (15) are

$$P_x = 0, \tag{19}$$

$$Q_x - P_t - [P,R] = 0, \tag{20}$$

$$R_x - Q_t - [Q,R] = 0. \tag{21}$$

Note that Eqs. (19)–(21) are invariant if P, Q, and R all undergo the same constant similarity transformation. Hence if P is independent of t it can be put into canonical form. In particular, if it is diagonalizable we can, without loss of generality, assume it has the form

$$P = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix},$$

for some constant k. From Eq. (20) we see that Q_x must have zero diagonal part. Hence we can take

$$Q = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}.$$

Equation (20) gives the parametrization of the off-diagonal elements of R in terms of q and r. So up to a constant (which can be gauged away), Eq. (21) gives

$$R = \frac{1}{2k}\begin{pmatrix} qr & q_x \\ r_x & -qr \end{pmatrix},$$

together with the equations

$$2kq_t = q_{xx} + 2q^2 r,$$

$$2kr_t = -r_{xx} - 2qr^2.$$

Choosing $k=i/2$, $r=\pm q^*$, gives the nonlinear Schrödinger equation

$$iq_t=q_{xx}\pm 2|q|^2 q.$$

B. The Korteweg–de Vries equation

Mason and Sparling[40] also considered the above reduction for the case in which P is not diagonalizable. Let P take the canonical form

$$P=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We then find that

$$Q=\begin{pmatrix} v & 1 \\ w & -v \end{pmatrix},$$

where $w=v_x-v^2$ (up to an arbitrary function of t),

$$R=\frac{1}{8}\begin{pmatrix} 4(v_x-v^2)_x & -8v_x \\ v_{xxx}-4vv_{xx}-2v_x^2+4v^2v_x & -4(v_x-v^2)_x \end{pmatrix}$$

and $u=-v_x$ satisfies the Korteweg–de Vries equation

$$u_t=\tfrac{1}{4}u_{xxx}+3uu_x.$$

C. The sine-Gordon equation

Suppose that the A_μ's depend on $x=\sigma$ and $t=\tilde{\sigma}$ only. If we use a gauge in which $A_{\tilde{\sigma}}=0$, then the linear problem (16) and (17) for the SDYM equations becomes

$$\partial_x \mathbf{\Psi}=(A_\sigma+\zeta A_{\tilde{\tau}})\mathbf{\Psi}, \quad \partial_t\mathbf{\Psi}=-\frac{1}{\zeta}A_\tau\mathbf{\Psi}.$$

Here we choose the Lie algebra to be $\mathfrak{su}(2)$ so that the A_μ's are skew-Hermitian. We introduce the parametrization

$$A_\sigma=-\frac{i}{2}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \quad A_\tau=-\frac{i}{2}\begin{pmatrix} 0 & a-ib \\ a+ib & 0 \end{pmatrix}, \quad A_{\tilde{\tau}}=-\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(Ref. 14), where a,b,c are real functions of x and t. The SDYM equations are equivalent to

$$\frac{\partial a}{\partial x}=-bc, \quad \frac{\partial b}{\partial x}=ac, \quad \frac{\partial c}{\partial t}=-b. \tag{22}$$

It follows from the first two equations of (22) that a^2+b^2 is independent of x. We choose $a^2+b^2=1$ and introduce the parametrization $a=\cos u$, $b=\sin u$. This gives $c=\partial u/\partial x$. The third equation in (22) then becomes the sine-Gordon equation

$$u_{xt}=-\sin u.$$

D. The *N*-wave equations

Following Chakravarty and Ablowitz,[14] we consider the case in which the A_μ's are functions of σ and τ only and the Lie algebra is $\mathfrak{su}(n;\mathbb{R})$. In this case, from the self-duality equation $F_{\tilde\sigma\tilde\tau}=0$, it follows that $[A_{\tilde\sigma},A_{\tilde\tau}]=0$. We take

$$A_{\tilde\sigma}=\mathrm{diag}(a_1,a_2,\ldots,a_n),\quad A_{\tilde\tau}=\mathrm{diag}(b_1,b_2,\ldots,b_n),$$

where the a_j's and b_j's are constants (we can use a gauge transformation to make them constant). Using the parametrization $A_\sigma=[A_{ij}]$ and $A_\tau=[B_{ij}]$, the vanishing of $F_{\sigma\tilde\sigma}+F_{\tau\tilde\tau}$ implies that $[A_\sigma,A_{\tilde\sigma}]+[A_\tau,A_{\tilde\tau}]=0$, which gives

$$A_{ij}=\lambda_{ij}B_{ij},$$

where

$$\lambda_{ij}=-\frac{b_i-b_j}{a_i-a_j},\quad i\neq j,$$

and we have assumed that the a_j's are distinct. Finally, the vanishing of $F_{\sigma\tau}$ gives the N-wave interaction equations,

$$\frac{\partial B_{ij}}{\partial\sigma}-\lambda_{ij}\frac{\partial B_{ij}}{\partial\tau}=\sum_{k=1}^{n}(\lambda_{ik}-\lambda_{kj})B_{ik}B_{kj}. \tag{23}$$

In the case $N=3$ and in which B is skew-Hermitian, Eq. (23) becomes the three-wave interaction equation.[66] The equation (23) for arbitrary N was studied by Ablowitz and Haberman.[3]

E. The chiral field equations

A number of important reductions of the SDYM equations come directly from Eq. (18). If J depends only on $x=\sigma+\tilde\sigma$ and $t=\tau+\tilde\tau$, then Eq. (18) becomes the chiral field equation

$$(J^{-1}J_x)_x+(J^{-1}J_t)_t=0. \tag{24}$$

Using $\sigma=(1/\sqrt{2})(x^1+ix^2)$, $\tilde\sigma=(1/\sqrt{2})(x^1-ix^2)$, Ward[61,60] obtained a $2+1$-dimensional generalization of Eq. (24) by considering a reduction of Eq. (18) in which J depends on $t=\tau+\tilde\tau$ and σ and $\tilde\sigma$. This gives

$$(J^{-1}J_t)_t+(J^{-1}J_\sigma)_{\tilde\sigma}=0, \tag{25}$$

which has been studied by Manakov and Zakharov[38] and Villarroel.[56] More generally, Ward obtains the equation

$$\begin{aligned}-(J^{-1}J_t)_t+(J^{-1}J_x)_x+(J^{-1}J_y)_y+a[(J^{-1}J_y)_x-(J^{-1}J_x)_y]+b[(J^{-1}J_t)_y-(J^{-1}J_y)_t]\\ +c[(J^{-1}J_x)_t-(J^{-1}J_t)_x]=0,\end{aligned} \tag{26}$$

where (a,b,c) is spacelike $(-a^2+b^2+c^2=1)$ or timelike $(-a^2+b^2+c^2=-1)$. If (a,b,c) is timelike, then Eq. (26) can be transformed to (25).

F. The Ernst equation

Following L. Witten,[62] we let J in Eq. (18) be a function of $\rho=\sqrt{(x^1)^2+(x^2)^2}$ and $z=x^3$ only. That is, $\rho^2=2\sigma\tilde\sigma$ and $z=(i/\sqrt{2})(\tau-\tilde\tau)$. Equation (18) becomes

$$\rho^{-1}(\rho J^{-1}J_\rho)_\rho+(J^{-1}J_z)_z=0. \tag{27}$$

When J is a real symmetric matrix in $\mathrm{SL}(2;\mathbb{R})$, we can parametrize J as

$$J=\frac{1}{f}\begin{pmatrix}1 & g\\ g & f^2+g^2\end{pmatrix}.$$

In terms of this parametrization, Eq. (27) becomes

$$f\Delta f=\nabla f\cdot\nabla f-\nabla g\cdot\nabla g,$$

$$f\Delta g=2(\nabla f)\cdot(\nabla g),$$

where $\nabla:=(\partial_\rho,\partial_z)$ and $\Delta:=\partial_\rho^2+\rho^{-1}\partial_\rho+\partial_z^2$ are the axisymmetric forms of the gradient and Laplacian on $\mathbb{R}^3$, respectively, in cylindrical-polar coordinates. Introducing the variable $\mathcal{E}:=f+ig$, this system can be written compactly as

$$\Re(\mathcal{E})\Delta\mathcal{E}=\nabla\mathcal{E}\cdot\nabla\mathcal{E}. \tag{28}$$

Equation (28) is called the Ernst equation and describes stationary axisymmetric space–times in general relativity.[30] The function $\mathcal{E}$ is known as the Ernst potential.

G. Toda molecule

In this reduction we choose the Lie algebra to be simple and of rank N [e.g., $\mathfrak{sl}(N+1;\mathbb{C})$]. We use the basis $\{H_j,E_j^+,E_j^-\}_{j=1}^N$, which satisfies

$$\begin{gathered}[H_i,H_j]=0,\quad [E_i^+,E_j^-]=\delta_{ij}H_j,\\ [H_i,E_j^+]=K_N^{ji}E_j^+,\quad [H_i,E_j^-]=-K_N^{ji}E_j^-,\end{gathered} \tag{29}$$

where $K_N=[K_N^{ij}]$ is a Cartan matrix. Recall that an $N\times N$-matrix K is called a Cartan matrix if it satisfies the following properties:

(1) $K^{ii}=2$.
(2) K^{ij} is a nonpositive integer if $i\neq j$.
(3) $K^{ij}=0$ if and only if $K^{ji}=0$.
(4) K is positive definite with rank N.

We choose the A_μ's to be functions of σ and $\tilde{\sigma}$ only and of the form

$$A_\sigma=\sum_{k=1}^N u_k(\sigma,\tilde{\sigma})H_k,\quad A_{\tilde{\sigma}}=\sum_{k=1}^N v_k(\sigma,\tilde{\sigma})H_k,$$

$$A_\tau=\sum_{k=1}^N w_k(\sigma,\tilde{\sigma})E_k^+,\quad A_{\tilde{\tau}}=\sum_{k=1}^N w_k(\sigma,\tilde{\sigma})E_k^-.$$

Substituting the above A_μ's into the self-duality equations and using the commutation relations (29), a straightforward calculation shows that the functions u_k and v_k, $k=1,2,\ldots,N$ can be eliminated from the resulting equations, which then yield

$$\Delta\phi_i=-\sum_{j=1}^N K_N^{ij}\exp(\phi_j),\quad i=1,2,\ldots,N, \tag{30}$$

where $w_i=\exp(\phi_i/2)$ and $\Delta\phi=\phi_{\sigma\tilde{\sigma}}$. Equations (30) are known as the Toda molecule equations.[46] The case $N=1$ corresponds to the Liouville equation $\Delta\phi_1=-2\exp(\phi_1)$.

The above analysis can be repeated for the case in which a basis $\{H_j, E_j^+, E_j^-\}_{j=0}^N$ satisfies the relations (29) in which $\tilde{K}_N$ is now taken to be an extended Cartan matrix. An $(N+1)\times(N+1)$ matrix is said to be an extended Cartan matrix if and only if it is of rank N and satisfies properties 1–3 above. Note that, in particular, an extended Cartan matrix possesses a zero eigenvalue because it is not of maximal rank. This gives the Toda lattice equation

$$\Delta\phi_i = -\sum_{j=1}^{N} \tilde{K}_N^{ij} \exp(\phi_j), \quad i = 0,1,\ldots,N. \tag{31}$$

In particular, if we take

$$\tilde{K}_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

then Eq. (31) becomes the two-dimensional Darboux–Toda equation

$$\Delta\phi_i = \exp(\phi_{i-1}) - 2\exp(\phi_i) + \exp(\phi_{i+1}), \tag{32}$$

where $\phi_i = \phi_{i+N+1}$. The system (32) was known to Darboux[28] in the nineteenth century.

H. Infinite-dimensional algebras and 2+1 equations

In Ref. 9, Ablowitz, Chakravarty, and Takhtajan considered reductions of the SDYM equations in which the Lie algebra is the infinite-dimensional Lie algebra of formal matrix differential operators in an auxiliary variable y. They then considered reductions in which the A_μ's depend only on two space–time variables. The resulting equations then depends on three variables.

1. The 2+1-dimensional N-wave equation

First consider a reduction in which the A_μ's are functions of $x=\sigma$ and $t=\tau$ only. Then the resulting SDYM linear system is given by

$$\partial_x\Psi = (A_\sigma + \zeta A_{\tilde{\tau}})\Psi,$$

$$\partial_t\Psi = (A_\tau - \zeta A_{\tilde{\sigma}})\Psi,$$

where Ψ is a function of x,t,y and ζ. Choose a gauge in which $A_{\tilde{\tau}}=B$ and $A_{\tilde{\sigma}}=C$, where B and C are constant, commuting $n\times n$ matrices. We take the remaining gauge potentials of the form $A_\sigma = U + B\partial_y$ and $A_\tau = V + C\partial_y$, where $U(x,y,t), V(x,y,t) \in \mathfrak{sl}(n,\mathbb{C})$. By taking $\Psi = Ge^{-\zeta y}$, we obtain a simpler, reduced linear system from above,

$$\partial_x G = (U + B\partial_y)G, \tag{33}$$

$$\partial_t G = (V + C\partial_y)G, \tag{34}$$

in terms of the function $G(x,y,t)$. The system (33) and (34) is the standard linear system related to the $2+1$-dimensional N-wave equations. The compatibility of (33) and (34) is

$$\partial_t U - \partial_x V + [U,V] - C\partial_y U + B\partial_y V = 0, \quad [B,V] = [C,U],$$

which gives

$$[B,\partial_t Q]-[C,\partial_x Q]-C[B,\partial_y Q]+B[C,\partial_y Q]+[[B,Q],[C,Q]]=0,$$

where $U=[B,Q]$ and $V=[C,Q]$. In the case when B and C are diagonal and Q is off-diagonal, these equations can be transformed to the $2+1$ N-wave equations

$$\frac{\partial Q_{ij}}{\partial t}=a_{ij}\frac{\partial Q_{ij}}{\partial x}+b_{ij}\frac{\partial Q_{ij}}{\partial y}+\sum_{k=1}^{n}(a_{ik}-a_{kj})Q_{ik}Q_{kj},$$

where a_{ij}, b_{ij} are suitable constants depending on the matrix elements of B and C.

2. KP, mKP, and DS equations

Chakravarty, Kent, and Newman[21] also obtained the Kadomtsev–Petviashvili, modified Kadomtsev–Petviashvili, and Davey–Stewartson equations directly as reductions of the SDYM equations with an infinite-dimensional Lie algebra of formal matrix differential operators. If the A_μ's are assumed to depend on $x=\sigma+\tilde{\sigma}$ and $t=\tau$ only, then the linear problem (16) and (17) becomes

$$\partial_x \boldsymbol{\Psi}=(A_\sigma+\zeta A_{\tilde{\tau}})\boldsymbol{\Psi}, \tag{35}$$

$$\partial_t \boldsymbol{\Psi}=(A_\tau+\zeta[A_\sigma+A_{\tilde{\sigma}}]+\zeta^2 A_{\tilde{\tau}})\boldsymbol{\Psi}. \tag{36}$$

The connection components are taken to be of the form

$$A_\sigma=U_0+U_1\partial_y,\quad A_{\tilde{\sigma}}=-(B_0+B_1\partial_y),\quad A_\tau=V_0+V_1\partial_y+V_2\partial_y^2,\quad A_{\tilde{\tau}}=A,$$

where the coefficients are 2×2 matrix-valued functions of x and t. In order to simplify the integrability conditions of (35) and (36), we demand (as in the $2+1$ N-wave case) that the spectral parameter ζ be eliminated from this system after a change of variable of the form $\boldsymbol{\Psi}=G\mathrm{e}^{-\zeta y}$. It can be shown that this requirement implies that $U_1=V_2=B_1+A$ and $V_1=U_0+B_0$. Subsequently, the system (35) and (36) becomes

$$\partial_x G=(U_0+A\partial_y)G, \tag{37}$$

$$\partial_t G=(V_0+[U_0+B_0]\partial_y+A\partial_y^2)G. \tag{38}$$

Appropriate choices of the matrices A, B, U, and V give the linear problem for the KP, mKP, and DS equations (see Ref. 21).

Another notable reduction with an infinite-dimensional gauge algebra was considered in Ref. 39 by Mason and Newman. They showed that the SDYM equations with the Lie algebra of the Lie group of volume preserving diffeomorphisms on a four-manifold, in which the A_μ's are independent of the space-time coordinates σ, $\tilde{\sigma}$, τ, and $\tilde{\tau}$, are equivalent to the self-dual Einstein equations. In particular, the reduction to the Plebanski heavenly equation is given in Ref. 22. This equation in turn has a reduction to the Monge–Ampere equation.

I. The SDYM hierarchy

In Ref. 9, the authors studied an infinite hierarchy of equations whose first member was the SDYM system. Each member of the hierarchy has the same underlying spectral problem, and the higher flows are derived from an infinite sequence of nonlocal conservation laws associated with the SDYM equations. Furthermore, many well known integrable hierarchies in $1+1$- and $2+1$-dimensions are derived from the symmetry reductions of the self-dual hierarchy.

In an appropriate gauge, the kth member of the SDYM hierarchy is given by

$$\partial_{\tau_k}A_\sigma-\partial_\sigma\Phi_{k-1}+[A_\sigma,\Phi_{k-1}]=0, \tag{39}$$

where Φ_n is given recursively by

$$(\partial_{\tilde{\tau}} - \operatorname{ad} A_{\tilde{\tau}})\Phi_{n+1} = -(\partial_\sigma - \operatorname{ad} A_\sigma)\Phi_n - \partial_{\tilde{\sigma}} A_\sigma \delta_{n0}, \quad n = 0,1,\ldots,$$

and $\Phi_0 = -A_{\tilde{\sigma}}$. Equation (39) is the compatibility condition for the system

$$D_1\Psi = \mathcal{A}_1\Psi, \quad D_2\Psi = \mathcal{A}_2\Psi, \tag{40}$$

where, for $k = 2,3,\ldots,$

$$D_1 = \partial_\sigma + \zeta\partial_{\tilde{\tau}}, \quad \mathcal{A}_1 = A_\sigma + \zeta A_{\tilde{\tau}},$$

$$D_k = \partial_{\tau_k} - \zeta^{k-1}\partial_{\tilde{\sigma}}, \quad \mathcal{A}_k = (\zeta^{k-1}\Phi)_+, \quad \Phi = \sum_{n=0}^{\infty} \Phi_n \zeta^{-n},$$

and $(F)_+$ denotes the power series part of the Laurent expansion of F about $\zeta = 0$. Note that if $k = 2$, then $D_2 = \partial_\tau - \zeta\partial_{\tilde{\sigma}}$ and $\mathcal{A}_2 = A_\tau - \zeta A_{\tilde{\sigma}}$, where $\tau = \tau_2$ and we have identified A_τ with Φ_1. In this case the linear problem (40) becomes the standard linear problem for the SDYM equations.

As mentioned above, the reduction of the SDYM equations to the $1+1$- and $2+1$-dimensional integrable equations can be extended to a reduction of the SDYM hierarchy to the corresponding hierarchies. In this way, we can obtain, for example, both the $1+1$- and $2+1$-dimensional N-wave hierarchies. In particular, the Davey–Stewartson equation (DS) can be obtained as the second member of the $2+1$-dimensional N-wave hierarchy. See Ref. 9 for details.

IV. ODE REDUCTIONS

A. Integrable tops

The equations of motion for a spinning top have played a fundamental role in the early development of the theory of integrable systems.

1. The Euler–Arnold–Manakov top

In this reduction, following Chakravarty, Ablowitz, and Clarkson,[18,19] we take the Lie algebra to be $\mathfrak{sl}(n;\mathbb{C})$ and we assume that the A_μ's are functions of $t = \sigma$ only. The vanishing of $F_{\tilde{\sigma}\tilde{\tau}}$ demands that $A_{\tilde{\sigma}}$ and $A_{\tilde{\tau}}$ commute. We take these matrices to be diagonal and of the form

$$A_{\tilde{\sigma}} = \operatorname{diag}(a_1, a_2, \ldots, a_n), \quad A_{\tilde{\tau}} = \operatorname{diag}(b_1, b_2, \ldots, b_n),$$

where the a_j's and b_j's are constants. The equation $F_{\sigma\tilde{\sigma}} + F_{\tau\tilde{\tau}} = 0$ is now the algebraic equation $[A_\sigma, A_{\tilde{\sigma}}] + [A_\tau, A_{\tilde{\tau}}] = 0$, which gives the elements of $A_\sigma = [A_{ij}]$ in terms of the elements of $A_\tau = [B_{ij}]$ as

$$A_{ij} = -\frac{b_j - b_i}{a_j - a_i} B_{ij}, \quad i \neq j,$$

provided $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$.

Choosing $a_i = b_i^2$, $i = 1,\ldots,n$, and A_σ and A_τ to be skew-symmetric, we have $A_{ij} = -(b_i + b_j)^{-1} B_{ij}$ and the vanishing of $F_{\sigma\tau}$ gives

$$\frac{dB_{ij}}{dt} = \sum_{k=1}^{n} \left(\frac{1}{b_j + b_k} - \frac{1}{b_k + b_i} \right) B_{ik} B_{kj}. \tag{41}$$

Equations (41) were first considered by Manakov[37] and Arnold[12] and are the equations of motion for a free n-dimensional rigid body about a fixed point. In the case $n = 3$, we obtain Euler's equations for a free spinning body about a fixed point,

$$\frac{dL_1}{dt}=\gamma_1 L_2 L_3, \quad \frac{dL_2}{dt}=\gamma_2 L_3 L_1, \quad \frac{dL_3}{dt}=\gamma_3 L_1 L_2, \tag{42}$$

where

$$L_1=B_{23}, \quad L_2=B_{31}, \quad L_3=B_{12}, \quad I_1=-(b_2+b_3), \quad I_2=-(b_3+b_1), \quad I_3=-(b_1+b_2),$$

and

$$\gamma_1=\frac{I_2-I_3}{I_2 I_3}, \quad \gamma_2=\frac{I_3-I_1}{I_3 I_1}, \quad \gamma_3=\frac{I_1-I_2}{I_1 I_2}.$$

In Eq. (42), (L_1,L_2,L_3) is the angular momentum in the body frame and the I_k's are the principal moments of inertia. Reductions to other integrable tops, including the Kowalevsaya top, are described in Refs. 18 and 19.

B. The Painlevé equations

With the exception of the reduction to the Ernst equation, all of the above reductions have been with respect to translational symmetries. In other words, the reductions have all resulted from considering A_μ's that depend only on one or more linear combination of the variables σ, $\tilde{\sigma}$, τ, and $\tilde{\tau}$. In this section we follow Mason and Woodhouse[41,42] and obtain reductions to the Painlevé equations by considering reductions by conformal symmetries.

The Painlevé equations are the following six classically known ODEs:

$$P_I \quad u''=6u^2+t,$$

$$P_{II} \quad u''=2u^3+tu+\alpha,$$

$$P_{III} \quad u''=\frac{1}{u}u'^2-\frac{1}{t}u'+\frac{1}{t}(\alpha u^2+\beta)+\gamma u^3+\frac{\delta}{u},$$

$$P_{IV} \quad u''=\frac{1}{2u}u'^2+\frac{3}{2}u^3+4tu^2+2(t^2-\alpha)u+\frac{\beta}{u},$$

$$P_V \quad u''=\left\{\frac{1}{2u}+\frac{1}{u-1}\right\}u'^2-\frac{1}{t}u'+\frac{(u-1)^2}{t^2}\left(\alpha+\frac{\beta}{u}\right)+\frac{\gamma u}{t}+\frac{\delta u(u+1)}{u-1},$$

$$P_{VI} \; u''=\frac{1}{2}\left\{\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-t}\right\}u'^2-\left\{\frac{1}{t}+\frac{1}{u-1}+\frac{1}{u-t}\right\}u'$$
$$+\frac{u(u-1)(u-t)}{t^2(t-1)^2}\left\{\alpha+\frac{\beta t}{u^2}+\frac{\gamma(t-1)}{(u-1)^2}+\frac{\delta t(t-1)}{(u-t)^2}\right\}.$$

These equations have played a very important role in integrable systems. Indeed they arise from similarity reductions of classical soliton equations and as monodromy preserving deformation equations associated with linear systems of ODEs with rational coefficients.

The SDYM equations are invariant under the group of conformal transformations (transformations that preserve the metric up to an over all factor). The metric on $\mathbf{R}^4$ in null coordinates, $d\sigma\, d\tilde{\sigma}+d\tau\, d\tilde{\tau}$, is proportional to $\varepsilon_{\alpha\beta\gamma\delta}dx^{\alpha\beta}dx^{\gamma\delta}$, where $[x^{\alpha\beta}]$ is the skew-symmetric singular matrix

$$[x^{\alpha\beta}]=\begin{pmatrix} 0 & \lambda & \sigma & \tilde{\tau} \\ -\lambda & 0 & -\tau & \tilde{\sigma} \\ -\sigma & \tau & 0 & 1 \\ -\tilde{\tau} & -\tilde{\sigma} & -1 & 0 \end{pmatrix}, \tag{43}$$

where $\lambda=\sigma\tilde{\sigma}+\tau\tilde{\tau}$.

Consider a mapping of the form $x\mapsto y=gxg^T$, where $g\in \mathrm{GL}(4;\mathbb{C})$. Then the mapping $x\mapsto y/y^{23}$ generates a conformal transformation of $\mathbb{C}^4$ as it maps the space of matrices of the form (43) into itself. In fact, every proper conformal transformation arises in this way. The generators of the conformal transformations are called conformal Killing vectors. In order to calculate the associated conformal Killing vectors we consider a one-parameter family of transformations given by $g=\exp(\epsilon K)=I+\epsilon K+O(\epsilon^2)$, where $K\in \mathfrak{gl}(4;\mathbb{C})$. Hence $y=x+\epsilon(Kx+xK^T)+O(\epsilon^2)$. Consider the case in which the matrix K has a 1 in the $\mu\nu$ component and zeros elsewhere. In components we have

$$y^{\alpha\beta}=x^{\alpha\beta}+\epsilon(x^{\alpha\nu}\delta^{\beta\mu}-x^{\beta\nu}\delta^{\alpha\mu})+O(\epsilon^2).$$

So

$$x^{\alpha\beta}\mapsto\tilde{x}^{\alpha\beta}=y^{\alpha\beta}/y^{23}=x^{\alpha\beta}+\epsilon(x^{\alpha\nu}\delta^{\beta\mu}-x^{\beta\nu}\delta^{\alpha\mu}+x^{\alpha\beta}q^{\mu\nu})+O(\epsilon^2),$$

where $q^{\mu\nu}=x^{3\nu}\delta^{2\mu}-x^{2\nu}\delta^{3\mu}$. By considering the appropriate components of $[x^{\alpha\beta}]$ we find

$$\sigma\mapsto\sigma+\epsilon(x^{0\nu}\delta^{2\mu}-x^{2\nu}\delta^{0\mu}+\sigma q^{\mu\nu})+O(\epsilon^2),$$

$$\tilde{\sigma}\mapsto\tilde{\sigma}+\epsilon(x^{1\nu}\delta^{3\mu}-x^{3\nu}\delta^{1\mu}+\tilde{\sigma} q^{\mu\nu})+O(\epsilon^2),$$

$$\tau\mapsto\tau+\epsilon(x^{2\nu}\delta^{1\mu}-x^{1\nu}\delta^{2\mu}+\tau q^{\mu\nu})+O(\epsilon^2),$$

$$\tilde{\tau}\mapsto\tilde{\tau}+\epsilon(x^{0\nu}\delta^{3\mu}-x^{3\nu}\delta^{0\mu}+\tilde{\tau} q^{\mu\nu})+O(\epsilon^2).$$

It follows that the conformal Killing vector $X_{\mu\nu}$ associated with the matrix K which is one in the $\mu\nu$-entry and zero elsewhere, is

$$X_{00}=\sigma\partial_\sigma+\tilde{\tau}\partial_{\tilde{\tau}}, \quad X_{20}=-\sigma\tilde{\tau}\partial_\sigma-\tilde{\sigma}\tilde{\tau}\partial_{\tilde{\sigma}}+\sigma\tilde{\sigma}\partial_\tau-\tilde{\tau}^2\partial_{\tilde{\tau}},$$

$$X_{01}=\tilde{\sigma}\partial_{\tilde{\tau}}-\tau\partial_\sigma, \quad X_{21}=\tau\tilde{\tau}\partial_\sigma-\tilde{\sigma}^2\partial_{\tilde{\sigma}}-\tau\tilde{\sigma}\partial_\tau-\tilde{\sigma}\tilde{\tau}\partial_{\tilde{\tau}},$$

$$X_{10}=\tilde{\tau}\partial_{\tilde{\sigma}}-\sigma\partial_\tau, \quad X_{30}=\sigma^2\partial_\sigma-\tau\tilde{\tau}\partial_{\tilde{\sigma}}+\sigma\tau\partial_\tau+\sigma\tilde{\tau}\partial_{\tilde{\tau}},$$

$$X_{11}=\tilde{\sigma}\partial_{\tilde{\sigma}}+\tau\partial_\tau, \quad X_{31}=-\sigma\tau\partial_\sigma-\tilde{\sigma}\tau\partial_{\tilde{\sigma}}-\tau^2\partial_\tau+\sigma\tilde{\sigma}\partial_{\tilde{\tau}},$$

$$X_{02}=\partial_{\tilde{\tau}}, \quad X_{22}=-\tilde{\sigma}\partial_{\tilde{\sigma}}-\tilde{\tau}\partial_{\tilde{\tau}},$$

$$X_{03}=-\partial_\sigma, \quad X_{23}=\tilde{\tau}\partial_\sigma-\tilde{\sigma}\partial_\tau,$$

$$X_{12}=\partial_{\tilde{\sigma}}, \quad X_{32}=-\tau\partial_{\tilde{\sigma}}+\sigma\partial_{\tilde{\tau}},$$

$$X_{13}=\partial_\tau, \quad X_{33}=-\sigma\partial_\sigma-\tau\partial_\tau.$$

Note that there are 15 independent conformal Killing vectors since $X_{00}+X_{11}+X_{22}+X_{33}=0$. This corresponds to the fact that we could have taken $g\in \mathrm{SL}(4;\mathbb{C})$, which would mean that the trace of K is zero.

The Painlevé equations correspond to reductions of the SDYM equations when the Lie algebra is $\mathfrak{sl}(2;\mathbb{C})$. The conformal Killing vectors of these reductions correspond to the following four-parameter subgroups of $\mathrm{GL}(4;\mathbb{C})$. Mason and Woodhouse[42] call these the Painlevé groups:

$$P_I,\ \ P_{II}\quad \begin{pmatrix} a_4 & a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & a_4 & a_3 \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

$$P_{III}\quad \begin{pmatrix} a_4 & a_2 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & a_3 & a_1 \\ 0 & 0 & 0 & a_3 \end{pmatrix},$$

$$P_{IV}\quad \begin{pmatrix} a_3 & a_2 & a_1 & 0 \\ 0 & a_3 & a_2 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

$$P_{V}\quad \begin{pmatrix} a_2 & a_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

$$P_{VI}\quad \begin{pmatrix} a_4 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix}.$$

The conformal Killing vectors associated with the Painlevé equations are discussed below.

$\mathbf{P_I}$, $\mathbf{P_{II}}$: The conformal Killing vectors associated with the Painlevé subgroup given above are of the form

$$a_1X_{03}+a_2(X_{02}+X_{13})+a_3(X_{01}+X_{12}+X_{23})+a_4(X_{00}+X_{11}+X_{22}+X_{33}).$$

In this case the vector multiplying a_4 is zero. More generally for the reductions described for other Painlevé equations below, the conformal Killing vector associated with a_4 is a linear combination of the conformal Killing vectors associated with a_1, a_2, and a_3.

The vectors multiplying a_1, a_2, and a_3 are, respectively,

$$\tilde{X}_1=-\partial_\sigma,$$

$$\tilde{X}_2=\partial_\tau+\partial_{\tilde{\tau}},$$

$$\tilde{X}_3=(\tilde{\tau}-\tau)\partial_\sigma+\partial_{\tilde{\sigma}}-\tilde{\sigma}\partial_\tau+\tilde{\sigma}\partial_{\tilde{\tau}}.$$

We now choose new variables w^1, w^2, w^3, and t, such that

$$\tilde{X}_i(w^j)=\delta_i^j,\quad \tilde{X}_i(t)=0,\quad i,j=1,2,3. \tag{44}$$

A particular choice is

$$w^1=-\sigma+\tilde{\sigma}(\tilde{\tau}-\tau)-\tfrac{2}{3}\tilde{\sigma}^3,\quad w^2=\tilde{\tau}-\tfrac{1}{2}\tilde{\sigma}^2,\quad w^3=\tilde{\sigma},\quad t=\tilde{\tau}-\tau-\tilde{\sigma}^2.$$

Using the gauge freedom, we let the one-form A have the form

$$A=W_j dw^j,$$

where the W_j's are functions of t only. That is, the gauge freedom has been used to choose the coefficient of dt to be zero. Hence,

$$A=-W_1 d\sigma+([\tilde{\tau}-\tau-2\tilde{\sigma}^2]W_1-\tilde{\sigma}W_2+W_3)d\tilde{\sigma}-\tilde{\sigma}W_1 d\tau+(\tilde{\sigma}W_1+W_2)d\tilde{\tau},$$

from which we can read the values of A_σ, $A_{\tilde{\sigma}}$, A_τ, and $A_{\tilde{\tau}}$. The SDYM equations are

$$W_1'=0,\quad W_2'=[W_1,W_3],\quad W_3'=[W_2,tW_1+W_3].$$

From these equations it follows that three conserved quantities are $l=\mathrm{Tr}(W_1W_2)$, $m=\mathrm{Tr}(W_3W_1+\tfrac{1}{2}W_2^2)$, and $n=\mathrm{Tr}(W_2W_3)$.

Using the residual gauge freedom, W_1 can be put into one of the canonical forms,

$$\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where k is a constant. The first case leads to P_{II} while the second leads to P_I. If we choose y to be one of the roots of the gauge-invariant equation

$$\det([W_1,yW_2-W_3])=0,$$

we find that (up to simple rescalings) y solves the appropriate Painlevé equation.

For the other Painlevé equations, we list the conformal Killing vectors, the choices for the w^j's and t, and the reduced SDYM equations.

P_{III}

$$\tilde{X}_1=\tilde{\tau}\partial_\sigma-\tilde{\sigma}\partial_\tau,$$

$$\tilde{X}_2=-\tau\partial_\sigma+\tilde{\sigma}\partial_{\tilde{\tau}},$$

$$\tilde{X}_3=-\sigma\partial_\sigma-\tilde{\sigma}\partial_{\tilde{\sigma}}-\tau\partial_\tau-\tilde{\tau}\partial_{\tilde{\tau}}.$$

$$w^1=-\tau/\tilde{\sigma},\quad w^2=\tilde{\tau}/\tilde{\sigma},\quad w^3=-\log\tilde{\sigma},\quad t=\tilde{\sigma}^{-1}\sqrt{\sigma\tilde{\sigma}+\tau\tilde{\tau}}.$$

$$W_1'=0,\quad tW_2'=2[W_3,W_2],\quad W_3'=2t[W_1,W_2].$$

$\mathbf{P_{IV}}$

$$\tilde{X}_1 = \partial_{\tilde{\tau}},$$

$$\tilde{X}_2 = -\tau\partial_\sigma + \partial_{\tilde{\sigma}} + \tilde{\sigma}\partial_{\tilde{\tau}},$$

$$\tilde{X}_3 = \sigma\partial_\sigma + \tau\partial_\tau .$$

$$w^1 = \tilde{\tau} - \tfrac{1}{2}\tilde{\sigma}^2, \quad w^2 = -\sigma/\tau, \quad w^3 = \log\tau, \quad t = \tilde{\sigma} + \sigma/\tau.$$

$$W_1' = 0, \quad W_2' = [tW_2 + W_3, W_1], \quad W_3' = [W_3, W_2]. \tag{45}$$

$\mathbf{P_{V}}$

$$\tilde{X}_1 = -\tau\partial_\sigma + \tilde{\sigma}\partial_{\tilde{\tau}},$$

$$\tilde{X}_2 = \sigma\partial_\sigma + \tilde{\sigma}\partial_{\tilde{\sigma}} + \tau\partial_\tau + \tilde{\tau}\partial_{\tilde{\tau}},$$

$$\tilde{X}_3 = -\tilde{\sigma}\partial_{\tilde{\sigma}} - \tilde{\tau}\partial_{\tilde{\tau}} .$$

$$w^1 = \tilde{\tau}/\tilde{\sigma}, \quad w^2 = \log([\sigma\tilde{\sigma} + \tau\tilde{\tau}]/\tilde{\sigma}), \quad w^3 = \log(\tau/\tilde{\sigma}), \quad t = \sigma/\tau + \tilde{\tau}/\tilde{\sigma}.$$

$$W_1' = 0, \quad W_2' = [W_3, W_1], \quad tW_3' = [tW_1 + W_2, W_3].$$

$\mathbf{P_{VI}}$

$$\tilde{X}_1 = -\sigma\partial_\sigma - \tau\partial_\tau,$$

$$\tilde{X}_2 = -\tilde{\sigma}\partial_{\tilde{\sigma}} - \tilde{\tau}\partial_{\tilde{\tau}}$$

$$\tilde{X}_3 = \tilde{\sigma}\partial_{\tilde{\sigma}} + \tau\partial_\tau .$$

$$w^1 = -\log\sigma, \quad w^2 = -\log\tilde{\tau}, \quad w^3 = \log(\tilde{\sigma}/\tilde{\tau}), \quad t = -(\tau\tilde{\tau})/(\sigma\tilde{\sigma}).$$

$$W_1' = 0, \quad tW_2' = [W_2, W_3], \quad t(1-t)W_3' = [W_3, tW_1 + W_2].$$

1. Reduction of the linear problem

Note that each of the symmetry reductions of the SDYM equations to one of the Painlevé equations extends to a reduction of the linear problem (16) and (17). However, in finding a reduction of the linear problem, the spectral parameter ζ must also be transformed. The symmetries of the field equations are lifted to symmetries of the linear problem. For P_{VI} we extend the reduction above to the linear problem (16) and (17) by restricting $\mathbf{\Psi}$ to have the form $\mathbf{\Psi}(\sigma,\tau,\tilde{\sigma},\tilde{\tau};\zeta) = \psi(t;\lambda)$, where $\lambda = \tilde{\tau}/(\sigma\zeta)$. Note that then

$$\tau\partial_\tau\mathbf{\Psi} = -\tilde{\sigma}\partial_{\tilde{\sigma}}\mathbf{\Psi} = t\partial_t\psi, \quad \text{and} \quad \tilde{\tau}\partial_{\tilde{\tau}}\mathbf{\Psi} = -\sigma\partial_\sigma\mathbf{\Psi} = t\partial_t\psi + \lambda\partial_\lambda\psi.$$

The linear problem (16) and (17) then becomes

$$\partial_\lambda\psi = -\left[\frac{W_2}{\lambda - 1} - \frac{W_1 + W_2 + W_3}{\lambda - 1} + \frac{W_3}{\lambda - t}\right]\psi, \quad \partial_t\psi = \frac{W_3}{\lambda - t}\psi. \tag{46}$$

The system of equations (46) is the isomonodromy problem for P_{VI}. The compatibility of (46) gives Eqs. (45), which are equivalent to P_{VI}. Isomonodromy problems for the above reductions to $P_I - P_V$ can be obtained in the same way. Indeed, it is often easier to identify a reduction to one of the Painlevé equations from the form of the isomonodromy problem. By comparing the isomonodromy problem to those in the literature, we can identify the component that will satisfy the appropriate Painlevé equation.

V. THE DARBOUX–HALPHEN SYSTEM

In this section we consider a reduction of the SDYM equations to an integrable generalization of the classical Darboux–Halphen system. Its general solution is densely branched and contains a movable natural barrier.

Consider the reduction of the SDYM equations in which the A_μ's are functions of $t := -x^0$ only. This gives the well known Nahm equations[43]

$$\dot{A}_1 = [A_2, A_3], \quad \dot{A}_2 = [A_3, A_1], \quad \dot{A}_3 = [A_1, A_2], \tag{47}$$

where we have chosen a gauge in which $A_0 \equiv 0$.

Using $\mathfrak{diff}(S^3)$, the infinite-dimensional Lie algebra of vector fields on S^3, we choose the components of the connection to be of the form

$$A_i(t) = \sum_{j,k=1}^{3} O_{ij} M_{jk}(t) X_k . \tag{48}$$

The X_k's are divergence-free vector fields on S^3 and satisfy the $\mathfrak{su}(2)$ commutation relations

$$[X_i, X_j] = \sum_{k=1}^{3} \varepsilon_{ijk} X_k , \tag{49}$$

where ε_{ijk} is totally antisymmetric and $\varepsilon_{123} = 1$. The SO(3) matrix $[O_{ij}]$ is used to represent the points of S^3 (see, e.g., Ref. 55) and the action of the vector fields X_i on O_{jk} is given by[6]

$$X_i O_{jk} = \sum_{l=1}^{3} \varepsilon_{ikl} O_{jl} . \tag{50}$$

Substituting Eq. (48) into Eq. (47) and using (49) and (50) together with the identities

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk} O_{ip} O_{jq} O_{kr} = \varepsilon_{pqr} , \quad \sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},$$

yields

$$\dot{M} = (\operatorname{Adj} M)^T + M^T M - (\operatorname{Tr} M) M, \tag{51}$$

where $(\operatorname{Adj} M) := (\det M) M^{-1}$ is the adjoint of M and the dot denotes differentiation with respect to t. Equation (51) was first derived in Ref. 20. Equation (51) was also derived in Ref. 34 where it was shown to represent an SU(2) invariant hypercomplex four-manifold. Since the Weyl curvature of a hypercomplex four-manifold is self-dual, Eq. (51) describes a class of self-dual Weyl Bianchi IX space–times with Euclidean signature.[15]

In order to solve Eq. (51) we first introduce a simple factorization. If the eigenvalues of the symmetric part, M_s, of M are distinct, then M_s can be diagonalized using a (complex) orthogonal matrix P. In this case we may write

$$M = M_s + M_a = P(d + a) P^{-1},$$

where $d := \mathrm{diag}\,(\omega_1,\omega_2,\omega_3)$, $\omega_i \neq \omega_j$, $i \neq j$, and the nonzero elements of the skew-symmetric matrix a are denoted as $a_{12} = -a_{21} = \tau_3$, $a_{23} = -a_{32} = \tau_1$ and $a_{31} = -a_{13} = \tau_2$. Using the above factorization of M, Eq. (51) can be transformed into

$$\begin{aligned}\dot{\omega}_1 &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + \tau^2,\\ \dot{\omega}_2 &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) + \tau^2,\\ \dot{\omega}_3 &= \omega_1\omega_2 - \omega_3(\omega_1+\omega_2) + \tau^2,\end{aligned} \tag{52}$$

where $\tau^2 := \tau_1^2 + \tau_2^2 + \tau_3^2$ and

$$\dot{\tau}_1 = -\tau_1(\omega_2+\omega_3), \quad \dot{\tau}_2 = -\tau_2(\omega_3+\omega_1), \quad \dot{\tau}_3 = -\tau_3(\omega_1+\omega_2), \tag{53}$$

together with the linear equation

$$\dot{P} + Pa = 0, \tag{54}$$

for the matrix P. The system (52) with $\tau^2 = 0$ is the classical Darboux–Halphen system which appeared in Darboux's analysis of triply orthogonal surfaces[27] and was later solved by Halphen.[32] Halphen also studied and solved Eqs. (52)–(56),[31] which are linearizable in terms of Fuchsian differential equations with three regular singular points.

Taking the differences between the various equations in (52) results in

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\ln(\omega_2-\omega_3), \quad \omega_2 = -\frac{1}{2}\frac{d}{dt}\ln(\omega_3-\omega_1), \quad \omega_3 = -\frac{1}{2}\frac{d}{dt}\ln(\omega_1-\omega_2).$$

Together with Eqs. (53), these equations show that

$$\begin{aligned}\alpha^2 &:= \frac{\tau_1^2}{(\omega_1-\omega_2)(\omega_3-\omega_1)},\\ \beta^2 &:= \frac{\tau_2^2}{(\omega_2-\omega_3)(\omega_1-\omega_2)},\\ \gamma^2 &:= \frac{\tau_3^2}{(\omega_3-\omega_1)(\omega_2-\omega_3)}\end{aligned} \tag{55}$$

are constants. Without loss of generality we choose α, β, and γ to have nonnegative real parts. Hence, provided the symmetric part of M has distinct eigenvalues, Eq. (51) reduces to the third-order system (52), where

$$\tau^2 = \alpha^2(\omega_1-\omega_2)(\omega_3-\omega_1) + \beta^2(\omega_2-\omega_3)(\omega_1-\omega_2) + \gamma^2(\omega_3-\omega_1)(\omega_2-\omega_3), \tag{56}$$

together with the linear equation (54) for P. In Refs. 47 and 33, solutions of equations (52)–(56) for special choices of α, β, and γ were determined in terms of automorphic forms.

Note that the system (52) with τ^2 as in (56) is invariant under the transformation

$$t \mapsto \mu(t) := \frac{at+b}{ct+d}, \quad \omega_j(t) \mapsto \frac{\omega_j(\mu(t))}{(ct+d)^2} + \frac{c}{ct+d}, \quad ad-bc=1.$$

We introduce the μ-invariant function

$$s := \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3}. \tag{57}$$

Differentiating Eq. (57) and using the system (52) yields $\dot{s} = 2s(\omega_1 - \omega_2)$. Repeating this process gives $\ddot{s} = 2(\dot{s} - 2\omega_3 s)(\omega_1 - \omega_2)$. Solving these two equations together with (57) for the ω's gives

$$\begin{aligned} \omega_1 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s(s-1)}, \\ \omega_2 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s-1}, \\ \omega_3 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s}. \end{aligned} \tag{58}$$

Substituting the parametrization (58) into any of the equations in (52) shows that $s(t)$ must satisfy

$$\{s,t\} + \frac{\dot{s}^2}{2}V(s) = 0, \tag{59}$$

where

$$\{s,t\} := \frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2$$

is the Schwarzian derivative and V is given by

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}. \tag{60}$$

The general solution of Eq. (59) is given implicitly by

$$t(s) = \frac{u_1(s)}{u_2(s)}, \tag{61}$$

where $u_1(s)$ and $u_2(s)$ are two independent solutions of the Fuchsian differential equation

$$\frac{d^2u}{ds^2} + \frac{1}{4}V(s)u = 0 \tag{62}$$

with regular singular points at 0, 1, and ∞. The transformation

$$u(s) = s^{c/2}(1-s)^{(a+b-c+1)/2}\chi(s) \tag{63}$$

maps Eq. (62) to the Gauss hypergeometric equation

$$s(1-s)\frac{d^2\chi}{ds^2} + [c-(a+b+1)s]\frac{d\chi}{ds} - ab\chi = 0, \tag{64}$$

where $a = (1+\alpha-\beta-\gamma)/2$, $b = (1-\alpha-\beta-\gamma)/2$, and $c = 1-\beta$ (see, e.g., Refs. 45 and 2). Note that from the general solution $s(t)$ of Eq. (59) we can reconstruct the ω_j's from Eqs. (58) and the τ_j's then follow from (55).

From Eq. (61), if α, β, and γ are non-negative real numbers such that $\alpha+\beta+\gamma<1$, then the upper- (or lower-) half s-plane is mapped to a triangular region in the t-plane whose sides are the arcs of circles and whose vertices subtend angles of $\alpha\pi$, $\beta\pi$, and $\gamma\pi$. Moreover, if α, β, and γ are either 0 or reciprocals of integers, then s is an analytic function of t on the interior of a circle on the complex sphere $\mathbb{C}\cup\infty$ but cannot be analytically extended across any point of the circle. That is, the circle is a natural barrier for the function $s(t)$.

The solution procedure just outlined allows us to obtain explicit expressions for the conserved quantities for the generalized Darboux–Halphen system (52) and (56). In Ref. 36, it was shown that the classical Darboux–Halphen system admits no meromorphic first integrals. In Ref. 17, the first integrals for the full system (52) and (56) were found and shown to be branched and transcendental involving hypergeometric functions. The existence of these integrals is consistent with Ref. 36 because even in the classical case they are branched despite the fact that the general solution is single-valued.

Fix two linearly independent solutions u_1 and u_2 of Eq. (62) with Wronskian

$$W(u_1,u_2)=u_1u_2'-u_2u_1'=1, \tag{65}$$

where prime denotes differentiation with respect to s. Then the general solution of Eq. (59) is given implicitly by

$$t(s)=\frac{J_2u_1(s)-J_1u_2(s)}{I_2u_1(s)-I_1u_2(s)}, \tag{66}$$

where I_α and J_α, $\alpha=1,2$, are constants satisfying $I_1J_2-I_2J_1=1$. Differentiating Eq. (66) twice and using (65) gives

$$I_2u_1-I_1u_2=\dot{s}^{1/2}, \quad I_2u_1'-I_1u_2'=\tfrac{1}{2}\dot{s}^{-3/2}\ddot{s}. \tag{67}$$

Solving the linear equations (67) for I_1 and I_2 gives

$$I_\alpha=\frac{d\phi_\alpha}{dt}, \quad \phi_\alpha=\dot{s}^{-1/2}u_\alpha(s), \quad \alpha=1,2. \tag{68}$$

The constants J_1 and J_2 are then obtained from Eqs. (66) and (68) together with the normalization $I_1J_2-I_2J_1=1$. They are given by

$$J_\alpha=tI_\alpha-\phi_\alpha, \quad \alpha=1,2\,.$$

So, the I_α and J_α, taken to be functions of $t,s,\dot{s}$ and $\ddot{s}$ are first integrals for the Schwarzian equation. In terms of the Darboux–Halphen variables, these quantities are

$$\phi_\alpha=\sqrt{2r(\omega_i)}\,u_\alpha(s(\omega_i))\,, \quad I_\alpha=\sqrt{\frac{2}{r(\omega_i)}}\,u_\alpha'(s(\omega_i))-(\omega_1-\omega_2-\omega_3)\sqrt{\frac{r(\omega_i)}{2}}\,u_\alpha(s(\omega_i))\,,$$

where $r(\omega_i)=\sqrt{(\omega_2-\omega_3)/(\omega_1-\omega_2)(\omega_1-\omega_3)}$ and $s(\omega_i)$ is given by Eq. (57).

In terms of these variables, the Darboux–Halphen system (52) and (56) can be written as the Hamiltonian system

$$\dot{\phi}_\alpha=\frac{\partial H}{\partial I_\alpha}=I_\alpha\,, \quad \dot{I}_\alpha=-\frac{\partial H}{\partial\phi_\alpha}=0\,, \quad H=\frac{I_1^2+I_2^2}{2}, \quad \alpha=1,2, \tag{69}$$

subject to the algebraic constraint

$$\phi_1I_2-\phi_2I_1=W(u_1,u_2)=1. \tag{70}$$

The canonical coordinates $\{I_\alpha,\phi_\alpha\}$ are analogs of the action-angle variables for the Darboux–Halphen system. The phase space dynamics of the system is restricted to the constraint subspace given by Eq. (70). This represents a three-dimensional complex quadric. Poisson–Nambu structures for the generalized Darboux–Halphen system (52) and (56) are presented in Ref. 17 which are similar to those for rigid body dynamics in three dimensions.[44,54] The system is also written as a gradient flow in Ref. 16.

A. The Chazy equation

Let ω_1, ω_2, ω_3 be a solution of (52) with $\tau=0$ and define $y:=-2(\omega_1+\omega_2+\omega_3)$. Then y is a solution of the equation[17,18]

$$\frac{d^3y}{dt^3}=2y\frac{d^2y}{dt^2}-3\left(\frac{dy}{dt}\right)^2, \tag{71}$$

which was studied by Chazy.[23–25] Furthermore, given a solution y of the Chazy equation (71), let ω_1, ω_2, and ω_3 be the three roots of the cubic equation

$$\omega^3+\frac{1}{2}y\omega^2+\frac{1}{2}\frac{dy}{dz}\omega+\frac{1}{12}\frac{d^2y}{dt^2}=0.$$

If the ω_j's are distinct, then they solve the classical Darboux–Halphen system [i.e., the system (52) with $\tau=0$].

The general solution of the Chazy equation is given by

$$y(t(s))=6\frac{d}{dt}\ln\chi_1(s), \quad t(s)=\chi_2(s)/\chi_1(s), \tag{72}$$

where χ_1 and χ_2 are two independent solutions of the special hypergeometric equation

$$s(1-s)\frac{d^2\chi}{ds^2}+\left(\frac{1}{2}-\frac{7}{6}s\right)\frac{d\chi}{ds}-\frac{1}{144}\chi=0.$$

On replacing χ_1 and χ_2 with the independent linear combinations $a\chi_1+b\chi_2$ and $c\chi_1+d\chi_2$, $ad-bc=1$, it can be seen from Eq. (72) that the Chazy equation admits the symmetry

$$y(t)\mapsto\tilde{y}(t)=(ct+d)^{-2}y\left(\frac{at+b}{ct+d}\right)-\frac{6c}{ct+d}, \quad ad-bc=1. \tag{73}$$

As well as having a general solution in terms of special hypergeometric functions as described above, the Chazy equation (71) is related to the theory of modular functions.[8,53] Indeed, a particular solution of (71) is given by

$$y(t):=i\pi E_2(t), \tag{74}$$

where

$$E_2(t):=1+\frac{6}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(mt+n)^2} \tag{75}$$

is the second Eisenstein series. [Note that the series (75) is not absolutely convergent, so the order of the sum is important.] The second Eisenstein series can also be written as the Fourier series

$$E_2(t)=1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \quad q=e^{2\pi it},$$

where $\sigma_1(n)$ is the sum of the divisors of n.

Furthermore, the solution (74) can be written in terms of a special logarithmic potential,

$$y(t)=\frac{1}{2}\frac{d}{dt}\ln\Delta(t), \tag{76}$$

where Δ is the discriminant cusp form of weight 12, which satisfies

$$\Delta(\mu(t))=(ct+d)^{12}\Delta(t), \qquad \mu\in \mathrm{PSL}(2;\mathbf{Z}). \tag{77}$$

This shows that there is a deep connection between the Chazy equation and the theory of modular forms.

The discriminant modular form has the well known representation

$$(2\pi)^{-12}\Delta(t)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n,$$

where $q=\mathrm{e}^{2\pi it}$ and the coefficient function $\tau(n)$ is called the Ramanujan τ-function (see, e.g., Ref. 52). From Eqs. (76) and (74) it can be shown that Δ satisfies the homogeneous ODE of degree 4;

$$\Delta^3\frac{d^4\Delta}{dt^4}-5\Delta^2\frac{d\Delta}{dt}\frac{d^3\Delta}{dt^3}-\frac{3}{2}\Delta^2\left(\frac{d^2\Delta}{dt^2}\right)^2+12\Delta\left(\frac{d\Delta}{dt}\right)^2\frac{d^2\Delta}{dt^2}-\frac{13}{2}\left(\frac{d\Delta}{dt}\right)^4=0.$$

Rankin[51] first showed that the discriminant cusp form satisfies this equation. Since Δ has no zeros or poles and satisfies a homogeneous equation it is the natural analog of the τ function that appears in Hirota's method (see, e.g., Sec. 3.3 of Ref. 4).

Note that the characterization of the Ramanujan coefficients $\tau(n)$ is a major problem in number theory. These famous numbers arise naturally as the Fourier coefficients of $\Delta(t)$ when we write the Chazy equation in the above homogeneous form.

Furthermore, there is another important correspondence between the Chazy equation and Ramanujan's work. In 1916 Ramanujan[50] proved that the functions

$$P(q)=1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

$$Q(q)=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$R(q)=1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

where $\sigma_k(n)=\Sigma_{d|n}d^k$, $k=1,3,5$ (sum of the divisors of n to the kth power), satisfy

$$q\frac{dP}{dq}=\frac{1}{12}(P^2-Q),$$

$$q\frac{dQ}{dq}=\frac{1}{3}(PQ-R), \tag{78}$$

$$q\frac{dR}{dq}=\frac{1}{2}(PR-Q^2).$$

Using $q=e^{2\pi it}$, $\tau=\pi t$, the equations (78) take the form

$$\frac{dP}{d\tau}=\frac{i}{6}(P^2-Q),$$

$$\frac{dQ}{d\tau}=\frac{2i}{3}(PQ-R), \tag{79}$$

$$\frac{dR}{d\tau}=i(PR-Q^2).$$

Using the first of the above equations to find Q: $Q=P^2+6i\,dP/d\tau$, the second equation implies $R=-9(d^2P/d\tau^2)+9iP\,(dP/d\tau)+P^3$. Then the last of the above equations yields

$$\frac{d^3\tilde{y}}{d\tau^3}=2y\frac{d^2\tilde{y}}{d\tau^2}-3\left(\frac{d\tilde{y}}{d\tau}\right)^2,$$

where $P(q)=-i\tilde{y}(\tau)$. Finally, in terms of $\tilde{y}(\tau)=\pi^{-1}y(t)$, y satisfies the Chazy equation (71). Thus the special solution (74) yields $y(t)=i\pi E_2(t)=i\pi P(q)$. Knowing $P(q)$, from (78) we can obtain the other functions $Q(q)$ and $R(q)$ directly. Moreover, since we know the general solution of the Chazy equation, we know the general solution of the equations of Ramanujan. Note that Q and R are also called the normalized Eisenstein series E_4 and E_6 (see, e.g., Ref. 35).

As an historical postscript we note that Chazy and Ramanujan both worked on the same equation at nearly the same time, but apparently they did not know this!

B. The generalized Chazy equation

Let $(\omega_1,\omega_2,\omega_3)$ be a solution of (52)–(56). Ablowitz, Chakravarty, and Halburd[5,7] showed that

$$y:=-2(\omega_1+\omega_2+\omega_3)=-2\,\mathrm{Tr}\,M \tag{80}$$

solves

$$\frac{d^3y}{dt^3}-2y\frac{d^2y}{dt^2}+3\left(\frac{dy}{dt}\right)^2=\frac{4}{36-n^2}\left(6\frac{dy}{dt}-y^2\right)^2, \tag{81}$$

if and only if either $\alpha=\beta=\gamma=:2/n$ or exactly one of the parameters α, β, γ is $2/n$ and the other two are $\frac{1}{3}$. Equation (81) was also studied by Chazy[23–25] and is usually referred to as the *generalized Chazy equation* [to contrast it with the classical Chazy equation (71), which is the special case $n=\infty$].

It follows from Eqs. (58) and (80) that the general solution of (81) is given by

$$y(t)=\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}^6}{s^4(s-1)^4}. \tag{82}$$

In Chazy's analysis of Eq. (81) he showed that its solution is given by

$$y(t)=\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{J}^6}{J^4(J-1)^3}, \tag{83}$$

where the Schwarz function J solves Eq. (59) with (60) and $\alpha=1/n$, $\beta=\frac{1}{3}$, $\gamma=\frac{1}{2}$. The function J, and hence y, is single-valued if n is an integer greater than 1. The choice $n=\infty$ again corresponds to the classical Chazy equation (71).

Equations (82) and (83) suggest that there is a relationship between J and the special Schwarzian triangle functions s described above. In the case when s corresponds to the choice $\alpha=\beta=\gamma=2/n$, it can be shown that

$$J=\frac{4}{27}\frac{(s^2-s+1)^3}{s^2(s-1)^2},$$

and, similarly, when $\alpha=2/n$, $\beta=\gamma=\frac{1}{3}$, we have

$$J=-4s(s-1)$$

(see Ref. 5).

VI. SUMMARY AND DISCUSSION

The SDYM equations are a rich source of integrable systems. The classical soliton equations in $1+1$ dimensions and the well known Painlevé quations P_I–P_{VI} are reductions of the SDYM equations with finite-dimensional Lie algebras. Reductions of the SDYM equations using infinite-dimensional algebras are of particular interest. They yield the classical $2+1$-dimensional soliton equations, the Chazy equations and a ninth-order generalization of the Darboux–Halphen system.

ACKNOWLEDGMENTS

This work was supported by NSF Grant No. DMS-007092 and Royal Society Grant No. 22204.

[1] Ablowitz, M. J. and Clarkson, P. A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Vol. 149 of *Lond. Math. Soc. Lecture Note Series* (Cambridge University Press, Cambridge, 1991).

[2] Ablowitz, M. J. and Fokas, A. S., *Complex Variables: Introduction and Applications* (Cambridge University Press, Cambridge, 1997).

[3] Ablowitz, M. J. and Haberman, R., "Resonantly coupled nonlinear evolution equations," J. Math. Phys. **16**, 2301–2305 (1975).

[4] Ablowitz, M. J. and Segur, H., *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[5] Ablowitz, M. J., Chakravarty, S., and Halburd, R., "The generalized Chazy equation and Schwarzian triangle functions," Asian J. Math. **2**, 619–624 (1998).

[6] Ablowitz, M. J., Chakravarty, S., and Halburd, R., "On Painlevé and Darboux-Halphen type equations," in *The Painlevé Property, One Century Later*, edited by R. Conte, CRM Series in Mathematical Physics (Springer, Berlin, 1998).

[7] Ablowitz, M. J., Chakravarty, S., and Halburd, R., "The generalized Chazy equation from the self-duality equations," Stud. Appl. Math. **103**, 75–88 (1999).

[8] Ablowitz, M. J., Chakravarty, S., and Takhtajan, L. A., "Integrable systems, self-dual Yang-Mills and connections with modular forms," in *Nonlinear Problems in Engineering and Science—Numerical and Analytical Approach*, edited by S. T. Xiao and X. C. Hu (Science, Beijing, 1992), pp. 1–15.

[9] Ablowitz, M. J., Chakravarty, S., and Takhtajan, L. A., "A self-dual Yang-Mills hierarchy and its reductions to integrable systems in $1+1$ and $2+1$ dimensions," Commun. Math. Phys. **158**, 289–314 (1993).

[10] Ablowitz, M. J., Kaup, D. J., Newell, A. C., and Segur, H., "Nonlinear evolution equations of physical significance," Phys. Rev. Lett. **31**, 125–127 (1973).

[11] Ablowitz, M. J., Kaup, D. J., Newell, A. C., and Segur, H., "The inverse scattering transform—Fourier analysis for nonlinear problems," Stud. Appl. Math. **53**, 249–315 (1974).

[12] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Vol. 60 of *Graduate Texts Math* (Springer-Verlag, Berlin, 1978).

[13] Atiyah, M. F. and Ward, R. S., "Instantons and algebraic geometry," Commun. Math. Phys. **55**, 117–124 (1977).

[14] Chakravarty, S. and Ablowitz, M. J., "On reduction of self-dual Yang-Mills equations," in *Painlevé Transcendents, their*

Asymptotics and Applications, edited by D. Levi and P. Winternitz (Plenum, London, 1992).
[15] Chakravarty, S. and Ablowitz, M. J., ''Integrability, monodromy, evolving deformations, and self-dual Bianchi IX systems,'' Phys. Rev. Lett. **76**, 857–860 (1996).
[16] Chakravarty, S. and Halburd, R. G., ''First integrals and gradient flow for a generalized Darboux-Halphen system,'' in *The Legacy of the Inverse Scattering Transform in Applied Mathematics*, edited by J. Bona, R. Choudhury, and D. Kaup (AMS, Providence, 2002).
[17] Chakravarty, S. and Halburd, R. G., ''First integrals of a generalized Darboux-Halphen system,'' J. Math. Phys. **44**, 1751–1762 (2003).
[18] Chakravarty, S., Ablowitz, M. J., and Clarkson, P. A., ''Reductions of self-dual Yang-Mills fields and classical systems,'' Phys. Rev. Lett. **65**, 1085–1087 (1990).
[19] Chakravarty, S., Ablowitz, M. J., and Clarkson, P. A., ''One dimensional reductions of self-dual Yang-Mills fields and classical systems,'' in *Recent Advances in General Relativity: Essays in Honor of Ted Newman (Pittsburg, 1990)*, edited by A. I. Janis and J. R. Porter (Birkhäuser, Boston, 1991).
[20] Chakravarty, S., Ablowitz, M. J., and Takhtajan, L. A., ''Self-dual Yang-Mills equation and new special functions in integrable systems,'' in *Nonlinear Evolution Equations and Dynamical Systems*, edited by M. Boiti, L. Martina, and F. Pempinelli (World Scientific, Singapore, 1992).
[21] Chakravarty, S., Kent, S. L., and Newman, E. T., ''Some reductions of the self-dual Yang-Mills equations to integrable systems in $2+1$ dimensions,'' J. Math. Phys. **36**, 763–772 (1995).
[22] Chakravarty, S., Mason, L., and Newman, E. T., ''Canonical structures on anti-self-dual four-manifolds and the diffeomorphism group,'' J. Math. Phys. **32**, 1458–1464 (1991).
[23] Chazy, J., ''Sur les équations différentielles dont l'intégrale générale est uniforme et admet des singularities essentielles mobiles,'' C. R. Acad. Sci. Paris, **149**, 563–565 (1909).
[24] Chazy, J., ''Sur les équations différentielles dont l'intégrale générale possède une coupure essentielle mobile,'' C. R. Acad. Sci. Paris, **150**, 456–458 (1910).
[25] Chazy, J., ''Sur les équations différentielles du troisième et d'ordre supérieur dont l'intégrale générale à ses points critiques fixés,'' Acta Math. **34**, 317–385 (1911).
[26] Corrigan, E. F., Fairlie, D. B., Yates, R. G., and Goddard, P., ''Construction of self-dual solutions to $SU(2)$ gauge theory,'' Commun. Math. Phys. **58**, 223–240 (1978).
[27] Darboux, G., ''Sur la théorie des coordonnées curvilignes et les systémes orthogonaux,'' Ann. Ec. Normale Supér. **7**, 101–150 (1878).
[28] Darboux, G., *Lecons sur la Théorie Générale des Surfaces* (Gauthier-Villars, Paris, 1887).
[29] Donaldson, S. K., ''An application of gauge theory to the topology of 4-manifolds,'' J. Diff. Geom. **18**, 269–316 (1983).
[30] Ernst, F. J., ''New formulation of the axially symmetric gravitational field problem,'' Phys. Rev. **167**, 1175–1178 (1968).
[31] Halphen, G., ''Sur certains systèmes d'équations différentielles,'' C. R. Acad. Sci. Paris **92**, 1404–1406 (1881).
[32] Halphen, G., ''Sur un système d'équations différentielles,'' C. R. Acad. Sci. Paris **92**, 1101–1103 (1881).
[33] Harnad, J. and McKay, J., ''Modular solutions to equations of generalized Halphen type,'' Proc. R. Soc. London, Ser. A **456**, 261–294 (2000).
[34] Hitchin, N., ''Hypercomplex manifolds and the space of framings,'' in *The Geometric Universe (Oxford, 1996)*, (Oxford University Press, Oxford, 1998), pp. 9–30.
[35] Koblitz, N., *Introduction to Elliptic Curves and Modular Forms* (Springer-Verlag, New York, 1993).
[36] Maciejewski, A. J. and Strelcyn, J., ''On the algebraic non-integrability of the Halphen system,'' Phys. Lett. A **201**, 161–166 (1995).
[37] Manakov, S. V., ''A remark on the integration of the Eulerian equations of the dynamics of an n-dimensional rigid body,'' Funct. Anal. Appl. **10**, 93–94 (1976).
[38] Manakov, S. V. and Zakharov, V. E., ''Three-dimensional model of relativistic-invariant field theory, integrable by the inverse scattering transform,'' Lett. Math. Phys. **5**, 247–253 (1981).
[39] Mason, L. J. and Newman, E. T., ''A connection between the Einstein and Yang-Mills equations,'' Commun. Math. Phys. **121**, 659–668 (1989).
[40] Mason, L. J. and Sparling, G. A. J., ''Nonlinear Schrödinger and Korteweg-de Vries are reductions of self-dual Yang-Mills,'' Phys. Lett. A **137**, 29–33 (1989).
[41] Mason, L. J. and Woodhouse, N. M. J., ''Self-duality and the Painlevé transcendents,'' Nonlinearity **6**, 569–581 (1993).
[42] Mason, L. J. and Woodhouse, N. M. J., *Integrability, Self-Duality, and Twistor Theory*, LMS Monograph, New Series 15 (Oxford University Press, Oxford, 1996).
[43] Nahm, W., ''The construction of all self-dual multimonopoles by the ADHM method,'' in *Monopoles in Quantum Field Theory* (World Scientific, Singapore, 1982), pp. 87–94.
[44] Nambu, Y., ''Generalized Hamiltonian dynamics,'' Phys. Rev. D **7**, 2405–2412 (1973).
[45] Nehari, Z., *Conformal Mapping* (McGraw-Hill, New York, 1952).
[46] Ohyama, Y., ''The symmetries of Dynkin diagrams and the reduction of Toda field equations,'' Nucl. Phys. B **215**, 470–494 (1983).
[47] Ohyama, Y., ''Systems of nonlinear differential equations related to second order linear equations,'' Osaka J. Math. **33**, 927–949 (1996).
[48] Pohlmeyer, K., ''On the Lagrangian theory of anti-self-dual fields in four-dimensional Euclidean space,'' Commun. Math. Phys. **72**, 37–47 (1980).
[49] Rajaraman, R., *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1989).
[50] Ramanujan, S., ''On certain arithmetical functions,'' Trans. Cambridge Philos. Soc. **22**, 159–184 (1916).

[51] Rankin, R. A., ''The construction of automorphic forms from the derivatives of a given form,'' J. Indian Math. Soc. **20**, 103–116 (1956).
[52] Serre, J. P., *A Course in Arithmetic*. Graduate Texts in Mathematics, Vol. 7 (Springer-Verlag, New York, 1973).
[53] Takhtajan, L. A., ''Modular forms as τ-functions for certain integrable reductions of the Yang-Mills equations,'' in *The Verdier Memorial Conference on Integrable Systems*, edited by O. Babelon, P. Cartier, and Y. Kosmann-Schwarzbach, (Birkhäuser, Boston, 1993).
[54] Takhtajan, L. A., ''On foundations of the generalized Nambu mechanics,'' Commun. Math. Phys. **160**, 295–315 (1994).
[55] Vilenkin, N. J., *Special Functions and the Theory of Group Representations* (AMS, Providence, RI, 1968) (translated from the 1965 Russian edition by V. N. Singh).
[56] Villarroel, J., ''Scattering data for the self-duality equations,'' Inverse Probl. **5**, 1157–1162 (1989).
[57] Wadati, M., Konno, K., and Ichikawa, Y. H., ''A generalization of the inverse scattering method,'' J. Phys. Soc. Jpn. **46**, 1965–1966 (1979).
[58] Ward, R. S., ''On self-dual gauge fields,'' Phys. Lett. A **61**, 81–82 (1977).
[59] Ward, R. S., ''Integrable and solvable systems, and relations among them,'' Philos. Trans. R. Soc. London, Ser. A **315**, 451–457 (1985).
[60] Ward, R. S., ''Integrability of chiral equations with torsion term,'' Nonlinearity **1**, 671–679 (1988).
[61] Ward, R. S., ''Soliton solutions in an integrable chiral model in $2+1$ dimensions,'' J. Math. Phys. **29**, 386–389 (1988).
[62] Witten, L., ''Static axially symmetric solutions of self dual $SU(2)$ gauge fields in Euclidean four-dimensional space,'' Phys. Rev. D **19**, 718–720 (1979).
[63] Woodhouse, N. M. J. and Mason, L. J., ''The Geroch group and non-Hausdorff twistor spaces,'' Nonlinearity **1**, 73–114 (1988).
[64] Yang, C. N., ''Conditions of self-duality for $SU(2)$ gauge fields on Euclidean four-dimensional space,'' Phys. Rev. Lett. **38**, 1377–1379 (1979).
[65] Yang, C. N. and Mills, R. L., ''Conservation of isotopic spin and isotopic gauge invariance,'' Phys. Rev. **96**, 191–195 (1954).
[66] Zakharov, V. E. and Manakov, S. V., ''Resonant interaction of wave packets in nonlinear media,'' JETP Lett. **18**, 243–245 (1973).