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The Geroch group and non-Hausdorff twistor spaces

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Abstract. By reducing the Ward correspondence, we show that there is a correspondence between stationary axisymmetric solutions of the vacuum Einstein equations and a class of holomorphic vector bundles over a reduced twistor space, which is a compact one-dimensional, but non-Hausdorff, complex manifold. We show that the solutions generated by Ward's ansätze correspond to bundles which have a simple behaviour on the 'real axis' in the reduced space. We identify the Geroch group (Kinnersley and Chitre's 'group K') with a subgroup of the loop group of $GL(2, \mathbb{C})$ and we describe its orbits. We also identify some of the subgroups which preserve asymptotic flatness.

1. Introduction

Einstein's equations for a stationary, axisymmetric gravitational field are equivalent to a form of the self-dual Yang–Mills equations. This was observed by Witten (1979), who suggested that one could generate solutions of Einstein's equations by using twistor methods to construct self-dual Yang–Mills fields with appropriate symmetries.

Witten's idea was taken up by Ward (1983). Ward used a different, and more direct, correspondence with the Yang–Mills system to obtain a series of ansätze for gravitational fields with predetermined behaviour on the symmetry axis.

Our purpose in this paper is to extend Ward's work by giving a complete 'holomorphic' description of the general solution of the stationary axisymmetric field equations in terms of vector bundles over a 'reduced' twistor space, and by showing how it relates to the solution generation techniques of Ehlers (1962), Geroch (1972) and Kinnersley and co-workers (Kinnersley 1977, Kinnersley and Chitre 1977, 1978a,b, Hoenselaers 1979, Hoenselaers *et al* 1979). Some preliminary steps in this direction were taken by Woodhouse (1987) and by Mason *et al* (1987).

That such a description should exist is not surprising: several mathematicians have already considered reduced forms of the Ward correspondence for Yang–Mills fields with various one- and two-dimensional symmetry groups (Hitchin 1982, 1983, Ward 1983, Atiyah 1987) and Ward has suggested that there is a fundamental

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connection between complete integrability (and hence the existence of large groups of 'hidden symmetries') on the one hand and the reduction of the Yang–Mills equations and their solution in terms of holomorphic objects on twistor space on the other (Ward 1985).

However, the details of the reduction in our case show some new features: in particular, the reduced twistor space is a compact, but non-Hausdorff, one-dimensional complex manifold.

1.1. Yang's equation

Ward's work begins with the following observation. Suppose that (M, g) is a vacuum solution of Einstein's equations with an orthogonally transitive symmetry group (Kramer 1980, p 175) generated by two commuting Killing vectors: a timelike vector field $\partial/\partial t$ and a spacelike vector field $\partial/\partial\theta$ (which generates rotations about the symmetry axis). Then the metric can be written

$$\mathrm{d}s^2 = -\Omega^2(\mathrm{d}z^2 + \mathrm{d}r^2) + J_{11}\,\mathrm{d}\theta^2 + 2J_{12}\,\mathrm{d}\theta\,\mathrm{d}t + J_{22}\,\mathrm{d}t^2 \tag{1.1}$$

where r and z are coordinates on the space of group orbits. Here $J = (J_{ij})$ is a symmetric matrix, J and Ω are functions of r and z alone, and $\det(J) = -r^2$ (we have assumed that $\det(J)$ is not constant). The coordinates r, θ, z, t are Weyl's canonical coordinates: in Minkowski spacetime, they are cylindrical polar coordinates with r measuring the distance from the symmetry axis (r is often denoted ρ).

Einstein's equations are

$$\partial_r(rJ^{-1}\,\partial_r J) + r\,\partial_z(J^{-1}\,\partial_z J) = 0 \tag{1.2}$$

$$4\mathrm{i}\,\partial_w \log\Omega = r\,\mathrm{Tr}[(\partial_w J^{-1})(\partial_w J)] - \frac{1}{r} \tag{1.3}$$

where $w = z + \mathrm{i}r$. The first of these is Yang's equation for a static, axisymmetric self-dual Yang–Mills field. The second determines the conformal factor Ω in terms of J; its integrability is an automatic consequence of the first equation. In effect, therefore, the problem of finding stationary axisymmetric solutions of Einstein's equations is the problem of finding symmetric $\mathrm{GL}(2, \mathbb{R})$-valued solutions $J(z, r)$ of equation (1.2), subject to the constraint $\det(J) = -r^2$.

Witten's original approach was different: he started not with J, but with a potential for J called the *Ernst potential* (Ernst 1968), which is a complex-valued function $\mathscr{E}(z, r)$ constructed from J by integrating a closed differential form. A paraphrase of his result is that the real and imaginary parts of $\mathscr{E}$ can be used to construct a 2×2 matrix J', satisfying $\det(J') = 1$; and that Einstein's equations are again equivalent to (1.2) on J'.

Ward's† J and Witten's J' are related by a discrete symmetry of Yang's equation which is essentially the same as the Neugebauer–Kramer (NK) transformation of the Ernst equation (Neugebauer and Kramer 1969). This transformation plays a key role in the construction of the 'hidden symmetries': the group K considered by Kinnersley and Chitre (the 'Geroch group') is generated by conjugating the obvious $\mathrm{SL}(2, \mathbb{R})$ symmetries of (1.2) by NK transformations.

† In fact Ward normalised J by dividing by r (so as to set $\det(J) = -1$). This normalisation is not, however, convenient for our purposes.

1.2. Outline

Our principal results are these: we show that there is a one-to-one correspondence between $\mathrm{GL}(2,\mathbb{C})$-valued solutions of (1.2) on a region V in the zr plane (modulo $J \mapsto AJB^{-1}$, A, $B \in \mathrm{GL}(2,\mathbb{C})$) and rank-2 holomorphic bundles over 'reduced twistor space'—a compact, non-Hausdorff Riemann surface R_V associated with V. The real symmetric solutions correspond to bundles which satisfy additional symmetry and reality conditions.

The reduced twistor space is obtained from two copies of the Riemann sphere S on which $w = z + ir$ is a stereographic coordinate by a 'cutting and pasting' construction. There is a natural projection $\Gamma : R_V \rightarrow S$ and, for each $w = z + ir \in S$, $\Gamma^{-1}(w)$ contains one point if $(z, r) \in V$ or $(z, -r) \in V$, and two points otherwise. Thus the domain on which the solution is defined is encoded in the geometry of R_V, and extending the domain of the solution is equivalent to extending the regions on the two copies of S which are pasted together. We show that if the 'pasting region' extends across the real axis $r = 0$, then the solution has a simple asymptotic behaviour as $r \rightarrow 0$ and the bundle over R_V is obtained by pasting together two holomorphic bundles over S (these are essentially the solutions generated by the Ward ansätze). If J is constructed from the Ernst potential of a spacetime which is regular at $r = 0$, then the two bundles over S are trivial and the bundle over R_V is characterised by a single holomorphic patching matrix which is equal to J at $r = 0$.

We also show that the extension of the 'pasting region' over $w = \infty$ for the bundle corresponding to the Ernst potential is equivalent to asymptotic flatness.

We show that the group K of hidden symmetries can be identified with a subgroup of the loop group $\mathrm{LGL}(2,\mathbb{C})$ of smooth maps $S^1 \rightarrow \mathrm{GL}(2,\mathbb{C})$, which acts on holomorphic bundles over R_V in a natural way by 'twisting' the fibres over the two copies of $w = \infty$ in R_V. This description of K enables us to give a straightforward parametrisation of its orbits. It also appears to provide a natural geometric setting for the transformation theory.

Our results add further weight to Ward's suggestion that the twistor approach makes it possible to investigate properties of solutions where the metric itself is too complicated to give any useful information.

In the next section we shall describe the geometry of R_V and its relation to Penrose's twistor space (Penrose and Rindler 1986). In §3 we shall derive the correspondence between holomorphic bundles over R_V and complex solutions of Yang's equation. In §4, we shall consider the symmetry and reality conditions. In §5, we shall describe the explicit representation of solutions in terms of the 'patching data' of the corresponding bundles and the connection with the Ward ansätze. In §6, we shall identify K with a subgroup of $\mathrm{LGL}(2,\mathbb{C})$ and in §7 we shall consider the behaviour of the solutions on the axis and at infinity and we shall identify some transformations that are known to preserve asymptotic flatness. Two appendices contain some general comments on non-Hausdorff complex manifolds and some indications of how our approach relates to others already published.

2. The reduced twistor space

'Reduced twistor space' is a compact, one-dimensional, but *non-Hausdorff* complex manifold R_W associated with a region W in the zr plane. We shall first describe a direct construction of R_W which, although it lacks obvious motivation, gives a concrete picture of its geometry. We shall then show how R_W can be obtained by projecting out two complex symmetries from the twistor space of four-dimensional Euclidean space $\mathbb{E}_4$.

2.1. Direct construction

We shall be interested in two cases. In the first, $W = V$, where V is a connected and simply connected open subset of the upper half H of the zr plane (that is $V \subset H = \{r > 0\}$); V will retain this meaning throughout the paper.

We put $w = z + ir$ and think of w as a stereographic coordinate on a Riemann sphere $S = \mathbb{CP}_1$. Let $\hat{V}$ be the image of V under the conjugation $w \mapsto \bar{w} = z - ir$.

We choose points c and $\bar{c}$ on the boundaries of V and $\hat{V}$ and make a cut C from c to $\bar{c}$. We then take two copies S_0 and S_1 of S (each with its own cut) and make the following identifications.

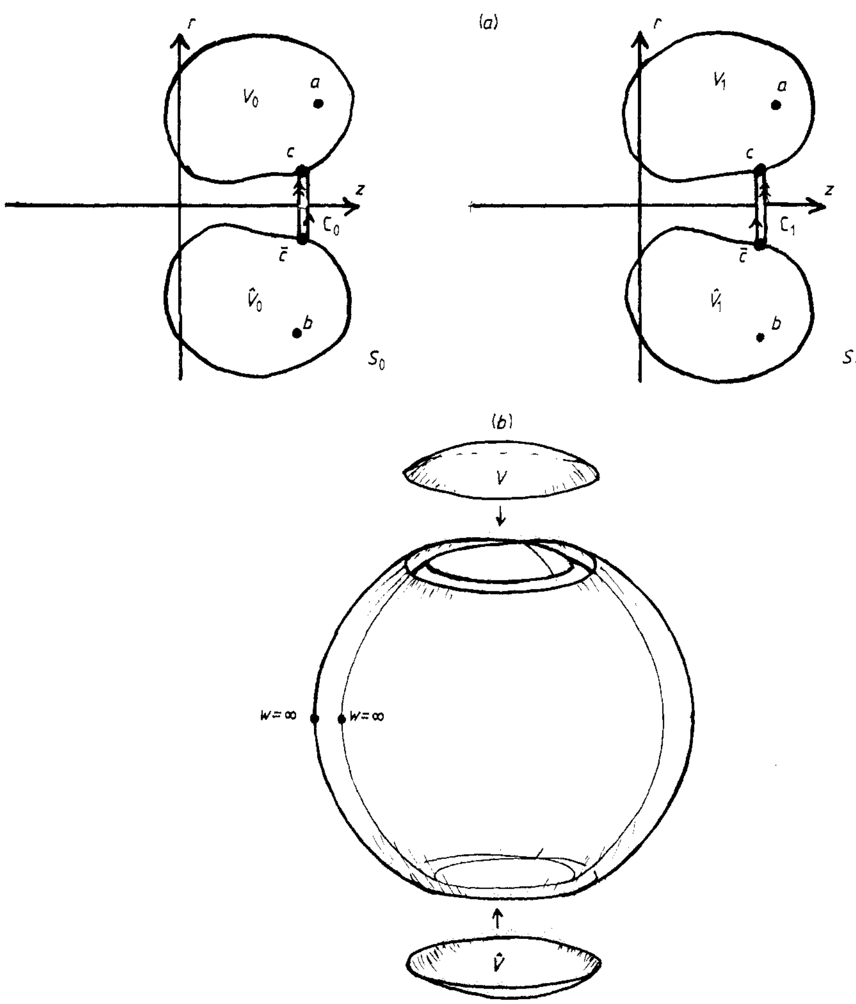

Figure 1. (*a*) The construction of R_V. The left side (right side) of the cut C_0 in S_0 is identified with the right side (left side) of C_1 in S_1. The point $w = a$ in V_0 is identified with $w = a$ in V_1, and $w = b$ in $\hat{V}_0$ is identified with $w = b$ in $\hat{V}_1$. (*b*) The geometry of R_V.

(1) We identify each point of $V_0 \subset S_0$ with the corresponding point of $V_1 \subset S_1$ and we identify each point of $\hat{V}_0 \subset S_0$ with the corresponding point of $\hat{V}_1 \subset S_1$ (V_0 is the copy of V in S_0, and so on).

(2) We identify each point on one side of the cut C_0 in S_0 with the corresponding point on the other side of the cut C_1 in S_1, just as in the construction of the Riemann surface of the square root of a quadratic.

The resulting space R_V is shown in figure 1. It is a compact complex manifold, but it is not Hausdorff, as one can see by considering the projection $\Gamma: R_V \to S$, given by $S \simeq S_0 \simeq S_1$. If $w \in S$, then $\Gamma^{-1}(w)$ contains one point if $w \in V$ or $w \in \hat{V}$, and two points otherwise. In particular, if w lies on the boundary of V or $\hat{V}$, then $\Gamma^{-1}(w)$ contains two points which cannot be separated by open subsets of R_V.

Because of the way in which the identifications are made along the cuts, $\Gamma^{-1}(S\backslash(V \cup \hat{V}))$ is a *connected* twofold covering space of $S\backslash(V \cup \hat{V})$: a path in $S\backslash(V \cup \hat{V})$ that begins and ends at W, and which makes a complete circuit of V, lifts to a path in $\Gamma^{-1}(S\backslash(V \cup \hat{V}))$ which joins the two points of $\Gamma^{-1}(w)$.

In the other case, which is rather simpler, $W = U$, where U is an open subset of S which intersects the real axis and which is mapped to itself under $w \mapsto \bar{w}$. Here we dispense with the cut and construct R_U by identifying the copies U_0 and U_1 of U in S_0 and S_1. Again, there is a projection $\Gamma: R_U \to S$. These spaces arise in the description of fields which are well behaved on the symmetry axis $r = 0$.

In both cases, we have two natural involutions $R_W \to R_W$. The first is the holomorphic map i which is given by interchanging S_0 and S_1; in the first case, it acts as the identity on $\Gamma^{-1}(V \cup \hat{V})$ and interchanges the two points in $\Gamma^{-1}(w)$ when $w \in S\backslash(V \cup \hat{V})$. In the second case, it acts as the identity on $\Gamma^{-1}(U)$, and interchanges the two points in $\Gamma^{-1}(w)$ when $w \in S\backslash U$.

The second involution is the antiholomorphic map $j: R_W \to R_W$ induced by the conjugations $S_0 \to S_0: w \mapsto \bar{w}$ and $S_1 \to S_1: w \mapsto \bar{w}$.

2.2. Euclidean twistor space

The construction of Penrose's twistor space from the point of view of four-dimensional Euclidean geometry has been described by Atiyah *et al* (1978) and by Atiyah (1979). Briefly, the essential ideas are as follows: first we attach a point at infinity to $\mathbb{E}_4$ to form the conformal compactification, the sphere S^4. We then construct the bundle $\mathbb{T}$ of right-handed two-component spinors on S^4 and let $\mathbb{PT}$ be the associated projective bundle (the fibres of $\mathbb{PT}$ are projective lines).

The two bundles are turned into complex manifolds by using the conformal structure on S^4 to define almost complex structures on $\mathbb{T}$ and $\mathbb{PT}$ and then showing that the almost complex structures are integrable. As a complex manifold, $\mathbb{T}$, which is called *twistor space,* is isomorphic to $\mathbb{C}^4$, and $\mathbb{PT}$, which is called *projective twistor space,* is isomorphic to $\mathbb{CP}_3$. The projection from $\mathbb{T}$ to $\mathbb{PT}$ is the standard holomorphic projection $\mathbb{C}^4 \to \mathbb{CP}_3$.

Let $\{t, x, y, z\}$ be a system of Cartesian coordinates on $\mathbb{E}_4 \subset S^4$ and let $\{t, r, \theta, z\}$ be the related system of cylindrical polar coordinates (that is, $x = r\cos\theta$, $y = r\sin\theta$). Then there are complex linear coordinates Z^α ($\alpha = 0, 1, 2, 3$) on $\mathbb{T} = \mathbb{C}^4$ such that Z^α is in the fibre over the point (t, r, θ, z) if and only if

$$Z^2 = (z + \mathrm{i}t)Z^0 - r\mathrm{e}^{-\mathrm{i}\theta}Z^1 \qquad Z^3 = (z - \mathrm{i}t)Z^1 + r\mathrm{e}^{\mathrm{i}\theta}Z^0 \tag{2.1}$$

(the conventions here are slightly different from those of Penrose and Rindler 1986).

On dividing through by Z^0, we obtain a holomorphic coordinate system on $\mathbb{PT}$ (less the plane $Z^0 = 0$)

$$\xi^1 = Z^1/Z^0 \qquad \xi^2 = Z^2/Z^0 \qquad \xi^3 = Z^3/Z^0 \tag{2.2}$$

in which (ξ^1, ξ^2, ξ^3) is in the fibre over (t, r, θ, z) if and only if

$$\xi^2 = z + \mathrm{i}t - \zeta r \mathrm{e}^{-\mathrm{i}\theta} \qquad \xi^3 = r\mathrm{e}^{\mathrm{i}\theta} + \zeta(z - \mathrm{i}t) \tag{2.3}$$

where $\zeta = \xi^1 = Z^1/Z^0$; ζ is a stereographic coordinate on the fibres.

2.3. Reduction of twistor space

We shall be interested in fields on $\mathbb{E}_4$ which are invariant under rotations in the xy plane and under translations in the t direction; that is, fields which are preserved by the action of the Abelian group of isometries $\mathrm{G} = S^1 \times \mathbb{R}$ generated by the two Killing vectors $\partial/\partial\theta$ and $\partial/\partial t$.

The action of G on $\mathbb{E}_4$ extends to a conformal action on S^4 and thence to biholomorphic actions on $\mathbb{T}$ and $\mathbb{PT}$. On $\mathbb{PT}$, the G action is generated by the real vector fields Θ and T, where

$$\Theta = \frac{\partial}{\partial\theta} + \mathrm{i}\zeta\frac{\partial}{\partial\zeta} - \mathrm{i}\bar{\zeta}\frac{\partial}{\partial\bar{\zeta}} \qquad T = \frac{\partial}{\partial t} \tag{2.4}$$

in the non-holomorphic coordinate system $\{t, r, \theta, z, \zeta, \bar{\zeta}\}$ or, equivalently, $\Theta = 2\,\mathrm{Re}(Y_\theta)$, $T = 2\,\mathrm{Re}(Y_t)$ where

$$Y_\theta = \mathrm{i}\xi^3\frac{\partial}{\partial\xi^3} + \mathrm{i}\xi^1\frac{\partial}{\partial\xi^1} \qquad Y_t = \mathrm{i}\frac{\partial}{\partial\xi^2} - \mathrm{i}\xi^1\frac{\partial}{\partial\xi^3} \tag{2.5}$$

in the holomorphic coordinates ξ^1, ξ^2, ξ^3.

Since Θ and T are the real parts of holomorphic vector fields, the action of G on $\mathbb{PT}$ is the 'real part' of the holomorphic action generated by Y_θ and Y_t of the complexified group $\mathrm{G}^{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}$. The orbits of $\mathrm{G}^{\mathbb{C}}$ are (parts of) the quadric surfaces Q_w on which

$$\xi^3/\xi^1 + \xi^2 = \zeta^{-1}r\mathrm{e}^{\mathrm{i}\theta} + 2z - \zeta r\mathrm{e}^{-\mathrm{i}\theta} = 2w \tag{2.6}$$

where w is a constant. These are everywhere tangent to Y_θ and Y_t.

A holomorphic object on some region $D \subset \mathbb{PT}$, such as a holomorphic function or a vector bundle, which is invariant along Θ and T must also be invariant along Y_θ and Y_t. One might expect, therefore, that it could be represented as the pull-back of a corresponding object on the space of orbits of $\mathrm{G}^{\mathbb{C}}$. There are, however, two problems: the first is that the vector fields Y_θ and Y_t have zeros, at which the projection onto the space of orbits is singular ($Y_\theta = 0$ on the two lines $L_0 = \{r = \zeta = 0\} = \{Z^3 = Z^1 = 0\}$ and $L_1 = \{r = 0, \zeta = \infty) = \{Z^2 = Z^0 = 0\}$, and $Y_t = 0$ on the fibre I over the point at infinity in S^4). The second is that, even if D avoids the zeros of Y_θ and Y_t, the orbits of $\mathrm{G}^{\mathbb{C}}$ can intersect D in disconnected sets. For example, a holomorphic function f such that $\Theta(f) = T(f) = 0$ must satisfy $Y_\theta(f) = Y_t(f) = 0$ and so must be locally constant on the orbits of $\mathrm{G}^{\mathbb{C}}$; but it can take different constant values of the different components of the intersection of an orbit with D and so it need not be the pull-back of a function on the space of orbits.

We shall avoid the first problem by choosing D so that it does not intersect L_0, L_1, or I. To get around the second problem, we look not at the space of orbits, but

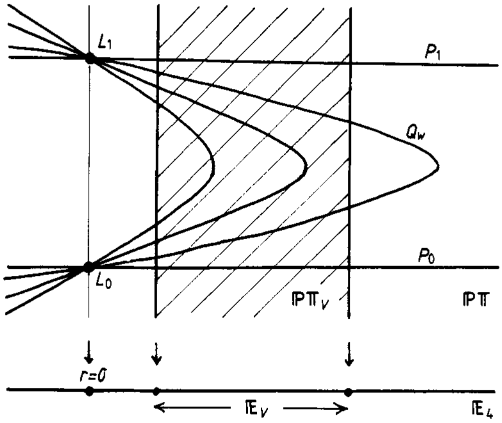

Figure 2. The geometry of $\mathbb{PT}$; The shaded region is $\mathbb{PT}_V$.

at the space R of leaves of the foliation F of D spanned by Y_θ and Y_t (F is non-singular provided D does not intersect L_0, L_1, or I). In other words, for each w, we count each connected component of $Q_w \cap D$ as a different point of R. Since the connectivity of $Q_w \cap D$ can change as w varies, R is generally a non-Hausdorff manifold.

To make the connection with the first construction (first case), let H and V be as before, let $\mathrm{pr}: \mathbb{E}_4 - \{r=0\} \to H$ be the projection $(t, r, \theta, z) \mapsto (z, r)$, and let $\mathbb{E}_V$ be an open subset of $\mathbb{E}_4$ such that $\mathrm{pr}(\mathbb{E}_V) = V$. (We shall usually assume that $\mathbb{E}_V$ is simply connected, which rules out the obvious choice $\mathbb{E}_V = \mathrm{pr}^{-1}(V)$.)

Let $\mathbb{PT}_V$ be the part of projective twistor space above $\mathbb{E}_V$. We want to show that if $D = \mathbb{PT}_V$, then $R = R_V$ (see figure 2).

Consider the map $\mathbb{C}^4 \to \mathbb{C}^2$ given by

$$(Z^0, Z^1, Z^2, Z^3) \mapsto (W^0, W^1) = (2Z^0Z^1, Z^0Z^3 + Z^1Z^2). \tag{2.7}$$

This induces a map $\mathbb{PT}\backslash(I \cup L_0 \cup L_1) \to S = \mathbb{CP}_1$

$$(\xi^1, \xi^2, \xi^3) \mapsto w = \tfrac{1}{2}((\xi^3/\xi^1) + \xi^2) = \tfrac{1}{2}\zeta^{-1} r e^{i\theta} + z - \tfrac{1}{2}\zeta r e^{-i\theta} \tag{2.8}$$

where $w = W^1/W^0$ is a stereographic coordinate on S. The surfaces of constant w make up the pencil of quadrics $\{Q_w\}$ and each quadric in the pencil is made up of orbits of $G^{\mathbb{C}}$ (see the remark below).

We fix a value of w. Let x be a point of $\mathbb{E}_V$ with coordinates (t, r, θ, z) and let $X \subset \mathbb{PT}$ be the fibre above x: X is a projective line, with stereographic coordinate ζ, which intersects Q_w at two points (which may coincide), given by the two roots ζ_+, ζ_- of

$$\tfrac{1}{2}\zeta^2 r e^{-i\theta} + (w - z)\zeta - \tfrac{1}{2} r e^{i\theta} = 0. \tag{2.9}$$

For each w, therefore, there are two points of R if the following condition holds, and one otherwise.

Condition. It is possible to change ζ_+ to ζ_- by continuously varying t, r, θ, and z so that x remains in $\mathbb{E}_V$.

The discriminant of (2.9) is $\Delta = (w - z)^2 + r^2$, which vanishes when $w = z \pm ir$. Now if $x \in \mathbb{E}_V$, then $(z, r) \in V \subset H$. Thus if we identify S with the Riemann sphere S in the

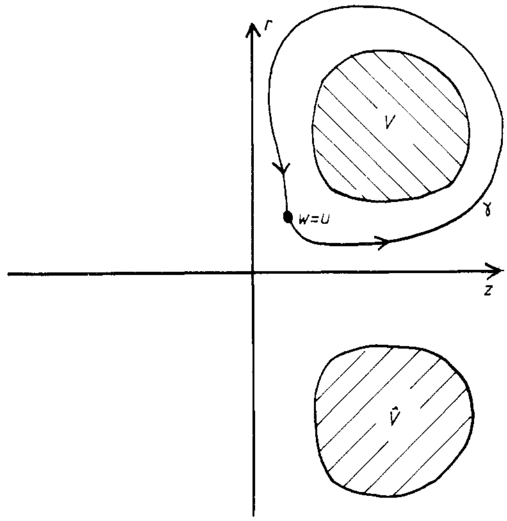

Figure 3. The path in S.

previous construction, then we have the following. For $w \notin V \cup \hat{V}$, the above condition does not hold since neither of the points $z \pm ir \in S$ can make a circuit of w without leaving V or $\hat{V}$ (it is here that the assumption that V is simply connected is important). But if, conversely, $w \in V \cup \hat{V}$, then we can deform ζ_+ into ζ_- by taking $z + ir$ (or $z - ir$) around a closed loop in V (or $\hat{V}$) which winds around w; and so the condition holds.

The surjection $\mathbb{PT}\backslash(I \cup L_0 \cup L_1) \to S$ gives a mapping $R \to S$, which we shall again denote by Γ. We have just shown that $\Gamma^{-1}(w)$ contains one or two points as w lies or does not lie in $V \cup \hat{V}$. To complete the argument that R_V is the same as R, therefore, we have only to show how the cut arises; or, put more precisely, we must show that $\Gamma^{-1}(S\backslash(V \cup \hat{V}))$ is a *connected* double cover of $S\backslash(V \cup \hat{V})$.

To do this, we fix $x \in \mathbb{E}_V$ and allow w to move round a closed path γ which makes a circuit of V in $S\backslash(V \cup \hat{V})$, beginning and ending at $w = u$ (see figure 3). Then ζ_+ changes continuously to ζ_-, and so γ lifts to a path in $\Gamma^{-1}(S\backslash(V \cup \hat{V})) \subset R$ joining the two points of $\Gamma^{-1}(u)$. Hence the structure of R is indeed the same as that of R_V.

The second case is similar: take $\mathbb{E}_U$ to be the set of points (t, r, θ, z) in $\mathbb{E}_4$ such that $(z, r) \in U$ and let $\mathbb{PT}_U$ be the portion of $\mathbb{PT}$ above $\mathbb{E}_U$ (note that $\mathbb{E}_U$ contains points on the symmetry axis). Then if $D = \mathbb{PT}_U\backslash(L_0 \cup L_1)$, we have $R = R_U$.

The two spheres S_0 and S_1 in R_U are the sets of leaves of F in D whose closures in $\mathbb{PT}_U$ intersect, respectively, L_0 and L_1. If $w \notin U$, then $Q_w \cap \mathbb{PT}_U$ has two components, one meeting L_0 and the other meeting L_1; while if $w \in U$, then $Q_w \cap \mathbb{PT}_U$ has one component which meets both L_0 and L_1. Hence the identification between the copies of U in S_0 and S_1.

We make some final remarks. (1) Given $w \in S$, we can write $w = z \pm ir$ $(r \geqslant 0)$ and so obtain a point (z, r) in the upper half-plane H or on the real axis $(r = 0)$. In geometric terms the correspondence is the following: w determines a quadric surface Q_w in $\mathbb{PT}$ which cuts the fibre X above a general $x \in \mathbb{E}_4$ in two points. But there are certain points in $\mathbb{E}_4$ for which X is tangent to Q_w. These points all have the same z and r coordinates and the correspondence associates w to the value of (z, r) for which X is tangent to Q_w.

(2) The quadric corresponding to $w \neq \infty$ is non-singular and so contains two families of lines. One is tangent to Y_θ (the 'θ family'); the other is tangent to Y_t (the 't family').

The θ family contains the two special lines L_0 and L_1 on which $Y_\theta = 0$. These both lie above the axis ($r = 0$) and they are common to all the quadrics in the pencil; L_0 is contained in the plane $P_0 = \{\zeta = 0\}$ and L_1 in the plane $P_1 = \{\zeta = \infty\}$. The plane pair (P_0, P_1) is the only singular quadric in the pencil; it is labelled by $w = \infty$.

Similarly the t family contains one special line, on which $Y_t = 0$. This is the fibre I above the point at infinity in S^4, along which the two planes in the plane pair intersect; again it is common to all the quadrics in the pencil.

For real w, the fibres above the axis points with coordinates $r = 0$ and $z = w$ are contained in the quadric, and belong to the θ family.

(3) The orbits of $\mathrm{G}^{\mathbb{C}}$ are: the two points $I \cap L_0$ and $I \cap L_1$, the complements of these points in the three lines I, L_0, and L_1, the complement of $I \cup L_0$ in the plane P_0; the complement of $I \cup L_1$ in the plane P_1 and, finally, the complement of the $L_0 \cup L_1 \cup I$ in each of the non-singular quadrics in the pencil.

(4) The picture is not radically different if one thinks of R_V as a reduction of the twistor space of Minkowski spacetime. Each Q_w gives rise (through its intersection with the null twistors) to a *Kerr congruence* (Newman and Winicour 1974), and one can describe R_V in terms of the geometry of these congruences.

(5) One can also obtain R_V by projecting out complex rotations about the z axis from 'minitwistor space', which is the two-dimensional complex manifold of directed lines in $\mathbb{E}_3$ (Hitchin 1982, Jones and Tod 1985). The orbit of a line under such complex rotations is the set of generators of a family of confocal hyperboloids of the form

$$\frac{r^2}{v^2 \cos^2 q} - \frac{(z-u)^2}{v^2 \sin^2 q} = 1 \tag{2.10}$$

where q labels the different members of the family and u and v are fixed ($u + iv$ is the coordinate w).

3. Bundles on reduced twistor space

Suppose that $J(z, r)$ is a $\mathrm{GL}(n, \mathbb{C})$-valued solution on $V \subset H$ of the static axisymmetric form of Yang's equation

$$\partial_z(J^{-1}\,\partial_z J) + r^{-1}\,\partial_r(rJ^{-1}\,\partial_r J) = 0. \tag{3.1}$$

We shall show that J determines a holomorphic bundle $E \to R_V$; and that, conversely, J can be recovered from E up to an equivalence transformation

$$J \mapsto AJB^{-1} \tag{3.2}$$

where A and B are constant elements of $\mathrm{GL}(n, \mathbb{C})$.

The first step in the route from J to E is essentially Yang's formulation of the anti-self-dual (ASD) field equations (Yang 1977); the second is Ward's extension to gauge fields of the Penrose transform, in a form due to Atiyah *et al* (1978). We shall describe these in just enough detail to establish our notation. The final step is the reduction of a bundle over $\mathbb{PT}_V$ to a bundle over R_V.

3.1. Reduction of the Ward correspondence

(1) We choose a connected, simply connected open subset $\mathbb{E}_V \subset \mathbb{E}_4$ such that $\mathrm{pr}(\mathbb{E}_V) = V$ and put $p = r\mathrm{e}^{\mathrm{i}\theta}$ and $q = z + \mathrm{i}t$. We then choose $H, \hat{H}: \mathbb{E}_V \to \mathrm{GL}(n, \mathbb{C})$ such that $J = H\hat{H}^{-1}$ and define a $\mathrm{gl}(n, \mathbb{C})$ connection $\mathrm{D} = \mathrm{d} + \Phi$ on the trivial bundle $\mathbb{E}_V \times \mathbb{C}^n$ by

$$\Phi = H^{-1}(\partial_{\bar{p}} H \, \mathrm{d}\bar{p} + \partial_{\bar{q}} H \, \mathrm{d}\bar{q}) + \hat{H}^{-1}(\partial_p \hat{H} \, \mathrm{d}p + \partial_q \hat{H} \, \mathrm{d}q) \tag{3.3}$$

(where J has been identified with $J \circ \mathrm{pr}$ and $\partial_p = \partial/\partial p$, and so on). Different choices of H and $\hat{H}$ (for fixed J) give gauge-equivalent connections. We shall take H and $\hat{H}$ to be functions of z and r alone. An obvious possibility is to take $H = J$, $\hat{H} = 1$; but in §6 we shall want to consider a decomposition in which H is upper triangular and $\hat{H}$ is lower triangular.

Equation (3.1) is equivalent to the condition that the curvature form $\mathrm{d}\Phi - \Phi \wedge \Phi$ should be anti-self-dual (ASD) (Yang 1977, Ward 1983).

(2) We define a $\mathrm{gl}(n, \mathbb{C})$-valued $(0, 1)$-form Ψ on $\mathbb{PT}_V$ by taking Ψ to be the $(0, 1)$ part of $\rho^*\Phi$ where $\rho: \mathbb{PT}_V \to \mathbb{E}_V$ is the projection. In the non-holomorphic coordinates $\{t, r, \theta, z, \zeta, \bar{\zeta}\}$,

$$\begin{aligned}\Psi = \frac{1}{1+\lambda^2}\{&(H^{-1}\,\partial_r H - \lambda H^{-1}\,\partial_z H)\,\bar{\partial} r + (H^{-1}\,\partial_z H + \lambda H^{-1}\,\partial_r H)\,\bar{\partial} z \\ &+ (\lambda^2 \hat{H}^{-1}\,\partial_r \hat{H} + \lambda \hat{H}^{-1}\,\partial_z \hat{H})\,\bar{\partial} r + (\lambda^2 \hat{H}^{-1}\,\partial_z \hat{H} - \lambda \hat{H}^{-1}\,\partial_r \hat{H})\,\bar{\partial} z\}\end{aligned} \tag{3.4}$$

where $\lambda = \mathrm{e}^{-\mathrm{i}\theta}\zeta$ and $\bar{\partial}$ is the $\bar{\partial}$ operator on $\mathbb{PT}_V$ (acting on functions); Ψ is a smooth form on $\mathbb{PT}_V$ even though the expression (3.4) appears to be singular at $\lambda = \pm\mathrm{i}$. Note that Ψ vanishes on restriction to the fibres of $\mathbb{PT}_V$, and that

$$\pounds_\Theta \Psi = 0 \qquad \pounds_T \Psi = 0. \tag{3.5}$$

Let $\Lambda_0 = P_0 \cap \mathbb{PT}_V$ and $\Lambda_1 = P_1 \cap \mathbb{PT}_V$. Then Λ_0 is the leaf of F given by $\zeta = \lambda = 0$ and Λ_1 is the leaf $\zeta = \lambda = \infty$.

On Λ_0, $\Psi = H^{-1}\,\bar{\partial} H$, and on Λ_1, $\Psi = \hat{H}^{-1}\,\bar{\partial}\hat{H}$.

The ASD condition on Φ is equivalent to

$$\bar{\partial}\Psi + \Psi \wedge \Psi = 0. \tag{3.6}$$

Hence Ψ determines a holomorphic structure on the trivial C^∞ bundle $B = \mathbb{PT}_V \times \mathbb{C}^n$, in which the local holomorphic sections of B are the $\mathbb{C}^n$-valued solutions of $\bar{\partial}\boldsymbol{b} + \Psi\boldsymbol{b} = 0$.

(3) Equations (3.5) and (3.6) imply that it is possible to find local *invariant holomorphic sections* of B, by which we mean local $\mathbb{C}^n$-valued functions $\boldsymbol{b}$ on $\mathbb{PT}_V$ which satisfy

$$\bar{\partial}\boldsymbol{b} + \Psi\boldsymbol{b} = 0 \qquad \Theta(\boldsymbol{b}) = 0 = T(\boldsymbol{b}) \tag{3.7}$$

(so that, in particular, $\boldsymbol{b}$ depends only on r, z, and λ). These conditions are compatible. They imply that

$$Y_\theta(\boldsymbol{b}) - (\bar{Y}_\theta \lrcorner \Psi)\boldsymbol{b} = 0 = Y_t(\boldsymbol{b}) - (\bar{Y}_t \lrcorner \Psi)\boldsymbol{b}. \tag{3.8}$$

Since the leaves of F are simply connected, equations (3.8), together with the complex conjugate equations, uniquely determine a solution $\boldsymbol{b}$ over the whole of a leaf Λ in terms of its value at any one point of Λ. For each Λ, the space E_Λ of solutions of (3.8) on Λ is an n-dimensional complex vector space.

The holomorphic bundle $E \to R_V$ is defined by taking the fibre of E at $\Lambda \in R_V$ to be E_Λ; and by identifying the local holomorphic sections of E with the local invariant holomorphic sections of B.

The columns of $H^{-1} \circ \rho \mid \Lambda_0$ and $\hat{H}^{-1} \circ \rho \mid \Lambda_1$ belong to E_{Λ_0} and E_{Λ_1} respectively. Thus the construction picks out preferred frames in the fibres of E at Λ_0 and Λ_1. We shall denote these f^0 and f^1.

The triple (E, f^0, f^1) is independent of the choice of H and $\hat{H}$, in the sense that if (E', f'^0, f'^1) is constructed by using a different decomposition of J, then $E' = E$ by an isomorphism which takes f'^0 to f^0 and f'^1 to f^1.

Remark. It is interesting to note that if we were to replace the rotation Killing vector $\partial/\partial\theta$ by $\partial/\partial x$ (where $x = r\cos\theta$), then we should have to replace Y_θ by

$$Y_x = \mathrm{i}\xi^1 \frac{\partial}{\partial \xi^1} + \mathrm{i}\frac{\partial}{\partial \xi^3} \tag{3.9}$$

which also commutes with Y_t. However, Y_x and Y_t are not independent everywhere on $\mathbb{PT}_V$ since $Y_t = -\mathrm{i}Y_x$ at $\zeta = \mathrm{i}$ and $Y_t = \mathrm{i}Y_x$ at $\zeta = -\mathrm{i}$. If these points are excised from $\mathbb{PT}_V$, then the fibres over points of $\mathbb{E}_V$ are no longer compact and the correspondence fails. In the seemingly simpler case of spacetimes with two translational Killing vectors, our 'holomorphic' solution of Yang's equation does not work.

Example. Take $n = 1$ and $J = r$. Then, with $H = r$, $\hat{H} = 1$

$$\Psi = \frac{1}{r(1+\lambda^2)}(\bar{\partial} r + \lambda\, \bar{\partial} z). \tag{3.10}$$

By using the relations (2.3) between the holomorphic coordinates ξ^1, ξ^2, ξ^3 and the non-holomorphic coordinates t, r, θ, z, ζ, and $\bar{\zeta}$, one can show that $\Psi = \frac{1}{2}\bar{\partial}\log(r\mathrm{e}^{-\mathrm{i}\theta})$. The logarithm is well defined provided we choose $\mathbb{E}_V$ so that it is impossible to make a circuit of the symmetry axis. Then Ψ is $\bar{\partial}$-exact, and so the (line) bundle $B \to \mathbb{PT}_V$ is trivial not only as a C^∞ bundle, but also as a holomorphic bundle. The information is carried by the $\mathrm{G}^{\mathbb{C}}$ action, which is non-trivial in the sense that the trivialisation of B as a holomorphic bundle is incompatible with the $\mathrm{G}^{\mathbb{C}}$ action since there is no global smooth function f on $\mathbb{PT}_V$ such that $\Psi = \bar{\partial} f$ and $\Theta(f) = 0 = T(f)$.

The local invariant holomorphic sections of B are of the form

$$\boldsymbol{b} = (r\lambda g(w))^{-1/2} \tag{3.11}$$

where w is given by (2.6) and $g(w)$ is holomorphic. There are no non-vanishing global holomorphic sections.

3.2. A direct approach

Before describing the reverse construction, we shall look at a more direct route for passing from J to E. If we take $H = J$ and $\hat{H} = 1$, then (3.7) reduces to

$$\begin{aligned} &(\partial_r + \lambda\,\partial_z + (\lambda/r)\,\partial_\lambda)\boldsymbol{b} + (J^{-1}\,\partial_r J)\boldsymbol{b} = 0 \\ &(-\lambda\,\partial_r + \partial_z + (\lambda^2/r)\,\partial_\lambda)\boldsymbol{b} + (J^{-1}\,\partial_z J)\boldsymbol{b} = 0. \end{aligned} \tag{3.12}$$

Thus if Λ is a leaf contained in the quadric Q_w, then E_Λ is the space of solutions $\boldsymbol{s} = \boldsymbol{s}(z, r)$ of

$$\begin{aligned}(\partial_r + \lambda\, \partial_z)\boldsymbol{s} + (J^{-1}\, \partial_r J)\boldsymbol{s} &= 0\\ (-\lambda\, \partial_r + \partial_z)\boldsymbol{s} + (J^{-1}\, \partial_z J)\boldsymbol{s} &= 0\end{aligned} \tag{3.13}$$

in which $\lambda = e^{-i\theta}\zeta$ is defined as a function of z and r by

$$r\lambda^2 - 2(z - w)\lambda - r = 0. \tag{3.14}$$

It can be checked directly that this is a linear system for the static, axisymmetric form of Yang's equation: integrability for all constant values of w is equivalent to (3.1) on J. (This linear system and others closely related to it appear throughout the literature on stationary axisymmetric solutions of Einstein's equations. Almost exactly this form can be found in Belinsky and Zakharov (1978) and in Maison (1979).)

The fact that λ is double valued reflects the geometry of the foliation of $\mathbb{PT}_V$. For each w, the quadric Q_w intersects the fibre $X \subset \mathbb{PT}_V$ above a general $x \in \mathbb{E}_V$ in two points. If $w \notin V \cup \hat{V}$, then these points cannot be interchanged by moving x around in $\mathbb{E}_V$ and $Q_w \cap \mathbb{PT}_V$ is made up of two leaves of F, one corresponding to each root of (3.14). If, on the other hand $w \in V \cup \hat{V}$, then the points can be interchanged and so the two roots cannot be distinguished as (z, r) varies over V. In this case, the intersection of Q_w with $\mathbb{PT}_V$ is connected: it is a single leaf of F.

The fibre of E over a leaf $\Lambda \subset Q_w$ is the space of solutions of (3.13). In the first case ($w \notin V \cup \hat{V}$), there are two distinct functions $\lambda(z, r)$ satisfying (3.14) and we must choose the one corresponding to Λ: the other choice gives the fibre over the second leaf of F in Q_w. In the second case ($w \in V \cup \hat{V}$), the function $\lambda(r, z)$, and hence also the solutions of (3.12), are double valued on the complement in V of the singular point at which $z + ir = w$ or $\bar{w}$. The solutions make up a single vector space, which is the fibre of E above the corresponding point of R_V.

3.3. The recovery of J from E

Most of the steps follow Ward's treatment of the ASD equations (Ward 1977, 1981, Atiyah and Ward 1977), and so we shall not spell out all the details.

Let $E \to R_V$ be a given holomorphic vector bundle of rank n and let B be its pull-back to $\mathbb{PT}_V$. We shall assume that E is such that B satisfies the *triviality condition* (T):

for every $x \in \mathbb{E}_V$, $B \mid X$ is a trivial holomorphic bundle.

(This is less restrictive than it appears: if it is satisfied at one point x, then it is satisfied in some open neighbourhood of x.) We choose frames f^0 and f^1 in the fibres of E at Λ_0 and Λ_1 and pull these back to Λ_0 and Λ_1 to get two frame fields: s^0 for $B \mid \Lambda_0$ and s^1 for $B \mid \Lambda_1$.

Because (T) holds, it is possible to find a smooth global frame field $b = \{\boldsymbol{b}_i\}$ ($i = 1, 2, \ldots, n$) for B such that the $\boldsymbol{b}_1$ are holomorphic sections of $B \mid X$ on each fibre $X \subset \mathbb{PT}_V$. In the corresponding global trivialisation of B (as a C^∞ bundle), the holomorphic structure Ψ on B is given by a gl$(n, \mathbb{C})$-valued $(0, 1)$-form on $\mathbb{PT}_V$ which is equal to the $(0, 1)$ part of $\rho^*\Phi$, where Φ is a gauge potential with ASD curvature.

There is considerable freedom in the choice of b, since we can replace b by $bg = \{\boldsymbol{b}_i g_{ij}(x)\}$ where g is a $\mathrm{GL}(n, \mathbb{C})$-valued function of t, r, θ and z alone. This is equivalent to making a gauge transformation of Φ.

We can fix the gauge by requiring that $b = s^1$ on Λ_1. Then $b = s^0 J$ on Λ_0 for some $J : \Lambda_0 \to \mathrm{GL}(n, \mathbb{C})$ which is uniquely determined by E, f^0 and f^1. At points of Λ_1, $\Psi = 0$, and at points of Λ_0,

$$\Psi = J^{-1}\,\tilde{\partial} J \tag{3.15}$$

from which it follows that in this gauge, Φ is related to J by (3.3), with $H = J$ and $\hat{H} = 1$. (We are not distinguishing here between J and $J \circ \rho$.)

In a general gauge, $b = s^0 H$ on Λ_0 and $b = s^1 \hat{H}$ on Λ_1, where $H\hat{H}^{-1} = J$, and Φ is given by (3.3).

It is also true that J depends only on z and r and so is a solution of (3.1). This is a consequence of $G^{\mathbb{C}}$ invariance, but it can also be seen more simply by following through the details of the following procedure for calculating J from E.

Pick a point $x \in \mathbb{E}_V$ with coordinates (t, r, θ, z) and let $X \subset \mathbb{PT}_V$ be the fibre above x. Then X is a copy of $\mathbb{CP}_1$ and we have holomorphic maps

$$X \xrightarrow{\pi} R_V \xrightarrow{\Gamma} S \tag{3.16}$$

where π is the restriction to X of the projection from $\mathbb{PT}_V$ to R_V and Γ was defined in §2. We shall use $\lambda = \mathrm{e}^{-\mathrm{i}\theta}\zeta$ as the stereographic coordinate on X. Then the composition $\Gamma \circ \pi$ is the Joukowsky transformation $\lambda \mapsto w$, where

$$w = \tfrac{1}{2} r(\lambda^{-1} - \lambda) + z. \tag{3.17}$$

Note that both $\lambda = 0$ and $\lambda = \infty$ are mapped to $w = \infty$ on S.

If we represent R_V in terms of the 'direct construction', then π is completely determined by $\Gamma \circ \pi$ provided that we label the spheres S_0 and S_1 so that the point $w = \infty$ on S_0 is the image of $\lambda = 0$ on X and the point $w = \infty$ on S_1 is the image of $\lambda = \infty$ on X. That is, the leaf Λ_0 is a point of S_0 and Λ_1 is a point of S_1.

By the hypothesis (T), $\pi^*(E)$ is the trivial holomorphic bundle on X and therefore has a natural global trivialisation, which is unique up to transformation by a constant element of $\mathrm{GL}(n, \mathbb{C})$. In such a trivialisation, the frames $s^0 = \pi^*(f^0)$ at $\lambda = 0$ and $s^1 = \pi^*(f^1)$ at $l = \infty$ become matrices and J is given by $s^1 = s^0 J$.

Since it is clear from (3.17) that the map $\pi \circ \Gamma$ from the λ Riemann sphere to the w Riemann sphere depends only on z and r, it follows that J also depends only on z and r.

To turn this into a practical procedure for generating solutions, we use an adaptation of Ward's splitting construction. We start by choosing open covers $\{X_\alpha\}$ of X and $\{R_\alpha\}$ of R_V by contractible Stein manifolds such that $\pi(X_\alpha) \subset R_\alpha$. Here $\alpha = 0, 1, \ldots$ is an index labelling the sets of the covers. We shall see in the examples that $\{R_\alpha\}$ must contain at least four sets, so our splitting procedure is more complicated than the original version in Ward (1977), where two sets were sufficient.

On each R_α, there is a holomorphic frame field e^α for E; and on the overlaps $e^\beta = e^\alpha P_{\alpha\beta}$ where the patching matrices

$$P_{\alpha\beta} : R_\alpha \cap R_\beta \to \mathrm{GL}(n, \mathbb{C}) \tag{3.18}$$

are holomorphic (note that there is no summation convention for the Greek indices which label the sets in the covers). The $P_{\alpha\beta}$ determine E, and are determined by E

up to
$$P_{\alpha\beta} \mapsto L_\alpha P_{\alpha\beta} L_\beta^{-1} \tag{3.19}$$
where the $L_\alpha : R_\alpha \to \mathrm{GL}(n, \mathbb{C})$ are holomorphic and generate frame transformations $e^\alpha \mapsto e^\alpha L_\alpha^{-1}$.

We limit this freedom as follows. We label the sets in the cover $\{R_\alpha\}$ so $\Lambda_0 \in R_0$ and $\Lambda_1 \in R_1$ and choose the frames e^0 and e^1 so that $e^0 = f^0$ at Λ_0 and $e^1 = f^1$ at Λ_1. Then the L_α in (3.19) are restricted by the condition that $L_0 = 1$ at Λ_0 and $L_1 = 1$ at Λ_1.

On pulling back the e^α to X, we obtain holomorphic frames for $\pi^*(E)$ on each X_α, related by the patching matrices $Q_{\alpha\beta} = P_{\alpha\beta} \circ \pi$. The $Q_{\alpha\beta}$ are holomorphic functions of λ on the overlaps $X_\alpha \cap X_\beta$.

By hypothesis (T), $\pi^*(E)$ is trivial and so the $Q_{\alpha\beta}$ can be split; that is,
$$Q_{\alpha\beta} = K_\alpha (K_\beta)^{-1} \tag{3.20}$$
where K_α is a holomorphic function of λ on X with values in $\mathrm{GL}(n, \mathbb{C})$.

The K_α can be made to depend smoothly on r and z as x varies. Then $b = e^\alpha K_\alpha$ is a global smooth frame field for B, which is holomorphic on restricting the fibres (note that $e^\alpha K_\alpha = e^\beta K_\beta$ on $X_\alpha \cap X_\beta$). In the corresponding global trivialisation of B as a smooth bundle
$$\Psi = (K_\alpha)^{-1} \bar{\partial} K_\alpha \qquad (\alpha = 0, 1, 2, \ldots) \tag{3.21}$$
(these expressions agree on the overlaps of their domains of validity). It follows that $J = H\hat{H}^{-1}$, where $H = K_0$ at $\lambda = 0$ and $\hat{H} = K_1$ at $\lambda = \infty$; and that $\Psi = \rho^* \Phi$ where Φ is given in terms of H and $\hat{H}$ by (3.3).

The only freedom in this recovery procedure is the choice of the frames at $\Lambda_0, \Lambda_1 \in R_V$. The effect of changing these is to replace J by AJB^{-1} where A and B are constant elements of $\mathrm{GL}(n, \mathbb{C})$. It is clear, moreover, that if we construct (E, f^0, f^1) from J, then the recovery procedure will give us J back again. Hence we have the following.

Proposition 3.1. There is a one-to-one correspondence between

(1) solutions $J(r, z)$ of (3.1) on $V \subset H$ and

(2) holomorphic rank-n vector bundles $E \to R_V$ satisfying (T), together with a choice of frames in the fibres of E over Λ_0 and Λ_1.

We shall use the shorthand 'framed bundle' to mean a holomorphic vector bundle $E \to R_V$ satisfying (T), in which frames have been picked out in the fibres over the two points Λ_0 and Λ_1. A framed bundle will be denoted (E, f^0, f^1).

4. Symmetry and reality conditions

There are some obvious ways of generating new solutions of (4.1) from old ones (as well as some less obvious ones that we shall look at in §7). These correspond to simple operations on (E, f^0, f^1), some of which we shall need to express the reality and symmetry of J as constraints on (E, f^0, f^1).

In the following list, J is the $\mathrm{GL}(n, \mathbb{C})$-valued solution generated by the framed bundle (E, f^0, f^1) on R_V. The proofs of the correspondences involve no more than a straightforward examination of the splitting procedure.

(1) If $A, B \in \mathrm{GL}(n, \mathbb{C})$, then AJB^{-1} is the solution generated by (E, f^0A, f^1B) (f^0A denotes the frame $\{f_i^0 A_{ij}\}$).

(2) The complex conjugate $\bar{J}$ is the solution generated by $(j^*(\bar{E}), j^*(\bar{f}^0), j^*(\bar{f}^1))$, which we shall denote $\bar{j}^*(E, f^0, f^1)$. Here $j: R_V \to R_V$ is the antiholomorphic map defined in §2, and $j^*(\bar{E})$ is the holomorphic bundle obtained by pulling back $\bar{E}$ by j).

(3) The inverse J^{-1} is the solution generated by $i^*(E, f^0, f^1) = (i^*(E), i^*(f^1), i^*(f^0))$, where $i: R_V \to R_V$ is the holomorphic map defined in §2; it interchanges Λ_0 and Λ_1).

(4) The inverse transpose $(J^{-1})^T$ is the solution generated by $(E, f^0, f^1)^* = (E^*, f^{0*}, f^{1*})$, where E^* is the holomorphic dual of E and f^{0*} and f^{1*} are the frames dual to f^0 and f^1.

(5) Let $u(z, r)$ be a scalar ($n = 1$) solution of Yang's equation; in other words, $\log u$ is an axisymmetric harmonic function on $\mathbb{E}_3$. Then u is generated by a framed line bundle (L, l^0, l^1). The solution uJ of Yang's equation is generated by $(L \otimes E, l^0 \otimes f^0, l^1 \otimes f^1)$ where $l^0 \otimes f^0$ denotes the frame $(l^0 \otimes f_i^0)$.

(6) The solution $u = \det(J)$ of the scalar equation is generated by the framed line bundle $\det(E, f^0, f^1) = (\wedge^n E, \wedge^n f^0, \wedge^n f^1)$, where $\wedge^n f^0 = f_1 \wedge f_2 \wedge \ldots \wedge f_n$ and so on.

(7) If $V' \subset V$, then there is a natural projection $R_{V'} \to R_V$: in the 'direct' representation of $R_{V'}$, it is given by identifying the two points of $\Gamma^{-1}(w)$ whenever w is in the complement of $V' \cup \hat{V}'$ in $V \cup \hat{V}$. The restriction of J to V' is generated by the pull-back of (E, f^0, f^1) to $R_{V'}$.

On putting together (2), (3), and (4), we have the following criteria on (E, f^0, f^1) for the reality and symmetry of J.

Proposition 4.1. $(E, f^0, f^1) = \bar{j}^*(E, f^0, f^1)$ if and only if $J = \bar{J}$.

Proposition 4.2. $(E, f^0, f^1)^* = i^*(E, f^0, f^1)$ if and only if $J = J^T$.

We shall say the (E, f^0, f^1) is *real* when $(E, f^0, f^1) = \bar{j}^*(E, f^0, f^1)$ and symmetric when $(E, f^0, f^1)^* = i^*(E, f^0, f^1)$. We thus have a one-to-one correspondence between real symmetric solutions of Yang's equation on V and real symmetric framed bundles over R_V.

There are two technical results that we shall need in order to transfer the reality and symmetry conditions from (E, f^0, f^1) to its patching matrices.

(1) Suppose that (E, f^0, f^1) is symmetric. Then the isomorphism between E^* and $i^*(E)$ can be thought of as a family of invertible linear maps

$$\rho_\Lambda : E_\Lambda \to E^*_{i(\Lambda)} \tag{4.1}$$

which depend holomorphically on $\Lambda \in R_V$. But $i^2 = 1$, so

$$\rho_{i(\Lambda)} : E_{i(\Lambda)} \to E^*_\Lambda. \tag{4.2}$$

Let

$$\rho^*_{i(\Lambda)} : E_\Lambda \to E^*_{i(\Lambda)} \tag{4.3}$$

be the map dual to $\rho_{i(\Lambda)}$ and put

$$\sigma_\Lambda = (\rho^*_{i(\Lambda)})^{-1} \circ \rho_\Lambda. \tag{4.4}$$

Then $\sigma : E_\Lambda \to E_\Lambda$, and so we have a holomorphic section σ of $\mathrm{Aut}(E) = E \otimes E^*$. The first result is that σ equal to the identity everywhere on R_V.

The proof is as follows. With $\Lambda = \Lambda_0$, ρ_Λ is the map that sends the frame f^0 to the dual of the frame f^1 at $i(\Lambda) = \Lambda_1$; and with $\Lambda = \Lambda_1$, ρ_Λ is the map that sends f^1 to the dual of f_0. Therefore σ is the identity at Λ_0 and Λ_1.

Now let X be a fibre of $\mathbb{PT}_V$ and let $B = \pi^*(E)$ be the pull-back of E to X. Then $\pi^*(\sigma)$ is a section of $\mathrm{Aut}(B)$ which is equal to the identity at $\lambda = 0$ and $\lambda = \infty$. But B, and hence also $\mathrm{Aut}(B)$, is a trivial holomorphic bundle. Therefore $\pi^*(\sigma)$ is the identity everywhere on X; hence σ is the identity everywhere on R_V.

(2) The second result is an analogous statement for a real framed bundle (E, f^0, f^1). In this case we have a linear map $\tau_\Lambda : E_\Lambda \to \bar{E}_{j(\Lambda)}$ for each Λ and essentially the same argument establishes that

$$\bar{\tau}_{j(\Lambda)} \circ \tau_\Lambda : E_\Lambda \to E_\Lambda \tag{4.5}$$

is the identity map for all Λ.

The practical consequence of the first result is that if (E, f^0, f^1) is symmetric and if f and f' are frames at Λ and $i(\Lambda)$ respectively, then f is mapped to the dual of f' under the isomorphism between E and $(i^*(E))^*$, if and only if f' is mapped to the dual of f.

The second result (together with some complex analysis†) has the following implication: suppose that $R \subset R_V$ is an open set which is biholomorphically equivalent to an open set D in the complex w plane. If R is invariant under j and (E, f^0, f^1) is real, then there is a holomorphic frame field on R which is mapped to its complex conjugate under the isomorphism between $\bar{E}$ and $j^*(E)$.

5. Description in terms of patching matrices

We shall characterise a general holomorphic bundle over R_V in terms of a standard cover by four open sets, each of which is Hausdorff and can be identified with an open set in the Riemannphere S.

5.1. The standard cover

Let U be an open set in the w plane joining V to $\hat{V}$, as in figure 4(a) where U is the diagonally shaded region. The exact choice of U is unimportant, but it must be invariant under $w \mapsto \bar{w}$; and $U \backslash V$, $U \backslash \hat{V}$, and U itself must be connected and simply connected. Let U' be a neighbourhood of $w = \infty$ which is invariant under $w \mapsto \bar{w}$ and which intersects U in an annular region A; A is the union of two simply connected subsets, A_- (shaded horizontally in figure 4(b)) and A_+ (shaded vertically). Note that $A_- \cap A_+ = (A \cap V) \cup (A \cap \hat{V})$.

We shall picture R_V in terms of the direct construction, with the cut C in the w plane in the position shown in figure 4(b).

Consider $\Gamma^{-1}(U')$. This is made up of two copies of U' which intersect in $\Gamma^{-1}(A_+ \cap A_-)$. One copy, which we shall denote R_0, is a neighbourhood of $w = \infty$ in

† By using the fact that D is a Stein manifold, the statement can be deduced from the following: suppose that j acts on D by $w \mapsto \bar{w}$. Let $g : D \to \mathrm{GL}(n, \mathbb{C})$ be a holomorphic map such that $g(w)\bar{g}(w) = 1$. Then there exists a nowhere-vanishing holomorphic map $v : D \to \mathbb{C}^n$ such that $v(w) = g(w)\bar{v}(w)$. Although not immediately obvious, this is relatively straightforward to prove.

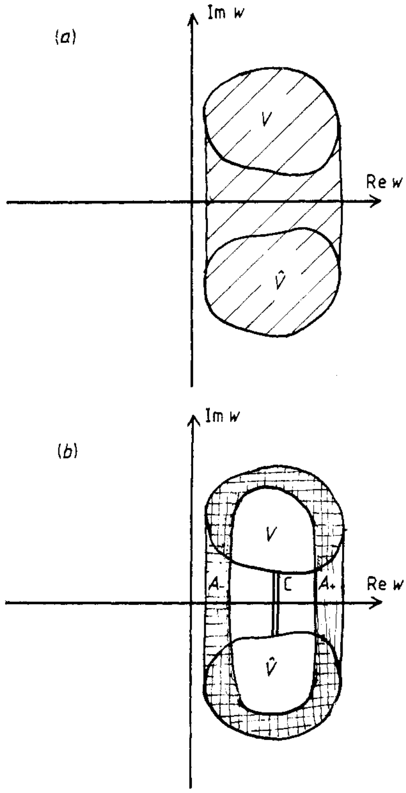

Figure 4. (a) The shaded region is V. (b) The shaded region is A.

S_0; the other, R_1, is a neighbourhood of $w = \infty$ in S_1. Thus $\Gamma^{-1}(U') = R_0 \cup R_1$ and $R_0 \cap R_1$ can be identified with $A_+ \cap A_-$.

Now consider $\Gamma^{-1}(U)$. This is made up of two copies of U, which intersect in $\Gamma^{-1}(V \cup \hat{V})$. We shall denote these R_2 and R_3, with the labelling fixed by

$$R_0 \cap R_2 \subset \Gamma^{-1}(A_-) \qquad R_0 \cap R_3 \subset \Gamma^{-1}(A_+) \tag{5.1}$$

(see figure 5). Thus a point moving from infinity in the positive direction along the negative real axis in R_0 passes from R_0 to R_2, and a point moving from infinity in the negative direction along the positive real axis in R_0 passes from R_0 to R_3.

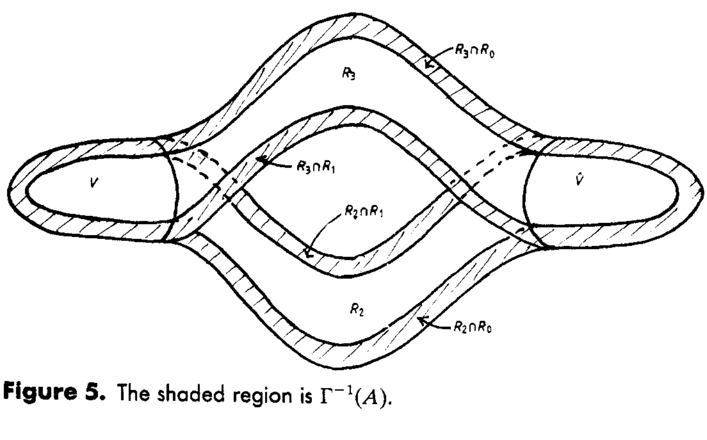

Figure 5. The shaded region is $\Gamma^{-1}(A)$.

5.2. Patching data for real symmetric framed bundles

Let (E, f^0, f^1) be a real, symmetric framed bundle on R_V. We shall identify $\bar{E}$ with $j^*(E)$ and E^* with $i^*(E)$.

Since the R_α are Stein manifolds, we can choose holomorphic frames e^α on each R_α such that $e^0 = f^0$ at Λ_0 and $e^1 = f^1$ at Λ_1 and we can characterise (E, f^0, f^1) by the patching matrices $P_{\alpha\beta}$, where $e^\beta = e^\alpha P_{\alpha\beta}$. These can be represented as holomorphic functions of w on regions in S:

$$\begin{aligned} &P_{01}\colon A_+ \cap A_- \to \mathrm{GL}(n, \mathbb{C}) &&P_{12}\colon A_+ \to \mathrm{GL}(n, \mathbb{C}) \\ &P_{02}\colon A_- \to \mathrm{GL}(n, \mathbb{C}) &&P_{13}\colon A_- \to \mathrm{GL}(n, \mathbb{C}) \\ &P_{03}\colon A_+ \to \mathrm{GL}(n, \mathbb{C}) &&P_{23}\colon V \cup \hat{V} \to \mathrm{GL}(n, \mathbb{C}). \end{aligned} \tag{5.2}$$

However, many of the data here are redundant. We can drop P_{01}, which can be found from P_{02} and P_{12} by using one of the cocycle relations $P_{\alpha\beta}P_{\beta\gamma}P_{\gamma\alpha} = 1$; and we can impose the conditions

$$\begin{aligned} &j^*(e^\alpha) = \bar{e}^\alpha \qquad &&i^*(e^1) = e^{0*} \qquad &&i^*(e^0) = e^{1*} \\ &i^*(e^2) = e^{3*} &&i^*(e^3) = e^{2*} \end{aligned} \tag{5.3}$$

where $\bar{e}^\alpha$ is the frame for $\bar{E} \mid R_\alpha$ conjugate to e^α, and $e^{\alpha*}$ is the frame for $E^* \mid R_\alpha$ dual to e^α (we have used here the technical results derived at the end of the last section). Then the $P_{\alpha\beta}$ are real (in the sense that $P_{\alpha\beta}(w) = \overline{P_{\alpha\beta}(\bar{w})}$) and $P_{02} = (P_{31})^T$, $P_{03} = (P_{21})^T$, $P_{23} = (P_{23})^T$.

We can therefore characterise (E, f^0, f^1) in terms of three holomorphic maps:

$$P\colon V \to \mathrm{GL}(n, \mathbb{C}) \qquad F\colon A_- \to \mathrm{GL}(n, \mathbb{C}) \qquad G\colon A_+ \to \mathrm{GL}(n, \mathbb{C}). \tag{5.4}$$

P is the restriction of P_{23} to V; $\bar{P}$ (where $\bar{P}(w) = \overline{P(\bar{w})}$) is the restriction of P_{23} to $\hat{V}$; $F = P_{02} = (P_{31})^T$ and $G = P_{12} = (P_{30})^T$.

The data (P, F, G) can be chosen arbitrarily, subject to

$$PG^TF = 1 \text{ on } A \cap V \qquad F = \bar{F} \qquad G = \bar{G} \qquad P = P^T. \tag{5.5}$$

The 'gauge freedom' in the representation of (E, f^0, f^1) by patching matrices appears here as

$$(P, F, G) \simeq (LPL^T, KFL^{-1}, (K^T)^{-1}GL^{-1}) \tag{5.6}$$

where $L\colon U \to \mathrm{GL}(n, \mathbb{C})$ and $K\colon U' \to \mathrm{GL}(n, \mathbb{C})$ are holomorphic and real; and $K(\infty) = 1$.

Thus we have a one-to-one correspondence between real symmetric solutions of Yang's equation and classes of equivalent triples (P, F, G) satisfying (5.5), but for one awkward point which we have glossed over: the triviality condition (T). It is certainly true that every solution on V can be obtained from a triple (P, F, G). But when we go in the other direction, and construct E from (P, F, G), it is not obvious that the pull-back $\pi^*(E)$ of E to a fibre X in $\mathbb{PT}_V$ will be a trivial holomorphic bundle. It is necessary both that $\pi^*(E)$ should be topologically trivial, which will happen if and only if its first Chern class vanishes, and that a more subtle analytic invariant should vanish. It is not hard to see that the topological condition is automatically satisfied; the analytic condition may fail at points of V, at which the solution of Yang's equation will have singularities.

5.3. Recovery of J from the patching data

The actual process recovery of J from (P, F, G) is as described in §3: we choose a cover $\{X_\alpha\}$ of X such that $\pi(X_\alpha) \subset R_\alpha$ and write down the patching matrices $Q_{\alpha\beta}$ for π^*E by substituting

$$w = \tfrac{1}{2}r(\lambda^{-1} - \lambda) + z \tag{5.7}$$

into the corresponding expression for $P_{\alpha\beta}$.

The obvious choice is to take $X_\alpha = \pi^{-1}(R_\alpha)$. This cover is shown in figure 6.

(To obtain this picture, note first that $\pi^{-1}(V)$ and $\pi^{-1}(\hat{V})$ are two open sets surrounding the points i and $-$i in the λ plane, on which π is two to one; and that the image of the unit circle under π is the line segment joining $w = z - ir$ to $w = z + ir$. If we remove the copies of V and $\hat{V}$ from R_V, then λ becomes a single-valued function on R_V: it is fixed by the conditions $\lambda = 0$ at $w = \infty$ in R_0 and $\lambda = \infty$ at $w = \infty$ in R_1. The set X_α is the image of $R_\alpha \backslash (V \cup \hat{V})$ under λ, together with appropriate pieces of $\pi^{-1}(V \cup \hat{V})$.)

Remarks. (1) From some points of view the use of the triples (P, F, G) as data is rather awkward because of the constraint $PG^TF = 1$. Given any symmetric $P: V \to \mathrm{GL}(n, \mathbb{C})$, it is always possible to find suitable F and G; but it may be difficult to do it explicitly. One can instead concentrate the patching data on the boundary of $\Gamma^{-1}(V \cup \hat{V})$ in R_V: this is made up of two closed curves which are double covers of the boundaries of V and $\hat{V}$ in S. Although it appears that it is possible to use this to characterise solutions in terms of unconstrained data, we have not yet managed to understand the details; also such data do not seem to be well adapted to the action of the hidden symmetry group.

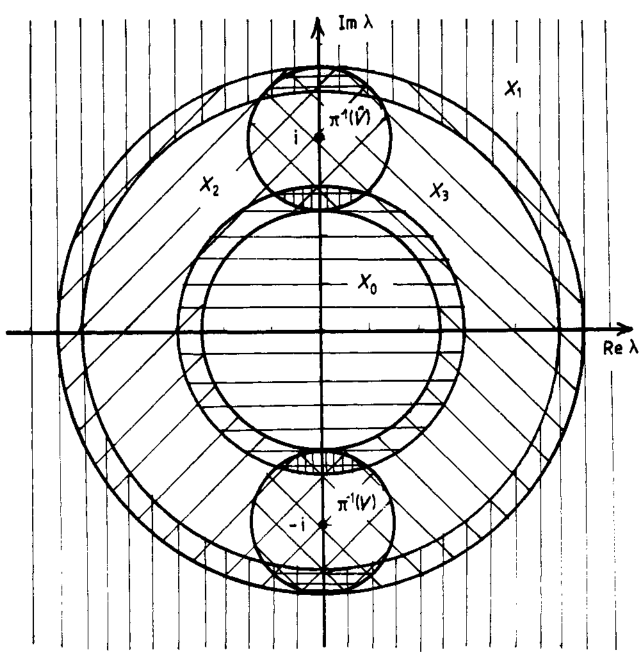

Figure 6. The covering of X: X_2 and X_3 are shaded diagonally: X_0 is shaded horizontally and X_1 is shaded vertically.

(2) The holomorphic bundle $E \mid \Gamma^{-1}(U)$ is trivial if and only if P has an analytic continuation to $P: U \to \mathrm{GL}(n, \mathbb{C})$ such that $P(w) = \bar{P}(w)$. We shall see that the non-triviality of $E \mid \Gamma^{-1}(U)$ measures how badly behaved J is on the symmetry axis, and that the 'hidden symmetry' transformations leave P and hence $E \mid \Gamma^{-1}(U)$ unchanged, but act transitively on F and G.

Example. $n = 1$, $V = H$, and $J = r$; we take U to be the whole of S, minus the point at infinity and U' to be the whole of S less the origin. Then A is the whole w plane, apart from the origin; A_- is the whole plane, apart from a cut along the positive real axis, and A_+ is the whole plane, apart from a cut along the negative real axis. The four sets R_α are all copies of the Argand plane, and $\Gamma^{-1}(U)$ consists of two copies of the w plane which are 'glued together' everywhere but on the real axis; R_V itself is a Riemann sphere, except that its real axis (a great circle) has been removed and replaced by a connected double cover.

Let λ_+ and λ_- be the positive and negative roots in the λ plane of $w(\lambda) = 0$ (both roots are real). Let C_0, C_1, C_2, and C_3 be cuts along the real axis:

C_0 from λ_- to λ_+ through $\lambda = \infty$

C_1 from λ_+ to λ_- through $\lambda = 0$

C_2 from $\lambda = 0$ to $\lambda = \infty$ along the positive real axis

C_3 from $\lambda = 0$ to $\lambda = \infty$ along the negative real axis.

Then X_α is the complement of C_α for $\alpha = 0, 1, 2, 3$.

Let $\sqrt{w}$ be the square root function defined on A_+ (the complement of the negative real axis) by $\sqrt{1} = 1$. Put

$$P(w) = -\mathrm{i} \qquad F(w) = \sqrt{-2w} \qquad G(w) = 1/\sqrt{2w}. \tag{5.8}$$

Then

$$\begin{aligned} Q_{02} &= \sqrt{(-r(\lambda^{-1} - \lambda) - 2z)} \qquad & Q_{03} &= \sqrt{(r(\lambda^{-1} - \lambda) + 2z)} \\ Q_{12} &= [\sqrt{(r(\lambda^{-1} - \lambda) + 2z)}]^{-1} \qquad & Q_{13} &= [\sqrt{(-r(\lambda^{-1} - \lambda) - 2z)}]^{-1}. \end{aligned} \tag{5.9}$$

Suitable splitting functions are

$$\begin{aligned} K_0 &= \sqrt{(r(1 - \lambda^2) + 2\lambda z)} \qquad & K_2 &= \sqrt{-\lambda} \\ K_1 &= [\sqrt{(r(1 - \lambda^{-2}) - 2\lambda^{-1} z)}]^{-1} \qquad & K_3 &= \sqrt{\lambda}. \end{aligned} \tag{5.10}$$

Hence the corresponding solution of Yang's equation is $J = K_0(0) K_1(\infty)^{-1} = r$.

Remark. In this case, $\bar{P} = \mathrm{i}$, which is not equal to the analytic continuation of P to the lower half-plane, so $E \mid \Gamma^{-1}(U)$ is not trivial.

The solution r^k ($k \in \mathbb{R}$) is generated by the kth power of (E, f^0, f^1), for which the data are $P = \mathrm{i}^{-k}$, $F = (\sqrt{(-2w)})^k$, $G = (\sqrt{(2w)})^{-k}$. When k is an even integer, P is a real constant and $P = \bar{P}$. In the terminology introduced below, the framed bundle corresponding to r^k is axis simple if and only if k is an even integer.

5.4. Axis-simple case

There is an obvious projection $\eta : R_V \to R_U$ which identifies the two points in $\Gamma^{-1}(w)$ for each $w \in U \backslash (V \cup \hat{V})$. We shall say that (E, f^0, f^1) is *axis simple* whenever

$$(E, f^0, f^1) = \eta^*(E', f'^0, f'^1) \tag{5.11}$$

for some bundle $E' \to R_U$ with frames f'^0 and f'^1 at $\eta(\Lambda_0)$ and $\eta(\Lambda_1)$; here η^* is the pull-back map on vector bundles. The terminology reflects the fact that the corresponding J has, at worst, simple poles in r on the axis.

The condition is equivalent to the triviality of $E \mid \Gamma^{-1}(U)$; or, for a real symmetric (E, f^0, f^1) with data (P, F, G), to the condition that P should have an analytic continuation to U, which is real on the real axis.

Now $R_U = S_0 \cup S_1$; and so $E' \to R_U$ is formed by patching together the two bundles $E' \mid S_0$ and $E' \mid S_1$. But $E' \mid S_0$ is a rank-n holomorphic vector bundle over a Riemann sphere and so must be a direct sum of line bundles of the form $L^p \oplus L^q \oplus \ldots \oplus L^r$ where L is the hyperplane section line bundle and $p, q, \ldots, r$ are integers (see, for example, Field 1982, vol II, p 186); $E' \mid S_1$ is a similar direct sum, which in the symmetric case must be $E' \mid S_1 = L^{-p} \oplus L^{-q} \oplus \ldots \oplus L^{-r}$.

It follows that a real symmetric framed bundle (E, f^0, f^1) is axis simple if and only if its patching data can be put in the form (P, F, G) where $P : U \to \mathrm{GL}(n, \mathbb{C})$ is holomorphic, with $P(w) = \bar{P}(w)$ and $P(w) = P(w)^T$, and

$$F = g\,\mathrm{diag}((2w)^p, (2w)^q, \ldots, (2w)^r) \qquad G = (F^T)^{-1}P^{-1} \tag{5.12}$$

where $p, q, \ldots, r \in \mathbb{Z}$, $g \in \mathrm{GL}(n, \mathbb{R})$.

Without essential loss of generality, one can set $g = 1$ by changing f^0 and f^1. Thus the 'free data' for a real symmetric axis-simple solution are the patching matrix P and the integers $p, q, \ldots, r$ (which determine the behaviour of J as $r \to 0$).

5.5. The Ward ansätze

Suppose that $n = 2$ and that (E, f^0, f^1) is real, symmetric and axis simple. Then the pull-back $\pi^*(E, f^0, f^1)$ under $\pi : X \to R_V$, where X is a fibre of $\mathbb{PT}_V$, has patching matrices

$$Q_{02} = \begin{pmatrix} (2w)^p & 0 \\ 0 & (2w)^q \end{pmatrix} \qquad Q_{12} = \begin{pmatrix} (2w)^{-p} & 0 \\ 0 & (2w)^{-q} \end{pmatrix} P(w)^{-1} \tag{5.13}$$

relative to a three-set cover, X_0 (a neighbourhood of $\lambda = 0$), X_1 (a neighbourhood of $\lambda = \infty$) and $X_2 = \{\lambda;\, w(\lambda) \in U\}$ (a neighbourhood of the unit circle in the λ plane). Here w is given as a function of λ, z, and r by (5.7).

Let $m_0 = 2\lambda w / r$ and $m_1 = -2w/\lambda r$ and put

$$M_0 = \begin{pmatrix} m_0^{-p} & 0 \\ 0 & m_0^{-q} \end{pmatrix} \qquad M_1 = \begin{pmatrix} m_1^p & 0 \\ 0 & m_1^q \end{pmatrix} \qquad M_2 = \begin{pmatrix} r^p\lambda^{-p} & 0 \\ 0 & r^q\lambda^{-q} \end{pmatrix}. \tag{5.14}$$

Since $M_\alpha : X_\alpha \to \mathrm{GL}(2, \mathbb{C})$ is holomorphic and $M_0(0) = 1 = M_1(\infty)$, the value of $J(z, r)$ is unchanged by replacing $Q_{\alpha\beta}$ by $\tilde{Q}_{\alpha\beta} = M_\alpha Q_{\alpha\beta}(M_\beta)^{-1}$. But $\tilde{Q}_{02} = 1$. Thus we can replace the three-set cover by the two sets $X'_0 = X_0 \cup X_2$ (a neighbourhood of $\lambda = 0$) and $X'_1 = X_1$ (a neighbourhood of $\lambda = \infty$). We then have a single patching matrix

$$Q'_{01} = \begin{pmatrix} r^p\lambda^{-p} & 0 \\ 0 & r^q\lambda^{-q} \end{pmatrix} P(w) \begin{pmatrix} (-r\lambda)^p & 0 \\ 0 & (-r\lambda)^q \end{pmatrix}. \tag{5.15}$$

On taking $p=-q=-k/2$, where k is an even integer, and imposing the additional constraint $\det(P)=-1$, which is equivalent to $\det(J)=-1$, we have Ward's 'ansatz k' (Ward 1983); Ward's ζ is our λ and his γ is our w.

5.6 The Weyl solutions

A special case is the class of axis-regular Weyl solutions in which $J=\mathrm{diag}(-r^2e^{-\chi}, e^{\chi})$. Here $p=1$, $q=0$, and $P=\mathrm{diag}(f^{-1}, f)$; $\chi(z,r)$ is an axisymmetric harmonic function on $\mathbb{E}_3$ which is non-singular at $r=0$. It is related to f by $f(z)=\exp(\chi(z,0))$ for real z; and also by Ward's contour integral formula

$$\chi(z,r)=\frac{1}{2\pi\mathrm{i}}\oint q\left(z-\frac{r}{2}(\lambda-\lambda^{-1})\right)\frac{\mathrm{d}\lambda}{\lambda} \qquad q=\log f \tag{5.16}$$

(Ward 1983). Particular examples are

Schwarzschild solution: $f=(w-m)/(w+m)$

Zipoy solution (oblate spheroidal case); $f=[(1+\mathrm{i}w)/(1-\mathrm{i}w)]^{\mathrm{i}m/a}$

Zipoy–Vorhees solution (prolate spheroidal case, two mass points separated by a strut): $f=[(w-1)/(w+1)]^{\beta}$.

The second solution raises an interesting global issue: our construction generates the metric only on simply connected domains in the (z,r) plane. But in this case the complete solution is defined not a subset of H, but on a covering space of $H\backslash\{w=\frac{1}{2}i\}$ (Zipoy 1966). It should be possible to build this behaviour into the structure of reduced twistor space in a natural way.

5.7. Kerr and NUT solutions

As final examples, the Kerr solution and the NUT (Newman–Unti–Tamburino) solution are given by $p=1$, $q=0$ and

$$P=\frac{1}{w^2+a^2-m^2}\begin{pmatrix}(w+m)^2+a^2 & 2am\\ 2am & (w-m)^2+a^2\end{pmatrix} \tag{5.17}$$

and

$$P=\frac{1}{w^2-l^2-m^2}\begin{pmatrix}(w+m)^2+l^2 & 2lw\\ 2lw & (w-m)^2+l^2\end{pmatrix} \tag{5.18}$$

respectively.

Remarks. (1) The solutions generated by Ward's ansätze for odd k are not axis simple according to our definition—for the same reason that $J=r$ is not axis simple; however, they can be converted into axis-simple solutions, and therefore obtained from patching matrices of our form, by multiplying by r.

(2) Ward also found a relationship between the value of $P(w)$ and its derivatives on the real axis and the asymptotic behaviour of $J(z,r)$ as $r\to 0$; he gives the formula in the case $k=1$ in Ward (1983). In our notation, the general statement is this. Suppose that $p-q=m>0$ and that

$$P(w)=\begin{pmatrix}g & -g\Omega\\ -g\Omega & g\Omega^2+\hat{g}^{-1}\end{pmatrix} \tag{5.19}$$

where g, $\hat{g}$, and Ω are functions of w, with g and $\hat{g}$ non-vanishing on the real axis. Then, as $r\to 0$,

$$J(z,r)=\begin{pmatrix}r^p & 0\\ 0 & r^q\end{pmatrix}\left[\begin{pmatrix}h(z) & -h(z)\Lambda\\ -h(z)\Lambda & h(z)\Lambda^2+\hat{h}(z)^{-1}\end{pmatrix}+\mathrm{O}(r^{m+1})\right]\begin{pmatrix}r^p & 0\\ 0 & r^q\end{pmatrix} \tag{5.20}$$

where

$$\Lambda = \frac{(-1)^p r^m}{2^m m!}\left[\frac{\mathrm{d}^m}{\mathrm{d}w^m}\Omega\right]_{w=z} \qquad h(z) = (-1)^p g(z) \qquad \hat{h}(z) = (-1)^q \hat{g}(z) \tag{5.21}$$

In particular, when $p = q = 0$, $J(z, r) = P(z) + \mathrm{O}(r)$. (We shall give a proof in the case $p = q = 0$ in §7.)

Ward's formula allows one to read off the patching data for axis-simple solutions from the leading-order terms in the expansion of J as $r \to 0$. It also allows one to relate the maximal domain of J to the domain of analyticity of the patching data on the axis (although the relationship is not quite as straightforward as it appears because of the triviality condition (T)).

6. Transformation theory

The reduced Einstein equations have a large number of 'hidden symmetries'. We shall look at those in the Geroch group (Kinnersley and Chitre's 'group K'), which includes Ehlers' 'gravitational duality rotation' (Ehlers 1962, Geroch 1972, Kinnersley 1977, Kinnersley and Chitre 1977).

We shall show that K has a straightforward action on the corresponding holomorphic bundles on the reduced twistor space R_V. The idea is a development of that described in Woodhouse (1987).

6.1. The infinitesimal action

Kinnersley and Chitre introduce their symmetry group by describing the action of its Lie algebra, which is the infinite-dimensional *loop algebra* k of $\mathrm{sl}(2, \mathbb{R})$. That is, the algebra of Laurent polynomials

$$\gamma(t) = \ldots + \gamma^{(-1)}t^{-1} + \gamma^{(0)} + \gamma^{(1)}t + \gamma^{(2)}t^2 + \ldots \tag{6.1}$$

where the $\gamma^{(k)}$ are elements of $\mathrm{sl}(2, \mathbb{R})$, all but a finite number equal to zero, and t is a variable; k is a Lie algebra under the obvious bracket operation.

The zero degree elements act by the infinitesimal version of the $\mathrm{SL}(2, \mathbb{R})$ action $J \mapsto gJg^T$, $g \in \mathrm{SL}(2, \mathbb{R})$. Thus

$$\gamma^{(0)}(J) = \gamma^{(0)}J + J\gamma^{(0)T}. \tag{6.2}$$

The extension to the rest of k is obtained by conjugating by a discrete symmetry of Yang's equation, the NK *transformation* (Neugebauer and Kramer 1969). We shall work with a slightly more general version of the NK transformation, which is the following.

Suppose that $J(z, r)$ is a $\mathrm{GL}(2, \mathbb{C})$-valued solution of (3.1), with $J_{22} \neq 0$. We write

$$J = \begin{pmatrix} f\omega\hat{\omega} + \hat{f}^{-1} & -f\omega \\ -f\hat{\omega} & f \end{pmatrix} = \begin{pmatrix} 1 & -f\omega \\ 0 & f \end{pmatrix}\begin{pmatrix} \hat{f} & 0 \\ \hat{f}\hat{\omega} & 1 \end{pmatrix}^{-1} \tag{6.3}$$

where ω, $\hat{\omega}$, f and $\hat{f}$ are complex-valued functions of z and r. Then $\det(J) = f/\hat{f}$ and (3.1) is equivalent to

$$\nabla^2 \log f - f\hat{f}(\partial_r\omega\,\partial_r\hat{\omega} + \partial_z\omega\,\partial_z\hat{\omega}) = 0 \tag{6.4}$$

$$\partial_r(rf\hat{f}\,\partial_r\omega) + \partial_z(rf\hat{f}\,\partial_z\omega) = 0 \tag{6.5}$$

together with the same pair of equations with f and $\hat{f}$ interchanged and ω and $\hat{\omega}$ interchanged. Here $\nabla^2 = r^{-1}\,\partial_r(r\,\partial_r) + \partial_z^2$ is the three-dimensional Laplacian acting on axisymmetric functions.

Equation (6.5) and the similar equation for $\hat{\omega}$ imply the existence of functions ψ and $\hat{\psi}$ such that

$$\begin{aligned} \partial_z\psi &= rf\hat{f}\,\partial_r\hat{\omega} \qquad & \partial_z\hat{\psi} &= rf\hat{f}\,\partial_r\omega \\ \partial_r\psi &= -rf\hat{f}\,\partial_z\hat{\omega} \qquad & \partial_r\hat{\psi} &= -rf\hat{f}\,\partial_z\omega. \end{aligned} \tag{6.6}$$

When J is the metric on the group orbits of a stationary axisymmetric spacetime, the functions are all real, $\omega = \hat{\omega}$, $\hat{f} = -f/r^2$ and we can take $\psi = \hat{\psi}$. Then $\mathscr{E} = f + \mathrm{i}\psi$ is the *Ernst potential.* (The labelling of the metric elements is based on that in Cosgrove (1980).)

The discrete symmetry is the map† ι which depends J to the matrix $\iota(J)$ defined by replacing f, $\hat{f}$, ω, and $\hat{\omega}$ by $f' = -(r^2\hat{f})^{-1}$, $\hat{f}' = f^{-1}$, $\omega' = \psi$ and $\hat{\omega}' = \hat{\psi}$. That is

$$\iota(J) = -\frac{1}{r^2\hat{f}}\begin{pmatrix} \psi\hat{\psi} - r^2 f\hat{f} & -\psi \\ -\hat{\psi} & 1 \end{pmatrix}. \tag{6.7}$$

Note that $\det(\iota(J)) = -\det(J)/r^2$ and that $\iota^{-1} = -r^2\iota$. Because $\iota(J)$ carries precisely the same information as $\mathscr{E}$ in the spacetime case, we shall call $\iota(J)$ the *Ernst potential* of J.

The next step in the construction of the k action is to lay down that the element

$$\gamma(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \tag{6.8}$$

should act by

$$J \mapsto \iota^{-1}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\iota(J) + \iota(J)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]. \tag{6.9}$$

Now the zero-degree elements, together with the $\gamma(t)$ in (6.8), generate the whole of k^+ (the subalgebra of k of Laurent polynomials in which all the negative-degree terms vanish). Thus (6.2) and (6.8) determine the action of k^+, but for one difficulty, which is also the reason that we must allow negative-degree elements in k: the discrete symmetry ι is not uniquely determined by (6.7) since we are free to add constants to ψ and $\hat{\psi}$; the Ernst potential $\iota(J)$ is only determined by J up to

$$\iota(J) \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\iota(J)\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \qquad a, b \in \mathbb{C}. \tag{6.10}$$

Thus in order to carry out repeated applications of (6.9), we must in some way single out a particular Ernst potential not only for J itself, but for all the intermediate J that we obtain at each stage of the process.

Geroch (1972) and Kinnersley and Chitre (1977) get around this difficulty by introducing an infinite chain of potentials for J which can be combined into a single generating function, the $F_{AB}(t)$ in Kinnersley and Chitre (1978a). The generating function encodes all the Ernst potentials and the algebra k acts on the $F_{AB}(t)$ rather than on the J themselves, with the negative degree elements generating 'gauge'

† It is related to the map $J \to I(J)$ in Woodhouse (1987) by $I(J) = r\iota(J)$; I is itself related to the KN transformation in Cosgrove (1980) by a complex equivalence transformation.

transformations. In particular, the element

$$\gamma(t)=\begin{pmatrix} 0 & 0 \\ t^{-1} & 0 \end{pmatrix} \tag{6.11}$$

acts on the Ernst potential according to

$$\iota(J)\mapsto\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\iota(J)+\iota(J)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.12}$$

We shall do what amounts to the same thing (at least at a formal level); we shall make k act not on framed bundles (i.e. solutions of Yang's equation) but on objects of the form (E, e^0, e^1) where E is a holomorphic rank-2 bundle over R_V (satisfying the triviality condition) and e^0 and e^1 are holomorphic frame fields defined on neighbourhoods of Λ_0 and Λ_1 respectively. When e^0 and e^1 are expanded in Taylor series about Λ_0 and Λ_1, the zero-degree terms turn E into a framed bundle, and so give a solution of Yang's equation. The remaining terms carry the same information as the generating function $F_{AB}(t)$ (see appendix 2).

6.2. The action of the loop group

Let γ be a simple closed curve γ in the (z, r) plane which is mapped to itself under $w\mapsto\bar{w}$ (as usual, the (z, r) plane is identified with the complex w plane by $w=z+\mathrm{i}r$). Let U be the open set inside γ and let D be the open set on the Riemann sphere S outside γ. We shall suppose that V and $\hat{V}$ are contained in U. Then $\Gamma^{-1}(\gamma)$ consists of two closed curves on R_V: one, γ_0, surrounds Λ_0; the other, γ_1, surrounds Λ_1. Both will be parametrised by $w\in\gamma$. Let D_0 and D_1 be the respective neighbourhoods of Λ_0 and Λ_1 bounded by γ_0 and γ_1 (so that $D_0\cup D_1=\Gamma^{-1}(D)$).

We shall deal first with a larger group than K, which preserves the symmetry, but not the reality of J. The larger group acts on triples (E, e^0, e^1), in which E is a rank-n holomorphic bundle over R, and e^0 and e^1 are holomorphic frames for E on D_0 and D_1, respectively, with smooth extensions to γ_0 and γ_1. The set of all such triples will be denoted $T(n, V, \gamma)$. By allowing a general value of n, we have not introduced any extra complication.

Let $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ be the *loop group* of smooth maps

$$g:\gamma\to\mathrm{GL}(n,\mathbb{C}):w\mapsto g(w) \tag{6.13}$$

and let K^γ be the subgroup of maps which are real in the sense that $\overline{g(w)}=g(\bar{w})$ and which satisfy $\det(g)=1$. The Lie algebra of $\mathrm{L}_\gamma\mathrm{GL}(2,\mathbb{C})$ is the set of Laurent polynomials with values in $\mathrm{gl}(n,\mathbb{C})$ (Kac 1985); the Lie algebra k of K^γ is the subalgebra of Laurent polynomials with values in $\mathrm{sl}(n,\mathbb{R})$.

We claim that $\mathrm{L}_\gamma\mathrm{GL}(n,\mathbb{C})$ (the 'larger group') has a natural action on $T(n, V, \gamma)$:

$$g:(E, e^0, e^1)\mapsto g(E, e^0, e^1) \tag{6.14}$$

and that, in the case $n=2$, the action of the subgroup K^γ coincides with that generated by the Kinnersley and Chitre infinitesimal transformations.

First the action of $\mathrm{L}_\gamma\mathrm{SL}(n,\mathbb{C})$. The idea is quite simple. We should like to extend $\{D_0, D_1\}$ to a cover $\{D_\alpha\}$ of R_V and make $\mathrm{L}_\gamma\mathrm{GL}(n,\mathbb{C})$ act on the corresponding

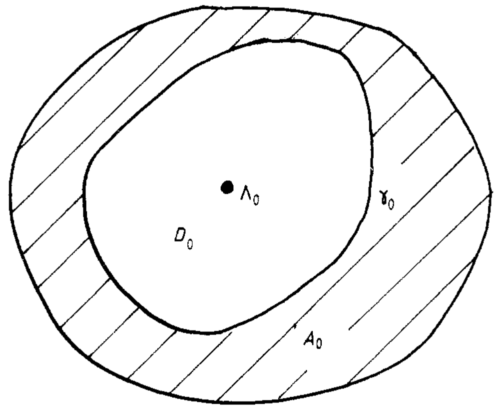

Figure 7. The shaded region is A_0.

patching matrices $P_{\alpha\beta}$ of E by

$$P_{0\alpha} \mapsto gP_{0\alpha} \qquad P_{1\alpha} \mapsto (g^{-1})^T P_{1\alpha} \qquad (\alpha = 2, 3, \ldots). \tag{6.15}$$

The fact that we are given frames e^0 and e^1 on D_0 and D_1 restricts the freedom in the choice of patching matrices by forcing $L_0 = 1 = L_1$ in (3.19). Thus the action on (E, e^0, e^1) is independent of the choice of trivialisations of E over $D_2, D_3, \ldots$. The difficulty is that to make sense of (6.15) for a general element of $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$, we must make analytic continuations of g from γ_0 and γ_1 to $D_0 \cap D_\alpha$ and $D_1 \cap D_\alpha$, and this is not possible for a general element of $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$.

We shall get around this by borrowing from Pressley and Segal (1986) a device based on Birkhoff's theorem, which is also useful because it gives us a parametrisation of the orbits of K^γ.

Let $E \mid R^\gamma$ be the restriction of E to the (non-Hausdorff) manifold with boundary $R^\gamma = R_V \backslash (D_0 \cup D_1)$. The boundary of R^γ is made up of the two curves γ_0 and γ_1 and the bundle $E \mid R^\gamma$ is holomorphic on R^γ.

Let $A_0 \subset \Gamma^{-1}(U)$ be an annular region surrounding Λ_0 with γ_0 as its inner boundary (figure 7) and let A_1 be an annular region surrounding Λ_1 with γ_1 as its inner boundary. Choose holomorphic frame fields k^0 on A_0 and k^1 on A_1 which extend smoothly to γ_0 and γ_1 respectively. This is possible since A_0 and A_1 are Stein manifolds.

On γ_0 and γ_1 respectively,

$$k^0 = e^0 Q_0 \qquad k^1 = e^1 Q_1 \tag{6.16}$$

where $Q_0 : \gamma_0 \to \mathrm{GL}(n, \mathbb{C})$ and $Q_1 : \gamma_1 \to \mathrm{GL}(n, \mathbb{C})$ are smooth. The Q_α $(\alpha = 0, 1)$ are determined by (E, e^0, e^1) up to $Q_\alpha \mapsto Q_\alpha L_\alpha$ where $L_\alpha : A_\alpha \to \mathrm{GL}(n, \mathbb{C})$ is holomorphic with smooth boundary values on γ_α (remember that the summation convention does not apply to the Greek labels $\alpha, \beta, \ldots$).

Conversely, given $E \mid R^\gamma$, k^0, k^1, Q_0, and Q_1, we can reconstruct (E, e^0, e^1) as follows. First we use Birkhoff's theorem to write $Q_\alpha = Q_\alpha^- Q_\alpha^+$, where, for $\alpha = 0, 1$, Q_α^+ is holomorphic on A_α, Q_α^- is holomorphic on D_α, and both are smooth on γ_α. Then extend $E \mid R^\gamma$ to a holomorphic bundle E on R_V by using Q_0^+ and Q_1^+ to attach the trivial bundles $R_0 \times \mathbb{C}^n$ and $R_1 \times \mathbb{C}^n$ to $E \mid R^\gamma$, where $R_0 = D_0 \cup \gamma_0 \cup A_0$ and $R_1 = D_1 \cup \gamma_1 \cup A_1$. In other words, let s^0 and s^1 be the standard frames for the trivial bundles on R_0 and R_1 and make the identifications

$$k^0 = s^0 Q_0^+ \qquad k^1 = s^1 Q_1^+ \tag{6.17}$$

Finally, we define e^0 and e^1 by

$$s^0 = e^0 Q_0^- \qquad s^1 = e^1 Q_1^- \tag{6.18}$$

on $D_0 \cup \gamma_0$ and $D_1 \cup \gamma_1$ respectively.

We can now define the action of $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ by

$$Q_0 \mapsto gQ_0 \qquad Q_1 \mapsto (g^{-1})^T Q_1 \tag{6.19}$$

which is equivalent to (6.15) when g has an analytic continuation to the appropriate neighbourhood of γ.

6.3. Orbits of $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ and K^γ

If $g: \gamma \to \mathrm{GL}(n, \mathbb{C})$ is the boundary value of a holomorphic map $D \to \mathrm{GL}(n, \mathbb{C})$, then g leaves E invariant but changes e^0 and e^1; such transformations are 'pure gauge'.

The restricted bundle $E \mid R^\gamma$ is unchanged by the action of any element of $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$. Also, if $g \in \mathrm{K}^\gamma$, and if (E, e^0, e^1) is real and symmetric†, then $g(E, e^0, e^1)$ is also real and symmetric; and $\det(g(E, e^0, e^1)) = \det(E, e^0, e^1)$.

Suppose, conversely, that we have two real symmetric triples (E, e^0, e^1) and (E', e'^0, e'^1) such that

$$E \mid R^\gamma = E' \mid R^\gamma \qquad \det(E, e^0, e^1) = \det(E', e'^0, e'^1) \tag{6.20}$$

by isomorphisms that respect the actions of i and j and which are compatible with each other in the sense that the nth exterior power of the first isomorphism is the same as the restriction of the second to R^γ. Then we can find an element of $\mathrm{K}^\gamma \subset \mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ which maps (E, e^0, e^1) onto (E', e'^0, e'^1)—since under these conditions we have $Q_0' Q_0^{-1} = (Q_1, Q_1'^{-1})^T$, with both sides real and taking values in $\mathrm{SL}(n, \mathbb{C})$.

Let $Y(n, V, \gamma)$ denote the subset of $T(n, V, \gamma)$ of real symmetric triples. Then we have the following.

Proposition 6.1. Suppose that $V \cup \hat{V} \subset U$ where U is the open set bounded by a simple closed curve γ. Then the loop group $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ has a natural action on $T(n, V, \gamma)$. The subgroup K^γ maps $Y(n, V, \gamma) \subset T(n, V, \gamma)$ to itself. The orbits of K^γ in $Y(n, V, \gamma)$ are labelled by the invariants $E \mid R^\gamma$ and $\det(E, e^0, e^1)$.

Remarks. (1) Suppose that (E, e^0, e^1) has patching data (P, F, G) relative to the standard cover introduced in §5, that the set U of §5 is the same as the U introduced above and that g has an analytic continuation to A. Then $g(E, e^0, e^1)$ has patching data $(P, gF, (g^T)^{-1}G)$.

(2) If g has an analytic continuation to a neighbourhood of γ in $\mathbb{C}$, and if γ' is some other simple closed curve which surrounds V and $\hat{V}$ and which is contained in the neighbourhood, then we can continue g from γ to obtain an element g' of $\mathrm{L}_{\gamma'}\mathrm{GL}(n, \mathbb{C})$. It follows from (6.15) that the actions of g and g' coincide on triples in their common domains.

† That is $(E, e^0, e^1)^* = i^*(E, e^0, e^1)$ and $(\bar{E}, \bar{e}^0, \bar{e}^1) = j^*(E, e^0, e^1)$.

6.4. Identification of K^γ with K

Suppose that we are given a solution $J(z, r)$ of Yang's equation on V. Then we have a (unique) corresponding framed bundle (E, f^0, f^1) over R_V. We can extend f^0 and f^1 to holomorphic frames on $D_0 \cup \gamma_0$ and $D_1 \cup \gamma_1$ (which, as we have remarked, is equivalent in the spacetime case to Kinnersley and Chitre's procedure of choosing a generating function $F_{AB}(t)$) and we can make $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ act on the resulting triple (E, e^0, e^1) to generate new triples and hence new solutions of Yang's equation. We claim that when $n=2$, the action of the subgroup K^γ coincides at the infinitesimal level with the action of the loop algebra k. This is an almost immediate consequence of the following propositions, the first of which we have already noted in §4.

Proposition 6.2. The constant loops $g \in \mathrm{GL}(2, \mathbb{C})$ act on J by $J \mapsto gJg^T$.

Proposition 6.3. The element

$$g_\iota = \begin{pmatrix} 0 & 1 \\ (2w)^{-1} & 0 \end{pmatrix} \tag{6.21}$$

of $\mathrm{L}_\gamma\mathrm{GL}(2, \mathbb{C})$ generates the discrete symmetry ι.

Proof. Let $J(z, r)$ be a $\mathrm{GL}(2, \mathbb{C})$-valued solution on V, given by (6.3), and let (E, f^0, f^1) be the corresponding framed bundle over R_V.

Choose $(E, e^0, e^1) \in T(2, V, \gamma)$ such that $e^0 = f^0$ at Λ_0 and $e^1 = f^1$ at Λ_1 and put $(E', e'^0, e'^1) = g_\iota(E, e^0, e^1)$. Then, if (E, e^0, e^1) has patching matrices $P_{\alpha\beta}$, (E', e'^0, e'^1) has patching matrices $P'_{\alpha\beta}$ where, for $\alpha, \beta, \ldots = 2, 3, \ldots$,

$$P'_{0\alpha} = \begin{pmatrix} 0 & 1 \\ (2w)^{-1} & 0 \end{pmatrix} P_{0\alpha} \qquad P'_{1\alpha} = \begin{pmatrix} 0 & 1 \\ 2w & 0 \end{pmatrix} P_{1\alpha} \qquad P'_{\alpha\beta} = P_{\alpha\beta}. \tag{6.22}$$

Here we are using the frames e^0 and e^1 to trivialise E over D_0 and D_1 and we have chosen the cover $\{D_\alpha\}$ of R_V so that Λ_0 lies only in D_0 and Λ_1 lies only in D_1.

Let $x \in \mathbb{E}_V$ be a point with coordinates (t, r, θ, z) and let $K_\alpha(\lambda)$ be splitting matrices for the pull-back of E to X, as in (3.20); these are determined by z, r, and the $P_{\alpha\beta}$ up to

$$K_\alpha \mapsto K_\alpha h(z, r) \tag{6.23}$$

where $h: V \to \mathrm{GL}(n, \mathbb{C})$. Put $H(z, r) = K_0(0)$ and $\hat{H}(z, r) = K_1(\infty)$.

We can fix h by imposing

$$H = \begin{pmatrix} 1 & \cdot \\ 0 & \cdot \end{pmatrix} \qquad \hat{H} = \begin{pmatrix} \cdot & 0 \\ \cdot & 1 \end{pmatrix}. \tag{6.24}$$

Then, on putting $J(z, r) = H\hat{H}^{-1}$ and recalling (6.3), we see that

$$\begin{aligned} K_0 &= \begin{pmatrix} 1 + \mathrm{O}(\lambda) & -\omega f + \mathrm{O}(\lambda) \\ -r^{-1}\lambda\varphi + \mathrm{O}(\lambda^2) & f + \mathrm{O}(\lambda) \end{pmatrix} \\ K_1 &= \begin{pmatrix} \hat{f} + \mathrm{O}(\lambda^{-1}) & -(r\lambda)^{-1}\hat{\varphi} + \mathrm{O}(\lambda^{-2}) \\ \hat{\omega}\hat{f} + \mathrm{O}(\lambda^{-1}) & 1 + \mathrm{O}(\lambda^{-1}) \end{pmatrix} \end{aligned} \tag{6.25}$$

as $\lambda \to 0$ and $\lambda \to \infty$ respectively, where φ and $\hat{\varphi}$ are functions of z and r.

Now we have two expressions for the matrix-valued form Ψ on $\mathbb{PT}_V$: equations (3.4) and (3.21). These must agree for all values of λ. By taking $\alpha = 0$ in (3.21) and comparing the 2,1-entries at the first order in λ, we find that

$$\varphi = \psi + \text{constant} \tag{6.26}$$

(by using $\bar{\partial}(\lambda/r) = O(\lambda^2)$); and by taking $\alpha = 1$ and comparing the 1,2-entries at the first order in $1/\lambda$, we find

$$\hat{\varphi} = \hat{\psi} + \text{constant}. \tag{6.27}$$

Since K_0 and K_1 have the special forms (6.25),

$$\begin{aligned} K_0' &= \begin{pmatrix} 0 & 1 \\ (2w)^{-1} & 0 \end{pmatrix} K_0 \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \qquad K_1' = \begin{pmatrix} 0 & 1 \\ 2w & 0 \end{pmatrix} K_1 \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \\ K_\alpha' &= K_\alpha \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{6.28}$$

split the patching matrices of the pull-back of E' to X. The only potential awkwardness arises at $\lambda = 0$ and $\lambda = \infty$; but K_0' and K_1' are non-singular there since $w \to \infty$ as $\lambda \to 0$ and as $\lambda \to \infty$.

Now at $\lambda = 0$, $2\lambda w = r$, and at $\lambda = \infty$, $2w/\lambda = -r$. Therefore the solution generated by (E', e'^0, e'^1) is

$$J'(z, r) = K_0'(0)K_1'(\infty)^{-1} = -\frac{1}{r^2 \hat{f}} \begin{pmatrix} \varphi\hat{\varphi} - r^2 f\hat{f} & -\varphi \\ -\hat{\varphi} & 1 \end{pmatrix}. \tag{6.29}$$

Apart from the constants in (6.26) and (6.27), therefore, $J' = \iota(J)$. We can fix the constants by replacing e^0 and e^1 by

$$e^0 \begin{pmatrix} 1 & 0 \\ \dfrac{a}{2w} & 1 \end{pmatrix} \quad \text{and} \quad e^1 \begin{pmatrix} 1 & \dfrac{-\hat{a}}{2w} \\ 0 & 1 \end{pmatrix} \tag{6.30}$$

($a, \hat{a} \in \mathbb{C}$). This leaves (E, f^0, f^1), and hence J unchanged, but adds a to φ and $\hat{a}$ to $\hat{\varphi}$.

Remarks. (1) Note that the Ernst potential is encoded in the first order terms in the expansions of e^0 and e^1 about Λ_0 and Λ_1, as we anticipated. When $g \in \mathrm{K}^\gamma$ acts on (E, e^0, e^1), it generates not only a new solution $J'(z, r)$, but also a new chain of potentials.

(2) Note that g_ι is not an element of K^γ: although it is real (in our usual sense that $g_\iota(w) = \bar{g}_\iota(w)$), it does not take values in $SL(2, \mathbb{C})$.

Proposition 6.4. The actions of the two one-parameter subgroups of K^γ

$$g^{(1)}(s) = \begin{pmatrix} 1 & 2ws \\ 0 & 1 \end{pmatrix} \qquad g^{(-1)}(s) = \begin{pmatrix} 1 & 0 \\ s(2w)^{-1} & 1 \end{pmatrix} \tag{6.31}$$

(where $s \in \mathbb{R}$ is the parameter) on $J(z, r)$ and its potentials transform the Ernst potential according to

$$\begin{array}{ll} (1) & \iota(J) \mapsto \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \iota(J) \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \\ (-1) & \iota(J) \mapsto \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \iota(J) \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \end{array} \tag{6.32}$$

Proof. The proposition is an immediate consequence of

$$\begin{aligned} g_\iota g^{(1)}(s) g_\iota^{-1} &= \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \\ g_\iota g^{(-1)}(s) g_\iota^{-1} &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{6.33}$$

Remarks. (1) Since $g^{(-1)}$ is the boundary value of a holomorphic map that extends across $w = \infty$, it generates 'gauge' transformations: it has no effect on J itself, but adds constants to ψ and $\hat{\psi}$. The subgroup $g^{(1)}$, on the other hand, does change J. It is the group of *gravitational duality rotations*, or *Ehlers transformations.*

(2) It is clear from the proofs of the propositions that an element of $\mathrm{L}_\gamma\mathrm{GL}(n, \mathbb{C})$ which can be expressed as a finite product of constant loops, KN transformations and inverse KN transformations (that is, any loop given by a Laurent polynomial) will preserve the triviality condition (T) and hence the domain of the solution.

If we identify the variable t in (6.1) with $2w$, then $g^{(1)}$ becomes the one-parameter subgroup generated by the element $\gamma(t)$ of k defined in (6.8); and $g^{(-1)}$ becomes the one-parameter subgroup generated by the $\gamma(t)$ in (6.11). It follows that K^γ has the same action at the infinitesimal level as that already defined for the loop algebra k, and hence that we can identify the 'group K' with K^γ.

Examples. (1) An element g of K^γ itself determines an (axis-simple) bundle on R_V (by applying g to the trivial bundle), and hence a solution J_0 of Yang's equation. When g is diagonal, $J_0 = \mathrm{diag}(u^{-1}, u)$ where, at worst, u or u^{-1} has a singularity of the form r^{-2k} as $r \to 0$ for some $k \in \mathbb{Z}^+$. The effect of g on the Weyl solutions is

$$\begin{pmatrix} -r^2 e^{-\chi} & 0 \\ 0 & e^{\chi} \end{pmatrix} \mapsto \begin{pmatrix} -r^2 u^{-1} e^{-\chi} & 0 \\ 0 & u e^{\chi} \end{pmatrix} \tag{6.34}$$

(see (5) in the list in §4). The orbit of $J = \mathrm{diag}(-r^2 e^{-\chi}, e^{\chi})$ under such transformations consists of Weyl solutions $J' = (-r^2 e^{-\chi}, e^{\chi})$ such that $\chi' - \chi$ behaves like $2k \log r$ as $r \to 0$ for some $k \in \mathbb{Z}$. Very roughly, the space of orbits of Weyl solutions is parametrised by one real function of one variable (see (4)).

(2) The KN transformation takes axis-simple bundles to axis-simple bundles. It changes (p, q) to $(p-1, q)$ and replaces P by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.35}$$

(3) The action of K^γ leaves $E \mid R^\gamma$ unchanged. An obvious way to construct orbits for which $E \mid R^\gamma$ is not trivial is to use patching data (P, F, G) for which P

extends to U, but is not real on the real axis. This gives a class of orbits parametrised (again very roughly) by two free real functions of one variable: the eigenvalues for real w of $\mathrm{Im}(P(w))$ with respect to $\mathrm{Re}(P(w))$. The condition $\det(J) = -r^2$ cuts this down to one real function.

(4) Suppose that $n = 1$ and that $J = e^{\chi}$ is generated by (P, F, G) where $P(z)\bar{P}(z)^{-1} = e^{2i\pi\alpha(z)}$ for real z. Then as $r \to 0$ for fixed z, J behaves like $r^{2(\alpha(z)+k)}$ for some integer k. The function $\alpha(z)$ (mod $\mathbb{Z}$) characterises the orbit of J.

7. Behaviour on the axis and at infinity

In a spacetime which is regular on the symmetry axis $r = 0$, we must have $\det(J) = -r^2$ and

$$J_{11} = g_{\theta\theta} = \mathrm{O}(r^2) \qquad J_{22} = g_{tt} = \mathrm{O}(1) \tag{7.1}$$

as $r \to 0$. These certainly hold if J is constructed from an axis-simple bundle over R_V with $p = 1$, $q = 0$ (§5), in which case the Ernst potential

$$\iota(J) = \begin{pmatrix} f^{-1}\psi^2 + f & -f^{-1}\psi \\ -f^{-1}\psi & f^{-1} \end{pmatrix} \tag{7.2}$$

corresponds to an axis-simple bundle with $p = q = 0$—an observation that motivates the following definition.

7.1. Axis-regular Ernst potentials

Let $n = 2$ and let J be a positive-definite real symmetric solution of Yang's equation on an open set $V \subset H$, with $\det(J) = 1$. Let (E, f^0, f^1) be the corresponding framed bundle. Suppose that $V = H \cap U$, where $U \subset S$ is a connected open set invariant under $w \mapsto \bar{w}$. Let I be the intersection of U with the real axis and let $\eta: R_V \to R_U$ be the projection (recall that R_U is defined by identifying the copies of U in the two Riemann spheres S_0 and S_1).

Definition. J is an *axis-regular Ernst potential* if $(E, f^0, f^1) = \eta^*(E', f'^0, f'^1)$ where (E', f'^0, f'^1) is a framed bundle over R_U such that $E' \mid S_0$ and $E' \mid S_1$ are trivial.

We shall also say that J is an axis-regular spacetime when $\iota(J)$ is an axis-regular Ernst potential.

The patching data for an axis-regular Ernst potential J can be put (uniquely) into the form (P, F, G) where $P: U \to \mathrm{SL}(2, \mathbb{C})$ is real (in the usual sense that $P(\bar{w}) = \overline{P(w)}$), symmetric, and holomorphic, $F = 1$, and $G = (P^{-1})^T$. We shall call P the patching function of J.

The frames f'^0 and f'^1 extend uniquely to holomorphic frames for E' on $S_0 \subset R_U$ and $S_1 \subset R_U$; and P is also equal to the corresponding patching matrix $P_{01}: S_0 \cap S_1 = U \to \mathrm{SL}(2, \mathbb{C})$.

Proposition 7.1. Let J be a positive-definite real symmetric solution on V with $\det(J) = 1$. Suppose that $J(z, r)$ has an extension to U which can be written as a smooth function of z and r^2. Then (1) J is an axis-regular Ernst potential, (2) J is analytic on U and (3) $J(z, 0) = P(z)$ for every $z \in I$, where P is the patching function of J.

Remarks. We shall prove this by taking 'smooth' to mean infinitely differentiable. Then the hypothesis is simply that J has a smooth extension from V to $V \cup I$ (since if J is non-singular and smooth (as a function of z and r) at $r=0$, then Yang's equation implies that all the odd partial derivatives of J with respect to r vanish at $r=0$; so if we extend J to U by taking $J(z,-r)=J(z,r)$, then J will be a smooth function of z and r^2). The step up from smoothness to analyticity (proved originally in a different way by Hauser and Ernst (1981)) depends, however, on the Newlander–Nirenberg theorem, which is true under much less stringent conditions than infinite differentiability (see Kobayashi and Nomizu (1969, appendix 8)). A close examination of the way the theorem is used (implicitly) in the following shows that smoothness can be taken to mean C^3.

The fact that axis values of the Ernst potential determine the solution is also not new: it is, for example, a special case of Ward's (1983) asymptotic formula which we stated in §5.

Proof. The proof makes use of the geometry of $\mathbb{PT}$. Let $\mathbb{E}_U$, $\mathbb{PT}_U$, and R_U be as in §2. Then $R_U = D/F$, where $D = \mathbb{PT}_U \backslash (L_0 \cup L_1)$; and $R_U = S_0 \cup S_1$, where the spheres S_0 and S_1 are the sets of leaves of F in D whose closures in $\mathbb{PT}_U$ intersect, respectively, L_0 and L_1. Let $\pi : \mathbb{PT}_U \to R_U$ denote the projection and let $\mathbb{P}_0 = \pi^{-1}(S_0)$ and $\mathbb{P}_1 = \pi^{-1}(S_1)$.

Suppose that J satisfies the hypothesis of the proposition and let (E, f^0, f^1) be the corresponding framed bundle on R_V. Then E is the reduction of the trivial C^∞ bundle $B = \mathbb{PT}_V \times \mathbb{C}^2$ with the holomorphic structure Ψ, where Ψ is the $(0,1)$ part of

$$\rho^*(J^{-1}\,\partial_{\bar{p}} J\,\mathrm{d}p + J^{-1}\,\partial_{\bar{q}} J\,\mathrm{d}\bar{q}) \tag{7.3}$$

(take $H=J$, $\hat{H}=1$ in (3.3)). But J is smooth on $\mathbb{E}_U$ since r^2 (unlike r) is a smooth function on $\mathbb{E}_U$. Hence Ψ extends smoothly to $\mathbb{PT}_U$ and so B extends as a holomorphic bundle to $\mathbb{PT}_U$.

To show that J is axis regular, it is enough to show that there exist smooth maps $K_\alpha : \mathbb{P}_\alpha \to \mathrm{GL}(2,\mathbb{C})$ $(\alpha = 0, 1)$ such that on $\mathbb{P}_\alpha$,

$$\Psi = K_\alpha^{-1}\,\bar{\partial} K_\alpha \qquad \Theta(K_\alpha) = 0 = T(K_\alpha) \tag{7.4}$$

since $P = K_0 K_1^{-1}$ then provides a patching function for the required bundle over R_U (the condition on the determinant bundle will follow from $\det(J)=1$).

Put $M_0 = L_0 \cap \mathbb{PT}_U$ and $M_1 = L_1 \cap \mathbb{PT}_U$. We shall, in fact, define K_0 and K_1 on $\mathbb{P}_0 \cup M_0$ and $\mathbb{P}_2 \cup \mathrm{M}_1$.

We choose a point on M_0 and pick a Stein neighbourhood $N_0 \subset \mathbb{PT}_U$ of the point which is invariant under the S^1 action generated by Θ. Since $\pounds_T \Psi = 0$ and since T does not vanish on N_0, it is possible to find $K_0 : N_0 \to \mathrm{GL}(2,\mathbb{C})$ such that $\Psi = K_0^{-1}\,\bar{\partial} K_0$ and $T(K_0) = 0$ on N_0. The problem is the condition $\Theta(K_0) = 0$. We can arrange that that also holds, however, by averaging an initial choice of K_0 over the S^1 action: the resulting matrix will have non-vanishing determinant on $M_0 \cap N_0$ (which is fixed by the S^1 action), and hence on N_0 (since the determinant is holomorphic and constant along Θ and T). Finally, we can use the conditions $\Theta(K_0) = 0 = T(K_0)$ to extend K_0 from N^0 to $\mathbb{P}_0$. We construct K_1 in the same way.

The second and third statements can be now be deduced from the following converse proposition.

Proposition 7.2. Let P be the patching function of an axis-regular Ernst potential J on V. Then J is analytic on U and $J(z, 0) = P(z)$ for real z.

Proof. With the same notation as above: let (E', f'^0, f'^1) be the framed bundle on R_U which generates J and let B be the pull-back of E' to D. Then B is trivial as a holomorphic bundle on $\mathbb{P}_0$ and $\mathbb{P}_1$ and has patching matrix $P \circ \pi$ on $\mathbb{P}_0 \cap \mathbb{P}_1$. Thus we can extend B to $\mathbb{PT}_U$ by taking B to be trivial over $\mathbb{P}_0 \cup M_0$ and $\mathbb{P}_1 \cup M_1$, with the same patching matrix on $(\mathbb{P}_0 \cup M_0) \cap (\mathbb{P}_1 \cup M_1) = \mathbb{P}_0 \cap \mathbb{P}_1$.

Then J is obtained from $B \to \mathbb{PT}_U$, and its patching matrix $P \circ \pi$, by the standard splitting procedure; J is analytic since it can be continued to complex values of z and r by taking X to be a general line in $\mathbb{PT}_U$ (as opposed to a fibre of $\mathbb{PT}_U$); and $J(z, 0) = P(z)$ for $z \in I$ since if X is the fibre above a point with coordinate z on the symmetry axis in $\mathbb{E}_U$, then the patching matrix for $B|_X$ is the constant matrix $P(z)$.

Remarks. (1) If γ is the boundary of U in the complex w plane, then K^γ acts transitively on axis-regular Ernst potentials on U which are smooth on the boundary of U. For if J (with $n = 2$) has patching data $(P, 1, (P^{-1})^T)$, where $P(w)$ is real, symmetric and positive definite for real w, and if $g : \gamma \to \mathrm{SL}(2, \mathbb{C})$ is an element of K^γ equal to the boundary value of a holomorphic map on U, then $g(P, 1, (P^{-1})^T) = (P, g, (P^{-1}g^{-1})^T)$, which is equivalent to $(P', 1, G')$ where $P' = gPg^T$ and $G' = (P'^{-1})^T$. But for any holomorphic map $P : U \to \mathrm{SL}(2, \mathbb{C})$, which is real symmetric and positive definite on the real axis, we can find a real holomorphic $g : U \to \mathrm{SL}(2, \mathbb{C})$ such that $gPg^T = 1$. Thus J can be mapped to $J = 1$ by an element of K^γ. this is effectively the form of Geroch's (1972) conjecture proved by Hauser and Ernst (1981).

(2) By using (3.12), we can calculate the patching matrix of an axis-regular Ernst potential by integration in the (z, r) plane, as follows. We fix a point $a \in I$, choose a value of w and choose a contour $\gamma : [0, 1] \to V \cup I$, with $\gamma(0) = \gamma(1) = a$, which winds once around the point w. We define $\lambda(t)$ on γ by choosing the root continuously in (3.14), with $\lambda \to 0$ as $t \to 0$ (then $\lambda \to \infty$ as $t \to 1$) and let $S_w(t) \in \mathrm{GL}(2, \mathbb{C})$ be the solution along γ of the parallel transport equation

$$\frac{\mathrm{d}S_w}{\mathrm{d}t} = \frac{1}{1 + \lambda^2}\, \dot{\gamma}(t) \lrcorner (\lambda J^{-1}\, \mathrm{d}^*J - J^{-1}\, \mathrm{d}J) S_w \tag{7.5}$$

with $S_w(0) = J^{-1}$. Here $\dot{\gamma}(t)$ is the tangent to γ, $\mathrm{d}J = (\partial_r J)\, \mathrm{d}r + (\partial_z J)\, \mathrm{d}z$ and $\mathrm{d}^*J = (\partial_z J)\, \mathrm{d}r - (\partial_r J)\, \mathrm{d}z$. Then $P(w) = S_w(1)^{-1}$, independently of a and contour.

7.2. Asymptotic flatness

Let R and φ be the polar coordinates near $w = \infty$ defined by $r = R^{-1} \sin \varphi$, $z = R^{-1} \cos \varphi$. Cosgrove (1980) defines a stationary axisymmetric spacetime to be asymptotically flat if its Ernst potential has an appropriate expansion in powers of R as $R \to 0$. It is unclear exactly how this relates to the standard definition in terms of the existence of conformal null infinity, but Cosgrove's definition is certainly wider in some respects, in that it is satisfied by the NUT metric. We shall adopt a definition which, although apparently weaker, is in fact equivalent to Cosgrove's.

Definition. Let J be a solution defined on $V \subset H \subset S$ where the closure of V in S intersects the real great circle in a closed interval $\bar{K}$, where K is an open interval containing the point $w = \infty$. We shall say that J is an asymptotically flat Ernst potential if it is axis regular and

(1) J has a limit at $w = \infty$ and

(2) J is a C^1 function of R and φ on $R = 0$.

We want to show that this is equivalent to the condition that J can be obtained by pulling back to R_V a holomorphic bundle over R_W, where $W = V \cup \hat{V} \cup K$. At first sight, this looks exactly like the situation that we have already considered; but it is in fact crucially different because W contains the point at infinity; and most of the previous argument breaks down if $\mathbb{PT}_U$ is allowed to contain points on the line at infinity in $\mathbb{PT}$.

So we must proceed in a slightly different way. Let (E, f^0, f^1) be the framed bundle on R_V corresponding to J. We write $K = I_+ \cup \{\infty\} \cup I_-$ where I_+ and I_- are unbounded open intervals in the positive and negative real axes (excluding $w = \infty$) and put $U_+ = V \cup I_+ \cup \hat{V}$, $U_- = V \cup I_- \cup \hat{V}$, and $W' = W\backslash\{\infty\} = U_+ \cup U_-$. We shall show that E is the pull-back of a bundle on R_W by first using axis regularity to show that it is the pull-back of a bundle on $R_{W'}$.

Consider $\Gamma^{-1}(W') \subset R_V$. This consists of two copies of W', which we shall denote W_0' and W_1', which are identified at corresponding values w, except when w is real.

Now W' is a Stein manifold (it is a neighbourhood of $w = \infty$, except for the point $w = \infty$ itself), and so $E \mid W_0'$ and $E \mid W_1'$ are both trivial. Thus the restriction of E to $\Gamma^{-1}(W') \subset R_V$ is characterised by a single patching matrix $Q: W_0' \cap W_1' = W'\backslash K \rightarrow \mathrm{SL}(2, \mathbb{C})$. By taking U and I in the discussion above to be first U_+ and I_+, and then U_- and I_-, we can deduce first that Q extends holomorphically across I_+ and then that it also extends across I_-. It follows that $E \mid W'$ is trivial, and hence that E is the pull-back of a bundle $E' \rightarrow R_{W'}$. Moreover, E' must be trivial on restriction to S_0 and S_1.

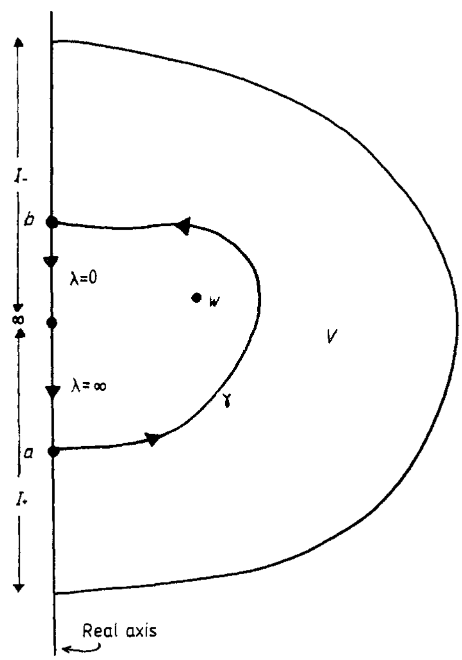

Figure 8. The contour in the zr plane.

So we can construct $E' \to R_{W'}$ from a single patching matrix $P: W' \to \mathrm{SL}(2, \mathbb{C})$, which is equal to J on I_+ and I_-.

We want to show that $P(w)$ is bounded, and hence holomorphic, at $w = \infty$. Because J is asymptotically flat, the R and φ components of $\mathrm{d}J$ and $\mathrm{d}^*J = -R(\partial_R J)\,\mathrm{d}\varphi + R^{-1}(\partial_\varphi J)\,\mathrm{d}R$ are bounded as $R \to 0$, and the φ components vanish at $R = 0$. We can therefore take part of the contour in (7.5) to be along the real axis through $w = \infty$, as shown in figure 8. The contribution from the real axis integral from b to a is then the same for all w (since $\lambda = 0$ or $\lambda = \infty$ on the real axis, whatever the value of w). Hence $P(w) = S_w(b)^{-1}A$, where A is a fixed element of $\mathrm{SL}(2, \mathbb{R})$. But as $w \to \infty$, the integral from a to b, and hence also $P(w)$, remains bounded. It follows that P extends to a holomorphic map $P: W \to \mathrm{SL}(2, \mathbb{C})$. Therefore E' is the pull-back of a bundle on R_W.

Conversely, if E' is the pull-back of a bundle on R_W, then P extends to $w = \infty$ and an examination of the limit of the splitting procedure shows that J is asymptotically flat.

Proposition 7.3. An axis-regular Ernst potential J is asymptotically flat if and only if it corresponds to a framed bundle (E, f^0, f^1) where E is the pull-back of a holomorphic bundle over R_W for some $W \subset S$ containing the point at infinity.

7.3. Transformations that preserve asymptotic flatness

Now let us return to the previous situation and suppose that we have an axis regular Ernst potential J defined on an open set $V \subset H$ where the closure of V contains an open interval I on the real axis. Let (E, f^0, f^1) be the corresponding framed bundle on R_V and put $U = V \cup \hat{V} \cup I$, as before. Then we know that $(E, f^0, f^1) = \eta^*(E', f'^0, f'^1)$, where (E', f'^0, f'^1) is a framed bundle on R_U, determined by a single patching function $P: U \to \mathrm{SL}(2, \mathbb{C})$.

Clearly J is asymptotically flat if and only if P can be continued analytically to $w = \infty$. This can be expressed in an invariant form. The symmetry condition on J implies that $E'^* = i^*(E')$ (§4); but on $\Gamma^{-1}(U)$, i acts as the identity. Hence $E'^* \mid \Gamma^{-1}(U) = E' \mid \Gamma^{-1}(U)$. This isomorphism determines a holomorphic bilinear form b on the fibres of E' over $\Gamma^{-1}(U)$.

The frame f^0 at Λ_0 extends uniquely to a global frame for $E' \mid S_0$ and the matrix of b in this frame is P; so, in particular, b is symmetric. Asymptotic flatness is simply the condition that b can be continued further to $\Gamma^{-1}(W)$ where $W \subset S$ contains U and $w = \infty$. Suppose that this condition holds and that $\gamma \subset W$. Suppose also that $J(\infty) = 1$, which does not involve any essential loss of generality. Then we can pick a triple $(E, e^0, e^1) \in T(2, V, \gamma)$ which generates J and which has the property that b has matrix 1 in the frame e^0. We shall say that such a triple is in *standard form.* (As a condition on patching data, this is $F = G$.)

Let (E, e^0, e^1) be in standard form and suppose that $g \in \mathrm{K}^\gamma$ takes values in $\mathrm{SO}(2, \mathbb{C})$. Let $(E_g, e_g^0, e_g^1) = g(E, e^0, e^1)$ and let b_g be the bilinear form on the fibres of E_g. Then b_g also extends to infinity since it is equal to b wherever it is defined on $E_g \mid R^\gamma = E \mid R^\gamma$; and b_g can be extended to infinity by taking its matrix to be 1 in the frame e^0.

We should like to deduce that (E_g, e_g^0, e_g^1) generates an asymptotically flat Ernst potential J_g. The problem is that it is not necessarily true that J_g is axis regular, although it must be axis simple. It is clear that E_g is the pull-back of a bundle on R_U, but not that that bundle has a trivial restriction to $S_0 \subset R_U$. For that, it is necessary that g should have a decomposition of the form $g_- g_+$ where g_- is holomorphic on the outside of γ in the w plane (including $w = \infty$) and g_+ is holomorphic inside γ. Now $\mathrm{SO}(2, \mathbb{C}) = \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$; and by applying Birkhoff's theorem to $\mathrm{GL}(1, \mathbb{C})$, we can deduce that g has such a splitting if and only if its winding number in $\mathbb{C}^*$ vanishes.

Hence the subgroup O^γ of $\mathrm{L}_\gamma \mathrm{SO}(2, \mathbb{C}) \cap \mathrm{K}^\gamma$ consisting of loops with zero winding number acts on triples in standard form in such a way as to preserve asymptotic flatness. On transferring the action of O^γ to the corresponding spacetimes by conjugating with g_ι, O^γ becomes Kinnersley and Chitre's (1978b) 'group B', which is generated by elements of k of the form

$$\gamma(t) = \begin{pmatrix} 0 & -t^{m+1} \\ t^{m-1} & 0 \end{pmatrix} \tag{7.6}$$

($m = 0, 1, 2, \ldots$).

Example. An asymptotically flat Weyl solution has patching function $P = \mathrm{diag}(f, f^{-1})$, where $f(\infty) = 1$. We choose γ so that f is holomorphic and non-vanishing outside γ. Then suitable patching data in standard form are

$$P = \begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix} \qquad F = \begin{pmatrix} f^{-1/2} & 0 \\ 0 & f^{1/2} \end{pmatrix} = G. \tag{7.7}$$

For $\varepsilon \ll 1$,

$$g = \begin{pmatrix} 1 & \varepsilon w \\ -\varepsilon w & 1 \end{pmatrix} \tag{7.8}$$

is an element of O^γ; it sends (P, F, G) to $(P, gF, (g^T)^{-1}G)$, which is equivalent to $(P, F^{-1}gF, F(g^T)^{-1}G)$ (see (5.6)).

To get this back to the Ward form, we must find the unique K and L which are holomorphic outside and inside γ, respectively, and which satisfy

$$F^{-1}gF = \begin{pmatrix} 1 & \varepsilon w f \\ -\varepsilon w f^{-1} & 1 \end{pmatrix} + \mathrm{O}(\varepsilon^2) = K^{-1}L \tag{7.9}$$

together with $K(\infty) = 1$. Then $(P, gF, (g^T)^{-1}G)$ is equivalent to $(P', 1, (P'^T)^{-1})$ where $P' = LPL^T$; and so the transformed Ernst potential has patching function LPL^T.

If we write $wf = w + c + g(w)$ and $w/f = w - c + \tilde{g}(w)$ where $g(\infty) = \tilde{g}(\infty) = 0$ and $c \in \mathbb{C}$, then to the first order in ε,

$$K = \begin{pmatrix} 1 & -\varepsilon g \\ -\varepsilon \tilde{g} & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & \varepsilon(w+c) \\ -\varepsilon(w-c) & 1 \end{pmatrix}. \tag{7.10}$$

So in the case of the Schwarzschild solution, for example, where $f = (w - m)/(w + m)$ and $c = -2m$,

$$LPL^T = \begin{pmatrix} \dfrac{w-m}{w+m} & \dfrac{4m^3\varepsilon}{w^2 - m^2} \\ \dfrac{4m^3\varepsilon}{w^2-m^2} & \dfrac{w+m}{w-m} \end{pmatrix} \tag{7.11}$$

which is the patching function of the Kerr solution with $a/m = -2m\varepsilon \ll 1$.

Finally, some remarks about the group of *HKX transformations* (Hoenselaers *et al* 1979). In a formal sense, these are generated by elements of k of the form

$$\gamma_s(t) = \sum_0^\infty s^k t^k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (1-st)^{-1} \\ 0 & 0 \end{pmatrix} \tag{7.12}$$

where s is a parameter. However, (7.12) is not a Laurent polynomial, and so is not, in fact, in k. But we can interpret the corresponding transformations to be elements of K^γ provided that we now take γ to be a small loop around $w = 1/2s$ (so that K^γ acts on triples (E, e^0, e^1) where e^0 and e^1 are holomorphic frames defined in neighbourhoods of the two copies of $1/2s$ in R_V). Such transformations can easily be seen to preserve axis regularity and asymptotic flatness.

Appendix 1. Non-Hausdorff complex manifolds

A non-Hausdorff complex manifold X is an object that satisfies the standard definition of a complex manifold in terms of the existence of an atlas with biholomorphic transition maps (e.g. Morrow and Kodaira 1971, p 7, Field 1982, vol I p 134), except that it is not required to satisfy the Hausdorff separation axiom. In principle, therefore, X can have an extremely complicated topological structure. The examples that we consider, however, fail to satisfy the Hausdorff axiom only in very simple ways: they all admit finite atlases.

A holomorphic vector bundle $E \to X$ is defined in the usual way in terms of holomorphic patching matrices (Morrow and Kodaira 1971, p 62). The total space of E can be Hausdorff even when X itself is non-Hausdorff.

Example 1. We construct X by taking two copies $\mathbb{C}_0$ and $\mathbb{C}_1$ of $\mathbb{C}$ and identifying $z \in \mathbb{C}_0$ with $z \in \mathbb{C}_1$ except when $z = 0$. Thus X is the Argand plane with an extra origin. Then

(1) $E = X \times \mathbb{C}$ is a trivial holomorphic vector bundle with a non-Hausdorff total space, but

(2) $E' = (\mathbb{C}_0 \times \mathbb{C}) \cup (\mathbb{C}_1 \times \mathbb{C})/\sim$, where $\sim$ is the equivalence relation

$$(z, w) \in \mathbb{C}_0 \times \mathbb{C} \sim \left(z, \frac{1}{z} w\right) \in \mathbb{C}_1 \times \mathbb{C} \qquad z \neq 0 \tag{A1.1}$$

is a non-trivial bundle, but has a total space which is Hausdorff.

The principal effect of dropping the Hausdorff axiom is that it is no longer true that X admits partitions of unity subordinate to arbitrary locally finite covers. Consequently many familiar results that equate holomorphic objects to smooth objects satisfying 'Cauchy–Riemann equations' are no longer valid. For example the Čech cohomology group $\mathrm{H}^1(X, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions, is not isomorphic in general to the first cohomology of the Dolbeault complex.

Example 2. For the manifold X in the first example, the smooth forms on X are the same as the smooth forms on $\mathbb{C}$ since a smooth form cannot take different values at the two copies of the origin. Thus the first cohomology of the Dolbeault complex vanishes.

However $f_{01} = z^{-1}$ is a representative cocycle relative to the cover $\{\mathbb{C}_0, \mathbb{C}_1\}$ of a non-zero element of $\mathrm{H}^1(X, \mathcal{O})$.

Remark. Bailey (1985) discusses the various cohomology groups on a non-Hausdorff manifold. He shows that if X is obtained by identifying the complements of a closed set F in two copies of a complex manifold M, then, for a 'well-behaved' sheaf $\mathcal{O}$,

$$H^1(X, \mathcal{O}) = H'(M, \mathcal{O}) \oplus H^1_F(M, \mathcal{O}) \tag{A1.2}$$

where the subscript denotes relative cohomology. This simple decomposition is no longer valid, however, when the construction of X involves cuts and twists, as in the case of the reduced twistor space R_V. There appears to be no straightforward 'classical' description of cohomology groups and vector bundles in terms of 'Hausdorff' objects in such cases.

Appendix 2. Comparison with other approaches

Our approach to the transformation theory in §6 was similar in spirit to Geroch's original treatment (Geroch 1972). We built up the 'group K' by passing back and forth between the metric J matrix and the Ernst potential $\iota(J)$, making an $\mathrm{SL}(2, \mathbb{R})$ transformation at each stage.

More recent treatments, however, make use of the fact that Yang's equation is the integrability condition for various linear systems, all of which are equivalent to (3.13). We shall explain how some of these other approaches—particularly that of Hauser and Ernst (1980)—relate to twistor theory.

The key is the following. Suppose that $K_\alpha(z, r, \lambda)$ $(\alpha = 0, 1, \ldots)$ are the splitting matrices in (3.20). The two matrices K_0 and K_1 are determined by (E, e^0, e^1) up to $K_0 \mapsto K_0 C$, $K_1 \mapsto K_1 C$, where $C \in \mathrm{GL}(2, \mathbb{C})$ is a function only of z and r. By choosing C appropriately, we can impose $K_1 = 1$ at $\lambda = \infty$. Then the columns of K_0^{-1} and K_1^{-1} are solutions of our linear system (3.13), and

$$F = (F_{AB}(z, r, t)) = \frac{\lambda^2}{1+\lambda^2}\left[1 - \frac{\mathrm{i}}{\lambda r} J\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] K_0^{-1} \tag{A2.1}$$

(where $t = (2w)^{-1}$ and λ is expressed as a function of z, r, and w by (3.14)) is a generating function in the sense of Kinnersley and Chitre (1978b). This is the basis of our remark in §6 that a choice of a holomorphic frame e^0 is the same as the choice of a generating function.

A solution of the linear system of Kramer and Neugebauer (1984) is given by

$$\Phi = \begin{pmatrix} rf^{-1} & \omega \\ 0 & 1 \end{pmatrix} K_0^{-1}. \tag{A2.2}$$

The transformations to the other linear systems then follow from the work of Cosgrove (1980), Kramer and Neugebauer (1984), and Breitenlohner and Maison (1986). The twistor linear system is distinguished by the fact that it is well behaved for all (z, r, λ) except $(z, 0, 0)$ and $(z, 0, \infty)$.

Hauser and Ernst (1980) (see also Hauser 1984) reduce the problem of finding the action of an element of K to the following steps. We choose a generating function (F potential) for the initial solution J, subject to certain conditions. We construct from F and $g \in \mathrm{K}^{\gamma}$ a function $Z(z, r, w)$ which for each z and r is holomorphic in w in an annular region in the w plane. Then we solve a Riemann–Hilbert ('splitting') problem of the form $Z = BA^{-1}$, where B extends as a holomorphic function to the interior of the annulus, and A to the exterior (including $w = \infty$), and we finally write down the new solution J' in terms of F and B.

Their technique can be understood in terms of twistor theory as follows. We shall work in the 'Ernst picture'. In other words, we shall start with a positive definite real symmetric solution $J(z, r)$ such that $\det(J) = 1$.

Suppose that J is generated by a triple (E, e^0, e^1) with patching matrices $P_{\alpha\beta}(w)$ (obtained from patching data (P, F, G)). We choose $H(z, r)$ such that $\det(H) = 1$ and $J = HH^T$. We fix (z, r) and let $K_\alpha(\lambda)$ be the splitting matrices on the corresponding Riemann sphere X.

By making a different choice of C to the one above, we can ensure that $K_0(0) = H$. Then K_0 and K_1 are unique up to right multiplication by a constant element of $\mathrm{SO}(2, \mathbb{R})$. Moreover

$$K_0(\lambda)K_1(-\lambda^{-1})^T = 1 \qquad K_0(\bar{\lambda}) = \overline{K_0(\lambda)} \tag{A2.3}$$

and $S = (K_0H^T)^{-1}$ is a matrix-valued solution of the linear system (3.13).

Now consider the effect on J of $g(w) \in \mathrm{K}^{\gamma}$ where g has an analytic continuation to the interior of γ in the w plane (there is little loss of generality here since we can almost always reduce g to this form by combining it with a transformation of e^0). The new solution is obtained by first finding splitting matrices K'_α for $Q'_{\alpha\beta} = P'_{\alpha\beta} \circ \pi$, where $P'_{\alpha\beta}$ are the new patching matrices given by (6.15). We shall also impose (A2.3) on the new splitting matrices.

To get to Hauser and Ernst's formulation, we make the ansatz $K'_0 = K_0M_0$, $K'_1 = K_1M_1$, $K'_\alpha = K_\alpha M$ $(\alpha = 2, 3, \ldots)$, where M_0, M_1, and M, have unit determinant, are real (in the sense that $M(\bar{\lambda}) = \overline{M(\lambda)}$) and satisfy

$$M_0(\lambda)M_1(-\lambda^{-1})^T = 1 = M(\lambda)M(-\lambda^{-1})^T \tag{A2.4}$$

and

$$M_0M^{-1} = K_0^{-1}g(w)K_0 \qquad M_1M^{-1} = K_1^{-1}(g(w)^T)^{-1}K_1 \tag{A2.5}$$

where w is expressed as a function of λ through (3.14). Thus (M_0, M_1, M) is the solution of the following rather complicated Riemann–Hilbert problem: given K_0, K_1 and g, find real $M_0(\lambda)$ (holomorphic in a neighbourhood of $\lambda = 0$), $M_1(\lambda)$ (holomorphic in a neighbourhood of $\lambda = \infty$) and $M(\lambda)$ (holomorphic away from $\lambda = 0$ and $\lambda = \infty$) such that (A2.4) and (A2.5) hold. If a solution exists, then it is unique up to right multiplication by a constant element of $\mathrm{SO}(2, \mathbb{R})$.

There are really only two unknowns here since M_1 is determined by M_0 through (A2.4) and, in fact, we can reduce the problem to a standard Riemann–Hilbert problem by putting

$$\begin{gathered} M = \Lambda^{-1}A(w)\Lambda \qquad M_0 = \Lambda^{-1}B(w)\Lambda \\ Z(w) = \Lambda K_0^{-1}gK_0\Lambda^{-1} = [\Lambda H^TS]g[\Lambda H^TS]^{-1} \end{gathered} \tag{A2.6}$$

where

$$\Lambda = \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix}. \tag{A2.7}$$

The problem now becomes one of finding $B(w)$ (holomorphic in a neighbourhood of $w=\infty$ and real for real w) and $A(w)$ (holomorphic away from $w=\infty$ and real for real w) such that $BA^{-1}=Z$. We can then recover M_0, M_1 and M from (A2.6) and (A2.4); moreover M will be non-singular at $\lambda=\pm\mathrm{i}$ provided we impose $A(z+\mathrm{i}r)=1$, which fixes A and B uniquely. The new Ernst potential J' is given by

$$\begin{aligned} J'(z,r) &= K_0(0)M_0(0)M_1(\infty)^{-1}K_1(\infty)^{-1} \\ &= HB(0)B(0)^T H^T. \end{aligned} \tag{A2.8}$$

To summarise, to obtain J' from J and g, we first choose an $\mathrm{SL}(2,\mathbb{R})$-valued solution $S(z,r,\lambda)$ of (3.13) (in some neighbourhood of $\lambda=0$) such that $S(z,r,\bar{\lambda})=\overline{S(z,r,\lambda)}$ and $S(z,r,0)=J^{-1}$. Then we choose $H(z,r)$ such that $J=HH^T$ and $\det(H)=1$. We fix (z,r), write down $Z(w)$ and solve the Riemann–Hilbert problem $BA^{-1}=Z$ (subject to $A(z+\mathrm{i}r)=1$, $A(\bar{w})=\overline{A(w)}$, $B(\bar{w})=\overline{B(w)}$). Finally we obtain $J'(z,r)$ from (A2.7). Apart from differences of 'gauge', this is the method of Hauser and Ernst.

Bäcklund transformations. Kramer and Neugebauer (1984) describe some Bäcklund transformations of (3.1). They begin with a 'seed metric' J and a matrix-valued solution Φ of one of the associated linear systems; Φ is then transformed into Φ', which is a solution of the linear system of a new solution J'. (A typical example of this process appears in the proof of proposition 6.3; see also Mason *et al* (1987).)

One such transformation, due to Cosgrove (1982) and Neugebauer (1979), has a simple interpretation in twistor theory. In the axis-regular case, it is constructed by treating r as a field. In other words, by writing Yang's equation in the form

$$\mathrm{d}^*(rJ^{-1}\,\mathrm{d}J)=0 \qquad \mathrm{d}^*\,\mathrm{d}r=0 \tag{A2.9}$$

where d and d* are the exterior derivative and its dual acting on forms on the Poincaré disc. In this guise, it is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$ on the disc. Under Cosgrove and Neugebauer's transformation, the value of J on the boundary is kept fixed, but r is replaced by $r'=r[(z-a)^2+r^2]^{-1}$, where a is a real parameter.

From the twistor point of view, the transformation is effected by keeping fixed the triple (E,e^0,e^1) fixed, but by replacing the point $w=\infty$ by the point $w=a$ in the recovery construction. The only element in the correspondence between framed bundles and solutions of Yang's equations that breaks the $\mathrm{SL}(2,\mathbb{R})$ symmetry is the choice of the point $w=\infty$ on the real axis.

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