AIP Journal of Mathematical Physics

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Citation: J. Math. Phys. 30, 1081 (1989); doi: 10.1063/1.528379
View online: http://dx.doi.org/10.1063/1.528379
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v30/i5
Published by the American Institute of Physics.

Symmetries of the self-dual Einstein equations. I. The infinite-dimensional symmetry group and its low-dimensional subgroups

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(Received 26 April 1988; accepted for publication 19 October 1988)

This is the first of two papers in which the authors give a complete classification of symmetry reduced solutions of Plebanski's potential equation for self-dual Einstein spaces. In this first part the infinite pseudogroup of symmetries of Plebanski's equation is described, and the conjugacy classes of all local subgroups of dimensions one, two, and three over both the real and complex numbers are classified. Then in the second paper, this classification is used to obtain all symmetry-reduced solutions.

I. INTRODUCTION

The purpose of this investigation is twofold. On the one hand, we plan to apply, in a systematic manner, the method of symmetry reduction, to obtain group invariant solutions of the Euclidean signature self-dual Einstein equations. On the other hand, a first step in this program is to obtain the relevant local symmetry group of local point transformations and then give a complete classification of its low-dimensional local subgroups. Since this treatment is entirely local, we work infinitesimally with the corresponding Lie algebras.

In the present paper (Part I of a series of two) we show that the infinite-dimensional symmetry group is essentially one of Cartan's infinite-dimensional primitive groups[1] and provide a classification of its complex and real subalgebras of dimension less than four. It is precisely the real subalgebras that correspond to groups having orbits of codimension k, with $1 \leqslant k \leqslant 3$ in the underlying space of independent variables $\mathbb{R}^4$, and that hence provide a reduction of the considered equations to lower-dimensional ones.

Actually we do not deal with the self-dual Einstein equations[2,3] per se but with the potential equation obtained by Plebanski,[4]

$$\Omega_{x\bar{x}}\Omega_{y\bar{y}} - \Omega_{x\bar{y}}\Omega_{\bar{x}y} = 1, \qquad (*)$$

where $x, y \in \mathbb{C}$, Ω is a real function, and subindices indicate partial derivatives. This Monge–Ampere type of equation is of independent interest, having a rich history. Calabi[5–7] has studied the n-dimensional analog that he calls the Levi invariant equation. The function Ω appears as a local Kähler potential on some domain of $\mathbb{C}^2$, and gives rise to what has become known in the literature as Calabi–Yau spaces. These spaces are not only Kähler but hyperkähler,[7] that is, there is a two sphere's worth of complex structures on $\mathbb{R}^4$ that are compatible with the metric and the metric is Kähler with respect to all of these complex structures. In all that follows here we shall make a choice of complex structure and hence, a choice of Kähler potential.

It has been known for some time that Eq. $(*)$ has an infinite-dimensional symmetry "group" although it does not seem to have appeared in the literature. (The symmetry group of a related equation was determined in Ref. 8.) It is at least partially the purpose of the present paper to not only rectify this but also to show that the "symmetry group" of $(*)$ is an Abelian extension of one of the infinite primitive pseudogroups of Cartan, namely the pseudogroup of all biholomorphic maps from domains in $\mathbb{C}^2$ to itself with constant Jacobian determinant.

In Part II of this series we shall use the results obtained here to systematically apply the method of symmetry reduction[9,10] to Eq. $(*)$ to find all solutions that can be obtained by this method. From these solutions we can then write down the corresponding self-dual Einstein metrics that should be of interest from the point of view of gravitational instantons.[11–13] In the process we show how Eq. $(*)$ is related to many other interesting partial (and ordinary) differential equations.

The outline of the present paper is as follows: In this section we give a brief discussion of self-dual Einstein spaces and their relationship with Eq. $(*)$. In Sec. II we determine the Lie algebra of infinitesimal symmetries of Eq. $(*)$. Section III consists of relevant comments about the classification of subalgebras. Sections IV and V make up the heart of the present article. They give the classification of conjugacy classes of complex and real subalgebras, respectively, of dimension less than or equal to three under the relevant pseudogroup of transformations. Finally Sec. VI gives a brief conclusion of our results as well as a preview of things to come.

Let us recall briefly how the Ω equation $(*)$ arises from the self-dual Einstein equations. This was first derived in this context for the complex self-dual Einstein equations by Plebanski.[4] Let (M,g) be a four-dimensional Riemannian manifold. The Levi-Civita connection Γ has values in the Lie algebra so(4). But there is a well-known isomorphism $\mathrm{so}(4) \simeq \mathrm{su}(2)_+ \oplus \mathrm{su}(2)_-$, where we have labeled the two copies of su(2) by $\pm$ for convenience. Thus the connection splits $\Gamma = \Gamma_+ + \Gamma_-$ and we let $R_\pm = \mathrm{curv}\, \Gamma_\pm$. The Riemannian manifold (M,g) is called self-dual if $R_- = 0$. It is not difficult to see by writing things out explicitly that $R_- = 0$ or $R_+ = 0$ implies the vacuum Einstein equations, in fact, the Ricci tensor vanishes identically. Thus $R_- = 0$ is

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called the self-dual Einstein equation. Now $R_-=0$ means that Γ_- is a flat connection. Here $\Gamma_\pm$ are connections on the (locally defined) spin bundles $V_\pm M$. Although $V_\pm M$ may not be defined globally as bundles on M, their second-order symmetric tensor product bundles $S^2_\pm M$ are globally defined bundles on M. Furthermore, $\Gamma_\pm$ induce connections, also denoted $\Gamma_\pm$, on $S^2_\pm M$. There is another important splitting related to the Lie algebra splitting described above, namely, the splitting induced by Hodge's $*$ operator on two-forms. In four-dimensional Riemannian geometry $*$ is an involution on two-forms $\Lambda^2 M$ split into the plus and minus eigenspaces of $*$, viz., $\Lambda^2 M=\Lambda^2_+ M\oplus\Lambda^2_- M$. Furthermore, this splitting is compatible with the Lie algebra splitting and this gives rise to bundle isomorphisms $S^2_\pm M\simeq\Lambda^2_\pm M$. Thus $\Gamma_\pm$ can be considered as connections in $\Lambda^2_\pm M$, the bundles of self-dual $(+)$ and anti-self-dual $(-)$ two-forms.

We now describe the consequences of having a flat connection Γ_- on $\Lambda^2_- M\simeq S^2_- M$. From now on our considerations will be entirely local. We use the standard dotted and undotted spinor indices that are to be raised and lowered by the symplectic form ϵ_{AB} according to the convention $\psi_A=\epsilon_{AB}\psi^B$, where $A,B=1,2$ and the Einstein summation convention is used on repeated indices. In spinor notation an orthonormal moving coframe is written as $\theta^{A\dot A}$. We define an almost complex structure on T^*M by requiring $\bar\theta^{A\dot A}=\theta_{A\dot A}$. Then a basis (locally) for $\Lambda^2_- M$ is given by

$$S^{\dot A\dot B}=\tfrac12\epsilon_{AB}\theta^{A\dot A}\wedge\theta^{B\dot B},\tag{1.1}$$

where $S^{\dot B\dot A}=S^{\dot A\dot B}$ and $\bar S^{\dot A\dot B}=S_{\dot A\dot B}$. Now as mentioned previously $R_-=0$ if and only if Γ_- is flat, that is, there is a choice of orthonormal frame such that the connection coefficients $\Gamma_{\dot A\dot B}$ with respect to this frame vanish. It follows that the two-forms $S^{\dot A\dot B}$ are all closed, i.e.,

$$dS^{\dot A\dot B}=0.\tag{1.2}$$

Considering first $dS^{\dot1\dot1}=0$, by Darboux's theorem there are local coordinates $\{q^A; A=1,2\}$ such that

$$S^{\dot1\dot1}=dq^1\wedge dq^2=\tfrac12 dq^A\wedge dq_A.\tag{1.3}$$

Moreover, $S^{\dot2\dot2}=S_{\dot1\dot1}=\bar S^{\dot1\dot1}=d\bar q'\wedge d\bar q^2$. Now let us consider $S^{\dot1\dot2}$. First, $S^{\dot1\dot2}=S^{\dot2\dot1}=-S_{\dot1\dot2}=-S_{\dot2\dot1}$. Second, this form is nondegenerate since an explicit computation shows

$$2S^{\dot1\dot2}\wedge S^{\dot1\dot2}=S^{\dot1\dot1}\wedge S^{\dot2\dot2}=S^{\dot1\dot1}\wedge\bar S^{\dot1\dot1}\tag{1.4}$$

and the latter is proportional to the volume element on M. Since $S^{\dot1\dot2}$ is closed and nondegenerate g is a Kähler metric on M and $S^{\dot1\dot2}$ is the Kähler form. So locally there is a smooth function Ω such that

$$S^{\dot1\dot2}=\Omega_{q^A\bar q^B}\,dq^A\wedge d\bar q^B,\tag{1.5}$$

where $\Omega_{q^A\bar q^B}=\partial\Omega/\partial q^A\,\partial\bar q^B$. Now since the Levi-Civita connection $\Gamma=\Gamma_+$ has values in su(2), the local holomony group must be SU(2) or a subgroup thereof. It follows that under a change of SU(2) frame, the volume element $S^{\dot1\dot1}\wedge S^{\dot2\dot2}$ is preserved, thus about every point of M there are coordinates such that (1.3)–(1.5) hold. Plugging (1.5) into (1.4) and using (1.3) gives the Ω equation $(*)$.

II. THE SYMMETRY GROUP OF THE Ω EQUATION

In order to apply the method of symmetry reduction to $(*)$ we need to compute its symmetry group. We shall show that this group is an Abelian extension of one of the six infinite primitive pseudogroups of Cartan (Ref. 1, Theorem IX). More precisely, we compute the Lie algebra of infinitesimal symmetries; but rather than work with the differential equation itself, we write it as an equivalent Pfaffian system. First, notice that $(*)$ can be written as a conservation law, namely,

$$\partial_{\bar q_B}(\partial_{q_A}\Omega_{q^A\bar q^B}-\bar q_B)=0\tag{2.1a}$$

or equivalently, the complex conjugate equation

$$\partial_{q_A}(\Omega_{\bar q_B}\Omega_{q^A\bar q^B}-q_A)=0.\tag{2.1b}$$

Equation (2.1a) is the integrability condition of the existence of a local complex-valued smooth function Σ satisfying

$$\Sigma_{\bar q^B}=\Omega_{q_A}\Omega_{q^A\bar q^B}-\bar q_B.\tag{2.2}$$

Similarly, we get the complex conjugate equation arising from (2.1b). In order to write the corresponding Pfaffian system we first construct the contact form θ_0 for Ω and then add the one-forms describing (2.2) and its complex conjugate. We arrive at

$$\begin{aligned}\theta_0&=d\Omega-p_A\,dq^A-\bar p_A\,d\bar q^A,\\ \theta_1&=d\Sigma-\tfrac12 p_A\,dp^A-s_A\,dq^A+\tfrac12\bar q_A\,d\bar q^A,\\ \bar\theta_1&=d\bar\Sigma-\tfrac12\bar p_A\,d\bar p^A-\bar s_A\,d\bar q^A+\tfrac12 q_A\,dq^A,\end{aligned}\tag{2.3}$$

which is a Pfaffian system $\mathscr{I}$ on the two-jet $J^2(\mathbb{C}^2,\mathbb{R}\times\mathbb{C})$ (actually on a certain submanifold $\mathbb{R}^{15}$ of the two-jet) where q^A, $A=1,2$ are complex coordinates on $\mathbb{C}^2$ and $(\Omega,\Sigma)\in\mathbb{R}\times\mathbb{C}$. The integral submanifolds $N\hookrightarrow J^2(\mathbb{C},\mathbb{R}\times\mathbb{C})$ which annihilate $\mathscr{I}$ and satisfy the independence condition $dq^1_A\wedge dq^2_A\wedge d\bar q^1_A\wedge d\bar q^2\neq0$ are precisely the solutions of $(*)$. Thus any infinitesimal contact transformation symmetry of $(*)$ must map $\mathscr{I}$ to $\mathscr{I}$. So we consider the vector fields on $J^2(\mathbb{C}^2,\mathbb{R}\times\mathbb{C})$ which satisfy

$$L_X\theta_i=\sum\lambda_{ij}\theta_j,\quad i,j=0,\pm1,\tag{2.4}$$

where $\theta_{-1}=\bar\theta_1$ and $\lambda^{\,i}_{\,j}$ are smooth functions. Of course, if $L_X\theta_1\in\mathscr{I}$ then $L_X\bar\theta_1\in\mathscr{I}$. The method for solving (2.4) is by now quite standard so we omit the details, just presenting the infinitesimal symmetries and vector fields on $\mathbb{R}^{15}$,

$$\begin{aligned}X^{q^A}&=\tfrac12 aq_A+\alpha_{q^A},\\ X^{p^A}&=\tfrac12\bar ap_A+\alpha_{q^Aq^B}p^B+\beta q^A,\\ X^{\Omega}&=\tfrac12(a+\bar a)\Omega+\beta+\bar\beta,\\ X^{s^A}&=(-\tfrac12+\bar a)s_A+\alpha_{q^Aq^B}s^B\\ &\quad+\alpha_{q^Aq^Bq^C}p^Bp^C+\beta_{q^Aq^B}p^B+\gamma_{q^A},\\ X^{\Sigma}&=\bar a\Sigma+\tfrac12\alpha_{q^Aq^B}p^Ap^B+\tfrac12\beta_{q^A}p^A+\tfrac12\bar\alpha_{\bar q^A}\bar q^A+\bar\alpha+\gamma,\end{aligned}\tag{2.5}$$

and their complex conjugates. Here, α, β, and γ are holomorphic functions on open sets of the origin in $\mathbb{C}^2$.

Let us now discuss the structure of the symmetry algebra $\widetilde T$ of the Pfaffian system (2.3). There are three arbitrary holomorphic functions of the complex variables $\{q^1,q^2\}$, namely, α, β, and γ, and one complex parameter a. Furthermore, the vector fields

$$\begin{aligned}\widetilde{X}_1 &= [\beta(a) + \bar{\beta}(\bar{q})]\partial_\Omega,\\ \widetilde{X}_2 &= \mathrm{Re}[\gamma(q)\partial_\Sigma],\\ \widetilde{X}_3 &= \mathrm{Im}[\gamma(q)\partial_\Sigma],\end{aligned} \tag{2.6}$$

generate an infinite-dimensional Abelian ideal $\widetilde{A}$ in $\widetilde{T}$. Moreover, $\widetilde{X}_2$ and $\widetilde{X}_3$ generate q-dependent translations in Σ and $\bar{\Sigma}$, the first integrals of Ω. Therefore, they do not appear as infinitesimal symmetries of the Ω equation $(*)$. Thus the full infinitesimal symmetry algebra $\widehat{T}$ of the Ω equation is generated by two holomorphic functions α and β and one complex parameter a. In order to discuss further this symmetry algebra we introduce an affine bundle (not a vector bundle) $A \to \mathbb{C}^2$ where $\{q^A: A = 1,2\}$ are complex coordinates on $\mathbb{C}^2$ and a local section of A is given by the graph $(q^1,q^2) \mapsto (q^1,q^2,\Omega(q,\bar{q}))$, so the fibers of A are real affine lines, i.e., there is no fixed origin in the fibers. In fact, the transformation generated by $\widetilde{X}$ sends the section Ω to $\Omega + \beta(q) + \bar{\beta}(\bar{q})$, which is a base point dependent translation in the fibers of A. With this in mind the representation of $\widehat{T}$ given by the vector fields (1.2b) is the prolongation to the one-jet $J^1(A,\mathbb{R})$ of fiber preserving local transformations of $A \to \mathbb{C}^2$. However, we shall see later that from the point of view of symmetry reduction the Abelian ideal A generated by $\widetilde{X}_1$ plays no role whatsoever. It is thus only the factor algebra $T = \widehat{T}/A$ which is of interest to us, and T is easily identified with one of the transitive primitive algebras of Carãn,[1] namely the Lie algebra of holomorphic vector fields on $\mathbb{C}^2$ with constant divergence. It should also be mentioned that the transformations of T fix the zero section of A. Thus $A \simeq \mathbb{C}^2 \times \mathbb{R}$ can now be viewed as a trivial vector bundle over $\mathbb{C}^2$, i.e., as a product.

Let us summarize our results as the following theorem.

Theorem 2.1: The infinitesimal symmetry algebra $\widehat{T}$ of infinitesimal contact transformations of the Ω equation $(*)$ is the prolongation to $J^2(A,\mathbb{R})$ of infinitesimal transformations $\psi: A \to A$ preserving the affine structure of A. Furthermore, $\widehat{T}$ is isomorphic to an Abelian extension of the Lie algebra T of holomorphic vector fields on $\mathbb{C}^2$ with constant divergence by the infinite-dimensional Abelian ideal generated by $\widetilde{X}_1$. Moreover, the infinitesimal transformations of T preserve the product structure $A \simeq \mathbb{C}^2 \times \mathbb{R}$.

As mentioned previously, it is the factor algebra T that is of interest for the purposes of symmetry reduction, and an important consequence of Theorem 2.1 is that as a complex Lie algebra, T consists of holomorphic vector fields. We shall be interested mainly in real subalgebras L of T and we shall associate to such subalgebras a complex invariant, namely, the complex divergence of the complexification X^C of vector fields X in L. To this end we consider the simple subalgebra $T_0 \subset T$ consisting of all holomorphic vector fields on $\mathbb{C}^2$ with zero divergence. Then we have the following lemma.

Lemma 2.2: Let L be a real subalgebra of T and L^1 its derived algebra, then $L^1 \subset T_0$.

Proof: Let E denote the one-dimensional complex Lie algebra generated by the Euler vector field on $\mathbb{C}^2$. Then T_0 is an ideal in T and we have an exact sequence of Lie algebras

$$0 \to T_0 \to T \to E \to 0.$$

It follows that $T^1 \subset T_0$. Moreover, as a real Lie algebra E is a two-dimensional Abelian Lie algebra, so for any real subalgebra $L \subset T$, $L^1 \subset T^1$ and this proves the lemma.

Now any infinite-dimensional Lie algebra T of vector fields on $\mathbb{C}^n$ generates a pseudogroup of transformations on $\mathbb{C}^2$ as follows[14]: Let $\{X_\alpha\}$ by any collection of (locally defined) vector fields in T. We can integrate these vector fields locally to get a collection of local one-parameter groups $\{\phi_{t_\alpha}\}$. The set of all such local diffeomorphisms generates a pseudogroup $P(T)$ (the pseudogroup generated by T). In our case the Lie algebra T generates the pseudogroup of local biholomorphic diffeomorphisms of $\mathbb{C}^2$ with constant Jacobian determinant P, whereas T_0 generates the subpseudogroup with unit Jacobian determinant P_0. Now given any subalgebra $L \subset T$ we will make use of the subpseudogroup $N(L) \subset P$ that normalizes the subalgebra L.

Finally to end this section we mention the connection between the infinitesimal symmetries of $(*)$ and Killing vector fields on a given solution of $(*)$. In Ref. 15 the homothetic Killing equations were integrated and a standard form for any homothetic Killing vector field was given. It is easy to see that any $X \in T$ is a homothetic Killing vector field. However, the pseudogroup S of allowed transformations for the homothetic Killing equations is larger. In general, S does not preserve the complex structure, but $P \subset S$. The action of S on the coordinates is induced by the action of SO(3) on $\Lambda^2\_ M \simeq S^2 V\_M$, which leaves (2.2) invariant. The unit sphere bundle in $\Lambda^2\_ M$ can be identified with the bundle of complex structures on M. So any homothetic Killing vector field will fix a complex structure and thus is equivalent to a vector field in T. However, there is an action of SO(3) which permutes all complex structures and so this is not equivalent to any subgroup of P. This gives a class of Bianchi IX solutions[16] of $(*)$ that cannot be obtained by symmetry reduction of $(*)$.

III. GENERAL COMMENTS ON THE CLASSIFICATION OF SUBALGEBRAS

In order to perform a symmetry reduction for the Ω equation we need to know all low-dimensional subgroups of the symmetry pseudogroup P of this equation. More precisely, we need a classification of all local subgroups that will have generic orbits of dimension $d = 1, 2, or\ 3$ in the underlying four-dimensional Euclidean space-time. Our procedure will be an algebraic one: we shall classify subalgebras of the symmetry algebra, namely the algebra of holomorphic vector fields in two complex variables, having constant divergence. The classification will be under the action of the pseudogroup P of holomorphic transformations with constant Jacobian determinant.

Lie in his classical lecture notes[16] (see also Cartan[16]) has solved a related problem, namely that of classifying all continuous groups of point transformations in two complex variables. He obtained an exhaustive list of representatives of Lie algebras that can be realized in terms of holomorphic vector fields in two complex variables. The vector fields do not necessarily have constant divergence. Furthermore, Lie's classifying group is correspondingly larger: the transformations do not necessarily have constant Jacobian determinant. Lie's results cannot be directly adapted to our case, mainly because the value of the divergence of a vector field is coordinate dependent and is thus not invariant under arbitrary transformations.

We shall be interested in low-dimensional Lie algebras, realized in terms of either complex or real vector fields. However, before discussing the general procedure we shall make some simplifications. First, we show that for symmetry reductions it suffices to consider the factor algebra $T \simeq \widehat{T}/A$. Moreover, according to Theorem 2.1, we may restrict ourselves to vector fields that represent infinitesimal symmetry transformations of the affine bundle A. Such complex vector fields may be written as

$$X = (aq^A + \alpha_{q_A})\partial_{q^A} + a\Omega\partial_\Omega + \beta\partial_\Omega. \tag{3.1}$$

The corresponding real vector fields are obtained by simply taking the real part of (3.1). If a and α_{q^A} do not both vanish, we can make the q-dependent translation $\Omega \mapsto \Omega + \gamma$ and remove β by choosing γ to be a solution of the first-order inhomogeneous partial differential equation

$$(aq^A + \alpha_{q_A})\gamma_{q^A} - a\gamma + \beta = 0. \tag{3.2}$$

On the other hand, if both a and α_{q^A} vanish, the invariant is an arbitrary function $F(q^1,q^2,\bar{q}^1,\bar{q}^2)$ which is independent of Ω and thus does not give rise to symmetry reduced solutions of the Ω equation $(*)$. Thus we may restrict our considerations to the Lie algebra T. Furthermore, notice that the projection $\pi: A \to \mathbb{R}^4$ induces a Lie algebra $\pi_* T$ that is isomorphic to T. So in order to simplify notations in Secs. IV and V and especially in the tables, we will drop the terms involving ∂_Ω in the vector fields. The notation X and X^R below will be used for the projections of the vector fields onto the base manifold, i.e., elements of $\pi_* T$. The full vector fields are always recovered by adding $\frac{1}{2}(\operatorname{div} X)\Omega\partial_\Omega$, or $\frac{1}{2}(\operatorname{div} X + \operatorname{div} \overline{X})\Omega\partial_\Omega$, to the corresponding complex or real vector fields.

The complex vector fields under consideration have the form

$$X = f(x,y)\partial_x + g(x,y)\partial_y \tag{3.3a}$$

satisfying

$$\operatorname{div} X \equiv f_x + g_y = \alpha = \text{const}. \tag{3.3b}$$

Real vector fields that we are dealing with have the form

$$X^R = f(x,y)\partial_x + g(x,y)\partial_y + \bar{f}(\bar{x},\bar{y})\partial_{\bar{x}} + \bar{g}(\bar{x},\bar{y})\partial_{\bar{y}}. \tag{3.4a}$$

We have

$$\operatorname{div} X^R = f_x + g_y + \bar{f}_{\bar{x}} + \bar{g}_{\bar{y}} = \alpha + \bar{\alpha}. \tag{3.4b}$$

Thus we may have $\operatorname{div} X^R = 0$, but $\operatorname{div} X = \alpha = -\bar{\alpha} \neq 0$.

Vector fields and algebras of vector fields will be classified under coordinate transformations of the pseudogroup P which we write explicitly as

$$\xi = F(x,y), \quad \eta = G(x,y), \tag{3.5a}$$

with constant Jacobian determinant,

$$\det J \equiv J_0 = F_x G_y - F_y G_x. \tag{3.5b}$$

Let us first present some general results that will be used below in the subalgebra classification.

Lemma 3.1: Let X be a vector field of the type (3.1) in coordinates (x,y) and $\widetilde{X}$ the same vector field in coordinates (ξ,η) of (3.5a). The divergences of X and $\widetilde{X}$ are related by

$$\operatorname{div} \widetilde{X} = \operatorname{div} X + X \ln(\det J). \tag{3.6}$$

In particular, if $\det J = J_0 = \text{const}$, then $\operatorname{div} X$ is invariant under (3.5).

Proof: The transformed vector field is

$$\widetilde{X} = (fF_x + gF_y)\partial_\xi + (fG_x + gG_y)\partial_\eta. \tag{3.7}$$

A simple calculation yields (3.6). Q.E.D.

Lemma 3.2: If A, B, and C are three vector fields, satisfying $A = [B,C]$ and $\operatorname{div} B = \beta$, $\operatorname{div} C = \gamma$, where β and γ are constants, then $\operatorname{div} A = 0$.

Proof: A simple calculation yields

$$\operatorname{div} A = B \operatorname{div} C - C \operatorname{div} B. \tag{3.8}$$

Hence

$$\operatorname{div} A = B\gamma - C\beta = 0. \quad \text{Q.E.D.}$$

Remark: Lemma 2.2 is a simple consequence of Lemma 3.2. We shall make use of the known classification of two- and three-dimensional Lie algebras into isomorphism classes.[18,19] They can be summed up in two lemmas.

Lemma 3.3: Any two-dimensional Lie algebra over either $\mathbb{C}$ or $\mathbb{R}$ is isomorphic to one of the following ones: (1) Abelian: $2A_1$,

$$[X_1,X_2] = 0. \tag{3.9a}$$

(2) Solvable, non-Abelian: $A_{2,1}$,

$$[X_1,X_2] = X_1. \tag{3.9b}$$

Lemma 3.4: Any three-dimensional Lie algebra over either $\mathbb{C}$ or $\mathbb{R}$ is isomorphic to one of the following ones: (1) Abelian: $3A_1$,

$$[X_1,X_2] = [X_2,X_3] = [X_3,X_1] = 0. \tag{3.10a}$$

(2) Decomposable, non-Abelian: $A_{2,1} \oplus A_1$,

$$[X_1,X_3] = X_1, \quad [X_2,X_3] = 0, \quad [X_1,X_2] = 0. \tag{3.10b}$$

(3) Indecomposable, nilpotent: $A_{3,1}$,

$$[X_2,X_3] = X_1, \quad [X_1,X_3] = 0, \quad [X_1,X_2] = 0. \tag{3.10c}$$

(4) Indecomposable, solvable, non-nilpotent, with a two-dimensional Abelian ideal $\{X_1,X_2\}$,

$$\begin{pmatrix} [X_1,X_3] \\ [X_2,X_3] \end{pmatrix} = M \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad [X_1,X_2] = 0, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{3.10d}$$

Over $\mathbb{C}$ the matrix M has one of the following forms:

$$A^{\alpha}_{3,2}: \; M^C_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

$$0 \leqslant |\alpha| \leqslant 1, \quad \alpha \in \mathbb{C}, \quad \text{if } |\alpha| = 1,$$

$$\text{then } 0 \leqslant \arg \alpha \leqslant \pi, \tag{3.10e}$$

$$A_{3,3}: \; M^C_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Over $\mathbb{R}$ the matrix has one of the following forms:

$$A^a_{3,2}:\ M^R_1=\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix},\quad a\in\mathbb{R},\quad 0<|a|\leqslant 1,$$

$$A_{3,3}:\ M^R_2=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\tag{3.10f}$$

$$A^a_{3,4}:\ M^R_3=\begin{pmatrix}a & -1\\ 1 & a\end{pmatrix},\quad a\in\mathbb{R},\quad 0\leqslant a.$$

(5) Simple.

Over $\mathbb{C}$ the only possibility is $\mathrm{sl}(2,\mathbb{C})$,

$$A_{3,5}:\ [X_1,X_2]=X_1,\quad [X_2,X_3]=X_3,\quad [X_1,X_3]=-2X_2.\tag{3.10g}$$

Over $\mathbb{R}$ there are two cases: (a) $\mathrm{sl}(2,\mathbb{R})\sim\mathrm{su}(1,1)\sim\mathrm{o}(2,1)$ with commutation relations (3.10g), and (b) $\mathrm{su}(2)\sim\mathrm{o}(3)$ with commutation relations

$$A_{3,6}:\ [X_i,X_k]=\epsilon_{ikl}X_l,\quad \{i,k,l\}=\{1,2,3\}.\tag{3.10h}$$

Two useful lemmas that can be proven by direct calculations are the following.

Lemma 3.5: The most general transformation of the type (3.5) leaving the vector space $\{X_1=\partial_x\}$ invariant is

$$\xi=\lambda x+H(y),\quad \eta=\beta y+\nu,\quad \lambda\mu\neq 0,\tag{3.11}$$

where $\lambda,\mu,\nu\in\mathbb{C}$ are constants.

Lemma 3.6: The most general transformation of the type (3.5) leaving the vector space $\{X_1=x\partial_x\}$ invariant is

$$\xi=\lambda x/\dot{G}(y),\quad \eta=G(y),\quad \lambda\neq 0,\quad \dot{G}(y)\neq 0.\tag{3.12}$$

We restrict our classification to subalgebras of dimension d with $1\leqslant d\leqslant 3$. The justification for this is that the $d=3$ subalgebras already provide reductions of the Ω equation to ordinary differential equations, or to algebraic ones. Higher-dimensional algebras will always contain at least one three-dimensional subalgebra, so the algebras of dimension $d\geqslant 4$ will not provide any new reductions. Indeed, every simple Lie algebra except $\mathrm{su}(1,1)$ contains at least one class of $\mathrm{su}(2)$ subalgebras. Every solvable Lie algebra of dimension n has subalgebras (and ideals) of all dimensions $1\leqslant d\leqslant n$. Algebras that are neither simple nor solvable have a Levi decomposition in which the semisimple part is either $\mathrm{su}(2)$ or $\mathrm{su}(1,1)$, or contains at least one of them as a subalgebra.

The problem of classifying subalgebras of the Lie algebra of holomorphic vector fields with constant divergence into conjugacy classes and the action of the pseudogroup P is conceptually similar to that of classifying the subalgebras of a finite-dimensional Lie algebra under the action of the group of inner automorphisms. The classification methods have been developed and applied in a series of earlier papers.[20–22]

IV. CLASSIFICATION OF LOW-DIMENSIONAL COMPLEX SUBALGEBRAS

A. One-dimensional subalgebras

We are given a vector field (3.1) satisfying (3.2) and perform a change of variables (3.5). The vector field transforms into (3.7). We set

$$fG_x+gG_y=0,\tag{4.1a}$$

$$fF_x+gF_y=-(g/G_x)J_0.\tag{4.1b}$$

Relation (4.1a) amounts to a choice of the function $G(x,y)$ that can always be made; (4.1b) is then a consequence of (4.1a) and (3.5b). The vector field X in the coordinates (ξ,η) is now

$$X=-(g/G_x)J_0\partial_\xi.\tag{4.2}$$

By assumption, J_0 is constant; if we require that $-g/G_x$ be constant then (4.1a) implies the compatibility condition $f_x+g_y=0$. If this is satisfied, i.e., if we have $\operatorname{div}X=0$, then X is conjugate to $\widetilde{X}=\partial_\xi$. If on the other hand, we have $\operatorname{div}X=\alpha\neq 0$, we set $-g(J_0/G_x)=\alpha F$, and obtain equations for G and F that are always compatible.

We arrive at the following theorem.

Theorem 4.1: An arbitrary one-dimensional subalgebra $\{X\}$ of the symmetry algebra is conjugate to one of the two following Lie algebras:

$$L_{1,1}(\mathbb{C})=\{\partial_x\},\tag{4.3a}$$

$$L_{1,2}(\mathbb{C})=\{x\partial_x\}.\tag{4.3b}$$

They are distinguished by the fact that $\operatorname{div}X=0$ for $X\in L^{\mathbb{C}}_{1,1}$ and $\operatorname{div}X\neq 0$ for $X\in L^{\mathbb{C}}_{1,2}$. □

B. Two-dimensional subalgebras

Consider a Lie algebra $\{X_1,X_2\}$ with a commutation relation as in (3.9). We shall assume that X_1 has already been transformed to standard form as in (4.3); X_2, on the other hand, is left general, as in (3.1). The procedure is first to implement the commutation relation, then simplify X_2, using the normalizer subpseudogroup, N or $P\{X_1\}$ of X_1 in P.

1. Abelian algebras

(a) Take X_1 as in (4.3a) and require $[X_1,X_2]=0$, $\operatorname{div}X_2=\alpha=\text{const}$. We obtain

$$X_1=\partial_x,\quad X_2=f(y)\partial_x+(\alpha y+\beta)\partial_y.\tag{4.4a}$$

The normalizer pseudogroup of X_1 is given in (3.11) and it transforms the vector fields into

$$X_1=\lambda\partial_\xi,$$
$$X_2=[\lambda f(y)+(\alpha y+\beta)\dot{H}(y)]\partial_\xi+(\alpha y+\beta)\mu\partial_\eta.\tag{4.4b}$$

If $(\alpha,\beta)\neq(0,0)$ we choose $H(y)$ to satisfy $(\alpha y+\beta)\dot{H}+\lambda f(y)=0$. If $\alpha\neq 0$ we put $\alpha\nu=\beta\mu$. If $\alpha=0$, $\beta\neq 0$ we put $\beta\mu=1$. Finally, if $(\alpha,\beta)=(0,0)$ no simplification occurs.

(b) Take X_1 as in (4.3b). Imposing $[X_1,X_2]=0$ and replacing X_2 by $X_2-\alpha X_1$ we obtain

$$X_1=x\partial_x,\quad X_2=-\dot{g}(y)x\partial_x+g(y)\partial_y.\tag{4.5}$$

Performing a transformation (3.12) with $G(y)=\alpha/g(y)$ we reduce the algebra (4.5) to $X_1=\xi\partial_\xi$, $X_2=\partial_\eta$.

2. Non-Abelian algebras

The commutation relation is given by (3.9b). According to Lemma 3.2 we have $\operatorname{div}X_1=0$. With no loss of generality we can take $X_1=\partial_x$. From (3.9b) and (3.2) we obtain

$$X_1 = \partial_x, \quad X_2 = [x + h(y)]\partial_x + (\alpha y + \beta)\partial_y. \quad (4.6)$$

Performing a transformation of the form (3.11) with $H(y)$ satisfying $h(y)\lambda + (\alpha y + \beta)\dot{H} = H$ we reduce (4.6) to

$$X_1 = \partial_\xi, \quad X_2 = \xi\partial_\xi + (\alpha\eta - \alpha\nu + \beta\mu)\partial_\eta. \quad (4.7)$$

If $\alpha \neq 0$ we choose ν and μ so that $\beta\mu - \alpha\nu = 0$. If $\alpha = 0$, $\beta \neq 0$ we choose μ so that $\beta\mu = 1$.

The results of this subsection are summarized in Table I. We have denoted V the vector space spanned by the vectors $\{X_1,X_2\}$ at any generic point (x, y). The three Abelian algebras $L_{2,1}(\mathbb{C})$, $L_{2,2}(\mathbb{C})$, and $L^f_{2,3}(\mathbb{C})$ are distinguished from each other by the invariants in columns 5 and 6, namely $\operatorname{div} X$, the divergence of the general element $X = \mu_1 X_1 + \mu_2 X_2$, and the dimension of V. For the $A_{2,1}$ type algebras these invariants coincide for $L_{2,4}(\mathbb{C})$ and $L^\alpha_{2,5}(\mathbb{C})$ if $\alpha \neq 0$, $\alpha \neq -1$. In this case X_1 is uniquely defined as the vector field spanning the derived algebra. The value of α itself is an invariant under the action of the isotropy group of X_1, i.e., the transformation (3.11).

The algebras $L^f_{2,3}(\mathbb{C})$ are somewhat exceptional. They depend on one arbitrary function $f(y)$.

In order to avoid redundancy in Table I and in other subalgebra lists we establish the following equivalence relation. The two sets of linearly independent functions

$$\{f_1(y),\dots,f_n(y)\} \quad \text{and} \quad \{g_1(y),\dots,g_n(y)\},$$

are equivalent, if constants $\lambda \neq 0$, μ, and a matrix $\rho \in \mathrm{GL}(n,\mathbb{C})$ exist, such that

$$g_i(y) = \sum_{k=1}^{n} \rho_{ik} f_k(\lambda y + \mu). \quad (4.8)$$

Using this equivalence concept we arrive at the following theorem.

Theorem 4.2: Every complex two-dimensional subalgebra of the algebra g of holomorphic vector fields with constant divergence is conjugate under the pseudogroup P to an algebra in Table I. Two algebras in Table I are mutually conjugate if and only if they are in the classes $L^f_{2,3}(\mathbb{C})$ and $L^g_{2,3}(\mathbb{C})$, and the pairs $(1, f(y))$ and $(1, g(y))$ are equivalent under relation (4.8).

C. Three-dimensional subalgebras

It follows from Lemma 3.4 that all complex three-dimensional Lie algebras except sl(2,$\mathbb{C}$) have a two-dimensional Abelian ideal. We choose it to be $\{X_1,X_2\}$. For solvable non-nilpotent Lie algebras this ideal is unique. Our procedure will be to assume that $\{X_1,X_2\}$ is in one of the standard forms $L_{2,1}(\mathbb{C})$, $L_{2,2}(\mathbb{C})$, or $L^f_{2,3}(\mathbb{C})$, whereas X_3 has the general form (3.1) satisfying (3.2). We then impose the commutation relations

$$\begin{pmatrix} [X_1,X_3] \\ [X_2,X_3] \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4.9)$$

for each standard form of M. Finally, we standardize M, using the normalizer pseudogroup of the ideal $\{X_1,X_2\}$. For Abelian and nilpotent algebras we must afterwards weed out possible redundancies, due to the uniqueness of the Abelian ideal.

Simple subalgebras, i.e., sl(2,$\mathbb{C}$) subalgebras, will be treated separately. In this case use will be made of the fact that sl(2,$\mathbb{C}$) contains a subalgebra of the type $A_{2,1}$, that is, however, not an ideal.

1. Solvable subalgebras

(a) Ideal $L_{2,1}(\mathbb{C}) = \{\partial_x,\partial_y\}$. We have

$$\begin{pmatrix} [X_1,X_3] \\ [X_2,X_3] \end{pmatrix} = \begin{pmatrix} f_x\partial_x + g_x\partial_y \\ f_y\partial_x + g_y\partial_y \end{pmatrix}. \quad (4.10)$$

Combining (4.9) and (4.10), we obtain

$$f = ax + cy + p, \quad g = bx + dy + q. \quad (4.11)$$

Replacing X_3 by $X_3 - pX_1 - qX_2$ we effectively set $p = q = 0$. Performing linear transformation of variables ($\xi = \mu x + \nu y, \eta = \rho x + \sigma y, \mu\sigma - \rho\nu \neq 0$) we can transform the matrix M to its standard form.

The Abelian case (3.10a) is excluded, since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

implies $X_3 = 0$. The decomposable case occurs if $a = 1$, $b = c = d = 0$ in M. This yields the decomposable algebra $\{\partial_x,\partial_y,x\partial_x\}$. The nilpotent case (3.10c) occurs for $c = 1$, $a = b = d = 0$ and leads to $\{\partial_x,\partial_y, y\partial_x\}$. The diagonalizable case (3.10e) corresponds to $a = 1$, $d = \alpha(0 \leqslant \alpha \leqslant 1)$ and yields $\{\partial_x,\partial_y,x\partial_x + \alpha y\partial_y\}$. Finally, the Jordan case (3.10e) corresponds to $a = d = 1$, $c = 1$, $b = 0$ and the algebra $\{\partial_x,\partial_y, (x+y)\partial_x + y\partial_y\}$.

(b) Ideal $L_{2,2} = \{\partial_x, y\partial_y\}$. According to Lemma 3.2, $X_2 = y\partial_y$ cannot figure in the derived algebra. Hence $b = d = 0$ in (4.9). Moreover, we have $f_x = a$, $yf_y = c$, $g_x = 0$, $yg_y - g = 0$. Taking an appropriate linear combination of X_3 with X_1, and X_2, we obtain $X_3 = (ax + c\ln y)\partial_x$. The transformation in P that leaves $L_{2,2}$ invariant is

$$\xi = \alpha x + \gamma \ln y + \mu, \quad \eta = \nu y, \quad \alpha\nu \neq 0. \quad (4.12)$$

The vector fields transform into

$$X_1 = \alpha\partial_\xi, \quad X_2 = \gamma\partial_\xi + \eta\partial_\eta,$$
$$X_3 = [a\xi + (-a\gamma + \alpha c)\ln\eta + (a\gamma - \alpha c)\ln\nu - a\mu]\partial_\xi. \quad (4.13)$$

If $a \neq 0$ we choose $\gamma = \alpha c/a$, $\mu = 0$ and obtain a decomposable algebra $\{\partial_x, y\partial_y, x\partial_x\}$. If $a = 0$ we obtain a nilpotent Lie algebra $\{\partial_x, y\partial_y, \ln y\partial_x\}$.

TABLE I. Two-dimensional complex subalgebras.

N_0	Type	Basis X_1	X_2	$\operatorname{div} X$, $X = \mu_1 X_1 + \mu_2 X_2$	$\dim V$
$L_{2,1}(\mathbb{C})$	$2A_1$	∂_x	∂_y	0	2
$L_{2,2}(\mathbb{C})$	$2A_1$	∂_x	$y\partial_y$	μ_2	2
$L^f_{2,3}(\mathbb{C})$	$2A_1$	∂_x	$f(y)\partial_x$, $f(y) \neq 0$	0	1
$L_{2,4}(\mathbb{C})$	$A_{2,1}$	∂_x	$x\partial_x + \partial_y$	μ_2	2
$L^\alpha_{2,5}(\mathbb{C})$	$A_{2,1}$	∂_x	$x\partial_x + \alpha y\partial_y$	$\mu_2(1+\alpha)$, 0 if $\alpha = -1$	2 if $\alpha \neq 0$, 1 if $\alpha = 0$

(c) Ideal $L_{2,3}^{\phi}(\mathbb{C}) = \{\partial_x, \phi(y)\partial_x; \dot{\phi}(y) \neq 0\}$. The commutation relations in this case are

$$\begin{pmatrix}[X_1,X_3]\\ [X_2,X_3]\end{pmatrix} = \begin{pmatrix}f_x\partial_x + g_x\partial_y\\ (\phi f_x - g\dot{\phi})\partial_x + \phi g_x\partial_y\end{pmatrix} = \begin{pmatrix}[a + b\phi(y)]\partial_x\\ [c + d\phi(y)]\partial_x\end{pmatrix}. \quad (4.14)$$

Taking linear combinations of X_1 and X_2 and performing an appropriate change of variables we can simultaneously assume that $\{X_1,X_2\}$ remain in their standardized form $X_1 = \partial_x$, $X_2 = \phi(y)\partial_x$ and that the matrix M is in its standard form (as in Lemma 3.4).

From (4.14) we obtain

$$g = g(y), \quad f = [a + b\phi(y)]x + \psi(y), \quad (4.15)$$

$$g\dot{\phi} = b\phi^2 + (a - d)\phi - c. \quad (4.16)$$

Further, div $X_3 = \lambda$ implies

$$a + b\phi(y) + g(y) = \lambda. \quad (4.17)$$

The transformation (3.5) that leaves the ideal $\{X_1,X_2\}$ invariant is

$$\xi = x + H(y), \quad \eta = py + q, \quad p = 0. \quad (4.18)$$

It transforms the algebra to

$$X_1 = \partial_\xi, \quad X_2 = \phi\partial_\xi,$$
$$X_3 = [(a + b\phi)\xi + \psi + g\dot{H} - (a + b\phi)H]\partial_\xi + pg\partial_\eta. \quad (4.19)$$

Let us now run through all standard forms of M.

(i) Abelian algebras: $a = b = c = d = 0$. Since $\dot{\phi}(y) \neq 0$, (4.16) implies $g = 0$ and hence $\lambda = 0$. We obtain the Lie algebra $\{\partial_x, \phi(y)\partial_y, \psi(y)\partial_y\}$ where 1, ϕ, and ψ are linearly independent.

(ii) Decomposable non-Abelian algebras: We take $a = b = c = 0$, $d = 1$. Equation (4.17) implies $g = \lambda y + \nu$. If $\lambda = \nu = 0$ we reobtain the Abelian algebra considered above. Hence we have $g(y) \neq 0$ and we can put $\dot{H} = -\psi g^{-1}$. Two cases arise, namely, the Lie algebra $\{\partial_x\, y^p\partial_x, -(1/p)y\partial_y\}$ if $\lambda \neq 0$ (we have put $p = -\lambda^{-1}$) and $\{\partial_x, e^y\partial_x, -\partial_y\}$ if $\lambda = 0$ (we put $p = -\nu^{-1}$).

(iii) Nilpotent algebras: We have $a = b = d = 0$ and $c = 1$. Equation (4.17) implies $g = \lambda y + \mu$ and (4.16) tells us that $\lambda = \mu = 0$ is excluded. Integrating (4.16) for $\lambda \neq 0$ and for $\lambda = 0$, $\mu \neq 0$ we get two nilpotent algebras that are not new: they have appeared above in cases (a) and (b) respectively.

(iv) Algebras of type $A_{3,2}^{\alpha}$: We have $a = 1$, $b = c = 0$, $d \equiv \alpha$, $0 < |\alpha| \leqslant 1$. From (4.16) and (4.17) we find

$$g = (\lambda - 1)y + \mu, \quad [(\lambda - 1)y + \mu]\dot{\phi} = (1 - \alpha)\phi.$$

For $\lambda = 1$, $\mu = 0$, $\alpha = 1$ we obtain the algebra $\{\partial_x, \phi(y)\partial_x, x\partial_x\}$; $\lambda \neq 1$, $\alpha = 1$ is not allowed; $\lambda \neq 1$, $\alpha \neq 1$ yields $\{\partial_x, py\partial_x, x\partial_x + [(1 - \alpha)/p]y\partial_y\}$ with $p \neq 0$. It is possible to transform p into $-p(p + 1)^{-1}$, hence we restrict to $-2 \leqslant p \leqslant 0$. Finally $\lambda = 1$, $\mu \neq 0$ leads to $\{\partial_x, e^y\partial_x, x\partial_x + (1 - \alpha)y\partial_y\}$.

(v) Algebras of type $A_{3,3}$: In this case $a = c = d = 1$, $b = 0$. Solving (4.16) we again have $g = (\lambda - 1)y + \mu$. For $\lambda \neq 1$ we obtain $\{\partial_x, [1/(1 - a)]\ln y\partial_x, x\partial_x + (a - 1)y\partial_y, a \neq 1\}$, for $\lambda = 1$, $\mu \neq 0$ we find another algebra, namely, $\{\partial_x, y\partial_x, x\partial_x - \partial_y\}$.

This completes the enumeration of all conjugacy classes of three-dimensional solvable Lie subalgebras over $\mathbb{C}$. To complete this section let us now construct the simple three-dimensional subalgebras of the considered algebra of holomorphic vector fields with constant divergence. According to Lemma 3.4 all such algebras over $\mathbb{C}$ must be isomorphic to sl(2,$\mathbb{C}$).

2. The sl(2,$\mathbb{C}$) subalgebras

We choose a basis $\{X_1,X_2,X_3\}$ of sl(2,$\mathbb{C}$) such that the commutation relations are

$$[X_1,X_2] = X_1, \quad [X_2,X_3] = X_3, \quad [X_3,X_1] = 2X_2. \quad (4.20)$$

We can assume that the subalgebra $\{X_1,X_2\}$ is already in standard form, namely $L_{2,4}(\mathbb{C})$ or $L_{2,5}^{\alpha}(\mathbb{C})$ of Table I. Since sl(2,$\mathbb{C}$) is simple its derived algebra equals the original sl(2,$\mathbb{C}$) algebra. It follows from Lemma 3.2 that we must have div $X_i = 0$, $i = 1,2,3$. Hence the only allowed $A_{2,1}$ subalgebra as a candidate for $\{X_1,X_2\}$ is $L_{2,5}^{\alpha = -1}(\mathbb{C})$. We have

$$X_1 = \partial_x, \quad X_2 = x\partial_x - y\partial_y,$$
$$X_3 = f(x,y)\partial_x + g(x,y)\partial_y, \quad f_x + g_y = 0. \quad (4.21)$$

Imposing the commutation relations (4.20) we find

$$f = -x^2 + \alpha/y^2, \quad g = 2xy + \beta, \quad (4.22)$$

where α and β are constants. Transformations with constant Jacobian determinant leaving the subalgebra $\{X_1,X_2\}$ invariant are

$$\xi = rx + s/y - p, \quad \eta = qy,$$
$$rq \neq 0, \quad r,s,p,q = \text{const.} \quad (4.23)$$

Using (4.23) to simplify (4.21) and (4.22) we find that two cases must be distinguished. For $\alpha = \beta^2/4$ in (4.22) we can transform X_3 into

$$X_3 = -x^2\partial_x + 2xy\partial_y, \quad (4.24)$$

where $\alpha \neq \beta^2/4$ and we can transform X_3 into

$$X_3 = (-x^2 + 1/y^2)\partial_x + 2xy\partial_y. \quad (4.25)$$

An alternative form of these two sl(2,$\mathbb{C}$) algebras is obtained by a coordinate transformation

$$x = v/u, \quad y = u^2. \quad (4.26)$$

The resulting representatives of conjugacy classes of sl(2,$\mathbb{C}$) algebras are

$$X_1 = u\partial_v, \quad X_2 = \tfrac{1}{2}(-u\partial_u + v\partial_v), \quad X_3 = -v\partial_u, \quad (4.27)$$

and

$$X_1 = u\partial_v, \quad X_2 = \tfrac{1}{2}(-u\partial_u + v\partial_v),$$
$$X_3 = (1/u^3)\partial_v + v\partial_u. \quad (4.28)$$

The classification of three-dimensional subalgebras over $\mathbb{C}$ is summarized in Table II. In the first column we list the isomorphism class of each subalgebra, following Lemma 3.4. For solvable algebras, namely, $L_{3,1},\ldots,L_{3,4}$, $\{X_1,X_2\}$ is an Abelian ideal. This ideal is uniquely defined in all isomorphism classes except $3A_1$ and $A_{3,1}$ (Abelian and nilpotent, respectively). For solvable algebras we also give the matrix

TABLE II. Three-dimensional complex subalgebras.

Type	N_0	Basis X_1	X_2	X_3	M	div X	div X_I	dim V	dim V_I
$3A_1$	$L^{f_1 f_2}_{3,1}(\mathbb{C})$	∂_x	$f_1(y)\partial_x$	$f_2(y)\partial_x$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	0	0	1	1
	$(1, f_1, f_2$ linearly independent)								
$A_1 \oplus A_2$	$L_{3,2}(\mathbb{C})$	∂_x	∂_y	$y\partial_y$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	μ_3	0	2	2
$A_2 = \{X_2, X_3\}$									
	$L_{3,3}(\mathbb{C})$	$y\partial_y$	∂_x	$x\partial_x$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\mu_1+\mu_3$	μ_1	2	2
	$L^{\alpha}_{3,4}(\mathbb{C})$ $(\alpha\neq 0)$	∂_x	$y^{\alpha}\partial_x$	$-(1/\alpha)y\partial_y$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$-(1/\alpha)\mu_3$	0	2	1
	$L_{3,5}(\mathbb{C})$	∂_x	$e^{-y}\partial_x$	∂_y	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	0	0	2	1
$A_{3,1}$	$L_{3,6}(\mathbb{C})$	∂_x	∂_y	$y\partial_x$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	0	0	2	2
	$L_{3,7}(\mathbb{C})$	∂_x	$y\partial_y$	$\ln y\partial_x$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	μ_2	μ_2	2	2
$A^{\alpha}_{3,2}$	$L^{\alpha}_{3,8}(\mathbb{C})$	∂_x	∂_y	$x\partial_x + \alpha y\partial_y$	$\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}$	$\mu_3(1+\alpha)$	0	2	2
$0<\lvert\alpha\rvert\leqslant 1$									
	$L^{\alpha,p}_{3,9}(\mathbb{C})$	∂_x	$y^{p}\partial_x$	$x\partial_x + [(1-\alpha)/p]y\partial_y$	$\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}$	$\mu_3(1+(1-\alpha/p))$	0	2	1
	$-2\leqslant p<0,\quad \alpha\neq 1$								
	$L^{\alpha}_{3,10}(\mathbb{C})$ $(\alpha\neq 1)$	∂_x	$e^{y}\partial_x$	$x\partial_x + (1-\alpha)\partial_y$	$\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}$	μ_3	0	2	1
	$L^{f}_{3,11}(\mathbb{C})$	∂_x	$f(y)\partial_x$	$x\partial_x$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	μ_3	0	1	1
	$\dot f(y)\neq 0$								
$A_{3,3}$	$L_{3,12}(\mathbb{C})$	∂_x	∂_y	$(x+y)\partial_x + y\partial_y$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$2\mu_3$	0	2	2
	$L^{\alpha}_{3,13}(\mathbb{C})$ $(\alpha\neq 0)$	∂_x	$\alpha\ln y\partial_x$	$x\partial_x - (1/\alpha)y\partial_y$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$-(1/\alpha)\mu_3$	0	2	1
	$L_{3,13}(\mathbb{C})$	∂_x	$y\partial_x$	$x\partial_x - \partial_y$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	μ_3	0	2	1
$A_{3,5}$,(sl(2,$\mathbb{C}$))	$L_{3,15}(\mathbb{C})$	$x\partial_y$	$\frac{1}{2}(-x\partial_x + y\partial_y)$	$-y\partial_x$		0	0	2	
	$L_{3,16}(\mathbb{C})$	$x\partial_y$	$\frac{1}{2}(-x\partial_x + y\partial_x)$	$x^{-3}\partial_y + y\partial_x$		0	0	2	

M of Lemma 3.4 in the fourth column. In the last columns $X = \sum_{i=1}^{3}\mu_i X_i$ is a general element of the Lie algebra, $X_I = \sum_{\mu=1}^{2}\mu_i X_i$ a general element of the ideal, V is the vector space spanned by elements of the Lie algebra at a generic point (x, y), V_I the vector space spanned by elements of the ideal at a generic point.

We have arrived at the following theorem.

Theorem 4.3: Every complex three-dimensional subalgebra of the algebra T is conjugate under the pseudogroup P to an algebra in Table III. Two algebras in Table III are mutually conjugate precisely in one of the two following cases:

TABLE III. One-dimensional real subalgebras.

N_0	Basis element X^R	Complex form	div X^R	div X ($p\in\mathbf{R}, \lambda, \eta, H(y)\in\mathbb{C}$)	Normalizer in P
$L_{1,1}(\mathbf{R})$	$i(\partial_x - \partial_{\bar x})$	$L^{\mathbb{C}}_{1,1}$	0	0	$\xi = px + H(y),\ \eta = \lambda y + \mu,\ \lambda p\neq 0$
$L_{1,2}(\mathbf{R})$	$i(x\partial_x - \bar x\partial_{\bar x})$	$L^{\mathbb{C}}_{1,2}$	0	$\neq 0$	$\xi = \lambda\dot H^{-1}(y)x, \eta = H(y), \dot H(y)\neq 0$
$L^{a}_{1,3}(\mathbb{C})$	$x\partial_x + \bar x\partial_{\bar x} + ia(x\partial_x - \bar x\partial_{\bar x})$	$L_{1,2}(\mathbb{C})$	$\neq 0$	$\neq 0$	$\xi = \lambda\dot H^{-1}(y)x, \eta = H(y), \dot H(y)\neq 0$

$$L_{3,1}^{f_1 f_2}(\mathbb{C}) \sim L_{3,1}^{g_1 g_2}(\mathbb{C}),$$

if $\{1, f_1, f_2\}$ and $\{1, g_1, g_2\}$ are equivalent under relation (4.8),

$$L_{3,11}^{f}(\mathbb{C}) \sim L_{3,11}^{g}(\mathbb{C}),$$

if $\{1, f\}$ and $\{1, g\}$ are equivalent under (4.8).

V. CLASSIFICATION OF LOW-DIMENSIONAL REAL SUBALGEBRAS

We are interested in algebras of vector fields of the form (3.3) satisfying $\operatorname{div} X = \text{const}$, i.e., the divergence of the complexification of X^R is constant. We proceed as in the complex case, remembering that the ground field now is $\mathbb{R}$ rather than $\mathbb{C}$. Whenever possible we make use of the results of Sec. IV.

A. One-dimensional real subalgebras

Starting with a vector field X^R in the form (3.4), we perform the general transformation (3.5), taking X into (3.7). The choice (4.1) takes X into (4.2) and hence X^R into

$$X^R = -(J_0 g/G_x)\partial_\xi - (\bar{J}_0 \bar{g}/\bar{G}_x)\partial_{\bar{\xi}}. \tag{5.1}$$

Requiring that the divergence be constant we obtain

$$\operatorname{div} X = -(J_0 g/G_x)_\xi - (\bar{J}_0 \bar{g}/\bar{G}_x)_{\bar{\xi}} = \alpha + \bar{\alpha}. \tag{5.2}$$

Differentiating with respect to ξ we obtain $(-J_0 g/G_x)_{\xi\xi} = 0$ and hence

$$-J_0 g/G_x = \alpha\xi + \beta. \tag{5.3}$$

The vector field X^R in the coordinates (ξ,η) now is

$$X^R = (\alpha\xi + \beta)\partial_\xi + (\bar{\alpha}\bar{\xi} + \bar{\beta})\partial_{\bar{\xi}}. \tag{5.4}$$

We can multiply X^R by an arbitrary nonzero real constant and also translate ξ. We find three different classes, corresponding to $\alpha = 0$, α pure imaginary and $\operatorname{Re}\alpha \neq 0$, respectively. We thus arrive at the following result.

Theorem 5.1: An arbitrary one-dimensional real subalgebra $\{X^R\}$ of the Lie algebra T of constant divergence holomorphic vector fields is conjugate under the pseudogroup P to precisely one of the algebras in Table III. □

The value of a in $L_{1,3}^{a}(\mathbb{R})$ is invariant under P, since we have $\operatorname{div} X = 1 + ia$ and Theorem 3.1 tells us that the divergence of a vector field X is not changed by holomorphic transformations with constant Jacobian determinant.

We shall need the normalizers of the one-dimensional subalgebras in P, i.e., the transformations (3.5) leaving $L_{1,i}$ $(i = 1,2,3)$ invariant. They are easy to calculate and are given in the sixth column of Table III. In the third column we give the complex Lie algebra, generated by X (rather than X^R).

B. Two-dimensional real subalgebras

Similarly as in the complex case, we shall assume that one element, X_1^R, is already in its standard form, namely one given in Table III. The other basis element X_2^R is in the general form (3.3). We first impose the commutation relation, then simplify X_2^R, using the normalizer of X_1^R in P (given in the sixth column of Table III).

1. Abelian subalgebras

(A) $X_1^R = i(\partial_x - \partial_{\bar{x}})$. Requiring $[X_1^R, X_2^R] = 0$ and using $\operatorname{Nor}_p L_{1,1}(\mathbb{R})$, we obtain

$$X_2^R = \{f(y)p + \dot{H}(y)[(\alpha/\lambda)(y-\mu) + \beta]\}\partial_x + [\alpha y - \mu\alpha + \beta\lambda]\partial_y + \text{c.c.}, \tag{5.5}$$

where $f(y)$, α, and β are given and p, λ, μ, and $H(y)$ are our choices. Depending on the original values of α, β, and $f(y)$, the following possibilities occur:

(A_1) $X_2^R = y\partial_y + \bar{y}\partial_{\bar{y}} + ia(y\partial_y - \bar{y}\partial_{\bar{y}})$,

(A_2) $X_2^R = i(y\partial_y - \bar{y}\partial_{\bar{y}})$,

(X_3) $X_2^R = i(\partial_y - \partial_{\bar{y}})$,

(X_4) $X_2^R = \partial_x + \partial_{\bar{x}}$,

(X_5) $X_2^R = f(y)\partial_x + \bar{f}(\bar{y})\partial_{\bar{x}}$, $\dot{f}(y) \neq 0$.

(B) $X_1^R = i(x\partial_x - \bar{x}\partial_{\bar{x}})$. Requiring $[X_1^R, X_2^R] = 0$ using $\operatorname{Nor}_p(L_{1,2}(\mathbb{R}))$ we obtain

$$X_2^R = [a - \dot{g}(y) - [\ddot{G}(y)/\dot{G}(y)]g(y)]x\partial_x + g\dot{G}(y)\partial_y + \text{c.c.}, \tag{5.6}$$

where a and $g(y)$ are given and $G(y)$ is our choice. Two possibilities occur, namely, $g(y) \neq 0$ and $g(y) = 0$,

(B_6) $X_2^R = y\partial_y + \bar{y}\partial_{\bar{y}}$,

(B_7) $X_2^R = x\partial_x + \bar{x}\partial_{\bar{x}}$.

(C) $X_1^R = x\partial_x + \bar{x}\partial_{\bar{x}} + ia(x\partial_x - \bar{x}\partial_{\bar{x}})$. Proceeding as above we obtain

$$X_2^R = [ib - \dot{g}(y) - (g/\dot{G})\ddot{G}]x\partial_x + g(y)\dot{G}\partial_y + \text{c.c.} \tag{5.7}$$

For $g \neq 0$, $b \neq 0$ we obtain an algebra conjugate to B_6. For $q \neq 0$, $b = 0$ we obtain an algebra conjugate to A_1. For $g = 0$ we reobtain B_7.

2. Non-Abelian subalgebras

Since the vector fields X_1 and X_2 satisfy $[X_1, X_2] = X_1$ it follows from Lemma 3.2 that $\operatorname{div} X = 0$. Hence we can always put

$$X_1^R = i(\partial_x - \partial_{\bar{x}}).$$

Using the commutation relation and the normalizer $\operatorname{Nor}_p L_{1,1}$ we obtain

$$X_2^R = [x - H(y) + ph + \{[(\alpha y - \mu)/\lambda] + \beta\}\dot{H}]\partial_x + (\alpha y_\alpha \mu + \beta\lambda)\partial_y + \text{c.c.} \tag{5.8}$$

Here $h(y)$, α, and β are given, p, λ, μ, and $H(y)$ are our choices. We distinguish three cases, namely, $\alpha = 0$, $\beta \neq 0$; $\alpha = 0$, $\beta = 0$; and $\alpha \neq 0$. We obtain

(D_8) $X_2^R = x\partial_x + \bar{x}\partial_{\bar{x}} + i(\partial_y - \partial_{\bar{y}})$,

(D_9) $X_2^R = x\partial_x + \bar{x}\partial_{\bar{x}}$,

(D_{10}) $X_2^R = x\partial_x + \bar{x}\partial_{\bar{x}} + a(y\partial_y + \bar{y}\partial_{\bar{y}}) + ib(y\partial_y - \bar{y}\partial_y)$, $(a,b) \neq (0,0)$.

For each algebra $L_{2,k}$ $(k = 1,\ldots,10)$, we calculate its normalizer $\operatorname{Nor}_p L_{2,k}(\mathbb{R})$ in the pseudogroup P. All results are summarized in Table IV. In the fourth column we give

TABLE IV. Two-dimensional real subalgebras.

Type	N_0	Basis X_1^R	X_2^R	Complexification	div X $X=p_1X_1+p_2X_2$	$\mathrm{Nor}_P L_{2,k}(\mathbb{R})$
$2A_1$	$L_{2,1}(\mathbb{R})=A_4$	$i(\partial_x-\partial_{\bar{x}})$	$\partial_x+\partial_{\bar{x}}$	$L_{1,1}(\mathbb{C})$	0	$\xi=\lambda x+H(y),\ \eta=\mu y+\nu,\ \lambda\mu\neq 0$
	$L_{2,2}(\mathbb{R})=B_7$	$i(x\partial_x-\bar{x}\partial_{\bar{x}})$	$x\partial_x+\bar{x}\partial_{\bar{x}}$	$L_{1,2}(\mathbb{C})$	ip_1+p_2	$\xi=[\lambda/H(y)]x,\ \eta=H(y),\ \lambda\dot{H}(y)\neq 0$
	$L_{2,3}(\mathbb{R})=A_3$	$i(\partial_x-\partial_{\bar{x}})$	$i(\partial_y-\partial_{\bar{y}})$	$L_{2,1}(\mathbb{C})$	0	$\xi=px+qy+\lambda,\ \eta=rx+sy+\nu,$ $ps-qr\neq 0$
	$L_{2,4}(\mathbb{R})=A_2$	$i(\partial_x-\partial_{\bar{x}})$	$i(y\partial_y-\bar{y}\partial_{\bar{y}})$	$L_{2,2}(\mathbb{C})$	ip_2	$\xi=px+q\ln y+\mu,\ \eta=\lambda y,\ p\lambda\neq 0$
	$L_{2,5}^a(\mathbb{R})=A_1$	$i(\partial_x-\partial_{\bar{x}})$	$y\partial_y+\bar{y}\partial_{\bar{y}}+ia(y\partial_y-\bar{y}\partial_{\bar{y}})$	$L_{2,2}(\mathbb{C})$	$(1+ia)p_2$	$\xi=px+[iq/(1+ia)]\ln y+\mu,\ \eta=\lambda y,\ p\lambda\neq 0$
	$L_{2,6}(\mathbb{R})=B_6$	$i(x\partial_x-\bar{x}\partial_{\bar{x}})$	$y\partial_y+\bar{y}\partial_{\bar{y}}$	$L_{2,2}(\mathbb{C})$	ip_1+p_2	$\xi=\alpha x,\ \eta=\beta y,\ \alpha\beta\neq 0$
	$L_{2,7}^f(\mathbb{R})=A_5$ $\dot{f}(y)\neq 0$	$i(\partial_x-\partial_{\bar{x}})$	$f(y)\partial_x+f(\bar{y})\partial_{\bar{x}}$	$L_{2,3}^f(\mathbb{C})$	0	$\xi=[\alpha/\dot{H}(y)]x+K(y),\ \eta=H(y),\ \alpha\dot{H}(y)\neq 0$ $i\beta f(y)f[H(y)]+\{-af(y)+\delta f[H(y)]\}$ $+ic=0,\ a\delta-\beta c=0$
$A_{2,1}$	$L_{2,8}(\mathbb{R})=D_8$	$i(\partial_x-\partial_{\bar{x}})$	$x\partial_x+\bar{x}\partial_{\bar{x}}+i(\partial_y-\partial_{\bar{y}})$	$L_{2,4}(\mathbb{C})$	p_2	$\xi=px+\lambda e^{-iqy}+ir,\ \eta=y+\mu,\ p\neq 0$
	$L_{2,9}(\mathbb{R})=D_9$	$i(\partial_x-\partial_{\bar{x}})$	$x\partial_x+\bar{x}\partial_{\bar{x}}$	$L_{2,5}^0(\mathbb{C})$	p_2	$\xi=px+q,\ \eta=\lambda x+\mu,\ p\lambda\neq 0$
	$L_{2,10}^{a,b}(\mathbb{R})=D_{10}$ $(a,b)\neq(0,0)$	$i(\partial_x-\partial_{\bar{x}})$	$x\partial_x+\bar{x}\partial_{\bar{x}}+a(y\partial_y+\bar{y}\partial_{\bar{y}})$ $+ib(y\partial_y-\bar{y}\partial_{\bar{y}})$	$L_{2,5}^{a+ib}(\mathbb{C})$	$p_2(1+a+ib)$	$\xi=x+\mu y[p/(a+ib)]-iq,\ \eta=\lambda y,\ \lambda\neq 0$

the corresponding complex algebra of Table I. The normalizers are in the sixth column.

We arrive at the following statement.

Theorem 5.2: Every two-dimensional real subalgebra of the Lie algebra T of constant divergence holomorphic vector fields is conjugate under the pseudogroup P to an algebra in Table IV. Two algebras in Table IV are mutually conjugate if and only if they are of the type $L^f_{2,7}(\mathbb{R})$ and $L^g_{2,7}(\mathbb{R})$, where $\{1,f(y)\}$ and $\{1,g(y)\}$ are equivalent under the relation (4.8).

Notice that a complication occurs for the algebras $L^f_{2,7}(\mathbb{R})$: The function $H(y)$ in the normalizer (see Table IV) satisfies a functional relation

$$i\beta f(y)f(H(y))+\{-af(y)+\delta f[H(y)]\}+ic=0. \tag{5.9}$$

Thus four numbers a, $c\in\mathbb{R}$, β, $\delta\in\mathbb{C}$, satisfying $a\delta-\beta c\neq 0$ must exist, such that $H(y)$ satisfies (5.9).

C. Three-dimensional real subalgebras

As in the case of complex three-dimensional subalgebras, we start with the solvable ones and assume that their two-dimensional Abelian ideal $\{X_1^R,X_2^R\}$ is in one of the standard forms $L_{2,i}(\mathbb{R})$ $(i=1,...,7)$ of Table IV. We take X_3^R in the general form (3.4). We first impose the commutation relations

$$\begin{pmatrix}[X_1^R,X_3^R]\\ [X_2^R,X_3^R]\end{pmatrix}\times\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}X_1^R\\ X_2^R\end{pmatrix},\quad a,b,c,d\in\mathbb{R}, \tag{5.10}$$

and then simplify X_3^R using the appropriate normalizer $\mathrm{Nor}_p\, L_{2,i}(\mathbb{R})$, listed in the sixth column of Table IV.

1. Solvable subalgebras

(A) Ideal $L_{2,1}(\mathbb{R})$. Implementing the commutation relations (5.10) in this case we find

$$X_3^R=[(a+ib)x+\phi(y)]\partial_x+(\alpha y+\beta)\partial_y+\text{c.c.} \tag{5.11}$$

and moreover

$$c=-b,\quad d=a, \tag{5.12}$$

in (5.9). Using the normalizer $\mathrm{Nor}_P(\mathbb{R})$ we transform X_3^R to

$$\begin{aligned}X_3^R=\{&(a+ib)\xi+[(\alpha\eta-\alpha\nu+\beta\mu)(\dot{H}/\mu)\\ &-(a-ib)H+\lambda\phi]\}\partial_\xi\\ &+(\alpha\eta-\alpha\nu+\beta\mu)\partial_\eta+\text{c.c.},\end{aligned} \tag{5.13}$$

where λ, μ, ν, and $H(y)$ can be chosen at will $(\lambda\mu\neq 0)$.

The following possibilities occur. (i) Abelian algebras: $a=b=0$.

(A_1) $X_3^R=y\partial_y+\bar{y}\partial_{\bar{y}}+ip(y\partial_y-\bar{y}\partial_{\bar{y}})$,

(A_2) $X_3^R=i(y\partial_y-\bar{y}\partial_{\bar{y}})$,

(A_3) $X_3^R=i(\partial_y-\partial_{\bar{y}})$,

(A_4) $\phi(y)\partial_x+\bar{\phi}(\bar{y})\partial_{\bar{x}}$, $\dot{\phi}(y)\neq 0$.

(ii) Algebras $A^1_{3,2}$: $a=1$, $b=0$.

(A_5) $X_3^R=x\partial_x+\bar{x}\partial_{\bar{x}}+i(\partial_y-\partial_{\bar{y}})$,

(A_6) $X_3^R=x\partial_x+\bar{x}\partial_{\bar{x}}+p(y\partial_y+\bar{y}\partial_{\bar{y}})$
$+iq(y\partial_y-\bar{y}\partial_{\bar{y}})$.

(iii) Algebras $A^a_{3,4}$: $b=1$.

(A_7) $X_3^R=a(x\partial_x+\bar{x}\partial_{\bar{x}})+i(x\partial_x-\bar{x}\partial_{\bar{x}})$
$+\partial_y+\partial_{\bar{y}}$.

(A_8) $X_3^R=a(x\partial_x+\bar{x}\partial_{\bar{x}})+i(x\partial_x-\bar{x}\partial_{\bar{x}})$
$+p(y\partial_y+\bar{y}\partial_{\bar{y}})+iq(y\partial_y-\bar{y}\partial_{\bar{y}})$.

(B) Ideal $L_{2,2}(\mathbb{R})$. In view of Lemma 3.2 neither of the operators $\{X_1^R,X_2^R\}\in L_{2,2}(\mathbb{R})$ can be in the derived algebra, hence any algebra containing $L_{2,2}$ as an ideal must be Abelian. We find

$$X_3^R=-x\dot{g}(y)\partial_x+g(y)\partial_y+\text{c.c.} \tag{5.14}$$

A transformation in $\mathrm{Nor}_P\, L_{2,2}(\mathbb{R})$ can be found, taking (5.14) into

$$(B_9)\quad X_3^R=i(\partial_y-\partial_{\bar{y}}). \tag{5.15}$$

(C) Ideal $L_{2,3}(\mathbb{R})$. For $L_{2,3}(\mathbb{R})$ to be an ideal X_3^R must have coefficients that are linear in x and y. We find

$$X_3^R = (ax+cy+r)\partial_x + (bx+dy+s)\partial_y + \text{c.c.} \qquad (5.16)$$

The normalizer $\mathrm{Nor}_P\, L_{2,3}(\mathbb{R})$ is then used to transform the matrix in (5.10) into its standard form. The following cases occur: (i) Abelian algebras: $a=b=c=d=0$.

$$X_3^R = r(\partial_x + \partial_{\bar{x}}) + s(\partial_y + \partial_{\bar{y}}).$$

The corresponding algebra is conjugate under P to the algebra A_3. (ii) Decomposable algebras: $a=1$, $b=c=d=0$. Depending on whether $s=0$ or $s\neq 0$, we obtain,

$$(C_{10})\ \ X_3^R = x\partial_x + \bar{x}\partial_{\bar{x}},$$

$$(C_{11})\ \ X_3^R = x\partial_x + \bar{x}\partial_{\bar{x}} + \partial_y + \partial_{\bar{y}}.$$

(iii) Nilpotent algebras: $a=b=d=0, c=1$. Two cases occur:

$$(C_{12})\ \ X_3^R = y\partial_x + \bar{y}\partial_{\bar{x}},$$

$$(C_{13})\ \ X_3^R = y\partial_x + \bar{y}\partial_{\bar{x}} + \partial_y + \partial_{\bar{y}}.$$

(iv) $A_{3,2}^p$ algebras: $a=1, d=\alpha, b=c=0$.

$$(C_{14})\ \ X_3^R = x\partial_x + \bar{x}\partial_{\bar{x}} + p(y\partial_y + \bar{y}\partial_{\bar{y}}),$$

$$-1\leqslant p\leqslant 1, \quad p\neq 0.$$

(v) $A_{3,3}$ algebras: $a=c=d=1, b=0$.

$$(C_{15})\ \ X_3^R = (x+y)\partial_x + (\bar{x}+\bar{y})\partial_{\bar{x}} + y\partial_y + \bar{y}\partial_{\bar{y}}.$$

(vi) $A_{3,4}^p$ algebras: $a=d=p, b=-c=1$.

$$(C_{16})\ \ X_3^R = p(x\partial_x + \bar{x}\partial_{\bar{x}} + y\partial_y + \bar{y}\partial_{\bar{y}})$$

$$-y\partial_x + x\partial_y - \bar{y}\partial_{\bar{x}} + \bar{x}\partial_{\bar{y}}.$$

(D) Ideal $L_{2,4}(\mathbb{R})$. Imposing the usual commutation relations and performing a transformation by the normalizer of $L_{2,3}(\mathbb{R})$, we find

$$X_3 = [ax + (-a\beta + c\alpha)\ln(y/\delta) - a\gamma + \beta q + p\alpha]\partial_x$$

$$+ qy\partial_y + \text{c.c.} \qquad (5.17)$$

where $a, c, p, q\in\mathbb{R}$ are given and $\alpha, \beta, \gamma, \delta$ are our choice. Notice that we have $b=d=0$ in (5.10), since $\operatorname{div} X_2\neq 0$ (Lemma 3.2).

The following possibilities occur: (i) Abelian algebra: $a=c=0$. We must have $q\neq 0$; we choose $\beta=-p\alpha/q$ and reobtain the algebra B_9 corresponding to $X_3^{R(9)}$ and the ideal $L_{2,2}(\mathbb{R})$. (ii) Decomposable algebra: $a=1$. Choosing β and γ appropriately we obtain

$$(D_{17})\ \ X_3^{R(17)} = x\partial_x + \bar{x}\partial_{\bar{x}} + p(y\partial_y + \bar{y}\partial_{\bar{y}}).$$

(iii) Nilpotent algebras: $a=0$, $c=1$. We obtain, for $q\neq 0$ and $q=0$, respectively,

$$(D_{18})\ \ X_3^{R(18)} = \ln y\partial_x + \ln\bar{y}\partial_{\bar{x}} + (y\partial_y + \bar{y}\partial_{\bar{y}}).$$

$$D_{19}:\ X_3^{R(19)} = \ln y\partial_x + \ln \bar{y}\partial_{\bar{x}}.$$

(E) Ideal $L_{2,5}^p(\mathbb{R})$. In this case $\operatorname{div} X_2 = (1+ip)\neq 0$, hence X_2 cannot be in the derived algebra and we have $b=d=0$ in the matrix M of (5.10). From (5.10) we obtain

$$X_3^R = (ax + [ic/(1+ip)]\ln y + r)\partial_x$$

$$+ isy\partial_y + \text{c.c.}, \quad a,c,r,s,p\in\mathbb{R}. \qquad (5.18)$$

Using the normalizer $\mathrm{Nor}_P\, L_{2,5}^p(\mathbb{R})$ we transform X_3^R into

$$X_3^R = \{a\xi + [i/(1+ip)](-ab_2 + cb_1)\ln(\eta/\beta)$$

$$-a\alpha + rb_1 - b_2 s/(1+ip)\}\partial_\xi + is\eta\partial_\eta + \text{c.c.},$$

where $b_1, b_2\in\mathbb{R}$ and $\alpha,\beta\in\mathbb{C}$ are at our disposal ($b_1\beta\neq 0$). The following cases occur: (i) Abelian algebras: $a=c=0$. If $s\neq 0$ we obtain an algebra conjugate to B_9, if $s=0$ we reobtain A_1. (ii) Decomposable algebras: $a=1, c=0$. We obtain one new algebra

$$E_{20}:\ X_3^R = x\partial_x + \bar{x}\partial_{\bar{x}} + is(y\partial_y - \bar{y}\partial_{\bar{y}}).$$

(iii) Nilpotent algebras: $a=0$, $c=1$. For $s=0$ we obtain a new algebra

$$E_{21}:\ X_3^R = [i/(1+ip)]\ln y\partial_x - [i/(1-ip)]\ln\bar{y}\partial_{\bar{x}}.$$

For $s\neq 0$ we obtain the algebra

$$X_1^R = i(\partial_x - \partial_{\bar{x}}), \quad X_2^R = (1+ia)y\partial_y + \text{c.c.},$$

$$X_3^R = iy\partial_y + [i/(1+ip)]\ln y\partial_x + \text{c.c.} \qquad (5.19)$$

Putting

$$X_1^R = i(\partial_x - \partial_{\bar{x}}),$$

$$\tilde{X}_2^R = X_3^R = iy\partial_y + [i/(1+ip)]\ln y\partial_x + \text{c.c.},$$

$$\tilde{X}_3^R = X_2^R - pX_3^R = y\partial_y - [ip/(1+ip)]\ln y\partial_x + \text{c.c.},$$

and transforming to new variables

$$\xi = x + H(y), \quad \eta = y, \ \text{with}\ y\dot{H} = -[1/(1+ip)]\ln y,$$

we show that (5.19) is conjugate to D_{18}.

(F) Ideal $L_{2,6}(\mathbb{R})$. Since $\operatorname{div} X_1\neq 0$, $\operatorname{div} X_2\neq 0$ any algebra with $L_{2,6}(\mathbb{R})$ as an ideal must be Abelian. Hence $a=b=c=d=0$ in (5.10). Imposing the commutation relations, we obtain a simple algebra, conjugate to B_9.

(G) Ideal $L_{2,7}^\phi(\mathbb{R})$. The ideal in this case depends on an arbitrary function that we shall here denote $\phi(y)$; both X_1 and X_2 can be present in the derived algebra. With no loss of generality we can assume that the matrix M in (5.10) is already in one of its standard forms. Imposing (5.10) we find

$$X_1^R = i(\partial_x - \partial_{\bar{x}}), \quad X_2^R = \phi(y)\partial_x + \bar{\phi}(\bar{y})\partial_{\bar{x}},$$

$$X_3^R = \{[a - ib\phi(y)]x + \psi(y)\}\partial_x + g(y)\partial_y + \text{c.c.}, \qquad (5.20)$$

with

$$g\dot{\phi} = -ib\phi^2 + (a-d)\phi - ic, \quad -ib\dot{\phi}(y) + \ddot{g}(y) = 0. \qquad (5.21)$$

Rather than use the normalizer of $L_{2,7}^\phi(\mathbb{R})$ in P, we use a simpler transformation

$$\xi = x + H(y), \quad \eta = \lambda y + \mu, \qquad (5.22)$$

that takes (5.20) into

$$X_1^R = i(\partial_\xi - \partial_{\bar{\xi}}), \quad X_2^R = \phi\partial_\xi + \text{c.c.},$$

$$X_3^R = [(a - ib\phi)\xi + \psi + g\dot{H} - (a - ib\phi)H]\partial_\xi + \lambda g\partial_\eta. \qquad (5.23)$$

In (5.23) we have actually put $\phi(y) = \phi((\eta-\mu)/\lambda) = \tilde{\phi}(\eta)$ and then dropped the $\sim$ sign; the same holds for $\psi(y)$, $g(y)$ and the auxiliary function $H(y)$ (all are now considered as functions of η). We now run through all possible types of algebras. In each case we must solve Eqs. (5.21) and then use $H(y)$, λ, and μ to simplify the result. (i) Abelian algebras: $a=b=c=d=0$. We obtain one type of algebra, namely

$$G_{22}\colon\ X_1^R = i(\partial_x - \partial_{\bar{x}}),$$
$$X_2^R = \phi_1(y)\partial_x + \bar{\phi}_1(\bar{y})\partial_{\bar{x}},$$
$$X_3^R = \phi_2(y)\partial_x + \bar{\phi}_2(\bar{y})\partial_{\bar{x}},$$

where 1, $\phi_1(y)$, and $\phi_2(y)$ are linearly independent. (ii) Decomposable algebras: $a=b=c=0, d=1$. From (5.21) we find $g=\alpha\eta+\beta$. For $\alpha\neq 0$ and $\alpha=0, \beta\neq 0$, we obtain, respectively,

$$G_{23}\colon\ X_1^R = i(\partial_x - \partial_{\bar{x}}),\quad X_2^R = y^\alpha\partial_x + \bar{y}^{\bar{\alpha}}\partial_{\bar{x}},$$
$$X_3^R = -(1/\alpha)y\partial_y - (1/\bar{\alpha})\bar{y}\partial_{\bar{y}}.$$
$$G_{24}\colon\ X_1^R = i(\partial_x - \partial_{\bar{x}}),$$
$$X_2^R = e^{-y}\partial_x + e^{-\bar{y}}\partial_{\bar{x}},\quad X_3^R = \partial_y + \partial_{\bar{y}}.$$

(iii) Nilpotent algebra: $a=b=d=0, c=1$. From (5.21) we again have $g=\alpha\eta+\beta$ and again two cases occur: $\alpha\neq 0$ or $\alpha=0, \beta\neq 0$,

$$X_1^R = i(\partial_x - \partial_{\bar{x}}),\quad X_2 = -i\ln y\partial_x + i\ln\bar{y}\partial_{\bar{x}},$$
$$X_3 = y\partial_y + \bar{y}\partial_{\bar{y}}$$

and

$$X_1^R = i(\partial_x - \partial_{\bar{x}}),\quad X_2 = y\partial_x + \bar{y}\partial_{\bar{x}},$$
$$X_3 = -i(\partial_y - \partial_{\bar{y}}).$$

Neither of these are new; the first coincides with a special case of E_{22}, the second with C_{12}. (iv) Algebras $A_{3,2}^p\colon a=1, b=c=0, d=p, -1\leqslant p<1, p\neq 0$. Standard calculations lead to three new types of algebras, namely,

$$G_{25}^{p,\alpha}\colon\ X_1 = i(\partial_x - \partial_{\bar{x}}),\quad X_2 = y^\alpha\partial_x + \bar{y}^{\bar{\alpha}}\partial_{\bar{x}},$$
$$X_3 = x\partial_x + \bar{x}\partial_{\bar{x}} + [(1-p)/\alpha]y\partial_y$$
$$+ [(1-a)/\bar{\alpha}]\bar{y}\partial_{\bar{y}},\quad \alpha\in\mathbb{C},\quad \alpha\neq 0,$$
$$G_{26}^{p}\colon\ X_1 = i(\partial_x - \partial_{\bar{x}}),\quad X_2 = e^y\partial_x + e^{\bar{y}}\partial_{\bar{x}},$$
$$X_3 = x\partial_x + \bar{x}\partial_{\bar{x}} + (1-p)\partial_y + (1-p)\partial_{\bar{y}},$$
$$G_{27}^{\phi}\colon\ X_1 = i(\partial_x - \partial_{\bar{x}}),\quad X_2 = \phi(y)\partial_x + \bar{\phi}(\bar{y})\partial_{\bar{y}},$$
$$X_3 = x\partial_x + \bar{x}\partial_{\bar{x}},\quad p=1,\ \dot{\phi}(y)\neq 0.$$

(v) Algebras $A_{3,3}$: $a=c=d=1, b=0$. Since $b=0$ we have $g(y)=\lambda y+\mu$ from (5.21). Depending on whether $\lambda=0$ or $\lambda\neq 0$, we obtain one of the following algebras:

$$G_{28}^{\lambda}\colon\ X_1 = i(\partial_x - \partial_{\bar{x}}),$$
$$X_2 = (-(i/\lambda)\ln y)\partial_x + ((i/\lambda)\ln\bar{y})\partial_{\bar{x}},$$
$$X_3 = x\partial_x + \bar{x}\partial_x + \lambda y\partial_y + \bar{\lambda}y\partial_{\bar{\lambda}},\quad \lambda\neq 0,$$
$$G_{29}\colon\ X_1 = i(\partial_x - \partial_{\bar{x}}),\quad X_2 = i(y\partial_x - \bar{y}\partial_{\bar{x}}),$$
$$X_3 = x\partial_x + \bar{x}\partial_{\bar{x}} - \partial_y - \partial_{\bar{y}}.$$

(vi) Algebras $A_{3,4}^a\colon d=a, b=-c=1$. In this case the first of Eqs. (5.21) is nonlinear and difficult to solve. To avoid solving it we first diagonalize the matrix $M=\begin{pmatrix} a & 1\\ -1 & a\end{pmatrix}$ over the field of complex numbers, solve (5.21) for that case, then transform back. This leads to two algebras, namely,

$$G_{30}^{a,\lambda}\colon\ X_1^R = (i+y^{2i/\lambda})\partial_x + (-i+\bar{y}^{-2i/\lambda})\partial_{\bar{x}},$$
$$X_2^R = (1+iy^{2i/\lambda})\partial_x + (1-i\bar{y}^{-2i/\lambda})\partial_{\bar{x}},$$
$$X_3^R = a(x\partial_x + \bar{x}\partial_{\bar{x}}) + i(x\partial_x - \bar{x}\partial_{\bar{x}})$$
$$+ \lambda y\partial_y + \bar{\lambda}\bar{y}\partial_{\bar{y}},$$

and

$$G_{31}\colon\ X_1^R = i(\partial_x - \partial_{\bar{x}}),$$
$$X_2 = \left(\frac{1+e^{-2iy}}{1-e^{-2iy}}\right)\partial_x + \left(\frac{1+e^{2iy}}{1-e^{2iy}}\right)\partial_{\bar{x}},$$
$$X_3^R = \left(a + \frac{1+e^{-2iy}}{1-e^{-2iy}}\right)x\partial_x$$
$$+ \left(a + \frac{1+e^{2iy}}{1-e^{2iy}}\right)\bar{x}\partial_{\bar{x}} + i(\partial_y - \partial_{\bar{y}}).$$

The ideal $\{X_1^R, X_2^R\}$ in the case G_{30} is not in its standard form $L_{2,7}^{\phi}(\mathbb{R})$. A transformation taking it into its standard form exists, but we have not constructed it explicitly.

The results obtained so far are summarized in Table V where the subalgebras are listed by isomorphism classes. The algebras $L_{3,2}(\mathbb{R})$, $L_{3,3}^{G}(\mathbb{R})$, and $L_{3,4}(\mathbb{R})$ all have the same complexification, namely the two-dimensional algebra $L_{2,2}(\mathbb{C})$. However, they are not conjugate to each other. Indeed, putting $X=\Sigma_{i=1}^3 p_i X_i$ we find $\operatorname{div} X = ip_3$ in $L_{3,2}(\mathbb{R})$, $\operatorname{div} X = (1+ia)p_3$ in $L_{3,3}(\mathbb{R})$ and $\operatorname{div} X = p_1 + ip_2$ in $L_{3,4}(\mathbb{R})$. For the algebras $A_{3,1}$ (nilpotent) we have weeded out all redundancies, due to the fact that in this case the Abelian ideal is not unique. In all other cases the Abelian ideal is unique. Once it is fixed, the only allowed transformations are in the normalizer of the ideal and these were used to the maximal possible degree in the text.

2. Simple subalgebras

Let us now turn to the simple three-dimensional real subalgebras of the algebra of holomorphic vector fields with constant divergence. Upon complexification such an algebra will turn into $\mathrm{sl}(2,\mathbb{C})$, i.e., into either $L_{3,15}(\mathbb{C})$ or $L_{3,16}(\mathbb{C})$ of Table II. We shall rewrite these two algebras as the real algebras o(3,1) and then pick out the corresponding o(3) and o(2,1) subalgebras [unique up to conjugacy under the corresponding o(3,1) group].

We start with the "linear" $\mathrm{sl}(2,\mathbb{C})$ algebra $L_{3,15}(\mathbb{C})$. Its o(3,1) realization is represented by

$$L_1 = (i/2)[+x\partial_x - y\partial_y - \bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}}],$$
$$L_2 = (i/2)[-x\partial_y - y\partial_x + \bar{x}\partial_{\bar{y}} + \bar{y}\partial_{\bar{x}}],$$
$$L_3 = \tfrac{1}{2}[x\partial_y - y\partial_x + \bar{x}\partial_{\bar{y}} - \bar{y}\partial_{\bar{x}}],$$
$$K_1 = \tfrac{1}{2}[-x\partial_x + y\partial_y - \bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}}],$$
$$K_2 = \tfrac{1}{2}[x\partial_y + y\partial_x + \bar{x}\partial_{\bar{y}} + \bar{y}\partial_{\bar{x}}],$$
$$K_3 = (i/2)[x\partial_y - y\partial_x - \bar{x}\partial_{\bar{y}} + \bar{y}\partial_{\bar{x}}]. \tag{5.24}$$

The commutation relations are the standard ones, namely,

$$[L_i, L_k] = \epsilon_{ikl}L_l,$$
$$[L_i, K_k] = \epsilon_{ikl}K_l,$$
$$[K_i, K_k] = -\epsilon_{ikl}L_l. \tag{5.25}$$

TABLE V. Three-dimensional real subalgebras.

Type	N_0	Basis X_1^R	X_2^R	X_3^R	Complexification
$3A_1$	$L_{3,1}(\mathbb{R}) = A_3$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$i(\partial_y - \partial_{\bar y})$	$L_{2,1}$(C)
	$L_{3,2}(\mathbb{R}) = A_2$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$i(y\partial_y - \bar y\partial_{\bar y})$	$L_{2,2}$(C)
	$L_{3,3}(\mathbb{R}) = A_1$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$y\partial_y + \bar y\partial_{\bar y} + ia(y\partial_y - \bar y\partial_{\bar y})$	$L_{2,2}$(C)
	$L_{3,4}(\mathbb{R}) = B_9$	$x\partial_x + \bar x\partial_{\bar x}$	$i(x\partial_x - \bar x\partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$L_{2,2}$(C)
	$L^f_{3,5}(\mathbb{R}) = A_4$, $(\dot f \neq 0)$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$f(y)\partial_x + \bar f(\bar y)\partial_{\bar x}$	$L^f_{2,3}$(C)
	$L^{f_1 f_2}_{3,6}(\mathbb{R}) = G_{22}$ ($1, f_1, f_2$ linearly independent)	$i(\partial_x - \partial_{\bar x})$	$f_1(y)\partial_x + \bar f_1(\bar y)\partial_{\bar x}$	$f_2(y)\partial_x + f_2(\bar y)\partial_{\bar x}$	$L^{f_1 f_2}_{3,1}$(C)
$A_1 \oplus A_2$	$L_{3,7}(\mathbb{R}) = C_{10}$	$i(\partial_x - \partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$y\partial_y + \bar y\partial_{\bar y}$	$L_{3,2}$(C)
$A_2 = \{X_2^R, X_3^R\}$	$L_{3,8}(\mathbb{R}) = C_{11}$	$i(\partial_x - \partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$y\partial_y + \bar y\partial_{\bar y} + \partial_x + \partial_{\bar x}$	$L_{3,2}$(C)
	$L^a_{3,9}(\mathbb{C}) = D_{17}$	$i(y\partial_y - y\partial_{\bar y})$	$i(\partial_x - \partial_{\bar x})$	$x\partial_x + \bar x\partial_{\bar x} + a(y\partial_y + \bar y\partial_{\bar y})$	$L_{3,3}$(C)
	$L^{ab}_{3,10}(\mathbb{R}) = E_{20}$	$y\partial_y + \bar y\partial_{\bar y} + ia(y\partial_y - \bar y\partial_{\bar y})$	$i(\partial_x - \partial_{\bar x})$	$x\partial_x + \bar x\partial_{\bar x} + ib(y\partial_y - \bar y\partial_{\bar y})$	$L_{3,3}$(C)
	$L^\alpha_{3,11}(\mathbb{R}) = G_{23}$, $(\alpha \neq 0)$	$i(\partial_x - \partial_{\bar x})$	$y^\alpha\partial_x + \bar y^{\bar\alpha}\partial_{\bar x}$	$-(1/\alpha)y\partial_y - (1/\bar\alpha)\bar y\partial_{\bar y}$	$L^\alpha_{3,4}$(C)
	$L_{3,12}(\mathbb{R}) = G_{24}$	$i(\partial_x - \partial_{\bar x})$	$e^{-y}\partial_x + e^{-\bar y}\partial_{\bar x}$	$\partial_y + \partial_{\bar y}$	$L_{3,5}$(C)
$A_{3,1}$	$L_{3,13}(\mathbb{R}) = C_{12}$	$i(\partial_x - \partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$y\partial_x + \bar y\partial_{\bar x}$	$L_{3,6}$(C)
	$L_{3,14}(\mathbb{R}) = C_{13}$	$i(\partial_x - \partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$y\partial_x + \bar y\partial_{\bar x} + \partial_y + \partial_{\bar y}$	$L_{3,6}$(C)
	$L_{3,15}(\mathbb{R}) = D_{19}$	$i(\partial_x - \partial_{\bar x})$	$i(y\partial_y - \bar y\partial_{\bar y})$	$\ln y\partial_x + \ln \bar y\partial_{\bar x}$	$L_{3,7}$(C)
	$L_{3,16}(\mathbb{R}) = D_{18}$	$i(\partial_x - \partial_{\bar x})$	$i(y\partial_y - \bar y\partial_{\bar y})$	$\ln y\partial_x + \ln \bar y\partial_{\bar x} + y\partial_y + \bar y\partial_{\bar y}$	$L_{3,7}$(C)
	$L^a_{3,17}(\mathbb{R}) = E_{21}$	$i(\partial_x - \partial_{\bar x})$	$y\partial_y + \bar y\partial_{\bar y} + ia(y\partial_y - \bar y\partial_{\bar y})$	$[i/(1+ia)]\ln y\partial_x - [i/(1-ia)]\ln \bar y\partial_{\bar x}$	$L_{3,7}$(C)
$A^a_{3,2}$: $-1 < a < 1$ $a \neq 0$	$L_{3,18}(\mathbb{R}) = A_5$, $(a=1)$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$x\partial_x + x\partial_{\bar x} + i(\partial_y - \partial_{\bar y})$	$L_{2,4}$(C)
	$L_{3,19}(\mathbb{R}) = A_6$, $(a=1)$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$x\partial_x + \bar x\partial_{\bar x} + p(y\partial_y + \bar y\partial_{\bar y}) + iq(y\partial_y - \bar y\partial_{\bar y})$	$L^a_{2,5}$(C)
	$L^a_{3,20}(\mathbb{R}) = C_{14}$	$i(\partial_x - \partial_{\bar x}$	$i(\partial_y - \partial_{\bar y})$	$x\partial_x + \bar x\partial_{\bar x} + a(y\partial_y + \bar y\partial_{\bar y})$	$L^a_{2,8}$(C)
	$L^{a\alpha}_{3,21}(\mathbb{R}) = G_{25}$, $(\alpha \neq 0)$	$i(\partial_x - \partial_{\bar x})$	$y^\alpha\partial_x + \bar y^{\bar\alpha}\partial_{\bar x}$	$x\partial_x + \bar x\partial_{\bar x} + [(1-a)/\alpha]\partial_y + [(1-a)/\bar\alpha]\partial_{\bar y}$	$L^{a\alpha}_{3,9}$(C)
	$L^a_{3,22}(\mathbb{R}) = G_{26}$	$i(\partial_x - \partial_{\bar x})$	$e^y\partial_x + e^{\bar y}\partial_{\bar x}$	$x\partial_x + \bar x\partial_{\bar x} + (1-a)\partial_y + (1-a)\partial_{\bar y}$	$L^a_{3,10}$(C)
	$L^f_{3,23}(\mathbb{R}) = G^f_{27}$	$i(\partial_x - \partial_{\bar x})$	$f(y)\partial_x + \bar f(\bar y)\partial_{\bar y}$	$x\partial_x + \bar x\partial_{\bar x}$	$L^f_{3,11}$(C)
$A_{3,3}$	$L_{3,24}(\mathbb{R}) = C_{15}$	$i(\partial_x - \partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$(x+y)\partial_x + (\bar x + \bar y)\partial_{\bar x} + y\partial_y + \bar y\partial_{\bar y}$	$L_{3,12}$(C)
	$L^\alpha_{3,25}(\mathbb{R}) = G_{28}$	$i(\partial_x - \partial_{\bar x})$	$\big(-(i/\alpha)\ln y\big)\partial_x + \big((i/\bar\alpha)\ln \bar y\big)\partial_{\bar x}$	$x\partial_x + \bar x\partial_{\bar x} + \alpha y\partial_y + \bar\alpha\bar y\partial_{\bar y}$	$L^\alpha_{3,13}$(C)
	$L^a_{3,26}(\mathbb{R}) = G_{29}$	$i(\partial_x - \partial_{\bar x})$	$i(y\partial_x - \bar y\partial_{\bar x})$	$x\partial_x + \bar x\partial_{\bar x} - \partial_y - \partial_{\bar y}$	$L_{3,14}$(C)
$A^a_{3,4}$ $0 < a < \infty$	$L_{3,27}(\mathbb{R}) = A^a_7$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$(a+i)x\partial_x + (a-i)\bar x\partial_{\bar x} + \partial_y + \partial_{\bar y}$	$L_{2,4}$(C)
	$L^{a,b,c}_{3,28}(\mathbb{R}) = A^a_8$	$i(\partial_x - \partial_{\bar x})$	$\partial_x + \partial_{\bar x}$	$(a+i)x\partial_x + (b+ic)y\partial_y$ + c.c.	$L^a_{2,5}$(C)
	$L^a_{3,29}(\mathbb{R}) = C^a_{16}$	$i(\partial_x - \partial_{\bar x})$	$i(\partial_y - \partial_{\bar y})$	$a(x\partial_x + y\partial_y) - y\partial_x + x\partial_y$ + c.c.	$L^a_{3,8}$(C)
	$L^{a\lambda}_{3,30}(\mathbb{R}) = G^{a\lambda}_{30}$, $(\lambda \neq 0)$	$(i + y^{2i/\lambda})\partial_x$ + c.c.	$(1 + iy^{2i/\lambda})\partial_x$ + c.c.	$(a+i)x\partial_x + \lambda y\partial_y$ + c.c.	$L^a_{3,9}$(C)
	$L^a_{3,31}(\mathbb{R}) = G^{a\lambda}_{31}$	$i(\partial_x - \partial_{\bar x})$	$[(1+e^{-2iy})/(1-e^{-2iy})]\partial_x$ + c.c.	$\big(a + (1+e^{-2iy})/(1-e^{-2iy})\big)x\partial_x$ + c.c.	$L^a_{3,10}$(C)
$A_{3,5}$ $(\mathrm{sl}(2,\mathbb{R}))$	$L_{3,32}(\mathbb{R})$	$\frac{1}{2}(-x\partial_x + y\partial_y$ + c.c.)	$\frac{1}{2}(x\partial_y + y\partial_x$ + c.c.)	$\frac{1}{2}(x\partial_y - y\partial_x$ + c.c.)	L_{15}(C)
	$L_{3,33}(\mathbb{R})$	$\frac{1}{2}(x\partial_x - y\partial_y$ + c.c.)	$\frac{1}{2}[(x + 1/x^3)\partial_y + y\partial_x$ + c.c.]	$\frac{1}{2}[(-x + 1/x^3)\partial_y + y\partial_x$ + c.c.]	L_{16}(C)
$A_{3,6}$ $(\mathrm{su}(2))$	$L_{3,34}(\mathbb{R})$	$(i/2)(-x\partial_x + y\partial_y)$ + c.c.	$(i/2)(x\partial_y + y\partial_x)$ + c.c.	$\frac{1}{2}(x\partial_y - y\partial_x$ + c.c.)	L_{15}(C)
	$L_{3,35}(\mathbb{R})$	$(i/2)(x\partial_x - y\partial_y)$ + c.c.	$(i/2)[(x + 1/x^2)\partial_y + y\partial_x]$ + c.c.	$\frac{1}{2}[(1 - x + 1/x^3)\partial_y + y\partial_x$ + c.c.]	L_{16}(C)

The o(3,1) realization of $L_{3,16}(\mathbb{C})$ is

$$\begin{aligned}
L_1 &= \tfrac{1}{2}[(-x + 1/x^3)\partial_y + y\partial_x \\
&\quad + (-\bar x - 1/\bar x^3)\partial_{\bar y} + \bar y\partial_{\bar x}], \\
L_2 &= (i/2)[-x\partial_x + y\partial_y + \bar x\partial_{\bar x} - \bar y\partial_{\bar y}], \\
L_3 &= \tfrac{1}{2}[-(x + 1/x^3)\partial_y - y\partial_x \\
&\quad + (\bar x + 1/\bar x^3)\partial_{\bar y} + \bar y\partial_{\bar x}], \\
K_1 &= \tfrac{1}{2}[(-x + 1/x^3)\partial_y \\
&\quad + y\partial_x - (-\bar x + 1/\bar x^3)\partial_{\bar y} - \bar y\partial_{\bar x}], \\
K_2 &= \tfrac{1}{2}[x\partial_x - y\partial_y + \bar x\partial_{\bar x} - \bar y\partial_{\bar y}], \\
K_3 &= \tfrac{1}{2}[(x + 1/x^3)\partial_y + y\partial_x \\
&\quad + (-\bar x - 1/\bar x^3)\partial_{\bar y} + \bar y\partial_{\bar x}],
\end{aligned} \tag{5.26}$$

and the commutation relations are again (5.25). We thus obtain two different o(3) [or su(2)] algebras, namely L_1, L_2, and L_3 in both cases. Convenient choices of the o(2,1) [i.e., $\mathrm{sl}(2,\mathbb{R})$] subalgebras are $\{K_1, K_2, L_3\}$ in the first case and $\{K_2, K_3, L_1\}$ in the second. Four further real subalgebras are thus obtained and they are included in Table V.

VI. CONCLUSIONS AND PREVIEW OF FUTURE ATTRACTIONS

The main results of this paper are summed up in Tables III–V providing representatives of the conjugacy classes of one-, two-, and three-dimensional *real* subalgebras of the algebra of holomorphic vector fields in two complex variables, having constant divergence. The classification is performed under the pseudogroup P of biholomorphic transformations with constant Jacobian determinant.

The stage is now set for performing the actual symmetry reduction of the Ω equation and obtaining solutions and metrics. To give an example of the type of application we have in mind, consider the subalgebra $L_{3,4}(\mathbb{R})$ of Table V. Calculating its invariants in a standard manner[20–23] we find the expression

$$\Omega(x,\bar{x},y,\bar{y}) = \sqrt{x\bar{x}}F(\xi), \quad \xi = y + \bar{y}. \tag{6.1}$$

Substituting into the Ω equation $(*)$ we obtain an equation for $F(\xi)$,

$$F\ddot{F} - \dot{F}^2 = 4. \tag{6.2}$$

This nonlinear ordinary differential equation is invariant under translations and dilations, and can hence easily be solved. Substituting the solution back into (6.1), we obtain

$$\Omega = (1/K\sqrt{x\bar{x}})\cosh 2K(y + \bar{y} - c), \tag{6.3}$$

where K and c are integration constants. From this expression for Ω we obtain the metric tensor

$$ds^2 = \cosh(y+\bar{y})\big((1/4\sqrt{x\bar{x}})dx\,d\bar{x} + \sqrt{x\bar{x}}\,dy\,d\bar{y}\big) + \sinh(y+\bar{y})\big(\tfrac{1}{2}\sqrt{(\bar{x}/x)}\,dx\,d\bar{y} + \tfrac{1}{2}\sqrt{(x/\bar{x})}\,d\bar{x}\,dy\big). \tag{6.4}$$

A straightforward curvature computation shows that this is the flat metric on $\mathbb{R}^4$. In Part II we shall use all obtained subalgebras in a similar manner and describe the metrics obtained. For example the su(2) algebra $L_{3,34}(\mathbb{R})$ gives the well-known Eguchi–Hanson metric.

ACKNOWLEDGMENTS

One of the authors (C.P.B.) was supported in part by NSF Grants DMS-8508950 and 8875581. The other author (P.W.) was supported in part by research grants from NSERC of Canada and the "Fonds FCAR du Gouvernment du Québec."

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