Symmetry and Group Actions

CommDSP Tutorial

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Abstract

We will see that Group Action on sets is the natural ways of modelling symmetry and studying invariants. We will set up the language and notation first.

Let X be a set and let $S(X)$ be the set of bijections on X. We write composition between maps by juxtaposition so that $(ST)x := S(Tx)$ for $x \in X$.

Definition 1. *A function $\sigma: X \to Y$ is said to be **invariant** under $T \in S(X)$ if $\sigma T = \sigma$. We also say σ is preserved by T, or that it is T-invariant.*

Example 1. Let X be the set of $n \times n$ complex matrices. Define $T(A) := \{T\,\mathrm{A}\mathrm{T}^{-1}\}$ for $A \in X$. Then $\sigma(A) := \mathrm{tr}(A)$ is invariant under T. Note that the eigenvalues, determinant, characteristic polynomial are also unaffected by the transformation T.

Example 2. Volume of a set is preserved by rotations and translations.

The invariants define a **equivalence relation** $\sim$ on X:

$$x \sim y \Leftrightarrow \sigma(x) = \sigma(y)$$

The **equivalence classes** of $x \in X$ is given by

$$[x] := \{y \in X: \sigma(x) = \sigma(y)\}$$

Note that $[x] \cap [y] \neq \phi \Rightarrow [x] = [y]$. So, these equivalence classes form a **partition** of X. The set of distinct equivalence classes is called the **quotient set** of $\sim$ and is denoted by by $X/\sim$. Any function (like σ) that is constant on each equivalence class is called a **class function**.

Definition 2. *Let us define the set of all σ-preserving transformations as $P(\sigma)$*

$$P(\sigma) := \{T \in S(X): \forall x \in X(\sigma(Tx) = \sigma(x))\}$$

Example 3. Let X be a normed space with norm N. Then, the set of all norm-preserving transformations is $P(N)$

Proposition 1. *Suppose T preserves σ. Then $\sigma(x) = \sigma(y) \Leftrightarrow \exists T \in P(\sigma)(y = Tx)$*

Proof. *($\Leftarrow$) follows directly from the definition. For the other implication, define T by*

$$T(z) := \begin{cases} x, z = y \\ y, z = x \\ z, \text{otherwise} \end{cases}$$

and note that $T \in P(\sigma)$ and $y = Tx$ $\square$

Proposition 2. *$\mathcal{G} := P(\sigma)$ satisfies the following properties (**Group Axioms**)*

i. $I \in \mathcal{G}$ (Existence of Identity)

ii. $T \in \mathcal{G} \Rightarrow T^{-1} \in \mathcal{G}$ (Existence of Inverse)

iii. $\mathcal{G}$ is closed under an associative binary operation (composition in this case)

Proof. *By combining Proposition(1) with the definition of* $\sim$, *we get*

$$x \sim y \Leftrightarrow \exists T \in \mathcal{G}(y = Tx)$$

The proof will follow from properties of the equivalence relation

i. *(Reflexivity)* $\forall x \in X(x \sim x) \Rightarrow I \in \mathcal{G}$

ii. *(Symmetry)* $\forall x, y \in X(x \sim y \Rightarrow y \sim x) \Rightarrow (T \in \mathcal{G} \Rightarrow T^{-1} \in \mathcal{G})$

iii. *(Transitivity)* $\forall x, y, z \in X(x \sim y \wedge y \sim z \Rightarrow x \sim z) \Rightarrow (S, T \in \mathcal{G} \Rightarrow ST \in \mathcal{G})$. *So,* $\mathcal{G}$ *is closed under composition. Note that composition of functions is automatically associative* □

We call any subset $\mathcal{G}$ of $P(\sigma)$ that is closed under inverses and composition a **Group**. We then say $\mathcal{G}$ is a **subgroup** of $P(\sigma)$. Note that any subgroup of $P(\sigma)$ preserves σ. Every class function of the equivalence relation defined by $x \sim y \Leftrightarrow \exists T \in \mathcal{G}(y = Tx)$ is an invariant under $\mathcal{G}$. For a general abstract group, multiplication operation on $\mathcal{G}$ need not be composition.

Proposition 3. $\pi: \mathcal{G} \times X \to X$ *defined by* $\pi(T, x) := Tx$ *(we will also denote it as* Tx*) satisfies the following properties (**Group Action of $\mathcal{G}$ on X**)*

i. $\forall x \in X\, (Ix = x)$

ii. $\forall S, T \in \mathcal{G}(\forall x \in X((ST)x = S(Tx))$

Proof. *Follows from definitions* □

Remark 1. If $\mathcal{G}$ is a group, X is a set and there exists $\pi: \mathcal{G} \times X \to X$ satisfying the group action axioms, we say **$\mathcal{G}$ acts on X**, or that **X is a $\mathcal{G}$-set**. The Elements of the $\mathcal{G}$ are sometimes called the called **Symmetries** of X. The definition above also applies to abstract groups – note that we denote the multiplication by juxtaposition, just like we do for composition

Most groups arise naturally as group actions. Here are some examples.

Example 4. The set of all bijections of a finite set X form a group $S(X)$ called the **Symmetric Group**. This acts on X by permuting the elements (Every bijection of a finite set is a permutation or a rearrangement). When $X := \{1, ..., n\}$, we denote it as S_n. Note that $|S_n| = n!$

Example 5. The set $\mathcal{G}$ of invertible $n \times n$ complex matrices under multiplication form a group. They act on vectors in $X = \mathbb{C}^n$. Note how the group operation of multiplication corresponds to composition of maps when it acts on the set X. This follows from the definition of Group Action.

Example 6. Every group G acts on itself by (left) multiplication. Define $\pi: G \times G \to G$ for g and h in G by $\pi(g, h) = g * h$, where $*$ is group multiplication. Check that it is a group action.

Example 7. Let X be a metric space with a metric ρ on $X \times X$. If $\mathcal{G}$ acts on X, note that $\mathcal{G}$ acts on $X \times X$ by $\hat{T}(x, y) := (Tx, Ty)$ where $T \in \mathcal{G}$ and $x, y \in X$. Then T is called an **isometry** if it preserves ρ. So, $\rho\hat{T} = \rho$ or $\forall x, y \in X(\rho(x, y) = \rho(Tx, Ty))$

Example 8. Another huge class of class of groups that arise naturally as group actions are **Symmetry groups**. They are isometries of a geometric figure that preserve the area occupied by them. This is a slight abuse of terminology. The isometries themselves only act on points $x \in X$ but they can be extended to sets $A \subset X$ by defining $T(A)$ to be the image of A under T.

Definition 3. *Suppose* $x \in X$ *and* $\mathcal{G}$ *acts on* X. *The **orbit** of* x *is* $\mathcal{G}x := \{y \in X: y = Tx, T \in \mathcal{G}\}$. *The **stabilizer** of* x *is* $\mathcal{G}_x = \{T \in \mathcal{G}: Tx = x\}$

The following are trivial consequences of the definition:

- Orbits define an equivalence relation on X given by $x \sim y \Leftrightarrow \exists T \in \mathcal{G}(y = Tx)$. In other words, X can be partitioned by the action of $\mathcal{G}$ into orbits. The set of orbits is the quotient set $X/\mathcal{G}x$.
- The Stabilizer of $x \in X$ is a subgroup of G. It is one of the equivalence classes of the equivalence relation $S \sim T \Leftrightarrow Sx = Tx$. The set of all these equivalence classes are called **cosets** of the subgroup $\mathcal{G}_x$ – these partition G