Collaboration Research Project

Computation of invariants of quasihomogeneous and Newton non-degenerate singularities

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1 Introduction

To study the singularities of a hypersurface one has a plethora of invariants that have diverse origins, but that they are related to each other. For example, multiplier ideals, the Bernstein-Sato polynomial, Hodge ideals or zeta functions. In general, these invariants are difficult to calculate, with high complexity of computation. Nonetheless, among the cases for which easier expressions are known, we find the hypersurfaces defined by quasi-homogeneous polynomials or non-degenerate polynomials with respect to their Newton polygon.

A good way to get started in the study of singularities is to understand the results that one can find in the literature regarding these examples. The tools used in these cases have a certain combinatorial flavor that makes them more affordable, but at the same time, these examples form a class of singularities general enough to observe certain interesting phenomena.

General methods for computing multiplier ideals, Bernstein-Sato polynomials or Hodge ideals have been implemented in packages of mathematical software such as Macaulay 2 or Singular, but they are very expensive from a computational point of view. Surprisingly, we find no implementation of formulas and algorithms specific to quasihomogeneous and Newton non-degenerate singularities. Understanding these algorithms and implementing them will be useful for developing a good bank of examples that would allow testing some conjectures.

2 Preliminaries

First of all, we begin by recalling basic notions of singularity theory, and set the default notation for this work, which will follow mostly the exposition in [GMG07].

Consider $f\colon U \to \mathbb{C}$ a holomorphic function defined on an open set $U \subset \mathbb{C}^n$, defining the hypersurface $X = f^{-1}(0)$. We call the *set of singular points* of X to the set

$$\mathrm{Sing}(X) \coloneqq \left\{ x \in U \mid f(x) = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\}$$

In particular, we will say that a point $x \in U$ is an *isolated singularity* if $x \in \mathrm{Sing}(X)$ and it is the only singularity in a small enough neighborhood $V \ni x$, i.e. $\mathrm{Sing}(X) \cap V = \{x\}$. Then, we also say that the germ $(X, x) \subset (\mathbb{C}^n, x)$ is an *isolated hypersurface singularity.*

By considering the Jacobian ideal in $\mathcal{O}_{\mathbb{C}^n}(U)$

$$J(f) \coloneqq \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \cdot \mathcal{O}_{\mathbb{C}^n}(U)$$

we can define the *Milnor* and the *Tjurina* algebras of f at x

$$M_{f.x} \coloneqq \mathcal{O}_{\mathbb{C}^n,x}/J(f)\,\mathcal{O}_{\mathbb{C}^n,x}, \qquad T_{f.x} \coloneqq \mathcal{O}_{\mathbb{C}^n,x}/(f, J(f))\,\mathcal{O}_{\mathbb{C}^n,x},$$

The dimension of these analytic algebras as $\mathbb{C}$-vector spaces are called, respectively, the *Milnor* and *Tjurina number* of f at x.

$$\mu(f, x) \coloneqq \dim_{\mathbb{C}} M_{f,x}, \qquad \tau(f, x) \coloneqq \dim_{\mathbb{C}} T_{f,x}$$

It is clear that $\mu(f, x) \neq 0$ if, and only if, $\frac{\partial f}{\partial x_i}(x) = 0$ for all i, so we can interpret μ as counting the singular points of the *function* f, each with multiplicity $\mu(f, x)$. Similarly, $\tau(f, x) \neq 0$ if, and only if, additionally $f(x) = 0$, so that τ counts the singular points of the *zero set* of f, each with multiplicity $\tau(f, x)$.

Throughout this work, we will consider for convenience that we have an isolated singularity at the origin and $f(0) = 0$. Since the context will be clear, from now on we will simply write M_f, T_f to denote the algebras, and μ_f, τ_f to denote the Milnor and Tjurina numbers, respectively.

Lemma 1 ([GMG07], Lemma 2.3) The following are equivalent

i) 0 is an isolated singularity of X

ii) $\mu_f < \infty$

iii) $\tau_f < \infty$

Even more, we have the following inequality involving both the Tjurina and Milnor numbers.

Theorem 1 ([Liu18], Thm. 1.1) Assume that $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. Then

$$\frac{\mu_f}{\tau_f} \leq n$$

with equality if, and only if, $\ker(f) = (f^{n-1})$.

Proof. Let $J(f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ be the Jacobian ideal, and consider the following exact sequence of $\mathbb{C}$-algebras

$$0 \to \ker(f) \to M_f \xrightarrow{f} M_f \to T_f \to 0$$

where $\dim \ker(f) = \tau_f$. We have that $f^n \in J(f)$ by a well-know result by Briançon and Skoda. Then,

$f^n = 0$ in M_f and thus, $(f^{n-1}) \subset \ker(f)$. Then, we have the finite decreasing chain

$$M_f \supset (f) \supset (f^2) \supset \cdots \supset (f^{n-1}) \supset (f^n) = 0$$

We now consider the exact sequence for each i from 1 to $n-1$,

$$0 \to \ker(f) \cap (f^i) \to (f^i) \xrightarrow{f} (f^i) \to (f^i)/(f^{i+1}) \to 0$$

Then $\dim\big((f^i)/(f^{i+1})\big) = \dim\big(\ker(f) \cap (f^i)\big) \leq \dim\ker(f) = \tau_f$. Therefore,

$$\mu_f = \dim(M_f) = \dim(T_f) + \sum_{i=1}^{n-1} \dim\Big((f^i)/(f^{i+1})\Big) \leq n \cdot \tau_f$$

as we wanted to show. Equality holds when for all $1 \leq i \leq n-1$, $\ker(f) \cap (f^i) = \ker(f)$, that is $\ker(f) \subset (f^i)$. Therefore, together with $(f^{n-1}) \subset \ker(f)$ we conclude $\ker(f) = (f^{n-1})$. □

To end the section, let us recall the definition of an invariant, and the notions of analytic and topological (see [GMG07, Def. 3.30]).

We say that two germs of isolated hypersurface singularities are *analytically equivalent* if there exists a local isomorphism mapping one to the other, and we refer to the corresponding equivalence classes as *analytic types*. On the other hand, we say that they are *topologically equivalent* if there exists a homeomorphism mapping one to the other, and we refer to the corresponding equivalence classes as *topological types*.

We call a number (or a set, or a group...) associated to a singularity an *analytic* (resp. *topological*) *invariant* if it remains unchanged within an analytic (resp. topological) equivalence class.

For example, the Milnor and Tjurina number introduced earlier are invariants associated to a singularity, and in the following sections we will introduce other invariants such as the Bernstein-Sato, the multiplier ideals and, ultimately, the Hodge ideals.

2.1 Bernstein-Sato polynomial

We now introduce the first invariant of singularities that we will mention, the Bernstein-Sato polynomial. It is a complex polynomial that arises from the existence of a given functional equation involving a local equation of the singularity considered. It is an analytical invariant, but not a topological one, and it is of particular interest to find relations among its roots and other invariants.

To introduce it, we first denote by $R \coloneqq \mathbb{C}[x_1, \ldots, x_n]$ the ring of complex polynomials in n variables. Also, let $\mathscr{D} \coloneqq R\langle \partial_1, \ldots, \partial_n \rangle$ be the *Weyl algebra*, where ∂_i are the partial derivatives operators with respect to x_i.

The notation in introducing the formal symbols ∂_i already suggests that the ring is non-commutative. Nonetheless, it is almost commutative, except for the relations $\partial_i x_i - x_i \partial_i = 1$. With this, it is easy to show that there exists a *normal form* $P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta$ to write any element of $\mathscr{D}$ as a finite sum. For a more gentle introduction and more details on the properties of the Weyl algebra, we refer to [CJ10].

Next, consider the polynomial ring $\mathscr{D}[s] \coloneqq \mathscr{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$, where we have introduced another variable s commuting with all x_i, ∂_i. In this case, any element of $\mathscr{D}[s]$ can be written as $P(s) = \sum_{i=0}^m s^i P_i$, where all $P_i \in \mathscr{D}$.

Lastly, we consider the localized ring $R_f[s] \coloneqq R[f^{-1}, s]$, and construct $R_f[s] \cdot f^s$, which is the free module generated by the formal symbol f^s. There is a natural structure of left $\mathscr{D}[s]$-module given by the product rule. Indeed, every element of the module can be written as $\frac{g}{f^k} \cdot f^s$ for some $g(x, s) \in R[s]$, and then the action of the partial derivatives is simply

$$\partial_i \cdot \left(\frac{g}{f^k} \cdot f^s\right) = \partial_i \cdot \left(\frac{g}{f^k}\right) \cdot f^s + \frac{sg}{f^{k+1}} \cdot \frac{\partial f}{\partial x_i} \cdot f^s$$

In this context, we introduce the Benstein-Sato functional equation. The theorem was first proved by Bernstein [Ber72] in the case of polynomials, and later by Kashiwara [Kas76] and Björk [Bjö73] for the case of holomorphic functions and formal power series, respectively. For a more detailed exposition on the history and results on the Bernstein-Sato, we refer the interested reader to [Gra10, ÀMJNB21].

Theorem 2 Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $b_{f,P}(s) \in \mathbb{C}[s]$ such that the relation

$$P(s)f^{s+1} = b_{f,P}(s)f^s \tag{2.1}$$

holds formally in the $\mathscr{D}$-module $R_f[s] \cdot f^s$.

Nonetheless, usually the linear differential operator $P(s)$ isn't even computed, as it is generally not relevant. Instead, it is more interesting to study the polynomial $b_{f,P}(s)$, which leads to the following definition.

Definition 1 (Bernstein-Sato polynomial) It can be seen that the set of polynomials $b_{f,P}(s)$ satisfying a functional equation as in (2.1) forms an ideal in $\mathbb{C}[s]$, and hence we can consider its monic generator. This polynomial $b_f(s)$ is called the Bernstein-Sato polynomial of f, and if the context is clear, the subscript is dropped.

If we were to consider $R = \mathbb{C}\{x_1, \ldots, x_n\}$ instead, we should refer to the *local* Bernstein-Sato polynomial denoted by $b_{f,p}(s)$ for a point $p \in X$. The *global* $b_f(s)$ and the *local* $b_{f,p}(s)$ polynomials can be related as follows (see [MNM91])

$$b_f(s) = \operatorname*{lcm}_{p \in \mathcal{V}(f)} b_{f,p}(s)$$

where one ought to keep in mind that for all smooth points, $b_{f,p}(s) = s + 1$.

Generally, it is very hard to compute the Bernstein-Sato polynomial for any $f \in R$. The first algorithm for that task was introduced by Oaku (see [Oak97]), using non-commutative Gröbner basis in the Weyl algebra, which gives a high complexity of computation and is not feasible to use in many examples. To show how the polynomials (and the differential operators) get increasingly difficult with already small examples, we next include some examples.

Example 1 (1) Let $f = x$ in $\mathbb{C}[x]$. Then, we have

$$\frac{\partial}{\partial x} f^{s+1} = (s+1)f^s$$

Therefore $b_f(s) \mid (s+1)$. In fact, $b_f(s) = (s+1)$ and this is the case for all smooth hypersurfaces. The reverse is also true, and the proof is due to Briançon and Maisonobe [BM96, Prop. 2.6].

(2) Let $f = x_1^2 + \cdots + x_n^2$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then, we have

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) f^{s+1} = 4(s+1)\left(s + \frac{n}{2}\right) f^s$$

Therefore $b_f(s) \mid (s+1)\left(s + \frac{n}{2}\right)$, and in fact we have equality.

(3) Let $f = x^2 + y^3$ in $\mathbb{C}[x, y]$. We have the following identity

$$\left[\frac{1}{12}\frac{\partial}{\partial x}\frac{\partial}{\partial y} y \frac{\partial}{\partial x} + \frac{1}{27}\left(\frac{\partial}{\partial y}\right)^3 + \frac{1}{4}\left(s + \frac{7}{6}\right)\left(\frac{\partial}{\partial x}\right)\right] f^{s+1} = (s+1)\left(s + \frac{5}{6}\right)\left(s + \frac{7}{6}\right) f^s$$

And it can be proved that $b_f(s) = (s+1)\left(s + \frac{5}{6}\right)\left(s + \frac{7}{6}\right)$.

(4) Let $f = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be a monomial in $\mathbb{C}[x_1, \ldots, x_n]$ Then

$$\frac{1}{\alpha_1^{\alpha_1} \ldots \alpha_n^{\alpha_n}} \left(\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\right) f^{s+1} = \prod_{i=1}^{n} \prod_{k=1}^{\alpha_i} \left(s + \frac{k}{\alpha_i}\right) f^s$$

In fact, $b_f(s) = \prod_{i=1}^{n} \prod_{k=1}^{\alpha_i} \left(s + \frac{k}{\alpha_i}\right)$.

(5) (Cayley) If $f = \det(x_{i,j}) \in \mathbb{C}[x_{i,j} \mid 1 \leq i, j \leq n]$, then

$$\det(\partial_{i,j}) f^{s+1} = (s+1)(s+2)\cdots(s+n) f^s$$

In fact, $b_f(s) = (s+1)(s+2)\cdots(s+n)$.

If f is not invertible, by setting $s = -1$ in the functional equation (2.1), we see that $(s+1)$ divides $b_f(s)$, hence we may define $\tilde{b}_f(s) = b_f(s)/(s+1)$ the *reduced Bernstein-Sato* polynomial of f. Also, as it can be already conjectured from Example 1, all the roots are negative rational numbers (see [Mal75, Kas76]). A set of candidates for the roots can be obtained from a resolution of singularities of f, and the values of the candidates will involve the multiplicities of the divisors.

Definition 2 (Minimal exponent) We call *minimal exponent* of f the greatest root of $\tilde{b}_f(s)$ and denote it by $\tilde{\alpha}_f$. By convention, if f is smooth and $\tilde{b}_f(s) = 1$ it is considered that $\tilde{\alpha}_f = \infty$.

This later number $\tilde{\alpha}_f$ will appear when we discuss the generation level of Hodge ideals.

Remark 1 Although we will define the *log canonical threshold* in the next section, we will use a well-known result (see, for example, [Kol97, Thm. 10.6]) which says that it coincides with the negative of the greatest root of $b_f(s)$, in order to remark the following relation

$$\operatorname{lct}(f) = \min\{\tilde{\alpha}_f, 1\}$$

In particular, they coincide if, and only if, $\tilde{\alpha}_f \leq 1$.

2.2 Multiplier ideals

The next invariant that we introduce are the multiplier ideals, which allow for both geometric and analytical definitions, following the exposition in [Laz03] and [ELSV04].

We first introduce the motivation that leads to the analytical definition of multiplier ideals, as a measure of the singularity by means of integration around the point (see [ELSV04, Rmk. 1.2] and references therein). Restricting for simplicity to the case of polynomial $f \in \mathbb{C}[x]$ singular at a point $p \in \mathcal{V}(f)$, we consider the following integral around a small closed ball around p

$$\int_{\overline{B}_\varepsilon(p)} \frac{|dx|^2}{|f|^{2\lambda}}$$

where $|dx| = dx\, d\overline{x}$. Clearly, this will converge for sufficiently small $\lambda \in \mathbb{R}_{>0}$, which leads to the definition of the *log-canonical-threshold*

$$\operatorname{lct}_p(f) = \sup\left\{\lambda \in \mathbb{R}_{>0} \;\middle|\; \exists \varepsilon \ll 1 \text{ st. } \int_{\overline{B}_\varepsilon(p)} \frac{|dx|^2}{|f|^{2\lambda}} < \infty\right\}$$

By means of resolution of singularities, we can better describe some conditions on λ such that the integral converges. Indeed, by a change of variables given by the resolution

$$\int_{\overline{B}_\varepsilon(p)} \frac{|dx|^2}{|f|^{2\lambda}} = \int_{\pi^{-1}\left(\overline{B}_\varepsilon(p)\right)} \frac{|d\pi|^2}{|\pi^* f|^{2\lambda}}$$

where the integration domain is still compact on the account that π is proper. Then, using the local monomial forms of the strict transform and the relative canonical divisor, the integral in one of the charts is expressed as

$$\int_{\mathcal{U}_\alpha} \frac{|u(z) z_1^{k_1} \cdots z_r^{k_r}|^2}{|z_1^{N_1} \cdots z_r^{N_r}|^{2\lambda}} |dz|^2$$

with $u(z)$ a unit, $r \leq n$ and $k_i, N_i \in \mathbb{Z}$. By Fubini's theorem, the given integral will be convergent if, and only if,

$$k_i - \lambda N_i > -1 \quad \forall i = 1, \ldots, r \qquad \Longleftrightarrow \qquad \lambda < \frac{k_i + 1}{N_i} \quad \forall i = 1, \ldots, r$$

With the ideas now introduced, the analytical definition of multiplier ideals arises naturally: they are the ideals consisting of polynomials which "can be used as multipliers to make the integral converge".

Definition 3 (Multiplier ideal) Let $f \in \mathbb{C}[x]$ and $p \in \mathcal{V}(f)$. The *multiplier ideal* of f at p associated with a rational number $\lambda \in \mathbb{Q}_{>0}$ is

$$\mathcal{J}(f^\lambda)_p = \left\{ g \in \mathbb{C}[x] \;\middle|\; \exists \varepsilon \ll 1 \text{ st. } \int_{\overline{B}_\varepsilon(p)} \frac{|g|^2}{|f|^{2\lambda}} |dx|^2 < \infty \right\}$$

The analytical definition can be extended for an ideal $\mathfrak{a} \subset \mathbb{C}[x]$, but we will not need it for our purposes. Rather, we now discuss the algebro-geometric approach in defining the multiplier ideals.

For that, consider X a smooth complex algebraic variety and $\mathcal{O}_X$ its structure sheaf. If $D = \sum_i a_i D_i$ is a $\mathbb{Q}$-divisor on X, we call the integral divisor $\lceil D \rceil = \sum_i \lceil a_i \rceil D_i$ the *round-up* of D.

Definition 4 The *multiplier ideal* sheaf associated to f and some rational number $\lambda \in \mathbb{Q}_{>0}$ is defined as

$$\mathcal{J}(X, f^\lambda) = \mathcal{J}(f^\lambda) \coloneqq \pi_* \mathcal{O}_Y \left(\lceil K_\pi - \lambda F_\pi \rceil \right)$$

where $\pi \colon Y \to X$ is a log-resolution of f, K_π denotes the canonical relative divisor, and F_π is the divisor associated to the strict transform.

Again, we can define them more generally for any ideal $\mathfrak{a} \subset \mathbb{C}[x]$. More relevant, it turns out that they are independent of the chosen log-resolution (see [Laz03, Thm. 9.2.18]). Also, Skoda's theorem provides a recurrent expression in λ to calculate them, which in the simplest case of a single polynomial states

$$\mathcal{J}(f^{\lambda+1}) = f \cdot \mathcal{J}(f^\lambda)$$

Therefore, we can restrict our attention as it is enough to calculate the multiplier ideals for $\lambda \in (0, 1]$.

Now, notice that

$$\lceil K_\pi - \lambda F_\pi \rceil \geq \lceil K_\pi - (\lambda + \varepsilon) F_\pi \rceil, \qquad \forall \varepsilon > 0$$

and that equality holds for $0 < \varepsilon \ll 1$. Then, that means there is an increasing discrete sequence of rational numbers $\lambda_i \coloneqq \lambda_i(f, p)$ with

$$0 = \lambda_0 < \lambda_1 < \dots$$

such that for every i we have

$$\begin{cases} \mathcal{J}(f^\lambda) = \mathcal{J}(f^{\lambda_i}) & \forall \lambda \in [\lambda_i, \lambda_{i+1}) \\ \mathcal{J}(f^{\lambda_{i+1}}) \subsetneq \mathcal{J}(f^{\lambda_i}) \end{cases}$$

In other words, there is a decreasing sequence of ideals

$$\mathcal{O}_X \supsetneq \mathcal{J}(f^{\lambda_1}) \supsetneq \mathcal{J}(f^{\lambda_2}) \supsetneq \dots$$

Definition 5 (Jumping numbers) The rational numbers $\lambda_i \coloneqq \lambda_i(f, p)$ are the *jumping numbers* associated to f at $p \in \mathcal{V}(f)$.

Furthermore, for a fixed f, we can relate the associated jumping numbers with the roots of its Bernstein-Sato polynomial.

Theorem 3 ([BS03, ELSV04]) Let $f \in \mathbb{C}[x]$ be a non-constant polynomial. Then if $\lambda_i = \lambda_i(f, p)$ is a jumping number of f in $(0, 1]$, then $b_f(-\lambda_i) = 0$.

Next, we define the concept of log-canonicity, which will be later generalized with the introduction of the Hodge ideals.

Definition 6 (Log-canonical) If D is a reduced effective divisor on the smooth variety X we say that the pair (X, D) is log-canonical if

$$\mathcal{J}(D) = \mathcal{O}_X$$

This is rather a rare situation, from the point of view of general singularity theory, as most singularities are not log canonical (see [Kol97]). Hence, the notion of the log-canonical-threshold is useful in extending this measure of triviality of the multiplier ideals, as we can interpret

$$\operatorname{lct}(D) = \inf\left\{\lambda \in \mathbb{R}_{>0} \;\middle|\; \mathcal{J}(D^{\lambda}) \neq \mathcal{O}_X\right\}$$

Lastly, although we haven't introduced the complex zeta function associated to a singularity, we can also mention that, in some sense, the log-canonical-threshold measures the extent to which the complex zeta function can be extended holomorphically.

3 Hodge ideals

Hodge ideals are another invariant in the theory of singularities, which in some sense generalize multiplier ideals. Roughly speaking, they are a sheaf of ideals that measure how different the Hodge filtration and the pole order filtration on a certain module are. This section gives the basic introduction to these objects, following the exposition by Mustaţă and Popa. For more details, we refer the interested reader to their work in [MP19a, MP19b, MP20a, MP20b].

We will first give the definition for an integral divisor D, and later for $\mathbb{Q}$-divisors αD with α a rational number, which will be the type of divisors for which we will be computing the Hodge ideals.

3.1 Preliminaries

We start with the first case, and we consider a smooth complex variety X of dimension n. To a reduced effective divisor D on X, we can associate the left $\mathscr{D}_X$-module of functions with poles along D

$$\mathcal{O}_X(*D) = \bigcup_{k\geq 0} \mathcal{O}_X(kD)$$

which is nothing else than the localization of $\mathcal{O}_X$ along D. For instance, if h is a local equation of D, then this is $\mathcal{O}_X[1/h]$, with the obvious action of differential operators. We now want to study the filtrations on this $\mathscr{D}_X$-module.

Definition 7 (Filtration) A *filtration* on the left $\mathscr{D}_X$-module $\mathcal{M}$ is a family $F = F_\bullet\mathcal{M}$ of coherent $\mathcal{O}_X$-modules, bounded from below and satisfying

(1) $F_k\mathcal{M} \subseteq F_{k+1}\mathcal{M}, \qquad \forall k \in \mathbb{Z}$

(2) $\bigcup_{k\in\mathbb{Z}} F_k\mathcal{M} = \mathcal{M}$

(3) $F_k\mathscr{D}_X \cdot F_l\mathcal{M} \subseteq F_{k+l}\mathcal{M}, \qquad \forall k, l \in \mathbb{Z}$

where $F_k\mathscr{D}_X$ is the sheaf of differential operators on X of order $\leq k$. We will denote this data by $(\mathcal{M}, F)$.

One ought to think about this definition as first providing $\mathscr{D}_X$ with the filtration $F_\bullet\mathscr{D}_X$ given by the order of the differential operator, and then defining a filtration $F_\bullet\mathcal{M}$ for a general $\mathscr{D}_X$-module $\mathcal{M}$ such that it is compatible with $F_\bullet\mathscr{D}_X$.

Definition 8 (Good filtration) The filtration is called *good* if the inclusion imposed in condition (3) is an equality for $l \gg 0$. This is, in turn, equivalent to the fact that the total associated graded module

$$\mathrm{gr}_\bullet^F\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^F\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

is finitely generated over $\mathrm{gr}_\bullet^F\mathscr{D}_X$.

3.2 Definition

The most obvious filtration we can endow to $\mathcal{O}_X(*D)$ is the pole order filtration, whose nonzero terms are given by

$$P_k\mathcal{O}_X(*D) = \mathcal{O}_X\left((k+1)D\right), \qquad \forall k \geq 0$$

Less obvious, there is the *Hodge filtration* $F_k\mathcal{O}_X(*D)$, again with nonzero terms for $k \geq 0$, whose existence is guaranteed by general results on Hodge modules (see [MP19a], Section 4). Furthermore, it is contained in the pole order filtration.

Theorem 4 ([Sai93], Prop. 0.9) For every $k \geq 0$, we have the inclusion

$$F_k\mathcal{O}_X(*D) \subseteq \mathcal{O}_X\left((k+1)D\right)$$

The idea in introducing the Hodge ideals is to measure how far these filtrations are for being equal, which is of relevant interest in the study of the singularities of D.

Definition 9 (Hodge ideals) From the above inclusion, we may introduce for each $k \geq 0$ a coherent sheaf of ideals $I_k(D)$ defined by

$$F_k\mathcal{O}_X(*D) = \mathcal{O}_X\left((k+1)D\right) \otimes I_k(D)$$

and it is called the k-th *Hodge ideal* associated to D.

So far, we have provided an algebraic definition of these objects. However, in order to obtain some nice properties about the Hodge ideals, it turns out that a geometric definition is more useful. This latter birrational approach using log-resolutions is introduced by Mustaţă and Popa (see [MP19b]), proving that the definitions are equivalent and that the ideals turn out to be independent of the log resolution (see [MP19a, Thm. 11.1]). Alternatively, Saito gives another characterization by studying the induced microlocal V-filtration on the structure sheaf (see [Sai17]).

Lastly, contrary to the situation with multiplier ideals, it is currently not known whether an analytical definition of the Hodge ideals exists.

3.3 Basic properties

An easy case to consider first is how do these ideals look when we have a smooth divisor.

Example 2 When D is smooth, then $I_k(D) = \mathcal{O}_X$ for all $k \geq 0$, which corresponds to equality between the Hodge filtration and the pole order filtration. We include a proof of the statement as a particular case of Proposition 7.

In fact, a converse statement also holds.

Theorem 5 ([MP19a], Thm. A) Let X be a smooth complex variety of dimension n, and D a reduced effective divisor on X. Then, the following are equivalent:

(1) D is smooth.

(2) the Hodge filtration and the pole order filtration on $\mathcal{O}_X(*D)$ coincide.

(3) $I_k(D) = \mathcal{O}_X$ for all $k \geq 0$.

(4) $I_k(D) = \mathcal{O}_X$ for some $k \geq \frac{n-1}{2}$.

And vice versa, the more singular D is, the smaller $I_k(D)$ is. Furthermore, the following inclusions are known among the ideals.

Proposition 1 ([MP19a], Prop. 13.1, Rmk. 14.3) For every reduced effective divisor D on a smooth variety X each $k \geq 1$, we have the following inclusions

(1) $I_{k-1}(D) \cdot \mathcal{O}_X(-D) \subseteq I_k(D)$

(2) $\mathcal{O}_X\left(-(k+1)D\right) \subseteq I_k(D)$

(3) $I_k(D) \subseteq I_{k-1}(D)$

3.4 The first Hodge ideals

In general, these objects are not easy to compute, and are yet to be fully understood. However, the 0^{th} Hodge ideal is easier to describe, as it is related to the multiplier ideals

$$I_0(D) = \mathcal{J}\left(X, (1-\varepsilon)D\right)$$

Namely, it is the multiplier ideal associated to the $\mathbb{Q}$-divisor $(1-\varepsilon)D$, with $0 < \varepsilon \ll 1$. Thus, we can think of the ideals $I_k(D)$ as a refinement of these multiplier ideals.

Also, with this identification we can express the condition of the pair (X, D) being log-canonical as $I_0(D) = \mathcal{O}_X$.

Next, for the 1^{st} Hodge ideal, we have some other known results.

Proposition 2 ([MP19a], Prop. 20.2) If X has dimension $n \geq 3$, and we consider an ordinary singularity, i.e. when $m = \mathrm{mult}_x(D) \geq 2$ and the projectivized tangent cone of D at x is smooth, then we have

(1) If $m \leq n/2$, then $I_1(D) = \mathcal{O}_X$

(2) If $n/2 \leq m \leq n-1$, then $I_1(D) = \mathfrak{m}_x^{2m-m}$

(3) If $m \geq n$, we have

$$\mathcal{O}_X(D) \cdot \mathfrak{m}_x^{m-n-1} + \mathfrak{m}_x^{2m-n} \subseteq I_1(D) \subseteq \mathcal{O}_X(D) \cdot \mathfrak{m}_x^{m-n-2} + \mathfrak{m}_x^{2m-n-1}$$

with $\dim_{\mathbb{C}} I_1(D)/\left(\mathcal{O}_X(D) \cdot \mathfrak{m}_x^{m-n-1} + \mathfrak{m}_x^{2m-n}\right) = m\binom{m-2}{n-2}$.

For the next Hodge ideals, there is, currently, only an analogue statement for (1), namely the following.

Proposition 3 ([MP19a], Prop 20.7, Thm. D) Suppose X has dimension $n \geq 3$, D is a reduced effective divisor on X, and $x \in X$ is a point such that $m = \mathrm{mult}_x(D) \geq 2$ and the projectivized tangent cone of D at x is smooth. Then, it holds

$$I_k(D) = \mathfrak{m}_x^{(k+1)m-n} \quad \text{around } x$$

with the convention that $\mathfrak{m}_x^j = \mathcal{O}_X$ if $j \leq 0$. In particular,

$$I_k(D) = \mathcal{O}_X \quad \text{around } x \quad \Longleftrightarrow \quad m \leq \frac{n}{k+1} \quad \Longleftrightarrow \quad k \leq \left\lfloor \frac{n}{m} \right\rfloor - 1$$

Returning to the study of singularities of the pair (X, D), the previous result suggests introducing a refinement of the notion of log-canonical singularity as follows.

Definition 10 (k-log-canonical) If D is a reduced effective divisor on the smooth variety X, we say that the pair (X, D) is k-log-canonical if

$$I_0(D) = I_1(D) = \cdots = I_k(D) = \mathcal{O}_X$$

Remark 2 From the chain of inclusions $\cdots \subseteq I_k(D) \subseteq \cdots \subseteq I_1(D) \subseteq I_0(D)$ described in Proposition 1, being k-log-canonical is equivalent to $I_k(D) = \mathcal{O}_X$.

Being log-canonical is of course equivalent to being 0-log-canonical in the above sense, while Theorem 5 says that $(n-1)/2$-log-canonical or higher is equivalent to D being smooth. One ought to think of the intermediate levels of log-canonicity as a refinement of the notion of rational singularities.

A bound on the level of log-canonicity of a singularity can be obtained from the minimal exponent (see Definition 2), as proved by Saito.

Proposition 4 ([Sai17], Cor. 1) Around a point x, the level of log-canonicity is $\lfloor \tilde{\alpha}_{f,x} \rfloor$, that is

$$I_k(D) = \mathcal{O}_X \quad \Longleftrightarrow \quad k \leq \tilde{\alpha}_{f,x} - 1$$

3.5 Surfaces

If we now focus on the case that X is a surface, we even have the following upper bounds.

Proposition 5 ([MP19a], Cor. 17.12) If X is a smooth surface and D is a reduced effective divisor on X such that $\mathrm{mult}_x(D) = m \geq 2$ for some $x \in X$, then

$$I_k(D) \subseteq \mathfrak{m}_x^{(k+1)(m-1)-1}, \qquad \forall k \geq 0$$

where $\mathfrak{m}_x$ is the maximal ideal defining x. Moreover, if $m = 2$, then

$$I_k(D) \subseteq \mathfrak{m}_x^{k+1}, \qquad \forall k \geq 0$$

unless the singularity at x is a node. In particular, if D is a singular divisor, $I_k(D) \neq \mathcal{O}_X$ for all $k \geq 1$.

Example 3 Although we will calculate examples of Hodge ideals in the following sections, we now include an example of when the above result is sharp. Let us consider the divisor $D = \mathcal{V}\left(xy(x+y)\right) \subset \mathbb{C}^2$, with a triple point at the origin, thus $m = 3$. The Hodge ideals in this case turn out to be

$$I_0(D) = (x, y) = \mathfrak{m}, \qquad I_1(D) = (x, y)^3 = \mathfrak{m}^{(1+1)(3-1)-1}$$

Note that in the case of surfaces, thanks to Proposition 5, it happens that unlike $I_0(D)$, for $k \geq 1$ the ideal $I_k(D)$ always detects singularities.

Also, still in the case of surfaces, it is known that the Hodge ideals can be defined as follows.

Proposition 6 ([MP19a], Cor. 17.8) If X is a smooth surface and D is a reduced effective divisor on X, then

$$F_k\mathscr{D}_X \cdot \mathcal{O}_X(D) = \mathcal{O}_X\left((k+1)D\right) \otimes I_k(D), \qquad \forall k \geq 0 \tag{3.1}$$

while for general X, only the inclusion $\subseteq$ is true.

3.6 Simple normal crossing divisors

We now consider the simple normal crossing (SNC) case. When E is a SNC divisor, the ideals $I_k(E)$ can be defined (see [MP19a, §8]) by the same expression as in the case of X being a surface, that is

$$F_k\mathscr{D}_X \cdot \mathcal{O}_X(E) = \mathcal{O}_X\left((k+1)E\right) \otimes I_k(E), \qquad \forall k \geq 0 \tag{3.2}$$

This includes the statement that $I_0(E) = \mathcal{O}_X$, and a simple calculation shows that $F_k\mathscr{D}_X \cdot \mathcal{O}_X(E) \subseteq \mathcal{O}_X\left((k+1)E\right)$.

Proposition 7 ([MP19a], Prop. 8.2) Suppose that around $p \in X$ we have coordiantes $x_1, \ldots x_n$ such that E is defined by $(x_1 \cdots x_r = 0)$. Then, for every $k \geq 0$, the ideal $I_k(E)$ is generated around p by

$$\left\{x_1^{a_1} \cdots x_r^{a_r} \mid 0 \leq a_i \leq k, \sum_i a_i = k(r-1)\right\}$$

In particular, if $r = 1$ (that is, when E is smooth), we have $I_k(E) = \mathcal{O}_X$ and if $r = 2$, then $I_k(E) = (x_1, x_2)^k$.

Proof. The assertions in the special cases $r = 1$ and $r = 2$ are clear. Then, for $r \geq 3$ notice that $F_k\mathscr{D}_X \cdot \mathcal{O}_X(E)$ is generated as an $\mathcal{O}_X$-module by

$$\left\{x_1^{-b_1} \cdots x_r^{-b_r} \mid b_i \geq 1, \sum_i b_i = r + k\right\}$$

According to (3.2), the expression $I_k(E)$ now follows by multiplying these generators by $(x_1 \cdots x_r)^{k+1}$.

$$-\sum_i b_i + r \cdot (k+1) = -(r+k) + r \cdot (k+1) = k \cdot (r-1). \qquad \square$$

It is relevant to mention this result, as our implemented code won't be able to calculate Hodge ideals for a single monomial.

3.7 Generation level of Hodge ideals

Definition 11 (Generated) Let D be a reduced effective divisor on the smooth n-dimensional variety X. We say that the filtration on $\mathcal{O}_X(*D)$ is *generate*d at level k if

$$F_l\mathscr{D}_X \cdot F_k\mathcal{O}_X(*D) = F_{k+l}\mathcal{O}_X(*D), \qquad \forall l \geq 0$$

Equivalently, the ideal $I_k(D)$ and the local equation of D determine all the higher Hodge ideals, i.e. for $l \geq 0$, by the formula

$$F_l\mathscr{D}_X \cdot (\mathcal{O}_X((k+1)D) \otimes I_k(D)) = \mathcal{O}_X((k+l+1)D) \otimes I_{k+l}(D)$$

Apart from the recursive expression obtained to generate the Hodge ideals up to a certain level, we also have an upper bound on the elements of these ideals.

Remark 3 ([MP19a], Rmk. 17.11) If we know that the filtration on $\mathcal{O}_X(*D)$ is generated at level l, then we have

$$I_{k+1}(D) \subseteq \mathcal{O}_X(-D) \cdot J(I_k(D)) + I_k(D) \cdot J(\mathcal{O}_X(-D)), \qquad \forall k \geq l$$

where J denotes the Jacobian ideal, generated by the partial derivatives of the elements considered.

Mustată and Popa prove a result to determine at which level the filtration is generated in the general case, in terms of the minimal exponent.

Theorem 6 ([MP20a], Thm. A) If X has dimension $n \geq 2$, the Hodge filtration on $\mathcal{O}_X(*D)$ for a singular divisor D is generated at level $n - 1 - \lceil \tilde{\alpha}_D \rceil$.

This is an improvement on a result by the same authors, where they showed that the filtration is generated at level $n-2$. Notice, however, that this is still far from being sharp.

Example 4 In the case of normal crossing divisor (see Proposition 7), the filtration on $\mathcal{O}_X(*D)$ is generated by its 0^{th} step, which implies that the generation level is 0.

Example 5 In the case of X being a smooth surface, since the dimension is $n = 2$ and $\tilde{\alpha}_f > 0$, from Theorem 6 we get that the generation level is also 0.

The expression in Theorem 6 was first pointed out by Saito, who proved the bound for isolated semi-quasihomogeneous singularities (where $\tilde{\alpha}_f$ can be computed explicitly). In Corollary 2 we will include the proof of a generalization of this result for $\mathbb{Q}$-divisors due to Zhang.

3.8 Hodge ideals of $\mathbb{Q}$-divisors

We now extend the study of Hodge ideals by considering an analogous theory for arbitrary $\mathbb{Q}$-divisors, following the birational approach in [MP19b].

Let Z be a reduced effective divisor on X, defined by $f \in \mathcal{O}_X$. Given a rational number γ, we can consider the $\mathscr{D}_X$-module

$$\mathcal{M}(f^\gamma) := \mathcal{O}_X(*Z)f^\gamma$$

This is a free $\mathcal{O}_X(*Z)$-module of rank 1 generated by the symbol f^γ, on which a derivation D of $\mathcal{O}_X$ acts by

$$D(rf^\gamma) = \left(D(r) + r\frac{\gamma \cdot D(f)}{f}\right) f^\gamma$$

We will keep the notation $\mathcal{O}_X(*Z)$ for $\mathcal{M}(f^0)$. The $\mathscr{D}_X$-modules $\mathcal{M}(f^\gamma)$ are regular holonomic, with quasi-unipotent monodromy.

Remark 4 If $\gamma_1 - \gamma_2 = d$ is an integer, then we have a canonical isomorphism of $\mathscr{D}_X$-modules

$$\mathcal{M}(f^{\gamma_1}) \xrightarrow{\cong} \mathcal{M}(f^{\gamma_2}), \qquad gf^{\gamma_1} \mapsto (gf^d)f^{\gamma_2}$$

Therefore, this justifies that we can restrict our attention to rational $\gamma \in (0, 1]$.

Next, consider an effective $\mathbb{Q}$-divisor D and $\alpha \in (0, 1]$ a rational number such that $D = \alpha Z$. We want to define the Hodge ideals similarly as before, but now considering the $\mathscr{D}$-module $\mathcal{M}(f^{-\alpha})$. However, it turns out to be more convenient to work with the $\mathscr{D}$-module $\mathcal{M}(f^{1-\alpha})$, and use the isomorphism noted in the previous Remark 4.

Definition 12 (Hodge ideals for $\mathbb{Q}$-divisors) It follows from [MP19b, Prop. 4.1] that for each $k \geq 0$ there is a unique coherent ideal $I_k(D)$ such that

$$F_k\mathcal{M}(f^{1-\alpha}) = I_k(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X((k+1)Z)\, f^{1-\alpha}$$

and it is called the k-th *Hodge ideal* associated to the $\mathbb{Q}$-divisor αZ.

We will now see that although we don't have as many nice properties as with reduced integral divisors, we can still get some relevant information from these invariants.

For example, we get *jumping coefficients* similar to the situation with multiplier ideals, by studying the behavior of the Hodge ideals $I_k(\alpha Z)$ when α varies. In the case of I_0, thanks to the connection with multiplier ideals, we have seen that the ideals get smaller as α increases, with changes occurring at the jumping coefficients λ_i. However, this is not the case for $k > 0$. For example, as we will calculate later, for the cusp $Z = \{x^2 + y^3 = 0\}$ and $\alpha = 1 - \varepsilon$ with $0 < \varepsilon \ll 1$ we have

$$I_2(\alpha Z) = (x^3, x^2y^2, xy^3, y^4 - (2\alpha + 1)x^2y)$$

and thus we obtain incomparable ideals. Nonetheless, thanks to a result by Mustată and Popa (see [MP20b, Cor. 5.6]), we have a similar situation if we consider images in $\mathcal{O}_Z$ instead.

Theorem 7 (Jumping coefficients) Given any $k \geq 0$, there exists a finite set of rational numbers $0 = c_0 < c_1 < \cdots < c_s < c_{s+1} = 1$ such that for each $0 \leq i \leq s$ and each $\alpha \in (c_i, c_{i+1}]$ we have

$$I_k(\alpha Z) \cdot \mathcal{O}_Z = I_k(c_{i+1}Z) \cdot \mathcal{O}_Z = \text{ constant}$$

and such that

$$I_k(c_{i+1}Z) \cdot \mathcal{O}_Z \subsetneq I_k(c_iZ) \cdot \mathcal{O}_Z$$

In fact, if Z is defined by a global equation f, the set of c_i is a subset of the set of jumping numbers associated to f.

Even though it can happen that we get incomparable ideals, we can still hope for some similar result to that of (3) in Proposition 1, where we had a decreasing sequence of ideals in the case of integral reduced divisors.

Proposition 8 ([MP20b], Cor. 5.1, Cor. 5.5) For every $p \geq 0$ we have

$$I_{p+1}(\alpha Z) + \mathcal{O}_X(-Z) \subseteq I_p(\alpha Z) + \mathcal{O}_X(-Z)$$

Also if $(X, \alpha Z)$ is $(p-1)$-log-canonical (which, by the above, is equivalent to $I_{p-1}(\alpha Z) = \mathcal{O}_X$), then

$$I_{p+1}(\alpha Z) \subseteq I_p(\alpha Z)$$

In particular, we always have $I_1(\alpha Z) \subseteq I_0(\alpha Z)$ when $\alpha \leq 1$.

Lastly, by considering $\mathbb{Q}$-divisors αZ we can also recover the roots (or at least its value shifted by an integer) of the Bernstein-Sato polynomial, where the following result can be understood as a generalization of Theorem 3.

Theorem 8 ([MP20b], Prop. 6.14) Let $Z \neq 0$ be a reduced, effective divisor on the smooth variety X and suppose that $\alpha \in (0, 1)$ is a rational number and $p \geq 0$ is an integer such that the pair $(X, \beta Z)$ is $(p-1)$-log canonical for some $\beta \in (\alpha, 1)$. If $I_p(\alpha Z) \neq I_p((\alpha + \varepsilon)Z)$ for $0 < \varepsilon \ll 1$, then we have $\tilde{b}_Z(-p - \alpha) = 0$.

Example 6 Let $Z \subset \mathbb{C}^2$ be the cusp, defined by $f = x^2 + y^3$. It is well known that

$$\tilde{b}_f(s) = \left(s + \frac{5}{6}\right)\left(s + \frac{7}{6}\right)$$

so that $\tilde{\alpha}_f = 5/6$ and $I_0(\alpha Z) = R$ for every $\alpha \leq 5/6$. On the other hand, we will see in later calculations that

$$I_1\left(\frac{1}{6}Z\right) \neq I_1\left(\left(\frac{1}{6} + \varepsilon\right)Z\right)$$

for $0 < \varepsilon \ll 1$. Thus, the 'other' root $-7/6 = -1 - 1/6$ is accounted for by the jumping number $1/6$ of I_1.

4 Quasihomogeneous

4.1 Definition

The adjective quasihomogeneous is used for polynomials whose monomials have the same degree after adding suitable weights to each of the variables. To state this definition more precisely, let us first write $w = (w_1, \dots, w_n) \in \mathbb{Q}^n_{>0}$ the weight vector, and $p = (p_1, \dots, p_n) \in \mathbb{N}^n$ the exponents of an arbitrary monomial. We will denote

$$|w| \coloneqq w_1 + \cdots + w_n, \qquad \langle w, p \rangle \coloneqq w_1 p_1 + \dots w_n p_n$$

Also, we will abbreviate in the standard monomial notation

$$x^p \coloneqq x_1^{p_1} \dots x_n^{p_n}$$

Then, we define

Definition 13 (w-degree) Let $w - \deg(x^p) \coloneqq \langle w, p \rangle$. Then, for any power series $g = \sum_{p \in \mathbb{N}^n} a_p x^p$, define

$$w - \deg(g) \coloneqq \min_{p \in \mathbb{N}^n} \{ w - \deg(x^p) \mid a_p \neq 0 \}$$

That is, the **minimum** scalar product against the exponents appearing in the expression of g.

Definition 14 (Quasihomogeneous) We say a polynomial $f = \sum_{p \in \mathbb{N}^n} a_p x^p \in \mathbb{C}[x]$ is a quasihomogeneous polynomial of weight $w = (w_1, \dots, w_n)$ if

$$\langle w, p \rangle = 1 \quad \forall p \in \mathbb{N}^n \text{ such that } a_p \neq 0$$

An isolated hypersurface singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ is called quasihomogeneous it there exists a quasi-homogeneous polynomial f such that $\mathcal{O}_{X,0} \cong \mathbb{C}\{x\}/(f)$ as analytic $\mathbb{C}$-algebras.

In some literature, f is also said to be *weighted homogeneous*. Also, the quantity $\langle w, p \rangle$ can be allowed to be any other positive integer, and it is then called the homogeneous degree of the polynomial.

There is a nice characterization of quasihomogeneous singularities, given by the following result of Saito.

Proposition 9 ([Sai71]) If $(X, x) \subset (\mathbb{C}^n, x)$ is an isolated hypersurface singularity, with $f \in \mathbb{C}[x]$ any local equation defining it, then

$$(X, x) \text{ is quasihomogeneous} \quad \iff \quad \mu_f = \tau_f$$

Furthermore, the implication to the right is interesting on its own, as it can be proved by noting that a quasihomogeneous polynomial f of weight w (and homogeneous degree 1) satisfies the *Euler relation*

$$f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i}$$

This relation also plays an important role in the proof of the expression of the Bernstein-Sato for this type of polynomials, which is the next Theorem 9. To state it, we first need to define a weight function as follows.

Definition 15 (Weight function ρ) For any weights $w = (w_1, \dots, w_n)$, we define a function $\rho \colon \mathbb{C}\{x\} \to \mathbb{Q}_{>0}$ for any $g = \sum_{p \in \mathbb{N}^n} a_p x^p$ by

$$\rho(g) = |w| + w - \deg(g) \qquad \left(= |w| + \min_{p \in \mathbb{N}^n} \{ \langle w, p \rangle \mid a_p \neq 0 \} \right)$$

and this defines a filtration on $\mathcal{O}_{X,0}$ by considering the sheaf ideals $\mathcal{O}^{\geq c} \coloneqq \{ g \in \mathbb{C}\{x\} \mid \rho(g) \geq c \}$.

In the case of quasihomogeneous polynomials, we obtain a very simple procedure to calculate the Bernstein-Sato polynomial.

Theorem 9 ([Gra10], Thm. 4.8) Let $f \in \mathbb{C}[x]$ be a quasihomogeneous polynomial of weight w. Choose a monomial basis $\{v_1, \dots, v_{\mu_f}\}$ for the Milnor algebra M_f as $\mathbb{C}$-vector space

$$M_f = \bigoplus_{i=1}^{\mu_f} \mathbb{C} \cdot v_i$$

Consider the set $\rho(V) = \{\rho(v_i) \mid i = 1, \dots, \mu_f\}$, where each number appears without repetitions. Then, the Bernstein-Sato polynomial of f is

$$b_f(s) = (s+1) \cdot \prod_{\lambda \in \rho(V)} (s + \lambda)$$

Although we will not include a proof of this result, we thoroughly describe the example of computing it for $f = x^2 + y^3$, which already gives the key ideas that are followed in the proof.

Example 7 Let $f = x^2 + y^3$, it is quasihomogeneous of degree 1 and weights $w = (1/2, 1/3)$. We consider the Euler operator

$$\chi = \frac{1}{2} x \frac{\partial}{\partial x} + \frac{1}{3} y \frac{\partial}{\partial y} = \frac{1}{2} \frac{\partial}{\partial x} x + \frac{1}{3} \frac{\partial}{\partial y} y - \frac{5}{6}$$

From a straightforward calculation and using $\chi f = f$, we get that $\chi f^s = s f^s$. Then,

$$\left(s + \frac{5}{6}\right) f^s = \left(\chi + \frac{5}{6}\right) f^s = \left(\frac{1}{2} \frac{\partial}{\partial x} x + \frac{1}{3} \frac{\partial}{\partial y} y\right) f^s = \frac{1}{4} \frac{\partial}{\partial x} \frac{\partial f}{\partial x} f^s + \frac{1}{3} \frac{\partial}{\partial y} y f^s$$

The monomial y is not yet in the ideal of partial derivatives, but we can treat the term yf^s in the same way as f^s, using $\chi(yf^s) = (s + 1/3) y f^s$

$$\left(s + \frac{7}{6}\right) y f^s = \left(s + \frac{1}{3} + \frac{5}{6}\right) y f^s = \left(\frac{1}{2} \frac{\partial}{\partial x} x + \frac{1}{3} \frac{\partial}{\partial y} y\right) y f^s = \frac{1}{6} \frac{\partial}{\partial x} y \frac{\partial f}{\partial x} f^s + \frac{1}{9} \frac{\partial}{\partial y} \frac{\partial f}{\partial y} f^s$$

Then, we multiply the first relation by $s + 7/6$, and replace the appearing $(s + 7/6) y f^s$ by the RHS of the second relation, so we obtain operators $B, C \in \mathscr{D}[s]$ such that

$$\left(s + \frac{5}{6}\right) \left(s + \frac{7}{6}\right) f^s = \left(B \frac{\partial f}{\partial x} + C \frac{\partial f}{\partial y}\right) f^s$$

Finally, multiplying by $s + 1$ and using the deep fact that $(s+1) \frac{\partial f}{\partial x} f^s = \frac{\partial}{\partial x} f^{s+1}$ yields the functional equation. The explicit result is

$$(s+1) \left(s + \frac{5}{6}\right) \left(s + \frac{7}{6}\right) f^s = \left[\frac{1}{12} \frac{\partial}{\partial x} \frac{\partial}{\partial y} y \frac{\partial}{\partial x} + \frac{1}{27} \left(\frac{\partial}{\partial y}\right)^3 + \frac{1}{4} \left(s + \frac{7}{6}\right) \left(\frac{\partial}{\partial x}\right)\right] f^{s+1}$$

The operator on the right can be modified by any element in the annihilator of f^s, which in this case is generated by $s - \chi$ and by

$$\frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$$

In particular, we can make $P(s)$ independent of s.

Also, we add the following remark, due to Saito, which will be used later to obtain the expression for the generation level of the Hodge ideals in the quasihomogeneous case.

Remark 5 ([Sai94], Rmk. 2.8) For f a quasihomogoneous polynomial with weight w, it holds

$$\max_{1 \le i \le \mu_f} \rho(v_i) = n - |w|$$

In particular, from the expression in Theorem 9 of the roots of the Bernstein-Sato polynomial, we have that the negative of the greatest root is $\tilde{\alpha}_f = |w|$, as we can always choose the first element of the monomial basis of M_f to be $v_1 = 1$, which has $\rho(1) = |w| + 0$. Thus,

$$\max_{1\leq i\leq \mu_f} \rho(v_i) = n - \tilde{\alpha}_f$$

We now include an example of a calculation of the weight ρ, and then a thorough example of the concepts introduced in this subsection.

Example 8 Take $w = (1/2, 1/3)$ and consider $g = x^2y + 4xy^3 - y^5$, then

$$\begin{aligned}\rho(g) &= (1/2 + 1/3) + \min\Big\{\langle(1/2,1/3),(2,1)\rangle, \langle(1/2,1/3),(1,3)\rangle, \langle(1/2,1/3),(0,5)\rangle\Big\}\\ &= 5/6 + \min\{4/3, 3/2, 5/3\}\\ &= 5/6 + 4/3 = 13/6\end{aligned}$$

Example 9 Let $f = x^2 + y^3$, quasihomogeneous of weight $w = (1/2, 1/3)$. Since $J(f) = (x, y^2)$, we can take the basis $\{1, y\}$ for the Milnor algebra

$$M_f = \mathbb{C}\{x,y\}/(x,y^2) \cong \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot y$$

and we see that $\mu_f = 2$. As expected, also $\tau_f = 2$, as we have $(f, J(f)) = J(f)$ on the account of the Euler relation

$$x^2 + y^3 = \frac{1}{2}x\frac{\partial}{\partial x}(x^2+y^3) + \frac{1}{3}y\frac{\partial}{\partial y}(x^2+y^3) = \frac{1}{2}2x^2 + \frac{1}{3}3y^2$$

More over, the weights of the chosen elements for the basis are

$$\begin{cases}\rho(1) = (1/2 + 1/3) + \min\Big\{\langle(1/2,1/3),(0,0)\rangle\Big\} = 5/6\\ \rho(y) = (1/2 + 1/3) + \min\Big\{\langle(1/2,1/3),(0,1)\rangle\Big\} = 7/6\end{cases}$$

From these we obtain the Bernstein-Sato polynomial $b_f(s) = (s+1)\,(s+5/6)\,(s+7/6)$, coinciding with the result in Example 7.

4.2 Hodge ideals

We now let Z be an integral reduced effective divisor, defined by f a quasihomogeneous polynomial with an isolated singularity at the origin. We begin by recalling the formula to the Hodge filtration on $\mathcal{O}_X(Z)$, due to Saito.

Theorem 10 ([Sai09], Thm. 0.7) Assume $0 \in Z$ is a germ of a quasihomogeneous isolated hypersurface singularity and choose a polynomial f as the local function of Z. Then, the Hodge filtration on $\mathcal{O}_X(Z)$ can be described by the following formula

$$F_k\mathcal{O}_X(Z) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \cdot \left(\frac{\mathcal{O}^{\geq i+1}}{f^{i+1}}\right), \qquad \forall k \in \mathbb{N}$$

In the work of Zhang, this is generalized to the divisor $D = \alpha Z$, where $0 < \alpha \leq 1$, and with the associated $\mathscr{D}_X$-module being the well-known twisted localization $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha}) := \mathcal{O}_X(Z)f^{1-\alpha}$ (see more details in [MP19a] for its construction of the Hodge filtration). The result in this case is the following.

Theorem 11 ([Zha21], Thm. 4.1.2) If $D = \alpha Z$, where $0 < \alpha \leq 1$ and Z is an integral reduced effective divisor locally defined by f, a quasihomogeneous polynomial with an isolated singularity at the origin, then we have

$$F_k\mathcal{M}(f^{1-\alpha}) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \cdot \left(\frac{\mathcal{O}^{\geq \alpha+i}}{f^{i+1}}f^{1-\alpha}\right), \qquad \forall k \in \mathbb{N}$$

where the action $\cdot$ of $\mathscr{D}_X$ on the right-hand side is the action on the left $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha})$ defined by

$$D \cdot (rf^{1-\alpha}) := \left(D(r) + r\frac{(1-\alpha)D(f)}{f} \right) f^{1-\alpha}, \qquad \forall D \in \mathrm{Der}_{\mathbb{C}}\mathcal{O}_X$$

The proof relies on the previous result by Saito, and an additional lemma giving an explicit formula for the Hodge filtration on the V-filtration of the algebraic microlocalization of $\iota_+\mathcal{O}_X$, where $\iota : X \to X \times \mathbb{C}$ is the graph embedding defined by $\iota(x) = (x, f(x))$.

As a consequence of the Theorem 11, a formula for the Hodge filtration is obtained, which is easier to use in practice.

Corollary 1 ([Zha21], Cor. 4.1.4) In the situation of Theorem 11, we have

$$\begin{aligned} &F_0\mathcal{M}(f^{1-\alpha}) = f^{-1} \cdot \mathcal{O}^{\geq \alpha} f^{1-\alpha} \\ &F_k\mathcal{M}(f^{1-\alpha}) = \big(f^{-1} \cdot \sum_{v_j \in \mathcal{O}^{\geq k+1+\alpha}} \mathbb{C}v_j\big) f^{1-\alpha} + F_1\mathscr{D}_X \cdot F_{k-1}\mathcal{M}(f^{1-\alpha}), \qquad \forall k \geq 1 \end{aligned}$$

where $\{v_1, \ldots, v_{\mu_f}\}$ is a monomial basis for the Milnor algebra $\mathcal{O}/J(f)$. Alternatively, in terms of the Hodge ideals, these formulas say

$$I_0(D) = \mathcal{O}^{\geq \alpha} \tag{4.1}$$

$$I_k(D) = \sum_{v_j \in \mathcal{O}^{\geq k+\alpha}} \mathbb{C}v_j + \sum_{\substack{1 \leq i \leq n \\ a \in I_{k-1}(D)}} \mathcal{O}_X \left(f\partial_i a - (\alpha + k - 1)a\partial_i f \right), \qquad \forall k \geq 1 \tag{4.2}$$

The result in this particular case allows for a simpler proof of the inclusion $I_{k+1}(D) \subseteq I_k(D)$ for all $k \in \mathbb{N}$. We already encountered this inclusion in Proposition 1 for the case of reduced integral divisors, but in general it is not clear whether it should hold for arbitrary $\mathbb{Q}$-divisors, as discussed for Proposition 8 (see also [MP19b, Rmk. 4.2] and [MP20b, Rmk. 5.2]).

Also, by a result by Saito the generation level of an isolated quasihomogeneous singularity is $\lfloor n - \tilde{\alpha}_f \rfloor - 1$ (see [Sai09, Thm. 0.7]), which in the work of Zhang is generalized to the case of $\mathbb{Q}$-divisors as follows.

Corollary 2 ([Zha21], Cor. 4.1.6) In the situation of Corollary 1, the generation level of $\mathcal{M}(f^{1-\alpha})$ is $\lfloor n - \tilde{\alpha}_f - \alpha \rfloor$.

Proof. Define

$$J_k(D)f^{1-\alpha} := F_1\mathscr{D}_X \cdot \left(\frac{I_{k-1}(D)}{f^k} f^{1-\alpha} \right) f^{k+1}$$

Since f is quasihomogeneous with weight $w = (w_1, \ldots, w_n)$

$$f = \sum_{i=1}^{n} w_i x_i \partial_i f$$

and we obtain, similarly as we obtained the formulas for the Hodge ideals from the result of the filtration, that

$$J_k(D) = (f)I_{k-1}(D) + \sum_{\substack{1 \leq i \leq n \\ a \in I_{k-1}(D)}} \mathcal{O}_X \left(f\partial_i a - (\alpha + k - 1)a\partial_i f \right)$$

Thus, $J_k(D) \subseteq J(f)$.

By Corollary 1 it is enough to show that

$$\sum_{v_j \in \mathcal{O}^{\geq \alpha+q}} \mathcal{O}_X v_j \subseteq J_q(D) \quad \iff \quad q \geq \lfloor n - \tilde{\alpha}_f - \alpha \rfloor + 1$$

where $\{v_1, \ldots, v_{\mu_f}\}$ is a monomial basis for the Milnor algebra $\mathcal{O}/J(f)$.

On the one hand, pick $q \leq \lfloor n - \tilde{\alpha}_f - \alpha \rfloor$, and we will show that the inclusion on the left does not hold. By Remark 5 the maximum of the values $\rho(v_i)$ is equal to $n - \tilde{\alpha}_f$. In particular, there exists v_s such that $\rho(v_s) = n - \tilde{\alpha}_f \geq q + \alpha$, by hypothesis, and therefore it satisfies $v_s \in \mathcal{O}^{\geq \alpha+q}$ and $v_s \notin J(f)$. On the account of the above-mentioned inclusion $J_q(D) \subseteq J(D)$, we conclude

$$\sum_{v_j \in \mathcal{O}^{\geq \alpha+q}} \mathcal{O}_X v_j \nsubseteq J_q(D)$$

On the other hand, for any $q \geq \lfloor n - \tilde{\alpha}_f - \alpha \rfloor + 1$, we get

$$\alpha + q \geq \lfloor n - \tilde{\alpha}_f - \alpha \rfloor + 1 + \alpha > n - \tilde{\alpha}_f - \alpha + \alpha = n - \tilde{\alpha}_f$$

Since the maximum of the values $\rho(v_i)$ is $n - \tilde{\alpha}_f < \alpha + q$, there is no such v_s satisfying $v_s \in \mathcal{O}^{\geq \alpha+q}$ and $v_s \notin J(f)$. Thus, the inclusion is satisfied and the proof is complete.

□

4.3 Algorithm

The algorithm is simply the implementation of the result of Corollary 1, and is justified by the results presented. First, it calculates the first Hodge ideal $I_0(D)$ from $\mathcal{O}^{\geq \alpha}$. Then, the following ideals are calculated using the recursive expression, until the desired level k_{lim}.

Algorithm 1: Hodge ideals for quasihomogeneous polynomials

`// Input:` A quasihomogeneous polynomial f and rational $\alpha \in (0, 1]$
`// Output:` Hodge ideals of f^α up to index k_{lim}.

1 Compute weights w, homogeneous degree d and minimal exponent $\tilde{\alpha}_f$
2 Compute Jacobian ideal $J(f)$
3 Compute monomial base for Milnor algebra $M_f \leftarrow \{v_1, \ldots, v_\mu\}$
4 **for** $j = 1, 2, \ldots, \mu$ **do**
5 | Compute $\rho(v_j)$
6 **end**
7 Compute $I_0 \leftarrow \mathcal{O}^{\geq \alpha}$
8 **for** $k = 1, 2, \ldots, k_{lim}$ **do**
9 | Initialize $I_k \leftarrow I_{k-1}$
  | `// First part`
10 | **for** $j = 1, 2, \ldots, \mu$ **do**
11 | | **if** $\rho(v_j) \geq k + \alpha$ **then**
12 | | | Add element $I_k \leftarrow (I_k, v_j)$
13 | | **end**
14 | **end**
  | `// Second part (loop only on a set of generators of` I_{k-1}`)`
15 | **for** $a \in I_{k-1}$ **do**
16 | | **for** $i = 1, 2, \ldots, n$ **do**
17 | | | Add element $I_k \leftarrow (I_k, f\partial_i a - (\alpha + k - 1)a\partial_i f)$
18 | | **end**
19 | **end**
20 **end**

Notice that the recursive calculation is separated in the two parts appearing in the expression, and that in the second term we only need to loop through a set of generators of the ideal. Altogether, all the steps in the algorithm are clear, except how to compute the filtration $\mathcal{O}^{\geq \alpha}$, required in Step 7. We discuss next how this is implemented.

The idea is to search the monomials that *first* surpass the required weight ρ of the target value α. Then, the ideal sheaf $\mathcal{O}^{\geq\alpha}$ is the one generated by these monomials.

The way to find these monomials is to iterate through the lattice of exponents. For a fixed monomial, the condition of $\rho(x^p) \geq \alpha$ is equivalent to

$$\rho(x^p) = |w| + \min_{q\in\mathbb{N}^n}\{\langle w, q\rangle \mid a_q \neq 0\} = |w| + \langle w, p\rangle \geq \alpha$$

where w is the fixed homogeneous weights of the polynomial f considered.

Hence, we are looking at the lattice points that are immediately above[1] the hyperplane defined by the equation $|w| + \langle w, x\rangle = \alpha$. Surely this could be implemented already somewhere, but a simple backtracking approach is enough for our purposes, which is described in Algorithm 2.

Algorithm 2: `get_lattice_points_QH(w,p,val,i,`β`)`

// Input: The weights w, the current coordinate vector p, the current value of val $= |w| + \langle w, p\rangle = \rho(x^p)$, the current index i to be edited, the target value β.
// Output: Coordinates of the lattice points immediately above the hyperplane defined by $\{|w| + \langle w, x\rangle = \beta\}$, with the first i coordinates fixed by the input p.

Initialize $L \leftarrow \emptyset$, the list to store results.

// If we have already fixed all coordinates except last
if $i = n$ **then**
    while val $< \beta$ **do**
        $p[n] \leftarrow p[n] + 1$
        val $\leftarrow$ val $+ w[n]$
    end
    $L \leftarrow p$
end

// Else, try both possibilities: increasing or not the i-th coordinate
else
    // Not increase $p[i]$
    $L \leftarrow L+$ `get_lattice_points_QH(w,p,val,i+1,`β`)`

    // Increase $p[i]$
    while val $< \beta$ **do**
        $p[i] \leftarrow p[i] + 1$
        val $\leftarrow$ val $+ w[i]$
        $L \leftarrow L+$`get_lattice_points_QH(w,p,val,i+1,`β`)`
    end
end

return L

And to use it initially, we will simply call `get_lattice_points_QH(`$w, (0, \ldots, 0), |w|, 1, \alpha$`)`[2]. Finally, these lattice points are read as exponents to obtain the monomials which are then used to generate the ideal sheaf $\mathcal{O}^{\geq\alpha}$.

It is clear that the algorithm terminates, and that it provides the first lattice points above the described hyperplane. Although the procedure could be optimized, it will be enough for our purposes.

4.4 Examples

Example 10 ([Zha21], Ex. 4.1.7) Let $f = x^2 + y^3$ be the cusp, quasihomogeneous with weights $w = (1/2, 1/3)$.

[1] meaning the closest, in the opposite side of the hyperplane as the origin.
[2] The indices in Singular start at 1, rather than 0.

To present the resulting Hodge ideals, we will first calculate the Bernstein-Sato polynomial associated to f, via the implemented routines in Singular. In this case, we have already seen that

$$b_f(s) = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$$

Then, for each root, we change its sign and shift this value to the interval $(0,1]$ (recall Remark 4), in this case obtaining the set of candidate values $\alpha \in \{1/6, 5/6, 1\}$. Indeed, by Theorem 8, we expect the changes in the Hodge ideals of $\mathbb{Q}$-divisors associated to f^α to (possibly) occur at these values.

α	$I_0(f^\alpha)$	$I_1(f^\alpha)$	$I_2(f^\alpha)$
$\frac{1}{6}$	R	(x,y)	(x^2, xy, y^3)
$\frac{5}{6}$	R	(x,y^2)	$(y^3-(2\alpha+1)x^2, xy^2, x^2y, x^3)$
$\frac{6}{6}$	(x,y)	(x^2,xy,y^3)	$(y^4-(2\alpha+1)x^2y, xy^3, x^2y^2, x^3)$

Table 1: Hodge ideals $I_k(f^\alpha)$, $k=0,1,2$, $\alpha \in \mathbb{Q}\cap(0,1]$ for $f = x^2+y^3$.

Example 11 ([Zha21], Ex. 4.2.8) Let $f = x^2+y^5$, quasihomogeneous with weights $w = (1/2, 1/5)$.

α	$I_0(f^\alpha)$	$I_1(f^\alpha)$	$I_2(f^\alpha)$
$\frac{1}{10}$	R	(x,y^2)	$(x^3, xy^2, x^2y, y^5-(2\alpha+1)x^2)$
$\frac{3}{10}$	R	(x,y^3)	$(x^3, xy^3, x^2y^2, y^5-(2\alpha+1)x^2)$
$\frac{7}{10}$	R	(x,y^4)	$(x^3, xy^4, x^2y^3, y^5-(2\alpha+1)x^2)$
$\frac{9}{10}$	(x,y)	(x^2,xy,y^5)	$(x^3, xy^5, x^2y^4, y^6-(2\alpha+1)x^2y)$
$\frac{10}{10}$	(x,y^2)	$(x^3,xy^2,x^2y,y^5-(2\alpha-1)x^2)$	$(x^4, x^3y, 3xy^5-(2\alpha-1)x^3, y^7-(2\alpha+1)x^2y^2)$

Table 2: Hodge ideals $I_k(f^\alpha)$, $k=0,1,2$, $\alpha \in \mathbb{Q}\cap(0,1]$ for $f = x^2+y^5$.

Example 12 ([Bla22], Ex. 3) Let $f = x^3+y^3+z^3+xyz$, homogeneous and, in particular, quasihomogeneous with weights $w = (1/3, 1/3, 1/3)$. The pair $(\mathbb{A}^3_{\mathbb{C}}, \operatorname{div}(f))$ is log-canonical, hence the multiplier ideals are trivial (see how the first column, representing the zeroth Hodge ideals, is trivial). Therefore, we can obtain a first non-trivial invariant of the singularity by considering the Hodge ideals.

α	$I_0(f^\alpha)$	$I_1(f^\alpha)$	$I_2(f^\alpha)$
$\frac{1}{3}$	R	(x,y,z)	$(x^2y, x^2z, xy^2, zy^2, yz^2, xz^2, y^3-z^3, x^3-z^3, xyz+3z^3, z^4)$
$\frac{2}{3}$	R	$(x,y,z)^2$	$(x,y,z)^4$
$\frac{3}{3}$	R	$(z^3, yz^2, xz^2,$ $xz+3y^2,$ $xy+3z^2,$ $yz+3x^2)$	$((3\alpha+56)x^3z+(3\alpha+56)y^3z-(53\alpha)xyz^2-(81\alpha+28)z^4,$ $(27\alpha)y^4+(18\alpha)xy^2z+(3\alpha+56)x^2z^2,$ $(27\alpha)xy^3+(9\alpha)x^2yz+(81\alpha+28)y^2z^2+(27\alpha)xz^3,$ $(3\alpha+56)x^2y^2+(18\alpha)xyz^2+(27\alpha)z^4,$ $(27\alpha)x^3y+(9\alpha)xy^2z+(81\alpha+28)x^2z^2+(27\alpha)yz^3,$ $(27\alpha)x^4+(18\alpha)x^2yz+(3\alpha+56)y^2z^2,$ $z^5, yz^4, xz^4, y^2z^3, xyz^3, x^2z^3, y^3z^2, xy^2z^2, x^2yz^2)$

Table 3: Hodge ideals $I_k(f^\alpha)$, $k=0,1,2$, $\alpha \in \mathbb{Q}\cap(0,1]$ for $f = x^3+y^3+z^3+xyz$.

Example 13 ([Bla22], Ex. 2) Let $f = (y^2 - x^3)(y^2 + \lambda x^3)$, quasihomogeneous with weights $w = (1/3, 1/2)$.

The parameter $\lambda \in \mathbb{C}^*$ is an analytical invariant, associated with the singularity at the origin. We can see that this parameter appears as well in some of the Hodge ideals computed.

α	$I_0(f^\alpha)$	$I_1(f^\alpha)$
$\frac{1}{12}$	R	(y^3, xy^2, x^3y, x^4)
$\frac{2}{12}$	R	(y^3, x^2y^2, x^3y, x^5)
$\frac{3}{12}$	R	$(y^4, xy^3, x^2y^2, (\lambda-1)x^3y + 2y^3, x^5)$
$\frac{4}{12}$	R	$(y^4, xy^3, (\lambda-1)x^3y + 2y^3, (2\lambda)x^5 + (-\lambda+1)x^2y^2)$
$\frac{5}{12}$	R	$(y^4, (\lambda-1)x^3y + 2y^3, x^2y^3, (2\lambda)x^5 + (-\lambda+1)x^2y^2)$
$\frac{7}{12}$	(x, y)	$(y^5, xy^4, x^2y^3, (\lambda-1)x^3y^2 + 2y^4, (\lambda-1)x^4y + 2xy^3, \lambda x^6 + y^4)$
$\frac{8}{12}$	(y, x^2)	$(y^5, x^2y^3, (\lambda-1)x^4y^2 + 2xy^4, x^5y,$ $\lambda x^6 + (\lambda-1)(2\alpha-1)x^3y^2 + (4\alpha-1)y^4)$
$\frac{9}{12}$	(y^2, xy, x^2)	$(y^5, x^2y^4, x^3y^3, (\lambda-1)x^4y^2 + 2xy^4, (\lambda-1)x^5y + 2x^2y^3, \lambda x^7 + xy^4)$
$\frac{10}{12}$	(y^2, xy, x^3)	$(y^5, x^2y^4, x^3y^3, x^5y^2, x^6y,$ $\lambda x^7 + (\lambda-1)(2\alpha-1)x^4y^2 + (4\alpha-1)xy^4)$
$\frac{11}{12}$	(y^2, x^2y, x^3)	$(y^6, xy^5, x^2y^4, ((\lambda^2+1)(\alpha-1) - 2\lambda\alpha)x^3y^3 + (\lambda-1)(2\alpha-1)y^5,$ $x^5y^2, ((\lambda^2+1)(\alpha-1) - 2\lambda\alpha)x^6y - 2(2\alpha-1)y^5, x^8)$
$\frac{12}{12}$	(y^3, xy^2, x^2y, x^4)	$(y^6, xy^5, x^3y^4, x^4y^3, x^6y^2, x^7y,$ $\lambda x^8 + (2\lambda-1)(\alpha-1)x^5y^2 + (4\alpha-1)x^2y^4)$

Table 4: Hodge ideals $I_k(f^\alpha)$, $k = 0, 1$, $\alpha \in \mathbb{Q} \cap (0, 1]$ for $f = (y^2 - x^3)(y^2 + \lambda x^3)$.

5 Newton non-degenerate

5.1 Definition

In this section, we still consider f with an isolated singularity at the origin, and $f(0) = 0$. For brevity, if $f = \sum_{p \in \mathbb{N}^n} a_p x^p$ we will define its support to be $\mathrm{supp}(f) = \{p \in \mathbb{N}^n \mid a_p \neq 0\}$.

Polynomials which are Newton non-degenerate are sometimes also called non-degenerate with respect to the Newton polytope, or simply non-degenerate. The condition that these polynomials satisfy has a more combinatorial flavor, as it is easier described by considering the following objects.

Definition 16 (Newton polytope, diagram) Let $f = \sum_{p \in \mathbb{N}^n} a_p x^p \in \mathbb{C}[x]$ be a polynomial with $f(0) = 0$. Then, the local Newton polytope $N(f, 0)$ is the convex hull of

$$\bigcup_{p \in \mathrm{supp}(f)} p + (\mathbb{R}_{\geq 0})^n$$

We define the Newton diagram $\Gamma(f, 0)$ of f as the union of all the compact faces of $N(f, 0)$. Moreover, we introduce for a face $\sigma \subset \Gamma(f, 0)$ the truncation

$$f^\sigma := \sum_{p \in \sigma \cap \mathrm{supp}(f)} a_p x^p$$

We illustrate the introduced concepts with the following example.

Example 14 Let us consider the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$, and we plot in the following Figure 1 its Newton diagram.

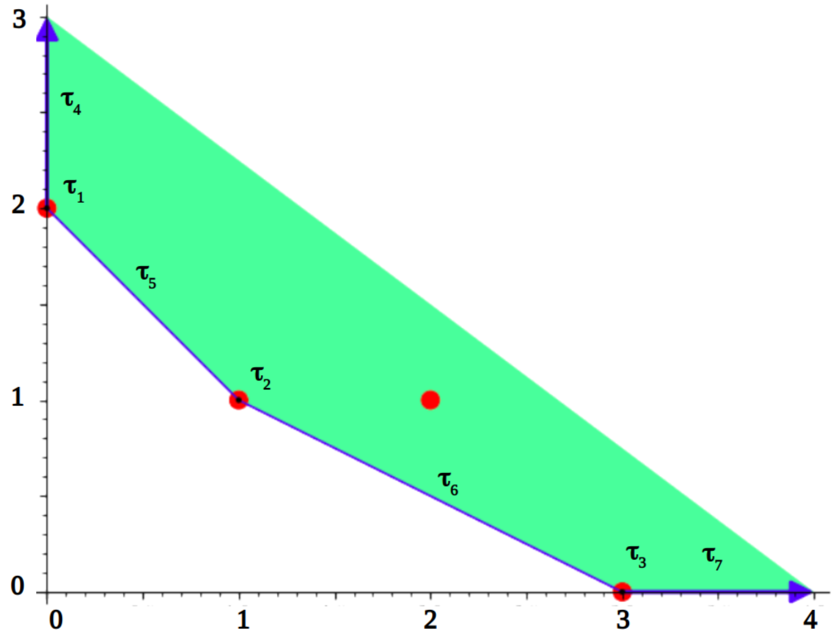

Figure 1: Newton polytope of the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$.

The points in red are the coordinates corresponding to the exponents of all the monomials appearing in the polynomial. Then, after taking the convex hull of

$$\left\{(0,2) + (\mathbb{R}_{\geq 0})^2\right\} \cup \left\{(1,1) + (\mathbb{R}_{\geq 0})^2\right\} \cup \left\{(2,1) + (\mathbb{R}_{\geq 0})^2\right\} \cup \left\{(3,0) + (\mathbb{R}_{\geq 0})^2\right\}$$

we obtain the Newton polytope, depicted in color green. The corresponding faces, and the associated truncation of the polynomial, are

- Faces of dimension $\dim \tau = 0$:

$$\begin{aligned} \tau_1 &= \{(0,2)\} & f^{\tau_1} &= -y^2 \\ \tau_2 &= \{(1,1)\} & f^{\tau_2} &= 4xy \\ \tau_3 &= \{(3,0)\} & f^{\tau_3} &= x^3 \end{aligned}$$

- Faces of dimension $\dim \tau = 1$:

$$\begin{aligned} \tau_4 &= \{(0,2) + \mathbb{R}_{\geq 0}(0,1)\} & f^{\tau_4} &= -y^2 \\ \tau_5 &= \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} & f^{\tau_5} &= -y^2 + 4xy \\ \tau_6 &= \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} & f^{\tau_6} &= x^3 + 4xy \\ \tau_7 &= \{(3,0) + \mathbb{R}_{\geq 0}(1,0)\} & f^{\tau_7} &= x^3 \end{aligned}$$

Definition 17 (Convenient) A polynomial $f \in\subset \mathbb{C}[x]$ with $f(0) = 0$ is called *convenient* if its Newton diagram $\Gamma(f, 0)$ meets all the coordinate axes. That is, if for each variable x_i, $1 \leq i \leq n$, there is a monomial $x_i^{n_i}$ appearing in the expression of f with a non-zero coefficient, for some $n_i \in \mathbb{N}$.

Sometimes, this condition is also called *comfortable*, which has a more intuitive meaning: the Newton diagram in this case rests nicely in the coordinate planes, i.e. it is in a comfortable position. This is the direct translation of the French word used originally in the paper by Kouchnirenko [Kou76], which is the main reference in introducing the concept of Newton non-degeneracy.

Definition 18 (Non-degenerate) We say that f is Newton non-degenerate at 0 if for any face $\sigma \subset \Gamma(f, 0)$, the hypersurface $f^\sigma = 0$ satisfies the condition

$$x_1 \frac{\partial f^\sigma}{\partial x_1} = \cdots = x_n \frac{\partial f^\sigma}{\partial x_n} = 0 \quad \Longrightarrow \quad x_1 \ldots x_n = 0$$

that is, the series $x_i \frac{\partial f^\sigma}{\partial x_i}$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

An example of how the information of the polynomial is encoded in the Newton diagram is the following.

Theorem 12 ([Kou76], Cor. 1.22) Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic map with an isolated singularity at the origin, and Newton non-degenerate. Then, for $n \neq 3$, the topological type of $f^{-1}(0)$ and the topological type of the Milnor fibration are determined by the Newton diagram $\Gamma(f)$.

Also relevant, even though we are restricting our attention to a subclass of all polynomials, these are still general enough. Indeed, a result by Kouchnirenko shows that, basically, the polynomials that are Newton non-degenerate are dense in the Zariski topology (if the base field is of characteristic zero).

Theorem 13 ([Kou76], Thm. 6.1) Let $\Delta \subset \mathbb{R}^k$ be any convex compact polyhedron with vertices in the lattice $\mathbb{Z}^k$, of dimension $q \leq k - 1$ and not entirely contained in a linear subspace of dimension q. Then, there exists a subset of $\mathbb{C}[\Delta] = \{f \in \mathbb{C}\{x\} \mid \operatorname{supp}(f) \subset \Delta\}$ open and dense in the Zariski topology, which consists of all Laurent polynomials $f \in \mathbb{C}[\Delta]$ such that the Laurent polynomials $x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n}$ do not vanish at the same time on $(\mathbb{C} \setminus 0)^n$. That is, it consists of all Newton non-degenerate Laurent polynomials with support in Δ.

Now, we proceed similarly as in the quasihomogeneous case, introducing a weight function. The idea is to notice that, the truncation at each of the faces is quasihomogeneous, so we can just evaluate the ρ weight defined for quasihomogeneous polynomials through all the faces.

Definition 19 (Weight function $\tilde{\rho}$) For any face $\sigma \in \Gamma(f, 0)$, f^σ is a quasihomogeneous polynomial of weight w^σ. Define

$$\tilde{\rho}^\sigma(g) := \rho_{w^\sigma}(g) = |w^\sigma| + w^\sigma - \deg(g)$$

Then, we define a weight function $\tilde{\rho}\colon \mathbb{C}\{x\} \to \mathbb{Q}_{>0}$ by

$$\tilde{\rho}(g) := \min\{\tilde{\rho}^\sigma(g) \mid \sigma \in \Gamma(f, 0), \dim \sigma = n - 1\}$$

Again, this defines a filtration on $\mathcal{O}_{X,0}$, by considering the sets $\tilde{\mathcal{O}}^{\geq c} := \{g \in \mathbb{C}\{x\} \mid \tilde{\rho}(g) \geq c\}$.

We give an example of the calculation of this weight function by continuing the earlier Example 14.

Example 15 Consider again the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$, with the Newton diagram depicted in Figure 1. First, notice that the only compact faces of maximum dimension are

$$\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} \qquad f^{\tau_5} = -y^2 + 4xy$$
$$\tau_6 = \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} \qquad f^{\tau_6} = x^3 + 4xy$$

and the corresponding homogeneous weights (remember we consider homogeneous degree 1) are

$$w^{\tau_5} = (1/2, 1/2) \qquad w^{\tau_6} = (1/3, 2/3)$$

Now, we calculate the weight $\tilde{\rho}(g)$ for $g = x^2 + y$, which has exponents $\{(2,0),(0,1)\}$. By definition,

$$\tilde{\rho}^{\tau_5}(g) = |w^{\tau_5}| + \min\{\langle w^{\tau_5}, (2,0)\rangle, \langle w^{\tau_5}, (0,1)\rangle\} = 1 + \min\{1, 1/2\} = 3/2$$
$$\tilde{\rho}^{\tau_6}(g) = |w^{\tau_6}| + \min\{\langle w^{\tau_6}, (2,0)\rangle, \langle w^{\tau_6}, (0,1)\rangle\} = 1 + \min\{2/3, 1/3\} = 4/3$$

Hence, we finally have

$$\tilde{\rho}(g) = \min\{3/2, 4/3\} = 3/2.$$

Remark 6 As a last remark, notice that the results in the cases that are being considered do not imply each other. Indeed, there are polynomials that are Newton non-degenerate but which are not quasihomogeneous, for example $f = x^5 + y^5 + x^2y^2$. In the contrary, $f = (x+y)^2 + xz + z^2$ is a quasihomogeneous polynomial, but it is degenerate.

Example 16 We can check indeed that $f = (x+y)^2 + xz + z^2 = x^2 + 2xy + xz + y^2 + z^2$ is degenerate on the face (in this case, of dimension 1, an edge) $\sigma := \overline{(2,0,0)(0,2,0)}$ of $\Gamma(f,0)$. The truncation is obtained by taking only the monomials in f with exponents (p_x, p_y, p_z) satisfying the equations $\{p_x + p_y = 2, p_z = 0\}$, which results in $f^\sigma = (x+y)^2$. Now, if we calculate the polynomials

$$\begin{cases} x\dfrac{\partial f^\sigma}{\partial x} = x \cdot 2(x+y) \\ y\dfrac{\partial f^\sigma}{\partial y} = y \cdot 2(x+y) \\ z\dfrac{\partial f^\sigma}{\partial z} = 0 \end{cases}$$

we can see that they all vanish on the points $(t, -t, s)$ for any $t, s \in \mathbb{C}$, in particular they vanish outside $(\mathbb{C} \setminus 0)^3$. Hence, the polynomial is degenerate with respect to its Newton polygon.

5.2 Hodge ideals

We now let Z be an integral reduced effective divisor, defined by f a Newton non-degenerate polynomial with an isolated singularity at the origin, and $f(0) = 0$. Then, we will consider the divisor $D = \alpha Z$, where $0 < \alpha \leq 1$, and reproduce the results in the previous case to the Hodge filtration on the $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha}) := \mathcal{O}_X(*Z)f^{1-\alpha}$.

Theorem 14 ([Zha21], Thm. 4.3.3) If $D = \alpha Z$, where $0 < \alpha \leq 1$ and Z is an integral reduced effective divisor locally defined by f, a Newton non-degenerate polynomial with an isolated singularity at the origin, then we have

$$F_k\mathcal{M}(f^{1-\alpha}) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \cdot \left(\frac{\tilde{\mathcal{O}}^{\geq \alpha+i}}{f^{i+1}} f^{1-\alpha}\right), \qquad \forall k \in \mathbb{N}$$

where the action $\cdot$ of $\mathscr{D}_X$ on the right-hand side is the action on the left $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha})$ defined by

$$D \cdot (rf^{1-\alpha}) := \left(D(r) + r\frac{(1-\alpha)D(f)}{f}\right) f^{1-\alpha}, \qquad \forall D \in \mathrm{Der}_{\mathbb{C}}\mathcal{O}_X$$

As a consequence of Theorem 14, a formula for the Hodge filtration is obtained, following the same reasoning as in the proof of Corollary 1 but with the filtration on the structure sheaf now given by $\tilde{\rho}$.

Corollary 3 In the situation of Theorem 14, we have that the Hodge ideals are given by

$$I_0(D) = \tilde{\mathcal{O}}^{\geq \alpha} \tag{5.1}$$

$$I_k(D) = \tilde{\mathcal{O}}^{\geq \alpha+k} + \sum_{\substack{1\leq i\leq n \\ a\in I_{k-1}(D)}} \mathcal{O}_X\left(f\partial_i a - (\alpha+k-1)a\partial_i f\right), \qquad \forall k \geq 1 \tag{5.2}$$

As already mentioned, the results are almost analogous to the quasihomogeneous case, but with the appropriate weight function defined for Newton non-degenerate polynomials. However, there is a key difference in the computation of the first term in the expression of $I_k(D)$.

Remark 7 (Differences between both cases) The difference between results Corollary 1 and Corollary 3 is when searching for monomials with weight ρ greater than $\alpha+k$, in the first term of the expression for $I_k(D)$. Indeed, in the case of quasihomogeneous we could restrict our search to the monomials of the Milnor algebra

$$\sum_{v_j\in\mathcal{O}^{\geq\alpha+k}} \mathbb{C}v_j$$

However, this turns out to be not enough in the case of Newton non-degenerate, and we have to look for monomials in all $\mathcal{O}$, that is to calculate

$$\tilde{\mathcal{O}}^{\geq\alpha+k}$$

This broader search would also suffice in the case of quasihomogeneous, but the difference is that there it can be justified (see the proof of Corollary 1 in [Zha21, Cor. 4.1.4]) that

$$\mathcal{O}^{\geq\alpha+k}\cap J(f) \subseteq \sum_{\substack{1\leq i\leq n \\ a\in I_{k-1}(D)}} \mathcal{O}_X\left(f\partial_i a - (\alpha+k-1)a\partial_i f\right)$$

Let us see with an example that indeed the expression (5.1) is the best we can hope for.

Example 17 Take Z the divisor defined by $f = x^5+y^5+x^2y^2$, which is Newton non-degenerate. For $\alpha = 3/10$ and $D = \alpha Z$, we obtain

$$I_0(D) = \tilde{\mathcal{O}}^{\geq 3/10} = R$$

as for example $\tilde{\rho}(1) = 5/10$. Then, also from the expression (5.1) we have

$$\begin{aligned}
I_1(D) &= \tilde{\mathcal{O}}^{\geq 3/10+1} + \sum_{\substack{1\leq i\leq n \\ a\in I_0(D)}} \mathcal{O}_X\left(f\partial_i a - (\alpha+k-1)a\partial_i f\right) \\
&= (x^4, x^3y, x^2y^2, xy^3, y^4) + (5x^4+2xy^2, 2x^2y+5y^4) \\
&= (x^4, x^2y, xy^2, y^4)
\end{aligned}$$

Instead, if we were to apply the exact analogous of expression (4.1), then we obtain the incorrect result

$$\begin{aligned}
I_1(D) &= \sum_{v_j\in\tilde{\mathcal{O}}^{\geq 3/10+1}} \mathbb{C}v_j + \sum_{\substack{1\leq i\leq n \\ a\in I_0(D)}} \mathcal{O}_X\left(f\partial_i a - (\alpha+k-1)a\partial_i f\right) \\
&= (x^3y, x^2y^2, xy^3) + (5x^4+2xy^2, 2x^2y+5y^4) \\
&= (5x^4+2xy^2, x^3y, x^2y^2, xy^3, 2x^2y+5y^4)
\end{aligned}$$

We can justify that this is indeed wrong by applying Theorem 8. Calculating Hodge ideals with the later incorrect expression, we check that $I_1\left(\frac{2}{10}Z\right) \neq I_1\left(\left(\frac{2}{10}+\varepsilon\right)Z\right)$. Since we can take $\beta = 1/2$ and $(X, 1/2Z)$ is 0-log-canonical, we deduce

$$\tilde{b}_Z\left(-1-\frac{2}{10}\right) = \tilde{b}_Z\left(-\frac{12}{10}\right) = 0$$

However, this is clearly false, given that the Bernstein-Sato polynomial of f is

$$b_Z(s) = \left(s+\frac{1}{2}\right)^2 \left(s+\frac{7}{10}\right) \left(s+\frac{9}{10}\right) (s+1)^2 \left(s+\frac{11}{10}\right) \left(s+\frac{13}{10}\right)$$

5.3 Algorithm

The algorithm is simply the implementation of the result of Corollary 3, and is justified by the results presented. First, it calculates the first Hodge ideal $I_0(D)$ from $\tilde{\mathcal{O}}^{\geq\alpha}$. Then, the following ideals are calculated using the recursive expression, until the desired level k_{lim}.

The algorithm is very similar to the quasihomogeneous case, but now including the required previous calculations of the Newton polygon, and using the weight function $\tilde{\rho}$ instead of ρ. Also, as in the first term of the recursive expression we are now searching among all monomials, there is no need to calculate a basis of the Milnor algebra.

Algorithm 3: Hodge ideals for Newton non-degenerate polynomials

`// Input: A Newton non-degenerate polynomial` f`, rational` $\alpha \in (0,1]$ `and index` k_{lim}
`// Output: Hodge ideals of` f^α `up to index` k_{lim}

1 Compute `polytope` the Newton polytope of f
2 Extract `facets` of `polytope` `// This includes the weights` w^σ `for each facet` σ`.`
3 Compute $I_0 \leftarrow \tilde{\mathcal{O}}^{\geq\alpha}$
4 **for** $k = 1, 2, \ldots, k_{lim}$ **do**
  `// First part`
5   Initialize $I_k \leftarrow \tilde{\mathcal{O}}^{\geq\alpha+k}$
  `// Second part (loop only on a set of generators of` I_{k-1}`)`
6   **for** $a \in I_{k-1}$ **do**
7   **for** $i = 1, 2, \ldots, n$ **do**
8   Add element $I_k \leftarrow (I_k, f\partial_i a - (\alpha+k-1)a\partial_i f)$
9   **end**
10   **end**
11   Save previous ideal $I_{k-1} \leftarrow I_k$
12 **end**

Notice that the recursive calculation is separated in the two parts appearing in the expression, and that in the second term we only need to loop through a set of generators of the ideal. Altogether, all the steps in the algorithm are clear, except how to compute the filtration $\tilde{\mathcal{O}}^{\geq\alpha}$, required in Steps 3 and 5. We discuss next how this is implemented.

Again, the idea is to search the monomials that *first* surpass the required weight $\tilde{\rho}$ of the target value α. Then, the ideal sheaf $\tilde{\mathcal{O}}^{\geq\alpha}$ is the one generated by these monomials.

The way to find these monomials is to iterate through the lattice of exponents, in the same spirit as the backtracking for the quasihomogeneous case. However, now the optimization used before to store the current value of $\tilde{\rho}(x^p)$ can not be copied, as it is not clear how the weight changes when increasing the exponents in the monomial $x^p \cdot x_i$. Instead, the weight function $\tilde{\rho}$ ought to be calculated each time. The implemented procedure is described in the following Algorithm 4.

And to use it initially, we will simply call `get_lattice_points_NND(facets,` $(0,\ldots,0), 1, \alpha$`)`. Finally, these lattice points are read as exponents to obtain the monomials which are then used to generate the ideal sheaf $\tilde{\mathcal{O}}^{\geq\alpha}$.

It is clear that the algorithm terminates, and that it provides the first lattice points to have $\tilde{\rho}$ weight higher than α. Although the procedure can surely be optimized, it will be enough for our purposes.

Algorithm 4: `get_lattice_points_NND(facets,p,i,`β`)`

// Input: Facets of the Newton polytope of f, the current coordinate vector p, the current index i to be edited, the target value β.

// Output: Coordinates of the minimal lattice points to have weight $\tilde{\rho}$ greater than β, with the first i coordinates fixed by the input p.

Initialize $L \leftarrow \emptyset$, the list to store results.

// If we have already fixed all coordinates except last

if $i = n$ **then**
  val $\leftarrow \tilde{\rho}(x^p)$
  while val $< \beta$ **do**
    $p[n] \leftarrow p[n] + 1$
    val $\leftarrow \tilde{\rho}(x^p)$
  end
  $L \leftarrow p$
end

// Else, try both possibilities: increasing or not the i-th coordinate

else
  // Not increase $p[i]$
  $L \leftarrow L+$ `get_lattice_points_NND(facets,p,i+1,`β`)`
  // Increase $p[i]$
  val $\leftarrow \tilde{\rho}(x^p)$
  while val $< \beta$ **do**
    $p[i] \leftarrow p[i] + 1$
    val $\leftarrow \tilde{\rho}(x^p)$
    $L \leftarrow L+$`get_lattice_points_NND(facets,p,i+1,`β`)`
  end
end

return L

5.4 Examples

Example 18 ([Bla22], Ex. 1) Let $f = x^5 + y^5 + x^2y^2$, which is possibly the simplest example of a plane curve which is not quasihomogeneous or a μ-constant deformation of a quasihomogeneous singularity.

α	$I_0(f^\alpha)$	$I_1(f^\alpha)$
$\frac{1}{10}$	R	(x^3, x^2y, xy^2, y^3)
$\frac{3}{10}$	R	(x^4, x^2y, xy^2, y^4)
$\frac{5}{10}$	R	$(5x^4 + 2xy^2, x^3y, x^2y^2, xy^3, 5y^4 + 2x^2y)$
$\frac{7}{10}$	(x, y)	$((5\alpha - 2)x^5 + (2\alpha - 1)x^2y^2, x^3y, xy^3, (5\alpha - 2)y^5 + (2\alpha - 1)x^2y^2)$
$\frac{9}{10}$	(x^2, xy, y^2)	$(x^6, x^4y, x^3y^2, x^2y^3, xy^4, y^6)$
$\frac{10}{10}$	(x^3, xy, y^3)	$(x^6 - (2\alpha - 1)x^3y^2, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6 - (2\alpha - 1)x^2y^3)$

Table 5: Hodge ideals $I_k(f^\alpha)$, $k = 0, 1$, $\alpha \in \mathbb{Q} \cap (0, 1]$ for $f = x^5 + y^5 + x^2y^2$.

Example 19 Let $f = x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$. In this case, the Bernstein-Sato polynomial associated to f is

$$b_f(s) = \left(s+\frac{11}{6}\right)\left(s+\frac{9}{5}\right)\left(s+\frac{12}{7}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{8}{5}\right)\left(s+\frac{11}{7}\right)\left(s+\frac{3}{2}\right)\left(s+\frac{10}{7}\right)\left(s+\frac{7}{5}\right)\cdot$$
$$\cdot\left(s+\frac{4}{3}\right)\left(s+\frac{9}{7}\right)\left(s+\frac{5}{4}\right)\left(s+\frac{6}{5}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{8}{7}\right)(s+1)^3\left(s+\frac{6}{7}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{4}{5}\right)\left(s+\frac{3}{4}\right)$$

and the Hodge ideals are summarized next.

α	$I_0(f^\alpha)$	$I_1(f^\alpha)$
$\frac{1}{7}$	R	$(z^2, yz, xz, y^2, xy, x^2)$
$\frac{1}{6}$	R	$(yz, xz, y^2, xy, x^2, z^3)$
$\frac{1}{5}$	R	$(xz, xy, x^2, z^3, yz^2, y^2z, y^3)$
$\frac{1}{4}$	R	$(z^3, yz^2, xz^2, y^2z, xyz, x^2z, y^3, xy^2, x^2y, x^3)$
$\frac{1}{3}$	R	$(yz^2, xz^2, y^2z, xyz, x^2z, y^3, xy^2, x^2y, x^3, z^4)$
$\frac{2}{5}$	R	$(xz^2, y^2z+yz^2, xyz, x^2z, xy^2, x^2y, x^3, z^4, yz^3, y^4)$
$\frac{3}{7}$	R	$(y^2z+yz^2, xyz, x^2z+xz^2, x^2y+xy^2, z^4, yz^3, xz^3, y^4, xy^3, x^4)$
$\frac{1}{2}$	R	$(y^2z+yz^2, xyz, x^2z+xz^2, x^2y+xy^2, yz^3, xz^3, y^4, xy^3, x^4, z^5)$
$\frac{4}{7}$	R	$(2xyz+y^2z+yz^2, x^2z-y^2z+xz^2-yz^2, x^2y+xy^2-y^2z-yz^2,$ $xz^3, y^2z^2+yz^3, y^3z-yz^3, xy^3, x^4, z^5, yz^4, y^5)$
$\frac{3}{5}$	R	$(2xyz+y^2z+yz^2, x^2z-y^2z+xz^2-yz^2, x^2y+xy^2-y^2z-yz^2,$ $xz^3, y^2z^2+yz^3, y^3z-yz^3, xy^3, x^4, yz^4, y^5, z^6)$
$\frac{2}{3}$	R	$(x^2z+2xyz+xz^2, x^2y+xy^2+2xyz, y^2z^2+yz^3, xyz^2, y^3z-yz^3,$ $xy^2z, 5x^4+2xyz+y^2z+yz^2, yz^4, xz^4, y^5, xy^4, z^6)$
$\frac{5}{7}$	R	$(x^2y+xy^2+2xyz, y^2z^2+yz^3, xyz^2, x^2z^2+xz^3, y^3z-yz^3, xy^2z,$ $x^3z-xz^3, 5x^4+2xyz+y^2z+yz^2, xz^4, 6y^5+x^2z+2xyz+xz^2, xy^4, z^6, yz^5)$
$\frac{3}{4}$	R	$(y^2z^2+yz^3, xyz^2, x^2z^2+xz^3, y^3z-yz^3, xy^2z, x^2yz, x^3z-xz^3, x^2y^2+xy^3, x^3y-xy^3,$ $5x^4+2xyz+y^2z+yz^2, xz^4, 6y^5+x^2z+2xyz+xz^2, xy^4, 7z^6+x^2y+xy^2+2xyz, yz^5)$
$\frac{4}{5}$	(x,y,z)	$(y^2z^2+yz^3, xyz^2, x^2z^2+xz^3, y^3z-yz^3, xy^2z, x^2yz, x^3z-xz^3,$ $x^2y^2+xy^3, x^3y-xy^3, xz^4, xy^4, x^5, yz^5, y^6, z^7)$
$\frac{5}{6}$	(x,y,z)	$(x^2z^2+2xyz^2+xz^3, xy^2z-xyz^2, (5\alpha-4)x^2yz+(15\alpha-11)xyz^2,$ $(5\alpha-4)x^3z+(-40\alpha+30)xyz^2+(-5\alpha+4)xz^3, x^2y^2+xy^3+2xyz^2,$ $(5\alpha-4)x^3y+(-5\alpha+4)xy^3+(-40\alpha+30)xyz^2, 5xz^4-2xyz^2-y^2z^2-yz^3,$ $y^2z^3+yz^4, xyz^3, y^3z^2-yz^4, y^4z+yz^4, 5xy^4-y^3z-2xyz^2-y^2z^2,$ $(5\alpha-4)x^5+(-4\alpha+3)xyz^2, yz^5, y^6, z^7)$
$\frac{6}{7}$	(x,y,z)	$((30\alpha-24)x^2yz+(30\alpha-25)xy^2z+(60\alpha-41)xyz^2, x^2y^2+xy^3+2xy^2z,$ $(15\alpha-12)x^3z+(-30\alpha+25)xy^2z+(15\alpha-12)x^2z^2+(-60\alpha+41)xyz^2,$ $(15\alpha-12)x^3y+(-15\alpha+12)xy^3+(-60\alpha+49)xy^2z+(-60\alpha+41)xyz^2,$ $5xz^4-2xyz^2-y^2z^2-yz^3, y^2z^3+yz^4, xyz^3, 5x^2z^3+2xyz^2+y^2z^2+yz^3,$ $y^3z^2-yz^4, xy^2z^2, y^4z+yz^4, xy^3z, 5xy^4-2xy^2z-y^3z-y^2z^2,$ $(30\alpha-24)x^5+(-6\alpha+5)xy^2z+(-18\alpha+13)xyz^2,$ $6yz^5+x^2z^2+2xyz^2+xz^3, 6y^6+xy^2z-xyz^2, z^7)$
1	(x,y,z)	$(x^3z+2x^2yz+x^2z^2, x^2y^2+xy^3+2xy^2z, x^3y-xy^3+2x^2yz-2xy^2z,$ $5xz^4-2xyz^2-y^2z^2-yz^3, y^2z^3+yz^4, xyz^3, 5x^2z^3+2xyz^2+y^2z^2+yz^3,$ $y^3z^2-yz^4, xy^2z^2, x^2yz^2, y^4z+yz^4, xy^3z, 5xy^4-2xy^2z-y^3z-y^2z^2,$ $(210\alpha-107)x^5+(84\alpha-55)x^2yz+(42\alpha-35)xy^2z+(42\alpha-36)xyz^2,$ $6yz^5+x^2z^2+2xyz^2+xz^3, (210\alpha-107)y^6+(35\alpha-28)x^2yz+(70\alpha-47)xy^2z+$ $+(35\alpha-30)xyz^2, (210\alpha-107)z^7+(30\alpha-24)x^2yz+(30\alpha-25)xy^2z+(60\alpha-41)xyz^2)$

Table 6: Hodge ideals $I_k(f^\alpha)$, $k=0,1$, $\alpha\in\mathbb{Q}\cap(0,1]$ for $f=x^5+y^6+z^7+x^2yz+xyz^2+xy^2z$.

This is the most complex example so far, due to the fact the Newton diagram of this polynomial has 7 compact faces of dimension 2, as can be seen in the following Figure 2.

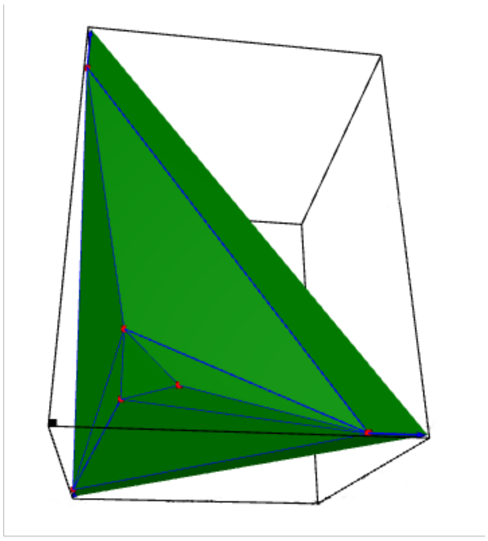

Figure 2: Newton polytope of the polynomial $f = x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$.

6 Conclusions

Hodge ideals are a relevant invariant in the study of singularities, and which are seen to generalize the multiplier ideals. For instance, Hodge ideals can provide a first non-trivial invariant in the case of a log-canonical singularity (see Example 12). Also, as mentioned in Example 3, in the case of surfaces the Hodge ideals $I_k(D)$ with $k \geq 1$ are always able to detect singularities, which is not true for example with multiplier ideals. Even more, it is a conjecture of Popa ([Zha21, Question 5.4.5]) that for plane curve singularities the first Hodge ideals of $\mathbb{Q}$-divisors determine the analytical type of the curve.

In this work, we have implemented the algorithms described in [Zha21] to calculate Hodge ideals in the cases of quasihomogeneous and Newton non-degenerate singularities. The final code can be found in

`https://github.com/baezaguasch/HodgeIdeals`

where the main routine computes the negative of the roots of the Bernstein-Sato polynomial, shifts its value to the interval $(0, 1]$, and then calculates the Hodge ideals at the associated $\mathbb{Q}$-divisors for each value, up to the specified level.

The implementation has been tested to reproduce the results known in literature, such as [Zha21] and [Bla22]. Furthermore, they have proven to result in very efficient and short calculation times. We include in the next Table 7 the time elapsed for executing the examples presented (up to the respective levels in each case). We compare the code developed in this work with the code for the general algorithm developed in [Bla22], which allows the computation for any polynomial (not only quasihomogeneous or Newton non-degenerate).

Polynomial	This work	[Bla22]
$x^2 + y^3$	< 0.1 s	0.3 s
$x^2 + y^5$	< 0.1 s	2.6 s
$x^3 + y^3 + z^3 + xyz$	< 0.1 s	6 days
$(y^2 - x^3)(y^2 + \lambda x^3)$	0.6 s	7.8 h
$x^5 + y^5 + x^2y^2$	0.3 s	3 days
$x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$	3 days $(*)$	—

Table 7: Execution time comparison for the examples presented.

Altogether, we can see that with the general algorithm, it is not feasible to compute many examples as soon as we increase the number of variables or the degree or the number of monomials of the polynomial. In the contrary, computations are much faster with the algorithm for the specialized cases, reducing the execution time from days down to seconds.

Hence, the computation time is basically determined by the time required to find the Bernstein-Sato polynomial. That is precisely what is happening in the last case $(*)$, where computing the roots of the Bernstein-Sato polynomial takes up to a few days, while then computing the Hodge ideals is a matter of seconds.

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