Monodromy conjecture for Newton non-degenerate hypersurfaces

Author: Oriol Baeza Guasch (FME, UPC)

Supervisor: Guillem Blanco Fernández (KU Leuven)

Co-supervisor: Josep Àlvarez Montaner (FME, UPC)

Contents

Introduction

Open problem in the theory of singularities, formulated by Igusa in the 70s.

Monodromy conjecture, topological version

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be non-constant. If s_0 is a pole of $Z_{\mathrm{top}}(f, \varphi; s)$, then

- *(standard)* $e^{2\pi\imath\Re(s_0)}$ is a monodromy eigenvalue of $f\colon \mathbb{C}^n \to \mathbb{C}$ at some point of $\{f = 0\}$.
- *(strong)* s_0 is a root of the Bernstein-Sato polynomial b_f.

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- *(strong)* s_0 is a root of the Bernstein-Sato polynomial b_f.

Known cases:

- Plane curves (Loeser '88)
- Newton non-degenerate polynomials* (Loeser '90)
- Some types of hyperplanes arrangements (Budur-Saito-Yuzvinsky '10, Walther '17, Bapat-Walters '15)
- Semi-quasihomogeneous singularities (Budur-Blanco-van der Veer '21)

Introduction

*The case of Newton non-degenerate polynomials requires some additional hypothesis on the *residue numbers*.

Theorem 5.5.1 [Loe90]

Let f be a comfortable polynomial verifying $f(0) = 0$, with Newton diagram $\Gamma(f)$, and Newton non-degenerate. Suppose that all compact faces τ_0 verify

1. $\frac{k(\tau_0)}{N(\tau_0)} < 1$,
2. For every face τ of codimension 1 of $\Gamma(f)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, the real parts of the poles of the zeta function of f are roots of the Bernstein-Sato polynomial of f.

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Question

Can this condition be removed? What is the situation for the cases where it does not hold?

Contents

Preliminaries — Resolution of singularities

- $f : U \to \mathbb{C}$ a holomorphic function defined on an open set $U \subset \mathbb{C}^n$
- Hypersurface $X = f^{-1}(0)$

Definition (Singularity)

We define the set of singular points of X by the set

$$\operatorname{Sing}(X) := \left\{ x \in U \mid f(x) = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\}$$

Additionally, if $x \in \operatorname{Sing}(X)$ is the only singularity in a small enough neighborhood $V \ni x$ we will say it is *isolated*.

Preliminaries — Resolution of singularities

Definition (Resolution)

A *resolution* of X is a proper morphism $\pi\colon Y \to X$ where

1. Y is a smooth variety.
2. The restriction outside the singular locus $\pi\,|_{Y\setminus\pi^{-1}(\mathrm{Sing}(X))}\colon Y \setminus \pi^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}(X)$ is an isomorphism.

Additionally, we will say that the resolution is *good* if also

3. For every singular point $p \in \pi^{-1}(\mathrm{Sing}(X))$, there exists an open neighborhood $U_p \subset Y$, and open $V \subset \mathbb{K}^n$ with a chart

$$
\begin{aligned}
\mathrm{y}\colon U_p &\xrightarrow{\cong} V\\
p &\longmapsto 0
\end{aligned}
$$

such that $U \cap \pi^{-1}(\mathrm{Sing}(X)) = \{y_{i_1} = \cdots = y_{i_r} = 0\}$ for certain indices $0 < i_1 < \cdots < i_r \leq n$.

Preliminaries — Resolution of singularities

Definition (Embedded resolution)

Let X be a smooth algebraic variety, $f\colon X \to \mathbb{K}$ a polynomial and abbreviate $S = \mathrm{Sing}(f^{-1}(0))$ be the set of singular points on the zero set of f. An *embedded resolution* of f is a proper morphism $\pi\colon Y \to X$ where

1. Y is a smooth variety.
2. The restriction outside the singular locus $\pi\,|_{Y \setminus \pi^{-1}(S)}\colon Y \setminus \pi^{-1}(S) \to X \setminus S$ is an isomorphism.
3. For every singular point $p \in \pi^{-1}(S)$, there exists an open neighborhood $U_p \subset Y$, and an open $V \subset \mathbb{K}^n$ with a chart

$$\begin{aligned} \mathrm{y}\colon U_p &\xrightarrow{\cong} V \\ p &\longmapsto 0 \end{aligned}$$

over which $\pi^* f = u(y)\, y_{i_1}^{N_1} \cdots y_{i_r}^{N_r}$, with $u(0) \neq 0$ a unit, and $N_i \geq 0$ integers.

Guaranteed in characteristic zero, thanks to a result by Hironaka [Hir64].

Preliminaries — Resolution of singularities

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial, and $\pi\colon X \to \mathbb{C}$ an embedded resolution of f.

- $(E_i)_{i \in J}$ the irreducible components of $\pi^{-1}\left(f^{-1}(0)\right)$. Locally $E_i\colon \{x_i = 0\}$.
- By the local expression of $\pi^* f$, we know it vanishes with order N_j on a generic point of E_j, and we may write globally

$$\operatorname{div}\left(\pi^* f\right) = \sum_{j \in J} N_j E_j$$

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Geometric setup

In the resolved space, the intersection of exceptional divisors is, at worst, like intersection of coordinate hyperplanes.

- Normal crossings: divisors are smooth and intersect transversely.
- Simple normal crossings: addtionally, no three intersect at the same point.

Preliminaries — Resolution of singularities

- To each divisor $E_j, j \in J$ we have already associated a numerical quantity N_j representing the order of vanishing of a generic point of $\pi^* f$ in E_j.
- Similarly, we define the integers k_j such that the order of vanishing of a generic point of the pullback of the standard volume form in E_j is $k_j - 1$. In particular, we can write the divisor

$$\operatorname{div}\left(\pi^*(\,\mathrm{d}x_1 \wedge \cdots \wedge \,\mathrm{d}x_n)\right) = \sum_{j \in J} (k_j - 1) E_j$$

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The quantities (N_j, k_j) are the *numerical data* associated to each divisor E_j. From these, we may define

- *Candidate value* associated to the divisor E_j

$$\sigma_j = \frac{k_j}{N_j}$$

- *Residue number* associated to divisors E_i and E_j

$$\varepsilon(i,j) = -N_j \sigma_i + k_j = -N_j \frac{k_i}{N_i} + k_j$$

Preliminaries — Zeta function

Consider $f \in \mathbb{R}[x_1, \ldots, x_n]$ a non-constant polynomial and a test function $\varphi \colon \mathbb{R}^n \to \mathbb{C}$. We construct the associated archimedean *zeta function* as

$$Z(s) = Z(f, \varphi; s) := \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) \, \mathrm{d}x$$

- Converges and is holomorphic in $\Re(s) > 0$.
- Question by I. Gel'fand in ICM 1954: possible meromorphic continuation and distribution of the poles?

Preliminaries — Zeta function

Consider $\pi\colon Y \to X = \mathbb{R}^n$ an embedded resolution

$$Z(s) = \int_Y |\pi^* f(y)|^s |\mathrm{Jac}_\pi(y)| (\pi^*\varphi))\, \mathrm{d}y$$

We only need a finite number of charts U_p, and with a partition of unity $\{\rho_p\}_p$ subordinate to it, we can write $Z(s)$ as a finite sum

$$Z(s) = \sum_p \int_{U_p} \rho_p(y) |u(y)|^s |v(y)| (\pi^*\varphi)(y) \prod_{i \in J_p} |y_i|^{N_i s + k_i - 1}\, \mathrm{d}y_i$$

Theorem 2.1 [Vey24]

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a non-constant polynomial and $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ a $\mathcal{C}^\infty$ function with compact support. Then $Z(f, \varphi; s)$ has a meromorphic continuation to $\mathbb{C}$, and its poles of the form $-\frac{k_j+\nu}{N_j}$, with $j \in J$ and ν a non-negative integer.

Preliminaries — Zeta function

Definition (Topological zeta function)

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial and choose an embedded resolution $\pi\colon Y \to \mathbb{A}^n_{\mathbb{C}}$ of $\{f = 0\}$. The *(global) topological zeta function* of f is

$$Z_{\mathrm{top}}(f; s) \coloneqq \sum_{I \subset J} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{k_i + N_i s}$$

and the *local topological zeta function* of f at $a \in \{f = 0\}$ is

$$Z_{\mathrm{top},a}(f; s) \coloneqq \sum_{I \subset J} \chi\left(E_I^\circ \cap \pi^{-1}\{a\}\right) \prod_{i \in I} \frac{1}{k_i + N_i s}$$

where in both cases I runs through all possible subsets of J.

- Introduced heuristically through the 'limit $p \to 1$' of an expression of the p-adic zeta function, later formalized by Denef and Loeser.
- Independent of the chosen embedded resolution [DL92, Thm. 3.2].

Preliminaries — Bernstein-Sato polynomial

- Bernstein [Ber72] in the case of polynomials
- Kashiwara [Kas76] in the case of holomorphic functions
- Björk [Bjö73] in the case of formal power series

Theorem

Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $e_{f,P}(s) \in \mathbb{C}[s]$ such that the relation

$$P(s)f^{s+1} = e_{f,P}(s)f^s$$

holds formally in the $\mathscr{D}$-module $R_f[s] \cdot f^s$.

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Example

Let $f = x_1^2 + \cdots + x_n^2$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then, we have

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) f^{s+1} = 4(s+1)\left(s + \frac{n}{2}\right) f^s$$

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Theorem

Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $b_{f,P}(s) \in \mathbb{C}[s]$ such that the relation

$$P(s)f^{s+1} = e_{f,P}(s)f^s$$

holds formally in the $\mathscr{D}$-module $R_f[s] \cdot f^s$.

Definition (Bernstein-Sato polynomial)

The Bernstein-Sato polynomial $b_f(s)$ is the monic generator of the ideal in $\mathbb{C}[s]$ consisting of polynomials $e_{f,P}(s)$ satisfying such a functional equation.

First algorithm to compute it introduced by Oaku in [Oak97], using non-commutative Gröbner basis in the Weyl algebra.

Preliminaries — Bernstein-Sato polynomial

Applying the functional equation to integrate by parts.

$$b_f(s)Z(s) = \int_{\mathbb{R}^n} b_f(s) f(x)^s \varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} P(s)\cdot f(x)^{s+1}\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)^{s+1}\underbrace{P^*(s)\cdot\varphi(x)}_{\varphi_1(x)}\,\mathrm{d}x$$

where P^* is the adjoint operator of P. Thus, we obtain a meromorphic continuation to the half-plane $\Re(s) > -1$ as

$$Z(s) = \frac{1}{b_f(s)}\int_{\mathbb{R}^n} f(x)^{s+1}\varphi_1(x)\,\mathrm{d}x$$

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Iterating this procedure for $r > 1$

$$Z(s) = \frac{1}{b_f(s+r-1)\cdots b_f(s+1)b_f(s)}\int_{\mathbb{R}^n} f(x)^{s+r}\varphi_r(x)\,\mathrm{d}x$$

Theorem 2.3 [Vey24]

Let $f \in \mathbb{R}[x_1,\ldots,x_n]$ be a non-constant polynomial and $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ a $\mathcal{C}^\infty$ function with compact support. Then $Z(f,\varphi;s)$ has a meromorphic continuation to $\mathbb{C}$, and its poles are of the form $\lambda-\nu$ for λ a root of b_f and ν a non-negative integer.

Preliminaries — Bernstein-Sato polynomial

Roots of the Bernstein-Sato polynomial

- -1 is always a root. We write $\tilde{b}_f(s) = b_f(s)/(s+1)$ for the *reduced* BS polynomial.
- All roots are negative rational numbers [Mal75; Kas76].
- A set of candidate roots can be obtained from a resolution of singularities [Kol97; Lic89].

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Theorem

With the notations introduced for the resolution of singularities, we have that every root of the Bernstein-Sato polynomial b_f is of the form

$$-\frac{k_j+\nu}{N_j}, \qquad j \in J, \nu \in \mathbb{Z}_{\geq 0}$$

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Question

Does this relation between poles and roots appear also in the topological setting?

Preliminaries — Strong Monodromy Conjecture

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function on the origin. Fix $0 < \delta < \varepsilon$ and denote T disk of radius δ, $T' = T \setminus \{0\}$ and

$$X = B_\varepsilon \cap f^{-1}(T), \qquad X' = X \setminus f^{-1}(0), \qquad X_t = f^{-1}(t) \cap X,\ t \in T$$

Theorem [Mil16, §4]

For small enough δ, ε, the restriction $f'\colon X' \to T'$ is a locally trivial smooth fiber bundle, and the diffeomorphism type of any fiber X_t is independent of δ, ε and t.

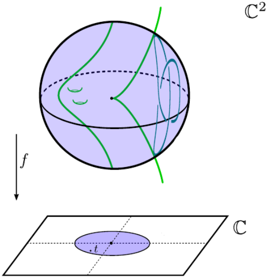

Figure: Adapted from [Viu21, Fig. 4].

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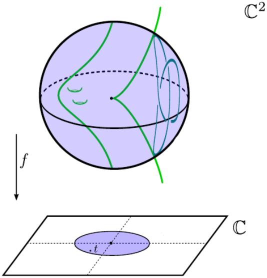

Figure: Adapted from [Viu21, Fig. 4].

For an isolated singularity

$$H_i(X_t, \mathbb{C}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^\mu, & i = n \\ 0, & \text{else} \end{cases}$$

Preliminaries — Strong Monodromy Conjecture

The action of $\pi_1(T', t) \cong \mathbb{Z}$ of the base of the fibration induces a diffeomorphism f on each fiber X_t, usually referred to as the *geometric monodromy*.

Also induces an automorphism on the homology and integer singular cohomology

$$h_*\colon H_*(X_t, \mathbb{C}) \longrightarrow H_*(X_t, \mathbb{C}), \qquad h^*\colon H^\bullet(X_t, \mathbb{C}) \longrightarrow H^\bullet(X_t, \mathbb{C})$$

These are usually referred as the *algebraic (complex) monodromy*

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Monodromy Theorem [RR06]

The operator h_* is quasi-unipotent, that is, there are p and q such that $(h_*^p - \mathrm{id})^q = 0$. In other words, the eigenvalues of the monodromy are roots of unity. Moreover, one can take $q = n + 1$.

Preliminaries — Strong Monodromy Conjecture

- Formulated by the Igusa in the late seventies, after some examples computed.
- Originally stated in terms of the p-adic zeta function, but has analogous statements in topological and motivic settings.

Monodromy conjecture, topological version

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Strong version implies the standard one

Proposition 7.1 [Mal83]

If $-\alpha$ is a root of $\tilde{b}_{f,0}(s)$, then $e^{2\pi\imath\alpha}$ is an eigenvalue of the monodromy of f at the origin.

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Remark

The strong version may even be formulated more precisely as stating that $b_f(s) \cdot Z_{\text{top}}(f, \varphi; s)$ is a polynomial.

Preliminaries — Periods of integrals

Fix $t = 1$ and choose a class of homology $\gamma(1) \in H_n(X_1, \mathbb{C})$, and from it we deduce a class $\gamma(t) \in H_n(X_t, \mathbb{C})$ in a neighborhood.
Consider $\pi \in \Omega^n$, with $\pi|_{X_t}$ holomorphic of maximal degree, thus closed. Define

$$I(t) := \int_{\gamma(t)} \pi$$

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$$I(t) := \int_{\gamma(t)} \pi \qquad\qquad I'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma(t)} \pi = \int_{\gamma(t)} \frac{\mathrm{d}\pi}{\mathrm{d}f}$$

- It is a holomorphic multivalued function on T'.
- If γ is a vanishing cycle (bounded and $\gamma(t) \xrightarrow[t \to 0]{} 0$), we have $\lim_{t \to 0} I(t) = 0$.
- Malgrange shows in [Mal74, p. 413] that we have a system of meromorphic differential equations satisfied by $I(t)$.

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Consider a vanishing cycle γ satisfying $(h - \lambda)^p \gamma(1) = 0$, with p minimal, then

$$I(t) = \sum_{\substack{\mu \in L(\lambda) \\ 0 \leq q \leq p-1}} c_{\mu,q}(\pi)\, t^{\mu} (\log t)^q$$

where $L(\lambda) = \{\mu > 0 \mid e^{2\pi\imath\mu} = \lambda\}$.

Next, consider $\omega \in \Omega^{n+1}$, and notice that there exists $\pi \in \Omega^n$ with $\mathrm{d}\pi = \omega$, on the account of Poincaré's lemma. Then, we have

$$I'(t) = \int_{\gamma(t)} \frac{\mathrm{d}\pi}{\mathrm{d}f} = \int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\substack{\mu \in L(\lambda)-1 \\ 0 \leq q \leq p-1}} d_{\mu,q}(\omega)\, t^{\mu} (\log t)^{q}$$

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Defining $\mu_k := \inf\{\mu \in L(\lambda) - 1 \mid \exists q \geq k-1, \omega \in \Omega^{n+1} \text{ with } d_{\mu,q}(\omega) \neq 0\}$.

Proposition 3.3 [Mal73]

The polynomial $(s + \mu_1) \ldots (s + \mu_p)$ divides $\tilde{b}$.

Preliminaries — Periods of integrals

Next, consider $\omega \in \Omega^{n+1}$, and notice that there exists $\pi \in \Omega^n$ with $\mathrm{d}\pi = \omega$, on the account of Poincaré's lemma. Then, we have

$$I'(t) = \int_{\gamma(t)} \frac{\mathrm{d}\pi}{\mathrm{d}f} = \int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\substack{\mu \in L(\lambda)-1 \\ 0 \leq q \leq p-1}} d_{\mu,q}(\omega)\, t^{\mu} (\log t)^q$$

Defining $\mu_k := \inf\{\mu \in L(\lambda) - 1 \mid \exists q \geq k-1, \omega \in \Omega^{n+1} \text{ with } d_{\mu,q}(\omega) \neq 0\}$.

Proposition 3.3 [Mal73]

The polynomial $(s + \mu_1) \ldots (s + \mu_p)$ divides $\tilde{b}$.

Conclusion

To apply this argument it only remains to show that there actually exists some non-zero homology class on $H_n(X_t, \mathbb{C})$. Or, equivalently, that a certain multivalued differential form defines a non-zero cohomology class.
Given by Loeser [Loe90, Thm. 3.7] for NND (toroidal version of [EV92]), and by Deligne and Mostow for plane curves [DM86, Prop. 2.14].

Contents

Plane curves — Definition

We think of a plane curve as a set of points with coordinates (x, y) in the complex plane, described as the zero locus of some equation $f(x, y) = 0$, with f a holomorphic function.

Remark

The need to work on a bigger space than that of polynomials arises naturally.

$$f(x,y) = x^2 - y^3 = 0 \quad \Longrightarrow \quad y = x^{2/3}$$

$$f(x,y) = x + y - xy \quad \Longrightarrow \quad y = -\sum_{r \geq 1} x^r$$

Nonetheless, thanks to Weierstrass preparation theorem, we may (and will) consider f to be a polynomial.

Each curve $C\colon f = 0$ can be decomposed in a unique way as a finite union of branches $B_j\colon g_j = 0$, arising from the factorization

$$f = \prod g_j^{a_j}, \qquad C = \sum a_j B_j$$

Plane curves — Resolution of singularities

Resolution of singularities for plane curves can be described as a composition of blowups.

- Only need a finite number of blowups to obtain smooth strict transform.
- Can be extended so that $\pi^{-1}(C)$ has simple normal crossings (all curves are smooth, with transverse intersection and no three meeting at a point).

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Resolution of singularities for plane curves can be described as a composition of blowups.

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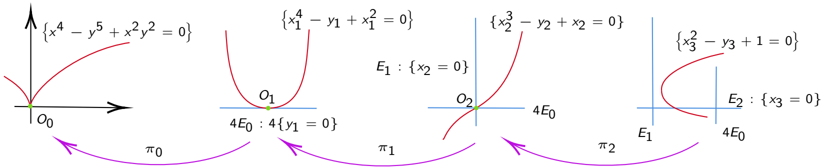
For example, a resolution of one branch of the curve $f = x^4 - y^5 + x^2y^2$ is given by

$$(x, y) \mapsto (x_1y_1, y_1), \qquad (x_1, y_1) \mapsto (x_2, x_2y_2), \qquad (x_2, y_2) \mapsto (x_3, x_3y_3)$$
$$\implies \quad \pi^* f = x_3^{10} y_3^4 (x_3^2 - y_3 + 1)$$



Definition (Dual graph)

Denote the *dual graph* of the curve C associated to the resolution π as $\Delta_\pi(C)$.

- Vertices V_i will correspond to the exceptional divisors E_i.
- Edges $V_i V_k$ whenever the curves E_i and E_k intersect.

Additionally, we will add arrowhead vertices W_j for each strict transform $B_j^{(N)}$, and call it *augmented dual graph* $\Delta_\pi^+(C)$.

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Constructing one blowup at a time, we can see that it is a tree.

1. If O_i is proximate only to O_{i-1}, add a new vertex V_i and a new edge $V_{i-1}V_i$.
2. If O_i is proximate to both O_{i-1} and some O_j with $j < i-1$, subdivide the existing edge $V_{i-1}V_j$ by adding a new vertex V_i.

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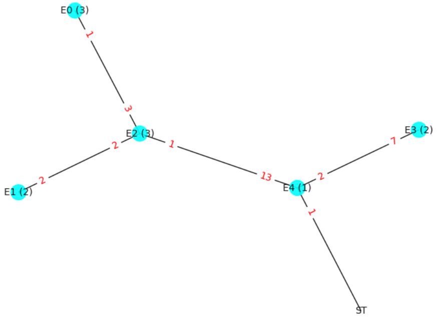
Remark

A sequence of proximity relations (or any equivalent information) about the curve determines the dual graph. The reverse is possible if we additionally label vertices.

Observation

If a branch has Puiseux characteristic exponents $(n; \beta_1, \dots, \beta_g)$, then Δ_π^+ consists of a single core chain of edges from the initial vertex to the arrowhead vertex, with g side edge branches.

For example, consider the polynomial $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7$, with Puiseux characteristic exponents $(4; 6, 7)$.



$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

$$P^t \cdot P = \begin{pmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

Plane curves — Topological zeta function

Proposition

The topological zeta function is independent of a choice of resolution.

For plane curves, its expression may be simplified to

$$Z_{\mathrm{top},0}(f;s) = \sum_{j\in J} \frac{\chi\left(E_j^\circ\right)}{k_j + N_j s} + \sum_{i\neq j\in J} \frac{\chi\left(E_i \cap E_j\right)}{(k_i + N_i s)(k_j + N_j s)}$$

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$$Z_{\mathrm{top},0}(f;s) = \sum_{j \in V(\Delta_\pi^+)} \frac{2 - v_i}{k_j + N_j s} + \sum_{(i,j) \in E(\Delta_\pi^+)} \frac{1}{(k_i + N_i s)(k_j + N_j s)}$$

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To study the final poles of the zeta function, we must study the quotient k/N over the divisors and the residue numbers $\varepsilon(i,j) = k_j - k_i \frac{N_j}{N_i}$

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Proposition II.3.1 [Loe90]

Let V_i be a vertex of the dual graph. Then, $-1 \leq \varepsilon(i,j) < 1$ for all $j \in \mathcal{N}(i)$. Moreover, equality holds if, and only if, it has valence $v_i = 1$.

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The vertices where the minimum of k/N can be attained form a connected chain, which is relevant to show

Theorem

The topological zeta function has at most one double pole, and the candidate value is $\min\{k_i/N_i\}$.

Plane curves — Topological zeta function

As for simple poles, we fix an exceptional divisor E_i and with partial fraction decomposition find its contribution to the zeta function as

$$\frac{1}{k_i + sN_i}\left(2 - v_i + \sum_{j\in\mathcal{N}(i)} \frac{1}{\varepsilon(i,j)}\right)$$

By the previous lemma and the recurrences over the dual graph, this quantity turns out to be zero for vertices V_i with valence $v_i = 1$ or 2.

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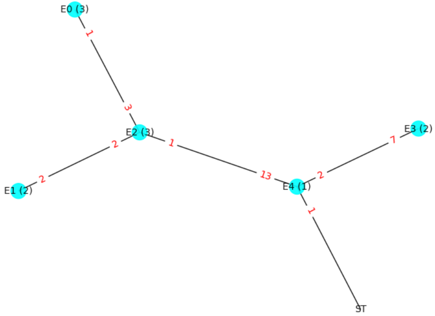
Theorem 8.4.5 [Wal04]

Any pole of the topological zeta function is of the form $s = -k_i/N_i$ for some vertex V_i which is either an arrowhead vertex or a rupture point.

Plane curves — Topological zeta function

Consider again the polynomial $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = 0$, with numerical data



Divisor	k	N	σ	v_i
E_0	2	4	$\frac{1}{2}$	1
E_1	3	6	$\frac{1}{2}$	1
E_2	5	12	$\frac{5}{12}$	3
E_3	6	13	$\frac{6}{13}$	1
E_4	11	26	$\frac{11}{26}$	3
ST	1	1	1	1

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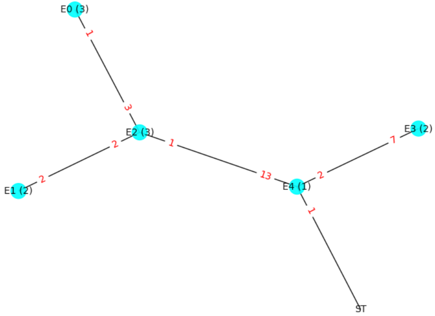

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$$E2\colon \frac{1}{5+12s}\left(2-3+\frac{1}{\varepsilon(2,0)}+\frac{1}{\varepsilon(2,1)}+\frac{1}{\varepsilon(2,4)}\right)=\frac{1}{5+12s}(-1+3+2+6)=\frac{5/6}{s+5/12}$$

$$E4\colon \frac{1}{11+26s}\left(2-3+\frac{1}{\varepsilon(4,2)}+\frac{1}{\varepsilon(4,3)}+\frac{1}{\varepsilon(4,ST)}\right)=\frac{1}{11+26s}\left(-1-13+2+\frac{26}{15}\right)=-\frac{77/195}{s+11/26}$$

$$ST\colon \frac{1}{1+s}\left(2-1+\frac{1}{\varepsilon(ST,4)}\right)=\frac{1}{1+s}\left(1-\frac{1}{15}\right)=\frac{14/15}{s+1}$$

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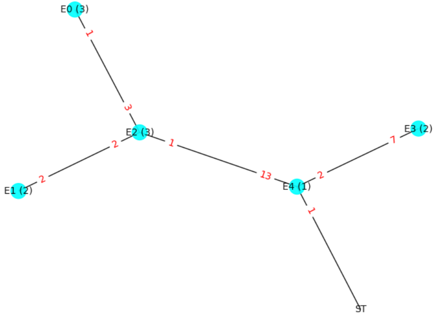

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Altogether

$$Z_0(s) = \frac{5/6}{s + 5/12} - \frac{77/195}{s + 11/26} + \frac{14/15}{s + 1}$$

Plane curves — Sketch of the proof

Proposition III.3.2 [Loe88]

If E_i is a rupture divisor, there exists a multivalued horizontal family $\gamma(t)$ of cycles in $H_1(X_t, \mathbb{C})$, $t \neq 0$, such that

$$\int_{\gamma(t)} \frac{\mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}f} = Ct^{-1+\frac{k_i}{N_i}} + o\left(t^{-1+\frac{k_i}{N_i}}\right)$$

with $C \neq 0$ a constant.

Theorem III.3.1 [Loe88]

Let $f\colon (\mathbb{C}^2, 0), \to (\mathbb{C}, 0)$ be the germ of an analytic function, and E_i a rupture divisor of the resolution of $\{f = 0\}$. Then

1. $-k_i/N_i$ is a root of $\tilde{b}_{f,0}$ the local reduced Bernstein-Sato polynomial of f.
2. If there exists $j \in \mathcal{N}(i)$ (a neighbor of V_i in the dual graph) such that $\varepsilon(i,j) = 0$, then $-k_i/N_i$ is a root of multiplicity two of $\tilde{b}_{f,0}$.

Contents

Newton non-degenerate — Definition

We consider a polynomial $f(x_1, \ldots, x_n) = \sum_{p \in \mathbb{N}^n} a_p \, x_1^p \ldots x_n^p$ such that $f(0) = 0$, and define the support $\mathrm{supp}(f) = \{p \in \mathbb{N}^n \mid a_p \neq 0\}$.

Definition (Newton polyhedron)

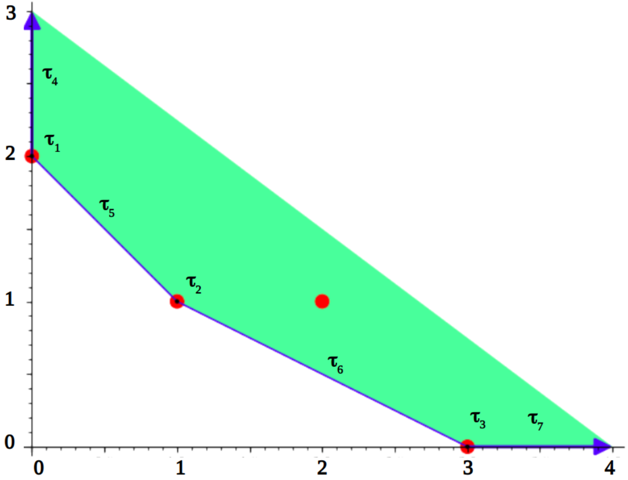
Let $f = \sum_{p \in \mathbb{N}^n} a_p x^p \in \mathbb{C}[x]$ with $f(0) = 0$. We define the *global Newton polyhedron* $\Gamma_{gl}(f)$ of f as the convex hull of $\mathrm{supp}(f)$. Also, we define the *local Newton polyhedron* $\Gamma(f)$ as the convex hull of the set

$$\bigcup_{p \in \mathrm{supp}(f)} p + (\mathbb{R}_{\geq 0})^n$$

In particular, it is immediate that $\Gamma(f) = \Gamma_{gl}(f) + (\mathbb{R}_{\geq 0})^n$.

Newton non-degenerate — Definition

Consider the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$.



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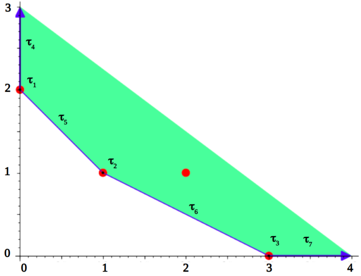

- Faces of dimension $\dim \tau = 0$:

$$\tau_1 = \{(0,2)\} \qquad f^{\tau_1} = -y^2$$
$$\tau_2 = \{(1,1)\} \qquad f^{\tau_2} = 4xy$$
$$\tau_3 = \{(3,0)\} \qquad f^{\tau_3} = x^3$$

- Faces of dimension $\dim \tau = 1$:

$$\tau_4 = \{(0,2) + \mathbb{R}_{>0}(0,1)\} \qquad f^{\tau_4} = -y^2$$
$$\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} \qquad f^{\tau_5} = -y^2 + 4xy$$
$$\tau_6 = \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} \qquad f^{\tau_6} = x^3 + 4xy$$
$$\tau_7 = \{(3,0) + \mathbb{R}_{>0}(1,0)\} \qquad f^{\tau_7} = x^3$$

Newton non-degenerate — Definition

Definition (Newton non-degenerate)

We say that f is Newton non-degenerate at 0 if for any face $\tau \subset \Gamma(f)$, the hypersurface $f^\tau = 0$ satisfies the condition

$$x_1 \frac{\partial f^\tau}{\partial x_1} = \cdots = x_n \frac{\partial f^\tau}{\partial x_n} = 0 \quad \Longrightarrow \quad x_1 \ldots x_n = 0$$

that is, the polynomials $x_i \frac{\partial f^\tau}{\partial x_i}$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

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that is, the polynomials $x_i \frac{\partial f^\tau}{\partial x_i}$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

For example, $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = (y^2 - x^3)^2 - 4yx^5 - x^7$ is degenerate: the truncation on the face (edge) τ with endpoints $(0,4)$ and $(6,0)$ is

$$f^\tau = (y^2 - x^3)^2 \quad \Longrightarrow \quad \begin{cases} x \cdot 2(y^2 - x^3) \cdot 3x^2 = 0 \\ y \cdot 2(y^2 - x^3) \cdot 2y = 0 \end{cases}$$

and the system considered has solutions outside $(\mathbb{C} \setminus 0)^2$.

Newton non-degenerate — Definition

Definition (N,k)

Let $\Gamma(f)$ be the Newton diagram of f as defined. For $a \in (\mathbb{R}^+)^n$, we define

$$N(a) := \inf_{x \in \Gamma(f)} \{\langle a, x\rangle\} = \min_{\substack{x \text{ vertex} \\ \text{of } \Gamma(f)}} \{\langle a, x\rangle\}$$

We may recover the face by considering $F(a) := \{x \in \Gamma(f) \mid \langle a, x\rangle = N(a)\}$ the first meet locus. Lastly, denote $k(a) := \sum_{i=1}^{n} a_i$.

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Definition (Dual cone)

For τ a face of $\Gamma(f)$, we define the *cone associated* to τ as

$$\Delta_\tau := \{a \in (\mathbb{R}_{>0})^n \mid F(a) = \tau\} / \sim$$

where the equivalence relation is given by $a \sim a' \iff F(a) = F(a')$.

Additionally, we will refer to the collection of these cones for all faces of the Newton polytope as the *dual fan*.

Newton non-degenerate — Definition

Notice that for every proper face τ we have

$$\tau = \bigcap_{\substack{\tau \subset \gamma \\ \dim \gamma = n-1}} \gamma$$

Moreover, for every face of $\Gamma(f)$ of codimension 1, there exists a unique integral primitive vector (meaning that all of its coordinates are relatively coprime) perpendicular to the face.

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Lemma

Let τ be a proper face of $\Gamma(f)$ and $\gamma_1, \ldots, \gamma_r$ the faces of $\Gamma(f)$ of dimension $n-1$ that contain it. Let $a_1, \ldots, a_r$ be the unique primitive normal vectors to $\gamma_1, \ldots, \gamma_r$, respectively. Then,

$$\Delta_\tau = \{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_{>0}\}$$

with $\dim \Delta_\tau = n - \dim \tau$.

Newton non-degenerate — Definition

Consider again $f = x^3 - y^2 + 4xy + 3x^2y$.

- $\tau_4 = \{(0,2) + \mathbb{R}_{>0}(0,1)\} \implies a_4 = (1,0)$
- $\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} \implies a_5 = (1,1)$
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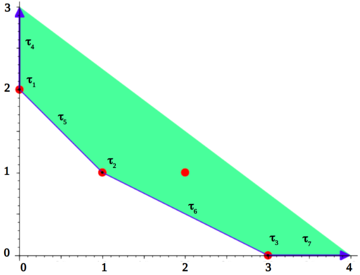
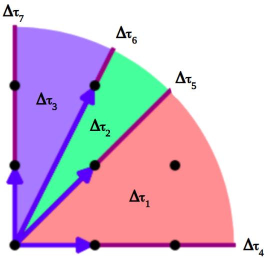
Then, the cones associated to each face

$$\Delta_{\tau_1} = \mathbb{R}_{>0}(1,0) + \mathbb{R}_{>0}(1,1), \qquad \Delta_{\tau_4} = \mathbb{R}_{>0}(1,0), \qquad \dots$$

Altogether, the associated dual fan is (right)





Definition (Cone)

A *convex polyhedral cone* is a set

$$C = \{\lambda_1 v_1 + \cdots + \lambda_s v_r \in V \mid \lambda_i \geq 0\} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$$

where V is an n-dimensional vector space over $\mathbb{R}$, and the vectors $\{v_i\}$ are called the *generators* of the cone.

We will say that a cone is...

- *simplicial* if its generating vectors $v_1, \ldots, v_r$ are linearly independent over $\mathbb{R}$.
- *regular* if $\{v_1, \ldots, v_r\}$ is a subset of a base of the $\mathbb{Z}$-module $\mathbb{Z}^n$.

Newton non-degenerate — Resolution of singularities

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Theorem

Each cone admits a partition in simplicial subcones. Moreover, they can be made regular cones by introducing suitable new generating rays.

Newton non-degenerate — Resolution of singularities

Definition (Cone)

A *convex polyhedral cone* is a set

$$C = \{\lambda_1 v_1 + \cdots + \lambda_s v_r \in V \mid \lambda_i \geq 0\} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$$

where V is an n-dimensional vector space over $\mathbb{R}$, and the vectors $\{v_i\}$ are called the *generators* of the cone.

We will say that a cone is...

- *simplicial* if its generating vectors $v_1, \ldots, v_r$ are linearly independent over $\mathbb{R}$.
- *regular* if $\{v_1, \ldots, v_r\}$ is a subset of a base of the $\mathbb{Z}$-module $\mathbb{Z}^n$.

Theorem

Each cone admits a partition in simplicial subcones. Moreover, they can be made regular cones by introducing suitable new generating rays.

Lemma 8.7 [AVG12]

There exists a regular fan subordinate to a Newton polyhedron.

Newton non-degenerate — Resolution of singularities

Definition (Toric blowup)

Consider a unimodular integral $n \times n$ matrix σ

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n} \end{pmatrix}$$

We define the *toric blowup* associated to σ as the birational morphism

$$\begin{aligned} \pi_\sigma \colon (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^n \\ (x_1, \dots, x_n) &\mapsto (x_1^{\sigma_{1,1}} x_2^{\sigma_{1,2}} \cdots x_n^{\sigma_{1,n}}, \dots, x_1^{\sigma_{n,1}} x_2^{\sigma_{n,2}} \cdots x_n^{\sigma_{n,n}}) \end{aligned}$$

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For a regular simplicial cone of maximum dimension in the subdivided dual fan $\Sigma^*(f)$ given by vectors $\{r_1, \ldots, r_n\}$, we can consider the matrix $\sigma = (r_1 \ \cdots \ r_n)$.

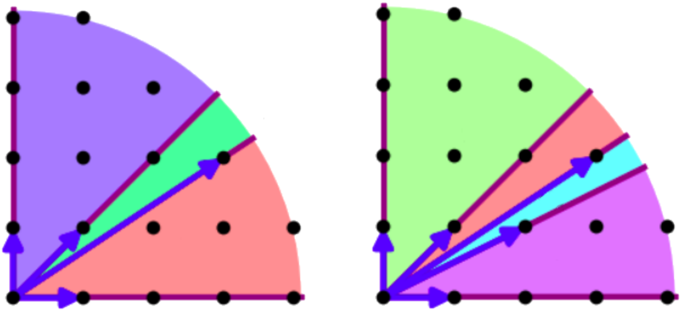
Gluing adequately these charts π_σ, we obtain a non-singular variety X, and a proper analytic map $\pi \colon X \to \mathbb{C}^n$: the toric blowup associated with $\Sigma^*(f)$.

Newton non-degenerate — Resolution of singularities

Theorem [Oka96, p. 101]

If f is Newton non-degenerate, then the associated toric blowup $\pi\colon X \to \mathbb{C}^n$ is a good resolution of the f as a germ at the origin.

Consider again $f = x^4 - y^5 + x^2y^2$.



Newton non-degenerate — Resolution of singularities

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Consider again $f = x^4 - y^5 + x^2y^2$.

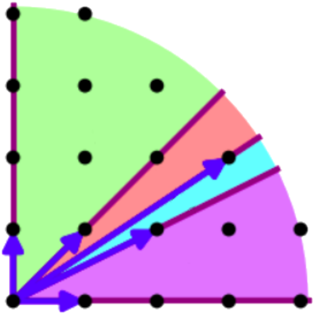
The rays of the regular subdivision are

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

taking one of the charts, we perform the toric blowup as

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \implies (x, y) \mapsto (z^3 w, z^2 w)$$

$$\pi^* f = z^{10} w^4 (z^2 - w + 1)$$



Newton non-degenerate — Topological zeta function

Let τ be a face in $\Gamma(f)$, and consider a decomposition of the associated cone $\Delta_\tau = \cup_{i=1}^{r} \Delta_i$ in simplicial cones of dimension $\dim \Delta_\tau = l$ such that $\dim(\Delta_i \cap \Delta_j) < l$, for all $i \neq j$. Then, define

$$J(\tau, s) := \sum_{i=1}^{r} J_{\Delta_i}(s), \qquad \text{with} \quad J_{\Delta_i}(s) = \frac{\operatorname{mult}(\Delta_i)}{(N(a_{i_1})s + k(a_{i_1}))\cdots(N(a_{i_l})s + k(a_{i_l}))}$$

being $a_{i_1}, \ldots, a_{i_l} \in \mathbb{N}^n$ the linearly independent primitive integral vectors that generate Δ_i. Lastly, if $\tau = \Gamma(f)$, we rather take $J(\tau, s) = 1$.

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being $a_{i_1}, \ldots, a_{i_l} \in \mathbb{N}^n$ the linearly independent primitive integral vectors that generate Δ_i. Lastly, if $\tau = \Gamma(f)$, we rather take $J(\tau, s) = 1$.

Theorem 5.3 [DL92]

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function Newton non-degenerate for $\Gamma_0(f)$, then

$$Z_0(s) = \sum_{\substack{\tau \text{ vertex} \\ \text{of } \Gamma(f)}} J(\tau, s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma(f) \\ \dim \tau \geq 1}} (-1)^{\dim \tau} (\dim \tau)! \operatorname{Vol}(\tau) J(\tau, s)$$

Newton non-degenerate — Sketch of the proof

We define the toric residue numbers, introduced by Loeser.

Definition (Toric residue numbers)

If τ, τ' are two distinct faces of codimension 1 of $\Gamma(f)$, we denote by $\beta(\tau, \tau')$ the greatest common divisor of the minors of order 2 of the matrix $\big(a(\tau), a(\tau')\big)$. Additionally, one defines

$$\lambda(\tau, \tau') = k(\tau') - \frac{k(\tau)}{N(\tau)} N(\tau'), \qquad \varepsilon(\tau, \tau') = \lambda(\tau, \tau')/\beta(\tau, \tau')$$

whenever $N(\tau) \neq 0$, which is the case if τ is a compact face.

- We are working with the original (not regularly subdivided) dual fan.
- The resulting variety might be singular.
- We lose some symmetry properties, but we gain 'uniqueness'.

Newton non-degenerate — Sketch of the proof

Theorem 4.2 [Loe90]

Let f be a germ of an analytic function, Newton non-degenerate at the origin. Let τ_0 be a compact face of codimension 1 of $\Gamma(f)$. Suppose that the following two conditions are verified

1. $\dfrac{k(\tau_0)}{N(\tau_0)} \notin \mathbb{Z}$,
2. For every face τ of codimension 1 of $\Gamma(f, 0)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, there exists a horizontal multiform section $\gamma(t)$ of the fibration H_n over T' such that

$$\lim_{t \to 0} t^{1-\frac{k(\tau_0)}{N(\tau_0)}} \int_{\gamma(t)} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{\mathrm{d}f} = C$$

with C a non-zero constant.

The proof consists of constructing a non-zero multivalued differential form ω, which requires that the monodromies are not the identity, hence the hypothesis.

Newton non-degenerate — Sketch of the proof

Theorem 5.5.1 [Loe90]

Let f be a comfortable polynomial verifying $f(0) = 0$, with Newton diagram $\Gamma(f)$, and Newton non-degenerate. Suppose that all compact faces τ_0 verify

i) $\frac{k(\tau_0)}{N(\tau_0)} < 1$,
ii) For every face τ of codimension 1 of $\Gamma(f)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, the real parts of the poles of the zeta function of f are roots of the Bernstein-Sato polynomial of f.

Remark 5.5.2.1 [Loe90]

If one replaces the condition $\frac{k(\tau_0)}{N(\tau_0)} < 1$ with $\frac{k(\tau_0)}{N(\tau_0)} \notin \mathbb{N}$, this is enough to prove the weak version of the conjecture.

Newton non-degenerate — Sketch of the proof

Question

Can the second hypothesis be relaxed?

Newton non-degenerate — Sketch of the proof

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Can the second hypothesis be relaxed?

It can be expected that non-positive residue numbers could be allowed to happen.

The idea for the negative integers is that we can argue the existence of a non-zero cohomology class, by calculating the degree of certain line bundles. For example, in the case of plane curves, a result in this spirit

Proposition 11.1 [Bla20]

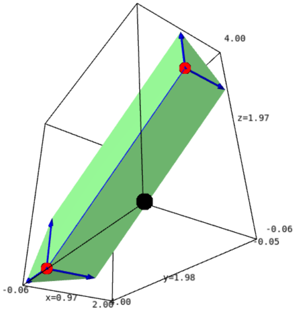
Let $\omega \in \Gamma(\mathbb{P}, \Omega^1(\sum \mu_s s - \sum \delta_x x)(L))$. Assume that $\sum_{s \in S} \mu_s \leq r-1$ and that $\alpha_s \neq 1$ for all $s \in S$. Then, ω defines a non-zero cohomology class in $H^1(\mathbb{P} \setminus S, L)$.

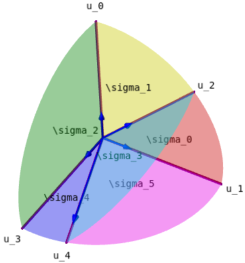
Contents

Example 1

Consider the polynomial $f = xz^3 + y^3 \in \mathbb{C}[x, y, z]$.

- Found via brute force search
- Positive integer residue numbers already with low degree polynomial
- SMConjecture holds





Example 1

Associated residue numbers to divisor...

- (0, 1, 1): {ST: -2/3 + 1, (1, 1, 0): 4/3, (0, 1, 0): 1, (2, 1, 0): 5/3, (3, 1, 0): 2, (0, 0, 1): 1, (1, 1, 1): 1}

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Observation

Now, we would like that the bad divisors with positive integer residue numbers didn't appear as poles in the zeta function.

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Local topological zeta function has poles $\{-1, -2/3\}$.

$$Z_0(s) = \frac{4/3}{s + 2/3} - \frac{1}{s+1}$$

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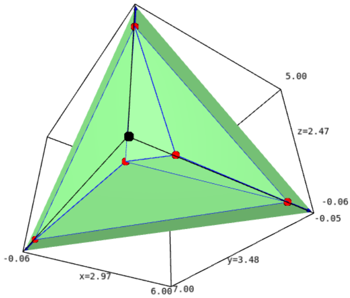
Associated toric residue numbers to divisor...

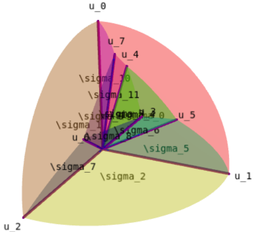
- (0, 1, 1): {ST: -2/3 + 1, (1, 0, 0): 1, (3, 1, 0): 2, (0, 0, 1): 1, (0, 1, 0): 1}

Example 2

Consider the convenient polynomial $f = x^5 + y^6 + z^4 + x^2yz + xy^2z \in \mathbb{C}[x, y, z]$.

- Built with geometric intuition.
- Two different integer residue numbers for a same divisor.
- SMConjecture holds.
- Toric residue numbers do satisfy the hypothesis.





Example 2

Regular subdivision requires over 60 new rays.

Observation

Computations for a regular subdivision rapidly increase in complexity, as the number of added rays increases largely.

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Associated residue numbers to divisor...

- (4, 7, 5): {ST: -4/5 + 1, ...}
- (1, 1, 2): {ST: -4/5 + 1, ...}
- (6, 5, 14): {ST: -5/6 + 1, ...}
- (5, 2, 3): {ST: -5/6 + 1, ...}
- (1, 1, 1): {ST: -3/4 + 1, ..., (8, 5, 10): 2, (6, 5, 12): 2, ...}

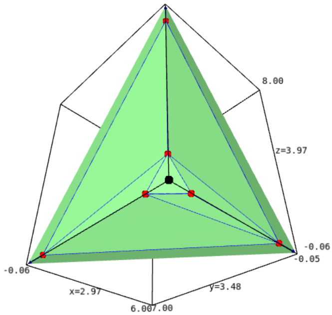
Local topological zeta function has poles $\{-3/4, -4/5, -5/6, -1\}$.

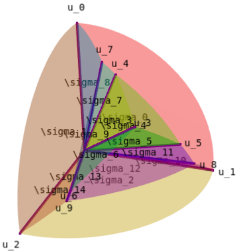
$$Z_0(s) = \frac{9}{s+3/4} - \frac{48/5}{s+4/5} - \frac{35/6}{s+5/6} + \frac{8}{s+1}$$

Example 3

Consider the convenient polynomial $f = x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$.

- Built with geometric intuition.
- Integer residue numbers and integer toric residue numbers.
- Bad divisors in each case are different, and arise from compact faces.
- SMConjecture holds
- Contributions of bad divisors to the zeta function are non-zero.





Example 3

Regular subdivision requires almost 400 new rays.

Associated residue numbers to divisor...

- (1, 1, 1): {ST: -3/4 + 1, (16, 6, 5): 3, (17, 11, 6): 4, (9, 7, 4): 2, (21, 15, 8): 5, (13, 15, 6): 4, (11, 5, 4): 2, (21, 7, 6): 4, (4, 10, 3): 2, (19, 17, 8): 5, (6, 5, 12): 2, ...}

Local topological zeta function has poles $\{-3/4, -4/5, -5/6, -6/7, -1\}$.

$$Z_0(s) = \frac{81/4}{s+3/4} - \frac{72/5}{s+4/5} - \frac{70/6}{s+5/6} - \frac{48/7}{s+6/7} + \frac{14}{s+1}$$

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- (1, 2, 1): {ST: -4/5 + 1, ...}
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Question

Is it possible that the pole arises from the good divisor? That is, the contribution of the bad divisor to $Z_0(s)$ has residue 0 at $s = -4/5$.

Example 3

Recall the expression for the local topological zeta function for NND polynomials

$$Z_0(s) = \sum_{\substack{\tau \text{ vertex} \\ \text{of } \Gamma(f)}} J(\tau, s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma(f) \\ \dim \tau \geq 1}} (-1)^{\dim \tau} (\dim \tau)! \mathrm{Vol}(\tau) J(\tau, s)$$

where

$$J(\tau, s) := \sum_{i=1}^{r} J_{\Delta_i}(s), \quad \text{with} \quad J_{\Delta_i}(s) = \frac{\mathrm{mult}(\Delta_i)}{(N(a_{i_1})s + k(a_{i_1})) \cdots (N(a_{i_l})s + k(a_{i_l}))}$$

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To extract the contribution of a single divisor $a = a(\tau)$, add up only the terms from cones Δ_i that contain a as one of its generating rays. Notice that this is not the same as simply taking the terms where a fraction of $\frac{1}{N(a)s+k(a)}$ appears.

Example 3

- Divisor $a_1 = (1,1,2)$, with $(k(a_1), N(a_1)) = (4,5)$.
- For $a_2 = (1,2,1)$ also $\sigma(a_2) = 4/5$, since $(k(a_2), N(a_2)) = (4,5)$.

We compute the residues of the contributions at the corresponding point

$$\operatorname*{Res}_{s=-4/5} Z_{0;(1,2,1)}(s) = -\frac{47}{5}, \qquad \operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2)}(s) = -\frac{47}{5}$$

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Remark

One should be careful not to conclude that the total residue is simply the sum of the residues given by all divisors with the given candidate value.

Rays $(1,1,2)$ and $(1,2,1)$ appear in a same cone: we are double counting a term!

$$\begin{aligned}\operatorname*{Res}_{s=-4/5} Z_0(s) = \frac{-72}{5} &= \frac{-47}{5} + \frac{-47}{5} - \frac{-22}{5} \\ &= \operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2)}(s) + \operatorname*{Res}_{s=-4/5} Z_{0;(1,2,1)}(s) - \operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2),(1,2,1)}(s)\end{aligned}$$

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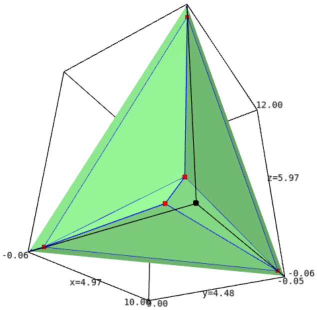
Observation

In a more general situation, where there are more divisors with the same candidate σ, an inclusion-exclusion expression should be used in order to equate the residues.

Example 4

Consider the convenient polynomial $f = x^9 + y^8 + z^{11} + xyz^2 + xy^2z$.

- Built with geometric intuition (same as Example 3).
- Integer residue numbers and integer toric residue numbers.
- Bad divisors in each case are different, and arise from compact faces.
- SMConjecture holds
- Contributions of bad divisors to the zeta function are non-zero and different.



Example 4

Associated toric residue numbers to divisor...

- (8, 9, 46): {ST: -7/8 + 1, (61, 11, 8): 3, ...}
- (5, 1, 1): {ST: -7/8 + 1, ...}

Local topological zeta function has poles $\{-1, -7/8, -10/11, -19/27\}$.

$$Z_0(s) = \frac{18}{s+1} - \frac{63/4}{s+7/8} - \frac{120/11}{s+10/11} + \frac{247/27}{s+19/27}$$

Now, we compute the individual contributions

$$\begin{aligned}\operatorname*{Res}_{s=-7/8} Z_0(s) &= \frac{-63}{4} = \frac{-535}{36} + \frac{-31}{4} - \frac{-247}{36} \\ &= \operatorname*{Res}_{s=-7/8} Z_{0;(5,1,1)}(s) + \operatorname*{Res}_{s=-7/8} Z_{0;(8,9,46)}(s) - \operatorname*{Res}_{s=-7/8} Z_{0;(5,1,1),(8,9,46)}(s)\end{aligned}$$

Monodromy conjecture for
Newton non-degenerate hypersurfaces

https://github.com/baezaguasch/MonodromyNND
oriol.baeza (at) estudiantat.upc.edu

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