Monodromy conjecture for Newton non-degenerate hypersurfaces

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Final bachelor's project

Bachelor's Degree in Mathematics

Bachelor's Degree in Aerospace Technology Engineering

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July 2024

Abstract

In this work, we study the Strong Monodromy Conjecture (SMC) in its topological setting, and the state of the art for some known cases of the conjecture. After introducing the concepts of resolution of singularities, Bernstein-Sato polynomial, topological zeta function, we study the cases of plane curves, and then that of Newton non-degenerate (NND) singularities. The proof of the SMC for both cases relies on the study of the asymptotics of certain periods of integrals and some technical cohomological results. However, the result for plane curves is simplified via the study of several invariants, while NND polynomials allow for a more combinatorial approach through the dual fan associated to its Newton polygon. We outline the proofs, and discuss the additional hypothesis on the residue numbers required for the NND case, essentially demanding that the residue numbers are not integers. Nonetheless, we compute some examples revealing that such integer values can appear and, furthermore, that the associated divisors can contribute to poles to the topological zeta function. This indicates that the current approach for NND singularities can't be generalised, and that a different approach is necessary to attack the general case.

Key words: Mondromy, Bernstein-Sato polynomial, resolution of singularities, plane curves, Newton non-degenerate.

MSC: 14B05, 14H20, 14J17, 32S40, 34M35

Resum

En aquest treball, estudiem la Strong Monodromy Conjecture (SMC) en el seu marc topològic i l'estat de l'art per a alguns casos coneguts de la conjectura. Després d'introduir els conceptes de resolució de singularitats, polinomi de Bernstein-Sato, funció zeta topològica, estudiem els casos de corbes planes, i després el de les singularitats Newton no degenerades (NND). La prova de l'SMC per a tots dos casos es basa en l'estudi de l'estudi asimptòtic de determinats períodes d'integrals i en alguns resultats tècnics cohomològics. Tanmateix, el resultat per corbes planes se simplifica mitjançant l'estudi de diverses invariants, mentre que els polinomis NND permeten un enfocament més combinatori mitjançant el ventall dual associat al seu polígon de Newton. Descrivim breument les demostracions i discutim la hipòtesi addicional sobre els nombres del residu necessària per al cas NND, exigint essencialment que els nombres del residu no siguin enters. No obstant això, calculem alguns exemples que revelen que aquests valors enters poden succeir i, a més, que els divisors associats poden contribuir als pols a la funció zeta topològica. Això indica que l'enfocament actual de les singularitats NND no es pot generalitzar i que cal un enfocament diferent per atacar el cas general.

Paraules clau: Mondromia, polinomi de Bernstein-Sato, resolució de singularitats, corbes planes, Newton no degenerat.

MSC: 14B05, 14H20, 14J17, 32S40, 34M35

Resumen

En este trabajo, estudiamos la Strong Monodromy Conjecture (SMC) en su marco topológico y el estado del arte para algunos casos conocidos de la conjetura. Después de introducir los conceptos de resolución de singularidades, polinomio de Bernstein-Sato, función zeta topológica, estudiamos los casos de curvas planas, y después el de las singularidades Newton no degeneradas (NND). La prueba de la SMC para ambos casos se basa en el estudio del estudio asintótico de determinados periodos de integrales y en algunos resultados técnicos cohomológicos. Aun así, el resultado para curvas planas se simplifica mediante el estudio de diversos invariantes, mientras que los polinomios NND permiten un enfoque más combinatorio mediante el abanico dual asociado a su polígono de Newton. Describimos brevemente las demostraciones y discutimos la hipótesis adicional sobre los números del residuo necesaria para el caso NND, exigiendo esencialmente que los números del residuo no sean enteros. Sin embargo, calculamos algunos ejemplos que revelan que estos valores enteros pueden suceder y, además, que los divisores asociados pueden contribuir a los polos a la función zeta topológica. Esto indica que el enfoque actual para las singularidades NND no se puede generalizar y que hace falta un enfoque diferente para atacar el caso general.

Palabras clave: Mondromía, polinomio de Bernstein-Sato, resolución de singularidades, curvas planas, Newton no degenerado.

MSC: 14B05, 14H20, 14J17, 32S40, 34M35

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Chapter I

Introduction

The monodromy conjecture is a long-standing problem in the theory of singularities, which remains open since its formulation by the Japanese mathematician Igusa in the late seventies. In its topological version, it predicts a precise relation between a polynomial arising from a differential equation satisfied by our singularity, the so called Bernstein-Sato polynomial, with the poles of a zeta function containing topological information of a resolution of singularities. Even though we will introduce with more detail all the required concepts, we include next the precise statement of the conjecture.

Conjecture (Monodromy conjecture, topological version)**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial. If s_0 is a pole of $Z_{\mathrm{top}}(f, \varphi; s)$, then*

1. (standard) $e^{2\pi\imath\Re(s_0)}$ *is a monodromy eigenvalue of $f\colon \mathbb{C}^n \to \mathbb{C}$ at some point of $\{f = 0\}$.*
2. (strong) s_0 *is a root of the Bernstein-Sato polynomial b_f.*

From now on, we will always refer to the strong version of the conjecture, and we will see that it implies the standard (or weaker) one.

Although the general case remains wide open, some progress has been made in proving the result for special cases. In particular, it is known in the case of plane curves (Loeser '88), in the case of Newton non-degenerate polynomials up to certain hypothesis on the so-called *residue numbers* appearing (Loeser '90), in some types of hyperplanes arrangements (Budur-Saito-Yuzvinsky '10, Walther '17, Bapat-Walters '15), and in the case of semi-quasihomogeneous singularities (Budur-Blanco-van der Veer '21).

However, from these approaches there does not appear to be a clear conceptual idea on why the result should be true, and the analogy with the complex zeta function is the only reason at all to expect the conjecture to hold (see [Vey24, Rmk. 2.13]). In fact, all proofs of known cases are basically 'ad hoc', deriving with hard work enough information from both sides of the problem: a description of the possible poles of the zeta function and a suitable set of candidates of roots for the Bernstein-Sato polynomial.

Nonetheless, it is still interesting to work on attempts at approaching the problem, as well as looking for interesting examples that lead to results of independent interest, or even to search possible counterexamples.

In this project, I will be studying the background required and the state of the art for some of the cases that are known. In particular, the main goal is to study the case of polynomials that are Newton non-degenerate (NND). These allow for a more combinatorial approach to the problem, that also make some heavy computations easier to interpret. However, we will rather start by studying the case of plane-curves, where the possibility to calculate many invariants allows for a more elementary proof. Although simpler, this case is still interesting and has intersection with the NND case, since plane curves whose branches all have one characteristic exponent are also NND.

For Newton non-degenerate singularities, the main result required for the proof is the following.

Theorem ([Loe90], Thm. 5.5.1)**.** *Let f be a comfortable polynomial verifying $f(0) = 0$, with Newton diagram $\Gamma(f)$, and Newton non-degenerate. Suppose that all compact faces τ_0 verify*

i) $\frac{k(\tau_0)}{N(\tau_0)} < 1$,

ii) For every face τ of codimension 1 of $\Gamma(f)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, the real parts of the poles of the zeta function of f are roots of the Bernstein-Sato polynomial of f.

As mentioned above, this includes two additional hypothesis. We will see that the relevant extra condition is the second one, which we understand as imposing that the (toric) residue numbers ε are not integers. In this work, we will find and study examples to determine whether this condition could be removed, or what is the situation for the cases where it does not hold.

Altogether, this thesis is organized as follows. First, in Chapter II, we study the required preliminaries for the monodromy conjecture such as resolution of singularities, the zeta function, the Bernstein-Sato polynomial, local systems, connections... and the main tools used to approach the problem. In Chapter III, we will focus our attention to the case of plane curves, while in Chapter IV, we will rather focus on Newton non-degenerate polynomials. Then, in Chapter V we thoroughly describe some NND examples, their Newton diagram, zeta function, residue numbers... and study the relevance of the required hypothesis for the result in this case. Lastly, in Chapter VI we discuss the conclusions and possible future lines of work.

Acknowledgements

I would like to thank Guillem Blanco Fernández for supervising this thesis and his guidance through the project, as well as Josep Àlvarez Montaner for his co-supervisor role and introducing me to the theory of singularities. I would also like to thank the financial support from CFIS, which has funded my stay in Leuven to undertake this project.

Chapter II

Preliminaries

1 Singularities

First of all, we begin by recalling basic notions of singularity theory, and set the default notation for this work, which will follow mostly the exposition in [GLS07].

Consider $f\colon U \to \mathbb{C}$ a holomorphic function defined on an open set $U \subset \mathbb{C}^n$, defining the hypersurface $X = f^{-1}(0)$. We call the *set of singular points* of X to the set

$$\operatorname{Sing}(X) := \left\{ x \in U \mid f(x) = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\}$$

In particular, we will say that a point $x \in U$ is an *isolated singularity* if $x \in \operatorname{Sing}(X)$ and it is the only singularity in a small enough neighborhood $V \ni x$, i.e. $\operatorname{Sing}(X) \cap V = \{x\}$. Then, we also say that the germ $(X, x) \subset (\mathbb{C}^n, x)$ is an *isolated hypersurface singularity.*

By considering the Jacobian ideal in $\mathcal{O}_{\mathbb{C}^n}(U)$

$$J(f) := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \cdot \mathcal{O}_{\mathbb{C}^n}(U)$$

we can define the *Milnor* and the *Tjurina* algebras of f at x

$$M_{f.x} := \mathcal{O}_{\mathbb{C}^n,x} / J(f)\, \mathcal{O}_{\mathbb{C}^n,x}, \qquad T_{f.x} := \mathcal{O}_{\mathbb{C}^n,x} / (f, J(f))\, \mathcal{O}_{\mathbb{C}^n,x},$$

The dimension of these analytic algebras as $\mathbb{C}$-vector spaces are called, respectively, the *Milnor* and *Tjurina number* of f at x.

$$\mu(f, x) := \dim_{\mathbb{C}} M_{f,x}, \qquad \tau(f, x) := \dim_{\mathbb{C}} T_{f,x}$$

It is clear that $\mu(f, x) \neq 0$ if, and only if, $\frac{\partial f}{\partial x_i}(x) = 0$ for all i, so we can interpret μ as counting the singular points of the *function* f, each with multiplicity $\mu(f, x)$. Similarly, $\tau(f, x) \neq 0$ if, and only if, additionally $f(x) = 0$, so that τ counts the singular points of the *zero set* of f, each with multiplicity $\tau(f, x)$.

Throughout this work, we will consider for convenience that we have an isolated singularity at the origin and $f(0) = 0$. Since the context will be clear, from now on we will simply write M_f, T_f to denote the algebras, and μ_f, τ_f to denote the Milnor and Tjurina numbers, respectively.

Lemma 1 ([GLS07], Lemma 2.3)**.** *The following are equivalent*

(i) 0 *is an isolated singularity of* X

(ii) $\mu_f < \infty$

(iii) $\tau_f < \infty$

To end the section, let us recall the definition of an invariant, and the notions of analytic and topological, as stated in [GLS07, Def. 3.30].

We say that two germs of isolated hypersurface singularities are *analytically equivalent* if there exists a local isomorphism mapping one to the other, and we refer to the corresponding equivalence classes as *analytic types*. On the other hand, we say that they are *topologically equivalent* if there exists a homeomorphism mapping one to the other, and we refer to the corresponding equivalence classes as *topological types.*

Definition 1 (Invariant)**.** *We call a number (or a set, or a group...) associated to a singularity an* analytic *(resp.* topological*)* invariant *if it remains unchanged within an analytic (resp. topological) equivalence class.*

2 Resolution of singularities

Consider X an algebraic variety over any algebraically closed field $\mathbb{K}$ (in our work, this will always be $\mathbb{C}$), and denote as introduced earlier $\mathrm{Sing}(X)$ the set of singular points of X. We will say that a variety is *smooth* whenever this set of singular points is empty.

Definition 2 (Resolution)**.** *A* resolution *of X is a proper morphism $\pi\colon Y \to X$ where*

(i) Y is a smooth variety.

(ii) The restriction outside the singular locus $\pi\,|_{Y\setminus\pi^{-1}(Sing(X))}\colon Y \setminus \pi^{-1}\,(Sing(X)) \to X \setminus Sing(X)$ is a birational isomorphism.

Additionally, we will say that the resolution is good *if also*

(iii) For every singular point $p \in \pi^{-1}\,(Sing(X))$, there exists an open neighborhood $U_p \subset Y$, and open $V \subset \mathbb{K}^n$ with a chart

$$\begin{aligned} \mathbf{y}\colon U_p &\xrightarrow{\cong} V \\ p &\longmapsto 0 \end{aligned}$$

such that $U \cap \pi^{-1}\,(Sing(X)) = \{y_{i_1} = \cdots = y_{i_r} = 0\}$ for certain indices $0 < i_1 < \cdots < i_r \leq n$.

In particular, if we want to resolve the singularities defined by the zero locus of a given polynomial, we may introduce the following notion of resolution.

Definition 3 (Embedded resolution)**.** *Let X be a smooth algebraic variety, $f\colon X \to \mathbb{K}$ a polynomial and abbreviate $S = Sing(f^{-1}(0))$ be the set of singular points on the zero set of f. An* embedded resolution *of f is a proper morphism $\pi\colon Y \to X$ where*

(i) Y is a smooth variety.

(ii) The restriction outside the singular locus $\pi\,|_{Y\setminus\pi^{-1}(S)}\colon Y \setminus \pi^{-1}\,(S) \to X \setminus S$ is a birational isomorphism.

(iii) For every singular point $p \in \pi^{-1}\,(S)$, there exists an open neighborhood $U_p \subset Y$, and an open $V \subset \mathbb{K}^n$ with a chart

$$\begin{aligned} \mathbf{y}\colon U_p &\xrightarrow{\cong} V \\ p &\longmapsto 0 \end{aligned}$$

over which $\pi^* f = u(y)\, y_{i_1}^{N_1} \cdots y_{i_r}^{N_r}$, with $u(0) \neq 0$ a unit, and $N_i \geq 0$ integers.

The existence of a resolution for fields of characteristic zero always exist, thanks to a result by Hironaka [Hir64] with very sophisticated tools and a sextuple induction. On the contrary, the existence is not known to be guaranteed for fields of positive characteristic. For a marvelous discussion on these topics, we refer the interest reader to the surveys by Hauser: [Hau03] for the proof in characteristic 0 and [Hau10] for why the problem is specially harder in characteristic $p > 0$.

Even more, the results by Hironaka imply that such a resolution can be described in terms of composition of simpler morphisms, known as *blowups.* We will describe with more detail this procedure in the case of plane curves in Section 2 of Chapter III, where a pictorial representation of the process is possible.

Also, analogously to the discussion above, we could consider resolutions of germs of holomorphic functions $f\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$. By an abuse of notation, we will consider a representative $f\colon B \to \mathbb{C}$

defined on a small open ball B around the origin, and consider the resolution of the germ f to be the resolution $\pi\colon X \to B$ of the representative.

2.1 Geometric point of view

Back to the definitions, it is usually more convenient to give a geometric point of view of the resolution. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial, and $\pi\colon X \to \mathbb{C}$ an embedded resolution of f.

We start by denoting $(E_i)_{i\in J}$ the irreducible components of $\pi^{-1}\left(f^{-1}(0)\right)$. Each divisor E_i is given in local coordinates by $\{x_i = 0\}$, respectively. Now, on the account of the local expression of the pullback of f, we know that $\pi^* f$ vanishes with order N_j on a generic point of E_j. Therefore, we may write globally

$$\operatorname{div}(\pi^* f) = \sum_{j\in J} N_j E_j$$

The idea on the setup of the resolved space is that, at worst, the intersection of divisors arising from the blowup of the singular points now look like intersections of coordinate hyperplanes. In particular, the divisors are smooth and intersect transversely, that is, they have *normal crossings*. Additionally, it is possible to impose that no three of them intersect at the same point, in which case we will say that they have *simple normal crossings*.

Lastly, it will be useful to set the notation of the two following objects

- $E_i^\circ \coloneqq E_i \setminus \bigcup_{j\neq i} E_j$, for $i \in J$.
- $E_I \coloneqq \bigcap_{i\in I} E_i$ and $E_I^\circ = E_I \setminus \bigcup_{j\notin I} E_j$ for $I \subset J$.

2.2 Numerical data

To each divisor $E_j, j \in J$ we have already associated a numerical quantity N_j representing the order of vanishing of a generic point of the pullback of f in E_j. We now introduce a second quantity, associated with the pullback of the standard volume form.

With a small enough chart U_p around a point $p \in Y$, as given in the definition for the embedded resolution, we can express the Jacobian determinant locally as

$$\mathrm{Jac}_\pi(y) = v(y) \prod_{j\in J_p} y_j^{k_j - 1}$$

with $v(0) \neq 0$ a unit, $p \in E_j$ precisely for $j \in J_p \subset J$ and k_j are integers. Thus, we can write the pullback of the standard volume form as

$$\pi^*(\,\mathrm{d}x_1 \wedge \cdots \wedge \,\mathrm{d}x_n) = v(y) \prod_{j\in J} y_j^{k_j-1} \,\mathrm{d}y_1 \wedge \cdots \wedge \,\mathrm{d}y_n$$

and globally the divisor

$$\operatorname{div}(\pi^*(\,\mathrm{d}x_1 \wedge \cdots \wedge \,\mathrm{d}x_n)) = \sum_{j\in J} (k_j - 1) E_j$$

The quantities (N_j, k_j) are the *numerical data* associated to each divisor $E_j, j \in J$. From these numbers, we introduce the following notation. First, we will write

$$\sigma_j = \frac{k_j}{N_j}$$

Second, we introduce the *residue number* associated to divisors E_i and E_j

$$\varepsilon(i,j) = -N_j \sigma_i + k_j = -N_j \frac{k_i}{N_i} + k_j$$

These residue numbers play an important role in the monodromy conjecture, as they will appear (in their toric version) in the extra hypothesis required for the proof of the Newton non-degenerate case.

3 Zeta function

3.1 Archimedean zeta functions

Let us introduce first an example that will motivate the definition of the zeta function, following the exposition by Veys in [Vey24]. First, let φ be a test function, that is, a complex $\mathbb{C}^{\infty}$ function with compact support. Then, we introduce the following integral, depending on a complex parameter s

$$I(\varphi; s) = \int_0^{\infty} x^s \varphi(x) \, \mathrm{d}x$$

It is an easy check that $I(s)$ converges for $\Re(s) > -1$, and it is holomorphic in that half-plane.

By separating the terms we may study a possible continuation

$$\begin{aligned} I(\varphi; s) &= \int_0^1 x^s \underbrace{(\varphi(x) - \varphi(0))}_{x\tilde{\varphi}(x)} \, \mathrm{d}x + \int_0^1 x^s \varphi(0) \, \mathrm{d}x + \int_1^{\infty} x^s \varphi(x) \, \mathrm{d}x \\ &= I(\tilde{\varphi}; s+1) + \frac{\varphi(0)}{s+1} + \int_1^{\infty} x^s \varphi(x) \, \mathrm{d}x \end{aligned}$$

The first summand is now converging for $\Re(s) > -2$, as $\tilde{\varphi}(x)$ is again a test function, while the third summand is holomorphic and convergent on $\mathbb{C}$. Therefore, we have meromorphically continued $I(\varphi; s)$ to the half-plane given by $\Re(s) > -2$. However, the price to pay is the (possible) introduction of -1 as a pole, coming from the second summand. By iterating this procedure, we obtain a meromorphic continuation of $I(\varphi; s)$ to the whole complex plane with possible (simple) poles in the negative integers.

More generally, we may consider any multivariate non-constant polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ and test function $\varphi \colon \mathbb{R}^n \to \mathbb{C}$ and construct the associated archimedean *zeta function*

$$Z(s) = Z(f, \varphi; s) := \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) \, \mathrm{d}x$$

Technically, we should consider these integrals as distributions on the space of test functions (see [Igu00] for details), since in the definition we are implicitly neglecting subsets of measure zero where the integrand is not defined, occurring when $f = 0$.

Now, it can be checked that $Z(s)$ converges and is holomorphic in the half-plane $\Re(s) > 0$. Its meromorphic continuation and the distribution of possible poles was then posed as a question by I. Gel'fand, in a talk at the International Congress of Mathematics in 1954.

> **Question.** [Gel54, §3.I] *It is necessary to prove that this is a meromorphic function of s (it would be natural to call it a ζ-function of the given polynomial), whose poles are located in points forming several arithmetical progressions, as well as to calculate the residues of this function.*

The answer to the problem would have to wait a few years, although it led to the development of some diverse approaches. On the one hand, it was proved via resolution of singularities by Bernstein and Gel'fand [BG69] and Atiyah [Ati70], while Bernstein [Ber72] gave another proof by introducing the so called b-function (now usually referred to as the Bernstein-Sato polynomial). Briefly, one ought to keep in mind that the big picture is that in the first case we are solving the integral via a change of variables, while in the second case we are using integration by parts.

With the concepts introduced in Section 2 we can already sketch the proof via resolution of singularities. If we consider $\pi \colon Y \to X = \mathbb{R}^n$ an embedded resolution, with the notations introduced we may write the changes of variables in the integral as

$$Z(s) = \int_Y |\pi^* f(y)|^s |\mathrm{Jac}_{\pi}(y)| (\pi^* \varphi)) \, \mathrm{d}y$$

Since π is proper, $\pi^* \varphi$ has compact support too. Therefore, we only need a finite number of charts U_p

to cover it, and with a partition of unity $\{\rho_p\}_p$ subordinate to it, we can write $Z(s)$ as a finite sum

$$Z(s) = \sum_p \int_{U_p} \rho_p(y)|u(y)|^s|v(y)|(\pi^*\varphi)(y) \prod_{i\in J_p} |y_i|^{N_i s+k_i-1}\,\mathrm{d}y_i$$

These integrals are a natural n-dimensional generalizations of the initial example presented, and thus it is immediate to obtain the following result.

Theorem 1 ([Vey24], Thm. 2.1). *Let $f \in \mathbb{R}[x_1,\ldots,x_n]$ be a non-constant polynomial and $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ a $\mathcal{C}^\infty$ function with compact support. Then $Z(f,\varphi;s)$ has a meromorphic continuation to $\mathbb{C}$, and its poles of the form $-\frac{k_j+\nu}{N_j}$, with $j \in J$ and ν a non-negative integer.*

3.2 p-adic zeta functions

We now briefly introduce the p-adic analogue of the archimedean zeta functions, obtained by replacing $\mathbb{R}$ with the field of p-adic numbers $\mathbb{Q}_p$, following the exposition in [Vey24; PV20].

Denoting by $\mathrm{ord}_p(\cdot)$ the p-order, we can consider the standard p-adic norm $|\cdot|_p = p^{-\mathrm{ord}_p(\cdot)}$. Next, we fix the standard Haar measure on $\mathbb{Q}_p$, normalized so that $\mathbb{Z}_p$, the ring of all elements with non-negative order, has measure 1.

Definition 4 (p-adic zeta function). *Let $f \in \mathbb{Q}_p[x_1,\ldots,x_n]$ be a non-constant polynomial and φ a test function on $\mathbb{Q}_p^n$ (meaning now a locally constant function with compact support). The* p-adic Igusa zeta function *associated to f and φ is defined for $s \in \mathbb{C}$ by*

$$Z_p(f,\varphi;s) := \int_{\mathbb{Q}_p^n} |f(x)|_p^s \varphi(x)\,\mathrm{d}x$$

Note that $Z_p(s)$ is in fact a function of p^{-s}, given the observation $|f(x)|_p^s = (p^{-s})^{\mathrm{ord}_p f(x)}$. Furthermore, the p-adic zeta function contains information about the number of solutions of the polynomial equation $f = 0$ over the finite rings $\mathbb{Z}/p^i\mathbb{Z}$ for $i > 0$.

Indeed, the zeta function satisfies the identity

$$Z_p(s) = \frac{(p^{-s}-1)P(p^{-s})+1}{p^{-s}}$$

where we introduce the formal power series

$$P(T) := 1 + \sum_{i>0} \#\{\text{solutions of } f=0 \text{ over } \mathbb{Z}/p^i\mathbb{Z}\}\, p^{-ni}T^i$$

In particular, the asymptotic behavior of the number of solutions as i tends to infinity is controlled by the poles of the zeta function (see [Seg12] for more details).

Remark 1 ([Vey24], Rmk. 1.2.10). Confusingly enough, in the literature the zeta function $Z_p(f,\varphi;s)$ is called either *p-adic*, *Igusa* and/or *local*. The local adjective refers to the fact that $\mathbb{Q}_p$ is a local field or sometimes, if it is clear from the context that we are working in the p-adic setting, to the fact that the integration domain is a smaller neighborhood (generally, that φ is the characteristic function of $(p^e\mathbb{Z}_p)^n$ for some $e \geq 1$).

Now, if we try to answer Gel'fand's question for the p-adic zeta function, we find the following. First, we can adapt the proof via change of variables, see for example [DL92, Thm. 2.2] for a reference on resolutions in this setting. In this case, we obtain that the integral equals a sum of products of elementary integrals of the form

$$\int_{p^e\mathbb{Z}_p} |y_j|_p^{N_j s+k_j+1}\,\mathrm{d}y_j = \frac{(1-p^{-1})p^{-e(N_j s+k_j)}}{1-p^{-(N_j s+k_j)}}$$

which are converging for $\Re(s) > -\frac{k_j}{N_j}$, so we conclude in the following result.

Theorem 2 ([Vey24], Thm. 1.2.11). *Let $f \in \mathbb{Q}_p[x_1, \ldots, x_n]$ be a non-constant polynomial and $\varphi\colon \mathbb{Q}_p^n \to \mathbb{C}$ a locally constant function with compact support. Then $Z_p(f, \varphi; s)$ is a rational function in p^{-s}, and it has a meromorphic continuation to $\mathbb{C}$ with poles contained in the locus where $p^s = p^{k_j/N_j}$ for some $j \in J$. Hence, in terms of s, the poles are of the form*

$$-\frac{k_j}{N_j} + \frac{2\pi\nu}{(\ln p)N_j} i$$

with $j \in J$ and $\nu \in \mathbb{Z}$.

As pointed out in [Vey24, p. 8]: "comparing with the real case, starting with the basic values $-\frac{k_j}{N_j}$ we have 'up and down' imaginary shifts instead of 'left' real shifts".

3.3 Topological zeta function

We now introduce the topological zeta function, which originally was defined from a heuristic manipulation on the p-adic zeta function introduced above. For that, first note that, unlike in the archimedean setting, an embedded resolution gives an explicit formula for the p-adic zeta function.

Theorem 3 ([Den87], Denef's formula). *Let $f \in \mathbb{Q}[x_1, \ldots, x_n]$ be a non-constant polynomial, and $\pi\colon Y \to \mathbb{A}^n_{\mathbb{Q}}$ be an embedded resolution of $\{f = 0\}$, that is defined over $\mathbb{Q}$. For any prime p, we can also consider f in $\mathbb{Q}_p[x_1, \ldots, x_n]$ and view Y and π as defined over $\mathbb{Q}_p$. Then, for all but finitely many p,*

$$Z_p(f; s) = \frac{1}{p^n} \sum_{I \subset J} \# \left(\overline{E_I}^{\circ} (\mathbb{F}_p) \right) \prod_{i \in I} \frac{p-1}{p^{k_i + N_i s} - 1} \tag{3.1}$$

The definition of the topological function can be obtained through a heuristic argument on this expression (3.1), by 'taking the limit as $p \to 1$'. This is rigorously formalized by Denef and Loeser in [DL92], and leads to the following definition.

Definition 5 (Topological zeta function). *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial and choose an embedded resolution $\pi\colon Y \to \mathbb{A}^n_{\mathbb{C}}$ of $\{f = 0\}$. The* (global) topological zeta function *of f is*

$$Z_{\text{top}}(f; s) := \sum_{I \subset J} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{k_i + N_i s}$$

and the local topological zeta function *of f at $a \in \{f = 0\}$ is*

$$Z_{\text{top},a}(f; s) := \sum_{I \subset J} \chi\left(E_I^\circ \cap \pi^{-1}\{a\}\right) \prod_{i \in I} \frac{1}{k_i + N_i s}$$

where in both cases I runs through all possible subsets of J.

A priori, this involves fixing an embedded resolution, but they are independent of it (see [DL92, Thm. 3.2]), and the topological zeta function is well-defined. Further developments prove this by using the *weak factorization theorem*, that reduces the problem to showing that $Z_{\text{top}}(f; s)$ is unchanged after an admissible blow-up in the resolution. On a side note, it should be mentioned that so far there does not exist an intrinsic definition of the topological zeta function.

4 Bernstein-Sato polynomial

We now introduce the Bernstein-Sato polynomial, a complex polynomial that arises from the existence of a given differential equation involving a local equation of the singularity considered. It is an analytical invariant, but not a topological one, and it is of particular interest to find relations between its roots and other invariants.

To introduce it, we first denote by $R := \mathbb{C}[x_1, \ldots, x_n]$ the ring of complex polynomials in n variables. Also, let $\mathscr{D} := R\langle \partial_1, \ldots, \partial_n \rangle$ be the *Weyl algebra*, where ∂_i are the partial derivatives operators with respect to x_i. The notation in introducing the formal symbols ∂_i already suggests that the ring is non-commutative. Nonetheless, it is almost commutative, except for the relations $\partial_i x_i - x_i \partial_i = 1$. With this, it is easy to show that there exists a *normal form* $P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta$ to write any element of $\mathscr{D}$ as a finite sum. For a more gentle introduction and more details on the properties of the Weyl algebra, we refer to [Cas10].

Next, consider the polynomial ring $\mathscr{D}[s] \coloneqq \mathscr{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$, where we have introduced another variable s commuting with all x_i, ∂_i. In this case, any element of $\mathscr{D}[s]$ can be written as $P(s) = \sum_{i=0}^m s^i P_i$, where all $P_i \in \mathscr{D}$.

Lastly, we consider the localized ring $R_f[s] \coloneqq R[f^{-1}, s]$, and construct $R_f[s] \cdot f^s$, which is the free module generated by the formal symbol f^s. There is a natural structure of left $\mathscr{D}[s]$-module given by the product rule. Indeed, every element of the module can be written as $\frac{g}{f^k} \cdot f^s$ for some $g(x,s) \in R[s]$, and then the action of the partial derivatives is simply

$$\partial_i \cdot \left(\frac{g}{f^k} \cdot f^s \right) = \partial_i \cdot \left(\frac{g}{f^k} \right) \cdot f^s + \frac{sg}{f^{k+1}} \cdot \frac{\partial f}{\partial x_i} \cdot f^s$$

In this context, we introduce the Benstein-Sato functional equation. The theorem was first proved by Bernstein [Ber72] in the case of polynomials, and later by Kashiwara [Kas76] and Björk [Bjö73] for the case of holomorphic functions and formal power series, respectively. For a more detailed exposition on the history and results on the Bernstein-Sato, we refer the interested reader to [Gra10; ÀJN21].

Theorem 4. *Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $b_{f,P}(s) \in \mathbb{C}[s]$ such that the relation*

$$P(s) f^{s+1} = b_{f,P}(s) f^s \tag{4.1}$$

holds formally in the $\mathscr{D}$-module $R_f[s] \cdot f^s$.

Nonetheless, usually the linear differential operator $P(s)$ isn't even computed, as it is generally not relevant. Instead, it is more interesting to study the polynomial $b_{f,P}(s)$, which leads to the following definition.

Definition 6 (Bernstein-Sato polynomial)**.** *It can be seen that the set of polynomials $b_{f,P}(s)$ satisfying a functional equation as in* (4.1) *forms an ideal in $\mathbb{C}[s]$, and hence we can consider its monic generator. This polynomial $b_f(s)$ is called the Bernstein-Sato polynomial of f, and if the context is clear, the subscript is dropped.*

If we were to consider $R = \mathbb{C}\{x_1, \ldots, x_n\}$ instead, we should refer to the *local* Bernstein-Sato polynomial denoted by $b_{f,p}(s)$ for a point $p \in X$. The *global* $b_f(s)$ and the *local* $b_{f,p}(s)$ polynomials can be related as follows (see [MN91])

$$b_f(s) = \operatorname*{lcm}_{p \in \mathcal{V}(f)} b_{f,p}(s)$$

where one ought to keep in mind that for all smooth points, $b_{f,p}(s) = s + 1$.

Generally, it is very hard to compute the Bernstein-Sato polynomial for any $f \in R$. The first algorithm for that task was introduced by Oaku in [Oak97], using non-commutative Gröbner basis in the Weyl algebra, which gives a high complexity of computation and is not feasible to use in many examples. To show how the polynomials (and the differential operators) get increasingly difficult with already small examples, we next include some examples.

Example 1. (1) Let $f = x$ in $\mathbb{C}[x]$. Then, we have

$$\frac{\partial}{\partial x} f^{s+1} = (s+1) f^s$$

Therefore $b_f(s) \mid (s+1)$. In fact, $b_f(s) = (s+1)$ and this is the case for all smooth hypersurfaces. The reverse is also true, and the proof is due to Briançon and Maisonobe in [BM96, Prop. 2.6].

(2) Let $f = x_1^2 + \cdots + x_n^2$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then, we have

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) f^{s+1} = 4(s+1) \left(s + \frac{n}{2} \right) f^s$$

Therefore $b_f(s) \mid (s+1)\left(s+\frac{n}{2}\right)$, and in fact we have equality.

(3) Let $f = x^2 + y^3$ in $\mathbb{C}[x,y]$. We have the following identity

$$\left[\frac{1}{12}\frac{\partial}{\partial x}\frac{\partial}{\partial y}y\frac{\partial}{\partial x} + \frac{1}{27}\left(\frac{\partial}{\partial y}\right)^3 + \frac{1}{4}\left(s+\frac{7}{6}\right)\left(\frac{\partial}{\partial x}\right)\right] f^{s+1} = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right) f^s$$

And it can be proved that $b_f(s) = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$.

(4) Let $f = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ be a monomial in $\mathbb{C}[x_1,\dots,x_n]$ Then

$$\frac{1}{\alpha_1^{\alpha_1}\dots\alpha_n^{\alpha_n}}\left(\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}\right) f^{s+1} = \prod_{i=1}^{n}\prod_{k=1}^{\alpha_i}\left(s+\frac{k}{\alpha_i}\right) f^s$$

In fact, $b_f(s) = \prod_{i=1}^{n}\prod_{k=1}^{\alpha_i}\left(s+\frac{k}{\alpha_i}\right)$.

(5) (Cayley) If $f = \det(x_{i,j}) \in \mathbb{C}[x_{i,j} \mid 1 \le i,j \le n]$, then

$$\det(\partial_{i,j}) f^{s+1} = (s+1)(s+2)\cdots(s+n) f^s$$

In fact, $b_f(s) = (s+1)(s+2)\cdots(s+n)$.

If f is not invertible, by setting $s=-1$ in the functional equation (4.1), we see that $(s+1)$ divides $b_f(s)$, hence we may define $\tilde{b}_f(s) = b_f(s)/(s+1)$ the *reduced Bernstein-Sato* polynomial of f.

Returning to the question of the possible continuation of the archimedean zeta function, the existence of the Bernstein-Sato polynomial and the associated functional equation provides an alternative proof, which is essentially using integration by parts.

Consider a non-constant polynomial $f \in \mathbb{C}[x_1,\dots,x_n]$, a test function φ, and the associated zeta function $Z(s)$. To ease the discussion, let us assume that $f(x) \ge 0$ on the domain of the test function, because otherwise we ought to split the positive and negative parts and apply the argument to follow twice. Using the functional equation (4.1), we compute

$$b_f(s)Z(s) = \int_{\mathbb{R}^n} b_f(s) f(x)^s \varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} P(s)\cdot f(x)^{s+1}\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)^{s+1}\underbrace{P^*(s)\cdot\varphi(x)}_{\varphi_1(x)}\,\mathrm{d}x$$

where the last inequality is given by integration by parts, and we have introduced the adjoint operator P^* of P. Moreover, it can be shown that the terms accompanying the power of f result in a new test function φ_1, so we may write

$$Z(s) = \frac{1}{b_f(s)}\int_{\mathbb{R}^n} f(x)^{s+1}\varphi_1(x)\,\mathrm{d}x$$

to obtain a meromorphic continuation to the half-plane given by $\Re(s) > -1$, with the possible addition of poles given by the roots of the polynomial b_f. Iterating this process, we obtain a meromorphic continuation on the whole complex plane, as taking arbitrary integer $r > 1$ we may write

$$Z(s) = \frac{1}{b_f(s+r-1)\cdots b_f(s+1)b_f(s)}\int_{\mathbb{R}^n} f(x)^{s+r}\varphi_r(x)\,\mathrm{d}x$$

We may summarize the conclusion in the following result.

Theorem 5 ([Vey24], Thm. 2.3)**.** *Let $f \in \mathbb{R}[x_1,\dots,x_n]$ be a non-constant polynomial and $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ a $\mathcal{C}^\infty$ function with compact support. Then $Z(f,\varphi;s)$ has a meromorphic continuation to $\mathbb{C}$, and its poles are of the form $\lambda - \nu$ for λ a root of b_f and ν a non-negative integer.*

Going back to the roots of the Bernstein-Sato polynomial, we describe next some well known properties. First, as it can be already conjectured from Example 1, all the roots are negative rational numbers (see [Mal75; Kas76]). Moreover, a set of candidates for the roots can be obtained from a resolution of singularities of f, and the values of the candidates will involve the multiplicities of the divisors (see [Kol97, Thm. 10.7] and [Lic89, Thm. 1]).

Theorem 6. *With the notations introduced for the resolution of singularities, we have that every root of the Bernstein-Sato polynomial b_f is of the form*

$$-\frac{k_j+\nu}{N_j}, \qquad j \in J, \nu \in \mathbb{Z}_{\geq 0}$$

Note that this set of candidate roots is consistent with the set of possible poles of the archimedean zeta function, as described in Theorem 1.

Lastly, following this archimedean case, we could try to answer the same question about the p-adic or the topological zeta functions using the Bernstein-Sato polynomial too. However, this completely breaks down in these settings, and it may appear that there no longer is a relation between the poles of these zeta functions and the roots of b_f. Nonetheless, thanks to several examples computed by Igusa, the following relation was conjectured: taking the real part of a pole results in a root of the Bernstein-Sato polynomial. This is basically the statement of the monodromy conjecture, which we have already presented in the introduction and which we will describe with more detail in Section 8.

5 Sheaves

Before that, we recall some definitions and the general properties of sheaves, which can be found in almost any algebraic geometry text (see for example *The Rising Sea: Foundations of Algebraic Geometry*, by Vakil).

5.1 Basic sheaf theory

Let X be a topological space, and $\mathbf{C}$ a category, which in our case will be the category of rings **Ring**.

Definition 7 (Presheaf)**.** *A* presheaf *$\mathscr{F}$ on X with values in $\boldsymbol{C}$ is a rule*

$$U \mapsto \mathscr{F}(U)$$

that assigns an object $\mathscr{F}(U)$ in $\boldsymbol{C}$ to ach open set $U \subset X$, and with every inclusion of open sets $V \subset U \subset X$ a restriction morphism

$$\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$$

in the category $\boldsymbol{C}$, satisfying the following properties

i) ρ^U_U is the identity for every open U in X.

ii) $\rho^U_W = \rho^V_W \circ \rho^U_V$ whenever $W \subset V \subset U$ are opens in X.

Purely in terms of categories, we can reformulate this description as follows. A presheaf $\mathscr{F}$ on X with values in $\mathbf{C}$ is nothing but a functor

$$\mathscr{F} : \mathbf{Open}^{op}_X \to \mathbf{C}$$

from the opposite category of $\mathbf{Open}_X$ (whose objects are the open subsets of X and the morphisms are the inclusion relations) to $\mathbf{C}$.

The categorical definition implies that presheaves on set-like categories on a fixed topological space X form a category, with a morphism of presheaves $\Phi : \mathscr{F} \to \mathscr{G}$ being a collection of maps $\Phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that for each $V \subset U$ open, the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\Phi_V"] \arrow[d, "(\rho_{\mathscr{F}})^U_V"'] & \mathscr{G}(V) \arrow[d, "(\rho_{\mathscr{G}})^U_V"] \\ \mathscr{F}(U) \arrow[r, "\Phi_U"] & \mathscr{G}(U) \end{tikzcd}$$

In its more general setting, the objects $\mathscr{F}(U)$ might not even be a set, but rather a collection of data that we are associating with the open set U. However, it is still sometimes useful to think of $\mathscr{F}(U)$ as a set of functions living on this open, and this will be the situation if the objects of the category $\mathbf{C}$ are sets, with possibly more structure, and we will assume so from now on.

In that case, we will denote by *section* of $\mathscr{F}$ in U any element f of $\mathscr{F}(U)$. Moreover, it is common to write the composition with the restriction morphisms simply as $f|_V$ rather than $\rho_V^U(f)$ for any open $V \subset U$. Additionally, an element of $\mathscr{F}(X)$ will be called a *global section.*

For a presheaf $\mathscr{F}$ and for every point $p \in X$ we can define the stalk $\mathscr{F}_p$ to be the set

$$\{(f, U) \mid f \in \mathscr{F}(U), p \in U \text{ open}\}/\sim$$

where the equivalence relation is given by

$$(f_1, U_1) \sim (f_2, U_2) \quad \text{if} \quad f_1|_V = f_2|_V \ \text{ for some open set } V \ni p, V \subseteq U_1 \cap U_2.$$

Again, we can give a description of the stalk in categorical terms, in this case they are the *colimit* of the diagram formed by all $\mathscr{F}(U)$ such that $p \in U$, that is

$$\mathscr{F}_p = \varinjlim_{U \ni p} \mathscr{F}(U)$$

Back to the first definition, we call the equivalence classes themselves germs, and we will denote them by $[f]_p$ or, usually with an abuse of notation if the base point is clear, again by f. As the notation suggests, it is rarely useful to remember the set U, since it can get arbitrarily small, and the germ should rather be though of as enriched values (while the stalk is the set of possible germs). Indeed, the germ of a global section at 0 lets us read the values $f'(0), f''(0)$,... since we have information of the values of points arbitrarily close to 0. In particular, if we consider the sheaf $\mathscr{F}$ on $\mathbb{C}$ of *holomorphic* functions, the germ determines the entire function.

The definition of a presheaf does not have an inherently local nature, and we might encounter problems when trying to glue information with respect to open covers. This leads to the definition of a *sheaf,* which is a presheaf for which the property we require on the functions returned is a *local* property. More formally, we have

Definition 8 (Sheaf)**.** *Let $\mathscr{F}$ be a presheaf on X with values in $\boldsymbol{C}$, a set-like category. We say $\mathscr{F}$ is a* sheaf *if it additionally satisfies*

i) (Local nature of equality) Let $U \in X$ be an open subset with $\{U_i\}_{\{i \in I\}}$ an open cover of U. For f, g elements of $\mathscr{F}(U)$, then $f = g$ if, and only if, $f|_{U_i} = g|_{U_i}$ for all $i \in I$.

ii) (Gluing) Let f_i be an element of $\mathscr{F}(U_i)$ for every $i \in I$. Then, there exists an element f in $\mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for every $i \in I$, and it is automatically unique by i).

iii) (Normalization) $\mathscr{F}(\emptyset)$ is a final object in $\boldsymbol{C}$.

The basic sheaf which we will work is the structure sheaf $\mathcal{O}_X$ on a differentiable manifold X, which is a presheaf of $\mathbb{R}$-algebras with the restriction morphisms given by the usual restrictions of functions to subsets of the domain.

One very useful procedure to construct sheaves is *sheafification.* We will not describe the details of this construction, which can be rather consulted in a usual algebraic geometry text. The idea behind it is that for a sheaf, sections correspond to sequences of compatible germs, so given an initial presheaf $\mathscr{F}$ we can define the sections of $\mathscr{F}^{sh}$ to be sequences of compatible $\mathscr{F}$-germs. When rather working with the definition via the universal property, it is immediate that this the sheafification is unique up to unique isomorphism if it exists, and the construction makes clear that we obtain a sheaf $\mathscr{F}^{sh}$. It is also relevant to note that the construction preserves stalks, that is for every point $x \in X$, we have $\mathscr{F}_x \cong (\mathscr{F}^{sh})_x$.

A usual example of this procedure is the construction of the constant sheaf, described next.

Example 2 (Constant sheaf)**.** For S a set, the *constant presheaf* on X with sections in S is the unique presheaf on X that maps every non-empty open of X to S and $\emptyset$ to a singleton, and such that every morphism between non-empty opens is the identity on S.

The *constant sheaf* on X with sections in S is the sheafification of this presheaf, and it is denoted by $\underline{S}_X$. In particular, it is easy to see that for very open $U \subset X$, the sections of $\underline{S}_X(U)$ may be interpreted as the continuous maps $U \to S$, where S is given the discrete topology.

In particular, taking the set to be $\mathbb{C}$, then the constant sheaf $\underline{\mathbb{C}}_X$ can be interpreted as the sheaf of continuous maps from X to $\mathbb{C}$ (because with the discrete topology this is the same as locally constant maps).

5.2 Direct and inverse image

Direct image

Let $h\colon Y \to X$ be a continuous map between topological spaces, and $\mathscr{G}$ a presheaf on Y. Then, we denote by $h_*\mathscr{G}$ *direct image* of $\mathscr{G}$ under the map h, that is the presheaf on X such that for every open $U \in X$ we have

$$h_*\mathscr{G} = \mathscr{G}\left(h^{-1}(U)\right)$$

and such that the restriction maps are induced by those of $\mathscr{G}$. Also, if $\mathscr{G}$ is a sheaf, then so is $h_*\mathscr{G}$. Furthermore, every morphism of (pre)sheaves $\mathscr{G} \to \mathscr{G}'$ on Y induces a morphism of (pre)sheaves $h_*\mathscr{G} \to h_*\mathscr{G}'$. This yields a functor h_* from the category of (pre)sheaves on Y to the category of (pre)sheaves on X, which is compatible with the composition of continuous maps: $(f \circ g)_* = f_* \circ g_*$.

Example 3 (Skyscraper sheaf)**.** Let $Y = \{x\}$ with $x \in X$ a point, and h the natural inclusion map. For any set S, the direct image $h_*\underline{S}_Y$ is called the *skyscraper sheaf* at x with sections in S. The name arises from the fact that $h_*\underline{S}_Y(U)$ is equal to S whenever U is an open set containing x, and it is equal to $\{0\}$ otherwise.

Inverse image

We can also define an *inverse image presheaf* $h^{-1}\mathscr{F}$ of a presheaf $\mathscr{F}$ on X under the map h. The definition is a bit more complicated, and it is better expressed in terms of a colimit, that is, for every open $V \subset Y$

$$h^{-1}\mathscr{F}(V) = \operatorname*{colim}_U \mathscr{F}(U)$$

where the colimit is taken over all open subsets $U \subset X$ such that $h(V) \subset U$, ordered by reverse inclusion. The restriction maps on $\mathscr{F}$ naturally induce those on $h^{-1}\mathscr{F}$.

Opposite to the direct image, even when $\mathscr{F}$ is a sheaf, the inverse image presheaf is usually not a sheaf. By taking its sheafification we can finally define the *inverse image* of $\mathscr{F}$, which is however still denoted by $h^{-1}\mathscr{F}$. The construction is again functorial in $\mathscr{F}$, with a functor h^{-1} from the category of (pre)sheaves on X to the category of (pre)sheaves on Y, and compatible with composition of continuous maps: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Example 4 (Retrieving the stalks)**.** If $Y = \{x\}$ is a single point $x \in X$ and h is the inclusion map, then $h^{-1}\mathscr{F}$ is simply the stalk $\mathscr{F}_x$.

More generally, when $h\colon Y \to X$ is an open embedding, the $h^{-1}\mathscr{F}$ is basically the *restriction* of $\mathscr{F}$ to Y, so by an abuse of notation it is custom to denote it simply by $\mathscr{F}|_Y$.

In general, the definition given for the inverse sheaf is not convenient to work with, and in practice they are handled thanks to the adjunction property. That is, the inverse image functor h^{-1} is *left adjoint* to the direct image functor h_*, which ultimately gives us a natural bijection between morphisms of (pre)sheaves $h^{-1}\mathscr{F} \to \mathscr{G}$ and morphisms of (pre)sheaves $\mathscr{F} \to h_*\mathscr{G}$.

Example 5 (Structure sheaves)**.** Let $h\colon Y \to X$ be a morphism between differentiable manifolds, and denote by $\mathscr{O}_X$, $\mathscr{O}_Y$ the respective structure sheaves. Since the composition of differentiable maps is still differentiable, h induces a morphism of sheaves of $\mathbb{R}$-algebras on X

$$\begin{aligned} h^{\#}\colon \mathscr{O}_X &\longrightarrow h_*\mathscr{O}_Y \\ f &\longmapsto f \circ h \end{aligned}$$

which by adjunction corresponds to a morphism of sheaves of $\mathbb{R}$-algebras $h^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ on Y.

6 Covers, locally constant sheaves and local systems

In this section, we recall the concepts of locally constant sheaves and local systems, which will allow us to describe the situation with monodromy.

To begin, let us give some motivation for the construction that we will require, and consider the differential equation

$$x\frac{\mathrm{d}}{\mathrm{d}x}y(x) - \alpha y(x) = 0, \qquad \alpha \in \mathbb{Q} \setminus \mathbb{Z}$$

The solutions in any neighborhood of the complex plane not containing the origin are of the form

$$y(x) = cx^{\alpha}$$

with $c \neq 0$ a constant. In particular, the local solutions form a vector space of dimension 1. However, we encounter a problem if we try to extend these solutions to the whole plane, for example, by transporting some solution around the origin. Indeed, if we start with x^{α} a local solution around the point $x = 1$ and try to continuously transform the solution going counterclockwise once around the unit circle, we rather end up with $e^{2\pi\imath\alpha}x^{\alpha}$.

This phenomenon is known as monodromy, and it is precisely its presence what does not allow for a global solution. The description of these local solutions which cannot be extended naturally leads to definition of the objects that will be introduced in this section, which allow for a formal treatment of what could be interpreted as *mutivalued solutions.* For more details and proofs of the results that we will mention, we refer to [Sza09].

6.1 Covers

Definition 9 (Cover)**.** *A* cover *of a topological space X is a space Y together with a projection map $p\colon Y \to X$ such that for every point $p \in X$ the following is satisfied: there exists an open neighborhood $V \ni p$ for which $p^{-1}(V)$ can be decomposed as a disjoint union of open subsets $U_i \subset Y$, such that $p|_{U_i}\colon U_i \xrightarrow{\simeq} V$ is a homeomorphism.*

For example, we may simply construct a *trivial cover* by considering a nonempty discrete topological space I and consider the product $X \times I$ with the map given by the projection $X \times I \to X$. In fact, it is easy to see that every cover is locally a trivial cover.

Given a cover $p\colon Y \to X$, the fibre $p^{-1}(x)$ over a point $x \in X$ carries a natural action by the group $\pi_1(X, x)$, a consequence of the existence of lifting of paths and homotopies (see [Sza09, Lemma 2.3.2]).

We construct such action as follows. Take a point y in the fiber $p^{-1}(x)$ and consider an element $\gamma \in \pi_1(X, x)$ represented by a loop $h\colon [0, 1] \to X$ with $h(0) = h(1) = x$. Next, take $\tilde{h}$ the unique lifting of this path such that $\tilde{h}(0) = y$ and define the action $\gamma y \coloneqq \tilde{h}(1)$. In other words, the action corresponds to taking the endpoint of the lifted loop. It can be seen that this does not depend on the choice of h, and that the resulting point still lies on the fiber of x by construction. It is immediate to check that this defines an action

$$(\gamma_1 \circ \gamma_2)y = \gamma_1(\gamma_2 y) \qquad \text{for } \gamma_1, \gamma_2 \in \pi_1(X, x).$$

and we call it the *monodromy* action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$.

This construction gives the basis to establish the equivalence of categories of covers of X with the category of left $\pi_1(X, x)$-sets (see [Sza09, Thm. 2.3.4]).

6.2 Locally constant sheaves

This equivalence can be reformulated in terms of locally constant sheaves too, which we introduce next.

Definition 10 (Locally constant sheaf)**.** *A sheaf $\mathscr{F}$ on a topological space X is said to be* locally constant *if for each point $x \in X$ there exists an open neighborhood $U \ni x$ such that $\mathscr{F}|_U$ is isomorphic to a constant sheaf (in the category of sheaves on U).*

From now on, we will work with locally connected spaces, and we will see that these are actually very familiar objects. First, we define the following notion.

Definition 11 (Section). *Let $p\colon Y \to X$ be a space over X, and $U \subset X$ an open set. A* section *of p over U is a continuous map $s\colon U \to Y$ such that $p \circ s$ is the identity on U.*

Then, given a covering $p\colon Y \to X$, we may define a presheaf $\mathscr{F}_Y$ of sets on X by taking $\mathscr{F}_Y(U)$ to be the set of sections of p over the open set U, with restriction maps $\mathscr{F}_Y(U) \to \mathscr{F}_Y(V)$ given by the restriction of sections for any $V \subset U$. It turns out that this is a sheaf and, in fact, a locally constant one (see [Sza09, Prop. 2.5.8]), and we will call it $\mathscr{F}$ the *sheaf of local sections* of the covering $p\colon Y \to X$.

Additionally, it can be seen that the rule $Y \mapsto \mathscr{F}_Y$ is a functor, and that it induces an equivalence between the category of covers of X and that of locally constant sheaves on X (see [Sza09, Thm 2.5.9]). Altogether, by the previous mentioned equivalence, we may state

Theorem 7 ([Sza09], Thm. 2.5.14). *Let X be a connected and locally simply connected topological space, and let x be a point in X. The category of locally constant sheaves of sets on X is equivalent to the category of sets endowed with a left action of $\pi_1(X, x)$, which is induced by the functor mapping a sheaf $\mathscr{F}$ to its stalk $\mathscr{F}_x$.*

6.3 Local systems

We now turn to the most interesting case of the construction required in proving the equivalence of categories stated above, and it is historically the case to arise first.

Definition 12 (Complex local system). *A* complex local system *on X is a locally constant sheaf of finite dimensional complex vector spaces. If X is connected, all stalks must have the same dimension, which is then called the* dimension *of the local system.*

Then, we can also state that the category of complex local systems on X is equivalent to the category of finite dimension left representations of $\pi_1(X, x)$, where X is a connected and locally simply connected topological space (see [Sza09, Cor. 2.6.2]). Thus, to give a local system on X is the same as a homomorphism $\pi_1(X, x) \to \mathrm{GL}(n, \mathbb{C})$ for some n, which we will call the *monodromy representation* of the local system.

We now give an example related to the motivating example at the beginning of the section (see also [Sza09, Ex. 2.6.4]). Consider, more generally, a complex domain $D \subset \mathbb{C}$, and the homogeneous linear differential equation of order n

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

with a_i holomorphic functions defined on D. We are interested in local holomorphic solutions over open subsets U of D. Since the equation is homogeneous, the local solutions form a $\mathbb{C}$-vector space $\mathcal{S}(U)$ and, by a classical result by Cauchy, for every point in D we can find an open neighborhood where U where $\mathcal{S}(U)$ has a finite basis $x_1, \ldots, x_n$. Moreover, the restrictions of these x_i to smaller open sets still form a basis for the solutions.

Thus, the local solutions of the equation form a subsheaf $\mathcal{S} \subset \mathcal{O}^n$, which is a sheaf of complex vector spaces and, in particular, a complex local system of dimension n.

By the equivalence of categories presented, the local system $\mathcal{S}$ is uniquely determined by an n-dimensional representation of $\pi_1(X, x)$, where $x \in D$ is a point. We can describe this representation more explicitly.

Take $f\colon [0,1] \to D$ representing an element $\gamma \in \pi_1(X, x)$ and take a $s \in \mathcal{S}_x$, that is the germ of a holomorphic vector field satisfying the differential equation. By the equivalence of categories between covers and locally constant sheaves, s is naturally a point of the fiber over x of the cover $p_{\mathcal{S}}\colon D_{\mathcal{S}} \to D$ associated with $\mathcal{S}$. By definition of the monodromy action, the action of γ on s is the following: $s \mapsto \tilde{f}(1)$ on the fiber $p_{\mathcal{S}}^{-1}(x) = \mathcal{S}_x$, where $\tilde{f}$ is the unique lifting of f to $D_{\mathcal{S}}$. Furthermore, it can be seen that γs is the analytic continuation of the holomorphic function germ s along the path f representing γ. Both the uniqueness and existence are guaranteed by the fact that $\mathcal{S}$ is a locally constant sheaf and hence $D_{\mathcal{S}} \to D$ is a cover.

Even more, let us consider now the case $n = 1$, that is, take $y'(x) = h(x)y(x)$ with h a meromorphic function at the origin and consider T' a punctured disk around the origin. Then, we can compute the monodromy representation $\pi_1(T', 1) \to \mathrm{GL}(1, \mathbb{C})$ of the associated local system, where we know that the fundamental group is $\pi_1(T', 1) \cong \mathbb{Z}$. Thus, we can simply determine a one-dimensional representation of it by giving the image $m \in \mathrm{GL}(1, \mathbb{C})$ of γ a counterclockwise loop around the unit circle. It can be seen that

$$m = e^{\int_\gamma h} = e^{2\pi\imath \mathrm{Res}(h,0)}$$

Hence, in the specific case presented at the beginning of the section, where $h(x) = \alpha/x$, we have $m = e^{2\pi\imath\alpha}$, coinciding with what we expect intuitively.

Conversely, it is possible to find a homogeneous differential equation whose local system has a given monodromy (see [Sza09, Rmk. 2.6.7]).

In the case of an isolated hypersurface singularity, Brieskorn [Bri70] gave an algorithm to compute the monodromy, by computing the connection matrix of the meromorphic Gauss-Manin connection, which we will discuss in the following section.

7 Connections

7.1 Holomorphic connections

The notion of connection completes the previous discussion of local systems, and we will require it to introduce the Gauss-Manin connection and the subsequent Gauss-Manin covariant derivative.

complex

First, let us recall the concept of a locally free sheaf. Start by considering a structure sheaf $\mathcal{O}_T$ of holomorphic functions on a connected open subset $T \subset \mathbb{C}$. Now, we say that a sheaf of $\mathcal{O}_T$-modules $\mathscr{F}$ is *locally free* if locally around any point we can find an open neighborhood $V \subset T$ where $\mathscr{F}|_V \cong \mathcal{O}_T^n|_V$ for some integer n, which we call *rank* of $\mathscr{F}$. Naturally, we will say that $\mathscr{F}$ is *free* of rank n if the isomorphism is global.

Next, we will also require the sheaf Ω^1_T of holomorphic 1-forms on T, which is actually a sheaf of $\mathcal{O}_T$-modules.

Definition 13 (Holomorphic connection)**.** *A* holomorphic connection *on T is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a locally free sheaf on T and $\nabla\colon \mathcal{E} \to \Omega^1_T \otimes_{\mathcal{O}_T} \mathcal{E}$ is the connection map: a morphism of sheaves of $\mathbb{C}$-vector spaces satisfying the* Leibniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

for f a section of $\mathcal{O}_T$ and s section of $\mathcal{E}$.

There is a nice relation between sections of a locally free sheaf that vanish under the connection map and functions satisfying a related system of differential equations (see, for example, [Sza09, Ex. 2.7.3]).

In general, a section $s \in \mathcal{E}(U)$ of a connection $(\mathcal{E}, \nabla)$ is called *horizontal* if it satisfies $\nabla(s) = 0$. We denoted by $\mathcal{E}^\nabla \subset \mathcal{E}$ the subsheaf of $\mathbb{C}$-vector spaces formed by horizontal sections, which turns out to be a local system of dimension equal to the rank of $\mathcal{E}$. In particular, this is used to prove that we have an equivalence between the category of holomorphic connections on T and that of complex local systems on T, given by the functor $(\mathcal{E}, \nabla) \mapsto \mathcal{E}^\nabla$ (see [Sza09, Prop. 2.7.5]). In the literature, this is also referred to as the 'Riemann-Hilbert correspondence'.

As a remark, this correspondence extends to higher dimensional complex manifolds, but there one must impose an additional condition that is automatic in dimension 1. Namely, we further require that the connection is *flat* or *integrable*, which we will not bother to discuss.

Also, in the general setting, given a connection we can define the notion of a *covariant derivative*. For that, denote by $\Theta_T \coloneqq \mathrm{Der}_\mathbb{C}(\mathcal{O}_T) = \mathrm{Hom}_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T)$ the sheaf of vector fields in the base space T, and let $\langle \cdot, \cdot \rangle$ be induced by the pairing $\Omega^1_T \times \Theta_T \to \mathcal{O}_T$. Then, the *covariant derivative* along a section v of Θ_T is given by

$$\nabla_v(\sigma) = \langle \nabla\sigma, v \rangle$$

Thus, $\nabla_v\colon \mathcal{E} \to \mathcal{E}$ is a $\mathbb{C}$-linear homomorphism still satisfying the Leibniz rule. A key remark when T is one-dimensional, as will happen in our work, is that giving a covariant derivative is the same as giving a connection, and we will use the terms indistinctly.

7.2 Meromorphic connections

Next, in our scenario, the base space will be the punctured disk T'. However, we would like to extend the connection to the origin as well, so that altogether it is defined on the whole T. For that, we will have to pay the price of allowing poles, and to consider instead meromorphic connections.

The first change is to consider instead $\mathbb{K} := \mathcal{O}_{T,0}[t^{-1}]$ the field of germs of meromorphic functions at the origin of T, and then consider V be a finite-dimensional $\mathbb{K}$-vector space. Then, let $\Omega^1_{T,0}$ naturally denote the sheaf of differential meromorphic forms at the origin.

Definition 14. *A* meromorphic connection ∇ *on* V *is a* $\mathbb{C}$*-linear map*

$$\nabla\colon V \to \Omega^1_{T,0} \otimes_{\mathcal{O}_{T,0}} V$$

satisfying the Leibniz rule

$$\nabla(fv) = \mathrm{d}f \otimes v + f\nabla v$$

We will see that a holomorphic connection can be uniquely extended to a meromorphic connection. For that purpose, we must first introduce the concept of a lattice.

Definition 15 (Lattice)**.** *A* lattice *on a* $\mathbb{K}$*-vector space* V *is a finitely generated* $\mathcal{O}_{T,0}$*-submodule* L *of* V *such that* $V = L \otimes_{\mathcal{O}_{T,0}} \mathbb{K}$*.*

Notice that L is torsion-free, on the account that $V = L \otimes_{T,0} \mathbb{K}$. Thus, L is a torsion-free finitely generated module over a principal ideal domain, so it is free of rank $\dim_{\mathbb{K}} V$. Also, it is a helpful property that for any pair of lattices L, L' we can find $p \in \mathbb{Z}$ such that $t^p L \subset L'$.

Proposition 1. *Let* V *be a finite dimensional* $\mathbb{K}$*-vector space and* ∇ *a holomorphic connection defined on a lattice* L*. Then* ∇' *extends to a unique covariant derivative of a meromorphic connection* (V, ∇)*, associated to* ∇'*.*

Proof. For each $v \in V$, since ∇' fulfills the Leibniz rule,

$$\nabla(t^{-k}v) := -kt^{-k-1} \otimes v + t^{-k} \otimes \nabla v$$

Thus, (V, ∇) is the unique meromorphic connection extending ∇'. □

We will now focus on *regular singularities*, which are an important subclass of meromorphic connections. They can be defined via the matrix of the associated meromorphic differential equation (see for instance [CL55, §4]), but we will rather follow an algebraic approach.

For that, fix the coordinate t in the base space, and consider the covariant derivative along section $\mathrm{d}/\mathrm{d}t$, which we will denote by

$$\partial_t := \nabla_{\mathrm{d}/\mathrm{d}t}$$

Then, it makes sense to apply the covariant derivative from one lattice to another, and we can give the following definition.

Definition 16 (Regular connection)**.** *A lattice* L *of* $\mathbb{K}$*-vector space* V *is called* saturated*, if it is stable under the operator* $t\partial_t$*, that is* $t\partial_t L \subseteq L$*. A meromorphic connection* $\nabla\colon V \to \Omega^1_{T,0} \otimes_{\mathcal{O}_{T,0}} V$ *is called* regular *if there exists a saturated lattice in* V*.*

This definition can be understood more naturally after introducing the concept of pole order of a connection.

Definition 17. *The* pole order *of a meromorphic connection* ∇ *on a lattice* L *is*

$$\mathrm{pol}_L \nabla := \min\left\{k \in \mathbb{Z} \mid t^k \partial_t L \subset L\right\}$$

More generally, we call the pole order *of* ∇ *to*

$$\mathrm{pol}\nabla := \min\{\mathrm{pol}_L\nabla \mid L \text{ lattice of } V\}$$

Then, the condition of regularity is simply $\mathrm{pol}\nabla \leq 1$.

If we are given a regular meromorphic connection ∇ on V, we can easily find such a saturated lattice. Indeed, take L any lattice of V and construct the following lattice $\widetilde{L}$ via *saturation*

$$\widetilde{L} := \sum_{p=0}^{\infty}(t\partial_t)^p L = L + (t\partial_t)L + (t\partial_t)^2 L + \dots$$

It is clear that is stable under the operator $t\partial_t$, but it is not obvious that this still defines a lattice. Nonetheless, the fact that $\mathcal{O}_{T,0} \cong \mathbb{C}\{t\}$ is noetherian guarantees that only a finite number of terms are required to construct the saturation.

7.3 Meromorphic Gauss-Manin connection

We now introduce the meromorphic Gauss-Manin connection of an isolated hypersurface singularity. As mentioned earlier, in our case it is indistinct to give the connection or the covariant derivative, which we denote by ∂_t.

Let $f\colon (X,x) = (\mathbb{C}^{n+1}, \mathbf{0}) \to (T,0)$ be the germ of an isolated hypersurface singularity, and choose local coordinates $\mathcal{O}_{X,x} = \mathbb{C}\{x_1,\dots,x_{n+1}\}$. Now, by the Hilbert Nullstelensatz we have

$$\sqrt{\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_{n+1}}\right)} = \mathcal{I}\left(\mathcal{V}\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_{n+1}}\right)\right) = \mathcal{I}(\{0\}) = (x_1,\dots,x_{n+1})$$

Hence, there exists a natural number k such that

$$(x_1,\dots,x_{n+1})^k \subset \left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_{n+1}}\right) =: (\partial_x f)$$

Thus, by the determinacy theorem we may assume that f is a polynomial after a holomorphic change of coordinates. This inclusion also proves the already mentioned fact that the Milnor algebra is finite-dimensional. Furthermore, from the inclusion we can define κ_f to be the minimal integer such that

$$f^{\kappa_f} \in (\partial_x f)$$

Next, denote by Ω_X the complex sheaf of holomorphic differentials, with d the usual differentiation. Then, introduce the *relative differential forms* on X at point x by

$$\Omega_f := \frac{\Omega_{X,x}}{\mathrm{d}f \wedge \Omega_{X,x}}$$

and consider d_f the differentiation induced by the quotient. With that, we introduce the *De Rham complex* $(\Omega_{X,x},\, \mathrm{d}f\wedge\cdot)$, on the account that $\mathrm{d}f\wedge\mathrm{d}f\wedge\Omega_{X,x} = 0$. It is exact except at the right-most term, but can be extended to an exact sequence of $\mathcal{O}_{X,x}$-modules by the De Rham lemma as

$$0 \to (\Omega_{X,x},\, \mathrm{d}f\wedge\cdot) \to \frac{\mathcal{O}_{X,x}}{(\partial_x f)} \to 0$$

Via the study of the study of these relative forms, Brieskorn defined the following lattices

$$\begin{aligned}
H_f &:= H^n(\Omega_f,\, \mathrm{d}_f)\\
H'_f &:= \operatorname{coker}\mathrm{d}_f^{n-1} = \frac{\Omega^n_{X,x}}{\mathrm{d}f\wedge\Omega^{n-1}_{X,x} + \mathrm{d}\Omega^{n-1}_{X,x}}\\
H''_f &:= \frac{\Omega^{n+1}_{X,x}}{\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{X,x}}
\end{aligned}$$

They are free $\mathcal{O}_{T,0}$-modules of rank μ, the Milnor Number, which was conjecture by Brieskorn, but proved later by Sebastiani [Seb70, Cor. 1]. Additionally, we have the inclusions

$$H_f \xhookrightarrow{\mathrm{id}} H'_f \xhookrightarrow{\mathrm{d}f\wedge\cdot} H''_f$$

while also

$$t^{\kappa_f} H''_f \subset \mathrm{d}f \wedge H'_f, \qquad t^{\kappa_f} H'_f \subset H_f$$

Moreover, we have that the following relation between quotients

$$\frac{H'_f}{H_f} = \frac{\Omega^n_f/\mathrm{im}\,\mathrm{d}^{n-1}_f}{\ker\,\mathrm{d}^n_f/\mathrm{im}\,\mathrm{d}^{n-1}_f} \cong \frac{\Omega^n_f}{\ker\,\mathrm{d}^n_f} \cong \mathrm{im}\,\mathrm{d}^n_f = \Omega^{n+1}_f = \frac{\Omega_{X,x}}{\mathrm{d}f\wedge\Omega^n_{X,x}} \cong \frac{\mathcal{O}_{X,x}}{(\partial_x f)} \cong \frac{H''_f}{\mathrm{d}f\wedge H'_f}$$

Furthermore, they are all lattices on the μ dimensional $\mathbb{K}$-vector space

$$V := H''_f \otimes_{\mathcal{O}_{T,0}} \mathbb{K}$$

Hence, we may define the connection ∇'_f by giving the covariant derivative $(\partial_t)'_f : \mathrm{d}f\wedge H'_f \to H''_f$ by

$$(\partial_t)'_f\,[\mathrm{d}f\wedge\omega] := [\mathrm{d}\omega]$$

and we know by Proposition 1 that it extends uniquely to a meromorphic connection on the whole V.

Definition 18 (Meromorphic Gauss-Manin connection)**.** *We call the associated connection of* ∇'_f *the* meromorphic Gauss-Manin connection of f*, and we will denote it by*

$$\nabla_f = (V, \nabla_f).$$

Brieskorn gives some nice properties about this connection in [Bri70], for example that it is regular, but also provides a procedure to compute the associated monodromy.

Lastly, we recall the classical result of Malgrange, that relates the concepts introduced in this subsection back to the Berntein-Sato polynomial.

Theorem 8 ([Mal75], Thm. 5.4)**.** *If* $f\colon (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ *is an isolated singularity, then the reduced local Bernstein-Sato polynomial* $\tilde{b}_{f,\mathbf{0}}(s)$ *of* f *is equal to the minimal polynomial of the endomorphism*

$$-\overline{\partial_t t}\colon \frac{\widetilde{H}''_n}{t\widetilde{H}''_n} \longrightarrow \frac{\widetilde{H}''_n}{t\widetilde{H}''_n}$$

of $\mathbb{C}$*-vector spaces of dimension* μ*, where* ∂_t *is the covariant derivative of the meromorphic Gauss-Manin connection, and we have introduced* $\widetilde{H}''_n$ *the saturation of the Brieskorn lattice*

$$\widetilde{H}''_n := \sum_{p=0}^{\infty} (t\partial_t)^p\, H''_n$$

We end the section with an important derived result, which will be relevant later in the statement of the conjecture.

Corollary 1 ([Mal83], Prop. 7.1)**.** *If* $-\alpha$ *is a root of* $\tilde{b}_{f,0}(s)$*, then* $e^{2\pi\imath\alpha}$ *is an eigenvalue of the monodromy of* f *at the origin. Conversely, each such eigenvalue is obtained this way.*

8 The monodromy conjecture

We now introduce with more detail the monodromy conjecture. The statement of the conjecture is already suggested by the discussion on the previous sections, giving a relation between the poles of the topological zeta function and the roots of the Bernstein-Sato polynomial. However, there is a weaker version of the conjecture, which involves the monodromy on the Milnor fibration, and the implication between both versions will be given precisely by Corollary 1.

8.1 Milnor fibration

First recall that a map $\pi: E \to B$ is (the projection of) a *fibration* if each point $b \in B$ has a neighborhood U such that there is a homeomorphism $\varphi: \pi^{-1}(U) \to F \times U$ whose second component is the restriction of π, and we call F the associated *fibre*. In particular, to have such homeomorphism it is only required to construct a map onto F, which may be defined simply by a map onto a fiber $F_b = \pi^{-1}(b)$.

Let $f: (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function on the origin. Fix real numbers $0 < \delta < \varepsilon$ and denote $T \subset \mathbb{C}$ the disk $|t| < \delta$ and $T' = T \setminus \{0\}$ the punctured disk. Also, denote $B_\varepsilon \subset \mathbb{C}^n$ the ball of radius ε, and $S_\varepsilon = \partial B_\varepsilon$ the sphere defining its boundary.

Furthermore, let us write

$$X = B_\varepsilon \cap f^{-1}(T), \qquad X' = X \setminus f^{-1}(0), \qquad X_t = f^{-1}(t) \cap X,\, t \in T$$

where, by an abuse of notation, f also denotes a representative of the germ. For a pictorial representation of the situation, see for example [Viu21, Fig. 4].

By a result of Milnor [Mil16, §4], for small enough δ, ε, the restriction $f': X' \to T'$ is a locally trivial smooth fiber bundle, and the diffeomorphism type of any fiber X_t is independent of δ, ε and t.

Moreover, a second fibration might be constructed, which turns out to be equivalent to the first one described, while making some computations simpler (see Example 6). If we write $K = f^{-1}(0) \cap S_\varepsilon$, then the map $f/|f|: S_\varepsilon \setminus K \to \mathbb{S}^1$ is the projection map of a smooth fiber bundle, equivalent to f'.

The complex homology $H_i(X_t, \mathbb{C})$ and cohomology $H^i(X_t, \mathbb{C})$ groups of the Milnor fiber, which are finite-dimensional vector spaces, vanish above degree n on the account that X_t has homotopy type of a finite CW-complex of dimension n (see [Mil16, Thm. 5.1]).

If we restrict our attention to the case of an isolated singularity, we can give a more explicit description of the fibers. In this case, we begin by defining the Milnor number μ. This quantity is introduced as a positive integer which measures the amount of degeneracy at the critical point. It is described as the multiplicity of the critical point (in our case, the origin) as solution to the collection of polynomial equations

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_{n+1}} = 0$$

that is,

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,\mathbf{0}}}{(\partial_x f)}$$

Notice that this is only a finite quantity if the point is indeed an isolated singularity. Moreover, μ also coincides with the degree of the fibration mappings described. Also, this number appears in the description of the fibers for the case of isolated singularities, as seen in the following result.

Theorem 9 ([Mil16], Thm 6.5)**.** *Each fiber has the homotopy type of a bouquet $\mathbb{S}^n \vee \cdots \vee \mathbb{S}^n$ of μ spheres.*

Therefore, we may write

$$H_i(X_t, \mathbb{C}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^\mu, & i = n \\ 0, & \text{else} \end{cases}$$

Example 6 ([Wal04], Lemma 6.3.3)**.** As an example, we include the computation of the Milnor number and fibre of the singularity defined by the transverse intersection of two smooth curves, which we will see is $\mu = 1$.

If $f = 0$ and $g = 0$ are the defining equations of these smooth curves, that means their differentials are linearly independent, and we may take f, g to be local coordinates. Hence, we reduce to the case of the plane curve $f(x, y) \equiv xy = 0$.

From the algebraic definition, we can already see that

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(x, y)} = \dim_{\mathbb{C}} \mathbb{C} = 1$$

Next, from the equivalence of the presented fibrations, let us consider the second one. Then, the fibre consists of points $(x, y) \in S_\varepsilon$ satisfying $\frac{f(x,y)}{|f(x,y)|} = 1$, which in our setup is equivalent to having $xy \in \mathbb{R}_{>0}$. We may parametrize this set as $(x, y) = (ae^{i\theta}, be^{-i\theta})$ with $a, b > 0$ reals such that $|(x, y)| = a^2 + b^2 = \varepsilon^2$, and $\theta \in [0, 2\pi)$ arbitrary. Thus, the Milnor fibre is homeomorphic to the product of a circle by an interval on the line, which has a first homology group of dimension 1, so again $\mu = 1$.

8.2 Monodromy

Now, we introduce the action of monodromy on the Milnor fibration $f' \colon X' \to T'$. To that purpose, consider two opens $U_1, U_2 \subset T'$ with non-empty intersection, $t \in U_1 \cap U_2$, and with corresponding homeomorphisms

$$\varphi_i \colon (f')^{-1}(U_i) \xrightarrow{\cong} X_t \times U_i, \qquad i = 1, 2$$

In general, it is not possible to arrange the two homeomorphisms of $(f')^{-1}(t)$ onto X_t (given by the respective restrictions of φ_1, φ_2) to agree. In the case of a fibration over $\mathbb{S}^1$, this leads to consider how much the fiber has changed after a full loop. More precisely, in our setup, the action of the fundamental group $\pi_1(T', t) \cong \mathbb{Z}$ of the base of the fibration induces a diffeomorphism f on each fiber X_t, usually referred to as the *geometric monodromy.*

On the other hand, this action induces an automorphism on the homology

$$h_* \colon H_*(X_t, \mathbb{C}) \longrightarrow H_*(X_t, \mathbb{C})$$

defined by a generator of $\pi_1(T', t)$, and analogously an automorphism on the integer singular cohomology

$$h^* \colon H^\bullet(X_t, \mathbb{C}) \longrightarrow H^\bullet(X_t, \mathbb{C})$$

These are usually referred as the *algebraic (complex) monodromy*, or as they were first introduced by Brieskorn [Bri70], the *Picard-Lefschetz monodromy.* This is a topological notion, which depends only on the singularity $(f^{-1}(0), 0)$ and determines its topology to a certain extend.

The following result describes the fundamental structure of the monodromy endomorphism.

Theorem 10 ([RR06], Monodromy)**.** *The operator h_* is quasi-unipotent, that is, there are p and q such that $(h_*^p - id)^q = 0$. In other words, the eigenvalues of the monodromy are roots of unity. Moreover, one can take $q = n + 1$.*

An even more precise description of the characteristic polynomial of the monodromies is given by A'Campo's formula. For that, we must first introduce the monodromy zeta function.

Definition 19. *Let $f \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ be a non-constant polynomial and fix a point $t \in \{f = 0\}$. Let X_t denote the Milnor fiber of f at t, and $P_i(s)$ the characteristic polynomial of the monodromy h^i acting on $H^i(X_t, \mathbb{C})$ for $i = 1, \ldots, n$. The* monodromy zeta function *of f at a is the alternating product*

$$\zeta_t(s) \coloneqq \prod_{i=0}^{n} P_i(s)^{(-1)^{i+1}} = \frac{P_1(s) \cdot P_3(s) \cdot \ldots}{P_0(s) \cdot P_2(s) \cdot \ldots}$$

Even more, A'Campo's formula allows for a more explicit description for the characteristic polynomial of the (only) monodromy h^n on an isolated singularity.

Theorem 11 ([ACa75], Thm. 4)**.** *Let $f \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ be a non-constant polynomial with an isolated singularity at t, and choose an embedded resolution $\pi \colon Y \to \mathbb{A}^{n+1}_{\mathbb{C}}$ of $\{f = 0\}$. Then,*

$$P_n(s) = \begin{cases} (s-1)\zeta_t(s) = (s-1) \displaystyle\prod_{j \in S_e} \left(s^{N_j} - 1\right)^{-\chi\left(E_j^\circ\right)}, & n+1 \text{ even} \\ \dfrac{1}{(s-1)\zeta_t(s)} = \dfrac{1}{s-1} \displaystyle\prod_{j \in S_e} \left(s^{N_j} - 1\right)^{\chi\left(E_j^\circ\right)}, & n+1 \text{ odd} \end{cases}$$

In particular, this allows to compute the eigenvalues of the monodromy after finding a resolution.

8.3 Conjecture

The monodromy conjecture is an open problem in the field of theory of singularities, which remains unsolved since its formulation by the Japanese mathematician Igusa in the late seventies. Thanks to some of the examples computed, he conjectured that, for s_0 a pole of the zeta function, $e^{2\pi\imath\Re(s_0)}$ resulted in a monodromy eigenvalue. Even more, on a stronger statement, that s_0 itself would be a root of the Bernstein-Sato polynomial.

The conjecture was originally stated in terms of the p-adic zeta function, but has been then analogously stated in the context of the topological function, as well as the motivic zeta function. In the topological setting, the conjecture can be stated as follows.

Conjecture 1 (Monodromy conjecture, topological version)**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial. If s_0 is a pole of $Z_{\mathrm{top}}(f, \varphi; s)$, then*

1. (standard) $e^{2\pi\imath\Re(s_0)}$ *is a monodromy eigenvalue of $f\colon \mathbb{C}^n \to \mathbb{C}$ at some point of $\{f = 0\}$.*
2. (strong) s_0 *is a root of the Bernstein-Sato polynomial b_f.*

Of course, the strong version implies the standard one, on the account of Corollary 1, and there is also a local version of the conjecture. A priori, there is no reason at all to expect these implications, except the possible analogy with the Archimedean case.

Remark 2. On a side note, the strong version may even be formulated more precisely as stating that $b_f(s) \cdot Z_{\mathrm{top}}(f, \varphi; s)$ is a polynomial. In other words, that the order of a pole of the topological zeta function is at most its multiplicity as a root of the Bernstein-Sato polynomial (see [Vey24, Rmk. 4.6]).

In the next section, we introduce the main tools that have been used to study the conjectured relation.

9 Periods of integrals

The goal of this section is to introduce the study of periods of certain integrals, introduced by Malgrange in [Mal73; Mal74]. With the study of these objects, we may transform the monodromy action in terms of asymptotics of integrals, which to this day is the main approach to prove the monodromy conjecture for the known cases.

We have already introduced the Milnor fibration, and the monodromy automorphism h, given by the action of $\pi_1(T', 1)$ on the homology group $H_n(X_1, \mathbb{C})$, where we are fixing $t = 1 \in T'$. We also know, by the monodromy Theorem 10 that the eigenvalues of h are roots of unity, and that the orders of the zeros of the minimal polynomial of h have order at most $n + 1$.

We now define the *bounded homology* as the cycles that are locally constant sections of $H_n(X_1, \mathbb{C})$. Also, we define the *vanishing homology* as cycles $\gamma(t)$ that are bounded and additionally $\gamma(t) \to 0$ as $t \to 0$. It is easy to check that the bounded homology H_n^b and the vanishing homology $\widetilde{H}_n$ groups are stable under the monodromy automorphism h.

Then, choose a class of homology $\gamma(1) \in H_n(X_1, \mathbb{C})$, and from it we deduce a class $\gamma(t) \in H_n(X_t, \mathbb{C})$ for t in a neighborhood of 1, thanks to the fact that $X' \to T'$ is a fibration. By continuation, one obtains a multivalued function (that is, a function defined on the universal covering of T'), which following the usual abuse of notation we will still denote by $\gamma(t)$ after a required choice of a branch of $\log t$.

Fix λ an eigenvalue of h, and suppose that $\gamma(1)$ is a generalized eigenvector of height p, that is

$$(h - \lambda)^p \gamma(1) = 0, \qquad (h - \lambda)^{p-1} \gamma(1) \neq 0$$

Now consider $\pi \in \Omega^n$, a differential form of degree n in $\mathrm{d}x_1, \dots, \mathrm{d}x_{n+1}$ with coefficients in the ring $\mathbb{C}[x_1, \dots, x_{n+1}]$. For $t \in T'$, the restriction $\pi\,|_{X_t}$ is a holomorphic form of maximal degree, thus closed. In particular, we can define

$$I(t) := \int_{\gamma(t)} \pi$$

Note that the action of the monodromy h translates to the integral $I(t)$ by the substitution $\log t \mapsto \log t + 2\pi i$. Malgrange states (see [Mal73, p. 4-5]) that $I(t)$ is a holomorphic (multivalued) function on T', and differentiation is given by

$$I'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma(t)} \pi = \int_{\gamma(t)} \frac{\mathrm{d}\pi}{\mathrm{d}f}$$

with $\frac{\mathrm{d}\pi}{\mathrm{d}f}$ being the usual relative form. More elementary, one can think of it as a family of forms over the fibers X_t for $t \in T'$, defined as follows. Over the points where $\{\frac{\mathrm{d}f}{\mathrm{d}x_i} \neq 0\}$, we can define $\frac{\mathrm{d}\pi}{\mathrm{d}f}(t)$ as the restriction to X_t of

$$\frac{\mathrm{d}\pi}{\mathrm{d}f}(t) = \left((-1)^{i-1} \left(\frac{\mathrm{d}f}{\mathrm{d}x_i} \right)^{-1} g \; \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \right)$$

where $\mathrm{d}\pi = g \; \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1}$.

It is now of interest to study the asymptotic behavior of this expression once we approach the origin. First, Deligne notes in [DM86] that there always exists an integer $k > 0$ such that, for some constant $C > 0$, on every angle $|\arg t| \leq C$ we have

$$\lim_{t \to 0} t^k I(t) = 0$$

If we additionally assume that γ is bounded, Malgrange gives a stronger result (see [Mal73, Prop. 3.1]), by saying that

$$\exists \lim_{t \to 0} I(t)$$

and furthermore if γ is vanishing, then this limit is zero.

Next, we can consider global sections on the sheaf of relative differentials such that the associated cohomology classes form a basis of the vector space where we defined the Gauss-Manin meromorphic connection. By computing the integrals of these forms, Malgrange shows in [Mal74, p. 413] that we have a system of meromorphic differential equations. Thus, by classical theory we may find an expression for the integral $I(t) = \sum c_{\mu,q}(\pi) \, t^\mu (\log t)^q$ for the basis elements, and extend it to any π by linearity.

If, moreover, we still consider a vanishing cycle γ satisfying $(h - \lambda)^p \gamma(1) = 0$, with p minimal, then by the monodromy Theorem 10 we can cut the terms at level p, so that we have the following expression

$$I(t) = \sum_{\substack{\mu \in L(\lambda) \\ 0 \leq q \leq p-1}} c_{\mu,q}(\pi) \, t^\mu (\log t)^q \tag{9.1}$$

where the coefficients $c_{\mu,q}$ are complex numbers. Also, we introduce $L(\lambda)$ the set of values $\mu > 0$ such that $e^{2\pi\imath\mu} = \lambda$. Recall that the eigenvalues of monodromy are roots of unity (again by Theorem 10), so that such μ are rational numbers.

As a remark, if γ is not vanishing, then the limit $\lim_{t \to 0} I(t) \neq 0$ is not necessarily zero, so a constant term should be added to the expression.

With the obtained expression, we immediately get an expression to compute the derivative of the considered integral. Indeed, consider $\omega \in \Omega^{n+1}$, and notice that there exists $\pi \in \Omega^n$ with $\mathrm{d}\pi = \omega$, on the account of Poincaré's lemma. Then, we have

$$I'(t) = \int_{\gamma(t)} \frac{\mathrm{d}\pi}{\mathrm{d}f} = \int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\substack{\mu \in L(\lambda)-1 \\ 0 \leq q \leq p-1}} d_{\mu,q}(\omega) \, t^\mu (\log t)^q$$

where we obtain some new coefficients $d_{\mu,q}$. Also notice that we have to subtract one from the values in the set $L(\lambda)$, which can be seen by taking the derivative of a single term in the sum

$$\frac{\mathrm{d}}{\mathrm{d}t} t^\mu (\log t)^q = q t^{(\mu-1)} (\log t)^{(q-1)} + \mu t^{(\mu-1)} (\log t)^q$$

Next, we are ready to define the following quantities. Take γ a vanishing cycle with the described properties then, for $1 \leq k \leq p$, let

$$\mu_k := \inf\{\mu \in L(\lambda) - 1 \mid \exists q \geq k-1, \omega \in \Omega^{n+1} \text{ with } d_{\mu,q}(\omega) \neq 0\} \tag{9.2}$$

Malgrange shows that this is well-defined (see [Mal73, p. 5]), as there exists ω and μ such that $d_{\mu,p-1} \neq 0$. Notice also that, since $L(\lambda)$ is defined by values $\mu > 0$, this implies that $\mu_k \in [-1, 0)$.

Moreover, we arrive at the following result, which ultimately gives us the relation between the asymptotics of the integral presented with the roots of the Bernstein-Sato polynomial.

Proposition 2 ([Mal73], Prop. 3.3)**.** *The polynomial* $(s+\mu_1)\dots(s+\mu_p)$ *divides* $\tilde{b}$.

The proof distinguishes the cases whether $\lambda = 1$ or not, but the reasoning is almost the same, with the case of equality requiring a bit more work. In the case $\lambda \neq 1$, it basically shows that

$$b(s)\int_0^1 t^s \,\mathrm{d}t \int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \int_0^1 t^{s+1}\,\mathrm{d}t \int_{\gamma(t)} \frac{P^*\omega}{\mathrm{d}f} + \int_{\gamma(1)} \omega_P \tag{9.3}$$

where b is the Bernstein-Sato polynomial, P^* is the adjoint operator of the differential operator appearing in the Bernstein-Sato functional equation, and $\omega_P \in \Omega^n[s]$ some suitable differential form.

Then, it supposes that $\mu_1 = \mu_2 = \cdots = \mu_k < \mu_{k+1}$, and it argues that the integral

$$\int_0^1 t^s \,\mathrm{d}t \int_{\gamma(t)} \frac{\omega}{\mathrm{d}f}, \quad \Re(s) > 1$$

can be continued to a meromorphic function in $s \in \mathbb{C}$, having $-\mu_1$ as a pole of order k. Moreover, the other terms can also be continued to a meromorphic function in $s \in \mathbb{C}$ without a pole in $-\mu_1$ and to a polynomial in s, respectively. Altogether, equality in (9.3) implies that $-\mu_1$ is a zero of order k of b. Proceeding similarly for the other values, the result follows.

Altogether, the discussion above shows what is the main approach to solve the monodromy conjecture in the known cases. Indeed, it only remains to show in each situation that there actually exists some non-zero homology class on $H_n(X_t, \mathbb{C})$. Since the pairing between homology and cohomology

$$H^n(X_t, \mathbb{C}) \times H_n(X_t, \mathbb{C}) \to \mathbb{C}$$

is non-degenerate, it is also sufficient to show that a certain multivalued differential form defines a non-zero cohomology class.

This is exactly the key step in the proof given by Loeser in [Loe88] for the case of Newton non-degenerate case, described in Proposition 11, and which is actually just a toroidal version of the result in general dimension, by Esnault and Viehwg [EV92]. Also relevantly, it is because of this result that the extra hypothesis on the residue numbers, mentioned in the introduction, are required.

This approach is also possible for the case of plane curves, with the cohomological result given here by Deligne and Mostow in [DM86, Prop. 2.14]. In this case, this is complemented by determining which divisors contribute to poles in the zeta function, through the study of the dual graph as described by Loeser in [Loe90].

Chapter III

Plane Curves

1 Definition and basic invariants

1.1 Generalities

We think of a plane curve as a set of points with coordinates (x, y) in the complex plane, described as the zero locus of some equation $f(x, y) = 0$. Usually, we will consider f to be simply a polynomial, but it is sometimes useful to work with convergent power series, formal series, or even fractional power series.

Remark 3. Already from elementary examples, the need to work on a bigger space than that of polynomials arises naturally. Indeed, if we are looking for a parametrization of already some simple looking curves, we encounter that we may require fractional powers

$$f(x, y) = x^2 - y^3 = 0 \qquad \Longrightarrow \qquad y = x^{2/3}$$

or to consider infinite series

$$f(x, y) = x + y - xy \qquad \Longrightarrow \qquad y = -\sum_{r\geq 1} x^r$$

The introduction of fractional powers leads to the introduction of the field of Puiseux series, which we will see is an algebraically closed field.

However, thanks to the following result, we can always consider that our curve is defined by a polynomial equation, which in practice is easier to work with.

Theorem 12 (Weierstrass' Preparation Theorem)**.** *Let $G \in \mathbb{C}\{x, y\}$ be regular of order s in y, that is $f(0, y) = y^s \tilde{f}(y)$ where $\tilde{f}(0) \neq 0$. Then, there exist $U \in \mathbb{C}\{x, y\}$ with non-zero constant term and $A_r \in \mathbb{C}\{x\}$ for $0 \leq r < s$ such that*

$$G(x, y) = U(x, y) \left\{ y^s + \sum_{r=0}^{s-1} A_r(x) y^r \right\}$$

Also relevant, this result serves to prove that $\mathbb{C}\{x, y\}$ and $\mathbb{C}[\![x, y]\!]$ are unique factorization domains.

By the above comments, we formalize the notion of a curve as follows. Namely, we have that a holomorphic function $f \in \mathbb{C}\{x, y\}$ with $f(0, 0) = 0$ determines a curve $C\colon f = 0$, and the converse is also true if we identify $f = 0, g = 0$ as the same curve whenever $g = Uf$ for some unit $U \in \mathbb{C}\{x, y\}$.

1.2 Branches, multiplicity and intersection

Then, each curve can be decomposed in a unique way as a finite union of branches, each of them having a distinct tangent direction at the point considered, each of them given by a parametrization $(x, y) = (\phi(t), \varphi(t))$. The branches composing the curve appear when considering the unique factorization (up to units) of

$$f = \prod g_j^{a_j}$$

with g_j being distinct factors and not admitting any non-trivial factorizations, and the a_j positive integers. By denoting $C\colon f = 0$ and $B_j\colon g_j = 0$, we may also write it in terms of curves as

$$C = \sum a_j B_j$$

We will say that the curve C is *reduced* if $a_j = 1$ for all indices. Next, we introduce the following definition.

Definition 20 (Multiplicity). *Let $C\colon f(x,y) = 0$ be a curve with $f \in \mathbb{C}\{x,y\}$. The multiplicity of f at point O is the order of f, and we denote it by* $\mathrm{mult}_O(C)$.

The multiplicity of a curve is well-defined, and it equals the sum of multiplicities of its branches, that is $\mathrm{mult}(C) = \sum a_j \mathrm{mult}(B_j)$. In particular, we will say that a curve is smooth at a point if, and only if, the multiplicity there is 1.

Then, we may define a *tangent line* to C at O as any of the linear factors of the homogeneous polynomial in x and y obtained by taking the terms of lower degree $n = \mathrm{mult}_O(C)$ in the series expansion of f.

Now, we must introduce the following definition, which is the type of parametrization that will allow us to correctly define the notion of intersection number.

Definition 21 (Good parametrization). *We say that a parametrization $(x,y) = (\phi(t), \varphi(t))$ of a plane curve is* good *if the map $t \mapsto (\phi(t), \varphi(t))$ is injective for $|t| < \varepsilon$ with $\varepsilon \ll 1$ small enough.*

For example, the cusp $x^2 - y^3$ might be parametrized by $(x,y) = (t^3, t^2)$, and this is a good parametrization. However, although also a valid alternative, $(x,y) = (u^6, u^4)$ won't be injective close to the origin. A good parametrization ensures that a general point of the curve corresponds to just one value of the parameter. Furthermore, from a good parametrization of a branch, we can recover a defining equation (see, for example, [EN85, p. 56-58]).

We now would like to establish the basic definitions of intersection numbers of two curves, which will allow us to quantify how *glued* together are two curves meeting at a common point. For Γ_1, Γ_2 curves, we will denote its intersection number at point P as $[\Gamma_1 \cdot \Gamma_2]_P$, but if the context is clear we may simply write $\Gamma_1 \cdot \Gamma_2$.

We begin by defining that, for smooth curves intersecting with distinct tangents at the origin, the intersection number at the origin is 1. In general, the idea to calculate the intersection for any curves is to apply small deformations to one (or both if required) of the curves so that we may obtain new intersection points, but each of them is smooth with different tangents. The most basic example is to take $\Gamma_1\colon y = 0$, $\Gamma_2\colon y - x^2 = 0$, a parabola intersecting the y-axis at the origin. Our intuition already suggests us that the intersection number should be 2. Indeed, if we deform the first curve to $\Gamma_1'\colon y = \varepsilon$, then the intersection points are now $\Gamma_1' \cap \Gamma_2 = \{(-\sqrt{\varepsilon}, \varepsilon), (\sqrt{\varepsilon}, \varepsilon)\}$, and at each of them the curves are smooth with distinct tangents.

The procedure to calculate this intersection number in the general case is to adapt our coordinates to be in the situation presented for the double point of the parabola, where one of the curves is simply $y = 0$ and we ought to calculate the vanishing order of the other one. More precisely, assuming $\Gamma_1\colon g(x,y) = 0$ and $\Gamma_2\colon (x,y) = (\phi(t), \varphi(t))$ is a good parametrization such that $\phi(0) = \varphi(0) = 0$, then we have

$$[\Gamma_1 \cdot \Gamma_2]_O = \mathrm{ord}_O\, g(\phi(t), \varphi(t))$$

The intersection number is well-defined, and it satisfies some decent properties (see [Wal04, Lemma 1.2.1]). For example, it is a positive integer, and the definition is symmetric, which can be seen from the purely algebraic definition of this quantity, take $\Gamma_1\colon f = 0$ and $\Gamma_2\colon g = 0$ then

$$[\Gamma_1 \cdot \Gamma_2]_O = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f,g)}$$

Finally, we state a useful property that will be key when computing intersection numbers of the divisors of a blow-up, as we will be able to 'interchange a pull-back by a push-forward' and compute the intersection in a simpler setup.

Lemma 2. *Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map around the origin with $F(O) = O$, let $\gamma_2\colon \mathbb{C} \to \mathbb{C}^2$ be a good parametrization with $\gamma(0) = O$ of a curve Γ_2 such that $F \circ \gamma_2$ is a good parametrization of $F(\Gamma_2)$. Also, let g_1 be the equation of a curve Γ_1, and we define a curve $F^{-1}\Gamma_1$ by the equation $g_1 \circ F$. Then $[\Gamma_1 \cdot F\Gamma_2] = [F^{-1}\Gamma_1 \cdot \Gamma_2]$.*

1.3 Puiseux's theorem

The theorem of Puiseux classically states the following.

Theorem 13 (Puiseux)**.** *Any equation $f(x,y)=0$, where $f \in \mathbb{C}[\![x,y]\!]$ is a series with zero constant term, admits at least one solution in formal power series of the form*

$$x = t^n, \qquad y = \sum_{r=1}^{\infty} a_r t^r, \qquad n \in \mathbb{N}$$

The history of the procedure involved in finding such a solution goes back to Euler's *rotating ruler*, described in his correspondences around 1676 (see [Tur+59, pp. 20-42, 110–163]), while Puiseux's own paper [Pui50] dates back to 1850. The algorithm consists in calculating successive terms in the power series until yielding a solution, although Newton didn't originally discuss its convergence.

The procedure requires introducing the Newton polygon associated to the polynomial $f = \sum_{r,s} a_{r,s} x^r y^s$, which consists in marking on the plane the points (r,s) such that $a_{r,s} \neq 0$ and taking the region above and to the right from each point. Although we will give a more formal definition of this object in the section of Newton non-degenerate polynomials, where we extend it to any dimension, we can already include an illustrating example next.

Example 7. Consider the polynomial $f = x^7 - x^6 + 2x^5y^2 + x^4y^4 + \pi x^3 y + x^2y^2 - exy^4 + y^5$. Then, the associated Newton polygon is depicted in the following Figure III.1. The exponents appearing in the expression of f are the points in red:

$$(7,0),(6,0),(5,2),(4,4),(3,1),(2,2),(1,4),(0,5)$$

and the blue segments are the ones relevant for the resolution. The interior of the polygon is painted in green, and should be thought as extending indefinitely to the right and up.

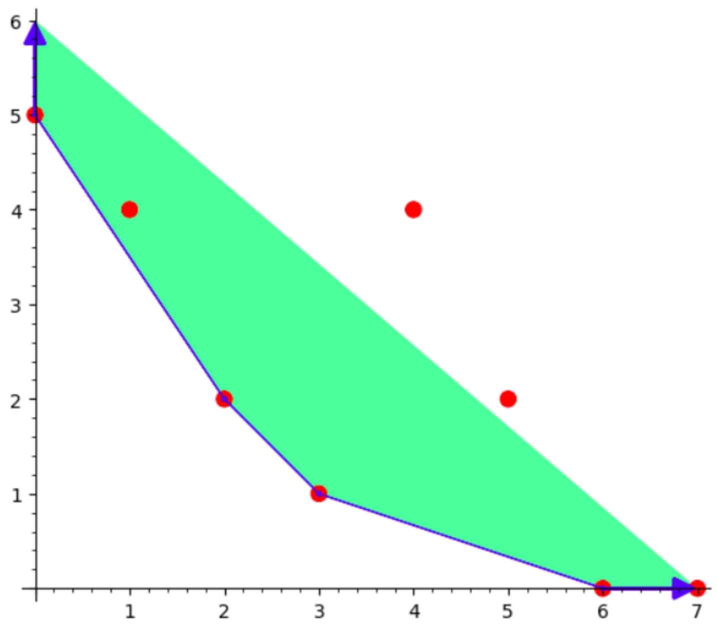


Figure III.1: Newton polygon of the polynomial $f = x^7 - x^6 + 2x^5y^2 + x^4y^4 + \pi x^3 y + x^2y^2 - exy^4 + y^5$.

More relevant, the union of the resulting segments not lying on the coordinate axis contain information about this resolution of $f = 0$, as described in [Wal04, Thm. 2.1.1].

Back to finding solutions, it can be shown that if the defining equation $f(x,y) = 0$ is holomorphic (so, a polynomial), then the resulting solutions are convergent power series. This is clear if the order of $f(0,y)$ is 1, thanks to the implicit function theorem, since this would imply the non-vanishing of $\partial f/\partial y$ at the origin.

For the general case, a more geometric approach is possible. However, we simply summarize the discussion in the following results, while noting that the solutions obtained through this argument are precisely the same as in the algorithmic procedure via the Newton polygon.

Theorem 14 ([Wal04], Thm. 2.2.6)**.**

(i) Any equation $f(x,y)=0$ where $f \in \mathbb{C}\{x,y\}$ with $f(0,0)=0$, $f(0,y)$ not identically zero, admits at least one solution of the form $y=g(x^{1/m_i})$ with $g \in \mathbb{C}\{z\}$.

(ii) If f is regular of order m in y, and we write $f=UF$ with U a unit and F a monic polynomial of degree m in y, there are m such solutions $g_j(x^{1/m_J})$, all distinct (unless the discriminant of F vanishes identically) and

$$F(y)=\prod_{j=1}^{m}\left(y-g_j(x^{1/m_j})\right)$$

Furthermore, the m solutions are partitioned into groups admitting a common parametrization $(x,y)=(t^{m_j},\sum_{r=1}^{\infty}a_{j,r}t^r)$, and this defines a *branch* of the curve $f(x,y)=0$. In general, it is easier to describe the structure of a single branch, as geometrically it can be regarded as a single solution of the given equation.

Altogether, we might understand the Puiseux' Theorem as stating that 'every equation has a root', but in a particular field that we now describe. Formally, for each $n \in \mathbb{N}$ introduce the variables x_n, subject to the relations $x_{mn}^m = x_n$ for all $m,n \in \mathbb{N}$. More intuitively, we should think of x_n as representing the fractional power $x^{1/n}$. Then, define the ring

$$\mathbb{P}[\![x]\!] \coloneqq \bigcup_{n\in\mathbb{N}} \mathbb{C}[\![x_n]\!]$$

and we make it into a field by adjoining the inverse of x, that is

$$\mathbb{P}[\![x]\!][x^{-1}]$$

And analogously, we could construct $\mathbb{P}\{x\}[x^{-1}]$. With this objects defined, we can state the theorem discussed as follows.

Theorem 15 (Puiseux)**.** *The fields $\mathbb{P}[\![x]\!][x^{-1}]$, $\mathbb{P}\{x\}[x^{-1}]$ are algebraically closed.*

1.4 Puiseux exponents and Zariski pairs

Consider a branch B of the germ of a holomorphic function at the origin, not being tangent to the y-axis. Then, we can choose a good parametrization

$$x=t^n, \quad y=\sum_{r\geq n} a_r t^r$$

In particular, imposing the injectivity implies that the greatest common factor of n and the indices r such that $a_r \neq 0$ is 1. This leads to the definition of the following parameters. First, let

$$\beta_1 = \min\{k \mid a_k \neq 0, n \nmid k\}, \qquad e_1 = \gcd(n,\beta_1)$$

Then, for all other indices $i>1$, let

$$\beta_i = \min\{k \mid a_k \neq 0, e_{i-1} \nmid k\}, \qquad e_i = \gcd(e_{i-1},\beta_i)$$

Also, as discussed, eventually we reach an index g with $e_g=1$, because the parametrization is assumed to be good. Then, we call the *Puixeux characteristic exponents* of B to the sequence

$$(n;\beta_1,\ldots,\beta_g)$$

In particular, for any Puiseux series $y(x)$ of f, we can write it as

$$y(x) = \sum_{\substack{1\leq j<\beta_1 \\ e_0|j}} a_j x^{j/n} + \sum_{\substack{\beta_1\leq j<\beta_2 \\ e_1|j}} a_j x^{j/n} + \cdots + \sum_{\substack{\beta_{g-1}\leq j<\beta_g \\ e_{g-1}|j}} a_j x^{j/n} + \sum_{j\geq\beta_g} a_j x^{j/n}$$

It turns out that these exponents are independent of the choice of coordinates, and also contain deep information about the branch. Also, by an analytical change of coordinates, if needed, we may assume that they are ordered increasingly $n < \beta_1 < \cdots < \beta_g$.

Now, we will express these quantities differently, as follows. Write $n_i = \frac{e_{i-1}}{e_i}$ for $i = 1, \ldots g$, so that we have $e_{i-1} = n_i n_{i+1} \cdots n_g$ after writing for for convenience put $n_0 = 0$. Also, by writing $e_0 = n$ we have $n = n_1 \cdots n_g$ too. Then, defining $m_i = \beta_i/e_i$, we may write

$$\left(\frac{\beta_1}{n}, \ldots, \frac{\beta_g}{n}\right) = \left(\frac{m_1}{n_1}, \frac{m_2}{n_1 n_2}, \ldots, \frac{m_g}{n_1 \cdots n_g}\right)$$

and we call the right quantities the *reduced characteristic exponents.*

Lastly, we introduce the *Zariski pairs* (q_i, n_i) for $i = 1, \ldots, g$, which are rather called *Puiseux pairs* in some literature, but we will avoid this term to avoid possible confusions. These quantities are defined by the Euclidean division

$$q_i = m_i - n_i m_{i-1}, \qquad i = 1, \ldots g$$

where, by convention, we take $m_0 = 0$ so that $q_1 = m_1$. Notice that $\gcd(q_i, n_i) = 1$ too, and that we can write the same Puiseux series $y(x)$ of f as

$$\begin{aligned} y(x) = & \sum_{1\leq j\leq\lfloor q_1/n_1\rfloor} a_{0,j} x^j \\ & + \sum_{0\leq j\leq\lfloor q_2/n_2\rfloor} a_{1,j} x^{(q_1+j)/n_1} \\ & + \sum_{0\leq j\leq\lfloor q_3/n_3\rfloor} a_{2,j} x^{q_1/n_1 + (q_2+j)/n_1 n_2} \\ & \vdots \\ & + \sum_{j\geq 0} a_{g,j} x^{q_1/n_1 + q_2/n_1 n_2 + \cdots + (q_g+j)/n_1\cdots n_g} \end{aligned}$$

Example 8 (Puiseux exponents and Zariski pairs)**.** Consider the curve $y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = 0$, whose Newton polygon consists of a single edge joining the points $(6, 0)$ and $(0, 4)$. B taking a resolution (via one's preferred method), we can find that a good parametrization is given by

$$(x, y) = (t^4, t^6 + t^7)$$

Then, in this case the Puiseux characteristic exponents are $(4; 6, 7)$. We can obtain the reduced quantities and write

$$\left(\frac{6}{4}, \frac{7}{4}\right) = \left(\frac{3}{2}, \frac{7}{2\cdot 2}\right)$$

Thus, we read $m_1 = 3, m_2 = 7$ and also $n_1 = 2, n_2 = 2$. With these, we compute $q_1 = m_1 = 3$ and $q_2 = m_2 - n_2 m_1 = 7 - 2\cdot 3 = 1$, so the corresponding Zariski pairs as $\{(3, 2), (1, 2)\}$.

1.5 Semigroup, conductor and the Milnor number

The Puiseux characteristic exponents, and related information, allow for a very neat description of the semigroup of a plane branch. The *semigroup* of the branch determined by f is defined by

$$\Gamma := \{\operatorname{ord}(g) \in \mathbb{Z}_{\geq 0} \mid \bar{g} \in \mathcal{O}_f \setminus \{0\}\}$$

where $\mathcal{O}_f$ is the ring obtained by localizing at f. More easily, we can compute the values in this set as the intersection number of our branch with different curves through the origin not containing the initial branch as a component (see, for example, [Wal04, Lemma 4.3.1]). Hence, we have that the semigroup contains the values $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,x}/(g,f)$.

Being $\mathbb{Z}_{\geq 0} \setminus \Gamma$ finite implies that we can find a set of minimal generators of the semigroup, that is to take $\bar{\beta}_i$ the minimum such that

$$\bar{\beta}_i \notin \langle \bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_{i-1} \rangle, \qquad \bar{\beta}_0 < \bar{\beta}_1 < \cdots < \bar{\beta}_i, \qquad \gcd(\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_i) = 1$$

Moreover, these values generating the semigroup arise as the intersection number of curves which are called *maximal contact elements*.

Again by [Zar06, §II.3], starting with $\bar{\beta}_0 = n, \bar{\beta}_1 = \beta_1$, the rest of the values can be obtained via the Puiseux information recursively

$$\bar{\beta}_i = n_{i-1}\bar{\beta}_{i-1} - \beta_{i-1} + \beta_i = e_i(n_i n_{i-1} \bar{m}_i + q_i), \qquad i = 2, \ldots, g$$

where we introduce $\bar{m}_i = \bar{\beta}_i / e_i$ the *reduced semigroup generators*, which are coprime to n_i. Also, notice that $\gcd(e_{i-1}, \bar{\beta}_i) = e_i$ and $e_{i-1} \nmid \bar{\beta}_i$.

On the other hand, since f is irreducible, it can be shown that there exists a minimum integer c such that every integer greater or equal to c belongs to the semigroup. We call this value $c(\Gamma)$ the *conductor*, and it can be computed from the Puiseux pairs via [Zar06, §II.3]

$$c(\Gamma) = n_g \bar{\beta}_g - \beta_g - (n-1)$$

To end the section, we present an expression to calculate the Milnor number of a single branch via the quantities we just introduced, (see [Wal04, Prop. 6.3.2]), that is

$$\mu = e_{g-1}\overline{\beta}_g - \beta_g - m + 1 = c(\Gamma) + 1$$

Then, we can extend it to compute the Milnor number for any curve, by using the following result.

Theorem 16 ([Wal04], Thm. 6.5.1)**.** *We have $\mu(C \cap C') = \mu(C) + \mu(C') + 2[C \cdot C'] - 1$. Hence, if C has irreducible components $B_1, \ldots, B_r$, we have*

$$\mu(C) = \sum_{i=1}^{r} \mu(B_i) + 2 \sum_{i<j} [B_i \cdot B_j] - r + 1$$

2 Resolution of plane curve singularities

2.1 Blowups

In the case of plane curves, we give a more explicit description of the blowup. Suppose we are given a curve C in a smooth algebraic surface S, and we want to resolve the singularity at a point P, which will be the center of the blowup[1]. Namely, we are looking to construct a new surface T and a suitable map $\pi \colon T \to S$ such that π gives an isomorphism between $T \setminus E \to S \setminus \{P\}$, where E is the curve given by $\pi^{-1}(P)$. The idea is that E is basically a projective line, which will parametrize the possible tangent directions of curves through that point.

To ease the notation and the description, we will consider local coordinates (x, y) on a neighborhood U of S around the point of interest P, so that it is expressed locally as $P = (0, 0)$. Now, consider the projective line $\mathbb{P}_1$ with coordinates $(\xi : \eta)$, and define the surface T as the subspace of $\mathbb{C}^2 \times \mathbb{P}_1$ satisfying

$$x\eta = y\xi$$

and the associated map $\pi \colon T \to S$ the projection to $\mathbb{C}^2$ composed with the biholomorphic mapping from neighborhood of the origin in $\mathbb{C}^2$ to the neighborhood U of P.

It is clear that any point different from the origin has a unique preimage, as the solution to $(\xi \colon \eta) = (x \colon y)$ is unique. However, $E = \pi^{-1}(P)$ is an entire projective line, that is $E \cong \mathbb{P}_1$ and we call it the

[1] In a more general setting, the blowup can be defined with center along any subvariety.

exceptional curve.

Next, recall that $\mathbb{P}_1$ can be covered by two affine coordinate charts $U_0\colon \xi \neq 0$ and $U_1\colon \eta \neq 0$. Hence, we may introduce the following coordinates X, Y as

$$\begin{cases} U_0\colon \xi \neq 0 & \implies & Y := \eta/\xi & \implies & V_0\colon \{y = xY\} \subset \mathbb{C}^2 \times \mathbb{P}_1 \\ U_1\colon \eta \neq 0 & \implies & X := \xi/\eta & \implies & V_1\colon \{x = Xy\} \subset \mathbb{C}^2 \times \mathbb{P}_1 \end{cases}$$

so that T can be covered by the coordinate charts V_0, V_1. More relevantly, this is the coordinates that one ought to keep in mind when doing the actual computations, as we will simply write $Y = y/x$ and $X = x/y$, each whenever we are in the appropriate chart.

From the geometric perspective, we have already mentioned the exceptional curve, which is the projective line appearing in the preimage of the center point. Additionally, we introduce the *total transform* of C as the pull-back of C by the map π, defined by the equation $\pi^* f$, and we write it by $\overline{C} = \pi^*(C)$. Also, we define the *strict transform* of C as the curve obtained by removing the component $n_P(C)E$ from the total transform $\overline{C}$, and we denote it by $\widetilde{C} = \overline{C} \setminus n_P(C)E$, where $n_P(C)$ is the multiplicity of the curve C at the center point P.

The blowup as described is intrinsically well-defined, and we now give some basic properties. First, if the curve C is smooth at the center point P, then the strict transform $\widetilde{C}$ meets the exceptional divisor E at a single point, with transverse intersection. Even more, $\widetilde{C} \cong C$ and the strict transform is still smooth. For a general curve, the intersection multiplicity of $\widetilde{C}$ with E of the blowup is equal to the multiplicity of C at the center point, which can be seen by analyzing the effect of a blowup on a local parametrization.

Also, it is important to study the geometry of the various exceptional divisors appearing, which is described by the following result.

Proposition 3 ([Wal04], Prop. 3.4.3)**.** *The exceptional curve $E_i \subset T_{i+1}$ intersects E_{i-1} and at most one curve E_j with $j < i-1$. All the intersections are transverse, and no three curves E_i pass through a common point.*

As we briefly mentioned in the general setting, such a configuration of curves (smooth, with transverse intersections and no three meeting in a point) is said to have *normal crossings.* A resolution π is said to be *good* if it gives an isomorphism between $T \setminus E \to S \setminus \{P\}$, where $E = \pi^{-1}(P)$ is a collection of exceptional curves, and the total transform $\pi^{-1}(C)$ has normal crossings.

For any plane curve, we only need to iterate the process of blowing up a finite number N of times to obtain $C^{(N)}$ smooth (see [Wal04, Thm. 3.3.1]). Furthermore, this process can be continued to obtain a good resolution. If we take the minimal N such that this is the case, we will say that the resolution is *minimal.*

We may write $\pi\colon T \to S$ as the final composition, where $\pi = \pi_0 \circ \pi_1 \cdots \circ \pi_{N-1}$ and $\pi_i\colon T_{i+1} \to T_i$ for $0 \leq i < N$, and we write $T_N = T$ and $T_0 = S$. Each step is a single blowup with center O_i, from which arises the exceptional curve E_i in the surface T_{i+1}. Technically, we should write $E_i^{(k)}$ to refer to the exceptional when lying in T_k, and similarly for the strict transforms $C^{(k)}$ of C.

Example 9 (Plane curve resolution via blowups)**.** Consider the polynomial $f = x^4 - y^5 + x^2y^2$. We now include the resolution via composition of blowups, where will be considering the two possible charts at each step.

For the first blowup, we start with $x^4 - y^5 + x^2y^2$ to obtain

$$0^{\text{th}} \text{ blowup} \begin{cases} (x,y) \mapsto (x_1, x_1y_1) & \implies & x_1^4 - x_1^5y_1^5 + x_1^2x_1^2y_1^2 = x_1^4(1 - x_1y_1^5 + y_1^2) \\ \\ (x,y) \mapsto (x_1y_1, y_1) & \implies & x_1^4y_1^4 - y_1^5 + x_1^2y_1^2y_1^2 = y_1^4(x_1^4 - y_1 + x_1^2) \end{cases}$$

Now, the first chart is not interesting, because the strict transform $1 - x_1y_1^5 + y_1^2$ does not go through the origin. Thus, from here we are not able to see the singular point, and we should perform a translation before continuing the blowup process. Instead, we choose the second chart, which introduces the divisor

$$4E_0\colon 4\{y_1 = 0\}$$

and we see on the total transform that the order of vanishing along E_0 is $N_0 = 4$. Also, we can find the effect on the standard volume form by calculating the Jacobian determinant of the map $(x_1, y_1) \mapsto (x/y, y)$

$$\begin{vmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial y_1}{\partial x} \\ \frac{\partial x_1}{\partial y} & \frac{\partial y_1}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & 0 \\ \frac{x}{y^2} & 1 \end{vmatrix} = \frac{1}{y} = \frac{1}{y_1}$$

so that the order of vanishing along E_0 is 1, thus $k_0 = 2$.

For the second blowup, we start with $y_1^4(x_1^4 - y_1 + x_1^2)$ to obtain

$$1^{\text{st}} \text{ blowup } \begin{cases} (x_1, y_1) \mapsto (x_2, x_2 y_2) \\ \qquad \implies \quad x_2^4 y_2^4 (x_2^4 - x_2 y_2 + x_2^2) = x_2^4 y_2^4 \left(x_2(x_2^3 - y_2 + x_2)\right) = x_2^5 y_2^4 (x_2^3 - y_2 + x_2) \\ (x_1, y_1) \mapsto (x_2 y_2, y_2) \\ \qquad \implies \quad y_2^4 (x_2^4 y_2^4 - y_2 - x_2^2 y_2^2) = y_2^4 \left(y_2(x_2^4 y_2^3 - 1 + x_2^2 y_2)\right) = y_2^5 (x_2^4 y_2^3 - 1 + x_2^2 y_2) \end{cases}$$

Here, it is the second chart that is not interesting. Instead, we choose the first chart, which has introduced the divisor

$$E_1 \colon \{x_2 = 0\}$$

and we can see on the total transform that the order of vanishing along E_1 is $N_1 = 5$. Also, we can find the effect on the standard volume form by calculating the Jacobian determinant of the composition of $(x_2, y_2) \mapsto (x_1, y_1/x_1)$ with the previous map, that is

$$\begin{vmatrix} \frac{\partial x_2}{\partial x} & \frac{\partial y_2}{\partial x} \\ \frac{\partial x_2}{\partial y} & \frac{\partial y_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial y_1}{\partial x} \\ \frac{\partial x_1}{\partial y} & \frac{\partial y_1}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -\frac{y_1}{x_1^2} \\ 0 & \frac{1}{x_1} \end{vmatrix} \frac{1}{y_1} = \frac{1}{x_1 y_1} = \frac{1}{x_2^2 y_2}$$

so that the order of vanishing along E_1 is 2, thus $k_1 = 3$.

For the third blowup, we start with $x_2^5 y_2^4 (x_2^3 - y_2 + x_2)$ to obtain

$$2^{\text{nd}} \text{ blowup } \begin{cases} (x_2, y_2) \mapsto (x_3, x_3 y_3) \\ \qquad \implies \quad x_3^5 x_3^4 y_3^4 (x_3^3 - x_3 y_3 + x_3) = x_3^9 y_3^4 \left(x_3(x_3^2 - y_3 + 1)\right) = x_3^{10} y_3^4 (x_3^2 - y_3 + 1) \\ (x_2, y_2) \mapsto (x_3 y_3, y_3) \\ \qquad \implies \quad x_3^5 y_3^5 y_3^4 (x_3^3 y_3^3 - y_3 + x_3 y_3) = x_3^5 y_3^9 \left(y_3(x_3^3 y_3^2 - 1 + x_3)\right) = x_3^5 y_3^{10} (x_3^3 y_3^2 - 1 + x_3) \end{cases}$$

Here, in both charts we have reached a strict transform that is smooth (notice they both have a monomial with exponent 1, so it is impossible to have both partial derivatives equal to 0). Let us consider the first chart, which will be easier to describe, since geometrically is simply a parabola. This chart has introduced the divisor

$$E_2 \colon \{x_3 = 0\}$$

and we can see on the total transform that the order of vanishing along E_2 is $N_2 = 10$. Also, we can find the effect on the standard volume from by calculating the Jacobian determinant of the composition of $(x_3, y_3) \mapsto (x_2, y_2/x_2)$ with the two previous maps, that is

$$\begin{vmatrix} \frac{\partial x_3}{\partial x} & \frac{\partial y_3}{\partial x} \\ \frac{\partial x_3}{\partial y} & \frac{\partial y_3}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_3}{\partial x_3} & \frac{\partial y_3}{\partial x_2} \\ \frac{\partial x_3}{\partial y_2} & \frac{\partial y_3}{\partial y_2} \end{vmatrix} \frac{1}{x_2^2 y_2} = \begin{vmatrix} 1 & -\frac{y_2}{x_2^2} \\ 0 & \frac{1}{x_2} \end{vmatrix} \frac{1}{x_2^2 y_2} = \frac{1}{x_2^4 y_2} = \frac{1}{x_3^3 y_3}$$

so that the order of vanishing along E_2 is 4, thus $k_2 = 5$.

Altogether, the resolution would be given by the composition $\pi_0 \circ \pi_1 \circ \pi_2$, with the change of coordinates acting as

$$(x/y, y^3/x^2) \xleftarrow{\pi_0} (x_1, y_1/x_1^2) \xleftarrow{\pi_1} (x_2, y_2/x_2) \xleftarrow{\pi_2} (x_3, y_3)$$
$$(x, y) \xmapsto{\pi_0^{-1}} (x_1 y_1, y_1) \xmapsto{\pi_1^{-1}} (x_2^2 y_2, x_2 y_2) \xmapsto{\pi_2^{-1}} (x_3^3 y_3, x_3^2 y_3)$$

Finally, we can represent pictorially the situation with the divisors in the following Figure III.2.

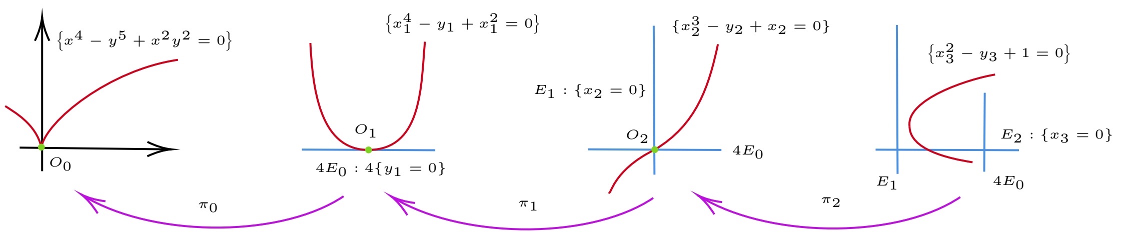


Figure III.2: Pictorial representation of the blowups of the resolution of one branch.

With this representation, it is clear that we have obtained a smooth curve already with the zeroth blowup. However, we additionally want to obtain normal crossings between the divisors. For instance, the strict transform is tangent to the exceptional divisor E_0, and hence we require another blowup. Then, we perform one last blowup to separate the the triple intersection point of the strict transform with the divisors E_0 and E_1.

2.2 Proximity relation and recurrences

We now focus on the case of a single branch B, and consider a resolution as a composition of blowups, following the notation introduced above.

First, we define an *infinitely near point* of first order to $O = O_0 \in \mathbb{C}^2$ as any point on the curve $E_0 \subset T_1$, in other words, that is mapped to O_0 under π_0. More generally, a point of $E_{r-1} \subset T_r$ is an infinitely near point of r^{th} order to O_0.

Note that O_i is always proximate to O_{i-1}, since we are choosing the center of the next blowup point along the last exceptional divisor obtained. The relations of proximity cannot be arbitrary, and are rather limited by the following conditions.

Proposition 4 ([Wal04], Prop. 3.5.1)**.** *Let O_i be the centers of the blowups of a resolution, then*

(i) For each i, there is at most one value $j < i-1$ such that O_i is proximate to O_j.

(ii) If O_i is proximate to O_j, then O_k is proximate to O_j for all intermediate indices $j < k < i$.

Geometrically, this is clear. Indeed, for the first condition recall that a divisor E_i intersects E_{i-1} and at most one other exceptional divisor E_j with $j < i-1$, by Proposition 3. Then, for the second, think that O_i being mapped to O_j implies

$$\pi_j \circ \cdots \circ \pi_{i-1} \circ \pi_i(O_i) = O_j \qquad \Longrightarrow \qquad \pi_j \circ \cdots \circ \pi_{i-1}(O_{i-1}) = O_j$$

and so on for all intermediate indices.

The proximity relations may be summarized in the form of a matrix, which we will call the *proximity matrix* of the branch. Namely, denote by $P(B)$ the $N \times N$ matrix with entries given by

$$p_{i,j} := \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } O_j \text{ is proximate to } O_i \text{ (necesssarily } i < j), \\ 0 & \text{otherwise.} \end{cases}$$

Next, we may use these relations to establish some recurrence expressions for the multiplicities and the numerical data of the divisors appearing in the resolution of a branch.

First, let us write $\mathrm{mult}_i(B)$ to denote the multiplicity at the point O_i of the strict transform $B^{(i)}$ of B in T_i, that is

$$\mathrm{mult}_i(B) := \mathrm{mult}_{O_i}(B^{(i)}).$$

In particular, $\mathrm{mult}_0(B) = n$ following the notation of the Puiseux information, since it is simply the multiplicity of the original curve B at the first center. Then, the successive values of the sequence can be read by describing the effects of a blowup on the characteristic exponents.

Theorem 17 ([Wal04], Thm. 3.5.5). *In terms of the Puiseux characteristic exponents, the blow up of a curve is given by*

$\beta_1 > 2n$	$(n; \beta_1 - n, \ldots, \beta_g - n)$
$\beta_1 < 2n, (\beta_1 - n) \nmid n$	$(\beta_1 - n; n, \beta_2 - \beta_1 + n, \ldots, \beta_g - \beta_1 + n)$
$(\beta_1 - n) \mid n$	$(\beta_1 - n; \beta_2 - \beta_1 + n, \ldots, \beta_g - \beta_1 + n)$

Hence, the Puiseux characteristic exponents of a branch determines the sequence of multiplicities and vice versa.

On the other hand, we also have the classical result that for a single branch B it holds

$$\mathrm{mult}_i(B) = \sum_{O_j \text{ proximate to } O_i} \mathrm{mult}_j(B)$$

where it should be remarked that the sum runs over points that are 'above' O_i, that is, they are mapped to O_i by $\pi_i \circ \cdots \circ \pi_{j-1} \circ \pi_j$, so in particular $i < j$. In the same spirit, we may find recurrence relations for the numerical data of divisors (see [Wal04, p. 199] for details), described next.

Proposition 5. *The following relations hold for any divisor E_i.*

(i) $N_i = \mathrm{mult}_i(B) + \sum_{O_i \text{ proximate to } O_j} N_j$

(ii) $(k_i - 1) = 1 + \sum_{O_i \text{ proximate to } O_j} (k_j - 1)$

where in this case the sum extends over points that are 'below' O_i, so in particular $i > j$, and in each sum there are at most two terms.

Remark 4. The numerical data N in Example 9 may seem to not satisfy such relations. However, in that case, one ought to keep in mind that the data is accounting for the two branches (which can be seen by plotting the original curve).

2.3 Homology of a blow-up

Now, we study the topology of the resolved surface and the curves obtained after the blowups lying in it.

Lemma 3 ([Wal04], Thm. 8.1.1). *The only non-vanishing (reduced) homology group of T is $H_2(T)$, which is free abelian on N generators, which may be taken as the homology classes $[E_i]$ of the exceptional curves E_i, for $0 \leq i < N$.*

We will use the word *cycle* to denote a formal linear combination (with integer coefficients) of irreducible curves. Any element of $H_2(T)$ given as a linear combination of $[E_i]$ will be called an *exceptional cycle.* Notice that the intersection numbers for cycles are well-defined, as they behave additively for unions of curves.

To begin, we may describe the behavior of a single blow-up. Consider the blowup with center $P \in S$ given by the map $\pi \colon T \to S$, and let E be the exceptional curve. Take C a curve through P defined locally by $f = 0$, and let n be the multiplicity of C (or f) at P. If we denote by $\widetilde{C}$ the strict transform of C, we have

$$\pi^*[C] = [\widetilde{C}] + n[E]$$

To calculate the intersection number of the divisors appearing, we first include the following result (see, for example, [Wal04, Lemma 8.1.2, Lemma 8.1.6]).

Lemma 4. *With the notation above,*

(i) $[E] \cdot [E] = -1$

(ii) $[\widetilde{C}] \cdot [\widetilde{C}] = [C] \cdot [C] - n^2$.

Moreover, for any curves F and G in S it is useful to remark that, thanks to Lemma 2, we can calculate intersection numbers in S or T as convenient since

$$\pi^*[F] \cdot \pi^*[G] = [F] \cdot [G]$$

In particular, if F is any curve through the blowup center, we may obtain that $[E]\cdot\pi^*[F] = \pi_*[E]\cdot[F] = 0$.

Next, we are interested in computing the intersection numbers of the exceptional divisors after the whole resolution. Although we will not include the details, by working with the negative dual basis, we may obtain that the auto-intersection matrix in the basis $\{[E_i]\}$ is given by

$$-P^t \cdot P = \Big([E_i] \cdot [E_j]\Big)_{i,j}$$

We can actually give a lot more information about the entries of such matrix. For instance, notice that for $i \neq j$ the intersection number $[E_i] \cdot [E_j]$ is zero if the divisors do not intersect, and 1 if they do, since we are considering a good resolution with normal crossings.

For the auto-intersection numbers, that is, the elements in the diagonal we have the following.

Lemma 5. *If there are r_j points O_i proximate to O_j, then $[E_j] \cdot [E_j] = -(r_j + 1)$.*

Notice that we have as an immediate consequence of Lemma 4, as the first appearance of $E_j^{(j)}$ has intersection number -1 and this is diminished by 1 each time a point on E_j is blown up.

Notation 1. We will denote by $\kappa_j \coloneqq -[E_j] \cdot [E_j]$ the negative of the *auto-intersection number* of the divisor E_j

2.4 Dual graph

We will now define the dual graph associated to the resolution of a plane curve singularity. Start by considering a curve C with branches B_j. We will take a (usually minimal) good resolution $\pi\colon T \to S$ and write the blowups in the composition as $\pi_i\colon T_{i+1} \to T_i$ for $0 \leq i < N$, with the convention that $T_N = T$ is the final surface, and $T_0 = S$. We will denote by E_i the exceptional curves, and $B_j^{(i)}$ the strict transforms in T_i.

We denote the *dual graph* of the curve C associated to the resolution π as $\Delta_\pi(C)$ and define it as follows. The vertices V_i will correspond to the exceptional divisors E_i, and we will draw an edge V_iV_k whenever the curves E_i and E_k intersect. Additionally, we will add vertices W_j for each strict transform $B_j^{(N)}$, which again will be joined by edges to any exceptional divisors intersecting them. Pictorially, we will represent this special edges with an arrow pointing to the strict transform vertex, and we will refer to them as arrow-head vertices. Moreover, we will cal this extension of the graph the *augmented dual graph*, and denote it by $\Delta_\pi^+(C)$.

We can construct the graph step by step, that is, one blowup at a time. Indeed, if we have the graph up to the curves in T_{i-1} and consider the additional blowup π_i with center O_i, two scenarios are possible.

(i) If O_i is proximate only to O_{i-1}, that is O_i belongs to the exceptional divisor E_{i-1} and not any to any other E_j, then we add a new vertex V_i and a new edge $V_{i-1}V_i$.

(ii) If O_i is proximate to both O_{i-1} and some O_j with $j < i - 1$, then E_{i-1} and E_j intersect in T_i, so there is already an edge $V_{i-1}V_j$. Then, we replace this by two edges $V_{i-1}V_i$ and V_iV_j, which can be understood as subdividing the previous edge $V_{i-1}V_j$ by adding a new vertex V_i.

This way, we can see inductively that the graph is a tree. Furthermore, for an irreducible curve with Puiseux characteristic exponents $(n; \beta_1, \ldots, \beta_g)$, the augmented dual graph consists of a single core chain of edges from the initial vertex to the arrowhead vertex, with g side branches, each a single chain attached to vertices in the core. Pictorially, this is represented in the following Figure III.3.

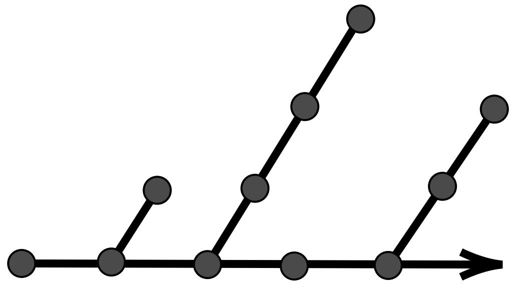


Figure III.3: Typical drawing of an augmented dual graph.

Indeed, one should think of the branching points as the points during the resolution where we are in the situation (ii) described, that is, the center of the blowup is the intersection of two exceptional divisors. Also, for a curve C with several branches, $\Delta_\pi^+(C)$ is in some sense the union of the trees $\Delta_\pi^+(B_j)$ corresponding to the different branches, overlapping as appropriate.

A sequence of proximity relations (or any equivalent information) about the curve clearly determines the abstract structure of the graph. However, the reverse is only possible if we additionally label the vertices with the integer corresponding to its position in the sequences of blowing ups — that is, the vertex V_r associated with the exceptional divisor E_r gets the label r.

Next, we will give an alternative expression to the recurrences given in Proposition 6. Indeed, the proximity relations might be a bit confusing, and it is easier to think about adjacency relations in the dual graph.

Then, the idea is to take the expressions and consider what happens if we compute the intersection number with an arbitrary $[E_i]$. Namely, for any cycle $[C] = \sum_j c_j [E_j]$ we obtain the identity

$$[C] \cdot [E_i] = c_i\, [E_i] \cdot [E_i] + \sum_{j \neq i} c_j\, [E_i] \cdot [E_j]$$

By recalling that the intersection number of two divisors equals 1 if the corresponding vertices are adjacent in the dual graph, and 0 otherwise, we find

$$\kappa_i c_i = -[C] \cdot [E_i] + \sum_{j \in \mathcal{N}(i)} x_j$$

Applying this to the previously mentioned expressions, we obtain the following (see also [Wal04, p. 199]).

Proposition 6. *The following relations hold for any vertex V_i of the dual graph.*

(i) $\kappa_i N_i = \sum_{j \in \mathcal{N}(i)} N_j$

(ii) $\kappa_i k_i = (2 - v_i) + \sum_{j \in \mathcal{N}(i)} k_j$

where the sum runs over $\mathcal{N}(i)$: the vertices adjacent to V_i in the augmented dual graph Δ_π^+.

Alternatively, this can be proved directly (see [Vey24, Lemma 4.1] for details) by computing in two different ways the intersection numbers of E_i with the divisor $\operatorname{div}(\pi^* f) = \sum_j N_j E_j = 0$, and E_i with the relative canonical divisor, plus the adjunction formula.

Example 10 (Dual graph)**.** Let us consider the polynomial $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7$, following Example 8 where we calculated the Puiseux exponents to be $(4; 6, 7)$. In particular, we expect the dual graph to have a core with two branches attached.

We take a minimal good resolution to obtain the proximity matrix and, from it, the autointersection matrix.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad P^t \cdot P = \begin{pmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

Thus, it is immediate to construct the dual graph as in the following Figure III.4. Indeed, the edges are identified with the entries -1, and the diagonal elements give minus the autointersection numbers of each exceptional divisor.

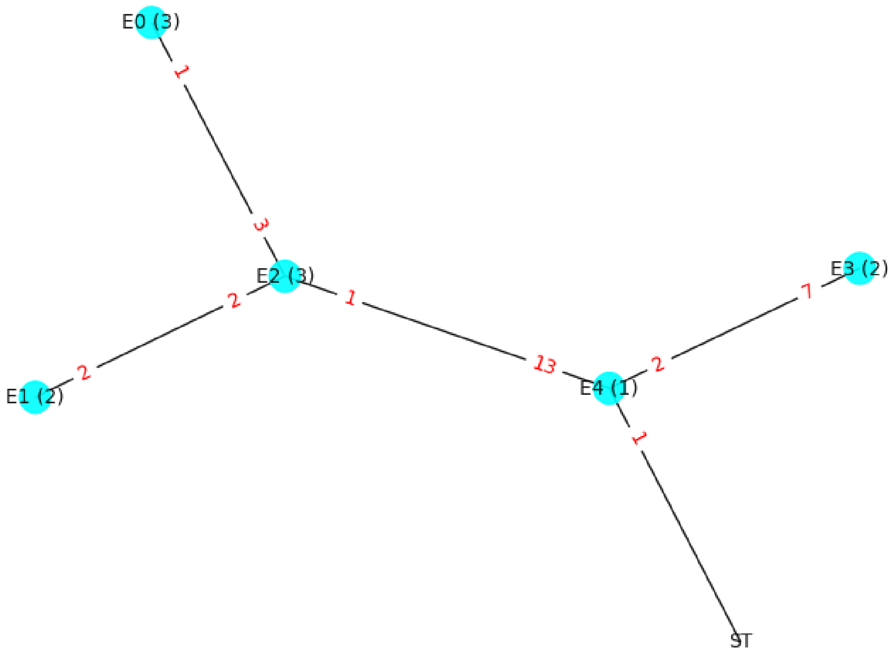


Figure III.4: Dual graph of a minimal resolution of the polynomial $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7$.

The points in blue are the vertices, representing an associated divisor, and the strict transform is the non-colored vertex with the label ST. Furthermore, in parentheses, we include the corresponding autointersection number for the exceptional divisors.

Additionally, the red numbers are the dual graph Eisenbud-Neumann decorations which we will not describe (see [EN85]), but rather mention that they appear in relation to splice diagrams and the alternative proof of a main result required for the bound on the residue numbers of plane curves.

We can also obtain the numerical data of each divisor from the resolution

$$(k_0, N_0) = (2, 4), \quad (k_1, N_1) = (3, 6), \quad (k_2, N_2) = (5, 12), \quad (k_3, N_3) = (6, 13), \quad (k_4, N_4) = (11, 26)$$

and it is an immediate check to verify that both the recurrences from the proximity relations (Proposition 5) and from the dual graph neighbors (Proposition 6) are satisfied.

For example, this task is easier if we write the recurrences in matrix form, taking advantage of the definition of the proximity matrix. For example, the first recurrences are

$$P \cdot \begin{pmatrix} N_0 \\ \vdots \\ N_r \end{pmatrix} = \begin{pmatrix} \mathrm{mult}_0(B) \\ \vdots \\ \mathrm{mult}_r(B) \end{pmatrix}, \qquad P \cdot \begin{pmatrix} k_0 - 1 \\ \vdots \\ k_r - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and indeed

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 12 \\ 13 \\ 26 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2-1 \\ 3-1 \\ 5-1 \\ 6-1 \\ 11-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

On the other hand, the recurrences involving the autointersection numbers can be written as

$$P^tP \cdot \begin{pmatrix} N_0 \\ \vdots \\ N_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad P^tP \cdot \begin{pmatrix} k_0 \\ \vdots \\ k_r \end{pmatrix} = \begin{pmatrix} 2-v_0 \\ \vdots \\ 2-v_r \end{pmatrix}$$

where we are omitting the arrow-head vertex in computing the valence of the divisor that it intersects, and the special last value 1 in the relation for N arises because of the strict transform as well. Finally, we check that indeed

$$\begin{pmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 12 \\ 13 \\ 26 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

3 Topological zeta function of plane curves

Recall the definition of the topological zeta function, given in Definition 5. Now, in the case of plane curves, we can take a good resolution having simple normal crossing divisors. In particular, we can impose that no three divisors intersect at a same point. Hence, we can restrict the summation to subsets I of size 1 or 2. In particular, we may write

$$Z_{\text{top}}(f;s) = \chi\left(\mathbb{C}^2 \setminus \cup_{j\in J} E_j\right) + \sum_{j\in J} \frac{\chi\left(E_j^\circ\right)}{k_j + N_j s} + \sum_{i\neq j\in J} \frac{\chi\left(E_i \cap E_j\right)}{(k_i+N_is)(k_j+N_js)}$$
$$Z_{\text{top},0}(f;s) = \sum_{j\in J} \frac{\chi\left(E_j^\circ\right)}{k_j + N_j s} + \sum_{i\neq j\in J} \frac{\chi\left(E_i \cap E_j\right)}{(k_i+N_is)(k_j+N_js)}$$

Recall that $\chi\left(\mathbb{C}^2 \setminus \cup_{j\in J} E_j\right) = \chi(\mathbb{C}^2 \setminus \{f=0\})$. Also, in the case of plane curves, each divisor E_i is isomorphic to a $\mathbb{P}_1$. Then, E_i° is precisely a projective line with a point removed for each other divisor intersecting it. In particular, this quantity is precisely the degree of the associated vertex V_i in the dual graph, say v_i. Hence, we have $\chi(E_i^\circ) = 2 - v_i$. Furthermore, the intersection of any two divisors is either empty or a single point, in which case $\chi(E_i \cap E_j) = 1$.

Then, for example with the local zeta function, we can write altogether that

$$Z_{\text{top},0}(f;s) = \sum_{j\in V(\Delta_\pi^+)} \frac{2-v_i}{k_j+N_js} + \sum_{(i,j)\in E(\Delta_\pi^+)} \frac{1}{(k_i+N_is)(k_j+N_js)} \tag{3.1}$$

We have already mentioned that the general topological zeta function is independent of the choice of resolution. Moreover, in the case of plane curves, this is notably easy to check. Indeed, thanks to the existence of a minimal resolution for plane curves, it is enough to check that the expression is unchanged after adding a single blowup.

- If we blowup a point lying on a single exceptional curve E_i, then in terms of the dual graph we are attaching a new edge V_iV_l, and notice that the new multiplicity is $\text{mult}_l = 0$. Hence, by the recurrence relations, we have $N_l = N_i$ and $k_l = k_i + 1$. Also, the valence of the vertex V_i increases by 1, and the effect on the zeta function is to add the terms

$$-\frac{1}{k_i + sN_i} + \frac{1}{k_i + 1 + sN_i} + \frac{1}{(k_i + sN_i)(k_i + 1 + sN_i)} = 0$$

- Alternatively, if we blowup the point $E_i \cap E_j$, we introduce a new vertex V_l of valence 2 that subdivides the existing edge V_iV_j. Hence, by the recurrence relations, we have $k_l = k_i + k_j$ and $N_l = N_i + N_j$ since again the multiplicity is $n_l = 0$. Thus, the effect to the zeta function is to add the terms

$$\frac{1}{(k_i + sN_i)(k_l + sN_l)} + \frac{1}{(k_j + sN_j)(k_l + sN_l)} - \frac{1}{(k_i + sN_i)(k_j + sN_j)} = 0$$

3.1 The quotient k/N

Next, to study the final poles of the zeta function, we must study the quotient k/N over the divisors, using the structure of the dual graph and the recurrence relations derived between neighbors in the previous section.

Let V_i be any vertex of Δ_π the dual graph with only one neighbor V_j, in particular with valence $v_i = 1$. Then, by the recurrences in Proposition 6, we have

$$\frac{k_i}{N_i} = \frac{\kappa_i k_i}{\kappa_i N_i} = \frac{1 + k_j}{N_j} > \frac{k_j}{N_j}$$

Else, if V_i has neighbors V_j, V_l, so in particular it has valence $v_i = 2$, we have

$$\frac{k_i}{N_i} = \frac{\kappa_i k_i}{\kappa_i N_i} = \frac{k_j + k_l}{N_j + N_l}$$

and since all coefficients are positive, either

$$\frac{k_{j_1}}{N_{j_2}} = \frac{k_i}{N_i} = \frac{k_{j_2}}{N_{j_2}}, \qquad \text{or} \qquad \frac{k_{j_1}}{N_{j_1}} < \frac{k_i}{N_i} < \frac{k_{j_2}}{N_{j_2}}$$

after possibly reordering of V_{j_1}, V_{j_2}. It also follows that the quantity

$$N_i k_{j_1} - N_{j_1} k_i = N_i k_{j_2} - N_{j_2} k_i$$

is constant along oriented edges $V_iV_{j_1}, V_iV_{j_2}$ where the intermediate vertex V_i has valence 2.

On the other hand, for any edge V_iV_j of the augmented dual graph Δ_π^+, recall the definition of the residue number associated with divisors E_i, E_j as

$$\varepsilon(i,j) = k_j - k_i \frac{N_j}{N_i}$$

and we have the next result.

Lemma 6. *For a vertex V_i of the dual graph, denote by $\mathcal{N}(i)$ its neighbors. Then,* $\sum_{j \in \mathcal{N}(i)} \varepsilon(i,j) = v_i - 2$.

The proof is a simple calculation using the recurrence in Proposition 6

$$\sum_{j \in \mathcal{N}(i)} \varepsilon(i,j) = \sum_{j \in \mathcal{N}(i)} k_j - \frac{k_i}{N_i} \sum_{j \in \mathcal{N}(i)} N_j = v_i - 2 + \kappa_i k_i - \frac{k_i}{N_i} \kappa_i N_i = v_i - 2$$

Alternatively, this can also be shown via the adjunction formula, as seen in [Vey91]. In any case, this is the main lemma required to prove the following bound the residue numbers.

Proposition 7 ([Loe90], Prop. II.3.1)**.** *Let V_i be a vertex of the dual graph. Then, $-1 \leq \varepsilon(i,j) < 1$ for all $j \in \mathcal{N}(i)$. Moreover, equality holds if, and only if, it has valence $v_i = 1$.*

Loeser proves it first for irreducible curves, and then extends it to the general case by showing that the residue numbers are a weighted sum of the residue numbers of the branches. The proof goes through the three possible cases, depending on the valence of the vertex, and using again Proposition 6. Also relevant, notice that the result holds for an arbitrary resolution, since for plane curves we can take a minimal resolution and show that the result still holds after any additional blowup.

Next, to study the possible double poles, we must see what happens for edges where $\varepsilon(i,j)=0$, since then $-k_i/N_i = -k_j/N_j$. The main result is the following.

Lemma 7 ([Wal04], Cor. 8.4.2)**.** *For any vertex V_i of the dual graph, there is at most one neighbor vertex N_j such that $\varepsilon(i,j)\leq 0$, or either the vertex has valence $v_i=2$ and both values vanish $\varepsilon(i,j_1)=\varepsilon(i,j_2)=0$.*

In other words, we may say that either $k_j/N_j > k_i/N_i$ holds for all but at most one neighbor V_j of V_i, or we have equality in the quotients for the two neighbors of V_i. An immediate consequence of this lemma is the following result.

Proposition 8 ([Wal04], Cor. 8.4.3)**.** *Denote by Y the set of vertices of V_i of the dual graph where the quotient k_i/N_i attains the minimum value. Then, Y is a connected chain, where intermediate vertices have valence 2, and every path starting and leaving Y strictly increases the quotient.*

Remark 5. Notice that for all edges V_iV_j contained in Y we have that $\varepsilon(i,j)=0$ and, even more, the converse is also true. Indeed, suppose $N_i/k_i = N_j/k_j$, then for the rest of the neighbors of j we must have $N_j/k_j \leq N_l/k_l$. In the case of equality, we stay in a set of vertices joined by edges with $\varepsilon=0$. Else we find a vertex with bigger quotient $N_j/k_j < N_l/k_l$ and, again by Lemma 7, this quotient cannot decrease for some other neighbor of V_l. Hence, since we can repeat the symmetric argument for the neighbors of V_i, we conclude that both V_i and V_j attain a minimum value of N/k, thus they are contained in Y.

We can give a more natural description of the possible shape of this set of vertices, as depicted in Figure III.5. For example, it could happen that Y is a single vertex, which in this case must be a rupture divisor (this happens, for example, when C is irreducible and we only have one branch). Else, it can be the set of vertices between two consecutive rupture divisors, these included. Lastly, if f is not reduced, these two cases can also happen replacing one of the rupture vertices by an arrowhead vertex, associated to the strict transform.

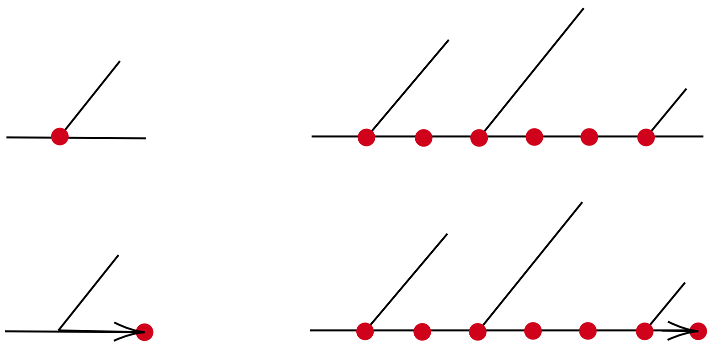


Figure III.5: Possible shape of the set Y, represented in the dual graph by the vertices marked in red.

3.2 Poles

First of all, we have that double poles arise from edges where (k_i+sN_i) and (k_j+sN_j) are proportional. This happens precisely if $k_iN_j - k_jN_i = 0$ or, equivalently, if $\varepsilon(i,j)=0$. Thus, such an edge belongs to the set Y described earlier. Since all factors corresponding to vertices in Y are proportional, we conclude

Theorem 18. *The topological zeta function has at most one double pole, and the candidate value is* $\min\{k_i/N_i\}$.

Also, as an immediate result of the theorem and the description of the set Y, we have the following.

Corollary 2. *If the curve has a single Puiseux characteristic exponent, then the dual graph has just one branch of edges attached to the core. Hence, there is only one rupture divisor and this will be the only vertex in the set Y. Therefore, in this case there is no double pole in the topological zeta function.*

To study which are the simple poles, we may now fix an exceptional divisor E_i, with the associated vertex V_i in the dual graph having valence v_i. Next, we can take the partial fraction decomposition

$$\frac{1}{(k_i+sN_i)(k_j+sN_j)} = \frac{1}{N_ik_j - N_jk_i}\left(\frac{N_i}{k_i+sN_i} - \frac{N_j}{k_j+sN_j}\right) = \frac{1}{\varepsilon(i,j)}\frac{1}{k_i+sN_i} - \frac{1}{\varepsilon(j,i)}\frac{1}{k_j+sN_j}$$

whenever $N_ik_j - N_jk_i \neq 0$ for some neighbor V_j. Then, we can regroup the terms in the expression of the topological zeta function, to find that the contribution of the divisor E_i is given by

$$\frac{1}{k_i+sN_i}\left(2 - v_i + \sum_{j\in\mathcal{N}(i)} \frac{1}{\varepsilon(i,j)}\right) \tag{3.2}$$

Now we analyze the different cases depending on the valence of V_i. If $v_i = 1$, we know that there is only one neighbor and for this one $\varepsilon(i,j) = -1$, thus

$$2 - v_i + \sum_{j\in\mathcal{N}(i)} \frac{1}{\varepsilon(i,j)} = 2 - 1 + \frac{1}{-1} = 0$$

Similarly, this also reduces to zero if $v_i = 2$, since

$$2 - v_i + \sum_{j\in\mathcal{N}(i)} \frac{1}{\varepsilon(i,j)} = 2 - 2 + \frac{\varepsilon(i,j_1)+\varepsilon(i,j_2)}{\varepsilon(i,j_1)\,\varepsilon(i,j_2)} = 0$$

on the account that the numerator is 0 by Lemma 6. Lastly, we see that the only contributing vertices to the zeta function are the ones with valence $v_i = 3$, that is, rupture divisors. Indeed, in this case

$$2 - v_i + \sum_{j\in\mathcal{N}(i)} \frac{1}{\varepsilon(i,j)} = 2 - 3 + \frac{1}{\varepsilon(i,j_1)} + \frac{1}{\varepsilon(i,j_2)} + \frac{1}{\varepsilon(i,j_3)} = \frac{(1-\varepsilon(i,j_1))\,(1-\varepsilon(i,j_2))\,(1-\varepsilon(i,j_3))}{\varepsilon(i,j_1)\,\varepsilon(i,j_2)\,\varepsilon(i,j_3)}$$

which does not vanish since the residue numbers do not take the value 1, as stated in Proposition 7. We conclude the discussion with the following statement.

Theorem 19 ([Wal04], Thm. 8.4.5)**.** *Any pole of the topological zeta function is of the form $s = -k_i/N_i$ for some vertex V_i which is either an arrowhead vertex or a rupture point.*

Example 11 (Topological zeta function of a plane curve)**.** Following Example 10, we compute the topological zeta function of the plane curve defined by $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = 0$, which has Puiseux characteristic exponents $(4; 6, 7)$.

Recall the numerical data of each divisor, summarized in the following Table III.1.

Divisor	k	N	σ	v_i
E_0	2	4	$\frac{1}{2}$	1
E_1	3	6	$\frac{1}{2}$	1
E_2	5	12	$\frac{5}{12}$	3
E_3	6	13	$\frac{6}{13}$	1
E_4	11	26	$\frac{11}{26}$	3
ST	1	1	1	1

Table III.1: Numerical data of a minimal resolution of $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = 0$.

The dual graph is plotted in Figure III.4. Notice that the minimum value $\sigma = k/N$ is achieved only at the rupture divisor $E2$, hence we won't have a double pole.

Now, we know that only rupture divisors, which in this case they are $E2$ with $\mathcal{N}(2) = \{0, 1, 4\}$ and $E4$ with $\mathcal{N}(4) = \{2, 3, ST\}$, and arrowhead vertices contribute to the zeta function. Hence, we begin by computing the associated residue numbers as

$$\varepsilon(2,0) = k_0 - \sigma_2 N_0 = \tfrac{1}{3} \qquad \varepsilon(2,1) = k_1 - \sigma_2 N_1 = \tfrac{1}{2} \qquad \varepsilon(2,4) = k_4 - \sigma_2 N_4 = \tfrac{1}{6}$$
$$\varepsilon(4,2) = k_2 - \sigma_4 N_2 = -\tfrac{1}{13} \qquad \varepsilon(4,3) = k_3 - \sigma_4 N_0 = \tfrac{1}{2} \qquad \varepsilon(4,ST) = 1 - \sigma_4 = \tfrac{15}{26}$$
$$\varepsilon(ST,4) = k_4 - \sigma_{ST} N_4 = -15$$

Then, following Equation (3.2), we find that the contributions of these divisors are

$$E2\colon \frac{1}{5+12s}\left(2-3+\frac{1}{\varepsilon(2,0)}+\frac{1}{\varepsilon(2,1)}+\frac{1}{\varepsilon(2,4)}\right) = \frac{1}{5+12s}\left(-1+3+2+6\right) = \frac{5/6}{s+5/12}$$
$$E4\colon \frac{1}{11+26s}\left(2-3+\frac{1}{\varepsilon(4,2)}+\frac{1}{\varepsilon(4,3)}+\frac{1}{\varepsilon(4,ST)}\right) = \frac{1}{11+26s}\left(-1-13+2+\frac{26}{15}\right) = -\frac{77/195}{s+11/26}$$
$$ST\colon \frac{1}{1+s}\left(2-1+\frac{1}{\varepsilon(ST,4)}\right) = \frac{1}{1+s}\left(1-\frac{1}{15}\right) = \frac{14/15}{s+1}$$

so, altogether

$$Z_0(s) = \frac{5/6}{s+5/12} - \frac{77/195}{s+11/26} + \frac{14/15}{s+1}$$

Alternatively, we can calculate all the terms following Equation (3.1), that is

$$Z_0(s) = \underbrace{\frac{2-1}{2+4s}}_{V_0} + \underbrace{\frac{2-1}{3+6s}}_{V_1} + \underbrace{\frac{2-3}{5+12s}}_{V_2} + \underbrace{\frac{2-1}{6+13s}}_{V_3} + \underbrace{\frac{2-3}{11+26s}}_{V_4} + \underbrace{\frac{2-1}{1+s}}_{ST}$$
$$+\underbrace{\frac{1}{(2+4s)(5+12s)}}_{V_0V_2} + \underbrace{\frac{1}{(3+6s)(5+12s)}}_{V_1V_2} + \underbrace{\frac{1}{(5+12s)(11+26s)}}_{V_2V_4} + \underbrace{\frac{1}{(11+26s)(6+13s)}}_{V_4V_3} + \underbrace{\frac{1}{(11+26s)(1+s)}}_{V_4ST}$$

and after simplifying, we obtain the same result as above.

4 Sketch of the proof of the conjecture

We are now ready to give the proof of the monodromy conjecture for the case of plane curves. The idea is, again, to construct a non-zero cohomology class to apply the arguments in Section 9 of Chapter II. In this case, the result for that purpose is the following.

Proposition 9 ([Loe88], Prop. III.3.2)**.** *If E_i is a rupture divisor, there exists a multivalued horizontal family $\gamma(t)$ of cycles in $H_1(X_t, \mathbb{C})$, $t \neq 0$, such that*

$$\int_{\gamma(t)} \frac{\mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}f} = Ct^{-1+\frac{k_i}{N_i}} + o\left(t^{-1+\frac{k_i}{N_i}}\right)$$

with $C \neq 0$ a constant.

With this, Loeser is able to give a positive answer to the strong version of the conjecture, namely the following result.

Theorem 20 ([Loe88], Thm. III.3.1)**.** *Let $f\colon (\mathbb{C}^2, \mathbf{0}), \to (\mathbb{C}, 0)$ be the germ of an analytic function, and E_i a rupture divisor of the resolution of $\{f = 0\}$. Then*

i) $-k_i/N_i$ is a root of $\tilde{b}_{f,0}$ the local reduced Bernstein-Sato polynomial of f.

ii) If there exists $j \in \mathcal{N}(i)$ (a neighbor of V_i in the dual graph) such that $\varepsilon(i,j) = 0$, then $-k_i/N_i$ is a root of multiplicity two of $\tilde{b}_{f,0}$.

The first part of the theorem was already proved by Lichtin in [Lic89] in the case of irreducible curves, using the fact that the monodromy endomorphism is semi-simple for plane curve singularities (see [Lê 72, Thm 3.3.1]).

Altogether, as we have discussed in the previous section, the key idea in the proof is to study the contributions of the different divisors, which in particular is found to be zero for those vertices that have valence 1 or 2 in the dual graph. This such detailed description of the resolution and the theory of the dual graph, however, does not generalize to higher dimensions.

Because of that reason, alternative proofs of the theorem have been looked for. Namely, the same results of this section introduced by Loeser can be used, but it would be interesting to show in a different manner that the residue numbers are bounded between -1 and 1. Indeed, this is ultimately the required hypothesis for this approach to work. The bound implies that the residue numbers are not integers, so that the monodromies are not the identity and a result like Proposition 9 holds. Furthermore, the case when a residue number is equal to -1 implies that the associated divisor has valence one in the dual graph, so in particular it is not a rupture divisor and we can discard it since it does not contribute to the poles of the zeta function.

With the intention of finding an alternative approach to the bound, Blanco shows in [Bla24] that it can be proved too by obtaining explicit formulae for the residue numbers from the information in the Zariski pairs. More explicitly, it makes use of some technical properties of *splice diagrams*, which are diagrams encoding these values. Also relevant, by the result [EN85, Thm. 20.1] these can be constructed from the dual graph, by adding decorations to the edges (see the red numbers appearing in Figure III.4) and then only keeping edges between rupture vertices or supporting arrowheads.

Chapter IV

Newton non-degenerate

1 Definition and properties

Polynomials which are Newton non-degenerate are sometimes also called non-degenerate with respect to the Newton polytope, or simply non-degenerate. The condition that these polynomials satisfy has a more combinatorial flavor, as it is easier described by considering the objects that we will introduce next.

We consider a polynomial $f(x_1, \ldots, x_n) = \sum_{p \in \mathbb{N}^n} a_p \, x_1^p \ldots x_n^p$ such that $f(\mathbf{0}) = 0$. For brevity, we will use multi-index notation when convenient $f(x) = \sum_{p \in \mathbb{N}^n} a_p \, x^p$, and we define its support to be $\operatorname{supp}(f) = \{p \in \mathbb{N}^n \mid a_p \neq 0\}$.

Definition 22 (Newton polyhedron). *Let $f = \sum_{p \in \mathbb{N}^n} a_p x^p \in \mathbb{C}[x]$ with $f(\mathbf{0}) = 0$. We define the* global Newton polyhedron $\Gamma_{gl}(f)$ *of f as the convex hull of* $\operatorname{supp}(f)$*. Also, we define the* local Newton polyhedron $\Gamma(f)$ *as the convex hull of the set*

$$\bigcup_{p \in \operatorname{supp}(f)} p + (\mathbb{R}_{\geq 0})^n$$

In particular, it is immediate that $\Gamma(f) = \Gamma_{gl}(f) + (\mathbb{R}_{\geq 0})^n$.

We will use the term *face* of $\Gamma(f)$ to refer to any convex subset τ that can be obtained by intersecting the Newton diagram with a hyperplane H of $\mathbb{R}^n$ such that $\Gamma(f)$ is contained in one of the half-spaces defined by H. Note that we also consider the total polyhedron as a face.

Definition 23 (Dimension). *We define the* dimension of a face τ *as the dimension of the subspace spanned by τ. In particular, we call* vertices *to the faces of dimension 0.*

Moreover, we introduce for a face $\tau \subset \Gamma(f)$ the truncation

$$f^\tau := \sum_{p \in \tau \cap \operatorname{supp}(f)} a_p x^p$$

which is a quasi-homogeneous polynomial for any proper face.

Example 12 (Newton polytope and truncations). Let us consider the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$, and we plot in the following Figure IV.1 its Newton diagram.

The points in red are the coordinates corresponding to the exponents of all the monomials appearing in the polynomial. Then, after taking the convex hull of

$$\left\{(0,2) + (\mathbb{R}_{\geq 0})^2\right\} \cup \left\{(1,1) + (\mathbb{R}_{\geq 0})^2\right\} \cup \left\{(2,1) + (\mathbb{R}_{\geq 0})^2\right\} \cup \left\{(3,0) + (\mathbb{R}_{\geq 0})^2\right\}$$

we obtain the Newton polytope, depicted in color green. The corresponding faces, and the associated truncation of the polynomial, are

- Faces of dimension $\dim \tau = 0$:

$$\begin{aligned} \tau_1 &= \{(0,2)\} & f^{\tau_1} &= -y^2 \\ \tau_2 &= \{(1,1)\} & f^{\tau_2} &= 4xy \\ \tau_3 &= \{(3,0)\} & f^{\tau_3} &= x^3 \end{aligned}$$

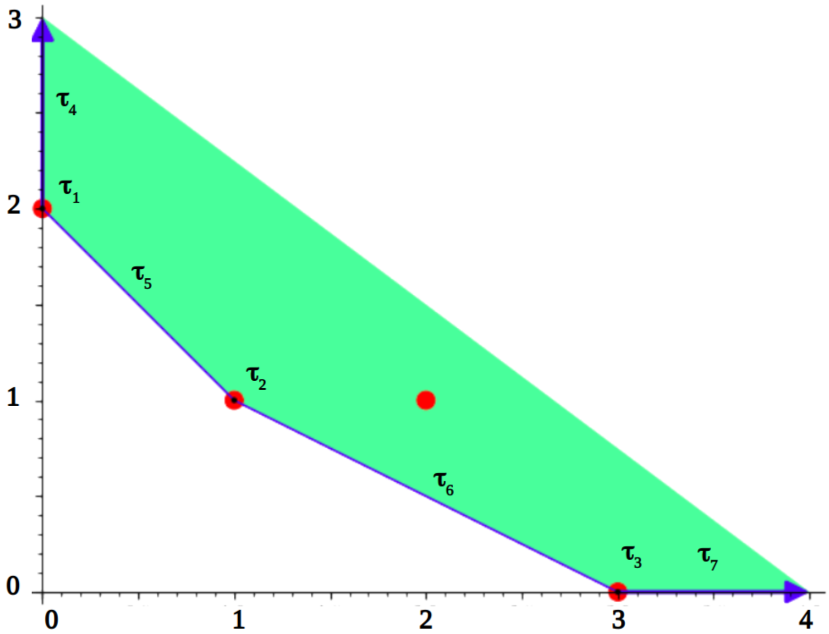


Figure IV.1: Newton polygon of the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$.

- Faces of dimension $\dim \tau = 1$:

$$
\begin{aligned}
&\tau_4 = \{(0,2) + \mathbb{R}_{>0}(0,1)\} && f^{\tau_4} = -y^2 \\
&\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} && f^{\tau_5} = -y^2 + 4xy \\
&\tau_6 = \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} && f^{\tau_6} = x^3 + 4xy \\
&\tau_7 = \{(3,0) + \mathbb{R}_{>0}(1,0)\} && f^{\tau_7} = x^3
\end{aligned}
$$

Definition 24 (Convenient)**.** *A polynomial $f \in \mathbb{C}[x]$ such that $f(\mathbf{0}) = 0$ is called* convenient *if its Newton diagram $\Gamma(f)$ meets all the coordinate axes. That is, if for each variable x_i, $1 \leq i \leq n$, there is a monomial $x_i^{n_i}$ appearing in the expression of f with a non-zero coefficient, for some $n_i \in \mathbb{N}$.*

Sometimes, this condition is also called *comfortable*, which has a more intuitive meaning: the Newton diagram in this case rests nicely in the coordinate planes, i.e. it is in a comfortable position. This is the direct translation of the French word used originally in the paper by Kouchnirenko [Kou76], which is the main reference in introducing the concept of Newton non-degeneracy.

Definition 25 (Non-degenerate)**.** *We say that f is Newton non-degenerate at 0 if for any face $\tau \subset \Gamma(f)$, the hypersurface $f^\tau = 0$ satisfies the condition*

$$x_1 \frac{\partial f^\tau}{\partial x_1} = \cdots = x_n \frac{\partial f^\tau}{\partial x_n} = 0 \quad \Longrightarrow \quad x_1 \dots x_n = 0$$

that is, the polynomials $x_i \frac{\partial f^\tau}{\partial x_i}$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

In other words, we are demanding that the zero locus of the truncation of the polynomial in each of the faces does not have singularities in $(\mathbb{C} \setminus 0)^n$.

Definition 26. *Let $\Gamma(f)$ be the Newton diagram of f as defined. For $a \in (\mathbb{R}^+)^n$, we define the function*

$$N(a) \coloneqq \inf_{x \in \Gamma(f)} \{\langle a, x \rangle\}$$

It is easy to show that the minimum is reached, and in particular it is reached on the set of vertices of the diagram. In particular, we may write

$$N(a) = \min_{p \in \operatorname{supp}(f)} \{\langle a, p\rangle\} = \min_{\substack{x \text{ vertex} \\ \text{of } \Gamma(f)}} \{\langle a, x\rangle\}$$

Although we will not require it now, we fix the following notation.

Notation 2. For $a \in (\mathbb{R}^+)^n$, we denote the sum of its components by

$$k(a) := \sum_{i=1}^{n} a_i = \langle a, (1, \ldots, 1)\rangle$$

Now, we may recover faces of the Newton diagram by considering the *first meet locus* of a given vector $a \in (\mathbb{R}^+)^n$, that is

$$F(a) := \{x \in \Gamma(f) \mid \langle a, x\rangle = N(a)\}$$

which is a proper face of $\Gamma(f)$ if $a \neq 0$, and $F(0)$ recovers the whole diagram.

Now we are ready to define the dual cone associated to a face of the Newton diagram.

Definition 27 (Associated cone)**.** *For τ a face of $\Gamma(f)$, we define the* cone associated *to τ as*

$$\Delta_\tau := \{a \in (\mathbb{R}_{>0})^n \mid F(a) = \tau\}/\sim$$

where the equivalence relation is given by

$$a \sim a' \iff F(a) = F(a')$$

that is, if two vectors are related if they define the same first meet locus.

Additionally, we will refer to the collection of these cones for all faces of the Newton polytope as the *dual fan.* Notice that in the case $n = 2$, the rays of a fan completely determine the cones, but this is not the case for higher dimensions.

We want to study the geometry of the cones in the dual fan, but first we must make two important remarks. Given that the Newton diagram is a polyhedron, we know that for every proper face τ

$$\tau = \bigcap_{\substack{\tau \subset \gamma \\ \dim \gamma = n-1}} \gamma$$

and that only a finite number of terms appear in the intersection.

Moreover, for every face of $\Gamma(f)$ of codimension 1, there exists a unique integral primitive vector (meaning that all of its coordinates are relatively coprime) in $\mathbb{N}^n \setminus \{\mathbf{0}\}$ perpendicular to the face. As usual, we will refer to this vector as the normal vector of the face.

Notation 3. By an abuse of notation, for a face τ of codimension 1 with associated integral primitive normal vector $a = a(\tau)$ (equivalently, $\tau = F(a)$), we will write whenever convenient

$$k(\tau) := k(a), \qquad N(\tau) := N(a)$$

Lemma 8. *Let τ be a proper face of $\Gamma(f)$ and $\gamma_1, \ldots, \gamma_r$ the faces of $\Gamma(f)$ of dimension $n-1$ that contain it. Let $a_1, \ldots, a_r$ be the unique normal vectors to $\gamma_1, \ldots, \gamma_r$, respectively. Then,*

$$\Delta_\tau = \{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_{>0}\}$$

with $\dim \Delta_\tau = n - \dim \tau$.

Example 13 (Dual fan)**.** Following Example 12, consider $f = x^3 - y^2 + 4xy + 3x^2y$, and we will describe its associated dual fan. First, we compute the normal vector of each proper face of codimension 1.

- $\tau_4 = \{(0,2) + \mathbb{R}_{>0}(0,1)\} \implies a_4 = (1,0)$
- $\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} \implies a_5 = (1,1)$
- $\tau_6 = \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} \implies a_6 = (1,2)$
- $\tau_7 = \{(3,0) + \mathbb{R}_{>0}(1,0)\} \implies a_7 = (0,1)$

Then, by Lemma 8, we can describe the cones associated to each face

$$\Delta_{\tau_1} = \mathbb{R}_{>0}(1,0) + \mathbb{R}_{>0}(1,1), \qquad \Delta_{\tau_2} = \mathbb{R}_{>0}(1,1) + \mathbb{R}_{>0}(1,2), \qquad \Delta_{\tau_3} = \mathbb{R}_{>0}(1,2) + \mathbb{R}_{>0}(0,1),$$
$$\Delta_{\tau_4} = \mathbb{R}_{>0}(1,0), \qquad \Delta_{\tau_5} = \mathbb{R}_{>0}(1,1), \qquad \Delta_{\tau_6} = \mathbb{R}_{>0}(1,2), \qquad \Delta_{\tau_7} = \mathbb{R}_{>0}(0,1)$$

Altogether, the associated dual fan is depicted in the following Figure IV.2.

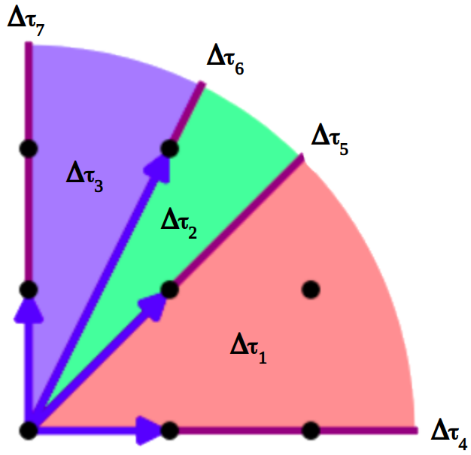


Figure IV.2: Dual fan of the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$.

2 Resolution of NND singularities

To introduce the description of the resolution of singularities in the case of Newton non-degenerate, we must first review some geometric concepts such as the notion of a cone, as well as regular simplicial subdivisions of fans. Finding such a subdivision for the dual fan of the Newton polygon is what ultimately will describe the resolution.

First, we have already seen the definition of the cone associated to a face, but we can write more generally the following.

Definition 28 (Cone)**.** *A* convex polyhedral cone*, or* cone *for short, is a set*

$$C = \{\lambda_1 v_1 + \cdots + \lambda_s v_r \in V \mid \lambda_i \geq 0\} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$$

where V is an n-dimensional vector space over $\mathbb{R}$, and the vectors $\{v_i\}$ are called the generators *of the cone. The dimension of C is defined to be the dimension of the smallest vector space containing it.*

We want now to add adjectives to the definition of a cone, referring to some conditions that we must require later.

Definition 29 (Simplicial, rational)**.** *We say that a cone, as defined above, is* simplicial *if its generating vectors $v_1, \ldots, v_r$ are linearly independent over $\mathbb{R}$. Moreover, we will say it is* simplicial rational *if on top of that, $v_1, \ldots, v_r \in \mathbb{Z}^n$.*

Definition 30 (Regular)**.** *We say that a cone, as defined above, is* regular *if $\{v_1, \ldots, v_r\}$ is a subset of a base of the $\mathbb{Z}$-module $\mathbb{Z}^n$. Equivalently, we can extend the set to have n integral vectors $\{v_1, \ldots, v_r, u_{r+1}, \ldots, u_n\}$ such that the matrix defined by these vectors has determinant 1.*

This condition is also referred to as being *simple*. Even though we shouldn't expect all cones to be this nice, the following result guarantees us that we can partition and subdivide (that is, to add new rays to) our cone in such a way that this holds.

Theorem 21. *Let Δ be a cone generated by vectors $v_1, \dots, v_r \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. There exists a finite partition of Δ in cones Δ_i, such that each cone is generated by a subset of linearly independent vectors of $\{v_1, \dots, v_r\}$. Moreover, if Δ is simplicial rational, a partition in regular cones can be obtained by introducing suitable new generating rays.*

We will not give a proof of this result (see [Kem+06, p. 32-25]), but rather give the basic of idea of the steps required in this procedure. For that, we may introduce the following concept.

Definition 31 (Multiplicity)**.** *Let $v_1, \dots, v_r \in \mathbb{Z}^n$ be linearly independent vectors over $\mathbb{R}$. We define the* multiplicity *of the cone they generate as the index of the lattice $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r$ in the group of integer points of the real vector space spanned by $\{v_1, \dots, v_r\}$.*

It is easy to verify the following properties, which give a more easy to compute description of the multiplicity.

Proposition 10 ([DH01], Prop. 2.11)**.** *Let $v_1, \dots, v_r \in \mathbb{Z}^n$ be linearly independent vectors over $\mathbb{R}$. Then*

1. *The multiplicity of $v_1, \dots, v_r$ equals the number of integer points contained in the set*

$$\left\{ \sum_{i=1}^{r} \lambda_i v_i \mid 0 \leq \lambda_i < 1 \right\}$$

2. *Let A be the matrix whose columns are the given vectors $v_1, \dots, v_r$. Then, their multiplicity equals the product of the elements in the diagonal of the Smith form of A.*

3. *The multiplicity of $v_1, \dots, v_r$ is 1 if, and only if, $\{v_1, \dots, v_r\}$ is a regular cone.*

Now, we are ready to describe the algorithm to subdivide a fan into a regular simplicial fan, while maintaining the total union of its cones (see also [AVG12, §8.2.2]).

The procedure consists of two stages, firstly breaking down the original cones arbitrarily into simplicial cones. Secondly, we must reduce the multiplicities of the maximum dimension cones down to 1, and the way to do it is with a double induction: on the number of cones with such maximum multiplicity and on the value of the maximum multiplicity.

The simplicial cones are broken by adding one-dimensional cones, that is, a new ray. In particular, if we find rational numbers $\alpha_1, \dots, \alpha_r \in (0, 1)$ such that

$$v = \alpha_1 v_1 + \cdots + \alpha_r v_r \quad \in \mathbb{Z}^n$$

is a primitive integer vector, and then we replace the previously existing cone C generated by $\{v_1, \dots, v_r\}$ by the cones C_i generated by $\{v_1, \dots, \widehat{v_i}, \dots, v_r, v\}$, whose multiplicity will satisfy (see [Kem+06, Lemma 3]) that

$$\operatorname{mult} C_i = \alpha_i \operatorname{mult} C$$

We call an inscribed fan a subdivision of the original fan. Moreover, we say that a fan is subordinate to a Newton polyhedron if it is a subdivision of the fan defined by the dual cones of its faces. Altogether, we summarize the previous discussion in the following result.

Theorem 22 ([AVG12], Lemma 8.7)**.** *There exists a regular fan subordinate to a Newton polyhedron.*

Notice, however, that we haven't claimed anything about uniqueness, as there can exist multiple valid subdivisions in simplicial regular cones.

Back to resolution of singularities, we have already mentioned that one can be obtained by an appropriate composition of blowups. In particular, this process is easily described in the case of plane curves, as discussed in the previous chapter. Now, we will see that for the case of Newton non-degenerate singularities, we can obtain the details of a resolution through the information contained in the dual fan of the Newton polygon.

Nonetheless, what we will obtain is rather a toroidal resolution, and for that we must introduce the concept of a toric blowup. For that, we will follow the exposition in [Oka96], see also [Mac06] for a more detailed approach to this precise topic. For a more complete treatment of toroidal varieties, we refer the interested reader to [Ful93; CLS11].

Usually, the process of composition of blowups is long, and the information at each step might not be easy to read. Instead, a toroidal resolution contains information perfectly suitable to determine the usual invariants, for example those introduced for plane curves. We now briefly introduce the definition of such a resolution and its main properties.

Definition 32 (Toric blowup)**.** *Consider a unimodular integral $n \times n$ matrix σ*

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n} \end{pmatrix}$$

We define the toric blowup *(or* modification*) associated to σ as the birational morphism*

$$\begin{aligned} \pi_\sigma \colon (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^n \\ (x_1, \ldots, x_n) &\mapsto (x_1^{\sigma_{1,1}} x_2^{\sigma_{1,2}} \cdots x_n^{\sigma_{1,n}}, \ldots, x_1^{\sigma_{n,1}} x_2^{\sigma_{n,2}} \cdots x_n^{\sigma_{n,n}}) \end{aligned}$$

If $\sigma_{1,j} \geq 0$ for all $j \geq 0$, this map can be extended to $x_1 = 0$, and analogously for the other variables. Since we are requiring that the matrices have determinant 1, we have

$$\pi_\sigma \circ \pi_\tau = \pi_{\sigma\tau}, \qquad (\pi_\sigma)^{-1} = \pi_{\sigma^{-1}}$$

We have already seen that any fan Σ can be subdivided to a regular (also called simple) fan Σ^*. In particular, if we have a regular simplicial cone in Σ^* of maximum dimension given by vectors $\{r_1, \ldots, r_n\}$, we can consider the matrix

$$\sigma = \begin{pmatrix} | & \cdots & | \\ r_1 & \cdots & r_n \\ | & \cdots & | \end{pmatrix}$$

and associate to it the birational map $\pi_\sigma : \mathbb{C}^n_\sigma \to \mathbb{C}^n$, which we ought to think about as one of the different charts of the resolution. With that, we construct a non-singular variety X as the quotient of the disjoint union $\sqcup_\sigma \mathbb{C}_\sigma$ over all the regular cones with the following identification. Two points $x \in \mathbb{C}^n_\sigma$ and $y \in \mathbb{C}^n_\tau$ are identified if, and only if, the birational map $\pi_{\tau^{-1}\sigma}$ is defined at the point x and $\pi_{\tau^{-1}\sigma}(x) = y$.

It can be verified that X is non-singular, and the maps $\{\pi_\sigma \colon \mathbb{C}^n_\sigma \to \mathbb{C}^n \mid \sigma \text{ regular simplicial cone}\}$ glue into a proper analytic map $\pi \colon X \to \mathbb{C}^n$.

Definition 33 (Associated toric blowup)**.** *The map $\pi \colon X \to \mathbb{C}^n$ is called the* toric blowup *(or modification)* associated *with Σ^* at the origin, where Σ^* is a regular simplicial cone subdivision of Σ and we have a coordinate system on $\mathbb{C}^n$ centered at the origin.*

Such map has the following properties. First of all, the *toric coordinates* charts of X are give precisely by the maps $\pi \colon \mathbb{C}^n_\sigma \to \mathbb{C}^n$. Second, as we would expect from a resolution,

$$\pi \colon X \setminus \pi^{-1}(\mathbf{0}) \to \mathbb{C}^n \setminus \{\mathbf{0}\}$$

is an isomorphism. Also, for each ray r in the fan, we can glue together the affine divisors obtained from various charts that are describing the same associated zero locus, and denote the resulting divisor by $E(r)$. Hence, these are precisely the irreducible components in $\pi^{-1}(\mathbf{0}) = \cup_r E(r)$.

It is of our particular interest to start with the fan given by the dual fan $\Sigma(f)$ of the Newton polygon, and in particular to the regular simplicial subdivision of it, guaranteed by Theorem 22 and which we will denote by $\Sigma^*(f)$. Associated with this fan, we have a toric blowing up, which we call *admissible toric blowing up* for f.

Notice that, so far, we haven't imposed that our considered polynomial is Newton non-degenerate, and the construction is valid in general. However, by having this additional hypothesis, we obtain the following result (see point 4. in [Oka96, p. 101]).

Theorem 23. *If f is Newton non-degenerate, then the associated toric blowup $\pi\colon X \to \mathbb{C}^n$ is a good resolution of the f as a germ at the origin.*

Example 14 (Toroidal resolution via Newton polygon)**.** We give an example of a toroidal resolution of the polynomial $f = x^4 - y^5 + x^2y^2$. This has already been presented in Example 9, where we present a resolution through a composition of blowups. However, here we will perform a toroidal resolution, via a regular subdivision of the associated dual fan.

Let us depict in the following Figure IV.3 the dual fan associated to the Newton polygon of f, and a regular subdivision obtained by adding rays.

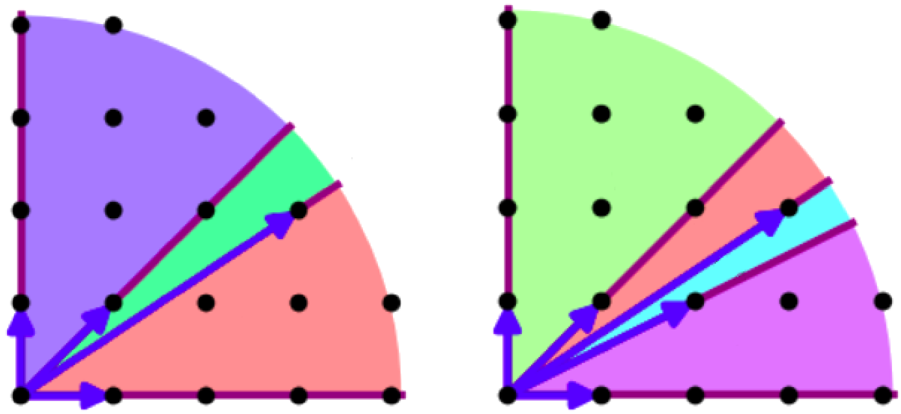


Figure IV.3: Dual fan (left) and regular subdivision of it (right), of the polynomial $f = x^4 - y^5 + x^2y^2$.

Originally, we have the rays

$$[(1,0),(3,2),(1,1),(0,1)]$$

and we can see that the cone $\langle(1,0),(3,2)\rangle$ is not regular, as $\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 2 \neq 1$. However, we can see by inspection that in this cone we can find the integral ray

$$(2,1) = \frac{1}{2}(1,0) + \frac{1}{2}(3,2)$$

Adding it to the fan, and writing all the rays like

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

it is clear that every cone defined by any two consecutive rays has volume 1.

Taking any of these cones provides with an associated toric blowup, which we ought to think as one of the charts of the resolution. However, we will see that the only 'interesting' charts are the ones associated with the original rays of the dual fan, and which are not coordinate rays.

To select the chart, we only need to choose a ray, and read the corresponding column and the one immediately to the right in the ray matrix above. We include next the associated toric maps and the total transform of $f = x^4 - y^5 + x^2y^2$.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \implies (x,y) \mapsto (zw^2, w) \implies \pi^* f = z^4w^8 - w^5 + z^2w^4w^2 = w^5(z^4w^3 - 1 + z^2w)$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \implies (x,y) \mapsto (z^2w^3, zw^2) \implies \pi^* f = z^8w^{12} - z^5w^{10} + z^4w^6z^2w^4 = z^5w^{10}(z^3w^2 - 1 + z)$$

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \implies (x,y) \mapsto (z^3w, z^2w) \implies \pi^* f = z^{12}w^4 - z^{10}w^5 + z^6w^2z^4w^2 = z^{10}w^4(z^2 - w + 1)$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \implies (x,y) \mapsto (z, zw) \implies \pi^* f = z^4 - z^5w^5 + z^2z^2w^2 = z^4(1 - zw^5 + w^2)$$

By comparing with Example 9, we can see that the chart given by cone $\langle(1,0),(2,1)\rangle$ corresponds with the second chart obtained in the 1st blowup. Also, the last chart, given by the cone $\langle(1,1),(0,1)\rangle$ corresponds to the first chart obtained in the 0th blowup. Instead, the charts given by the cones $\langle(2,1),(3,2)\rangle$ and $\langle(3,2),(1,1)\rangle$ correspond with the two final charts obtained in the 2nd blowup.

Notice that the comparison with the charts obtained via the regular resolution clearly shows that any toric resolution can be obtained as such a composition of simple blowups.

Remark 6 (Relation between NND and plane curves)**.** A plane curve is also Newton non-degenerate if, and only if, each branch has a single Puiseux characteristic exponent. We can check this is the case with the example above, as indeed we will see that we can take $(n;\beta_1) = (3;2)$.

Consider one the charts presented above, for example the one given by the original ray $(3,2)$ and the ray $(1,1)$ from the regular subdivision. In this chart, the pullback is

$$(x,y) \mapsto (z^3w, z^2w) \implies \pi^* f = z^{10}w^4(z^2 - w + 1)$$

so we can find a parametrization as

$$z^2 - w + 1 = 0 \implies (z,w) = (t, t^2+1) \implies (x,y) = (t^3(t^2+1), t^2(t^2+1))$$

Now, take $u = t(t^2+1)^{1/3}$ as a new parameter, and compute the expansion of

$$(t^2+1)^{1/3} = 1 + \frac{1}{3}u^2 + \frac{1}{9}u^4 - \frac{10}{81}u^6 \cdots = 1 + u^2h(u^2)$$

Hence, we may also write the good parametrization as

$$(x,y) = (u^3, u^2(t^2+1)^{1/3}) = \big(u^3, u^2(1 + u^2h(u^2))\big)$$

Thus, we can read the Puiseux characteristic exponents

$$(n;\beta_1) = (3;2) \implies \{(q_1,n_1)\} = \{(2,3)\}$$

and compute the Zariski pairs (which in the case of a single pair, consists of simply swapping the characteristic exponents).

Altogether, starting from the (original) ray associated to a face of the Newton polygon, we have found a good resolution from which we get a single Puiseux characteristic exponent and a single Zariski pair.

3 Topological zeta function of NND singularities

Recall from Definition 5 the definitions for the topological zeta function, which we will simply write by $Z(s)$ and $Z_0(s)$, for the global and local versions, respectively.

Next, we want to give a description of this zeta function in terms of the information retrieved from the

Newton polygon. For that, we first introduce the next concepts, following the exposition in [Viu12].

Definition 34 (Volume form). *For γ a convex polyhedron with vertices on $\mathbb{Z}^n$, we denote by ω_γ the* volume form *on the affine space $Aff(\gamma)$ generated by γ, such that the parallelepiped defined by a base of the $\mathbb{Z}$-module associated to $Aff(\gamma) \cap \mathbb{Z}^n$ has volume 1.*

With that, we are ready to give a notion of volume to the faces of the Newton polytope. For that, recall that $\Gamma_{gl}(f)$ is defined as the convex hull of the exponents of the monomials, appearing in the expression of f.

Definition 35 (Volume). *For τ a face of the Newton polytope, we define* $\mathrm{Vol}(\tau)$ *as the* volume *of $\tau \cap \Gamma_{gl}(f)$ over the volume form ω_τ. Also, if* $\dim \tau = 0$*, we will write* $\mathrm{Vol}(\tau) = 1$.

Next, to each face we want to associate a rational function $J(\tau, s)$, as follows.

Definition 36. *Let τ be a face in $\Gamma(f)$, and consider a decomposition of the associated cone $\Delta_\tau = \cup_{i=1}^r \Delta_i$ in simplicial cones of dimension* $\dim \Delta_\tau = l$ *such that* $\dim(\Delta_i \cap \Delta_j) < l$*, for all $i \neq j$. Then, define*

$$J(\tau, s) := \sum_{i=1}^{r} J_{\Delta_i}(s), \qquad \textit{with} \qquad J_{\Delta_i}(s) = \frac{\mathrm{mult}(\Delta_i)}{(N(a_{i_1})s + k(a_{i_1})) \cdots (N(a_{i_l})s + k(a_{i_l}))}$$

being $a_{i_1}, \ldots, a_{i_l} \in \mathbb{N}^n$ the linearly independent primitive integral vectors that generate Δ_i. Lastly, if $\tau = \Gamma(f)$, we rather take $J(\tau, s) = 1$.

It should be noted that this definition is independent of the choice of the decomposition of Δ_τ in simplicial cones (see [DL92, Lemme 5.1.1]), which can be proved by showing that

$$J(\tau, s) = \int_{r \in \Delta\tau} e^{-(N(r)s + k(r))} \omega_{\Delta_\tau}$$

Moreover, the definition shows already that the poles of $J(\tau, s)$ are of the form $-k(a)/N(a)$ with a a primitive integral vector generating Δ_τ. With this, we are ready to introduce the main result giving a combinatorial expression for the zeta function.

Theorem 24 ([DL92], Thm. 5.3). *Let $f \in \mathbb{C}[x]$ be a polynomial Newton non-degenerate for $\Gamma_{gl}(f)$, then*

$$Z(s) = \sum_{\substack{\tau \text{ vertex} \\ \text{of } \Gamma(f)}} J(\tau, s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \dim \tau \geq 1}} (-1)^{\dim \tau} (\dim \tau)! \mathrm{Vol}(\tau) J(\tau, s)$$

Let $f \colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function Newton non-degenerate for $\Gamma(f)$, then the expression for $Z_0(s)$ is analogous, except that the sum runs only over compact faces of $\Gamma(f)$.

As the poles of $Z(s)$ arise from the poles of the terms $J(\tau, s)$ in the sum, we see that they are still either -1 or of the form $-k(a)/N(a)$ (see also [Loe90, Thm. 5.3.1]). Comparing this pole candidates with the setting in Theorem 1, the choice of notation for $k(a), N(a)$ becomes clear.

Example 15 ([DL92], Ex. 5.4). Let $f = x_1^{\alpha_1} + \cdots + x_n^{\alpha_n} + \sum_{\substack{p \in \mathbb{N}^n \\ \langle w, p \rangle = 1}} a_p x^p$ be a convenient quasihomogeneous polynomial with weights $w = (1/\alpha_1, \ldots, 1/\alpha_n)$, where $\alpha_1, \ldots, \alpha_n \geq 1$ are given integers. Further, suppose it is Newton non-degenerate for $\Gamma_{gl}(f)$. Write $\rho = \sum_{i=1}^n 1/\alpha_i$ and

$$A = \sum_{I \subset \{\alpha_1, \ldots, \alpha_n\}} (-1)^{|I|} \frac{\prod_{\alpha \in I} \alpha}{\mathrm{lcm}(\{\alpha \in I\})^2}$$

Then

$$Z(s) = \frac{1}{s+1} - \frac{sA}{(s+1)(s+\rho)} = \left(1 + \frac{A}{\rho - 1}\right) \frac{1}{s+1} - \frac{\rho A}{\rho - 1} \frac{1}{s+\rho}$$

As a remark, the Bernstein-Sato polynomial for quasihomogeneous polynomials also has a simple expression (see, for example, [Gra10, §4] for a nice exposition). In particular, -1 and $-\rho$ are roots of it, and the strong monodromy conjecture is immediate in this case.

4 Sketch of the proof of the conjecture

In this subsection we go through the proof of the monodromy conjecture for the case of Newton non-degenerate polynomials (up to certain hypothesis), following the results introduced by Loeser in [Loe90].

4.1 Toric residue numbers

We begin by introducing the following quantities.

Definition 37. *If τ, τ' are two distinct faces of codimension 1 of the Newton polyhedron at the origin of f, we denote by $\beta(\tau, \tau')$ the greatest common divisor of the minors of order 2 of the matrix $\big(a(\tau), a(\tau')\big)$. Additionally, one defines*

$$\lambda(\tau, \tau') = k(\tau') - \frac{k(\tau)}{N(\tau)} N(\tau'), \qquad \varepsilon(\tau, \tau') = \lambda(\tau, \tau')/\beta(\tau, \tau')$$

whenever $N(\tau) \neq 0$, which is the case if τ is a compact face.

Remark 7. The definition of the residue numbers ε in the work of Loeser, which we will call *toric* residue numbers, differs from the usual definition by a dividing factor $\beta(\tau, \tau')$. Loeser requires to rather work with this quantities in his proof because, when considering the associated toroidal resolution with the original (non necessarily regular) dual fan, we may obtain a singular variety. Thus, some computations require special care, and this factor β appears as the degree of a finite morphism between singular toric varieties.

Also relevant, because we are working now without a regular simplicial subdivision, we may avoid the problems of non-uniqueness. Indeed, we have discussed that such a fan subdivision exists, but that it is not unique (already a simplicial subdivision may not be unique). However, on the contrary, we lose some symmetry properties. For example, let us consider faces τ_1, τ_2 of codimension 1 such that the divisors $E(\tau_1), E(\tau_2)$ have numerical data satisfying

$$\sigma(\tau_1) = \frac{k(\tau_1)}{N(\tau_1)} = \frac{k(\tau_2)}{N(\tau_2)} = \sigma(\tau_2)$$

Then, for any other divisor $E(\rho)$ intersecting both of them, we have the symmetry

$$\lambda(\tau_1, \rho) = -\sigma(\tau_1)N(\rho) + k(\rho) = -\sigma(\tau_2)N(\rho) + k(\rho) = \lambda(\tau_2, \rho)$$

but now, when introducing the factor $\beta(\tau_i, \rho)$, the equality may be lost.

Example 16. We will see this example with more detail in the detailed Example 2 of Chapter V, however, we simply include it here for completeness to illustrate the discussion above.

Consider $f = x^5 + y^6 + z^4 + x^2yz + xy^2z$, and the numerical data of rays of the dual fan (without subdividing) given by

$$\begin{cases} \tau_1 \colon 6x + 5y + 14z = 30 & \Longrightarrow \quad (k(\tau_1), N(\tau_1)) = (25, 30) \\ \tau_2 \colon 5x + 2y + 3z = 12 & \Longrightarrow \quad (k(\tau_2), N(\tau_2)) = (10, 12) \end{cases}$$

with same $\sigma(\tau_1) = \sigma(\tau_2) = 5/6$. Also, they both have the common neighbor (meaning they belong to one cone together, respectively) given by

$$\rho \colon 0x + 1y + 0z = 0 \quad \Longrightarrow \quad (k(\rho), N(\rho)) = (1, 0)$$

so that we have

$$\lambda(\tau_1, \rho) = \lambda(\tau_2, \rho) = -\frac{5}{6} \cdot 0 + 1 = 1$$

Now, we compute the Loeser β factors

$$\beta(\tau_1, \rho) = \gcd\left(\begin{vmatrix} 6 & 5 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 6 & 14 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 5 & 14 \\ 1 & 0 \end{vmatrix} \right) = \gcd(6, 0, 14) = 2$$

$$\beta(\tau_2,\rho) = \gcd\left(\begin{vmatrix}5 & 2\\0 & 1\end{vmatrix}, \begin{vmatrix}5 & 3\\0 & 0\end{vmatrix}, \begin{vmatrix}2 & 3\\1 & 0\end{vmatrix}\right) = \gcd(5,0,3) = 1$$

so it happens that

$$\varepsilon(\tau_1,\rho) = \frac{1}{2} \neq \frac{1}{1} = \varepsilon(\tau_2,\rho)$$

are not equal.

4.2 Main results

Next, we return to the main results obtained by Loeser.

Theorem 25 ([Loe90], Thm. 4.2)**.** *Let f be a germ of an analytic function, Newton non-degenerate at the origin. Let τ_0 be a compact face of codimension 1 of $\Gamma(f)$. Suppose that the following two conditions are verified*

(i) $\dfrac{k(\tau_0)}{N(\tau_0)} \notin \mathbb{Z}$,

(ii) For every face τ of codimension 1 of $\Gamma(f,0)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0,\tau) \notin \mathbb{Z}$.

Then, there exists a horizontal multiform section $\gamma(t)$ of the fibration H_n over T' such that

$$\lim_{t\to 0} t^{1-\frac{k(\tau_0)}{N(\tau_0)}} \int_{\gamma(t)} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{\mathrm{d}f} = C$$

with C a non-zero constant.

Although we will not reproduce the details of the proof, we will sketch the main ideas next. For that, take f verifying the hypothesis of the theorem, and we consider the associated toric modification, $\pi\colon X(\Sigma(f)) \to \mathbb{C}^n$ as defined in the previous subsection via the dual fan associated to the Newton polygon (without regular subdivision).

Let $\mathcal{T}_0$ be the set of faces τ of codimension 1 that intersect our face of interest τ_0. We denote $\alpha(\tau) = \varepsilon(\tau_0,\tau)$ if $\tau \in \mathcal{T}_0$, and $\alpha(C) = 1 - \frac{k(\tau_0)}{N(\tau_0)}$ for the strict transform C.

We now define a multivalued differential form ω on the smooth open $U = X \setminus D$, where $D = \sum_{\tau\in\mathcal{T}} E(\tau) + C$ is a complete toroidal divisor in X. For that, consider the multivalued differential form

$$\eta = \pi^*\left(f^{1-\frac{k(\tau_0)}{N(\tau_0)}}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\right)$$

and take its residue along $\pi^{-1}(f^{-1}(0))$, that is the form $\frac{\eta}{\pi^*(\mathrm{d}f)}$. The restriction to U finally defines

$$\omega = \frac{\eta}{\pi^*(\mathrm{d}f)}\,|_U$$

Notice that ω doesn't vanish anywhere, thanks to the choice of the exponent $1 - \frac{k(\tau_0)}{N(\tau_0)}$, as can be seen with a computation on local coordinates, recalling that $\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$ vanishes along U with order $k(\tau_0) - 1$.

Let us denote L the local system of rang 1 over U, such that ω is a section over U of $\Omega^n_U \otimes_{\mathbb{C}} L$. In [Loe90, Prop. 4.3], it is proven that in a neighborhood of a general point of the divisor $E(\tau)$ in X, given locally by $x = 0$, we have

$$\omega = x^{\alpha(\tau)}\frac{\mathrm{d}x}{x} \wedge \varphi \otimes e \tag{4.1}$$

with φ a holomorphic differential form that does not vanish, and e a multivalued local section of L. Similarly, around the divisor C

$$\omega = x^{\alpha(C)}\frac{\mathrm{d}x}{x} \wedge \varphi \otimes e \tag{4.2}$$

In particular, this implies that ω is meromorphic on X.

Also, we may take the local system L of rank 1 over U in a manner that the monodromies are given by $e^{-2\pi\imath\alpha(\tau)}$ around $E(\tau)$, and by $e^{-2\pi\imath\alpha(C)}$ around C. Thus, if conditions (i) and (ii) are satisfied, this ensures that these exponents are not integer multiples of $2\pi\imath$. Hence, the monodromies around the irreducible components of $X \setminus U$ are not the identity, which is the key step of the whole proof.

Indeed, after a technical check that $\Omega_X^{n-1}(\log D)$ is of maximum Kodaira dimension, one can apply the following result, which is the toroidal variant of the result by Esnault and Viehweg [VE87].

Proposition 11 ([Loe90], Thm. 3.7)**.** *Let X be a toroidal variety compact and irreducible of dimension n, D a toroidal complete divisor over X, such that $U = X \setminus |D|$ is smooth. Suppose that $\Omega_X^n(\log D)$ is an invertible sheaf of maximum Kodaira dimension and numerically effective. Let L be a local system of rank one over U for which all the monodromies around the irreducible components of D are different from the identity. Let $\omega \in \Gamma(U, \Omega_U^n \otimes_{\mathbb{C}} L)$ a multiform differential form over U. Further suppose that ω is meromorphic over X and that it does not vanish over U. Then, the cohomology class of the differential form ω over $H^n(U, L)$ is non-zero.*

Thus, we deduce that the cohomology class of the differential form ω in $H^n(U, L)$ is non-zero, and we can apply the arguments described in Section 9 of Chapter II to finally conclude in the following theorem.

Theorem 26 ([Loe90], Thm. 5.5.1)**.** *Let f be a comfortable polynomial verifying $f(0) = 0$, with Newton diagram $\Gamma(f)$, and Newton non-degenerate. Suppose that all compact faces τ_0 verify*

i) $\frac{k(\tau_0)}{N(\tau_0)} < 1$,

ii) For every face τ of codimension 1 of $\Gamma(f)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, the real parts of the poles of the zeta function of f are roots of the Bernstein-Sato polynomial of f.

4.3 Relaxing hypothesis

We start by discussing the first hypothesis. For this one, Loeser already points out in [Loe90, Rmk. 5.5.2.1] that if one replaces the condition $\frac{k(\tau_0)}{N(\tau_0)} < 1$ with $\frac{k(\tau_0)}{N(\tau_0)} \notin \mathbb{N}$, this is enough to prove the weak version of the conjecture. Namely, if α is the real part of a pole of the zeta function of f, then $e^{2\pi\imath\alpha}$ is an eigenvalue of the local monodromy of f at the origin.

Next, for the second hypothesis, although it is not clear if it is possible to remove, we can try to relax it. Indeed, one ought to expect that non-positive residue numbers could be allowed to happen. That is, that the real problems arise for positive integers.

The idea for the negative integers is that we can argue the existence of a non-zero cohomology class, as required in the described approach, by calculating the degree of certain line bundles. For example, in the case of plane curves, a result in this spirit is the following (which is a slight generalization of a result by Deligne and Mostow [DM86, Prop. 2.14]).

Proposition 12 ([Bla20], Prop. 11.1)**.** *Let $\omega \in \Gamma(\mathbb{P}, \Omega^1\left(\sum \mu_s s - \sum \delta_x x\right)(L)$. Assume that $\sum_{s \in S} \mu_s \leq r - 1$ and that $\alpha_s \neq 1$ for all $s \in S$. Then, ω defines a non-zero cohomology class in $H^1\left(\mathbb{P} \setminus S, L\right)$.*

Hence, a technical generalization to higher dimensions would allow omitting the negative integers in the hypothesis.

As for the value $\varepsilon = 0$, this determines a double pole. Hence, we might argue that for now we are restricting our study to examples only having simple poles, and then expect that this can be extended to any order by a decreasing induction on the maximum order of the poles of the zeta function.

Lastly, for the value $\varepsilon = 1$, the differential form being considered in (4.1) or (4.2) would have order $\varepsilon - 1 = 0$ at that point. In particular, it does not have neither a zero nor a pole so this case can be omitted altogether.

Chapter V

Examples

We now introduce some remarkable examples we have found, that aim to study the question posed in this work. Let us recall the result by Loeser that gives a positive answer to the monodromy conjecture in the case of Newton non-degenerate singularities.

Theorem ([Loe90], Thm. 5.5.1)**.** *Let f be a comfortable polynomial verifying $f(0) = 0$, with Newton diagram $\Gamma(f)$, and Newton non-degenerate. Suppose that all compact faces τ_0 verify*

i) $\frac{k(\tau_0)}{N(\tau_0)} < 1$,

ii) For every face τ of codimension 1 of $\Gamma(f)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, the real parts of the poles of the zeta function of f are roots of the Bernstein-Sato polynomial of f.

In particular, we have discussed that the important extra hypothesis here is the second one, demanding that the toric residue numbers are non-integers. Thus, we will search for examples of NND polynomials to see if this condition is met, and if not, if it is possible to discard the contributions of the faces not satisfying it to the poles of the topological zeta function.

Via the examples that we will describe, we will see that we can indeed encounter integer residue numbers, both with the usual definition and the toric one. Furthermore, we will then compute the individual contributions of the associated divisors to the topological zeta function, by checking its residue at the candidate values associated to the divisors. With that, we will see that it can happen that such divisors give a pole. Altogether, this allows us to confirm that we cannot remove the extra hypothesis with the current approach given to prove the conjecture for NND singularities.

Next, we are ready to describe thoroughly each of the examples. We will also briefly summarize why it is relevant to the purpose of our search and give some motivation on how it was found. In particular, only the first example was found via brute force, that is, with an exhaustive search over the possible Newton polygons of polynomials of degree less or equal than 4. In contrast, the other examples are built from a more intuitive approach following the geometry of its Newton polygon, which turns out to be a necessary approach after such an exhaustive search becomes computationally unfeasible for polynomials of larger degree.

1 Example 1

We begin by considering the polynomial $f = xz^3 + y^3$ in $\mathbb{C}[x, y, z]$, which has been found via a brute force search. In this case, we will find positive integer values for some of the residue numbers. However, we will check that both the strong and standard conjectures hold. This example is relevant because, already with very little number of terms and degree of the polynomial considered, we see that we can encounter bad residue numbers.

First, let us plot the Newton polygon and its associated dual fan, depicted in the following Figure V.1.

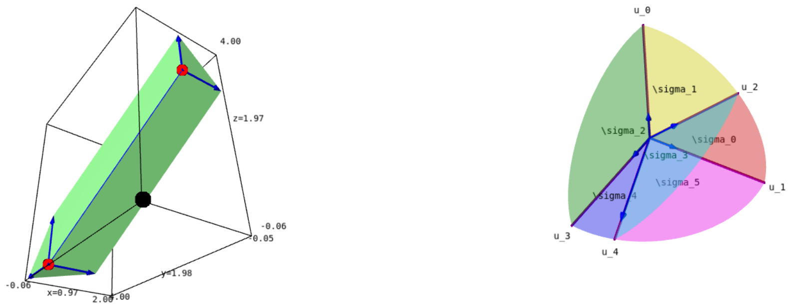


Figure V.1: Newton polygon and associated dual fan of the polynomial $f = xz^3 + y^3$.

The original rays in the fan are

$$[(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (3, 4, 3)]$$

and the charts to consider are precisely those that are not coordinate rays.

We will not plot in any of the 3 dimensional examples the regular subdivision of the dual fan, as the plots get messy very rapidly. Instead, they can be executed with the code in the annex, where the figure obtained is interactive and can be rotated and zoomed.

In any case, an example of a regular subdivsion can be obtained by adding the following rays

$$[(1, 2, 2), (2, 3, 2), (2, 3, 2), (1, 2, 1), (1, 1, 1), (2, 3, 3), (1, 2, 1)]$$

Thus, we can compute the residue numbers associated. We will present them as follows: for each ray representing an exceptional divisor, we include a list containing the divisors that it intersects and the corresponding residue number.

- Associated residue numbers to divisor (3, 1, 0):
  {ST: -4/3 + 1, (2, 1, 1): 0, (2, 1, 0): 1/3, (1, 0, 0): 1, (0, 0, 1): 1, (1, 1, 1): -1, (0, 1, 1): -2}
- Associated residue numbers to divisor (0, 1, 1):
  {ST: -2/3 + 1, (1, 1, 0): 4/3, (0, 1, 0): 1, (2, 1, 0): 5/3, (3, 1, 0): 2, (0, 0, 1): 1, (1, 1, 1): 1}

For example, in this case, we are saying that ray $(3, 1, 0)$ intersects the strict transform (ST) with residue number $\varepsilon((3, 1, 0), \mathrm{ST}) = -4/3 + 1$, that it also intersects the exceptional divisors associated with ray $(2, 1, 1)$ with residue number $\varepsilon((3, 1, 0), (2, 1, 1)) = 0$, and so on. Furthermore, we have marked in red the residue numbers that are integers values greater than 1.

Then, we can compute the (local) zeta function

$$Z_0(s) = \frac{1}{3}\frac{s+2}{(s+2/3)(s+1)} = \frac{4/3}{s+2/3} - \frac{1}{s+1}$$

and, in this case, the expression for the global zeta function is the same $Z(s) = Z_0(s)$. We can see that the poles are precisely -1 and $-2/3$. As we've mentioned at the beginning, the polynomial is quasihomogeneous with weights $(1/4, 1/3, 3/4)$. However, it is not convenient, so we can not directly apply the expression in Example 15 to obtain the topological zeta function.

Now, we would like that such divisors with positive integer residue numbers didn't appear as poles in the zeta function, because then, even if we could not apply the results to construct the non-zero cohomology class, we could argue that they do not contribute to the poles of the zeta function. However, we see that the candidate arising from the bad divisor actually appears as a pole, which may be problematic.

Moreover, if we compute the Bernstein-Sato polynomial

$$b_f(s) = (s+1)^2\left(s+\frac{5}{3}\right)\left(s+\frac{2}{3}\right)\left(s+\frac{4}{3}\right)$$

we see that the roots of the reduced polynomial are $\{-1, -5/3, -2/3, -4/3\}$, so the conjecture holds here.

We can also check that the weak version of the conjecture holds, by computing the eigenvalues of the monodromy. For that, we compute the monodromy zeta function as

$$\zeta(t) = (-1)(t-1)(t^2+t+1)$$

and then, via Theorem 11 we find the minimal polynomial of the monodromy and its roots are

$$\{1, e^{\pi\imath 2/3}, e^{\pi\imath 4/3}\}$$

Next, one would hope that if we go back and consider toric residue numbers instead, then the hypothesis are fulfilled. However, this turns out to be false as well.

- Associated toric residue numbers to divisor (3, 1, 0):
  {ST: -4/3 + 1, (1, 0, 0): 1, (0, 0, 1): 1, (0, 1, 0): 1/3, (0, 1, 1): -2}
- Associated toric residue numbers to divisor (0, 1, 1):
  {ST: -2/3 + 1, (1, 0, 0): 1, (3, 1, 0): 2, (0, 0, 1): 1, (0, 1, 0): 1}

In the next example, we look for a polynomial with also bad residue numbers, but being convenient.

2 Example 2

In this example, we consider the polynomial $f = x^5+y^6+z^4+x^2yz+xy^2z$, for which we will see that there exist two different bad residue numbers for a same divisor, but in this case toric residue number satisfy the hypothesis. Again, we will check that both the strong and standard conjectures hold. In contrast to the previous example, this polynomial is now convenient (it contains the monomials x^5, y^6, z^4), and has been constructed via geometric intuition which well be discussed together with the Newton polygon.

First, let us plot the Newton polygon and its associated dual fan, depicted in the following Figure V.2. From it, it is clear that the polynomial is not quasihomogeneous (in that case, all points should lie on a single hyperplane).

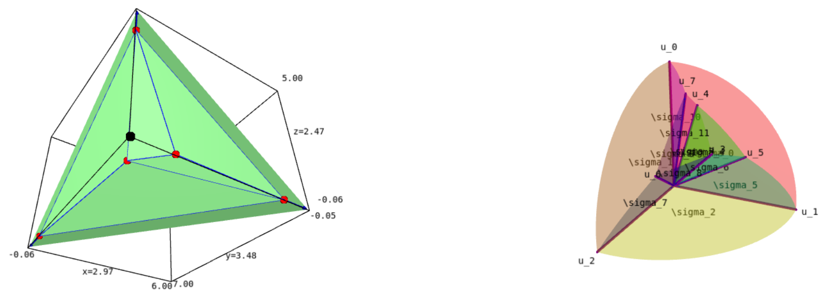


Figure V.2: Newton polygon and associated dual fan of the polynomial $f = x^5 + y^6 + z^4 + x^2yz + xy^2z$.

We now discuss the idea to build this example, and more generally the intuition behind the most relevant examples found. First, we begin constructing f by adding the individual monomials $x^p + y^q + z^r$

for some large enough integers p, q, r, which already ensures that the polynomial will be convenient. Next, by adding some mixed monomials of the type $x^s y^t z^u$ for some small integers s, t, u results in the formation of some small compact faces close to the origin. By adequately choosing the appearing monomials and exponents, we can construct these faces in the Newton polygon with a desired normal vector. From here, it is a matter of trial and error to find an appropriate combination of normal vectors (corresponding to rays defining the dual fan) that will lead to bad residue numbers.

In this case, the original rays in the fan are

$$[(0,0,1),(0,1,0),(1,0,0),(1,1,1),(1,1,2),(4,7,5),(5,2,3),(6,5,14)]$$

and we can give a regular subdivision by adding the following rays

$$\begin{aligned}
&[(2,4,3),(3,6,4),(2,1,2),(1,2,2),(3,1,2),(2,3,3),(1,2,2),(3,2,2),(3,5,4),(2,1,1),\\
&(2,3,3),(6,4,9),(2,1,2),(9,5,10),(8,5,10),(3,2,4),(5,3,6),(12,6,11),(3,5,4),(4,3,6),\\
&(4,3,8),(6,3,5),(4,3,8),(5,4,11),(5,4,11),(5,3,6),(3,2,5),(7,4,8),(7,4,8),(5,4,10),\\
&(11,6,11),(4,2,3),(3,2,5),(9,5,9),(13,7,13),(17,9,17),(5,4,10),(8,4,7),(6,5,12),\\
&(8,4,7),(4,3,7),(19,10,19),(6,5,13),(7,5,12),(6,5,13),(5,3,5),(10,5,9),(7,5,11),\\
&(9,5,9),(15,8,15),(5,4,9),(10,5,9),(9,7,18),(15,8,15),(5,4,8),(10,8,21),(3,2,3),\\
&(7,4,7),(5,3,5),(8,6,15),(8,6,15),(3,2,3),(7,4,7)]
\end{aligned}$$

We can see that the computations for a regular subdivision rapidly increase in complexity, as the number of added rays increases largely.

We will not include all the residue numbers associated to the subdivision, but rather include only the important information for the relevant divisors.

- Associated residue numbers to divisor (4, 7, 5): {ST: -4/5 + 1, ...}
- Associated residue numbers to divisor (1, 1, 2): {ST: -4/5 + 1, ...}
- Associated residue numbers to divisor (6, 5, 14): {ST: -5/6 + 1, ...}
- Associated residue numbers to divisor (5, 2, 3): {ST: -5/6 + 1, ...}
- Associated residue numbers to divisor (1, 1, 1): {ST: -3/4 + 1, ..., (8, 5, 10): 2, (6, 5, 12): 2, ...}

So the divisor associated with ray $(1,1,1)$ has two neighbors for which the residue number is a positive integer, and it has candidate value $\sigma = -3/4$.

Then, we can compute the (local) zeta function

$$Z_0(s) = \frac{47}{30}\frac{s^3 + \frac{371}{188}s^2 + \frac{251}{188}s + \frac{15}{47}}{(s+3/4)(s+4/5)(s+5/6)(s+1)} = \frac{9}{s+3/4} - \frac{48/5}{s+4/5} - \frac{35/6}{s+5/6} + \frac{8}{s+1}$$

Thus, the poles are precisely $\{-3/4, -4/5, -5/6, -1\}$, so we see that again the bad divisor appears as a pole.

Furthermore, we find the Bernstein-Sato polynomial

$$\begin{aligned}
b_f(s) = \left(s+\frac{9}{5}\right)\left(s+\frac{7}{4}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{8}{5}\right)\left(s+\frac{3}{2}\right)\left(s+\frac{7}{5}\right)\left(s+\frac{4}{3}\right)\cdot \\
\cdot\left(s+\frac{5}{4}\right)\left(s+\frac{6}{5}\right)\left(s+\frac{7}{6}\right)(s+1)^3\left(s+\frac{5}{6}\right)\left(s+\frac{4}{5}\right)\left(s+\frac{3}{4}\right)
\end{aligned}$$

and check that the strong conjecture holds.

Lastly, we find the monodromy zeta function

$$\zeta(t) = (-1)(t+1)^5(t-1)^7(t^2-t+1)(t^2+t+1)(t^2+1)^4(t^4+t^3+t^2+t+1)^2$$

and, proceeding as before, we end up with the monodromy eigenvalues

$$\{-1, 1, e^{\pi\imath/3}, e^{\pi\imath 2/3}, e^{-\pi\imath 2/3}, e^{-\pi\imath/3}, i, -i, e^{\pi\imath 2/5}, e^{\pi\imath 4/5}, e^{-\pi\imath 4/5}, e^{-\pi\imath 2/5}\}$$

It is then an immediate check that the weaker statement also holds.

Next, if we try going back and considering toric residue numbers instead, it happens in this case that there aren't any positive integers greater than 1.

3 Example 3

We now include a more complex example, by taking the NND convenient polynomial $f = x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$. We will see that we encounter positive integer residue numbers from a regular subdivision, but also with toric residue numbers. More interestingly, the bad divisors in each case will be different and they will be associated to compact faces of the Newton polygon. Nonetheless, we still check that the strong and standard conjectures hold. Lastly, we see that the contributions of these bad divisors to the zeta function is non-zero.

First, let us plot in the following Figure V.3 the Newton polygon and its associated dual fan.

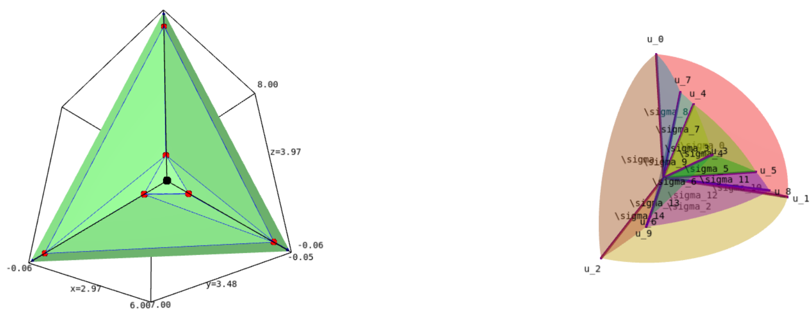


Figure V.3: Newton polygon and associated dual fan of $f = x^5 + y^6 + z^7 + x^2yz + xyz^2 + xy^2z$.

The original rays in the fan are:

$$[(0,0,1), (0,1,0), (1,0,0), (1,1,1), (1,1,2), (1,2,1), (3,1,1), (6,5,14), (7,18,5), (23,7,6)]$$

and a regular subdivision requires almost 400 new rays, which we won't list here.

Again, we won't include all the residue numbers, but rather highlight the bad ones.

- Associated residue numbers to divisor (1, 2, 1): {ST: -4/5 + 1, ...}
- Associated residue numbers to divisor (1, 1, 2): {ST: -4/5 + 1, ... }
- Associated residue numbers to divisor (7, 8, 15): {ST: -6/7 + 1, ...}
- Associated residue numbers to divisor (23, 7, 6): {ST: -6/7 + 1,... }
- Associated residue numbers to divisor (6, 5, 14): {ST: -5/6 + 1, ... }
- Associated residue numbers to divisor (3, 1, 1): {ST: -5/6 + 1, ... }
- Associated residue numbers to divisor (1, 1, 1):
  {ST: -3/4 + 1, (16, 6, 5): 3, (17, 11, 6): 4, (9, 7, 4): 2, (21, 15, 8): 5, (13, 15, 6): 4, (11, 5, 4): 2, (21, 7, 6): 4, (4, 10, 3): 2, (19, 17, 8): 5, (6, 5, 12): 2, ...}

Altogether, we can see that the residue numbers obtained via a regular subdivision appear to behave very undesirably, as it seems we can construct polynomials with arbitrarily many bad divisor intersections.

Then, we can compute the (local) zeta function

$$Z_0(s) = \frac{557}{420} \frac{s^4 + \frac{2987}{1114}s^3 + \frac{3177}{1114}s^2 + \frac{825}{557}s + \frac{180}{557}}{(s+3/4)(s+4/5)(s+5/6)(s+6/7)(s+1)} = \frac{81/4}{s+3/4} - \frac{72/5}{s+4/5} - \frac{70/6}{s+5/6} - \frac{48/7}{s+6/7} + \frac{14}{s+1}$$

with poles $\{-3/4, -4/5, -5/6, -6/7, -1\}$. So, in particular, we have that the bad divisor $(1,1,1)$ with $\sigma = -3/4$ appears as a pole, and it is the only divisor with such a candidate value.

Furthermore, we find the Bernstein-Sato polynomial

$$b_f(s) = \left(s+\frac{11}{6}\right)\left(s+\frac{9}{5}\right)\left(s+\frac{12}{7}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{8}{5}\right)\left(s+\frac{11}{7}\right)\left(s+\frac{3}{2}\right)\left(s+\frac{10}{7}\right)\left(s+\frac{7}{5}\right)\cdot$$
$$\cdot\left(s+\frac{4}{3}\right)\left(s+\frac{9}{7}\right)\left(s+\frac{5}{4}\right)\left(s+\frac{6}{5}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{8}{7}\right)(s+1)^3\left(s+\frac{6}{7}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{4}{5}\right)\left(s+\frac{3}{4}\right)$$

so we can check that the strong conjecture holds. For the standard statement, we compute again the monodromy zeta function

$$\zeta(t) = (-1)(t+1)^3(t-1)^7(t^2+1)(t^2-t+1)^2(t^2+t+1)^2(t^4+t^3+t^2+t+1)^3(t^6+t^5+t^4+t^3+t^2+t+1)$$

with monodromy eigenvalues:

$$\{-1, 1, i, -i, e^{\pi\imath/3}, e^{-\pi\imath/3}, e^{\pi\imath 2/3}, e^{-\pi\imath 2/3}, e^{\pi\imath 2/5}, e^{\pi\imath 4/5}, e^{-\pi\imath 4/5},$$
$$e^{-\pi\imath 2/5}, e^{\pi\imath 2/7}, e^{\pi\imath 4/7}, e^{\pi\imath 6/7}, e^{-\pi\imath 6/7}, e^{-\pi\imath 4/7}, e^{-\pi\imath 2/7}\}$$

Back to the residue numbers and checking the hypothesis, we return to the Loeser definitions. Indeed, we can hope again that the Loeser residue numbers (which are unique, in the sense that do not depend on a subdivision) may present a better situation. Nonetheless, we have that the residue numbers here are

- Associated toric residue numbers to divisor (1, 2, 1): {ST: -4/5 + 1, ...}
- Associated toric residue numbers to divisor (1, 1, 2): {ST: -4/5 + 1, (7, 18, 5): 2, ...}
- Associated toric residue numbers to divisor (7, 8, 15): {ST: -6/7 + 1, ...}
- Associated toric residue numbers to divisor (23, 7, 6): {ST: -6/7 + 1, ...}
- Associated toric residue numbers to divisor (6, 5, 14): {ST: -5/6 + 1, ...}
- Associated toric residue numbers to divisor (3, 1, 1): {ST: -5/6 + 1, ...}
- Associated toric residue numbers to divisor (1, 1, 1): {ST: -3/4 + 1, ...}

Also, we can point out that the bad divisor $(1,1,2)$ comes from a compact face and its bad neighbor ray also have associated faces that are compact. Moreover, the candidate value $-4/5$ indeed appears as a pole in the zeta function. It is also interesting to remark that this pole is simple, even though we have two divisors with this candidate value, so it could happen that a double pole arose from these contributions.

The two divisors with candidate $\sigma = -4/5$ are the bad divisor $(1,1,2)$ and also $(1,2,1)$. As discussed, one could expect that this candidate appears as a pole in the zeta function because of the second good divisor, instead of the bad one. To check this, we now compute the contributions of an individual divisor to the zeta function.

For that purpose, we must return to the expression given by Theorem 24, that is

$$Z_0(s) = \sum_{\substack{\tau \text{ vertex} \\ \text{of } \Gamma(f)}} J(\tau, s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ compact face of } \Gamma(f) \\ \dim\tau \geq 1}} (-1)^{\dim\tau}(\dim\tau)!\mathrm{Vol}(\tau)J(\tau, s)$$

where

$$J(\tau, s) := \sum_{i=1}^{r} J_{\Delta_i}(s), \qquad \text{with} \qquad J_{\Delta_i}(s) = \frac{\mathrm{mult}(\Delta_i)}{(N(a_{i_1})s + k(a_{i_1}))\cdots(N(a_{i_l})s + k(a_{i_l}))}$$

and $\Delta_\tau = \cup_{i=1}^r \Delta_i$ is a decomposition in simplicial cones of dimension $\dim \Delta_\tau = l$ such that $\dim(\Delta_i \cap \Delta_j) < l$, for all $i \neq j$. Notice that we only required a simplicial decomposition, which does not need adding any additional new rays, but rather reorganizing the existing rays into (possibly different) cones, such that the generators of each one are linearly independent.

Now, to extract the contribution of a single divisor with associated normal vector $a = a(\tau)$, we must add up only the terms from cones Δ_i that contain a as one of its generating rays. The important observation is that this is not the same as simply taking the terms where a fraction of $\frac{1}{N(a)s+k(a)}$ appears, as it may happen that another divisor a' gives rise to same candidate $-k(a)/N(a) = -k(a')/N(a')$. It is precisely the individual contributions of a divisor, rather than from a candidate σ, what we are interested in.

In our case, we pick $a_1 = (1,1,2)$, with $(k(a_1), N(a_1)) = (4,5)$. As discussed above, if we were to compute directly the residue of the zeta function at $s = -4/5$ we would not be selecting only the contribution from this divisor, as it happens that $a_2 = (1,2,1)$ also has the same candidate $\sigma(a_2) = 4/5$ as it has numerical data $(k(a_2), N(a_2)) = (4,5)$.

Then, for $(1,1,2)$ we obtain

$$\begin{aligned} Z_{0;(1,1,2)}(s) &= \frac{2}{5}\frac{5040s^5 + 18534s^4 + 28442s^3 + 23124s^2 + 10035s + 1851}{(7s+6)(6s+5)^2(5s+4)(4s+3)(s+1)} \\ &= \frac{44}{4s+3} - \frac{47}{5s+4} - \frac{126}{5(6s+5)} - \frac{7}{5(7s+6)} - \frac{54}{5(6s+5)^2} + \frac{16}{5(s+1)} \end{aligned}$$

and for $(1,2,1)$ we obtain

$$\begin{aligned} Z_{0;(1,2,1)}(s) &= \frac{1}{30}\frac{70560s^5 + 260316s^4 + 403664s^3 + 334581s^2 + 149083s + 28308}{(7s+6)^2(6s+5)(5s+4)(4s+3)(s+1)} \\ &= \frac{44}{4s+3} - \frac{47}{5s+4} - \frac{36}{5(6s+5)} - \frac{119}{5(7s+6)} - \frac{511}{30(7s+6)^2} + \frac{17}{5(s+1)} \end{aligned}$$

Thus, we cannot discard the contributions of the two divisors with candidate value $\sigma = 4/5$. In particular, the residues at the corresponding point are

$$\operatorname*{Res}_{s=-4/5} Z_{(1,2,1)}(s) = -\frac{47}{5}, \qquad \operatorname*{Res}_{s=-4/5} Z_{(1,1,2)}(s) = -\frac{47}{5}$$

We have seen that the pole $s = -4/5$ indeed appears in the total zeta function, so the total residue is not zero. However, one should be careful not to conclude that the total residue is simply the sum of the residues given by all divisors with the given candidate value. Indeed, in this case

$$\operatorname*{Res}_{s=-4/5} Z(s) = -\frac{72}{5} \neq \frac{-47}{5} + \frac{-47}{5}$$

Instead, what is happening is that the rays $(1,1,2)$ and $(1,2,1)$ appear in a same cone. Hence, we are double counting an extra term when adding both contributions. In that case, we ought to compute the contribution to the zeta function given by the cones Δ_i containing both, which turns out to be

$$\begin{aligned} Z_{0;(1,1,2),(1,2,1)}(s) &= \frac{1}{5}\frac{840s^4 + 2966s^3 + 4122s^2 + 2639s + 648}{(7s+6)(6s+5)(5s+4)(4s+3)(s+1)} \\ &= \frac{16}{4s+3} - \frac{22}{5s+4} - \frac{6}{5(6s+5)} - \frac{7}{5(7s+6)} + \frac{1}{s+1} \end{aligned}$$

with residue

$$\operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2),(1,2,1)}(s) = -\frac{22}{5}$$

So, now indeed it holds that

$$\begin{aligned} \operatorname*{Res}_{s=-4/5} Z_0(s) &= \frac{-72}{5} = \frac{-47}{5} + \frac{-47}{5} - \frac{-22}{5} \\ &= \operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2)}(s) + \operatorname*{Res}_{s=-4/5} Z_{0;(1,2,1)}(s) - \operatorname*{Res}_{s=-4/5} Z_{0;(1,1,2),(1,2,1)}(s) \end{aligned}$$

And in a more general situation, where there are more divisors with the same candidate σ, an inclusion-exclusion expression should be used in order to equate the total residue at the zeta function as a combination of the different residues of the partial contributions.

To sum up, we conclude the example by stating that even checking individual contributions of bad divisors to the zeta function, we should not expect that the residue is zero.

4 Example 4

Consider the polynomial $f = x^9 + y^8 + z^{11} + xyz^2 + xy^2z$, which is constructed in the same spirit as before. This case is included as a follow-up of the previous example, where a similar analysis is conducted. We find again that the residue at the pole introduced from a bad divisor is non-zero, but now additionally we no longer have an equality between the residues of different divisors with same candidate value.

First, we depict in the following Figure V.4 the Newton polygon.

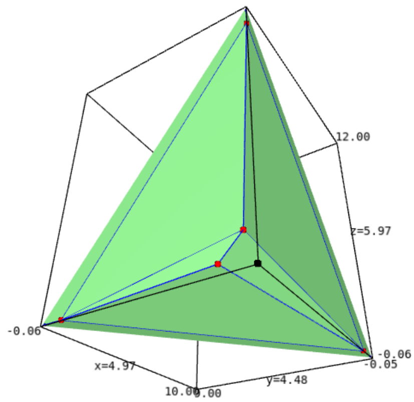


Figure V.4: Newton polygon of the polynomial $f = x^9 + y^8 + z^{11} + xyz^2 + xy^2z$.

The original rays on the associated dual fan are

$$[(1,0,0),(0,1,0),(0,0,1),(61,11,8),(8,9,46),(11,70,9),(5,1,1),(3,8,8)]$$

Then, the resulting toric residue numbers are

- Associated toric residue numbers to divisor (61, 11, 8): {ST: -10/11 + 1, ...}
- Associated toric residue numbers to divisor (11, 70, 9): {ST: -10/11 + 1, ...}
- Associated toric residue numbers to divisor (8, 9, 46): {ST: -7/8 + 1, (61, 11, 8): 3, ...}
- Associated toric residue numbers to divisor (5, 1, 1): {ST: -7/8 + 1, ...}
- Associated toric residue numbers to divisor (3, 8, 8): {ST: -19/27 + 1, ...}

where we find the bad divisor $(8, 9, 46)$, with candidate value $\sigma = -7/8$. Notice, we have a second divisor with this candidate, namely $(5, 1, 1)$.

The (local) zeta function is

$$Z_0(s) = \frac{581}{1188} \frac{s^3 + \frac{2297}{1162}s^2 + \frac{2609}{1162}s + \frac{95}{83}}{(s+19/27)(s+7/8)(s+10/11)(s+1)} = \frac{18}{s+1} - \frac{63/4}{s+7/8} - \frac{120/11}{s+10/11} + \frac{247/27}{s+19/27}$$

so, in particular, we have the total residue of

$$\operatorname*{Res}_{s=-7/8} Z_0(s) = -\frac{63}{4}$$

Now, we compute the individual contributions, first for $(5,1,1)$ as

$$\begin{aligned} Z_{0;(5,1,1)}(s) &= \frac{1}{72}\frac{470448s^4+1099296s^3+1528352s^2+1479247s+585567}{(27s+19)(11s+10)^2(8s+7)(s+1)} \\ &= \frac{91}{9(s+1)} - \frac{1070}{9(8s+7)} - \frac{5071}{72(11s+10)^2} - \frac{297}{8(11s+10)} + \frac{1809}{8(27s+19)} \end{aligned}$$

then for $(8,9,46)$ as

$$\begin{aligned} Z_{0;(8,9,46)}(s) &= \frac{1}{72}\frac{40392s^3+113425s^2+135066s+59369}{(27s+19)(11s+10)(8s+7)(s+1)} \\ &= -\frac{62}{8s+7} - \frac{187}{72(11s+10)} + \frac{777}{8(27s+19)} + \frac{37}{8(s+1)} \end{aligned}$$

and finally, the doubly counted contribution of

$$\begin{aligned} Z_{0;(5,1,1),(8,9,46)}(s) &= \frac{1}{72}\frac{21384s^3+56601s^2+75458s+37873}{(27s+19)(11s+10)(8s+7)(s+1)} \\ &= -\frac{494}{9(8s+7)} - \frac{11}{8(11s+10)} + \frac{81}{27s+19} + \frac{37}{9(s+1)} \end{aligned}$$

Thus, we compute the residues

$$\operatorname*{Res}_{s=-7/8} Z_{0;(5,1,1)}(s) = -\frac{535}{36}, \qquad \operatorname*{Res}_{s=-7/8} Z_{0;(8,9,46)}(s) = -\frac{31}{4}, \qquad \operatorname*{Res}_{s=-7/8} Z_{0;(5,1,1),(8,9,46)}(s) = -\frac{247}{36}$$

so, altogether, we can check that

$$\begin{aligned} \operatorname*{Res}_{s=-7/8} Z_0(s) &= \frac{-63}{4} = \frac{-535}{36} + \frac{-31}{4} - \frac{-247}{36} \\ &= \operatorname*{Res}_{s=-7/8} Z_{0;(5,1,1)}(s) + \operatorname*{Res}_{s=-7/8} Z_{0;(8,9,46)}(s) - \operatorname*{Res}_{s=-7/8} Z_{0;(5,1,1),(8,9,46)}(s) \end{aligned}$$

In comparison with the previous example, there it happened that the residues from both divisors with the same candidate where equal, but this is no longer true in this case. This shows that, not only the contribution of bad divisors may be different from zero, but that we should expect any simple comparison of the contributions of divisors with the same candidate.

5 Example 5

Consider the polynomial $f = x_1^4 + x_2^3 + x_3^3 + x_4^3 + x_1x_5^2$ in $\mathbb{C}[x_1,x_2,x_3,x_4,x_5]$, which we include for completeness to show that we can also encounter negative integer values for the residue numbers, even though we have already argued that this might not be a problem.

In this dimension, we will not be able to give a graphic representation of its Newton polygon and associated dual fan. However, we can describe that the Newton polygon has support points

$$[(4,0,0,0,0),(0,3,0,0,0),(0,0,3,0,0),(0,0,0,3,0),(1,0,0,0,2)]$$

and that it has 75 proper faces. The faces of maximum dimension are given by the following inequalities

- Facet 0: $6x_1 + 8x_2 + 8x_3 + 8x_4 + 9x_5 - 24 \geq 0$
- Facet 1: $3x_1 + x_2 + x_3 + x_4 - 3 \geq 0$
- Facet 2: $x_1 \geq 0$
- Facet 3: $x_2 \geq 0$
- Facet 4: $x_3 \geq 0$
- Facet 5: $x_4 \geq 0$
- Facet 6: $x_5 \geq 0$

As for the dual fan, the original rays are

$$[(0,0,0,0,1),(0,0,0,1,0),(0,0,1,0,0),(0,1,0,0,0),(1,0,0,0,0),(3,1,1,1,0),(6,8,8,8,9)]$$

with only two non-coordinate rays (corresponding with the two non coordinate facets).

A good resolution may be given by adding the rays

$$\begin{aligned}[&(1,1,1,1,1),(2,1,1,1,0),(2,3,3,3,3),(3,4,4,4,5),(1,1,1,1,0),(3,4,4,4,5),(3,3,3,3,4),\\ &(2,2,2,2,3),(3,3,3,3,4),(1,1,1,1,1),(3,4,4,4,5),(2,2,2,2,3),(3,2,2,2,2),(3,4,4,4,5),\\ &(2,2,2,2,3),(2,3,3,3,3),(3,2,2,2,2),(3,3,3,3,4),(2,1,1,1,1),(3,2,2,2,2),(1,1,1,1,1),\\ &(3,2,2,2,2),(3,2,2,2,2),(3,2,2,2,2),(2,1,1,1,1),(1,2,2,2,2),(1,1,1,1,2),(1,1,1,1,1),\\ &(4,4,4,4,5),(2,2,2,2,3),(1,1,1,1,2),(1,1,1,1,2),(4,3,3,3,3),(2,3,3,3,4),(2,2,2,2,3),\\ &(4,3,3,3,3),(4,4,4,4,5),(4,4,4,4,5),(5,5,5,5,6),(5,4,4,4,4),(2,3,3,3,4),(2,3,3,3,4),\\ &(1,1,1,1,2),(5,5,5,5,6),(4,3,3,3,3),(5,4,4,4,4),(2,1,1,1,1),(2,2,2,2,3),(1,1,1,1,2),\\ &(1,1,1,1,2),(5,5,5,5,6),(5,4,4,4,4)]\end{aligned}$$

This results in the residue numbers listed next

- Associated residue numbers to divisor (6, 8, 8, 8, 9):
  {ST: -13/8 + 1, (0, 0, 1, 0, 0): 1, (2, 3, 3, 3, 3): 1, (0, 1, 0, 0, 0): 1, (3, 4, 4, 4, 5): 1/2, (0, 0, 0, 1, 0): 1, (1, 1, 1, 1, 1): 1/8}
- Associated residue numbers to divisor (3, 1, 1, 1, 0):
  {ST: -2 + 1, (4, 3, 3, 3, 3): -2, (3, 2, 2, 2, 2): -1, (0, 0, 1, 0, 0): 1, (0, 0, 0, 0, 1): 1, (0, 1, 0, 0, 0): 1, (3, 4, 4, 4, 5): -4, (0, 0, 0, 1, 0): 1, (2, 1, 1, 1, 1): 0, (1, 0, 0, 0, 0): 1, (1, 1, 1, 1, 1): -1, (5, 4, 4, 4, 4): -3, (2, 1, 1, 1, 0): 1}

where we have highlighted in blue the negative integer values.

Chapter VI

Conclusions and future work

In this project we have studied the monodromy conjecture, in its strong version, and in particular the existing proofs in the cases of plane curve singularities and of Newton non-degenerate singularities.

To begin, we have reviewed the required preliminaries. First, we have described the basic concepts in resolution of singularities, then defined the zeta function and the problem of giving it a meromorphic continuation, as well as introduced the Bernstein-Sato polynomial as a tool to answer that question. Next, we have reviewed the concepts of sheaf theory, then stated the equivalence of categories of covers, locally constant sheaves and local systems, to finally define the notion of connection. With this, we have been able to describe the meaning and statement of the monodromy conjecture, in its strong and weaker versions. Lastly, we have introduced the study of asymptotics of periods of integrals, which are the current approach used to solve the cases that we will discuss.

Secondly, we have studied with detail the case of plane curves. Here, we have reviewed the concepts of branches, multiplicity and intersection number, and later discussed the theory of Puiseux's theorem, together with the associated invariants and Milnor fibration in this setting. Next, we have given a more concrete description of the resolution of singularities via blowups, and introduced the dual graph to represent the information in this process. Next, we have studied the expression of the zeta function and the contribution of the divisors to its poles, which we then use to give a proof of the conjecture.

Thirdly, we have studied with detail the case of Newton non-degenerate polynomials. We have begun by giving the definition of the Newton polygon and dual fan, and how this contains the information of a good resolution. Next, we have discussed the expressions of the zeta function in this case and its possible poles. Lastly, we have followed the approach by Loeser to prove the conjecture, under the conditions on the residue numbers discussed.

Lastly, we have included various examples of computations, where we have calculated the residue numbers (with regular subdivision and without it) and compared it with the hypothesis in the result of NND. Also, we have calculated the Bernstein-Sato polynomial of some examples, as well as the zeta function to check the relation between roots and poles.

To sum up, so far we have mentioned that the proof of the monodromy conjecture in the strong version for certain geometric configurations is basically to derive with hard work enough information from both sides of the problem. In particular, in the cases presented in this project, this is done via the study of asymptotics of certain integrals, after constructing a suitable non-zero cohomology class. We have studied this latter ideas, and checked the additional hypothesis on the residue numbers in the case of NND. Indeed, by computing several examples, we have found that it can happen that residue numbers are positive integers. Even worse, we have seen that divisors with such *bad* residue numbers can contribute with non-zero residue to the zeta function, so we can not discard its associated candidate as a pole.

In conclusion, we have confirmed that the current approach (of constructing a non-zero cohomology class via the mentioned results that require non-identity monodromies) does not allow removing the extra conditions in the proof for NND, and even less to extend it to the general case.

A possible future line of work would require to come up with a novel approach to tackle the problem, either by constructing such non-zero cohomology classes in another way, or rather by a completely different manner of showing that some candidate pole is a root of the Bernstein-Sato polynomial. Thus, there are many options to expand this work, but they are all currently out of the indented scope of the project.

References

[ACa75] Norbert A'Campo. "La fonction zeta d'une monodromie". In: *Commentarii Mathematici Helvetici* 50 (1975), pp. 233–248 (cit. on p. 21).

[ÀJN21] Josep Àlvarez Montaner, Jack Jeffries, and Luis Núñez-Betancourt. "Bernstein-Sato polynomials in commutative algebra". In: *Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of his 75th Birthday.* Springer, 2021, pp. 1–76 (cit. on p. 9).

[Ati70] Michael Francis Atiyah. "Resolution of singularities and division of distributions". In: *Communications on pure and applied mathematics* 23.2 (1970), pp. 145–150 (cit. on p. 6).

[AVG12] Vladimir Igorevich Arnold, Aleksandr Nikolaevich Varchenko, and Sabir Medzhidovich Gusein-Zade. *Singularities of differentiable maps: Volume II Monodromy and asymptotic integrals.* Vol. 83. Springer Science & Business Media, 2012 (cit. on p. 48).

[Ber72] Joseph Bernstein. "The analytic continuation of generalized functions with respect to a parameter". In: *Functional Analysis and its applications* 6.4 (1972), pp. 273–285 (cit. on pp. 6, 9).

[BG69] Joseph Bernstein and Sergei Gel'fand. "Meromorphy of the function $\mathrm{P}(\lambda)$". In: *Funkcional. Anal. i Prilozen* 3.1 (1969), pp. 84–85 (cit. on p. 6).

[Bjö73] Jan-Erik Björk. *Dimensions over algebras of differential operators.* Département de mathématiques, 1973 (cit. on p. 9).

[Bla20] Guillem Blanco. "Bernstein-Sato polynomial of plane curves and Yano's conjecture". PhD thesis. Universitat Politècnica de Catalunya, 2020 (cit. on p. 55).

[Bla24] Guillem Blanco. *Topological roots of the Bernstein-Sato polynomial of plane curves.* 2024. arXiv: `2406.09034 [math.AG]`. URL: `https://arxiv.org/abs/2406.09034` (cit. on p. 43).

[BM96] Joël Briançon and Philippe Maisonobe. "Caractérisation géométrique de l'existence du polynôme de Bernstein relatif". In: *Algebraic Geometry and Singularities.* Ed. by Antonio Campillo López and Luis Narváez Macarro. Basel: Birkhäuser Basel, 1996, pp. 215–236. ISBN: 978-3-0348-9020-5 (cit. on p. 9).

[Bri70] Egbert Brieskorn. "Die monodromie der isolierten singularitäten von hyperflächen". In: *Manuscripta mathematica* 2 (1970), pp. 103–161 (cit. on pp. 16, 19, 21).

[Cas10] Francisco J Castro Jiménez. "Modules over the Weyl algebra". In: *Algebraic Approach To Differential Equations.* World Scientific, 2010, pp. 52–118 (cit. on p. 9).

[CL55] Earl A Coddington and Norman Levinson. *Theory of ordinary differential equations.* Vol. 158. McGraw-Hill New York, 1955 (cit. on p. 17).

[CLS11] David A Cox, John B Little, and Henry K Schenck. *Toric varieties.* Vol. 124. American Mathematical Soc., 2011 (cit. on p. 49).

[Den87] Jan Denef. "On the degree of Igusa's local zeta function". In: *American Journal of Mathematics* 109.6 (1987), pp. 991–1008 (cit. on p. 8).

[DH01] Jan Denef and Kathleen Hoornaert. "Newton polyhedra and Igusa's local zeta function". In: *Journal of number Theory* 89.1 (2001), pp. 31–64 (cit. on p. 48).

[DL92] Jan Denef and François Loeser. "Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques". In: *Journal of the American Mathematical Society* 5.4 (1992), pp. 705–720 (cit. on pp. 7, 8, 52).

[DM86] Pierre Deligne and George Daniel Mostow. "Monodromy of hypergeometric functions and non-lattice integral monodromy". In: *Publications Mathématiques de l'IHÉS* 63 (1986), pp. 5–89 (cit. on pp. 23, 24, 55).

[EN85] David Eisenbud and Walter D Neumann. *Three-dimensional link theory and invariants of plane curve singularities.* 110. Princeton University Press, 1985 (cit. on pp. 26, 37, 43).

[EV92] Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*. Vol. 20. Springer, 1992 (cit. on p. 24).

[Ful93] William Fulton. *Introduction to toric varieties*. 131. Princeton university press, 1993 (cit. on p. 49).

[Gel54] Israel M Gel'fand. "Some aspects of functional analysis and algebra". In: *Proceedings of the International Congress of Mathematicians, Amsterdam*. Vol. 1. 1954, pp. 253–276 (cit. on p. 6).

[GLS07] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin. *Introduction to Singularities and Deformations*. Springer Berlin, Heidelberg, 2007. ISBN: 978-3-540-28419-2 (cit. on p. 3).

[Gra10] Michel Granger. "Bernstein-Sato polynomials and functional equations". In: *Algebraic Approach to Differential Equations*. World Scientific, 2010, pp. 225–291 (cit. on pp. 9, 52).

[Hau03] Herwig Hauser. "The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)". In: *Bulletin of the American Mathematical Society* 40.3 (2003), pp. 323–403 (cit. on p. 4).

[Hau10] Herwig Hauser. "On the problem of resolution of singularities in positive characteristic (or: a proof we are still waiting for)". In: *Bulletin of the American Mathematical Society* 47.1 (2010), pp. 1–30 (cit. on p. 4).

[Hir64] Heisuke Hironaka. "Resolution of singularities of an algebraic variety over a field of characteristic zero: II". In: *Annals of Mathematics* (1964), pp. 205–326 (cit. on p. 4).

[Igu00] Jun-ichi Igusa. *An introduction to the theory of local zeta functions*. Vol. 14. American Mathematical Soc., 2000 (cit. on p. 6).

[Kas76] Masaki Kashiwara. "b-functions and holonomic systems". In: *Inventiones mathematicae* 38.1 (1976), pp. 33–53 (cit. on pp. 9, 10).

[Kem+06] George Kempf et al. *Toroidal embeddings 1*. Vol. 339. Springer, 2006 (cit. on p. 48).

[Kol97] János Kollár. "Singularities of pairs". In: *Proceedings of Symposia in Pure Mathematics*. Vol. 62. American Mathematical Society. 1997, pp. 221–288 (cit. on p. 10).

[Kou76] Anatoli G Kouchnirenko. "Polyedres de Newton et nombres de Milnor". In: *Inventiones mathematicae* 32.1 (1976), pp. 1–31 (cit. on p. 45).

[Lê 72] Trang Lê Dung. "Sur les noeuds algébriques". In: *Compositio Math.* 25 (1972), pp. 281–321 (cit. on p. 42).

[Lic89] Ben Lichtin. "Poles of $|f(z,w)|^{2s}$ and roots of the b-function". In: *Arkiv för Matematik* 27.1 (1989), pp. 283–304 (cit. on pp. 10, 42).

[Loe88] François Loeser. "Fonctions d'Igusa p-adiques et pôlynomes de Berstein". In: *American Journal of Mathematics* 110.1 (1988), pp. 1–21 (cit. on pp. 24, 42).

[Loe90] François Loeser. "Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton." In: (1990) (cit. on pp. 1, 24, 39, 52–56).

[Mac06] James MacLaurin. "The Resolution of Toric Singularities". PhD thesis. School of Mathematics, The University of New South Wales, 2006 (cit. on p. 49).

[Mal73] Bernard Malgrange. "Sur les polynômes de IN Bernstein". In: *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi" Séminaire Goulaouic-Schwartz"* (1973), pp. 1–10 (cit. on pp. 22–24).

[Mal74] Bernard Malgrange. "Intégrales asymptotiques et monodromie". In: *Annales scientifiques de l'École normale supérieure*. Vol. 7. 3. 1974, pp. 405–430 (cit. on pp. 22, 23).

[Mal75] Bernard Malgrange. "Le polynôme de Bernstein d'une singularité isolée". In: *Fourier Integral Operators and Partial Differential Equations*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1975, pp. 98–119. ISBN: 978-3-540-37521-0 (cit. on pp. 10, 19).

[Mal83] Bernard Malgrange. "Polynômes de Bernstein-Sato et cohomologie évanescente". In: *Astérisque* 101.102 (1983), pp. 243–267 (cit. on p. 19).

[Mil16] John Milnor. *Singular Points of Complex Hypersurfaces (AM-61)*. Vol. 61. Princeton University Press, 2016 (cit. on p. 20).

[MN91] Zoghman Mebkhout and Luis Narváez-Macarro. "La théorie du polynôme de Bernstein-Sato pour les algebres de Tate et de Dwork-Monsky-Washnitzer". In: *Annales scientifiques de l'École normale supérieure*. Vol. 24. 2. 1991, pp. 227–256 (cit. on p. 9).

[Oak97] Toshinori Oaku. "Algorithms for the b-function and D-modules associated with a polynomial". In: *Journal of Pure and Applied Algebra* 117 (1997), pp. 495–518 (cit. on p. 9).

[Oka96] Mutsuo Oka. "Geometry of plane curves via toroidal resolution". In: *Algebraic geometry and singularities*. Springer, 1996, pp. 95–121 (cit. on pp. 49, 50).

[Pui50] Victor Puiseux. "Recherches sur les fonctions algébriques". In: *Journal de mathématiques pures et appliquées* 15 (1850), pp. 365–480 (cit. on p. 27).

[PV20] Naud Potemans and Willem Veys. "Introduction to p-adic Igusa zeta functions". In: (2020) (cit. on p. 7).

[RR06] Michèle Raynaud and Dock Sang Rim. *Groupes de Monodromie en Geometrie Algebrique: Seminaire de Geometrie Algebrique du Bois-Marie 1967-1969. (SGA 7 I)*. Vol. 288. Springer, 2006 (cit. on p. 21).

[Seb70] Marcos Sebastiani. "Preuve d'une conjecture de Brieskorn". In: *Manuscripta mathematica* 2 (1970), pp. 301–308 (cit. on p. 19).

[Seg12] Dirk Segers. *The asymptotic behaviour of the number of solutions of polynomial congruences*. 2012. arXiv: `1208.4704 [math.NT]`. URL: `https://arxiv.org/abs/1208.4704` (cit. on p. 7).

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups, Cambridge Uni.* Press, 2009 (cit. on pp. 14–16).

[Tur+59] Herbert Turnbull et al. "The Correspondence of Sir Isaac Newton". In: *Cambridge University Press* 1977 (1959), p. 253 (cit. on p. 27).

[VE87] Eckart Viehweg and Hélène Esnault. "A remark on a non-vanishing theorem of P. Deligne and GD. Mostow." In: (1987) (cit. on p. 55).

[Vey24] Willem Veys. *Introduction to the monodromy conjecture*. 2024. arXiv: `2403.03343 [math.AG]`. URL: `https://arxiv.org/abs/2403.03343` (cit. on pp. 1, 6–8, 10, 22, 36).

[Vey91] Willem Veys. "Relations between numerical data of an embedded resolution". In: *Amer. J. Math* 113.4 (1991), pp. 573–592 (cit. on p. 39).

[Viu12] Juan Viu-Sos. "Funciones zeta y poliedros de Newton: Aspectos teóricos y computacionales". In: *Trabajo Fin de Master, Director: Enrique Artal Bartolo. Universidad Zaragoza* (2012) (cit. on pp. 52, 76).

[Viu21] Juan Viu-Sos. "An introduction to p-adic and motivic integration, zeta functions and invariants of singularities". In: *Contemp. Math* 778 (2021) (cit. on p. 20).

[Wal04] Charles Terence Clegg Wall. *Singular points of plane curves*. 63. Cambridge University Press, 2004 (cit. on pp. 20, 26–28, 30, 31, 33, 34, 36, 40, 41).

[Zar06] Oscar Zariski. *The moduli problem for plane branches*. American Mathematical Soc., 2006 (cit. on p. 30).

Appendix A

Codes

All the codes and the `Jupyter Notebook` developed in this work will be included in the GitHub repository `MonodromyNND`: https://github.com/baezaguasch/MonodromyNND.

1 Plane curves

The following code allows to compute the basic invariants described in this project for plane curves, starting from the Puiseux characteristic exponents of a plane branch. It also allows to obtain the associated dual graph, to plot it with decorations and to plot the corresponding splice diagram.

It is based on the `Magma` library `SingularitiesDim2` developed by Guillem Blanco, which can be found in https://github.com/gblanco92/SingularitiesDim2.

PlaneCurves.py

```python
import numpy as np
from math import gcd
import networkx as nx
import matplotlib.pyplot as plt


def Beta_from_series_exp(n,Exps):
    """
    Returns Puiseux characteristic exponents Beta given the multiplicity n and the
    exponents Exps appearing in the Puiseux series of a good parametrization.
    """

    Beta = [n]
    e = [n]

    e_curr = n
    while e_curr != 1:
        index = 0
        while (Exps[index]%e_curr == 0):
            index += 1

        beta_curr = Exps[index]
        e_curr = gcd(e_curr, beta_curr)

        Beta.append(beta_curr)
        e.append(e_curr)

    return Beta, e


def blowup_charexp(Beta):
    """
    Computes a blowup via the Puiseux characteristic exponents.
    Check Thm. 3.5.5 in [Wal04] for details and proof.

    REFERENCES:
    - [Wal04] CTC Wall. Singular points of plane curves. 63.
              Cambridge University Press, 2004
    """

    new_Beta = []
    m = Beta[0]
    b1 = Beta[1]

    if b1 > 2*m:
        new_Beta.append(m)
        for i in range(1,len(Beta)):
            new_Beta.append(Beta[i] - m)
    elif m % (b1 - m) != 0:
        new_Beta.append(b1 - m)
        new_Beta.append(m)
        for i in range(2,len(Beta)):
            new_Beta.append(Beta[i] - Beta[i-1] + m)
    else:
        new_Beta.append(b1 - m)
        for i in range(2,len(Beta)):
```

```
            new_Beta.append(Beta[i] - Beta[i-1] + m)
    return new_Beta


def multipl_from_charexp(Beta):
    """
    Returns the multiplicities of the strict transform Mult from the Puiseux
    characteristic exponents, that is Beta = (n; beta1, ..., betag)
    """

    Mult = [Beta[0]]

    while len(Beta) > 1:
        Beta = blowup_charexp(Beta)
        Mult.append(Beta[0])

    # Extra 1?
    Mult.append(1)
    return Mult


def proximity_matrix_from_mult(Mult):
    """
    Returns the proximity matrix P given the multiplicities of the strict transform Mult.
    """

    s = len(Mult)

    P = np.eye(s, dtype=int)

    for i in range(0,s-1):
        sum_prox_m = 0

        j = i+1
        while sum_prox_m < Mult[i]:
            sum_prox_m += Mult[j]
            P[j][i] = -1
            j = j+1

    return P


def info_from_charexp(Beta):
    """
    Computes several information of the plane curve, given the Puiseux characteristic
    exponents Beta = (n; beta1, ..., betag).

    Returns the Zariski pairs {(q,n)}, the gcd quantities {e}, the reduced Puiseux
    exponents {m}, the semigroup generators {Semig}, its conductor {conductor} and the
    reduced quantities {m_bar} = Semig/e.
    """

    e = [Beta[0]]*len(Beta)
    for i in range(1, len(Beta)):
        e[i] = gcd( e[i-1], Beta[i] )

    n = [0] + [  int(e[i-1]/e[i]) for i in range(1, len(Beta)) ]

    m = [0] + [ int(Beta[i]/e[i]) for i in range(1, len(Beta)) ]

    q = [0] + [  m[i]-n[i]*m[i-1] for i in range(1, len(Beta)) ]

    Semig = [Beta[0], Beta[1]]
    for i in range(2, len(q)):
        Semig.append( n[i-1] * Semig[i-1] - Beta[i-1] + Beta[i] )

    conductor = n[-1]*Semig[-1] - Beta[-1] - (Beta[0]-1)

    mbar  = [ int(Beta[1]/e[1]) ] + [ int(Semig[i]/e[i]) for i in range(2, len(q)) ]

    #return list(zip(Q[1:],n[1:]),
    return q[1:], n[1:], e, m, Semig, conductor, mbar


def autointersection_matrix(P):
    """
    Given the proximity matrix P, returns the autointersection matrix P^t*P.
    """
    return np.matmul(P.transpose(),P)


def edge_decoration(A,i,j):
    """
    Computes edge decoration next to Ei, along edge Ei->Ej, where Ei, Ej are
    exceptional divisors and A represents the autointersection matrix.
    """

    seen = set()
    # visit j and all neighbors of j (except i) recursively
```

```python
    to_see = set([j])
    while to_see:
        node = to_see.pop()
        seen.add(node)

        # Get neighbors
        for n in range(0,len(A)):
            # !ATTENTION: +-1 according to criteria in definition A = Pt*P
            if (n != i and A[node][n] == -1):
                # If not seen, add them to_see
                if n not in seen:
                    to_see.add(n)

    indices = list(seen)#.sort()

    # Select submatrix to compute the determinant
    subA = A[np.ix_(indices,indices)]
    decor = int(abs(np.linalg.det(subA)))

    return decor


def dual_graph(A):
    """
    Returns the matrix representation G of the dual graph obtained from the
    information in the autointersection matrix A.
    """

    n_div_exc = len(A)

    G = nx.from_numpy_matrix(-A)

    return G

def dual_graph_ST(A):
    """
    Returns the matrix representation G of the dual graph obtained from the information
    in the autointersection matrix A, and adding the strict transform information.
    """

    n_div_exc = len(A)

    G = nx.from_numpy_matrix(-A)

    # Add strict transform through last vertex
    G.add_node(n_div_exc)
    G.add_edge(n_div_exc-1, n_div_exc)

    return G


def plot_decorated_dual_graph(A):
    """
    Plots the decorated dual graph from the information in the autointersection matrix A.
    Additionally, each exceptional divisor includes in parenthesis their autointersection
    number. The strict transform is represented by the label ST, instead of the usual arrow.

    The decorations and their properties are described in [Bla24], and are depicted in red.
    They are added for each directed egde, and placed closer to the initial edge considered.

    REFERENCES:
     - [Bla24] G Blanco Fernandez. Topological roots.
               Arxiv preprint: ?. 2024
    """

    n_div_exc = len(A)

    G = nx.from_numpy_matrix(-A)

    # Add strict transform passing through the last vertex
    G.add_node(n_div_exc)
    G.add_edge(n_div_exc-1, n_div_exc)

    pos = nx.spring_layout(G)
    plt.figure()

    vertex_labels = {}
    for node in G.nodes():
        if node != n_div_exc:
            vertex_labels[node] = f"E{node} ({A[node,node]})"
        else:
            vertex_labels[node] = "ST"

    node_sizes = [250]*(n_div_exc+1)
    node_sizes[-1] = 0

    nx.draw(
        G, pos, edge_color="black", width=1, linewidths=1,
        node_size=node_sizes, node_color="cyan", alpha=0.9,
```

```
        labels=vertex_labels, font_size=10
    )

    edge_labels = {}
    for n1, n2 in G.edges:
        if n2 == n_div_exc:
            edge_labels[(n1,n2)] = 1
        elif n1 != n2:
            edge_labels[(n1,n2)] = edge_decoration(A,n1,n2)
            edge_labels[(n2,n1)] = edge_decoration(A,n2,n1)


    nx.draw_networkx_edge_labels(
        G, pos, edge_labels=edge_labels, label_pos=0.8,
        font_color="red", font_size=10#, font_weight="bold"
    )

    plt.axis("off")
    plt.show()


def splice_diag_from_pairs(n,mbar):
    """
    Plots the splice diagram with decorations, obtained from the values (n, mbar),
    derived from the Zariski pairs {(q,n)}

    See [Bla24] for details, and how to merge splice diagrams for multiple branches.

    REFERENCES:
     - [Bla24] G Blanco Fernandez. Topological roots.
                Arxiv preprint: code. 2024
    """

    g = len(n)
    G = nx.Graph()

    # Add nodes and edges
    G.add_node(0)
    for i in range(g):
        G.add_node(2*i+1)
        G.add_node(2*i+2)

        G.add_edge(2*i, 2*i+2)
        G.add_edge(2*i+1, 2*i+2)

    # Add strict transform passing through the last vertex
    G.add_node(2*g+1)
    G.add_edge(2*g, 2*g+1)

    pos = nx.spring_layout(G)
    plt.figure()

    vertex_labels = {}
    vertex_labels[2*g+1] = "ST"
    node_sizes = [100]*(2*g+2)
    node_sizes[-1] = 0

    nx.draw(
        G, pos, edge_color="black", width=1, linewidths=1,
        node_size=node_sizes, node_color="red", alpha=1,
        labels=vertex_labels, font_size=10, font_color="red"
    )

    edge_labels = {}
    for i in range(g):
        edge_labels[(2*i+2, 2*i+1)] = n[i]
        edge_labels[(2*i+2, 2*i)] = mbar[i]
        if (i < g-1):
            edge_labels[(2*i+2, 2*i+4)] = 1
    edge_labels[(2*g, 2*g+1)] = 1

    nx.draw_networkx_edge_labels(
        G, pos, edge_labels=edge_labels, label_pos=0.8,
        font_color="red", font_size=10#, font_weight="bold"
    )

    plt.axis("off")
    plt.show()


def splice_diag_from_dual(A):
    """
    Plots the splice diagram with decorations, obtained from the information of
    the dual graph given in the form of the autointersection matrix A.

    The process consists of constructing the dual graph, computing the decorations,
    and then removing all edges except for those on rupture vertices or supporting
    arrowheads. See [EN85, Thm. 20.1] for details and proof of the construction.
```

```
    REFERENCES:
    - [EN85] D Eisenbud and WD Neumann. Three-dimensional link theory
              and invariants of plane curve singularities. 110.
              Princeton University Press, 1985
    """

    n_nodes = len(A)

    G = nx.from_numpy_matrix(-A)

    # Add strict transform passing through the last vertex
    G.add_node(n_nodes)
    G.add_edge(n_nodes-1, n_nodes)

    vertex_labels = {}
    vertex_labels[n_nodes] = "ST"

    def check_degree(node):
        deg = 0
        for k in range(len(A)):
            if k != node:
                deg -= A[node][k]
        if node == len(A)-1:
            deg += 1
        return deg

    edge_labels = {}
    labeled_nodes = set()
    for n1, n2 in G.edges:
        if n2 == n_nodes:
            edge_labels[(n1,n2)] = 1
        elif n1 != n2:
            if check_degree(n1) >= 3:
                labeled_nodes.add(n1)
                edge_labels[(n1,n2)] = edge_decoration(A,n1,n2)
            if check_degree(n2) >= 3:
                labeled_nodes.add(n2)
                edge_labels[(n2,n1)] = edge_decoration(A,n2,n1)
    print(edge_labels)
    print(A)

    # Prune
    check_nodes = set(G.nodes())
    while check_nodes:
        j = check_nodes.pop()

        # List of neighbors
        neighbors = [n for n in G.neighbors(j) if n != j]

        # If degree = 2, eliminate vertex
        # Skip the except divisor intersecting the strict transform
        if len(neighbors) == 2 and j != n_nodes-1:
            n1 = neighbors[0]
            n2 = neighbors[1]

            # Add edge and copy decorations
            G.add_edge(n1,n2)
            if n1 in labeled_nodes:
                edge_labels[(n1,n2)] = edge_labels.pop((n1,j))
            if n2 in labeled_nodes:
                edge_labels[(n2,n1)] = edge_labels.pop((n2,j))

            # Remove node
            G.remove_node(j)

            print(f"Deleted vertex {j}, who had neighbors {n1}, {n2}")
            print(A)
            print(edge_labels)
            print(check_nodes)

            # Restart search
            check_nodes = set(G.nodes())


    # Plot
    pos = nx.spring_layout(G)
    plt.figure()

    node_sizes = [100]*len(G.nodes())
    node_sizes[-1] = 0

    nx.draw(
        G, pos, edge_color="black", width=1, linewidths=1,
        node_size=node_sizes, node_color="red", alpha=1,
        labels=vertex_labels, font_size=10, font_color="red"
    )

    nx.draw_networkx_edge_labels(
        G, pos, edge_labels=edge_labels, label_pos=0.8,
```

```
        font_color="red", font_size=10#, font_weight="bold"
    )

    plt.axis("off")
    plt.show()



"""
# Example computation from exponents in good parametrization of a branch
n = 4
Exps = [6,7]
Beta, e = Beta_from_series_exp(n,Exps)
"""

# Example computation from Puiseux characteristic exponents of a branch
Beta = [4,6,7]

print(f"Puiseux characteristic exponents: {Beta} \n")


Mult = multipl_from_charexp(Beta)
print(f"Multiplicities of ST: {Mult} \n")


P = proximity_matrix_from_mult(Mult)
print(f"Proximity matrix P: \n{P}\n")

invP = np.linalg.inv(P).astype(int)
print(f"Inverse of proximity matrix P^(-1):\n {invP} \n")


print("Numerical data")
N = np.matmul(Mult, invP.transpose())
print(f"N: {N}")

k = np.matmul(np.ones(len(N), dtype=int), invP.transpose())
# Definicio Guillem
k += np.ones(len(N), dtype=int)
print(f"k: {k} \n")


sigma = [-(ki)/Ni for ki,Ni in zip(k,N)]
sigma_rat = [f"-{ki}/{Ni}" for ki,Ni in zip(k,N)]
print(f"sigma: \n {sigma} \n {sigma_rat} \n")


q, n, e, m, Semig, conductor, mbar = info_from_charexp(Beta)
print(f"Zariski pairs (q,n): \n {list(zip(q,n))} \n")

g = len(q)
print(f"Number of brances in dual graph g: {g} \n")

print(f"Semigroup: {Semig}")
print(f"Conductor: {conductor} \n")


A = autointersection_matrix(P)
print(f"Autointersection matrix P^t*P: \n {A} \n")

G = dual_graph(A)
n_div_exc = len(A)

neighbors = [ [n for n in G.neighbors(j) if n != j] for j in G.nodes ]
print(f"Neighbors dual graph (without ST): \n {neighbors} ")

degrees = [len(n) for n in neighbors]
print(f"Degrees dual graph (without ST): \n {degrees} \n")


epsilon = [ [ (N[j]*sigma[i] + k[j]) for j in neighbors[i] ] for i in G.nodes ]
# Add strict transform
epsilon[-1].append(Mult[n_div_exc -1]*sigma[n_div_exc -1])
print(f"Epsilons (with ST): \n {epsilon} \n")


# Plot of the decorated dual graph
plot_decorated_dual_graph(A)

# Plot of the splice diagram from Zariski pairs
splice_diag_from_pairs(n, mbar)

# Plot of the splice diagram from dual graph info
splice_diag_from_dual(A)
```

2 Topological zeta function contributions

The following code contains two methods that ought to be added to the `ZetaFunctions()` class in the library `ZetaFunctionsNewtonND.py` developed by Juan Viu, which can be found in https://github.com/jviusos/ZetaFunctionsNewtonND-Sagemath, and is part of his work in [Viu12].

The added functions allow to compute the separated contributions to the topological zeta function of Newton non-degenerate polynomials.

ZetaTopContributions.py

```
"""
This code adds routines to the class
    class ZetaFunctions():
in the code "ZetaFunctionsNewtonND.py" in
    https://jviusos.github.io/
by Juan Viu-Sos (juan.viusos@upm.es), from Universidad Politecnica de Madrid
"""

import ZetaFunctionsNewtonND

def top_zeta_contrib(self, d=1, local=False, weights=None, info=False, check="ideals", print_result=
    True):
    """
    Return the expression of the topological zeta function and the individual contributions:
    for each ray, runs through all cone containing it and adds up only these terms.
    The output is a dictionary where the key is the face and the value the contribution.

    INPUTS and WARNING are the same as the function ''topological_zeta''

    EXAMPLES:

        sage: R.<x,y,z> = QQ[]
        sage: zex = ZetaFunctions(x^5 + y^6 + z^7 + x^2*y*z + x*y*z^2 + x*y^2*z)
        sage: s = var("s")
        sage: contrib = zex.top_zeta_contrib(d=1, local=True, info=False)

        > Analyzing contribution of ray (1, 2, 1)
        > Which appears in face given by rays
        >       [(3, 1, 1), (23, 7, 6), (1, 1, 1), (1, 2, 1), (7, 18, 5)]
        > and contributes by:
        >   (464/11025) * (s + 6/7)^-2 * (s + 3/4)^-1 * (s + 4/5)^-1 *
                        * (s + 5/6)^-1 * (s^2 + 12163/7424*s + 1245/1856)
        > ...

    """
    f = self._f
    s = polygen(QQ, "s")
    ring_s = s.parent()
    P = self._Gammaf
    if check != "no_check":
        if local:
            if is_newton_degenerated(f, P, local=True, method=check, info=info):
                raise TypeError("degenerated wrt Newton")
        else:
            if is_global_degenerated(f, method=check):
                raise TypeError("degenerated wrt Newton")
    else:
        print("Warning: not checking the non-degeneracy condition!")
    result = ring_s.zero()
    if local:
        faces_set = compact_faces(P)
    else:
        faces_set = proper_faces(P)
        if d == 1:
            total_face = faces(P)[-1]
            dim_gamma = total_face.dim()
            vol_gamma = face_volume(f, total_face)
            result = (s / (s + 1)) * (-1) ** dim_gamma * vol_gamma
            if info:
                print("Gamma: total polyhedron")
                print("J_gamma = 1")
                print("dim_Gamma!*Vol(Gamma) = " + str(vol_gamma))
                print()

    faces_set = face_divisors(d, faces_set, P)

    dict_rays = dict()
    possible_rays = set()

    for tau in faces_set:
        dict_rays[tau] = [tuple(ray) for ray in cone_from_face(tau).rays()]
        if info:
            print(f"Face info: {face_info_output(tau)}")
            print(f"With rays: {dict_rays[tau]} \n")
        for ray in dict_rays[tau]:
```

```
                possible_rays.add(ray)

    if info:
        print(f"Possible rays: {possible_rays} \n")

    contrib = dict()

    for ray in possible_rays:
        if print_result:
            print(f"\nAnalyzing contribution of ray {ray}")
        result = 0
        for tau in faces_set:
            if ray in dict_rays[tau]:
                if print_result:
                    rays_face = [tuple(ray) for ray in cone_from_face(tau).rays()]
                    print(f"Which appears in face given by rays {rays_face}")
                J_tau, cone_info = Jtau(tau, P, weights, s)
                dim_tau = tau.dim()
                vol_tau = face_volume(f, tau)

                if d == 1:
                    if dim_tau == 0:
                        term = J_tau
                    else:
                        term = (s / (s + 1)) * ((-1) ** dim_tau) * vol_tau * J_tau
                else:
                    term = ((-1) ** dim_tau) * vol_tau * J_tau
                result += term
                result = simplify(expand(result))
                if result != 0:
                    result = result.factor()
                if print_result:
                    print(f"and contributes by: {result}\n")
        contrib[ray] = result

    return contrib


def top_zeta_contrib_containing_rays(self, cont_rays, d=1, local=False, weights=None, info=False,
     check="ideals", print_result=True):
    """
    Return the contribution to the topological zeta function only of the terms from faces whose
    associated dual cone contains the given ray(s) given in {cont_rays}.
    The output is a dictionary where the key is the face and the value the contribution.

    WARNING and INPUTS are the same as the function ''topological_zeta'', except for the additional:

    - ''cont_rays'' -- array of rays (each as a tuple) for which to compute the contribution

    EXAMPLES:

        sage: R.<x,y,z> = QQ[]
        sage: zex = ZetaFunctions(x^5 + y^6 + z^7 + x^2*y*z + x*y*z^2 + x*y^2*z)
        sage: s = var("s")
        sage: cont_rays = [(1,1,2), (1,2,1)]
        sage: contrib_selected = zex.top_zeta_contrib_containing_rays(cont_rays,
                 d=1, local=True, info=True)

        > Analyzing contribution of rays [(1, 1, 2), (1, 2, 1)]

        > Which appear(s) in face given by rays [(1, 1, 1), (1, 2, 1), (1, 1, 2)]
        > and contributes by: (1/100) * (s + 4/5)^-2 * (s + 3/4)^-1
        >  ...

    """

    f = self._f
    s = polygen(QQ, "s")
    ring_s = s.parent()
    P = self._Gammaf
    if check != "no_check":
        if local:
            if is_newton_degenerated(f, P, local=True, method=check, info=info):
                raise TypeError("degenerated wrt Newton")
        else:
            if is_global_degenerated(f, method=check):
                raise TypeError("degenerated wrt Newton")
    else:
        print("Warning: not checking the non-degeneracy condition!")
    result = ring_s.zero()
    if local:
        faces_set = compact_faces(P)
    else:
        faces_set = proper_faces(P)
        if d == 1:
            total_face = faces(P)[-1]
            dim_gamma = total_face.dim()
            vol_gamma = face_volume(f, total_face)
            result = (s / (s + 1)) * (-1) ** dim_gamma * vol_gamma
```

```
            if info:
                print("Gamma: total polyhedron")
                print("J_gamma = 1")
                print("dim_Gamma!*Vol(Gamma) = " + str(vol_gamma))
                print()

    faces_set = face_divisors(d, faces_set, P)

    dict_rays = dict()

    for tau in faces_set:
        dict_rays[tau] = [tuple(ray) for ray in cone_from_face(tau).rays()]

    if print_result:
        print(f"\n      Analyzing contribution of rays {cont_rays} \n")

    dict_contrib = dict()

    for tau in faces_set:
        result = 0

        check_face = True
        for ray in cont_rays:
            if not (ray in dict_rays[tau]):
                check_face = False

        if check_face:
            J_tau, cone_info = Jtau(tau, P, weights, s)
            dim_tau = tau.dim()
            vol_tau = face_volume(f, tau)

            if d == 1:
                if dim_tau == 0:
                    term = J_tau
                else:
                    term = (s / (s + 1)) * ((-1) ** dim_tau) * vol_tau * J_tau
            else:
                term = ((-1) ** dim_tau) * vol_tau * J_tau
            result += term
            result = simplify(expand(result))
            if result != 0:
                result = result.factor()

            dict_contrib[tau] = result

            associated_rays = [tuple(ray) for ray in cone_from_face(tau).rays()]

            if print_result:
                print(f"Which appear(s) in face given by rays {associated_rays}")
                print(f"and contributes by: \n {result}\n")

    return dict_contrib


ZetaFunctions.top_zeta_contrib = top_zeta_contrib
ZetaFunctions.top_zeta_contrib_containing_rays = top_zeta_contrib_containing_rays
```

3 Regular subdivison and residue numbers

The following code contains the basic functions required for computing a regular subdivision of a fan (including the computation of the required additional blowup rays), and the computation of the residue numbers (with and without the regular subdivision).

Again, it depends on some of the functions defined in the library `ZetaFunctionsNewtonND.py` developed by Juan Viu.

ResidueNumbersFromND.py

```
import ZetaFunctionsNewtonND

def backtracking(den, coef, index):
    """
    List of possible coefficients used to compute the blowup ray as a linear combination
    of the cone rays. It lists positive integer vectors whose coordiantes are <= den,
    and are obtained via a simple backtracking.
    """

    N = len(coef)
    if index == N:
        #print(coef)
        return [coef]

    L = []
    L = L + backtracking(den, coef, index + 1)
    i = 1
    while coef[index] + i < den:
        new_coef = coef.copy()
        new_coef[index] += i
        L = L + backtracking(den, new_coef, index + 1)
        i += 1
    return L


def blowup_ray(R):
    """
    Compute the blowup ray added to subdivide a non regular cone given by the columns of R
    Required: the vectors in R are primitive (gcd of its components = 1)
    """

    N = R.nrows()
    M = R.ncols()
    den = 2
    list_coefs = []

    while den < 100000:
        list_coefs = backtracking(den, [0]*N, 0)

        for coef in list_coefs[1:]:
            x = [0]*M
            for j in range(M):
                for k in range(N):
                    x[j] += coef[k]*R[k][j]/den

            # Check if all entries are integers
            all_integers = True
            for xj in x:
                if floor(xj) != xj:
                    all_integers = False
            # Return x in that case
            if all_integers:
                x_gcd = x[0]
                for i in range(M):
                    x_gcd = gcd(x_gcd, x[i])
                x_red = [xj/x_gcd for xj in x]
                return x_red
        # Try next denominator
        den += 1
        list_coefs = []

    # Error, not found for denominator up to 100000 (increase depth cutoff)
    return -1

def simple_partition(fan, printing=False):
    """
    Compute a simple (regular) partition of the given fan
    """

    P = fan.cone_lattice()
    added_rays = []

    # Ordered list of subcones of maximum dimension
    list_cones =  [ls for i in range(1, fan.dim() + 1) for ls in fan.cones(i)]
```

```
    # Make simplicial subdivision
    list_simplicial_cones = []
    for cone in list_cones:
        for c in simplicial_partition(cone):
            list_simplicial_cones.append(c)

    max_dim_cones = [c for c in list_simplicial_cones if c.dim() == fan.dim() ]

    # Set containing codes that are to be subdivided (ie. mult > 1)
    cones_to_divide = set()
    for c in max_dim_cones:
        mult = multiplicity(c)
        if mult > 1:
            cones_to_divide.add((c, mult))

    depth = 1
    while cones_to_divide and depth < 1000:
        c, mult  = cones_to_divide.pop()

        R = c.rays().matrix()
        N = R.nrows()

        if printing: print(f"Simplifying cone with mult = {mult}, given by rays \n{R}")

        # Construction of new ray x
        x = blowup_ray(R)
        list_simplicial_cones.append(Cone([x]))
        added_rays.append(tuple(x))
        if printing: print(f"New constructed x = {x}\n")

        # Remove non-simple cone and add subdivided cones
        list_simplicial_cones.remove(c)
        if printing: print(f"Removing cone =\n{R}\n")
        for j in range(N):
            new_c_list_rays = [R[i] for i in range(N) if i != j] + [tuple(x)]
            new_c = Cone(new_c_list_rays)

            # Add only if it is of maximum dimension
            if new_c.dim() == fan.dim():
                list_simplicial_cones.append(new_c)
                if printing: print(f"Adding cone = \n {new_c_list_rays}\n")

                # Check if subcones are simple
                new_mult = multiplicity(new_c)
                if new_mult > 1:
                    cones_to_divide.add((new_c,new_mult))
                    if printing: print(f"...also to divide, mult = {new_mult} \n")

        depth += 1

    return list_simplicial_cones, added_rays


def print_check_simple(max_dim_cones):
    """
    Prints a check that all the maximum dimension cones are simple: checks that the matrix defined
    by its rays has determinant equal to 1 in absolute value.
    """

    for c in max_dim_cones:
        print("Cone:")
        print(c.rays().matrix())
        print(f"... |det| = {abs(c.rays().matrix().det())}")

def plot_subdivided_fan(max_dim_cones):
    """
    Plots the subdivision of the dual fan, given the maximum dimension cones
    """

    subdividedFan = Fan(max_dim_cones)
    return subdividedFan.plot()


def is_ray_in_cone(r, c):
    """
    Checks if ray r is one of the generators of cone c
    """

    r = vector(r)
    c_matrix = c.rays().matrix()
    for ray in c_matrix:
        if ray == r:
            return True
    return False

def neighbor_divisors(r, max_dim_cones, printing=False):
    """
    Returns neighbors to ray r, that is, other rays of the fan that appear in a same cone with r
    """
```

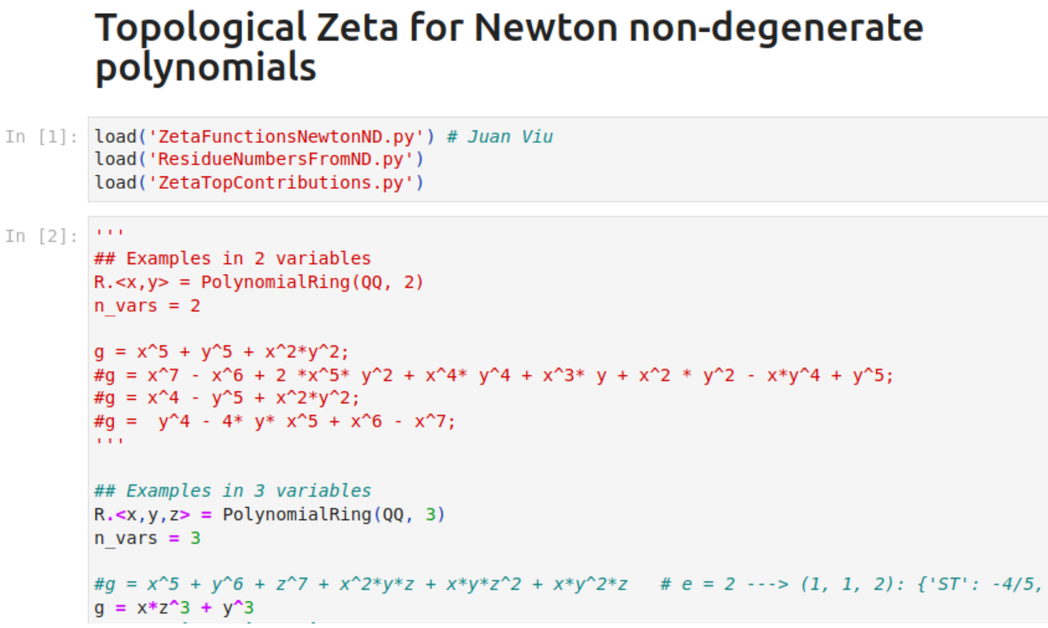
```
    r = vector(r)
    neighbors = set()
    for cone in max_dim_cones:
        if is_ray_in_cone(r, cone):
            cone_matrix = cone.rays().matrix()
            for ray in cone_matrix:
                if ray != r:
                    if printing: print(f"Adding neighbor ray: {ray}")
                    neighbors.add(tuple(ray))
    return neighbors

def dot_prod(a, b):
    return vector(a)*vector(b)

def get_k_N(r, g):
    """
    Returns numerical data k, N for ray r, in the dual fan of g
    """
    k = sum(r)

    list_possible_N = [dot_prod(exp, r)  for exp in g.exponents()]
    N = min(list_possible_N)

    return k, N

def all_neighbors_k_N(r, max_dim_cones, g):
    """
    Returns numerical data k, N for ray r and all of its neighbors in the dual fan of polynomial g
    """
    neighbors = neighbor_divisors(r, max_dim_cones)

    info_neighbors = []
    for ray in neighbors:
        k, N = get_k_N(ray, g)
        #print(f"Ray {ray} has (k,N) = ({k},{N})")
        info_neighbors.append( [ray, k, N] )
    return info_neighbors


def rays_k_N(g):
    """
    Returns the numerical data of all original rays of the Newton dual fan of the polynomial g
    """
    # Construct Newton polyhedron and dual fan
    P = newton_polyhedron(g)
    F = fan_all_cones(P)

    # Save original rays and coordinate rays
    original_rays = [tuple(r) for r in F.rays()]

    # Max dimension cones
    L = F.cones(F.dim())
    max_dim_cones = [l for l in L]

    data_ray = dict()
    for ray in original_rays:
        data_ray[ray] = get_k_N(ray, g)
    return data_ray


def get_epsilons(g, printing = True):
    """
    Computes the residue numbers (epsilons) for each divisor, corresponding to a ray of the
    orginal dual fan associated to the polynomial g, and its respective neighbor rays in the
    regular subdivided dual fan.

    Returns:
     - epsilons: a dictionary, for each ray gives another dictionary including
        the neighbor rays and the corresponding residue number
     - possible_val: a sete with the residue numbers found in the example
     - max_dim_cones: the list of maximum dimension cones, obtained after
        the regular subdivision (required for the subdivided fan plot)
     - original_rays: list of the original rays of the dual fan
     - added_rays: list of the added rays in the regular subdivision
    """

    zex = ZetaFunctions(g)
    zex.topological_zeta(local = True)
    s = var("s")

    # Construct Newton polyhedron and dual fan
    P = newton_polyhedron(g)
    F = fan_all_cones(P)

    # Save original rays and coordinate rays
    original_rays = [tuple(r) for r in F.rays()]
    Id = matrix.identity(n_vars)
    coordinate_rays = set( [tuple(Id[j]) for j in range(n_vars)] )
```

```
    # Simplicial regular (simple) subdivision
    L, added_rays = simple_partition(F, printing = False)
    # Max dimension cones of the subdivision
    max_dim_cones = [l for l in L if l.dim() == F.dim()]

    chart_rays = list(set(original_rays).difference(coordinate_rays))
    total_rays = original_rays + added_rays

    epsilons = dict()
    possible_val = set()

    for ray in chart_rays:
        k, N = get_k_N(ray, g)

        eps_ray = dict()
        eps_ray["ST"] = -k/N
        possible_val.add( -k/N )

        sum_eps = -k/N

        info_neighbors = all_neighbors_k_N(ray, max_dim_cones, g)
        #print(info_neighbors)

        for elem in info_neighbors:
            neigh = elem[0]
            k_neigh = elem[1]
            N_neigh = elem[2]
            eps = - N_neigh * k/N + k_neigh
            eps_ray[neigh] = eps
            possible_val.add(eps)
            sum_eps += eps

            ## debugging
            if eps > 1 and eps == round(eps):
                print(f"{g} HAS INTEGER EPSILON > 1: {eps}")

        epsilons[ray] = [eps_ray, sum_eps]

    if printing:
        print("")
        for key in epsilons:
            print(f"{key} has epsilons: {epsilons[key][0]} \n")

    print(f"\n Possible epsilons found: {possible_val}")

    return epsilons, possible_val, max_dim_cones, original_rays, added_rays



def gcd_menors(A, B):
    """
    Computes the gcd of the 2x2 minors of the matrix formed by taking vectors A,B as columns.
    This is the beta factor defined by Loeser in [p. 87, Loe90].

    REFERENCES
     - [Loe90] F. Loeser. "Fonctions d'Igusa p-adiques, polynomes de Bernstein,
               et polyedres de Newton" (1990).
    """

    n = len(A)
    pairs = [ [i,j] for i in range(n) for j in range(i+1,n) ]
    list_menors = [ abs(A[i]*B[j]-A[j]*B[i]) for i,j in pairs  ]

    return gcd(list_menors)


def get_loeser_epsilons(g, printing=True):
    """
    Computes the Loeser residue numbers (epsilons) for each divisor, corresponding to a ray
    of the orginal dual fan associated to the polynomial g, and its respective neighbor rays
    in the original dual fan (that is without regular subdvision!)

    Returns:
     - epsilons: a dictionary, for each ray gives another dictionary including
        the neighbor rays and the corresponding residue number
     - possible_val: a sete with the residue numbers found in the example
    """

    # Construct Newton polyhedron and dual fan
    P = newton_polyhedron(g)
    F = fan_all_cones(P)

    # Save original rays and coordinate rays
    original_rays = [tuple(r) for r in F.rays()]

    Id = matrix.identity(n_vars)
    coordinate_rays = set( [tuple(Id[j]) for j in range(n_vars)] )
```

```python
    chart_rays = list(set(original_rays).difference(coordinate_rays))

    # Max dimension cones
    L = F.cones(F.dim())
    max_dim_cones = [l for l in L]

    epsilons = dict()
    possible_val = set()

    for ray in chart_rays:
        k, N = get_k_N(ray, g)

        eps_ray = dict()
        eps_ray["ST"] = -k/N
        possible_val.add( -k/N )

        sum_eps = -k/N

        info_neighbors = all_neighbors_k_N(ray, max_dim_cones, g)
        #print(info_neighbors)

        for elem in info_neighbors:
            neigh = elem[0]
            k_neigh = elem[1]
            N_neigh = elem[2]
            eps = - N_neigh * k/N + k_neigh

            # Loeser factor beta
            beta = gcd_menors(ray, neigh)
            eps /= beta

            eps_ray[neigh] = eps
            possible_val.add(eps)
            sum_eps += eps

            ## debugging
            if eps > 1 and eps == round(eps):
                print(f"{g} HAS INTEGER EPSILON > 1: {eps}")

        epsilons[ray] = [eps_ray, sum_eps]

    if printing:
        print("")
        for key in epsilons:
            print(f"{key} has Loeser epsilons: {epsilons[key][0]} \n")

    print(f"\n Possible epsilons found: {possible_val}")

    return epsilons, possible_val
```

4 Full detailed example

Finally, we have a `Jupyter Notebook` that contains an interactive computation with full detail of the desired example. That is, it computes the Newton polygon and associated dual fan, then the residue numbers (both with and without regular subdivision), then the topological zeta function, the contributions and residues at the poles, and some additional information about the regular subdivision.



Topological Zeta for Newton non-degenerate polynomials

```
In [1]: load('ZetaFunctionsNewtonND.py') # Juan Viu
        load('ResidueNumbersFromND.py')
        load('ZetaTopContributions.py')

In [2]: '''
        ## Examples in 2 variables
        R.<x,y> = PolynomialRing(QQ, 2)
        n_vars = 2

        g = x^5 + y^5 + x^2*y^2;
        #g = x^7 - x^6 + 2 *x^5* y^2 + x^4* y^4 + x^3* y + x^2 * y^2 - x*y^4 + y^5;
        #g = x^4 - y^5 + x^2*y^2;
        #g =  y^4 - 4* y* x^5 + x^6 - x^7;
        '''

        ## Examples in 3 variables
        R.<x,y,z> = PolynomialRing(QQ, 3)
        n_vars = 3

        #g = x^5 + y^6 + z^7 + x^2*y*z + x*y*z^2 + x*y^2*z   # e = 2 ---> (1, 1, 2): {'ST': -4/5,
        g = x*z^3 + y^3
```

Figure A.1: Screenshot of the first cells of the `Jupyter Notebook`.

The notebook file with the code can be found in https://github.com/baezaguasch/MonodromyNND/blob/main/Main.ipynb, and `html` versions of a computed example can be found in the main repository.