Resolution of singularities via combinatorics of the Newton polygon

An introduction to toric varieties and the monodromy conjecture.

Oriol Baeza Guasch

Contents

Introduction

The monodromy conjecture is an open problem in the theory of singularities, formulated by Igusa in the 70s.

Monodromy conjecture, topological version

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be non-constant. If s_0 is a pole of $Z_{\text{top}}(f, \varphi; s)$, then

- *(standard)* $e^{2\pi \imath \Re(s_0)}$ is a monodromy eigenvalue of $f \colon \mathbb{C}^n \to \mathbb{C}$ at some point of $\{f = 0\}$.
- *(strong)* s_0 is a root of the Bernstein-Sato polynomial b_f.

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- *(strong)* s_0 is a root of the Bernstein-Sato polynomial b_f.

Known cases:

- Plane curves (Loeser '88)
- Newton non-degenerate polynomials* (Loeser '90)
- Some types of hyperplanes arrangements (Budur-Saito-Yuzvinsky '10, Walther '17, Bapat-Walters '15)
- Semi-quasihomogeneous singularities (Budur-Blanco-van der Veer '21)

Contents

Singularities

- $f : U \to \mathbb{C}$ a holomorphic function defined on an open set $U \subset \mathbb{C}^n$
- Hypersurface $X = f^{-1}(0)$

Definition (Singularity)

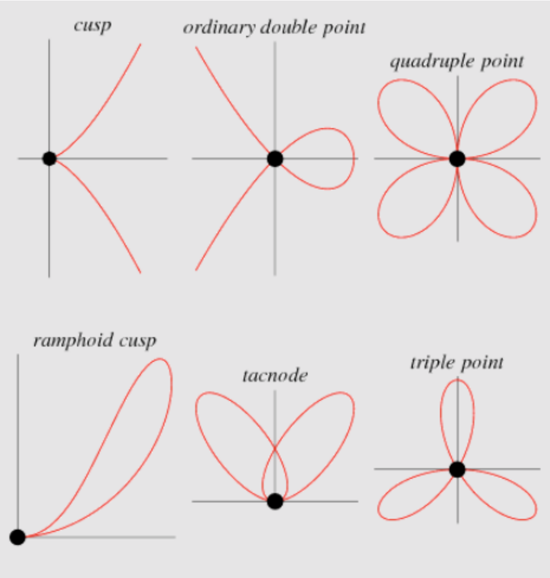
We define the set of singular points of X by the set

$$\mathrm{Sing}(X) := \left\{ x \in U \mid f(x) = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\}$$

Additionally, if $x \in \mathrm{Sing}(X)$ is the only singularity in a small enough neighborhood $V \ni x$ we will say it is *isolated*.

Examples of singularities I

Examples of singularities I



Examples of singularities II

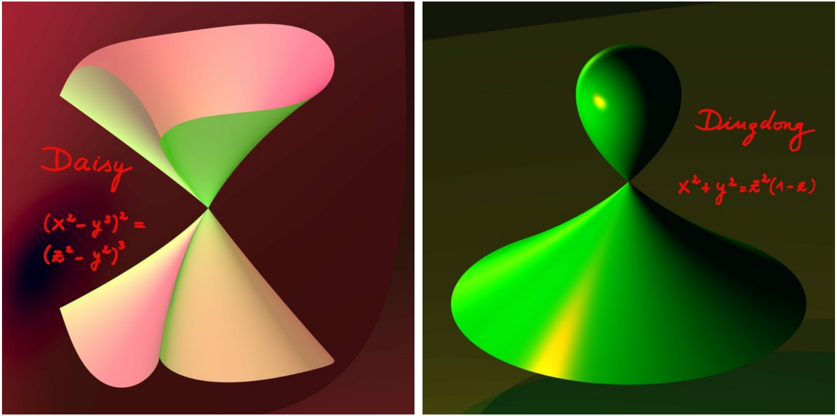


Figure: Herwig Hauser's algebraic surfaces. IMAGINARY exhibition (CC BY-NC-SA-3.0)

See more: `https://www.imaginary.org/gallery/herwig-hauser-classic`

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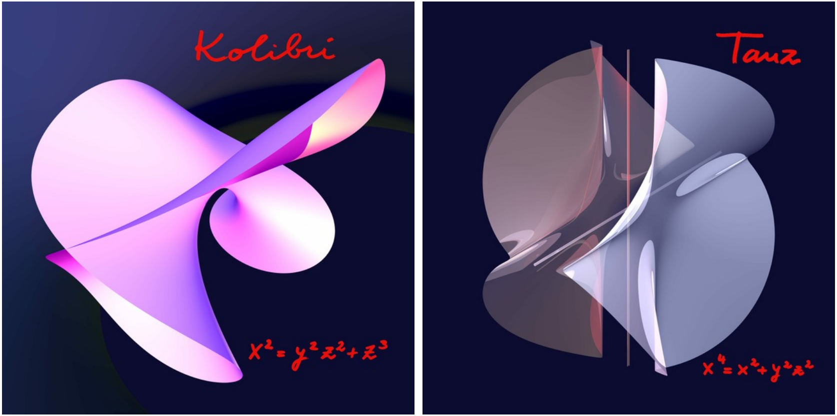

Figure: Herwig Hauser's algebraic surfaces. IMAGINARY exhibition (CC BY-NC-SA-3.0)

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Resolution of singularities

Definition (Resolution)

A *resolution* of X is a proper morphism $\pi\colon Y \to X$ where

1. Y is a smooth variety.
2. The restriction outside the singular locus $\pi\,|_{Y\setminus\pi^{-1}(\mathrm{Sing}(X))}\colon Y\setminus\pi^{-1}(\mathrm{Sing}(X)) \to X\setminus\mathrm{Sing}(X)$ is an isomorphism.

Additionally, we will say that the resolution is *good* if also

3. For every singular point $p \in \pi^{-1}(\mathrm{Sing}(X))$, there exists an open neighborhood $U_p \subset Y$, and open $V \subset \mathbb{K}^n$ with a chart

$$\begin{aligned} \mathrm{y}\colon U_p &\xrightarrow{\cong} V \\ p &\longmapsto 0 \end{aligned}$$

such that $U \cap \pi^{-1}(\mathrm{Sing}(X)) = \{y_{i_1} = \cdots = y_{i_r} = 0\}$ for certain indices $0 < i_1 < \cdots < i_r \leq n$.

Resolution of singularities

Definition (Embedded resolution)

Let X be a smooth algebraic variety, $f\colon X \to \mathbb{K}$ a polynomial and abbreviate $S = \mathrm{Sing}(f^{-1}(0))$ be the set of singular points on the zero set of f. An *embedded resolution* of f is a proper morphism $\pi\colon Y \to X$ where

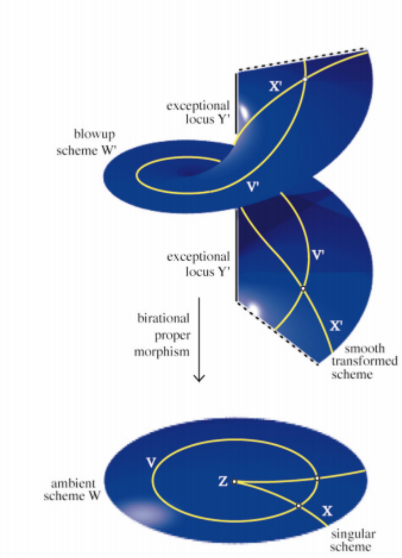
1. Y is a smooth variety.
2. The restriction outside the singular locus $\pi\,|_{Y\setminus\pi^{-1}(S)}\colon Y \setminus \pi^{-1}(S) \to X \setminus S$ is an isomorphism.
3. For every singular point $p \in \pi^{-1}(S)$, there exists an open neighborhood $U_p \subset Y$, and an open $V \subset \mathbb{K}^n$ with a chart

$$\begin{aligned} \mathrm{y}\colon U_p &\xrightarrow{\cong} V \\ p &\longmapsto 0 \end{aligned}$$

   over which $\pi^* f = u(y)\, y_{i_1}^{N_1} \cdots y_{i_r}^{N_r}$, with $u(0) \neq 0$ a unit, and $N_i \geq 0$ integers.

Guaranteed in characteristic zero, thanks to a result by Hironaka [Hir64].

Picture of a resolution



Geometric description

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be non-constant, $\pi \colon X \to \mathbb{C}$ an embedded resolution of f. Let $(E_i)_{i \in J}$ the irreducible components of $\pi^{-1}\left(f^{-1}(0)\right)$. Locally $E_i \colon \{x_i = 0\}$.

Geometric setup

In the resolved space, the intersection of exceptional divisors is, at worst, like intersection of coordinate hyperplanes.

- Normal crossings: divisors are smooth and intersect transversely.
- Simple normal crossings: addtionally, no three intersect at the same point.

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Numerical data

Denote the order of vanishing of $\pi^* f$ and $\pi^*(\mathrm{d}x)$ on a generic point of E_j by N_j, $k_j - 1$, respectively. We can write globally the divisors

$$\operatorname{div}(\pi^* f) = \sum_{j \in J} N_j E_j$$

$$\operatorname{div}(\pi^*(\,\mathrm{d}x_1 \wedge \cdots \wedge \,\mathrm{d}x_n)) = \sum_{j \in J} (k_j - 1) E_j$$

Plane curves

Given a curve C in a smooth algebraic surface S, singular at a point P. Construct surface T and map $\pi\colon T \to S$ such that π restricts to an isomorphism between $T \setminus E \to S \setminus \{P\}$, where $E = \pi^{-1}(P)$ is basically a projective line.

- Only need a finite number of blowups to obtain smooth strict transform.
- Can be extended so that $\pi^{-1}(C)$ has simple normal crossings.

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- Only need a finite number of blowups to obtain smooth strict transform.
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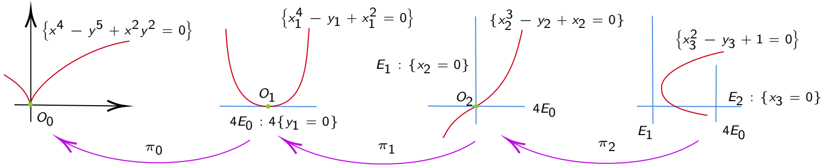
For example, a resolution of one branch of the curve $f = x^4 - y^5 + x^2y^2$ is given by

$$(x, y) \mapsto (x_1y_1, y_1), \qquad (x_1, y_1) \mapsto (x_2, x_2y_2), \qquad (x_2, y_2) \mapsto (x_3, x_3y_3)$$
$$\implies \quad \pi^* f = x_3^{10} y_3^4 (x_3^2 - y_3 + 1)$$



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Newton polyhedron

We consider a polynomial $f(x_1, \ldots, x_n) = \sum_{p \in \mathbb{N}^n} a_p \, x_1^p \ldots x_n^p$ such that $f(0) = 0$, and define the support $\operatorname{supp}(f) = \{p \in \mathbb{N}^n \mid a_p \neq 0\}$.

Definition (Newton polyhedron)

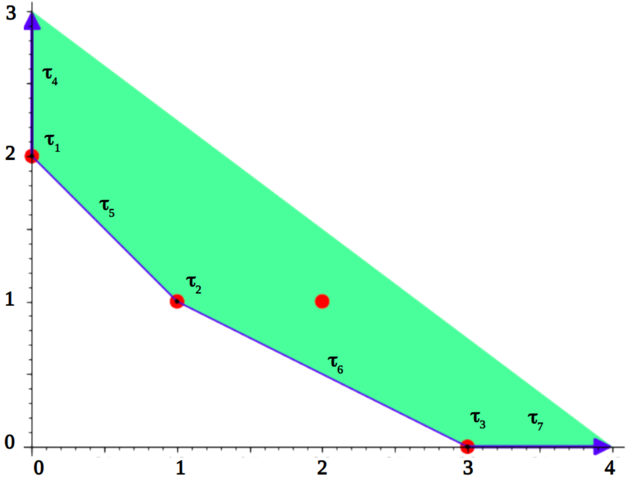
Let $f = \sum_{p \in \mathbb{N}^n} a_p x^p \in \mathbb{C}[x]$ with $f(0) = 0$. We define the *global Newton polyhedron* $\Gamma_{gl}(f)$ of f as the convex hull of $\operatorname{supp}(f)$. Also, we define the *local Newton polyhedron* $\Gamma(f)$ as the convex hull of the set

$$\bigcup_{p \in \operatorname{supp}(f)} p + (\mathbb{R}_{\geq 0})^n$$

In particular, it is immediate that $\Gamma(f) = \Gamma_{gl}(f) + (\mathbb{R}_{\geq 0})^n$.

Example

Consider the polynomial $f = x^3 - y^2 + 4xy + 3x^2y$.



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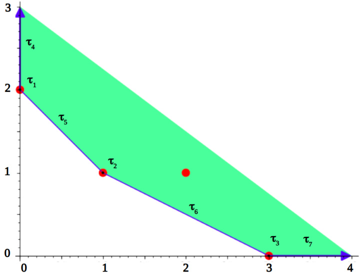

- Faces of dimension $\dim \tau = 0$:

$$\tau_1 = \{(0,2)\} \qquad f^{\tau_1} = -y^2$$
$$\tau_2 = \{(1,1)\} \qquad f^{\tau_2} = 4xy$$
$$\tau_3 = \{(3,0)\} \qquad f^{\tau_3} = x^3$$

- Faces of dimension $\dim \tau = 1$:

$$\tau_4 = \{(0,2) + \mathbb{R}_{>0}(0,1)\} \qquad f^{\tau_4} = -y^2$$
$$\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} \qquad f^{\tau_5} = -y^2 + 4xy$$
$$\tau_6 = \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} \qquad f^{\tau_6} = x^3 + 4xy$$
$$\tau_7 = \{(3,0) + \mathbb{R}_{>0}(1,0)\} \qquad f^{\tau_7} = x^3$$

Non-degeneracy

Definition (Newton non-degenerate)

We say that f is Newton non-degenerate at 0 if for any face $\tau \subset \Gamma(f)$, the hypersurface $f^\tau = 0$ satisfies the condition

$$x_1 \frac{\partial f^\tau}{\partial x_1} = \cdots = x_n \frac{\partial f^\tau}{\partial x_n} = 0 \quad \Longrightarrow \quad x_1 \dots x_n = 0$$

that is, the polynomials $x_i \frac{\partial f^\tau}{\partial x_i}$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

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that is, the polynomials $x_i \frac{\partial f^\tau}{\partial x_i}$ do not vanish at the same time in $(\mathbb{C} \setminus 0)^n$.

For example, $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = (y^2 - x^3)^2 - 4yx^5 - x^7$ is degenerate: the truncation on the face (edge) τ with endpoints $(0, 4)$ and $(6, 0)$ is

$$f^\tau = (y^2 - x^3)^2 \quad \Longrightarrow \quad \begin{cases} x \cdot 2(y^2 - x^3) \cdot 3x^2 = 0 \\ y \cdot 2(y^2 - x^3) \cdot 2y = 0 \end{cases}$$

and the system considered has solutions outside $(\mathbb{C} \setminus 0)^2$.

Numerical data

Definition (N,k)

Let $\Gamma(f)$ be the Newton diagram of f as defined. For $a \in (\mathbb{R}^+)^n$, we define

$$N(a) := \inf_{x \in \Gamma(f)} \{\langle a, x \rangle\} = \min_{\substack{x \text{ vertex} \\ \text{of } \Gamma(f)}} \{\langle a, x \rangle\}$$

We may recover the face by considering $F(a) := \{x \in \Gamma(f) \mid \langle a, x \rangle = N(a)\}$ the first meet locus. Lastly, denote $k(a) := \sum_{i=1}^{n} a_i$.

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Definition (Dual cone)

For τ a face of $\Gamma(f)$, we define the *cone associated* to τ as

$$\Delta_\tau := \{a \in (\mathbb{R}_{>0})^n \mid F(a) = \tau\}/ \sim$$

where the equivalence relation is given by $a \sim a' \iff F(a) = F(a')$.

Additionally, we will refer to the collection of these cones for all faces of the Newton polytope as the *dual fan*.

Computing the dual fan

Notice that for every proper face τ we have

$$\tau = \bigcap_{\substack{\tau \subset \gamma \\ \dim \gamma = n-1}} \gamma$$

Moreover, for every face of $\Gamma(f)$ of codimension 1, there exists a unique integral primitive vector (meaning that all of its coordinates are relatively coprime) perpendicular to the face.

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Lemma

Let τ be a proper face of $\Gamma(f)$ and $\gamma_1, \ldots, \gamma_r$ the faces of $\Gamma(f)$ of dimension $n-1$ that contain it. Let $a_1, \ldots, a_r$ be the unique primitive normal vectors to $\gamma_1, \ldots, \gamma_r$, respectively. Then,

$$\Delta_\tau = \{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_{>0}\}$$

with $\dim \Delta_\tau = n - \dim \tau$.

Example

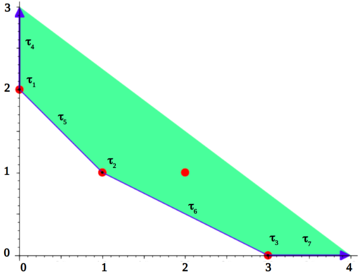
Consider again $f = x^3 - y^2 + 4xy + 3x^2y$.

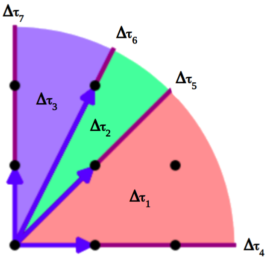
- $\tau_4 = \{(0,2) + \mathbb{R}_{>0}(0,1)\} \quad \Longrightarrow \quad a_4 = (1,0)$
- $\tau_5 = \{(1-\lambda)(0,2) + \lambda(1,1) \mid 0 < \lambda < 1\} \quad \Longrightarrow \quad a_5 = (1,1)$
- $\tau_6 = \{(1-\lambda)(1,1) + \lambda(3,0) \mid 0 < \lambda < 1\} \quad \Longrightarrow \quad a_6 = (1,2)$
- $\tau_7 = \{(3,0) + \mathbb{R}_{>0}(1,0)\} \quad \Longrightarrow \quad a_7 = (0,1)$

Then, the cones associated to each face

$$\Delta_{\tau_1} = \mathbb{R}_{>0}(1,0) + \mathbb{R}_{>0}(1,1), \qquad \Delta_{\tau_4} = \mathbb{R}_{>0}(1,0), \qquad \dots$$

Altogether, the associated dual fan is (right)





Cones and subdivisions

Definition (Cone)

A *convex polyhedral cone* is a set

$$C = \{\lambda_1 v_1 + \cdots + \lambda_s v_r \in V \mid \lambda_i \geq 0\} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$$

where V is an n-dimensional vector space over $\mathbb{R}$, and the vectors $\{v_i\}$ are called the *generators* of the cone.

We will say that a cone is...

- *simplicial* if its generating vectors $v_1, \ldots, v_r$ are linearly independent over $\mathbb{R}$.
- *regular* if $\{v_1, \ldots, v_r\}$ is a subset of a base of the $\mathbb{Z}$-module $\mathbb{Z}^n$.

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Theorem

Each cone admits a partition in simplicial subcones. Moreover, they can be made regular cones by introducing suitable new generating rays.

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Theorem

Each cone admits a partition in simplicial subcones. Moreover, they can be made regular cones by introducing suitable new generating rays.

Lemma 8.7 [AVG12]

There exists a regular fan subordinate to a Newton polyhedron.

Toric blowups

Definition (Toric blowup)

Consider a unimodular integral $n \times n$ matrix σ

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n} \end{pmatrix}$$

We define the *toric blowup* associated to σ as the birational morphism

$$\begin{aligned} \pi_\sigma \colon (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^n \\ (x_1, \ldots, x_n) &\mapsto (x_1^{\sigma_{1,1}} x_2^{\sigma_{1,2}} \cdots x_n^{\sigma_{1,n}}, \ldots, x_1^{\sigma_{n,1}} x_2^{\sigma_{n,2}} \cdots x_n^{\sigma_{n,n}}) \end{aligned}$$

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For a regular simplicial cone of maximum dimension in the subdivided dual fan $\Sigma^*(f)$ given by vectors $\{r_1, \ldots, r_n\}$, we can consider the matrix $\sigma = (r_1 \ \cdots \ r_n)$.

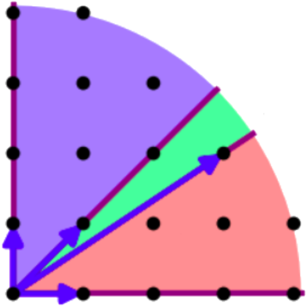
Gluing adequately these charts π_σ, we obtain a non-singular variety X, and a proper analytic map $\pi\colon X \to \mathbb{C}^n$: the toric blowup associated with $\Sigma^*(f)$.

Resolution of singularities

Theorem [Oka96, p. 101]

If f is Newton non-degenerate, then the associated toric blowup $\pi\colon X \to \mathbb{C}^n$ is a good resolution of the f as a germ at the origin.

Consider again $f = x^4 - y^5 + x^2y^2$.



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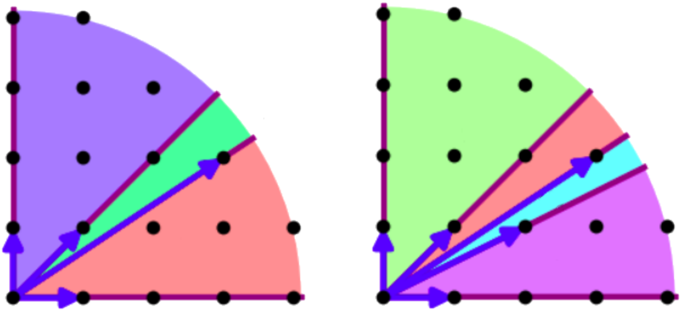

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Consider again $f = x^4 - y^5 + x^2y^2$.

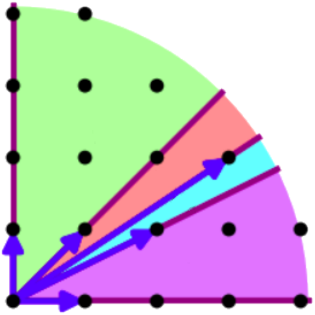
The rays of the regular subdivision are

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

taking one of the charts, we perform the toric blowup as

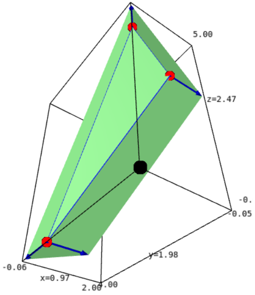
$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \implies (x, y) \mapsto (z^3 w, z^2 w)$$

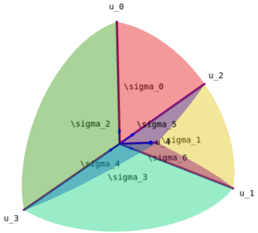
$$\pi^* f = z^{10} w^4 (z^2 - w + 1)$$

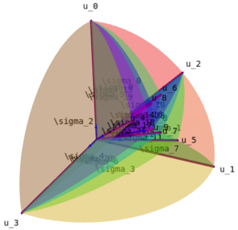


3 dimensional example

Consider $f = xz^3 + z^4 + y^3 \in \mathbb{C}[x, y, z]$. Construct Newton polyhedron, dual cone and a regular simplicial subdivision.







Take the chart $\{(1, 0, 0), (3, 4, 3), (1, 1, 1)\}$, then the resolution gives

$$(x, y, z) \xmapsto{\begin{pmatrix}1 & 0 & 0\\ 3 & 4 & 3\\ 1 & 1 & 1\end{pmatrix}} (u, u^3v^4z^3, uvw) \implies \pi^*f = u^4v^3w^3(u^5v^9w^6 + vw + 1)$$

Note that the partial derivatives vanish all only on $v = w = 0$, which are not points on the curve $\pi^*f = 0$.

Contents

Bernstein-Sato polynomial

- Bernstein [Ber72] in the case of polynomials
- Kashiwara [Kas76] in the case of holomorphic functions
- Björk [Bjö73] in the case of formal power series

Theorem

Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $e_{f,P}(s) \in \mathbb{C}[s]$ such that the relation

$$P(s)f^{s+1} = e_{f,P}(s)f^s$$

holds formally in the $\mathscr{D}$-module $R_f[s] \cdot f^s$.

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Let $f \in R$ be a polynomial. Then, there exists a polynomial $P(s) \in \mathscr{D}[s]$ and a polynomial $e_{f,P}(s) \in \mathbb{C}[s]$ such that the relation

$$P(s)f^{s+1} = e_{f,P}(s)f^s$$

holds formally in the $\mathscr{D}$-module $R_f[s] \cdot f^s$.

Definition (Bernstein-Sato polynomial)

The Bernstein-Sato polynomial $b_f(s)$ is the monic generator of the ideal in $\mathbb{C}[s]$ consisting of polynomials $e_{f,P}(s)$ satisfying such a functional equation.

First algorithm to compute it introduced by Oaku in [Oak97], using non-commutative Gröbner basis in the Weyl algebra.

Bernstein-Sato polynomial

Example

Let $f = x_1^2 + \cdots + x_n^2$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then, we have

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) f^{s+1} = 4(s+1)\left(s + \frac{n}{2}\right) f^s$$

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Roots of the Bernstein-Sato polynomial

- -1 is always a root. We write $\tilde{b}_f(s) = b_f(s)/(s+1)$ for the *reduced* BS.
- All roots are negative rational numbers [Mal75; Kas76].
- A set of candidate roots can be obtained from a resolution of singularities [Kol97; Lic89].

Theorem

With the notations introduced for the resolution of singularities, we have that every root of the Bernstein-Sato polynomial b_f is of the form

$$-\frac{k_j + \nu}{N_j}, \qquad j \in J, \nu \in \mathbb{Z}_{\geq 0}$$

Topological zeta function

Definition (Topological zeta function) [DL92, Thm. 3.2]

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial and choose an embedded resolution $\pi \colon Y \to \mathbb{A}^n_{\mathbb{C}}$ of $\{f = 0\}$. The *local topological zeta function* of f at $a \in \{f = 0\}$ is

$$Z_{\mathrm{top},a}(f; s) := \sum_{I \subset J} \chi\left(E_I^\circ \cap \pi^{-1}\{a\}\right) \prod_{i \in I} \frac{1}{k_i + N_i s}$$

where I runs through all possible subsets of J.

Monodromy conjecture

- Formulated by the Igusa in the late seventies, after some examples computed.
- Originally stated in terms of the p-adic zeta function, but has analogous statements in topological and motivic settings.

Monodromy conjecture, topological version

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be non-constant. If s_0 is a pole of $Z_{\text{top}}(f, \varphi; s)$, then

- *(standard)* $e^{2\pi\imath\Re(s_0)}$ is a monodromy eigenvalue of $f : \mathbb{C}^n \to \mathbb{C}$ at some point of $\{f = 0\}$.
- *(strong)* s_0 is a root of the Bernstein-Sato polynomial b_f.

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Strong version implies the standard one

Proposition 7.1 [Mal83]

If $-\alpha$ is a root of $\tilde{b}_{f,0}(s)$, then $e^{2\pi\imath\alpha}$ is an eigenvalue of the monodromy of f at the origin.

Topological zeta function for NND

Let τ be a face in $\Gamma(f)$, and consider a decomposition of the associated cone $\Delta_\tau = \cup_{i=1}^r \Delta_i$ in simplicial cones of dimension $\dim \Delta_\tau = l$ such that $\dim(\Delta_i \cap \Delta_j) < l$, for all $i \neq j$. Then, define

$$J(\tau, s) := \sum_{i=1}^{r} J_{\Delta_i}(s), \qquad \text{with} \quad J_{\Delta_i}(s) = \frac{\mathrm{mult}(\Delta_i)}{(N(a_{i_1})s + k(a_{i_1})) \cdots (N(a_{i_l})s + k(a_{i_l}))}$$

being $a_{i_1}, \ldots, a_{i_l} \in \mathbb{N}^n$ the linearly independent primitive integral vectors that generate Δ_i. Lastly, if $\tau = \Gamma(f)$, we rather take $J(\tau, s) = 1$.

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Theorem 5.3 [DL92]

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function Newton non-degenerate for $\Gamma_0(f)$, then

$$Z_0(s) = \sum_{\substack{\tau \text{ vertex} \\ \text{of } \Gamma(f)}} J(\tau, s) + \left(\frac{s}{s+1}\right) \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma(f) \\ \dim \tau \geq 1}} (-1)^{\dim \tau} (\dim \tau)! \operatorname{Vol}(\tau) J(\tau, s)$$

Sketch of the proof for NND

Theorem 4.2 [Loe90]

Let f be a germ of an analytic function, Newton non-degenerate at the origin. Let τ_0 be a compact face of codimension 1 of $\Gamma(f)$. Suppose that the following two conditions are verified

1. $\dfrac{k(\tau_0)}{N(\tau_0)} \notin \mathbb{Z}$,
2. For every face τ of codimension 1 of $\Gamma(f, 0)$, distinct of τ_0 and having non-empty intersection with τ_0, we have $\varepsilon(\tau_0, \tau) \notin \mathbb{Z}$.

Then, there exists a horizontal multiform section $\gamma(t)$ of the fibration H_n over T' such that

$$\lim_{t \to 0} t^{1 - \frac{k(\tau_0)}{N(\tau_0)}} \int_{\gamma(t)} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{\mathrm{d}f} = C$$

with C a non-zero constant.

Proposition 3.3 [Mal73]

The polynomial $(s + \mu_1) \ldots (s + \mu_p)$ divides $\tilde{b}$.

Resolution of singularities via combinatorics of the Newton polygon

An introduction to toric varieties and the monodromy conjecture.

https://github.com/baezaguasch/MonodromyNND
ob410 (at) cam.ac.uk

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