Q. Verify Rank-Nullity theorem for $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$T(x, y, z) = (x - y + z,\ x + 2y - z,\ 2x + y).$$

Soln: $\dim V = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$

$\quad\ \ \| $

$\dim \mathbb{R}^3 = 3.$

$$\operatorname{Ker} T = \{ \bar{x} \in \mathbb{R}^3 \mid T(\bar{x}) = \bar{0} \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = (0, 0, 0) \}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x-y+z=0 \\ x+2y-z=0 \\ 2x+y=0 \end{array} \right\}$$

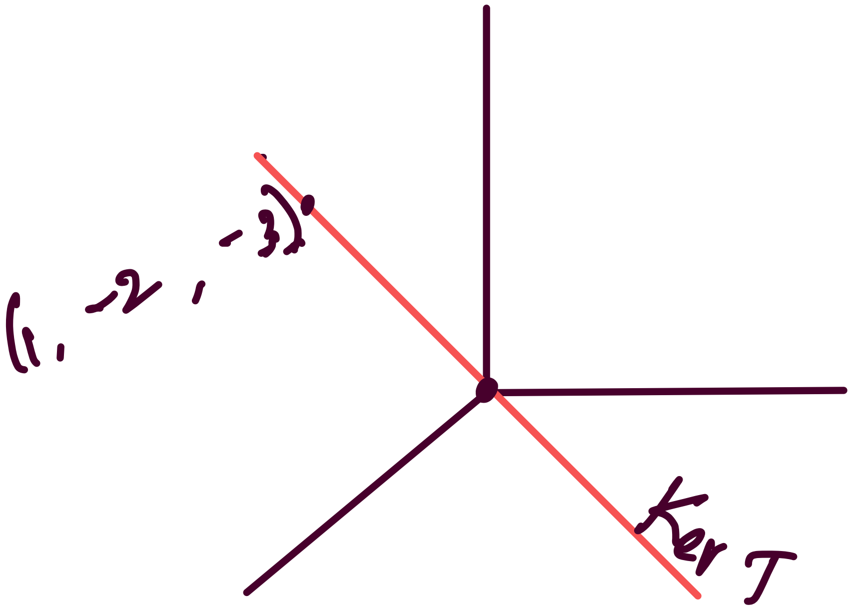
$$\Rightarrow y=-2x$$

$$x-y+z=0 \quad \Rightarrow \quad x+2x+z=0$$

$$\Rightarrow z=-3x.$$

$$\therefore \ \mathrm{Ker}\, T = \{ (x, -2x, -3x) \mid x \in \mathbb{R} \}$$

$$= \{ x(1, -2, -3) \mid x \in \mathbb{R} \}$$



$$\Rightarrow \dim(\operatorname{Ker} T) = n\left(\{(1,-2,-3)\}\right)$$

Basis

$$= 1.$$

$$\text{Im}\, T = \{ T(\bar{x}) \mid \bar{x} \in \mathbb{R}^3 \}$$

$$= \{ T(x, y, z) \mid x, y, z \in \mathbb{R} \}$$

$$= \{ (\underbrace{x - y + z}_{= a}, \underbrace{x + 2y - z}_{= b}, \underbrace{2x + y}_{= c}) \mid x, y, z \in \mathbb{R} \}$$

Solve

$$x - y + z = a$$

$$x + 2y - z = b$$

$$2x + y = c$$

$$\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{A} \quad B \\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 2 & -1 & b \\ 2 & 1 & 0 & c \end{array}\right] \end{array}$$

augmented matrix

$$R_2 \to R_2 - R_1 \quad ; \quad R_3 \to R_3 - 2R_1 \quad \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 3 & -2 & b-a \\ 0 & 3 & -2 & c-2a \end{array}\right]$$

$$R_3 \to R_3 - R_2 \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 3 & -2 & b-a \\ 0 & 0 & 0 & \underbrace{c-2a-b+a}_{=c-a-b} \end{array}\right]$$

For solution to exist,

$$2 = \text{rank}(A) = \text{rank}(A|B) = 2$$

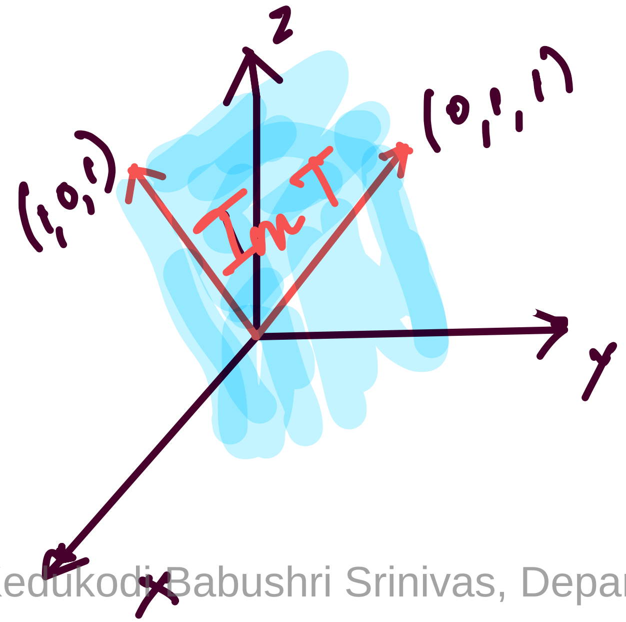
i.e., $c - a - b = 0$

$$\Rightarrow c = a + b.$$

$$\operatorname{Im} T = \{(a, b, a+b) \mid a, b \in \mathbb{R}\}$$

$$= \{(a, 0, a) + (0, b, b) \mid a, b \in \mathbb{R}\}$$

$$= \{a(1, 0, 1) + b(0, 1, 1) \mid a, b \in \mathbb{R}\}$$



Basis for $\operatorname{Im} T = \{(0,1,1), (1,0,1)\} = B'$

$\Rightarrow \dim \operatorname{Im} T = n(B') = 2.$

$\therefore \dim \operatorname{Im} T + \dim \operatorname{Ker} T = 2 + 1$

$= 3$

$= \dim V.$

Polynomials of degree upto n.

$P_n = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_0, a_1, \dots, a_n \in \mathbb{R}\}$

$$f+g = (a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n)$$

$$= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n.$$

$$\in P_n.$$

$$\alpha f = \alpha(a_0 + a_1 x + \cdots + a_n x^n)$$

$$= \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n \in P_n$$

$\therefore P_n$ is a vector space.

Q. $\dim P_n = ?$

Let $B = \{1, x, x^2, \dots, x^n\}$ $\rightarrow$ L.I.
$\rightarrow$ Spanning set

Suppose $\alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \dots + \alpha_n \cdot x^n = 0$

$\Rightarrow \alpha_0 = 0, \alpha_1 = 0, \dots, \alpha_n = 0$

$\Rightarrow B$ is L.I.

Take $f \in P_n$.

Then $f = \underbrace{a_0 + a_1 x + \cdots + a_n x^n}$

linear combination of vectors in B

$\Rightarrow f \in \text{span}(B)$

$\Rightarrow$ B is a spanning set.

$\therefore$ B is a basis for P_n.

$\Rightarrow \dim P_n = n(B) = \underline{\underline{n+1}}$.

Q. Prove that $T : P_n \to P_{n+1}$

$T(f) = f'$ is a L.T.

Verify the Rank-nullity theorem for T.

Soln: $f, g \in P_n$.

$$T(f+g) = (f+g)'$$

$$= \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} = f' + g'$$

$$= T(f) + T(g).$$

$\alpha \to$ scalar.

$$
\begin{aligned}
T(\alpha f) &= (\alpha f)' \\
&= \frac{d}{dx}(\alpha f) \\
&= \alpha \frac{df}{dx} \\
&= \alpha f' \\
&= \alpha T(f)
\end{aligned}
$$

$\therefore$ T is a L.T.

$$\operatorname{Ker} T = \{ f \in P_n \mid T(f) = 0 \}$$

$$= \{ f \in P_n \mid f' = 0 \}$$

$$= \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_1 + 2a_2 x + \cdots + n a_n x^{n-1} = 0 \}$$

$$= \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_1 = 0,\ a_2 = 0, \ldots, a_n = 0 \text{ and } a_0 \in \mathbb{R} \}$$

$$= \{ a_0 \mid a_0 \in \mathbb{R} \} = \{ a_0 \cdot 1 \mid a_0 \in \mathbb{R} \}$$

line

Basis for $\ker T = \{1\} = B$

$\Rightarrow \dim \ker T = n(B) = 1.$

Also, $\dim V = \dim P_n = n+1.$

$\operatorname{Im} T = \{ T(f) \mid f \in P_n \}$

$= \{ f' \mid f \in P_n \}$

$f = a_0 + a_1 x + \dots + a_n x^n \Rightarrow f' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1}.$

$= \{ a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} \mid a_i \in \mathbb{R} \}$

Basis for $\operatorname{Im} T = \{1, x, x^2, \ldots, x^{n-1}\}$

$\Rightarrow \dim \operatorname{Im} T = n$ $\Rightarrow \operatorname{Im} T = P_{n-1}$.

$\therefore \dim \ker T + \dim \operatorname{Im} T = 1+n$

$= \dim P_n$.

Eg: $T : P_3 \rightarrow P_2$

$T(f) = f'$.

Q. $m(T)$?

$$P_3 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R}\}.$$

Basis for $\underset{\hookrightarrow \text{domain}}{P_3} = \{1, x, x^2, x^3\} = B_1$

$$T(1) = 0$$

$$T(x) = 1$$

$$T(x^2) = 2x$$

$$T(x^3) = 3x^2$$

B_2 = basis for $P_2 = \{1, x, x^2\}$

$\uparrow$ codomain.

$$T(1) = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$T(x) = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$T(x^2) = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$

$$T(x^3) = 0\cdot 1 + 0\cdot x + 3\cdot x^2$$

$$\therefore\ m(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$