Propositional Calculus

Declaration
This note was prepared based on the book *Logic for Computer Science* by A. Singha.

1 Use of Metatheorems

Metatheorems can be used to show the existence of a proof rather than constructing an actual proof. As expected, this simplifies our work. For example, in the proof of RA(2), we have already used RA(1) showing $\neg\neg p \to p$.

As in Propositional Logic (PL), the deduction theorem is especially useful for propositions involving serial implications. See the following examples.

Example 1.1. *Show that $\Sigma \cup \{A\}$ is consistent if and only if $\Sigma \cup \{\neg\neg A\}$ is consistent.*

In the proof of RA(2), we proved one part; now using RA, we prove both:

$$\begin{aligned}\Sigma \cup \{A\} \textit{ is inconsistent } &\iff \Sigma \vdash \neg A \\ &\iff \Sigma \cup \{\neg\neg A\} \textit{ is inconsistent.}\end{aligned}$$

Example 1.2. *Show that $\vdash \neg\neg p \to p$ and $\vdash p \to \neg\neg p$.*

Due to the deduction theorem and RA(1),

$$\begin{aligned}\vdash \neg\neg p \to p &\iff \neg\neg p \vdash p \\ &\iff \{\neg\neg p, \neg p\} \textit{ is inconsistent.}\end{aligned}$$

The last statement holds as $\{\neg\neg p, \neg p\} \vdash \neg p$ and $\{\neg\neg p, \neg p\} \vdash \neg\neg p$.

Similarly, by the deduction theorem and RA(2),

$$\begin{aligned}\vdash p \to \neg\neg p &\iff p \vdash \neg\neg p \\ &\iff \{p, \neg p\} \textit{ is inconsistent.}\end{aligned}$$

The following theorems and derived rules of double negation are given:

(DN) $\vdash A \to \neg\neg A \quad \vdash \neg\neg A \to A$

$$\frac{\neg\neg A}{A}$$
$$\frac{A}{\neg\neg A}$$

Example 1.3.

$$(\neg p \to q) \to ((q \to \neg p) \to p), \quad (\neg p \to q) \to (q \to \neg p), \quad \neg p \to q \vdash p.$$

1. $\neg p \to q \quad P$
2. $(\neg p \to q) \to (q \to \neg p) \quad P$
3. $q \to \neg p \quad 1,2\ MP$
4. $(\neg p \to q) \to ((q \to \neg p) \to p) \quad P$
5. $(q \to \neg p) \to p \quad 1,4\ MP$
6. $p \quad 3,5\ MP$

Example 1.4. *Show that $\neg q \to ((p \to q) \to \neg p)$. By the deduction theorem and RA, we see that*

$$\neg q \to ((p \to q) \to \neg p) \iff \neg q \vdash ((p \to q) \to \neg p) \iff \{\neg q, p \to q\} \vdash \neg p$$

$$\iff \{\neg q, p \to q, p\} \text{ is inconsistent} \iff \{p \to q, p\} \vdash q.$$

The last one is proved by an application of MP.

The last example brings up another familiar rule. We write it as the derived rule of Modus Tolens:

$$\textbf{(MT)} \quad \frac{\neg B \quad A \to B}{\neg A}$$

Example 1.5. *Show that $\vdash (p \to q) \to (\neg q \to \neg p)$. By the deduction theorem,*

$$\vdash (p \to q) \to (\neg q \to \neg p) \iff \{p \to q, \neg q\} \vdash \neg p.$$

An application of MT does the job.

We have the following theorems and the derived rules of contraposition:

(CN) $\quad \vdash (\neg B \to \neg A) \to (A \to B) \quad \vdash (A \to B) \to (\neg B \to \neg A)$

$$\textbf{(CN)} \quad \frac{\neg B \to \neg A}{A \to B} \quad \frac{A \to B}{\neg B \to \neg A}$$

Example 1.6. *Show that* $\vdash \neg p \to (p \to q)$. *By the deduction theorem,* $\vdash \neg p \to (p \to q) \iff \{\neg p, p\} \vdash q$. *Now,* $\{\neg p, p\}$ *is inconsistent. By monotonicity,* $\{\neg p, p, \neg q\}$ *is inconsistent. By RA,* $\{\neg p, p\} \vdash q$.

Below is another alternative proof of the theorem:

1. $\neg p \quad P$
2. $\neg p \to (\neg q \to \neg p) \quad A1$
3. $\neg q \to \neg p \quad 1, 2\ MP$
4. $p \to q \quad 3,\ CN$

Example 1.7. *Show that* $\vdash p \to (\neg q \to \neg(p \to q))$.

$$p \to (\neg q \to \neg(p \to q)) \iff \{p, \neg q\} \vdash \neg(p \to q) \iff \{p, \neg q, p \to q\} \text{ is inconsistent}$$

$$\iff \{p, p \to q\} \vdash q, \text{ which is MP.}$$

Notice that the deduction theorem on MP proves $\vdash p \to ((p \to q) \to q)$. *We use this as a theorem in the following alternative proof of* $p \vdash \neg q \to \neg(p \to q)$*:*

1. $p \quad P$
2. $p \to ((p \to q) \to q) \quad Th\ (From\ MP)$
3. $(p \to q) \to q \quad 1, 2\ MP$
4. $\neg q \to \neg(p \to q) \quad 3,\ CN$

Example 1.8. *Show that* $\vdash (\neg q \to q) \to q$.

$$(\neg q \to q) \to q \iff \neg q \to q \vdash q \iff \{\neg q \to q, \neg q\} \text{ is inconsistent.}$$

Look at lines 1 and 3 in the following proof:

1. $\neg q \quad P$
2. $\neg q \to q \quad P$
3. $q \quad 1, 2\ MP$

Example 1.9. *Show that* $p \to q, \neg p \to q \vdash q$.

1. $p \to q \quad P$
2. $\neg q \quad P$
3. $\neg p \quad 1, 2\ MT$
4. $\neg p \to q \quad P$

5. $q \quad 3,4$ *MP*

To keep the axiomatic system simple, we have used only two connectives. The other connectives can be introduced with definitions. Remember that the definitions are, in fact, definition schemes. In the following, we use the symbol $\doteq$ for the expression "equal to by definition".

$$
\begin{array}{ll}
\textbf{(D1)} & p \wedge q \doteq \neg(p \to \neg q) \\
\textbf{(D2)} & p \vee q \doteq \neg p \to q \\
\textbf{(D3)} & p \leftrightarrow q \doteq \neg((p \to q) \to \neg(q \to p)) \\
\textbf{(D4)} & \top \doteq p \to p \\
\textbf{(D5)} & \bot \doteq \neg(p \to p)
\end{array}
$$

We also require some inference rules to work with the definitions. They would provide us with ways of how to use the definitions. We have the two rules of definition, written as one, as follows:

$$
\textbf{(RD)} \quad \frac{X \doteq Y \quad Z}{Z[X := Y]} \qquad \frac{X \doteq Y \quad Z}{Z[Y := X]}
$$

The notation $Z[Y := X]$ is the uniform replacement of Y by X in Z. The proposition $Z[Y := X]$ is obtained from the proposition Z by replacing each occurrence of the proposition Y by the proposition X. The rule RD says that if X and Y are equal by definition, then one can be replaced by the other wherever we wish. For instance, from $(p \vee q) \to r$ we can derive $(\neg p \to q) \to r$, and from $(\neg p \to q) \to r$, we may derive $(p \vee q) \to r$. In fact, given any consequence, we may apply this rule recursively replacing expressions involving connectives other than $\neg$ and $\to$ with the ones having only these two. Then the axiomatic system PC takes care of the consequence. Of course, the definitions and the rule RD do the job of eliminating as also introducing the other connectives.

Example 1.10. *(a)* $\vdash p \wedge q \to p$

(b) $\vdash p \wedge q \to q$

(c) $\vdash (p \to (q \to (p \wedge q)))$

Proof:

(a)

1. $\neg p \to (p \to q)$ *Th, Example 2.14*
2. $\neg(p \to q) \to \neg\neg p \quad 1,$ *CN*
3. $\neg\neg p \to p$ *DN*
4. $\neg(p \to q) \to p \quad 2, 3,$ *HS*
5. $p \wedge q \to p \quad 4,$ *RD*

(b)

$$\begin{array}{lll} 1. & \neg q \to (p \to \neg q) & A1 \\ 2. & \neg(p \to \neg q) \to \neg\neg q & 1, CN \\ 3. & \neg\neg q \to q & DN \\ 4. & \neg(p \to \neg q) \to q & 2, 3, HS \\ 5. & p \wedge q \to q & 4, RD \end{array}$$

(c)

$$\begin{array}{lll} 1. & p & P \\ 2. & q & P \\ 3. & p \to \neg q & P \\ 4. & \neg q & 1, 3, MP \end{array}$$

Thus, $\{p, q, p \to \neg q\}$ is inconsistent. By RA, $p, q \vdash \neg(p \to \neg q)$. Due to D1 and RD, $p, q \vdash (p \wedge q)$. By DT, $\vdash (p \to (q \to (p \wedge q)))$.

Example 1.11. *Show that $\vdash (p \to q) \to ((p \vee r) \to (q \vee r))$.*

$$\begin{array}{lll} 1. & p \to q & P \\ 2. & \neg q & P \\ 3. & \neg p & 1, 2, MT \\ 4. & \neg p \to r & P \\ 5. & r & MP \end{array}$$

Hence $p \to q, \neg p \to r, \neg q \vdash r$. By DT, $\vdash (p \to q) \to ((\neg p \to r) \to (\neg q \to r))$. By D1 and RD, we conclude that $\vdash (p \to q) \to ((p \vee r) \to (q \vee r))$.

Example 1.12. *(a) $\vdash p \to (p \vee q)$*

(b) $\vdash q \to (p \vee q)$

Proof:

(a)

$$\begin{array}{lll} 1. & p \to \neg\neg p & DN \\ 2. & \neg\neg p \to (\neg p \to q) & Th \\ 3. & p \to (\neg p \to q) & 1, 2, HS \\ 4. & p \to (p \vee q) & 3, RD \end{array}$$

Example 1.13. *Prove: Σ is inconsistent iff $\Sigma \vdash \perp$, for any set of propositions Σ.*

Let Σ be inconsistent. Then $\Sigma \vdash p$ and $\Sigma \vdash \neg p$ for some proposition p. Now, construct a proof by adjoining the proofs of $\Sigma \vdash p$, of $\Sigma \vdash \neg p$, and of $\{p, \neg p\} \vdash \neg(p \to p)$ (See Example 2.5.), in that order. Next, use D5 with RD to derive $\bot$.

Conversely, suppose $\Sigma \vdash \bot$. By D5 and RD, $\Sigma \vdash \neg(p \to p)$. Adjoin to its proof the proof $\vdash p \to p$. The new proof proves both $p \to p$ and $\neg(p \to p)$. Thus, Σ is inconsistent.

The result of the last example can be summarized as a derived inference rule, called the rule of Inconsistency.

$$\text{(IC):} \quad \frac{A \quad \neg A}{\bot}$$

Then the second part of Monotonicity can be restated as:

If $\Sigma \subseteq \Gamma$ and $\Sigma \vdash \bot$, then $\Gamma \vdash \bot$.

And, RA can be restated as follows:

1. $\Sigma \vdash w$ *iff* $\Sigma \cup \{\neg w\} \vdash \bot$.
2. $\Sigma \vdash \neg w$ *iff* $\Sigma \cup \{w\} \vdash \bot$.

Exercises

1. Show the following:

    1. $\vdash p \to \neg\neg p$
    2. $\vdash (p \to q) \to (\neg q \to \neg p)$
    3. $\vdash (p \vee q) \to ((p \to q) \to q)$
    4. $\vdash ((p \to q) \to q) \to (p \vee q)$
    5. $\vdash (p \to \neg q) \to (q \to \neg p)$
    6. $\vdash (\neg p \to q) \to (\neg q \to p)$
    7. $\vdash (p \to \neg q) \to ((p \to q) \to \neg p)$
    8. $\vdash (q \to \neg p) \to ((p \to q) \to \neg p)$
    9. $\vdash (p \to r) \to ((q \to r) \to (p \vee q \to r))$
    10. $\vdash (p \to q) \to ((p \to r) \to (p \to q \wedge r))$

2. For each $i \in \mathbb{N}$, let x_i be a proposition; and let $\Sigma = \{x_i \to x_{i+1} : i \in \mathbb{N}\}$. Show that $\Sigma \vdash x_0 \to x_n$ for each $n \in \mathbb{N}$.

2 Adequacy of PC to PL

We must see that the syntactic notion of proof matches with the semantic notion of validity. This is called the **adequacy** of PC to PL. Similarly, we would like to see that provability of consequences matches with the validity of consequences. This is called the **strong adequacy** of PC to PL.

The matching is two-way: each valid consequence has a proof and each provable consequence is valid. The former property is called the **strong completeness** of PC, and the latter property is called the **strong soundness** of PC with respect to PL. The adjective **strong** is used to say that the property holds for consequences and not only for theorems.

Observe that **strong soundness** of PC implies its **soundness**; **strong completeness** implies **completeness**; and **strong adequacy** implies **adequacy**. We thus prove **strong soundness** and **strong completeness**.

Notice that the axioms and the rules of inference of PC use only $\neg$ and $\rightarrow$. The other connectives are introduced to PC by way of **definitions**. For convenience, We will restrict **PL** to the fragment of **PROP** where we do not have the symbols $\top, \bot, \wedge, \vee$, and $\leftrightarrow$. Later, you will be able to see that **strong adequacy** holds for the full **PL** as well.

Theorem 2.1 (Strong Soundness of PC to PL)**.** *Let Σ be a set of propositions, and let w be a proposition.*

1. *If $\Sigma \vdash w$ in **PC**, then $\Sigma \models w$.*
2. *If Σ is **satisfiable**, then Σ is **PC-consistent**.*

Proof. (1) We apply **induction** on the lengths of proofs. In the basis step, if a proof of $\Sigma \vdash w$ has only one proposition, then it must be w. Now, w is either an **axiom** or a premise in Σ. Since the **axioms** are **valid propositions** (We checked it in the class.), $\Sigma \models w$.

Lay out the **induction hypothesis** that for every proposition v, if $\Sigma \vdash v$ has a proof of less than m propositions, then $\Sigma \models v$. Let P be a proof of $\Sigma \vdash w$ having m propositions.

If w is again an **axiom** or a premise in Σ, then clearly $\Sigma \models w$ holds. Otherwise, w has been obtained in P by an application of **Modus Ponens (MP)**.

Then, there are propositions v and $v \rightarrow w$ occurring earlier to w in P. By the **induction hypothesis**, $\Sigma \models v$ and $\Sigma \models v \rightarrow w$. Since $\{v, v \rightarrow w\} \models w$, it follows that $\Sigma \models w$.

(2) Let Σ be **inconsistent**, then $\Sigma \vdash u$ and $\Sigma \vdash \neg u$ for some proposition u. By (1), $\Sigma \models u$ and $\Sigma \models \neg u$. Hence, any model of Σ is a model of both u and $\neg u$. This is impossible, since the same interpretation cannot be a model of both u and $\neg u$. Therefore, Σ does not have a model; Σ is **unsatisfiable**. $\square$

A set of propositions is called a **maximally consistent set** iff it is **consistent** and each proper **superset** of it is **inconsistent**.

The set of all propositions (now, without $\vee, \wedge, \leftrightarrow, \perp$) is **countable**. Suppose the following is an **enumeration** of it:

$$w_0, w_1, w_2, \ldots, w_n, \ldots$$

Let Σ be a **consistent** set of propositions. Define a sequence of sets of propositions Σ_m inductively by:

$$\Sigma_0 = \Sigma;$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n, & \text{if } \Sigma_n \cup \{w_n\} \text{ is inconsistent} \\ \Sigma_n \cup \{w_n\}, & \text{if } \Sigma_n \cup \{w_n\} \text{ is consistent} \end{cases}$$

We see that each Σ_i is **consistent** and that if $i < j$, then $\Sigma_i \subseteq \Sigma_j$.

Next, take:

$$\Sigma^* = \bigcup_{m \in \mathbb{N}} \Sigma_m = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \ldots$$

The following lemma lists some interesting properties of Σ^*.

Lemma 2.1 (Maximal Consistency Properties)**.** *Let Σ be a **consistent** set of propositions. Let Σ^* be the set as constructed earlier, and let p, q be propositions.*

1. *If $\Sigma^* \vdash p$, then there exists $m \in \mathbb{N}$ such that $\Sigma_m \vdash p$.*
2. *Σ^* is consistent.*
3. *Σ^* is **maximally consistent**.*
4. *$q \in \Sigma^*$ iff $\Sigma^* \vdash q$.*
5. *Either $q \in \Sigma^*$ or $\neg q \in \Sigma^*$.*
6. *If $q \in \Sigma^*$, then $p \to q \in \Sigma^*$.*
7. *If $p \notin \Sigma^*$, then $p \to q \in \Sigma^*$.*
8. *If $p \in \Sigma^*$ and $q \notin \Sigma^*$, then $p \to q \notin \Sigma^*$.*

In fact, **Lindenbaum Lemma** can be proved even for propositional languages having an uncountable number of propositional variables, using Zorn's lemma.

Lemma 2.2 (**Lindenbaum**)**.** *Each consistent set of propositions has a maximally consistent extension.*

Theorem 2.2 (**Model Existence Theorem for PC**)**.** *Every PC-consistent set of propositions has a model.*

Proof. Let Σ be a consistent set of propositions. Let Σ^* be the maximally consistent set of Lemma 2.1 (or as in Lemma 2.2). Define a function i from the set of all propositions to $\{0, 1\}$ by

$$i(p) = \begin{cases} 1, & \text{if } p \in \Sigma^* \\ 0, & \text{otherwise} \end{cases}$$

The function i is a boolean valuation due to Lemma 2.1(5)-(8). Obviously, i is a model of Σ^*. Since $\Sigma \subseteq \Sigma^*$, i is a model of Σ as well. $\Box$

Model existence theorem says that every consistent set is satisfiable, or that every unsatisfiable set is inconsistent. Then, RA (in PC and in PL) gives the following result.

Theorem 2.3 (**Strong Completeness of PC to PL**). *Let Σ be a set of propositions, and let w be any proposition. If $\Sigma \vDash w$, then $\Sigma \vdash_{PC} w$.*

With this, we have proved the **Strong Adequacy of PC to PL**. Thus, the inherent circularity in the semantic method is eliminated. We may then approach any problem in PL through semantics or through PC-proofs.

If a proposition is a theorem, we may be able to show it by supplying a PC-proof of it. If it is not a theorem, then the mechanism of PC fails to convince us. Reason: *I am not able to construct a proof does not mean there exists no proof!* In such a case, we may resort to truth tables and try to supply a falsifying interpretation. That would succeed, at least theoretically, if the given proposition is not a theorem.

Exercises

1. Let Σ be a set of propositions, and let x, y, z be propositions. Show that if $\Sigma \vDash y \to z$, then $\Sigma \vDash ((x \to y) \to (x \to z))$.
2. A set Σ of propositions is called **negation complete** iff for each proposition w, either $\Sigma \vDash w$ or $\Sigma \vDash \neg w$. Show that each consistent set of propositions is a subset of a negation complete set.
3. Show that a consistent set of propositions is maximally consistent iff it is negation complete.

3 Compactness of PL

Each finite subset of the set of natural numbers $\mathbb{N}$ has a minimum. Also, $\mathbb{N}$ itself has a minimum. However, each finite subset of $\mathbb{N}$ has a maximum, but $\mathbb{N}$ does not have a maximum. The properties of the first type, which hold for an infinite set whenever they hold for all finite subsets of the infinite set, are called **compact properties**.

For example, in a vector space, if all finite subsets of a set of vectors are linearly independent, then the set of vectors itself is linearly independent. Thus, **linear independence** is a compact property.

In PC, **consistency** is a compact property (why?). We use this to show that satisfiability in PL is also a compact property.

Theorem 3.1 (Compactness of PL). *Let Σ be an infinite set of propositions, and let w be a proposition.*

1. *$\Sigma \vDash w$ iff Σ has a finite subset Γ such that $\Gamma \vDash w$.*

2. *Σ is unsatisfiable iff Σ has a finite unsatisfiable subset.*

3. *Σ is satisfiable iff each nonempty finite subset of Σ is satisfiable.*

Proof. 1. Suppose $\Sigma \vDash w$. By the **strong completeness** of PC, $\Sigma \vdash_{PC} w$. By finiteness, there exists a finite subset Γ of Σ such that $\Gamma \vdash w$. By the **strong soundness** of PC, $\Gamma \vDash w$. Conversely, if there exists a finite subset Γ of Σ such that $\Gamma \vDash w$, then by **monotonicity**, $\Sigma \vDash w$.

2. In (1), take $w = p_0$ and then take $w = \neg p_0$. We have finite subsets Γ_1 and Γ_2 of Σ such that $\Gamma_1 \vDash p_0$ and $\Gamma_2 \vDash \neg p_0$. By **monotonicity**, $\Gamma \vDash p_0$ and $\Gamma \vDash \neg p_0$, where $\Gamma = \Gamma_1 \cup \Gamma_2$ is a finite subset of Σ. Now, Γ is inconsistent.

3. This is a restatement of (2).

□

Exercise: Call a set of propositions **finitely satisfiable** iff every finite subset of it is satisfiable. Without using compactness, prove that if Σ is a finitely satisfiable set of propositions, and A is any proposition, then one of the sets $\Sigma \cup \{A\}$ or $\Sigma \cup \{\neg A\}$ is finitely satisfiable.

4 QUASI-PROOFS IN PL

First, we consider the following theorem without the proof.

Theorem 4.1 (Laws of PL)**.** *Let x, y, and z be propositions. Then the following laws hold in PL:*

1. ***Absorption***

$$x \wedge (x \vee y) \equiv x, \quad x \vee (x \wedge y) \equiv x.$$

2. ***Associativity***

$$x \wedge (y \wedge z) \equiv (x \wedge y) \wedge z, \quad x \vee (y \vee z) \equiv (x \vee y) \vee z,$$

$$x \leftrightarrow (y \leftrightarrow z) \equiv (x \leftrightarrow y) \leftrightarrow z.$$

3. ***Biconditional***

$$x \leftrightarrow y \equiv (x \to y) \wedge (y \to x),$$

$$x \leftrightarrow y \equiv (x \vee \neg y) \wedge (\neg x \vee y), \quad x \leftrightarrow y \equiv (x \wedge y) \vee (\neg x \wedge \neg y),$$

$$\neg(x \leftrightarrow y) \equiv \neg x \leftrightarrow y, \quad \neg(x \leftrightarrow y) \equiv x \leftrightarrow \neg y,$$

$$\neg(x \leftrightarrow y) \equiv (x \wedge \neg y) \vee (\neg x \wedge y), \quad \neg(x \leftrightarrow y) \equiv (x \vee y) \wedge (\neg x \vee \neg y).$$

4. ***Cases***

$$\text{If } x \models z \text{ and } y \models z, \text{ then } x \vee y \models z.$$

5. ***Clavius***

$$\neg x \to x \equiv x.$$

6. ***Commutativity***

$$x \wedge y \equiv y \wedge x, \quad x \vee y \equiv y \vee x, \quad x \leftrightarrow y \equiv y \leftrightarrow x.$$

7. ***Constants***

$$\neg\top \equiv \bot, \quad \neg\bot \equiv \top, \quad x \wedge \top \equiv x, \quad x \wedge \bot \equiv \bot,$$
$$x \vee \top \equiv \top, \quad x \vee \bot \equiv x, \quad x \to \top \equiv \top, \quad x \to \bot \equiv \neg x,$$
$$\top \to x \equiv x, \quad \bot \to x \equiv \top, \quad x \leftrightarrow \top \equiv x, \quad x \leftrightarrow \bot \equiv \neg x.$$

8. ***Contradiction***

$$x \wedge \neg x \equiv \bot, \quad x \leftrightarrow \neg x \equiv \bot,$$
$$(\neg x \to \neg y) \to ((\neg x \to y) \to x) \equiv \top.$$

9. ***Contraposition***

$$x \to y \equiv \neg y \to \neg x, \quad \neg x \to y \equiv \neg y \to x, \quad x \to \neg y \equiv y \to \neg x.$$

10. ***De Morgan's Laws***

$$\neg(x \wedge y) \equiv \neg x \vee \neg y, \quad \neg(x \vee y) \equiv \neg x \wedge \neg y.$$

11. ***Disjunctive Syllogism***

$$\{x \vee y, \neg x\} \models y.$$

12. ***Distributivity***

$$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z).$$

13. ***Double Negation***

$$\neg\neg x \equiv x.$$

14. ***Elimination***

$$x \wedge y \models x, \quad x \wedge y \models y.$$

15. ***Excluded Middle***

$$x \vee \neg x \equiv \top.$$

16. ***Exportation***

$$x \to (y \to z) \equiv (x \wedge y) \to z.$$

17. ***Golden Rule***

$$x \leftrightarrow y \leftrightarrow (x \wedge y) \leftrightarrow (x \vee y) \equiv \top.$$

18. ***Hypothesis Invariance***

$$x \to (y \to x) \equiv \top.$$

19. ***Hypothetical Syllogism***

$$\{x \to y, y \to z\} \vDash x \to z.$$

20. ***Idempotency***

$$x \wedge x \equiv x, \quad x \vee x \equiv x.$$

21. ***Identity***

$$x \equiv x, \quad x \leftrightarrow x \equiv \top, \quad x \to x \equiv \top.$$

22. ***Implication***

$$x \to y \equiv \neg x \vee y, \quad \neg(x \to y) \equiv x \wedge \neg y.$$

23. ***Introduction***

$$\{x, y\} \vDash x \wedge y, \quad x \vDash x \vee y, \quad y \vDash x \vee y.$$

24. ***Modus Ponens***

$$\{x, x \to y\} \vDash y.$$

25. ***Modus Tollens***

$$\{x \to y, \neg y\} \vDash \neg x.$$

26. ***Pierce's Law***

$$(x \to y) \to x \equiv x, \quad (x \to y) \to y \equiv x \vee y.$$

The laws mentioned in Theorem 4.1 give rise to derived rules with the help of the replacement laws. Using these, we may create **quasi-proofs**. Like proofs, quasi-proofs are finite sequences of propositions that are either valid propositions mentioned in the laws or are derived from earlier propositions by using some laws. From a quasi-proof, a formal PC-proof can always be developed.

We write a quasi-proof in three columns:

- The first column contains the line numbers.
- The second column is the actual quasi-proof.
- The third column is the justification column documenting the laws that have been used to derive the proposition in that line.

In the third column, we will write 'P' for a premise, and sometimes 'T' for a law, to hide details.

Moreover, the **Deduction Theorem** can be used inside a quasi-proof rather than outside. To prove $X \to Y$, we simply introduce X anywhere in the quasi-proof and mark its introduction by 'DTB', abbreviating *an application of the Deduction Theorem begins here.*

When we deduce Y later in the quasi-proof, the next line will have $X \to Y$, and it will be marked as **DTE**, abbreviating:

The application of the Deduction Theorem ends here.

This notation indicates that the extra assumption X has been removed as an assumption, and by the Deduction Theorem, the formula $X \to Y$ has been obtained.

The Deduction Theorem can be applied multiple times in a quasi-proof, and the pairs of DTB-DTE must be nested like parentheses. In such a case, we may write DTB_1-DTE_1, DTB_2-DTE_2, and so on.

Similarly, **Reductio ad Absurdum** can be used inside a quasi-proof by introducing a formula $\neg X$ anywhere in the quasi-proof. We document it as **RAB**, abbreviating:

An application of a proof by Reductio ad Absurdum begins here,

and then by deducing $\perp$ later. The next line to $\perp$ would be X, and the documentation would include **RAE**, abbreviating:

The application of the proof by Reductio ad Absurdum ends here.

Thereby, the conditionality of the extra premise $\neg X$ is considered over. Again, multiple applications of Reductio ad Absurdum are nested like parentheses. We will mark the pairs as RAB_1-RAE_1, RAB_2-RAE_2, etc.

When both the Deduction Theorem and Reductio ad Absurdum are used in a quasi-proof, the pairs DTB_m-DTE_m and RAB_n-RAE_n will be used as parentheses of different types. They can nest, but the nesting must not intersect.

If a quasi-proof employs premises from a set of propositions Σ and its last proposition is w, then we say that it is a quasi-proof of $\Sigma \vdash w$. When $\Sigma = \emptyset$, the quasi-proof does not use any premise; thus, its last proposition is valid. The validity of such a proposition follows from the adequacy of PC. Also, a quasi-proof for $\vdash w$ proves the validity of w.

Example 4.1. *Construct a quasi-proof to show that:*

$$\{p \to \neg q, r \to s, \neg t \to q, s \to \neg u, t \to \neg v, \neg u \to w\} \vdash p \wedge r \to \neg(w \to v).$$

Line	*Proposition*	*Justification*
1.	$p \wedge r$	*DTB*
2.	p	*1, T*
3.	$p \to \neg q$	*P*
4.	$\neg q$	*2, 3, T*
5.	$\neg t \to q$	*P*
6.	$\neg\neg t$	*4, 5, T*
7.	t	*6, T*
8.	$t \to \neg v$	*P*
9.	$\neg v$	*7, 8, T*
10.	r	*1, T*
11.	$r \to s$	*P*
12.	s	*10, 11, T*
13.	$s \to \neg u$	*P*
14.	$\neg u$	*12, 13, T*
15.	$\neg u \to w$	*P*
16.	w	*14, 15, T*
17.	$\neg\neg(w \to v)$	*RAB*
18.	$w \to v$	*17, T*
19.	v	*16, 18, T*
20.	$\bot$	*9, 19, T*
21.	$\neg(w \to v)$	*17, 20, RAE*
22.	$p \wedge r \to \neg(w \to v)$	*1, 21, DTE*

Document all instances of 'T' in the above quasi-proof by writing the exact law. For instance, the 'T' in Line 6 is Modus Ponens. In a quasi-proof the replacement laws work implicitly. They give rise to the instances of other laws mentioned as 'T'. Since a PC-proof can be constructed from a quasi-proof, we regard quasi-proofs as intermediary steps towards construction of a formal PC-proof.

Exercises

Give quasi-proofs of the following valid propositions:

1. $((p \to q) \leftrightarrow (p \leftrightarrow (p \wedge q)))$
2. $((p \to q) \leftrightarrow (q \leftrightarrow (p \vee q)))$
3. $(((p \leftrightarrow q) \leftrightarrow (p \vee q)) \leftrightarrow (p \leftrightarrow q))$
4. $(((p \leftrightarrow q) \leftrightarrow r) \leftrightarrow (p \leftrightarrow (q \leftrightarrow r)))$
5. $((p \to r) \wedge (q \to r) \to (p \vee q \to r))$
6. $((p \to q) \wedge (p \to r) \to (p \to q \wedge r))$
7. $((p \wedge q \to r) \wedge (p \to q \vee r) \to (p \to r))$