Another way to Compute P:

In the last class, we got

$$u_1 = \frac{1}{5}(3, 0, 4)$$

$$u_2 = \frac{1}{5}(-4, 0, 3)$$

$$U = [u_1 \ u_2] = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 0 & 0 \\ 4 & 3 \end{bmatrix}_{3\times 2}$$

$$V = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ -4 & 0 & 3 \end{bmatrix}_{2\times 3}$$

$$= U^T$$

Note that $U^T U = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ -4 & 0 & 3 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 0 & 0 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

$$P = UU^T = \frac{1}{25}\begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Q</u>: Find eigen values of $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol:

$$|P-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2(-\lambda) = 0$$

$\Rightarrow$ $\lambda = 0$ is eigenvalue of multiplicity 1 and $\lambda = 1$ is an eigenvalue of multiplicity 2.

Result: Eigenvalues of any projection matrix P are either 0 or 1.

Proof: $Px = \lambda x$; $x \neq 0$

$$\Rightarrow P(Px) = P\lambda x = \lambda P x$$
$$= \lambda(\lambda x)$$
$$= \lambda^2 x \quad \text{—— (1)}$$

$$LHS = P(Px) = P^2 x = Px = \lambda x \quad \text{—— (2)}$$

$(P^2 = P)$

Equating (1), (2) : $\lambda x = \lambda^2 x$

$$\Rightarrow (\lambda^2 - \lambda) x = 0 \Rightarrow \lambda(\lambda - 1) = 0$$

$(x \neq 0)$.

$$\Rightarrow \lambda = 0 \text{ or } \lambda = 1.$$

Q: Orthonormalise $\{1, x, x^2, x^3\}$ given

$$\{\underset{w_0}{1}, \underset{w_1}{x}, \underset{w_2}{x^2}, \underset{w_3}{x^3}\}$$

basis for P_3

$$\langle f(x), g(x)\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

S/o:

$$v_0 = w_0 = 1$$

$$v_1 = w_1 - \frac{\langle w_1, v_0\rangle v_0}{\|v_0\|^2}$$

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$\langle w_1, v_0 \rangle = \langle x, 1 \rangle$$

$$= \int_{-\pi}^{\pi} x \cdot 1 \, dx$$

$$= \underline{\underline{0}}$$

$$\therefore \; v_1 = w_1 = x$$

$$v_2 = w_2 - \frac{\langle w_2, v_0 \rangle v_0}{\| v_0 \|^2} - \frac{\langle w_2, v_1 \rangle v_1}{\| v_1 \|^2}$$

$$\langle w_2, v_1 \rangle = \langle x^2, x \rangle$$

$$= \int_{-\pi}^{\pi} x^2 \cdot x \cdot dx$$

odd function

$$= 0$$

$$\langle w_\alpha, v_0 \rangle = \int_{-\pi}^{\pi} x^2 \cdot 1 \, dx = 2\int_0^{\pi} x^2 \, dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^3}{3}$$

$$\|v_0\|^2 = \langle v_0, v_0 \rangle$$

$$= \int_{-\pi}^{\pi} 1 \cdot dx = x \Big]_{-\pi}^{\pi}$$

$$= \pi - (-\pi)$$

$$= \underline{\underline{2\pi}}$$

$$v_2 = x^2 - \frac{\frac{2\pi^3}{3}}{2\pi} \cdot 1$$

$$= \underline{\underline{x^2 - \pi^2/3}}$$

$$v_3 = w_3 - \frac{\langle w_3, v_0\rangle v_0}{\|v_0\|^2} - \frac{\langle w_3, v_1\rangle v_1}{\|v_1\|^2} - \frac{\langle w_3, v_2\rangle v_2}{\|v_2\|^2}$$

0 $\quad$ 0

$$\langle \underset{x^3}{w_3}, \underset{1}{v_0}\rangle = \int_{-\pi}^{\pi} x^3 \cdot 1\, dx = 0$$

$$\langle \underset{x^3}{w_3}, \underset{x^2}{v_2}\rangle = \int_{-\pi}^{\pi} x^3 \cdot x^2\, dx = 0$$

$$\langle \underset{x^3}{w_3}, \underset{x}{v_1} \rangle = \int_{-\pi}^{\pi} x^3 \cdot x \, dx$$

$$= 2\int_{0}^{\pi} x^4 \, dx$$

$$= 2\left[x^5/5 \right]_0^{\pi}$$

$$= \frac{2\pi^5}{5}$$

$$\|v_1\|^2 = \langle v_1, v_1 \rangle$$

$$= \int_{-\pi}^{\pi} x \cdot x \, dx = 2\int_0^{\pi} x^2 \, dx$$

$$= 2\left[x^3/3 \right]_0^{\pi}$$

$$= \frac{2\pi^3}{3}$$

$$\therefore \; v_3 = \frac{\left(x^3 - \frac{2\pi^5}{5}\right)}{\left(\frac{2\pi^3}{3}\right)} \cdot x$$

$$v_3 = x^3 - \frac{3\pi^2}{5}x$$

$v_0 = 1$

$v_1(x) = x$

$v_2(x) = x^2 - \pi^2/3$

$v_3(x) = x^3 - \frac{3\pi^2}{5} x$

Now, $\{v_0(x), v_1(x), v_2(x), v_3(x)\}$ is an orthogonal set. We need to make each vector as unit vector as follows:

$$\left\{ \frac{v_0(x)}{\|v_0(x)\|}, \frac{v_1(x)}{\|v_1(x)\|}, \frac{v_2(x)}{\|v_2(x)\|}, \frac{v_3(x)}{\|v_3(x)\|} \right\}.$$

Normalisation

$$u_0 = \frac{v_0}{\|v_0\|}$$

$$\|v_0\|^2 = \langle v_0, v_0 \rangle = \int_{-\pi}^{\pi} 1 \, dx$$

$$= x\Big]_{-\pi}^{\pi}$$

$$= \pi - (-\pi) = 2\pi$$

$$\therefore \|v_0\| = \sqrt{2\pi}$$

$$\therefore u_0 = \frac{1}{\sqrt{2\pi}}$$

$$u_1 = \frac{v_1}{\|v_1\|}$$

$$\|v_1\|^2 = \langle \overset{x}{v_1}, \overset{x}{v_1} \rangle = \int_{-\pi}^{\pi} x \cdot x \, dx$$

$$= 2\int_0^{\pi} x^2 \, dx$$

$$= 2\left[x^3/3 \right]_0^{\pi} = \frac{2\pi^3}{3}$$

$$u_1 = \frac{x}{\sqrt{2\pi^3/3}}$$

$$\|v_2\|^2 = \langle v_2, v_2 \rangle$$

$$= \int_{-\pi}^{\pi} \left(x^2 - \pi^2/3\right)^2 dx$$

$$= 2\int_{0}^{\pi} \left(x^4 - 2x^2\frac{\pi^2}{3} + \frac{\pi^4}{9}\right) dx$$

$$= 2\left[\frac{x^5}{5} - 2\frac{x^3\pi^2}{3} + \frac{\pi^4 x}{9}\right]_0^{\pi}$$

$$= 2\left[\frac{\pi^5}{5} - \frac{2\pi^5}{9} + \frac{\pi^5}{9}\right]$$

$$= 2\left[\frac{\pi^5}{5} - \frac{\pi^5}{9}\right]$$

$$u_2 = \frac{v_2}{\|v_2\|}$$

$$= \frac{v_2}{\sqrt{2\left(\frac{\pi^5}{5} - \frac{\pi^5}{9}\right)}}$$

$$= \frac{x^2 - \pi^2/3}{\sqrt{2\left(\frac{4\pi^5}{45}\right)}} = \sqrt{\frac{45}{8\pi^5}}\left(x^2 - \pi^2/3\right)$$

$$\|v_3\|^2 = \langle v_3, v_3 \rangle$$

$$= \int_{-\pi}^{\pi} \left(x^3 - \frac{3\pi^2}{5} x \right)^2 dx$$

$$= 2 \int_{0}^{\pi} \left(x^6 - 2 \cdot x^3 \cdot \frac{3\pi^2}{5} x + \frac{9\pi^4}{25} x^2 \right) dx$$

$$= 2 \int_{0}^{\pi} \left(x^6 - \frac{6}{5} x^4 \pi^2 + \frac{9}{25} \pi^4 x^2 \right) dx$$

$$K^2 = 2\left[\frac{x^7}{7} - \frac{6}{5}\frac{\pi^2 x^5}{5} + \frac{9\pi^4}{25}\frac{x^3}{3}\right]_0^{\pi}$$

$$K = \sqrt{2\left(\frac{\pi^7}{7} - \frac{6}{25}\pi^7 + \frac{3}{25}\pi^7\right)} = \sqrt{\frac{8}{175}\pi^7}$$

$$\therefore \quad u_3 = \frac{1}{K}\left(x^3 - \frac{3\pi^2}{5}x\right)$$

$$= \sqrt{\frac{175}{8\pi^7}}\left(x^3 - \frac{3\pi^2}{5}x\right).$$

$$u_0 = \frac{1}{\sqrt{2\pi}} \; ; \quad u_1 = \frac{x}{\sqrt{2\pi^3/3}}$$

$$u_2 = \sqrt{\frac{45}{8\pi^5}}\left(x^2 - \pi^2/3\right)$$

$$u_3 = \sqrt{\frac{175}{8\pi^7}}\left(x^3 - \frac{3\pi^2}{5}x\right)$$

$\{u_0, u_1, u_2, u_3\}$ is orthonormal set by the Gram-schmidt process.

orthonormal basis for P_3.

Exercise:

Q: Orthonormalize $\{1, x, x^2, x^3\}$ using the Gram-Schmidt process by taking the inner product as

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

Answers:

$$u_0 = \frac{1}{\sqrt{2}}$$

$$u_1 = \sqrt{3/2}\, x$$

$$u_2 = \sqrt{\frac{45}{8}}\left(x^2 - 1/3\right)$$

$$u_3 = \sqrt{\frac{175}{8}}\left(x^3 - 3/5\, x\right)$$

$\{u_0, u_1, u_2, u_3\}$ is an orthonormal set by the Gram-Schmidt process.