$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

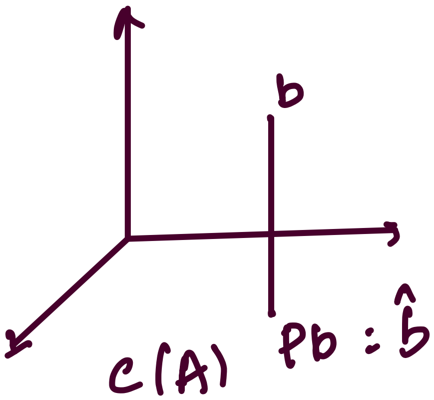
$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = X^T Y.$$

↪ vectorization of $\langle X, Y \rangle$.

$$\langle Y, X \rangle = y_1 x_1 + \cdots + y_n x_n = Y^T X = \langle X, Y \rangle.$$

Given a matrix A,



P : involves computation of inverses.

Algorithm.

Step 1 : $C(A) \rightarrow$ orthonormal basis $\{u_1, u_2, \dots, u_n\}$

$$\langle u_i, u_j \rangle = 0 \quad , \quad i \neq j$$

$$\langle u_i, u_i \rangle = \|u_i\|^2 = 1$$

Step 2 : $\hat{b} \in C(A)$

$$\Rightarrow \hat{b} = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

scalars / projections

$$= \frac{\langle u_1, b\rangle}{\|u_1\|} u_1 + \frac{\langle u_2, b\rangle}{\|u_2\|} u_2 + \cdots + \frac{\langle u_n, b\rangle}{\|u_n\|} u_n$$

$$= \langle u_1, b\rangle u_1 + \langle u_2, b\rangle u_2 + \cdots + \langle u_n, b\rangle u_n$$

(as $\|u_i\| = 1$, $1 \le i \le n$)

$$= u_1^T b\, u_1 + u_2^T b\, u_2 + \cdots + u_n^T b\, u_n$$

(vectorization)

Then $\hat{b} = \underbrace{u_1^T b}_{\text{Scalar}} u_1 + u_2^T b u_2 + \cdots + \underbrace{u_n^T b}_{\text{Scalar}} u_n$

$$= u_1 (u_1^T b) + u_2 (u_2^T b) + \cdots + u_n (u_n^T b)$$ (as $u_i^T b$ is a scalar)

$$= (u_1 u_1^T + u_2 u_2^T + \cdots + u_n u_n^T) b$$

$$= [u_1 \; u_2 \; \cdots \; u_n] \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} b$$ (vectorization)

$= \underbrace{UV}_{P} b$; where $U = [u_1 \; u_2 \cdots u_n]$, $V = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$.

$= Pb$; $P = UV$ (Found P without matrix inverse)

Q. Without using matrix inverse, find projection matrix; given $A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 4 & 7 \end{bmatrix}$.

$$\begin{array}{cc} \uparrow & \uparrow \\ w_1 & w_2 \end{array}$$

Soln:

$$\{w_1, w_2\} = \{\underbrace{(3,0,4)}_{w_1}, \underbrace{(-1,0,7)}_{w_2}\}.$$

Gram-Schmidt process: $v_1 = w_1 = (3,0,4)$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle v_1}{\|v_1\|^2}$$

$$= (-1,0,7) - \frac{\langle(-1,0,7),(3,0,4)\rangle}{(3^2+0^2+4^2)}(3,0,4)$$

$$v_2 = (-1,0,7) - \frac{(-1)(3)+0(0)+7(4)}{25}(3,0,4)$$

$$= (-1,0,7) - \frac{25}{25}(3,0,4)$$

$$= (-1,0,7) - (3,0,4)$$

$$= (-4,0,3)$$

$$\langle v_1, v_2 \rangle = 3(-4)+0(0)+4(3) = 0.$$

$\Rightarrow \{v_1, v_2\}$ is orthogonal.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(3,0,4)}{\sqrt{3^2+0+4^2}} = \frac{1}{5}(3,0,4).$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(-4, 0, 3)}{\sqrt{4^2+0^2+3^2}} = \frac{1}{5}(-4, 0, 3)$$

$\therefore \{u_1, u_2\}$ is orthonormal.

$$U = [u_1 \quad u_2]$$

$$V = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$$

$\therefore$ projection matrix $P = UV$

$$= [u_1 \quad u_2] \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$$

$$P = u_1 u_1^T + u_2 u_2^T$$

$$= \underbrace{\frac{1}{5}\underset{3\times 1}{\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}} \frac{1}{5}\underset{1\times 3}{\begin{bmatrix} 3 & 0 & 4 \end{bmatrix}}}_{\text{outer product of vectors}} + \frac{1}{5}\underset{3\times 1}{\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}} \frac{1}{5}\underset{1\times 3}{\begin{bmatrix} -4 & 0 & 3 \end{bmatrix}}$$

$$= \frac{1}{25}\begin{bmatrix} 9 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} + \frac{1}{25}\begin{bmatrix} 16 & 0 & -12 \\ 0 & 0 & 0 \\ -12 & 0 & 9 \end{bmatrix}.$$

$$= \frac{1}{25}\begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Q. Project $(5,6,7)$ on $C(A)$; given

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 4 & 7 \end{bmatrix}.$$

Soln: Take $b = (5,6,7)$.

projection matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$Pb = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}.$$