Sample questions with selected solutions

Ref: Complex variables & applications
— Brown & Churchill.

1) Show that
$z = x+iy$ is purely imaginary
$\Leftrightarrow \bar{z} = -z$

2) What are the values of

a) $e^{\pi i}$ b) $e^{i\pi/2}$ c) $e^{-i\pi/2}$ d) $e^{2\pi i}$

3) Show that $e^{\alpha + 2\pi n i} = e^{\alpha}$; $\alpha \in \mathbb{C}$

4) a) Show that $\arg z_1 + \arg z_2 = \arg(z_1 z_2)$. $z_1, z_2 \neq 0$

b) Is $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \mathrm{Arg}(z_1 z_2)$?

Arg z — denotes the principal argument of z.

5) For $z \neq 0$, $\arg z^{-1} = -\arg z$.

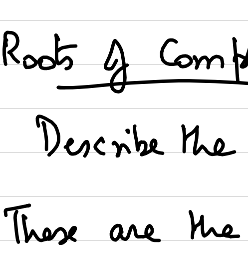
Soln: $z = re^{i\theta}, r>0 \Rightarrow z^{-1} = r^{-1}e^{-i\theta}$
$\Rightarrow \arg z^{-1} = \{-\theta + 2n\pi \mid n \in \mathbb{Z}\}$

However, $\arg z = \{\theta + 2n\pi \mid n \in \mathbb{Z}\}$
$\Rightarrow -\arg z = \{-\theta - 2n\pi \mid n \in \mathbb{Z}\}$
$= \{-\theta + 2m\pi \mid m \in \mathbb{Z}\}$
$= \arg z^{-1}$.

6) If $z_2, z_1 \neq 0$,
$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

7) Find $\mathrm{Arg}\left(\frac{-2}{1+\sqrt{3}i}\right)$

$$\arg\left(\frac{-2}{1+\sqrt{3}i}\right) = \arg(-2) - \arg(1+\sqrt{3}i)$$
$$= \{\pi + 2n\pi \mid n \in \mathbb{Z}\} - \{\tfrac{\pi}{3} + 2m\pi \mid m \in \mathbb{Z}\}$$



$$z_2 = 2\left(\frac{1+\sqrt{3}i}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\therefore \arg\left(\frac{-2}{1+\sqrt{3}i}\right) = \left\{\frac{2\pi}{3} + 2n\pi \mid n \in \mathbb{Z}\right\}$$
$$\therefore \mathrm{Arg}\left(\frac{-2}{1+\sqrt{3}i}\right) = \frac{2\pi}{3} + 2n\pi \text{ where}$$
$$-\pi < \frac{2\pi}{3} + 2n\pi \leq \pi$$
$$\Rightarrow n = 0 \Rightarrow \mathrm{Arg}\left(\frac{-2}{1+\sqrt{3}i}\right) = \frac{2\pi}{3}.$$

Roots of Complex Nos

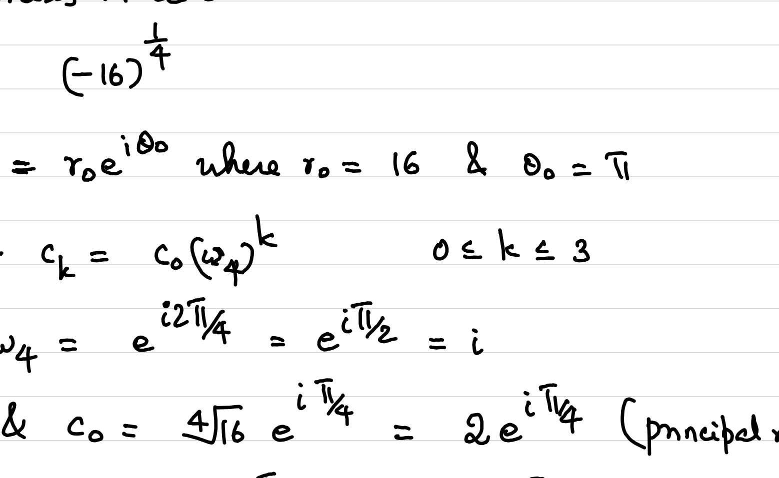
8) Describe the set $(2)^{1/3}$.

These are the three complex roots of $z^3 - 2 = 0$.

The distinct n^{th} roots of a non-zero complex no. $z_0 = r_0 e^{i\theta_0}$ are of the form
$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} \qquad 0 \leq k \leq n-1.$$

Here, $r_0 = 2$ & take $\theta_0 = 0 \Rightarrow$

$c_0 = \sqrt[3]{2}$
$c_1 = \sqrt[3]{2}\, e^{i2\pi/3}$
$c_2 = \sqrt[3]{2}\, e^{i4\pi/3}$



c_0 is the principal cube root of 2.

9) Exercises : Sec 10

$(-16)^{\frac{1}{4}}$

$z_0 = r_0 e^{i\theta_0}$ where $r_0 = 16$ & $\theta_0 = \pi$

$\therefore c_k = c_0 (\omega_4)^k \qquad 0 \leq k \leq 3$

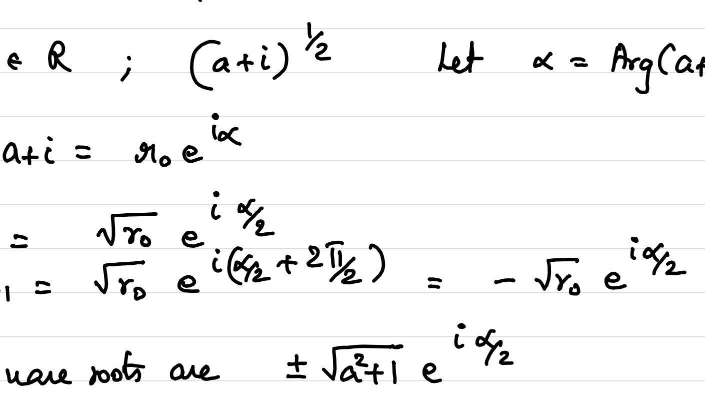
where $\omega_4 = e^{i2\pi/4} = e^{i\pi/2} = i$

& $c_0 = \sqrt[4]{16}\, e^{i\pi/4} = 2e^{i\pi/4}$ (principal root)

$c_1 = 2e^{i\pi/4}\omega_4 = 2e^{i3\pi/4}$

$c_2 = 2e^{i\left(\frac{3\pi}{4} + \frac{\pi}{2}\right)} = 2e^{i\frac{5\pi}{4}}$

$c_3 = 2e^{i\frac{7\pi}{4}}$



$c_0 = \sqrt{2}(1+i)$; $c_1 = \sqrt{2}(-1+i)$

$c_3 = -\sqrt{2}(1+i)$; $c_4 = \sqrt{2}(1-i)$

5) a) $a \in \mathbb{R}$; $(a+i)^{1/2}$ Let $\alpha = \mathrm{Arg}(a+i)$

Let $a+i = r_0 e^{i\alpha}$

$\therefore c_0 = \sqrt{r_0}\, e^{i\alpha/2}$
& $c_1 = \sqrt{r_0}\, e^{i(\alpha/2 + 2\pi/2)} = -\sqrt{r_0}\, e^{i\alpha/2}$

$\therefore$ Two square roots are $\pm\sqrt{a^2+1}\, e^{i\alpha/2}$

6) Zeroes of z^4+4 are the four complex roots of -4, ie $(-4)^{\frac{1}{4}}$.

$c_0 = \sqrt[4]{4}\, e^{i\pi/4} = \sqrt{2}\, e^{i\pi/4}$
$c_1 = \sqrt[4]{4}\, e^{i\left(\pi/4 + \frac{2\pi}{4}\right)} = \sqrt{2}\, e^{i3\pi/4}$
$c_2 = \sqrt[4]{4}\, e^{i\left(\pi/4 + \frac{4\pi}{4}\right)} = \sqrt{2}\, e^{-i3\pi/4}$
$c_3 = \sqrt[4]{4}\, e^{i\left(\frac{\pi}{4} + \frac{6\pi}{4}\right)} = \sqrt{2}\, e^{-i\pi/4}$

Here, $z_0 = c_0 = 1+i$ & c_3 is its conjugate
$\therefore (z-c_0)(z-c_3) = z^2 - (c_0+c_3)z + c_0 c_3$
$= z^2 - 2\,\mathrm{Re}(c_0)z + |c_0|^2$
$= z^2 - 2z + 2$

$c_1 = -1+i$

Ily, $(z-c_1)(z-c_2) = z^2 - 2\,\mathrm{Re}(c_1)z + |c_1|^2$
$= z^2 + 2z + 2$

$$\Rightarrow z^4 + 4 = \prod_{i=0}^{3}(z-c_i) = (z^2-2z+2)(z^2+2z+2)$$

Sec 8 (Q9)

a) $1 + z + z^2 + \ldots + z^n = \dfrac{1-z^{n+1}}{1-z}$, $z \neq 1$

Exercise

b) (Lagrange's trigonometric identity)

Let $z = e^{i\theta} \neq 1$.

$$\therefore 1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$$

$$\therefore 1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \mathrm{Re}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right).$$

$$\frac{\left(1-e^{i(n+1)\theta}\right)\left(1-e^{-i\theta}\right)}{|1-e^{i\theta}|^2} = \frac{1 - e^{i(n+1)\theta} - e^{-i\theta} + e^{in\theta}}{(1-\cos\theta)^2 + (\sin\theta)^2}$$

$\mathrm{Re}(\text{Numerator}) = 1 - \cos(n+1)\theta - \cos\theta + \cos n\theta$

$\cos n\theta - \cos(n+1)\theta = -2\sin\left(\frac{2n\theta + \theta}{2}\right)\sin\left(-\frac{\theta}{2}\right)$

& $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

Using this, arrive at the identity.

> **Aside**: In general is $\left(\frac{z}{a^n}\right)^{\frac{1}{n}} = \frac{z^{\frac{1}{n}}}{a}$ for $z, a \neq 0$?
> (For. eg is $(z)^{\frac{1}{2}} = \frac{z^{\frac{1}{2}}}{-1}$?) as sets

Note that
$\left(\frac{z}{a^n}\right)^{\frac{1}{n}}$ is the set of all distinct n^{th} roots of $\frac{z}{a^n}$.
However, $\frac{z^{\frac{1}{n}}}{a} = \left\{\frac{1}{a}c_0, \ldots, \frac{1}{a}c_{n-1}\right\}$ where

c_k for $0 \leq k \leq n-1$ are n distinct n^{th} roots of z.

Ans: YES.
Let $z = re^{i\theta}$ & $a = |a|e^{i\alpha}$

$\therefore \frac{z}{a^n} = \frac{r}{|a|^n}e^{i(\theta - n\alpha)} \Rightarrow \left(\frac{z}{a^n}\right)^{\frac{1}{n}} = \{c_0', c_1', \ldots, c_{n-1}'\}$ where

$c_0' = \sqrt[n]{\frac{r}{|a|^n}}\, e^{i\left(\frac{\theta - n\alpha}{n}\right)}$ $\qquad c_1' = \sqrt[n]{\frac{r}{|a|^n}}\, e^{i\left(\frac{\theta - n\alpha}{n}\right)} e^{2\pi i/n}$

$\vdots$

$c_k' = \sqrt[n]{\frac{r}{|a|^n}}\, e^{i\left(\frac{\theta - n\alpha}{n}\right)} e^{2\pi k i/n}$

$\vdots$

$c_{n-1}' = \sqrt[n]{\frac{r}{|a|^n}}\, e^{i\left(\frac{\theta - n\alpha}{n}\right)} e^{2\pi i\frac{(n-1)}{n}}$.

whereas
$c_k = \sqrt[n]{r}\, e^{i\theta/n} e^{2\pi k i/n}$

Note $c_k' = \sqrt[n]{\frac{r}{|a|^n}}\, e^{i\left(\frac{\theta - n\alpha}{n}\right)} e^{2\pi k i/n}$
$= \frac{\sqrt[n]{r}}{|a|} \frac{e^{i\theta/n} e^{2\pi k i/n}}{e^{i\alpha}}$
$= \frac{c_k}{a}$

Back to Sec 10: Q7 (Exercise)

Sec 10: Q8 a) $az^2 + bz + c = 0$, $a \neq 0$

ie, $a\left[z^2 + \frac{b}{a}z + \frac{c}{a}\right] = 0$

ie, $a\left[z^2 + 2z\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] = 0$

$\Rightarrow a\left(z + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$

$\Rightarrow a\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$

If $b^2 - 4ac = 0$, then $z = -\frac{b}{2a}$ is the only root (with multiplicity 2)

Otherwise, the solutions for the quadratic eqn $az^2+bz+c=0$ for $a, b, c \in \mathbb{C}$, $a \neq 0$ can be found if we know all the square roots of $\frac{b^2-4ac}{4a^2}$

Thus, complex solns of $az^2+bz+c=0$ when $a \neq 0$ & $b^2 - 4ac \neq 0$ are
$$-\frac{b}{2a} + \left(\frac{b^2-4ac}{4a^2}\right)^{\frac{1}{2}}. \quad — (1)$$
where $\left(\frac{b^2-4ac}{4a^2}\right)^{\frac{1}{2}}$ is the set of all square roots of the complex number $\frac{b^2-4ac}{4a^2}$.

However, solution given in the problem is of the form
$$\frac{-b + (b^2-4ac)^{\frac{1}{2}}}{2a} \quad — (2)$$
which means if $(b^2-4ac)^{\frac{1}{2}} = \{c_0, c_1\}$, then
$$\frac{-b + (b^2-4ac)^{\frac{1}{2}}}{2a} = \left\{\frac{-b+c_0}{2a}, \frac{-b+c_1}{2a}\right\}.$$

We established earlier that $\left(\frac{b^2-4ac}{4a^2}\right)^{\frac{1}{2}} = \frac{(b^2-4ac)^{\frac{1}{2}}}{2a}$ as sets. Hence, (distinct) solns are

$$z = \begin{cases} \dfrac{-b + (b^2-4ac)^{\frac{1}{2}}}{2a} & \text{if } a \neq 0 \;\&\; b^2-4ac \neq 0 \\[2ex] \underbrace{-\dfrac{b}{2a}}_{\text{repeated root}} & \text{if } a \neq 0 \;\&\; b^2-4ac = 0. \end{cases}$$