Vectors : Basic concepts (Lec 1-3)

Lec 1:- We recalled the notions of vectors & scalars with examples. We saw that two vectors in R^n are equal regardless of their initial & terminal points if they have the same magnitude & direction.

We saw the (triangular law & parallelogram law) addition operation in R^n & defined zero vector. Further, we defined the unique negative (inverse) of a vector. We saw that $(R^n, +)$ forms an abelian group under vector addition

Further, we saw scalar multiplication (by real no.) on R^n & by noting the distributive properties, we saw that R^n becomes a vector space over the field of real numbers R. (We demonstrated these vector space properties geometrically in the case $n=3$.)

Further, we recalled the Cartesian Coordinate system XYZ in R^3 & saw that if $P=(x_1,y_1,z_1)$ & $Q=(x_2,y_2,z_2)$ are two points in R^3 w.r.t this co-ordinate system, then x_2-x_1, y_2-y_1 & z_2-z_1 are called the components of the vector $\vec{PQ}$ in the X, Y & Z directions respectively.

Further, we defined the length or norm of the vector $\vec{PQ}$ as

$$\|\vec{PQ}\| := \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Notice that we use the symbol $\|\cdot\|$ instead of the usual symbol $|\vec{PQ}|$ to distinguish it from the absolute value symbol $|\cdot|$. This will keep us in distinguishing between lengths of vectors & lengths of scalars (which is their absolute value!)

For eg. if $\vec{v} = -2(\hat{i}+2\hat{j})$, then

$$\|\vec{v}\| = |-2| \cdot \|\hat{i}+2\hat{j}\|$$

$$\text{ie, } \|\vec{v}\| = 2\times\sqrt{5}$$

Inner product :-

What differentiates R^n from an arbitrary n dimensional vector space is the existence of concepts like inner products & angle between two vectors.

For e.g, it is not apriori understood what is the length of any polynomial in following 3 d vector space

$$V = \{a_0+a_1x+a_2x^2 \mid a_0,a_1,a_2 \in R\}$$

Let us take the case R^3. Here, inner product or dot product of two vectors $\vec{a}$ & $\vec{b}$ is

$$\vec{a}\cdot\vec{b} = \begin{cases} \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta & \text{if } \vec{a}\neq 0 \text{ \& } \vec{b}\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta \in [0,\pi]$ is the acute angle between $\vec{a}$ & $\vec{b}$. If we use the Cartesian coordinates

$$\vec{a} = (a_1,a_2,a_3)$$
$$\vec{b} = (b_1,b_2,b_3), \text{ then}$$

$$\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3.$$

Orthogonality

A vector $\vec{a}$ is orthogonal to $\vec{b}$ if $\vec{a}\cdot\vec{b}=0$. Since inner product commutes, here, we can say $\vec{b}$ is also orthogonal to $\vec{a}$ ie, $\vec{a}$ & $\vec{b}$ are orthogonal to each other.

Theorem

The inner product of two non-zero vectors is 0 iff these vectors are $\perp r$.

Proof:- "$\Rightarrow$" (only if part) (or necessary part)
Suppose $\vec{a}\cdot\vec{b}=0$, ie, $\|\vec{a}\|\cdot\|\vec{b}\|\cos\theta = 0$
$\Rightarrow \theta = \pi/2$ as $\vec{a}$ & $\vec{b}$ are non zero vectors

$\Leftarrow$ (if part) (sufficient part) Let $\vec{a}$ & $\vec{b}$ be $\perp r$ & non-zero.
Then, $\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos\pi/2 = 0$

Length & angle

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$$ (only defined if both $\vec{a}$ & $\vec{b}$ are non zero)

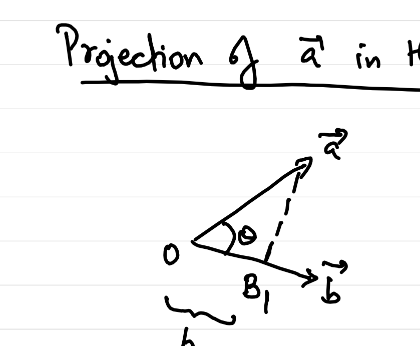
Cauchy-Schwarz inequality :-

$$|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|$$

Triangle inequality

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\|+\|\vec{b}\|$$

Projection of $\vec{a}$ in the direction of $\vec{b}$



Projection of $\vec{a}$ along a non zero vector $\vec{b}$ is

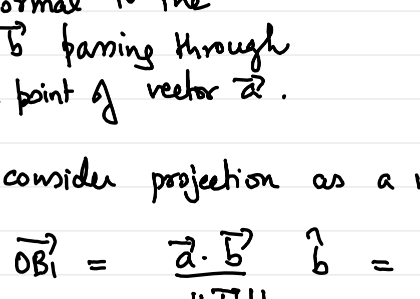
$$p = \|\vec{a}\|\cos\theta = \vec{a}\cdot\left(\frac{\vec{b}}{\|\vec{b}\|}\right) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$

Geometrically, it is the distance OB_1 where B_1 is the point of intersection of the normal to the vector $\vec{b}$ passing through terminal point of vector $\vec{a}$.

Some also consider projection as a vector. Then,

$$\vec{OB_1} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\hat{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}.$$

If θ is between 90 & 180, then projection p is a -ve value. as it is antiparallel to $\vec{b}$.

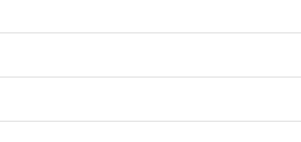


Vector Product

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(is a vector)

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\cdot\|\vec{b}\|\sin\theta$$



Direction: determined by right hand screw rule.

Scalar & vector fields

Let $f: X \longrightarrow Y$

X : domain of f
Y : co domain of f

We say f is a scalar function or scalar valued function if $f(P)$ is a scalar for every P in X. We also say that f defines a scalar field in that domain of definition of f.

Eg. a) Temperature field of a body or at any point on the Earth's surface.
b) Pressure field of the air in Earth's atmosphere.
c) Any function $f: R^2 \to R$ or $f: R^3 \to R$. like $f(x,y,z) = x^3y - z^2$
d) distance of any place in earth from your home. $P=(x,y,z)$ $H=(a,b,c)$

$$f(P) = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

e) Divergence of a vector field defines a scalar field. (to be defined later).

Vector fields :-

We say v is a vector function ie whose values $v(P)$ (for P in the domain of v) are vectors in the 3 dimensional space.

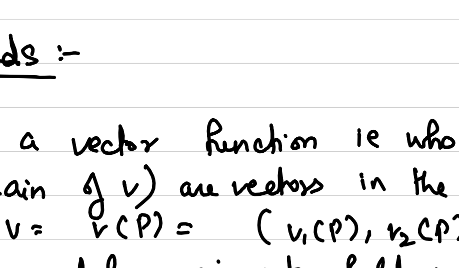
$$v = v(P) = (v_1(P), v_2(P), v_3(P))$$

A vector function v defines a vector field in its domain of definition.

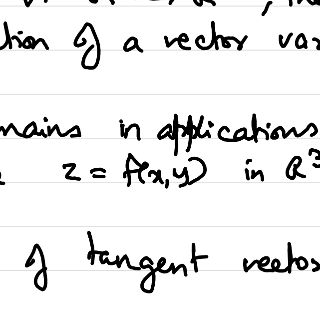
If $v: R^n \to R^m$, then we call v as a vector valued function of a vector variable. (usually when $n,m>1$).

Typical domains in applications are either R^3 or a surface $z=f(x,y)$ in R^3 or a curve in R^3.

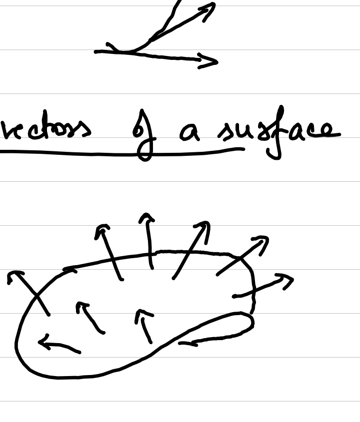
Eg. a) Field of tangent vectors of a differentiable curve



b) Normal vectors of a surface

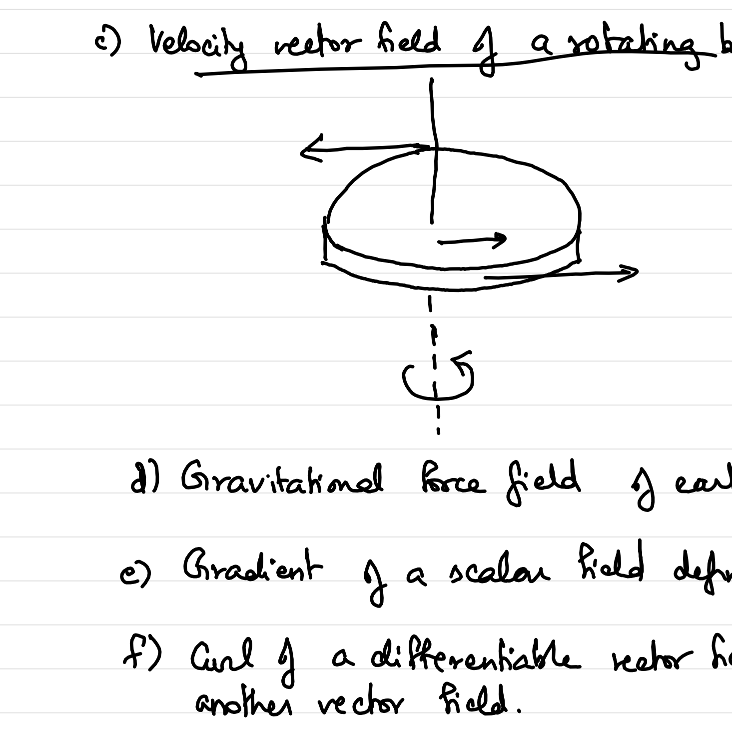


c) Velocity vector field of a rotating body



d) Gravitational force field of earth on any object
e) Gradient of a scalar field defines a vector field
f) Curl of a differentiable vector field defines another vector field.

Differentiation of univariate Functions



The slope of the secant line PQ is $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

As $\Delta x \to 0$ (ie as Q approaches P), the secant line will approach the tangent line l &

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$ will be the slope of l.

Thus, $f'(x_0)$ is the slope of the tangent to f at P.

Here, we assume the above limit exists.

We will see the notion of derivatives for vector valued functions in upcoming lectures.

Vector Calculus

Let $\vec{a_n}$ be an infinite sequence of vectors in $\mathbb{R}^3$. Further,

$$\vec{a_n} = a_{n,1}\hat{i} + a_{n,2}\hat{j} + a_{n,3}\hat{k}.$$

We say $\vec{a_n}$, $n \in \{1,2,..\}$ converges if $\exists\ \vec{a} \in \mathbb{R}^3$ such that

$$\lim_{n\to\infty} \|\vec{a_n} - \vec{a}\| = 0.$$

Here, $\|\vec{x}\| = \|x_1\hat{i} + x_2\hat{j} + x_3\hat{k}\| \overset{\text{defn}}{=} \sqrt{x_1^2 + x_2^2 + x_3^2}$ (called l_2 norm)

$\vec{a}$ is called the <u>limit vector</u> of $\vec{a_n}$ & we write

$$\operatorname{Lt}_{n\to\infty} \vec{a_n} = \vec{a}.$$

<u>Thm 1)</u> $\vec{a_n} \to \vec{a} \iff \operatorname{Lt}_{n\to\infty} a_{n,1} = a_1$

$\operatorname{Lt}_{n\to\infty} a_{n,2} = a_2$

& $\operatorname{Lt}_{n\to\infty} a_{n,3} = a_3$.

<u>Proof</u> :- "$\Rightarrow$"

<u>Given</u> : $\operatorname{Lt}_{n\to\infty} \|\vec{a_n} - \vec{a}\| = 0$

This implies $\operatorname{Lt}_{n\to\infty} \|\vec{a_n} - \vec{a}\|^2 = 0$ as $\|\vec{a_n} - \vec{a}\|^2 \le \|\vec{a_n} - \vec{a}\|$ for n sufficiently large

$$\|\vec{a_n} - \vec{a}\|^2 = |a_{n,1} - a_1|^2 + |a_{n,2} - a_2|^2 + |a_{n,3} - a_3|^2 \quad \text{—} ①$$

In particular, for each $r \in \{1,2,3\}$

$$|a_{n,r} - a_r|^2 \le \|\vec{a_n} - \vec{a}\|^2$$

$$\therefore \operatorname{Lt}_{n\to\infty} |a_{n,r} - a_r| \le \operatorname{Lt}_{n\to\infty} \|\vec{a_n} - \vec{a}\|$$

Thus,

$$\operatorname{Lt}_{n\to\infty} a_{n,r} = a_r \quad \text{for each } r \in \{1,2,3\}.$$

"$\Leftarrow$" Applying limits on both sides of $\sqrt{①}$, we see that

$$\operatorname{Lt}_{n\to\infty} \|\vec{a_n} - \vec{a}\| = \operatorname{Lt}_{n\to\infty} \sqrt{\left(|a_{n,1} - a_1|^2 + \operatorname{Lt}_{n\to\infty} |a_{n,2} - a_2|^2 + \operatorname{Lt}_{n\to\infty} |a_{n,3} - a_3|^2\right)}$$

$$= \left(\operatorname{Lt}_{n\to\infty} |a_{n,1} - a_1|^2 + \operatorname{Lt}_{n\to\infty} |a_{n,2} - a_2|^2 + \operatorname{Lt}_{n\to\infty} |a_{n,3} - a_3|^2\right)^{1/2}$$

$$\le \left(\operatorname{Lt}_{n\to\infty} |a_{n,1} - a_1| + \operatorname{Lt}_{n\to\infty} |a_{n,2} - a_2| + \operatorname{Lt}_{n\to\infty} |a_{n,3} - a_3|\right)^{1/2}$$

$\because |a_{n,r} - a_1|^2 \le |a_{n,r} - a_1|$ for n suff. large

$= 0.$

<u>Another way</u> :- $\Leftarrow$

$$\|(a_{n,1} - a_1)\hat{i} + (a_{n,2} - a_2)\hat{j} + (a_{n,3} - a_3)\hat{k}\|$$

$$\le |a_{n,1} - a_1| + |a_{n,2} - a_2| + |a_{n,3} - a_3| \quad (\Delta r \text{ inequality})$$

Applying limit $_{n\to\infty}$ on both sides, we get

$\operatorname{Lt}_{n\to\infty} \|\vec{a_n} - \vec{a}\| \le 0$ & we are done.

Open-Balls in $\mathbb{R}^n$, $n \ge 1$.

Let $\vec{a} \in \mathbb{R}^n$ & let $r > 0$. An open n-ball of radius r & center $\vec{a}$ is given by

$$B(a,r) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < r\}.$$

$n=1$: Open intervals $(a-r, a+r)$

$n=2$: Open (circular disks)

$n=3$: Open spherical solids

Limit of a functions

A vector function $v(t)$ of a real variable (say, $v: S \to \mathbb{R}^3$) is said to have the limit l as $t \to t_0$

ie, $\operatorname{Lt}_{t\to t_0} v(t) = l$ if

1) $v(t)$ is defined in $B(t_0, r) \setminus \{t_0\}$ for some $r > 0$ &
2) $\operatorname{Lt}_{t\to t_0} \|v(t) - l\| = 0.$

<u>Note</u> : l is a vector here

Continuity

A vector function $v(t)$ of a real variable (say, $v: S \to \mathbb{R}^3$) is continuous at $t_0 \in S$ if

1) v is defined in $B(t_0, r)$ for some $r > 0$ &
2) $\operatorname{Lt}_{t\to t_0} v(t) = v(t_0).$

(Verify) v is continuous at $t_0 \iff v_1(t), v_2(t)$ & $v_3(t)$ are continuous at t_0.

where

$$v(t) = [v_1(t), v_2(t), v_3(t)].$$

<u>Defn : Limit point</u>

Let $S \subseteq \mathbb{R}^n, n \ge 1$. Assume $\vec{a} \in \mathbb{R}^n$. $\vec{a}$ is called a limit point of S if every deleted neighbourhood of $\vec{a}$ contains a point of $S \setminus \{\vec{a}\}$. Here,

neighbourhood means an open n-ball $B(\vec{a}, r)$ of radius r around $\vec{a}$, and

deleted neighbourhood means $B(\vec{a}, r) \setminus \{\vec{a}\}$

$$= \{\vec{x} \in \mathbb{R}^n \mid 0 < \|x - \vec{a}\| < r\}$$

In other words, limit $\operatorname{Lt}_{t\to t_0} v(t)$ $\left(\text{or } \operatorname{Lt}_{h\to 0} v(t_0 + h)\right)$ makes sense only if t_0 is a <u>limit point of S.</u>

Derivative of Vector functions (vector valued function)

Let $S \subseteq \mathbb{R}$.

A vector function $v: S \to \mathbb{R}^3$ is said to be differentiable at a point $t_0 \in S$ if

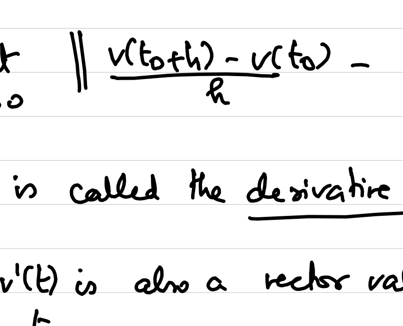
$$\operatorname{Lt}_{h\to 0} \frac{v(t_0 + h) - v(t_0)}{h} \text{ exists. Call } v'(t_0).$$

Equivalently,

$$\operatorname{Lt}_{h\to 0} \left\| \frac{v(t_0+h) - v(t_0)}{h} - v'(t_0) \right\| = 0.$$

Here, $v'(t_0)$ is called the <u>derivative of $v(t)$</u> at $t = t_0$.

Note that $v'(t)$ is also a vector valued function of a real variable t.



$v'(t_0)$ is called the tangent vector of the curve traced by the vector field v at $t = t_0$.

Ex 1) Show the following :- For u, v differentiable on $\mathbb{R}$ & c any scalar,

1) $\dfrac{d}{dt}(u+v) = \dfrac{du}{dt} + \dfrac{dv}{dt}$

2) $\dfrac{d}{dt}(cv) = c\dfrac{dv}{dt}$ c – scalar

Thus, we observe that on the vector space

$$V = \{f: \mathbb{R} \to \mathbb{R}^3 \mid f \text{ is differentiable everywhere on } \mathbb{R}\}$$

the derivative map $\frac{d}{dt}$ is a linear map

Normal vector to a plane

Pr 1) Find a unit vector $\perp r$ to the plane $4x + 2y + 4z = -7$. Further, find the distance of the plane from the origin.

Ans: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ denote the position vector of any pt on the plane. Then, $\vec{r} \cdot \vec{a} = -7$, where $\vec{a} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

Further, if $\vec{r_1}$ & $\vec{r_2}$ denote the p.v. of any 2 pts say P & Q on the plane, then

$$(\vec{r_2} - \vec{r_1}) \cdot \vec{a} = \vec{PQ} \cdot \vec{a} = 0$$

(Used linearity of dot product in the 1st variable)

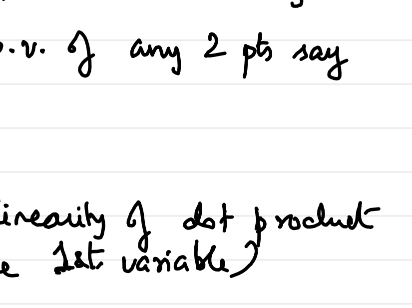
Thus, $\vec{a}$ is $\perp r$ to any vector on the plane.

Thus, $\vec{a}$ is normal to this plane

$$\& \quad \frac{\vec{a}}{\|\vec{a}\|} = \frac{4\hat{i} + 2\hat{j} + 4\hat{k}}{\sqrt{16 + 4 + 16}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}.$$

b)

Let the normal from the origin to the plane hit the plane at Q, without loss of generality. Thus, $\vec{OQ} \parallel \vec{a}$.



We need to find $\|\vec{OQ}\|$, ie, $\left|\,|\vec{OP}|\cos\theta\,\right|$

$$= \left| \vec{OP} \cdot \frac{\vec{OQ}}{|\vec{OQ}|} \right|$$

$$= \left| \vec{OP} \cdot \frac{\vec{a}}{|\vec{a}|} \right|$$

$$= \left| \frac{-7}{6} \right| = 7/6.$$

Solve qn 1) using cross-products.

Curves in $\mathbb{R}^3$ & their parametric representation

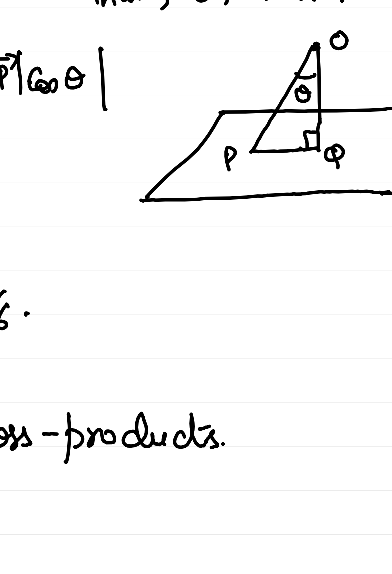
Bodies that move in space form paths that may be represented by curves C. It is often useful to use the following parametric representation for C

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Here, t is the parameter

It is customary to write

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}.$$



Eg 1) The circle $x^2 + y^2 = 4$, $z = 0$ can be represented as

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$$

where $0 \le t \le 2\pi$.

The direction along the curve C which we traverse as t increases (decreases) is called the direction in the positive (negative) sense on C.

Eg 2). The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $z = 0$ is represented as

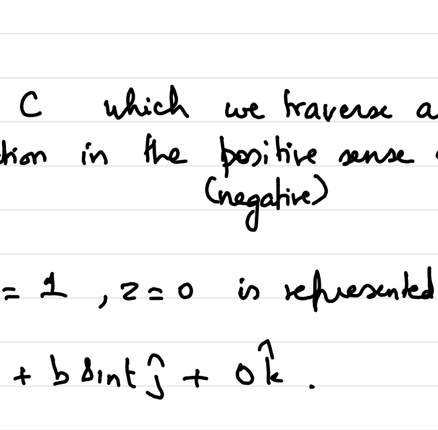
$$\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j} + 0\hat{k}.$$

Eg 3) <u>Straight line</u>

A straight line through a point P with position vector $\vec{a}$ in the direction of a constant vector $\vec{b}$ can be represented parametrically by

$$\vec{r}(t) = \vec{a} + t\vec{b}$$

$$-\infty < t < \infty$$



<u>Plane curve</u>

A curve that lies in a plane in space.

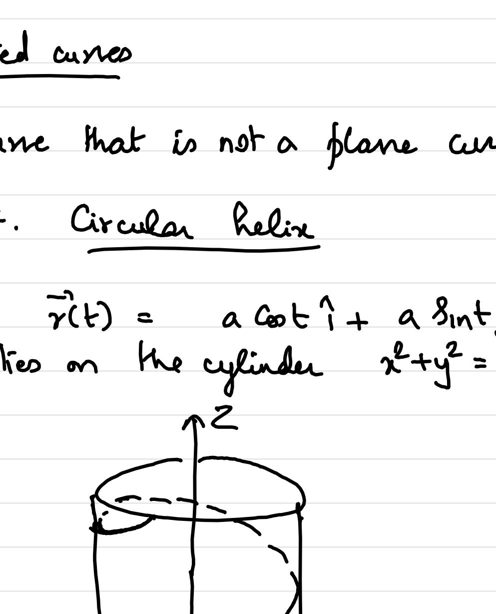
<u>Twisted curves</u>

A curve that is not a plane curve is called a twisted curve.

Eg. 4. <u>Circular helix</u>

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\,\hat{k}$$

It lies on the cylinder $x^2 + y^2 = a^2$.

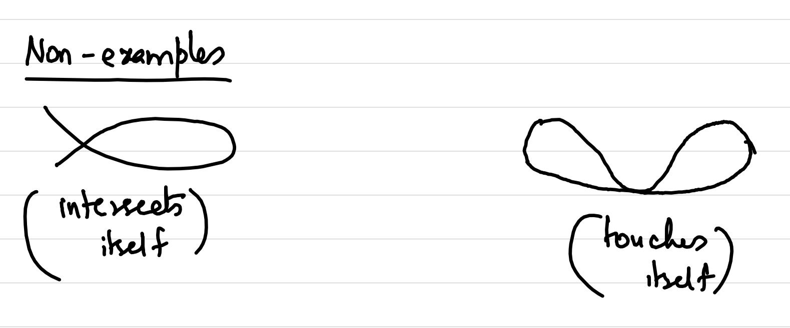


$c > 0$

Right handed circular helix.

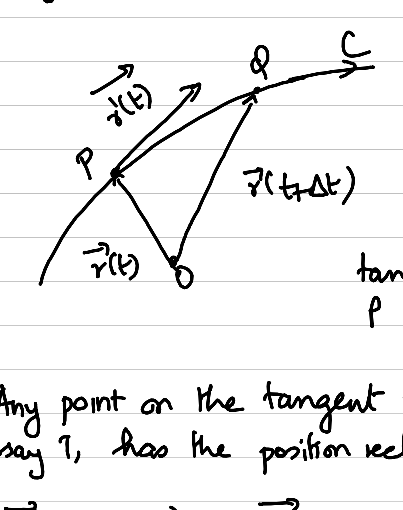
A <u>simple curve</u> is a curve without points at which the curve intersects or touches itself.

<u>Non-examples</u>



(intersects itself)   (touches itself)

<u>Tangent to a differentiable curve</u>



Recall

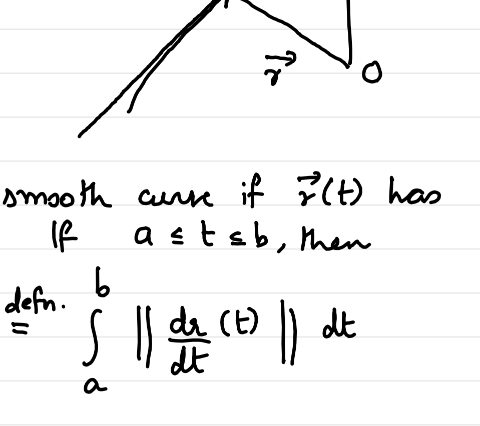
$$\left.\frac{d\vec{r}}{dt}\right|_P = \operatorname{Lt}_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

If $\vec{r}'(t) \ne 0$, we call $\vec{r}'(t)$ as the tangent vector of the curve C at P

Any point on the tangent vector at P, say T, has the position vector

$$\vec{q}(w) = \vec{r} + w\vec{r}'$$

Here, w is the parameter



A curve $\vec{r}(t)$ is called smooth curve if $\vec{r}(t)$ has a continuous derivative $\vec{r}'(t)$. If $a \le t \le b$, then

$$\text{Length of the smooth curve } C \overset{\text{defn.}}{=} \int_a^b \left\| \frac{dr}{dt}(t) \right\| dt$$

Work out Problem set 9.5 : Q 11, 12, Q14, 15, 16, 29

(Kreyszig)
10th edition
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Some problems

2) A particle moves along the curve (t-time)

$$x = 2t^2, \quad y = t^2 - 4t, \quad z = -t - 5$$

Find the components of its velocity & accelaration at $t = 1$ in the direction $\hat{i} - 2\hat{j} + 2\hat{k}$.

Ans: $\vec{r}(t) = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (-t-5)\hat{k}$

$$\vec{v} = \frac{d\vec{r}}{dt}(1) = 4t\,\hat{i} + (2t-4)\hat{j} - \hat{k}\,\Big|_{t=1}$$

$$= 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}(1) = 4\hat{i} + 2\hat{j}\,\Big|_{t=1} = 4\hat{i} + 2\hat{j}$$

<u>Components of $\vec{v}$ & $\vec{a}$ along $\hat{i} - 2\hat{j} + 2\hat{k} = \vec{b}$</u>

$$\hat{b} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} \quad (\text{Unit vector along } \vec{b})$$

$$\vec{v} \cdot \hat{b}\,\Big|_{t=1} = (4\hat{i} - 2\hat{j} - \hat{k}) \cdot \left(\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}\right)$$

$$= 2.$$

$$\vec{a} \cdot \hat{b}\,\Big|_{t=1} = (4\hat{i} + 2\hat{j}) \cdot \left(\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}\right)$$

$$= 0.$$

3) Find the tangent to the ellipse $\dfrac{x^2}{4} + y^2 = 1$

at $P = (\sqrt{2}, 1/\sqrt{2})$.

Ans : $\vec{r}(t) = 2\cos t\,\hat{i} + \sin t\,\hat{j}$

$\vec{r}'(t) = -2\sin t\,\hat{i} + \cos t\,\hat{j}$

Let $\vec{r}(t_0) = P \Rightarrow t_0 = \pi/4$

$$\Rightarrow \vec{r}'(t) = -2\sin\frac{\pi}{4}\,\hat{i} + \cos\frac{\pi}{4}\,\hat{j}$$

$$= -\sqrt{2}\,\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\Rightarrow \text{Tangent vector} : \vec{r} + w\vec{r}' = \sqrt{2}\,\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + w\left(-\sqrt{2}\,\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

Partial Derivatives of scalar fields :-

Let $f: \mathbb{R}^2 \to \mathbb{R}$

$\frac{\partial f}{\partial x}(x_1, y_1)$: The partial deriv of f wrt x at (x_1, y_1) is or

$$f_x(x_1,y_1) = \lim_{h\to 0} \frac{f(x_1+h, y_1) - f(x_1,y_1)}{h}$$

$|||^{ly}$, the partial deriv of f wrt y at (x_1, y_1) is

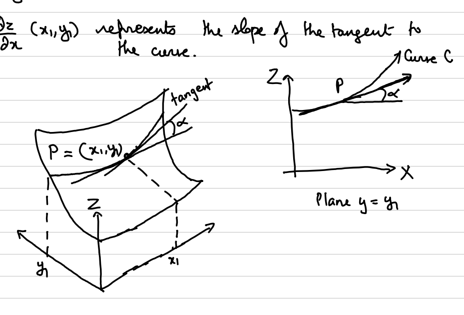
$$f_y(x_1,y_1) = \lim_{k\to 0} \frac{f(x_1, y_1+k) - f(x_1,y_1)}{k}$$

The partial derivatives f_x & f_y are also functions of x & y.

Geometric interpretation

Let $z = f(x,y)$ represent a surface in space. The plane $y = y_1$ (vertical plane) cuts the surface, intersecting it along a curve & the partial derivative

$\frac{\partial z}{\partial x}(x_1, y_1)$ represents the slope of the tangent to the curve.



$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \text{ may or may not be equal to}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}.$$

Two mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ & $\frac{\partial^2 f}{\partial y \partial x}$ are equal at a point (x_0, y_0) if

1) All first order partial derivatives f_x, f_y; second order partial derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ exists (here, $n=2 \Rightarrow$ open disc) on an n-ball containing (x_0, y_0), say B

2) $\frac{\partial^2 f}{\partial x \partial y}$ & $\frac{\partial^2 f}{\partial y \partial x}$ are continuous at (x_0, y_0). OR

2') Either $\frac{\partial^2 f}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial y \partial x}$ is continuous on B.

Ex 1). $f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Soln :- $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$. From defn, both 1st order partial derivatives of f exist at pts other than origin.

$$\therefore f_x(x,y) = y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy \times \left[\frac{(x^2+y^2)(2x) - (x^2-y^2) \times 2x}{(x^2+y^2)^2}\right]$$

if $(x,y) \neq (0,0)$. Hence, $f_x(0,k) = -k$ for $k \neq 0$.

$$\therefore (f_x)_y(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = -1.$$

Similarly,

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

$$f_y(x,y) = x\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy \times \left[\frac{(x^2+y^2)(-2y) - (x^2-y^2) \times 2y}{(x^2+y^2)^2}\right]$$

if $(x,y) \neq (0,0)$. Hence, $f_y(h,0) = h$ for $h \neq 0$.

$$\therefore (f_y)_x(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

GRADIENT OF A SCALAR FIELD :-

Let $f: S \to \mathbb{R}$ be a scalar field, where $S \subseteq \mathbb{R}^n$, $n \geq 1$. The gradient of f, written grad f or ∇f is defined as the vector function

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Useful in

a) finding the rate of change of $f(x,y,z)$ in any direction in space.

b) to obtain surface normal vector

c) in deriving vector fields from scalar fields.

We also introduce the differential operator ∇ as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

If you know Laplace eqn in PDE, then by noting $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, we can rewrite it as

$$\nabla^2 f = 0.$$

DIRECTIONAL DERIVATIVE : Generalisation of partial derivatives

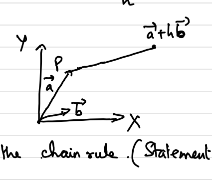
This addresses the question of rate of change of f in an arbitrary direction in space. (f → scalar function)

Let $f: S \to \mathbb{R}$ be a scalar function & $S \subseteq \mathbb{R}^n$, $n \geq 1$. The directional derivative of f at a point $P \in S$ along the unit vector $\hat{b}$ is defined by

$$(D_{\hat{b}} f)(P) := \lim_{h\to 0} \frac{f(\vec{a} + h\hat{b}) - f(\vec{a})}{h}$$

where $\vec{OP} = \vec{a}$.

Illustration $S = \mathbb{R}^2$



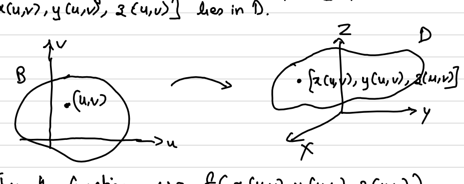
Now, let us recall the chain rule. (Statement only).

Theorem 1 (Chain Rule)

Let $w = f(x,y,z)$ be continuous and have continuous first order partial derivatives in a domain D in xyz space. Let

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

be functions that are continuous & have first order partial derivatives in a domain B in the uv plane, where B is such that for every $(u,v) \in B$, the corresponding point $[x(u,v), y(u,v), z(u,v)]$ lies in D.



Then, the function $w = f(x(u,v), y(u,v), z(u,v))$ is defined in B & has first order partial derivatives w.r.t u & v in B &

$$\left.\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}, \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}. \end{aligned}\right\} (1)$$

Special cases :- If we drop the variable z, ie, if $w = f(x,y)$ then, the third term in the above eqn (1) will be dropped.

If we drop the variable v, we get

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. \quad (1a)$$

If we drop both z & v, we get

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad (1b)$$

Finally, if we drop y, z & v, we get the familiar one variable chain rule. ie, if $z = f(x)$ & $x = g(t)$, then

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt}. \quad (1c)$$

3) Chain rule illustration :-

If $w = x^2 - y^2$ & we define polar co-ordinates r, θ by $x = r\cos\theta$, $y = r\sin\theta$, then, what is $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial \theta}$ in terms of r & θ ?

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= 2x \times \cos\theta + (-2y) \times \sin\theta = 2r\cos^2\theta - 2r\sin^2\theta \\ &= 2r\cos 2\theta. \end{aligned}$$

$|||^{ly}$,

$$\begin{aligned} \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} \\ &= 2x(-r\sin\theta) + (-2y)(r\cos\theta) \\ &= -2r(r\cos\theta \cdot \sin\theta + r\sin\theta\cos\theta) \\ &= -4r^2\sin\theta\cos\theta = -2r^2\sin 2\theta = -4xy \end{aligned}$$

Chain rule as matrix multiplication :-

Coming back to eqn (1), we see that

$$\frac{dw}{dt} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} \text{ in case of 1b}$$

OR

$$= \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \text{ in case of 1a}$$

More generally, ① can be rewritten as

$$\underbrace{\begin{bmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{bmatrix}_{1\times 2}}_{\text{Gradient of } w \text{ w.r.t } (u,v)} = \underbrace{\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}_{1\times 3}}_{\text{Gradient of } w \text{ wrt } (x,y,z)} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}_{3\times 2}$$

Interpretation of $D_{\vec{b}}(f)$ in terms of gradient

Let $S \subseteq \mathbb{R}^3$.

Lemma 2: If $f: S \to \mathbb{R}$ has continuous first order partial derivatives in S, then,

$$(D_{\hat{b}} f)(\vec{a}) = \nabla f(\vec{a}) \cdot \hat{b}$$

Proof :- Consider

$$\begin{aligned} g(t) &= f(\vec{a} + t\vec{b}) \\ \Rightarrow g(0) &= f(\vec{a}) \end{aligned}$$

$$\therefore \lim_{h\to 0} \frac{g(h) - g(0)}{h} = (D_{\hat{b}} f)(\vec{a}).$$

$$\Rightarrow g'(0) = \frac{dg}{dt}(0) = (D_{\hat{b}} f)(\vec{a}).$$

Let $\vec{r}(t) = \vec{a} + t\vec{b} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

where, $x(t) = a_1 + tb_1$, $y(t) = a_2 + tb_2$, $z(t) = a_3 + tb_3$.

$$\therefore g(t) = f(\vec{a} + t\vec{b}) \Rightarrow \frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

ie, $g'(0) = \frac{\partial f}{\partial x}(\vec{a}) \cdot b_1 + \frac{\partial f}{\partial y}(\vec{a})\, b_2 + \frac{\partial f}{\partial z}(\vec{a})\, b_3$

ie, $(D_{\hat{b}} f)(\vec{a}) = \nabla f(\vec{a}) \cdot \hat{b}$

This shows that the directional derivative is simply the component of the gradient vector in the direction of $\vec{b}$.

Q 1) Consider the scalar function

$$f(x,y,z) = x^2 + y^2 + xz.$$

a) Find grad f

b) Find grad f at the point $P = (2,-1,3)$.

c) Find the directional derivative of f in the direction of $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ at the point $P = (2,-1,3)$.

Soln :- a) $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

where $\frac{\partial f}{\partial x}(x,y,z) = 2x + z$

$\frac{\partial f}{\partial y}(x,y,z) = 2y$

$\frac{\partial f}{\partial z}(x,y,z) = x$

$\Rightarrow \nabla f(x,y,z) = [2x+z, 2y, x]$.

b) $\nabla f(P) = [2x+z, 2y, x]\big|_{(2,-1,3)}$

$= [7, -2, 2]$.

c) $(D_{\vec{b}} f)(P) = \nabla f(P) \cdot \left(\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}\right)$

$= (7\hat{i} - 2\hat{j} + 2\hat{k}) \cdot \left(\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}\right)$

$= \frac{7-4+2}{\sqrt{6}} = \frac{5}{\sqrt{6}}$.

The +ve sign indicates that f is increasing along $\vec{b}$.

2) Suppose $\phi(x,y,z) = xy^2z$ &

$\vec{A}(x,y,z) = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$.

Find $\frac{\partial^3}{\partial x^2 \partial z}(\phi A)$ at the point $(2,-1,1)$.

Ans : $4\hat{i} - 2\hat{j}$ (Exercise)

Next theorem tells you that gradient points in the direction of maximum increase of f. In other words, f undergoes its maximum rate of change in the direction of the gradient vector.

Let $S \subseteq \mathbb{R}^3$.

Theorem 3: Let $f(x,y,z)$ be a scalar function on S, having continuous first order partial derivatives in S. Then, ∇f exists in S & if $\nabla f(P) \neq \vec{0}$ at some point P, then $\nabla f(P)$ has the direction of maximum increase of f at P.

Proof :- Recall that

$$\begin{aligned} D_{\hat{b}}(f)(P) &= \nabla f(P) \cdot \hat{b}. \\ &= \|\nabla f(P)\| \cos\theta \end{aligned}$$

Thus, the directional derivative of f at P is maximum along the direction of the vector $\nabla f(P)$, provided $\nabla f(P) \neq \vec{0}$.

Pr 4) In which direction the directional derivative of $\phi(x,y,z) = x^2yz^3$ is maximum at $(2,1,-1)$ & find the magnitude of the maximum.

Soln :-

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla\phi(x,y,z) = (2xyz^3, x^2z^3, 3x^2yz^2)$$

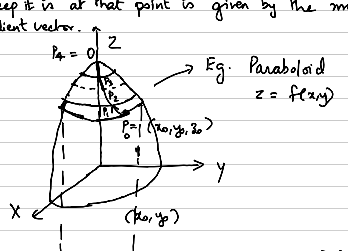
$$\Rightarrow \nabla\phi(2,1,-1) = (-4,-4,12)$$

The directional derivative of f at $P = (2,1,-1)$ is maximum when the direction is that of gradient at P

$$\therefore \hat{b} \text{ can be chosen at } \frac{-4\hat{i} - 4\hat{j} + 12\hat{k}}{\sqrt{16+16+144}} = \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}.$$

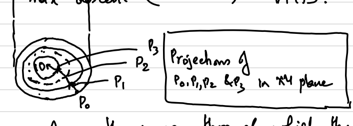
Magnitude of $D_{\overrightarrow{\nabla\phi(P)}} f(P) = \|\overrightarrow{\nabla\phi}(P)\| = 4\sqrt{11}$.

Illustration :- Consider a hill (surface) whose height above the sea level is a function of x & y, say $H(x,y)$. The gradient of H at a point P is a plane vector in the XY plane which points in the direction of steepest slope at P. How steep it is at that point is given by the magnitude of the gradient vector.

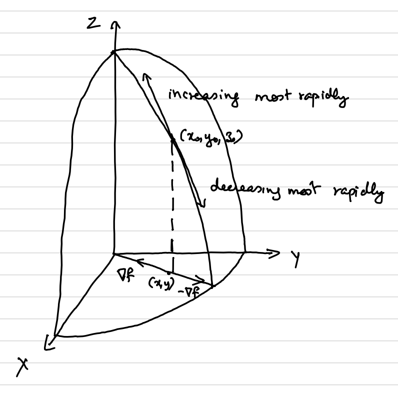


The direction of max ascent will be along $\vec{\nabla f}(P)$ &
(— " —) max descent (— " —) $-\vec{\nabla f}(P)$.

Top view



On the above surface, the curve through which the steepest ascent happens say $P_0 - P_1 - P_2 - \ldots - P_n = O$ (approximately) will have each segment's (say $P_i - P_{i+1}$) projection in the XY plane to be the gradient of f (at P_i), ie, normal to level curve passing through P_i.

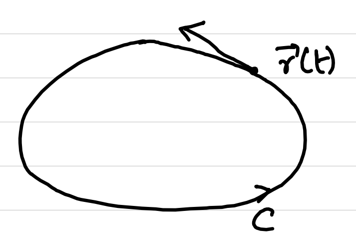


Directional derivatives, gradients & level sets

(Ref: Calculus Vol II - Tom Apostol Sec 8.15 - 8.16)

Defn :-
Directional derivative of $f: \mathbb{R}^n \to \mathbb{R}$ along a curve C at an arbitrary point $\vec{r}(t)$ on the curve is the directional derivative along its tangent $r'(t)$

$$D_{\vec{b}}(f)(\vec{r(t)}) \quad \text{where} \quad \vec{b} = \frac{d\vec{r}}{dt}(t).$$

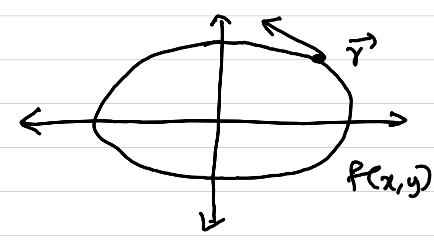


Problem 1)

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a non-constant scalar field having continuous first order partial derivatives everywhere in the plane. Let c be a constant. Then, the equation $f(x,y) = c$ describes a curve C in the plane. Assume that C has a tangent at each of its points. Prove that f has the following properties at each point of C :-

a) ∇f is normal to C.
b) The directional derivative of f is zero along C.
c) The directional derivative of f has its largest value in a direction normal to C.

Soln :- Consider the curve $f(x,y) = c$



Let $\vec{r}(t)$ be an arbitrary point on C
say, $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

(i) $f(x(t), y(t)) = c \quad \forall t$
Taking derivative w.r.t t (Chain rule)

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= \nabla f(\vec{r}) \cdot \vec{r'(t)} = 0$$
$$\Rightarrow \left(D_{\vec{r'(t)}} f\right)(\vec{r(t)}) = 0 \quad \text{— (1)}$$

$\therefore$ Gradient of f at any point $\vec{r}$ in the curve has to be perpendicular to the tangent vector of C at $\vec{r(t)}$.
Thus, (a) is proved.

(ii) Recall $D_{\vec{b}}(f)(\vec{r}) = \vec{\nabla f}(\vec{r}) \cdot \hat{b}$

$$(1) \Rightarrow \left(D_{\vec{r'(t)}} f\right)(\vec{r}) = \vec{\nabla f}(\vec{r}) \cdot \frac{\vec{r'(t)}}{\|\vec{r'(t)}\|} = 0$$

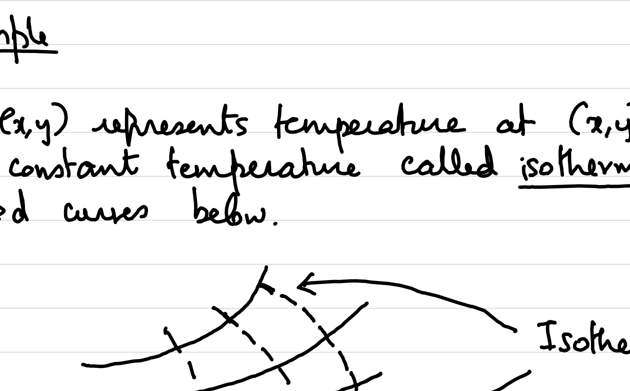
The directional derivative of f along C at $\vec{r}(t)$ is $D_{\vec{r'(t)}} f(\vec{r})$.
Hence, (b) is proved.

(iii) For which $\vec{b}$ is $(D_{\vec{b}} f)(\vec{r}(t))$ the maximum?

$$(D_{\vec{b}} f)(\vec{r}) = |\nabla f(\vec{r})| \quad \text{if} \quad \vec{b} \ \& \ \vec{\nabla f}(\vec{r})$$
are parallel, ie, $\vec{b}$ is along the gradient of f at $\vec{r}$.
Thus, the directional derivative of f has its largest value in a direction normal to C.

Example

If $f(x,y)$ represents temperature at (x,y), the curves of constant temperature called isothermals are the dotted curves below.



The flow of heat takes place in the direction of most rapid change in temperature. This direction is normal to the isothermals (level curves of $f(x,y)$, temperature function).

Level sets & tangent planes

Let f be a scalar field defined on a set S in $\mathbb{R}^n$ & let c be a constant. Define

$$L(c) = \{ \vec{x} \in S \mid f(x) = c \}.$$

This set $L(c)$ is called a level set of f.

$n = 2$: $L(c)$ is called a level curve
$n = 3$: level surface.

Now, let $f: S \to \mathbb{R}$, where S, open subset of $\mathbb{R}^3$ & let f have continuous first order partial derivatives in S. Consider its level surface

$$L(c) : \quad f(x,y,z) = c$$

Let $\vec{a} \in L(c)$ & consider a curve C which lies on $L(c)$ passing through $\vec{a}$. (Assume C can be parametrised by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ which has tangents at each of its points)

We shall prove that $\nabla f(\vec{a})$ is normal to C at $\vec{a}$

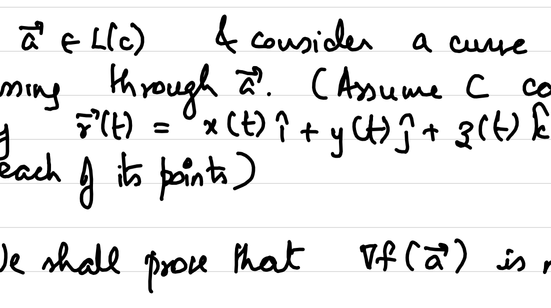
Let $\vec{r}(t) \in C \Rightarrow f(\vec{r}(t)) = f(x(t), y(t), z(t)) = c$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$
$$\Rightarrow \frac{df}{dt}(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r'(t)} = 0$$

In particular, if $\vec{a} = \vec{r}(t_1)$, then

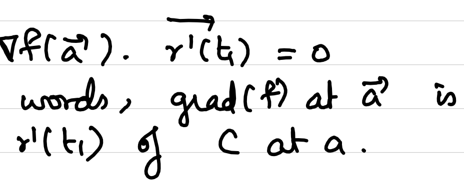
$$\nabla f(\vec{a}) \cdot \vec{r'(t_1)} = 0$$

In other words, grad(f) at $\vec{a}$ is $\perp$r to the tangent vector $r'(t_1)$ of C at a.



The choice of C passing through $\vec{a}$ on the surface $L(c)$ was arbitrary. So, if we take the family of curves on $L(c)$ that pass through $\vec{a}$, then, the tangent vectors $\vec{r}'(t_1)$ of all these curves are perpendicular to the gradient vector $\nabla f(\vec{a})$.

If $\vec{\nabla f}(\vec{a})$ is not the zero vector, then these tangent vectors form a plane known as tangent plane to the surface $L(c)$ at $\vec{a}$.



Thus, we obtain the following theorem :-

Theorem

Let f be a scalar field $f: S \to \mathbb{R}$ where S is an open subset of $\mathbb{R}^3$ such that f has continuous first order partial derivatives on S. If the gradient of f at a point P lying on $L(c)$ $(\vec{OP} = \vec{a})$ is not the zero vector, then it is a normal vector of $L(c)$ at P.

Equation of the tangent plane at $\vec{a}$

Any point (X, Y, Z) on the tangent plane at $\vec{a}$ satisfies

$$\nabla f(\vec{a}) \cdot (X - a_1, Y - a_2, Z - a_3) = 0$$

or
$$\boxed{\frac{\partial f}{\partial x}(\vec{a})(X - a_1) + \frac{\partial f}{\partial y}(\vec{a})(Y - a_2) + \frac{\partial f}{\partial z}(\vec{a})(Z - a_3) = 0}$$

Lec 10: Problem solving session on gradients.

Exercise problems

1) Find the equation of the tangent plane to the surface

$$x^2 + 2xy^2 - 3z^3 = 6$$

at the point $P = (1,2,1)$. This is a level surface of the form $F(x,y,z) = c$.

2) Suppose f & g are differentiable vector valued ($\mathbb{R}^3$) functions of a scalar t. Prove that

$$\frac{d}{dt}(f \cdot g) = f \cdot \frac{dg}{dt} + g \cdot \frac{df}{dt}.$$

3) Find the equation of the tangent plane & normal line to the surface

$$4z = x^2 - y^2$$

at the point $(3,1,2)$.

4) Show that

(i) $\nabla(f^n) = n f^{n-1} \nabla f$

(ii) $\nabla(fg) = f\nabla g + g \nabla f$

(iii) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$. (whenever valid)

5) For what points $P = (x,y,z)$ does ∇f with $f = 25x^2 + 9y^2 + 16z^2$ have the direction from P to the origin?

6) Find the angle between the surfaces

$$z = x^2+y^2 \quad \&$$

$$z = \left(x - \frac{1}{\sqrt{6}}\right)^2 + \left(y - \frac{1}{\sqrt{6}}\right)^2$$

at the point $P = \left(\frac{\sqrt{6}}{12}, \frac{\sqrt{6}}{12}, \frac{1}{12}\right)$.

Note that angle between 2 surfaces at any point of intersection is the angle between their normals.

Some questions from pre-requisites for the course :-

7) Find the angles that the vector $\vec{a} = 4\hat{i} - 8\hat{j} + \hat{k}$ makes with the coordinate axes.

8) Find the unit tangent vector to any point on the curve

$$x = t^2 - t$$
$$y = 4t - 3$$
$$z = 2t^2 - 8t.$$

Solns :- 1) Did in class.

2) $$\frac{d}{dt}(f \cdot g) = f \cdot \frac{dg}{dt} + \frac{df}{dt} \cdot g$$

Here, let $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{v} \to (f_1(\vec{v}), f_2(\vec{v}), f_3(\vec{v}))$$

& $$\vec{f(\vec{v})} \cdot \vec{g(\vec{v})} = f_1(\vec{v})g_1(\vec{v}) + f_2(\vec{v})g_2(\vec{v}) + f_3(\vec{v})g_3(\vec{v}).$$

$\vec{v}$ itself depends on t.

$$\frac{d}{dt}(f \cdot g)$$

$$= \frac{d}{dt}(f_1(\vec{v})g_1(\vec{v})) + \frac{d}{dt}(f_2(\vec{v})g_2(\vec{v})) + \frac{d}{dt}(f_3(\vec{v})g_3(\vec{v}))$$

$f_i(\vec{v})g_i(\vec{v})$ is a real valued function of the vector variable $\vec{v}$; Thus, applying product rule,

$$\frac{d}{dt}(f \cdot g)(\vec{v}) = f_1(\vec{v})\frac{d}{dt}g_1(\vec{v}) + g_1(\vec{v})\frac{df_1(\vec{v})}{dt}$$
$$f_2(\vec{v})\frac{d}{dt}g_2(\vec{v}) + g_2(\vec{v})\frac{d}{dt}f_2(\vec{v})$$
$$f_3(\vec{v})\frac{d}{dt}g_3(\vec{v}) + g_3(\vec{v})\frac{d}{dt}f_3(\vec{v})$$

$$= (f_1(\vec{v})\hat{i} + f_2(\vec{v})\hat{j} + f_3(\vec{v})\hat{k}) \cdot \frac{d}{dt}(g_1(\vec{v})\hat{i} + g_2(\vec{v})\hat{j} + g_3(\vec{v})\hat{k})$$
$$+ (g_1(\vec{v})\hat{i} + g_2(\vec{v})\hat{j} + g_3(\vec{v})\hat{k}) \cdot \frac{d}{dt}(f_1(\vec{v})\hat{i} + f_2(\vec{v})\hat{j} + f_3(\vec{v})\hat{k})$$

$$= f(\vec{v}) \cdot \frac{dg}{dt}(\vec{v}) + \frac{df}{dt}(\vec{v}) \cdot g(\vec{v})$$

4) Note $f^n(\vec{v}) = [f(\vec{v})]^n$ (let $f : \mathbb{R}^3 \to \mathbb{R}$)

Let $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$. Then,

$$\nabla(f^n) = \left(\frac{\partial}{\partial x}f^n(\vec{v}), \frac{\partial}{\partial y}f^n(\vec{v}), \frac{\partial}{\partial z}f^n(\vec{v})\right)$$

or Equivalently $= \frac{\partial}{\partial x}f^n(\vec{v})\hat{i} + \frac{\partial}{\partial y}f^n(\vec{v})\hat{j} + \frac{\partial}{\partial z}f^n(\vec{v})\hat{k}$.

$$\frac{\partial}{\partial x}[f^n(x,y,z)] = n f^{n-1}\frac{\partial f}{\partial x} \quad \text{(by Chain rule)}$$

|||ly, $$\frac{\partial}{\partial y}[f^n(x,y,z)] = n f^{n-1}\frac{\partial f}{\partial y}$$

& $$\frac{\partial}{\partial z}[f^n(x,y,z)] = n f^{n-1}\frac{\partial f}{\partial z}$$

$$\Rightarrow \nabla(f^n) = n f^{n-1}\left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right]$$
$$= n f^{n-1}\nabla f.$$

$\Rightarrow \boxed{P = (0,0,0)}$

Similarly, do other parts.

5) Ignore the question

6) Did in class.

7) Angle made by $\vec{a}$ w.r.t X axis θ_1

$$\cos\theta_1 = \frac{\vec{a} \cdot \hat{i}}{\|\vec{a}\|} = \frac{4}{\sqrt{16+64+1}} = \frac{4}{9} \quad \text{(in Q1)}$$

|||ly, $$\cos\theta_2 = \frac{\vec{a}\cdot\hat{j}}{\|\vec{a}\|} = \frac{-8}{9}. \quad (\theta_2 \text{ in Q2})$$

8) Let tangent vector be $\frac{d\vec{r}}{dt}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d}{dt}x(t)\hat{i} + \frac{d}{dt}y(t)\hat{j} + \frac{d}{dt}z(t)\hat{k}$$
$$= (2t-1)\hat{i} + 4\hat{j} + (4t-8)\hat{k}.$$

$$\therefore \text{unit tangent} = \frac{\frac{d\vec{r}}{dt}}{\sqrt{(2t-1)^2 + (4)^2 + (4t-8)^2}}.$$

Internal assessment - 1

Ref : Kreyszig : 10th edition (International student version)

1) Problem Set 9.7 (Q 28)

Direction of steepest ascent is along the gradient of the elevation $z(x,y) = 3000 - x^2 - 9y^2$

$$\nabla z\,(4,1) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\Big|_{(4,1)}$$

$$= (-2x, -18y)\,\big|_{(4,1)}$$

$$= (-8, -18)$$

In terms of unit vector, $\dfrac{-8\hat{i} - 18\hat{j}}{\sqrt{64+324}} = \dfrac{-4\hat{i} - 9\hat{j}}{\sqrt{97}}$

2) Prob. Set 9.8 (Q 7)

$\vec{v}$ is solenoidal if $\operatorname{div}(\vec{v}) = 0$

Note $\vec{v}(x,y,z) = e^y \sin x\,\hat{i} + e^y \cos x\,\hat{j} + v_3(x,y,z)\,\hat{k}$
We need to find v_3.

$$0 = \frac{\partial}{\partial x}(e^y \sin x) + \frac{\partial}{\partial y}(e^y \cos x) + \frac{\partial}{\partial z} v_3$$

$$\Rightarrow 0 = e^y \cos x + e^y \cos x + \frac{\partial v_3}{\partial z}$$

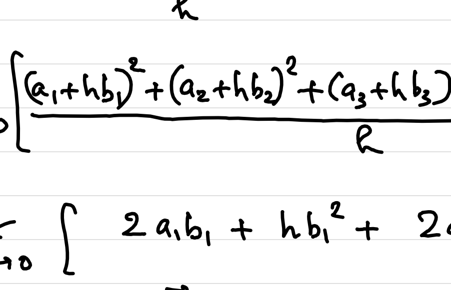
$$\frac{\partial v_3}{\partial z}(x,y,z) = -2e^y \cos x$$

$\Rightarrow v_3(x,y,z) = -2z\,e^y \cos x +$ any function of x & y

3) $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$

$f(\vec{x}) = x_1^2 + x_2^2 + x_3^2 \qquad f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$D_{\vec{b}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{b}) - f(\vec{a})}{h}$$



Let $\vec{b} = (b_1, b_2, b_3)$

$$\lim_{h \to 0} \frac{\|\vec{a} + h\vec{b}\|^2 - \|\vec{a}\|^2}{h}$$

$$= \lim_{h \to 0} \left[\frac{(a_1+hb_1)^2 + (a_2+hb_2)^2 + (a_3+hb_3)^2 - a_1^2 - a_2^2 - a_3^2}{h}\right]$$

$$= \lim_{h \to 0} \left[2a_1b_1 + hb_1^2 + 2a_2b_2 + hb_2^2 + 2a_3b_3 + hb_3^2\right]$$

$$= 2\vec{a}\cdot\vec{b}.$$

4) Let $\vec{v}(x,y,z) = (x^2 \cos y)\hat{i} + (y^2 \sin z)\hat{j} + (z^3 \sin^2 x)\hat{k}$

$$\nabla \cdot (\nabla \times \vec{v}) = \operatorname{div}(\operatorname{curl} \vec{v})$$

Since $\vec{v}$ is twice continuously differentiable, $\operatorname{div}(\operatorname{curl}\vec{v})$ is 0.
No need to do explicit calculation here. This theorem works for any $\vec{v}$ as above. Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\operatorname{curl}\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\Rightarrow \operatorname{div}(\operatorname{curl}\vec{v}) = \frac{\partial^3 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}.$$

Since v_1, v_2 & v_3 are C^2 smooth, $\dfrac{\partial^2 v_1}{\partial y \partial z} = \dfrac{\partial^2 v_1}{\partial z \partial y}$ & so on. Hence, $\operatorname{div}(\operatorname{curl}\vec{v}) = 0$.

5) Equation of the tangent plane at $P = (1, -3, 2)$

Verify : $P \in$ surface.

Let $f(x,y,z) = xz^2 + x^2y - z$ & consider the level surface $L(-1) = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = -1\}$.

$\vec{\nabla f}(P)$ is the surface normal to $L(-1)$ as P

$$\therefore \left((x-1)\hat{i} + (y+3)\hat{j} + (z-2)\hat{k}\right)\cdot\left(\vec{\nabla f}(P)\right) = 0$$

$$\vec{\nabla f}(x,y,z)\big|_P = (z^2 + 2xy)\hat{i} + x^2\hat{j} + (2xz - 1)\hat{k}\,\big|_P$$

$$= -2\hat{i} + \hat{j} + 3\hat{k}$$

$$\therefore -2(x-1) + (y+3) + 3(z-2) = 0$$

or $2x - y - 3z = -1$

Quiz-2

1) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (Let $\vec{r} \neq \vec{0}$).

$$\phi(x,y,z) = \ln\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2}\ln(x^2+y^2+z^2)$$

Let $r = \sqrt{x^2+y^2+z^2} \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \dfrac{x}{r}$

Also, $|||^{ly}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ & $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Here, $\phi(r,\theta) = \ln r \Rightarrow \dfrac{\partial \phi}{\partial r} = \dfrac{1}{r}$ & $\dfrac{\partial \phi}{\partial \theta} = 0$

$\Rightarrow \dfrac{\partial \phi}{\partial x} = \dfrac{1}{r} \times \dfrac{x}{r}$. $|||^{ly}$, $\dfrac{\partial \phi}{\partial y} = \dfrac{y}{r^2}$ ...

$$\therefore \nabla\phi(\vec{r}) = \left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$$

$$= \frac{\vec{r}}{r^2}$$

$$\therefore \vec{\nabla\phi}(2,0,0) = \frac{2\hat{i}}{4} = \frac{\hat{i}}{2}.$$

2) $\vec{v}(x,y,z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\vec{v} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(v_2 z - v_3 y) + \hat{j}(x v_3 - v_1 z) + \hat{k}(y v_1 - x v_2)$$

$$\operatorname{div}(\vec{v} \times \vec{r}) = \frac{\partial}{\partial x}(v_2 z - v_3 y) + \frac{\partial}{\partial y}(x v_3 - v_1 z) + \frac{\partial}{\partial z}(y v_1 - x v_2)$$

$$= z\frac{\partial v_2}{\partial x} - y\frac{\partial v_3}{\partial x} + x\frac{\partial v_3}{\partial y} - z\frac{\partial v_1}{\partial y} + y\frac{\partial v_1}{\partial z} - x\frac{\partial v_2}{\partial z}$$

$$= x\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + y\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + z\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Recall $\operatorname{curl}\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\therefore \operatorname{div}(\vec{v} \times \vec{r}) = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\operatorname{curl}\vec{v}$$

$$= \vec{r}\cdot\operatorname{curl}\vec{v}$$

$$\boxed{\operatorname{div}(\vec{v} \times \vec{r}) = \vec{r}\cdot\operatorname{curl}\vec{v}}$$

Further, if $\vec{v}$ is irrotational, $\operatorname{div}(\vec{v} \times \vec{r}) = 0$.

Further questions :-

Prove a) $\nabla \times (\nabla\phi) = \vec{0} \qquad \operatorname{curl}(\operatorname{grad}\phi) = \vec{0}$

b) $\nabla \cdot (\nabla \times \vec{v}) = 0 \qquad \operatorname{div}(\operatorname{curl}\vec{v}) = 0.$

Here, $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, & $\vec{v}$ is C^2 smooth

Did in class a) & b). b) is also done in Q4 above

1. Let $ln(\cdot)$ denote the natural logarithm to the base e. Also, let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Then, the gradient of $\phi(x, y, z) = ln\, r$, where, $r = \| \vec{r} \|$ at (2,0,0) is

a) $(ln\ 2)\hat{\imath}$ b) $\frac{\hat{\imath}}{2}$ c) $\frac{\hat{\jmath}+\hat{k}}{ln\ 2}$ d) $2\hat{\imath}$

b

2. Let $\vec{V}(x, y, z)$ be a differentiable vector field. Which of the following options is/are always true? The value of div($\vec{V} \times \vec{r}$) is

$a)$ 0 b) $cannot\ be\ determined$ from given information c) 0 if $\vec{V}$ $is\ irrotational$

d) $\vec{r}.\,curl\ \vec{V}$

c,d