Semester III BTech (Mathematics and Computing) End-Semester Examination

Discrete Mathematics – MAT 2138

Time: 3 Hours | 02/12/2024 | Max. Marks: 50

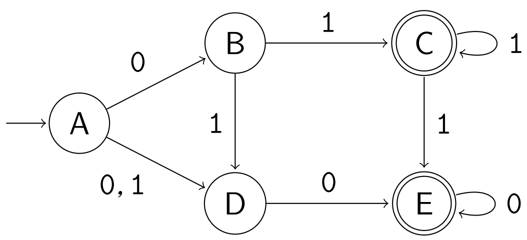
1A. Show that if A and B are finite sets, then the number of functions from A to B is $|B|^{|A|}$. Hence show, by a suitable choice of B, that the number of subsets of A is $2^{|A|}$. (3)

1B. Let $\mathbb{Z}$ be the set of integers and $P = \mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$. Define a relation $\sim$ on S as follows: For any $A, B \in P$, $A \sim B$ if and only if $A \triangle B$ is a set of even integers. Show that $\sim$ is an equivalence relation on P. (3)

1C. Show that the set $\mathbb{R}$ of real numbers is uncountable. (4)

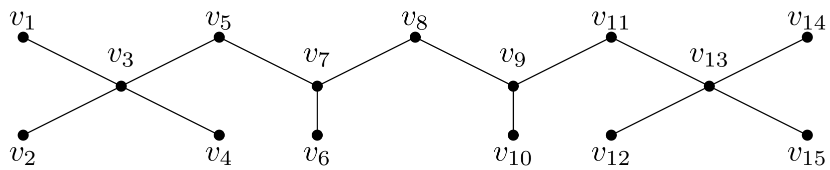
2A. Construct a DFA that accepts the language of all strings over $\{\mathrm{a}, \mathrm{b}\}$ that contain the substring aaa but does not contain aaaa. Also write the regular expression that generates this language. (5)

2B. Convert the NFA given below to a DFA using the subset construction: (5)



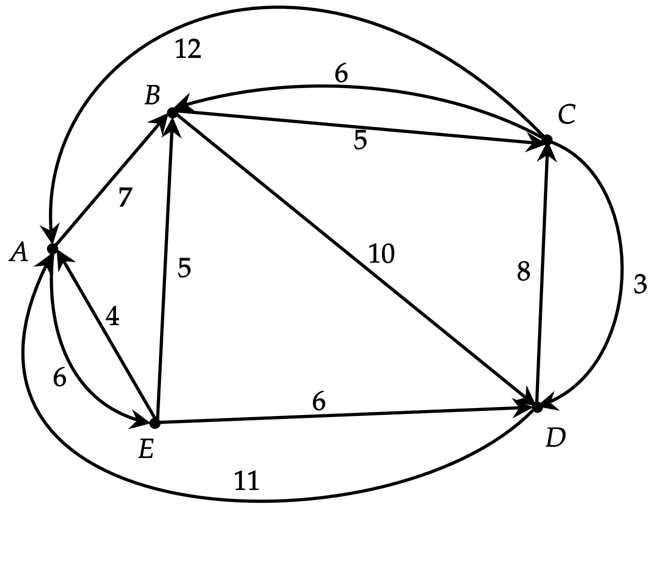
3A. Show that for any graph G, if $\operatorname{diam} G \geq 3$, then $\operatorname{diam} \overline{G} \leq 3$. (3)

3B. Show that the centre of any tree consists of either one vertex or two adjacent vertices. Find the centre of the tree given below. (3)



3C. Prove that a connected graph is bipartite if and only if all its cycles are even. (4)

4A. In the following network, find the shortest distances to the vertices A, B, D, E from the vertex C using Dijkstra's algorithm. (5)



4B. Prove that if A is the adjacency matrix of a graph G with vertices $v_1, \ldots, v_n$, then the (i,j)-entry of A^m is the number of walks of length m from v_i to v_j. Hence show that if $\operatorname{diam} G = k$, then $I + A + A^2 + \cdots + A^{k-1}$ has at least one zero entry. (5)

5A. Using generating functions, find the number of ways of distributing (3)

(i) 20 identical objects into 4 distinct boxes such that each box is non-empty.

(ii) 20 distinct objects into 4 distinct boxes such that each box is non-empty.

5B. Using generating functions, solve the following recurrence relation: (3)

$$a_n = 2a_{n-1} - a_{n-2}$$
$$a_0 = 0, \; a_1 = 1.$$

5C. Find the exponential generating function for the number of derangements from the recurrence relation: (4)

$$d_n = (n-1)(d_{n-1} + d_{n-2})$$
$$d_0 = 1, \; d_1 = 0.$$