Basis of $P_n = \{1, x, x^2, x^3, \cdots, x^n\}$

For $f, g \in P_n$,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

$$u_0 = \frac{1}{\sqrt{2\pi}}$$

$$u_1 = \frac{x}{\sqrt{\frac{2\pi^2}{3}}}$$

$$u_2 = \sqrt{\frac{45}{8\pi^5}}\left(x^2 - \frac{\pi^2}{3}\right)$$

$$u_3 = \sqrt{\frac{175}{8\pi^7}}\left(x^3 - \frac{3\pi^2}{5}x\right)$$

$$v(x) = \sin x \notin \text{span}\{u_0, u_1, u_2, u_3\} = U$$

$$\text{proj}_U\, v = \underbrace{\langle \underset{\sin x}{v}, \underset{1}{u_0}\rangle}_{\text{odd fn.} = 0} u_0 + \underbrace{\langle v, \overset{x}{u_1}\rangle}_{\text{projection}} u_1 + \underbrace{\langle v, \underset{x^2}{u_2}\rangle}_{\text{odd fn.} = 0} u_2 + \underbrace{\langle v, u_3\rangle}_{\text{projection}} u_3$$

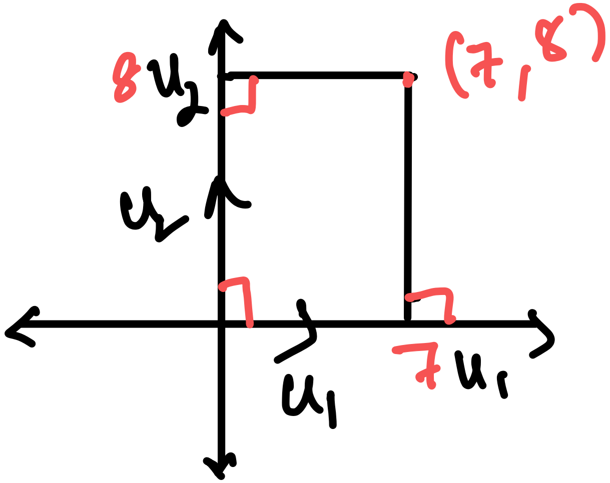
Note:

standard basis of $\mathbb{R}^2$

$$B = \{ \underset{u_1}{(1,0)}, \underset{u_2}{(0,1)} \}$$

$$(7,8) = 7(1,0) + 8(0,1)$$

coefficients are projections on the respective axis (orthogonal basis vectors)



$$(7,8) = 7u_1 + 8u_2$$

Earlier Exercise:

Q: Orthonormalize $\{1, x, x^2, x^3\}$ using the Gram-Schmidt process by taking the inner product as

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

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We get:

$$v_0(x) = 1$$

$$v_1(x) = x$$

$$v_2(x) = x^2 - 1/3$$

$$v_3(x) = x^3 - \frac{3}{5}x$$

$$\vdots$$

Another way of normalization:

$$p_n(x) = \frac{v_n(x)}{v_n(1)}$$

such that we get $p_n(1) = 1$.

$$P_0(x) = \frac{v_0(x)}{v_0(1)} = \frac{1}{1} = 1.$$

$$P_1(x) = \frac{v_1(x)}{v_1(1)} = \frac{x}{1} = x.$$

$$P_2(x) = \frac{v_2(x)}{v_2(1)} = \frac{x^2 - 1/3}{1 - 1/3} = \frac{3}{2}(x^2 - 1).$$

$$P_3(x) = \frac{v_3(x)}{v_3(1)} = \frac{x^3 - \frac{3}{5}x}{1 - \frac{3}{5}(1)} = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$\vdots$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \longrightarrow \text{solution of}$$

Legendre's differential eqn.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$\uparrow$

called

Legendre polynomials $\rightarrow \{P_0(x), P_1(x), P_2(x), \ldots\}$

Take $\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\, g(x)\, w(x)\, dx$

pdf

Suppose $w(x) = e^{-x^2/2}$;

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-x^2/2}\, dx \rightarrow$$

gives rise to Hermite polynomials

Suppose $w(x) = \dfrac{1}{\sqrt{1-x^2}}$;

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x) \frac{1}{\sqrt{1-x^2}}\, dx \rightarrow$$ yields Chebyshev's polynomials.

Substituting $n = 0, 1, 2$ in $P_n(x)$;

$$P_0(x) = \frac{1}{2^0\, 0!} \frac{d^0}{dx^0} (x^2-1)^0$$

$$= 1.$$

$$P_1(x) = \frac{1}{2^1\, 1!} \frac{d^1}{dx^1} (x^2-1)^1 = \frac{1}{2} \frac{d}{dx} (x^2-1)$$

$$= \frac{1}{2}\, 2x = x.$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{4 \cdot 2} \frac{d}{dx} \left(2(x^2-1)\, 2x \right) = \frac{1}{2}\left(3x^2 - 1\right).$$

$\vdots$

Gram-Schmidt process:

$$A = [\underbrace{w_1, w_2, \ldots, w_n}_{\text{independent columns of } A}]$$

QR Decomposition:

$$v_1 = w_1$$

$$u_1 = \frac{v_1}{\|v_1\|}.$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

To express columns of A interms of $u_1, u_2, \ldots$

where $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{w_1}{\|v_1\|}$

$$w_1 = \underbrace{\|v_1\|}_{r_{11}} u_1 \quad \longleftarrow (1)$$

$$u_2 = \frac{v_2}{\|v_2\|} \Rightarrow v_2 = \|v_2\| \, u_2 .$$

$$\vdots$$

$$w_2 = \frac{\langle w_2, v_1 \rangle v_1}{\|v_1\|^2} + v_2$$

$$= \left\langle w_2 , \frac{v_1}{\|v_1\|} \right\rangle \frac{v_1}{\|v_1\|} + \|v_2\| \, u_2 .$$

$$= \underbrace{\langle w_2, u_1 \rangle}_{r_{12}} u_1 + \underbrace{\|v_2\|}_{r_{22}} u_2 \quad \text{—— (2)}$$

$$w_3 = \underbrace{\langle w_3, u_1 \rangle}_{r_{13}} u_1 + \underbrace{\langle w_3, u_2 \rangle}_{r_{23}} u_2 + \overbrace{\|v_3\|}^{r_{33}} u_3 . \quad \text{—— (3)}$$

$$\vdots$$

Vectorize equations (1), (2), (3), ...

$$\underset{m\times n}{A} = [w_1, w_2, \dots]_{m\times n}$$

$$= \underbrace{[u_1, u_2, u_3, \dots]_{m\times n}}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots \\ 0 & r_{22} & r_{23} & \\ 0 & 0 & r_{33} & \\ \vdots & \vdots & 0 & \\ & & \vdots & \end{bmatrix}_{n\times n}}_{R} = QR$$

$A = QR$; where $Q^T Q = I_{n\times n}$.

$\hookrightarrow$ QR decomposition.

Take $A = QR$

$\Rightarrow Q^T A = Q^T Q R$

$= IR = R$

$\therefore \boxed{R = Q^T A.}$ $\leftarrow$ Formula to compute R.

Q. Find solution to

$x = 0$

$y = 0$

$x + y = 1$, by QR decomposition.

Soln:

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

Step 1: Find QR decomposition of A : $\begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \\ w_1 \quad w_2 \end{array}$

$$v_1 = w_1 = (1, 0, 1)$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (0, 1, 1) - \frac{\langle (0,1,1), (1,0,1) \rangle}{1^2 + 0^2 + 1^2} \cdot (1, 0, 1)$$

$$v_2 = (0,1,1) - \frac{0.1 + 1.0 + 1.1}{2} \cdot (1,0,1)$$

$$= (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right).$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,1)}{\sqrt{1^2+0^2+1^2}}$$

$$= \frac{1}{\sqrt{2}}(1,0,1).$$

$$u_2 = \frac{v_2}{\|v_2\|}$$

$$= \frac{(-1/2,\ 1,\ 1/2)}{\sqrt{(-1/2)^2 + 1^2 + (1/2)^2}} = \sqrt{2/3}\,(-1/2,\ 1,\ 1/2).$$

$$Q_2 = [u_1 \quad u_2]$$

↳ as column

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 0 & \sqrt{2/3} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6} \end{bmatrix}.$$

$$R = Q^T A$$

$$= \underset{2\times 3}{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & 1/\sqrt{6} \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{2/3} + 1/\sqrt{6} \end{bmatrix}$$

$$= \underset{2\times 2}{\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{bmatrix}}.$$

$$QR = \underset{3\times 2}{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2/3} \\ 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{bmatrix}}.$$

$$= A_{3\times 2}$$ (without using matrix inverses)

Consider $AX = b$

$\Rightarrow QRX = b$

$\Rightarrow Q^T QRX = Q^T b$

$$\Rightarrow IRX = Q^T b \qquad (\text{as } Q^T Q = I)$$

$$\Rightarrow RX = Q^T b.$$