Theorem The linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$, where $d = \gcd(a, b)$. If x_0, y_0 is any particular solution of this equation, then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)t \qquad y = y_0 - \left(\frac{a}{d}\right)t$$

where t is an arbitrary integer.

Proof. To establish the second assertion of the theorem, let us suppose that a solution x_0, y_0 of the given equation is known. If x', y' is any other solution, then

$$ax_0 + by_0 = c = ax' + by'$$

which is equivalent to

$$a(x' - x_0) = b(y_0 - y')$$

By the corollary to Theorem 2.4, there exist relatively prime integers r and s such that $a = dr, b = ds$. Substituting these values into the last-written equation and canceling the common factor d, we find that

$$r(x' - x_0) = s(y_0 - y')$$

The situation is now this: $r \mid s(y_0 - y')$, with $\gcd(r, s) = 1$. Using Euclid's lemma, it must be the case that $r \mid (y_0 - y')$; or, in other words, $y_0 - y' = rt$ for some integer t. Substituting, we obtain

$$x' - x_0 = st$$

This leads us to the formulas

$$x' = x_0 + st = x_0 + \left(\frac{b}{d}\right)t$$

$$y' = y_0 - rt = y_0 - \left(\frac{a}{d}\right)t$$

It is easy to see that these values satisfy the Diophantine equation, regardless of the choice of the integer t; for

$$\begin{aligned} ax' + by' &= a\left[x_0 + \left(\frac{b}{d}\right)t\right] + b\left[y_0 - \left(\frac{a}{d}\right)t\right] \\ &= (ax_0 + by_0) + \left(\frac{ab}{d} - \frac{ab}{d}\right)t \\ &= c + 0 \cdot t \\ &= c \end{aligned}$$

Thus, there are an infinite number of solutions of the given equation, one for each value of t.

Example Consider the linear Diophantine equation

$$172x + 20y = 1000$$

Applying the Euclidean's Algorithm to the evaluation of gcd(172, 20), we find that

$$\begin{aligned} 172 &= 8 \cdot 20 + 12 \\ 20 &= 1 \cdot 12 + 8 \\ 12 &= 1 \cdot 8 + 4 \\ 8 &= 2 \cdot 4 \end{aligned}$$

whence $\gcd(172, 20) = 4$. Because $4 \mid 1000$, a solution to this equation exists. To obtain the integer 4 as a linear combination of 172 and 20, we work backward through the previous calculations, as follows:

$$\begin{aligned} 4 &= 12 - 8 \\ &= 12 - (20 - 12) \\ &= 2 \cdot 12 - 20 \\ &= 2(172 - 8 \cdot 20) - 20 \\ &= 2 \cdot 172 + (-17)20 \end{aligned}$$

Upon multiplying this relation by 250, we arrive at

$$\begin{aligned} 1000 = 250 \cdot 4 &= 250[2 \cdot 172 + (-17)20] \\ &= 500 \cdot 172 + (-4250)20 \end{aligned}$$

so that $x = 500$ and $y = -4250$ provide one solution to the Diophantine equation in question. All other solutions are expressed by

$$\begin{aligned} x &= 500 + (20/4)t = 500 + 5t \\ y &= -4250 - (172/4)t = -4250 - 43t \end{aligned}$$

for some integer t.

A little further effort produces the solutions in the positive integers, if any happen to exist. For this, t must be chosen to satisfy simultaneously the inequalities

$$5t + 500 > 0 \qquad -43t - 4250 > 0$$

or, what amounts to the same thing,

$$-98\frac{36}{43} > t > -100$$

Because t must be an integer, we are forced to conclude that $t = -99$. Thus, our Diophantine equation has a unique positive solution $x = 5$, $y = 7$ corresponding to the value $t = -99$.

Example A customer bought a dozen pieces of fruit, apples and oranges, for \$1.32. If an apple costs 3 cents more than an orange and more apples than oranges were purchased, how many pieces of each kind were bought?

To set up this problem as a Diophantine equation, let x be the number of apples and y be the number of oranges purchased; in addition, let z represent the cost (in cents) of an orange. Then the conditions of the problem lead to

$$\underline{(z + 3)x + zy = 132}$$

x – no. of apples
y – no. of oranges
z – cost of orange (in cents)

or equivalently

$$3x + (\overbrace{x + y}^{12})z = 132$$

Because $x + y = 12$, the previous equation may be replaced by

$$3x + 12z = 132$$

which, in turn, simplifies to $x + 4z = 44$.

Stripped of inessentials, the object is to find integers x and z satisfying the Diophantine equation

$$x + 4z = 44 \tag{1}$$

Inasmuch as $\gcd(1, 4) = 1$ is a divisor of 44, there is a solution to this equation. Upon multiplying the relation $1 = 1(-3) + 4 \cdot 1$ by 44 to get

$$44 = 1(-132) + 4 \cdot 44$$

it follows that $x_0 = -132$, $z_0 = 44$ serves as one solution. All other solutions of Eq. (1) are of the form

$$x = -132 + 4t \qquad z = 44 - t$$

where t is an integer.

Not all of the choices for t furnish solutions to the original problem. Only values of t that ensure $\underline{12 \geq x > 6}$ should be considered. This requires obtaining those values of t such that

$$\underline{12 \geq -132 + 4t > 6}$$

Now, $12 \geq -132 + 4t$ implies that $t \leq 36$, whereas $-132 + 4t > 6$ gives $t > 34\frac{1}{2}$. The only integral values of t to satisfy both inequalities are $t = 35$ and $t = 36$. Thus, there are two possible purchases: a dozen apples costing 11 cents apiece (the case where $t = 36$), or 8 apples at 12 cents each and 4 oranges at 9 cents each (the case where $t = 35$).

Ex:- Determine all solutions in positive integers

(a) $18x + 5y = 48$

$$\gcd(18, 5) = 1 = 18 \times 2 + 5 \times (-7)$$

$$48 = 18 \times 96 + 5 \times (-336)$$

$$\therefore x_0 = 96, \quad y_0 = -336.$$

$$\therefore x = x_0 + 5t, \quad y = y_0 - 18t$$

$$x = 96 + 5t, \quad y = -336 - 18t$$

$x > 0, y > 0 \Rightarrow$

$$96 + 5t > 0 \Rightarrow t > -\frac{96}{5} = -19.2$$

$$-336 - 18t > 0 \Rightarrow t < \frac{-336}{18} = -18.6$$

$$-19.2 < t < -18.6$$

$$\Rightarrow t = \underline{\underline{-19}}$$

$$\therefore \underline{x = 1, \quad y = 6}$$

$\underline{\underline{Ex}}$:- $54x + 21y = 906$

$\gcd(54, 21) = 3$ & $3 \mid 906$

$54 \times 2 + 21 \times (-5) = 3$

$\Rightarrow 54 \times (2 \times 302) + 21 \times (-5 \times 302) = 3 \times 302$

$\Rightarrow 54 \times 604 + 21 \times (-1510) = 906$

$x = 604 + 7t$

$y = -1510 - 18t$

$604 + 7t > 0 \Rightarrow t > -\dfrac{604}{7}$

$= \underline{\underline{-86.28}}$

$-1510 - 18t > 0 \Rightarrow t < -83.88$

$\Rightarrow \underline{\underline{t = -84, -85, -86}}$

$\Rightarrow x = 16, y = 2$

$x = 9, \quad y = 20$

$x = 2, \quad y = 38$

HW (i) $123x + 360y = 99$

(ii) $158x - 57y = 7$.