(Kreyszig Pg 652) Cauchy's Integral theorem

(1)

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed contour C, then

$$\oint_C f(z)\,dz = 0$$

Eg. $\oint_C e^z\,dz = 0$; $\oint_C \cos z\,dz = 0$; $\oint_C z^n\,dz = 0$ $(n=0,1,\ldots)$

For any ~~closed~~ contour C in $\mathbb{C}$

Pr 1) Let C be the unit circle. Find a) $\oint_C \sec z\,dz$

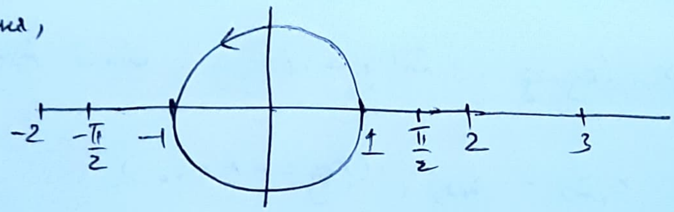
Skip this as trigonometric fns. have not been introduced extensively

$\sec z = \dfrac{1}{\cos z}$ is not analytic when $\cos z = 0$

ie, when $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$

ie, $\{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}$

However,



all such points can be brought outside the simply connected domain

$D = \{ z \in \mathbb{C} \ |z| < 1 + \frac{\pi}{4} \}$ where $\sec z$ is analytic

$\Rightarrow \oint_C \sec z\,dz = 0$

Do this

b) $\oint_C \dfrac{1}{z^2+4}\,dz$

Pr c) $\oint_C \bar{z}\,dz = \int_{-\pi}^{\pi} e^{-it}\, ie^{it}\,dt = i \times 2\pi$

$z(t) = e^{it}$

$-\pi \le t \le \pi$

$\dfrac{dz}{dt} = ie^{it}$

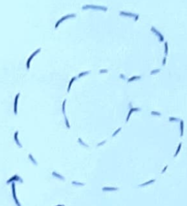
d) $\oint_C \frac{1}{z}\,dz = 2\pi i$

A domain that is not simply connected is called <u>multiply connected</u>.

Eg. <u>Annulus</u> : It is doubly connected ; $1 < |z-a| < 2$

A disc without the circle, say, $0 < |z| < 1$

Doubly connected



Triply connected

Simply connected

Now let us dissect the Cauchy's Theorem

1) Analyticity
2) Simply connectedness of the domain
3) closed

Pr 2) $\oint_C \frac{1}{z^2}\, dz$ with C = Unit Circle

$\frac{1}{z^2}$ is not analytic on any SCD containing C. However,

$$\oint_C \frac{1}{z^2} dz = 0$$

$z(t) = e^{it}$; $-\pi \le t \le \pi$ $\quad \frac{dz}{dt} = ie^{it}$

$$\Rightarrow \oint_C \frac{1}{z^2} dz = \int_{-\pi}^{\pi} e^{-2it}\, ie^{it}\, dt = i\left.\frac{e^{-it}}{-i}\right|_{-\pi}^{\pi} = e^{\pi i} - e^{-\pi i}$$

$$= -1-(-1) = 0$$

To clarify,

$$\begin{matrix} f \text{ analytic on } D \\ \& \ C \subseteq D \end{matrix} \Rightarrow \oint_C f(z)dz = 0$$

However if $\exists$ a SCD D containing C such that

~~However,~~ $\oint_C f(z)dz = 0$, it doesn't imply that f is analytic on D always (as in the above case)

Note that 1) Analyticity 2) Simply connectedness of the domain 3) "closed" contour are 3 key ~~Rosfarches~~ ingredients of the theorem. If we try to remove ~~them~~ 1, integral may or may not be zero.

For eg. if we consider Pr 1 c), integral is no longer zero as $\bar{z}$ is not analytic on ~~$\{z \in \mathbb{C} \mid |z| < 2\}$~~ any SCD that contains C.

However, if we consider $f(z) = \frac{1}{z^2}$, then $\oint_C \frac{1}{z^2}\,dz$ turns out to be zero even though $\frac{1}{z^2}$ is not analytic on any SCD that contains C.

__Calculation__ :- $\oint_C \frac{1}{z^2}\,dz \qquad C : z(t) = e^{it}, \quad -\pi \le t \le \pi$

$$= \int_{-\pi}^{\pi} e^{-2it}\, i e^{it}\,dt = \left. \frac{i e^{-it}}{-i} \right|_{-\pi}^{\pi} = \left. e^{-it} \right|_{\pi}^{-\pi} = e^{\pi i} - e^{-\pi i}$$
$$= (-1) - (-1) = 0$$

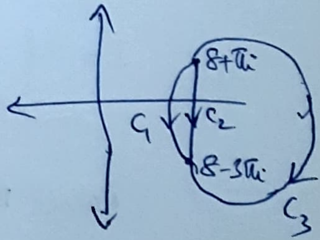
__Simply connectedness__ 2 Consider Pr 1 d)

If we replace SCD by a domain that is doubly connected say

$A = \{ z \in \mathbb{C} \mid \frac{1}{2} < |z| < 3/2 \}$, then $f(z) = \frac{1}{z}$ is analytic on A but

$$\oint_C \frac{1}{z}\,dz = 2\pi i \text{ still holds.}$$

Simply connectedness is necessary (so as to present a theorem where $\oint_C f(z)dz = 0$).



__Closed__ 3 Let us consider any contour C from

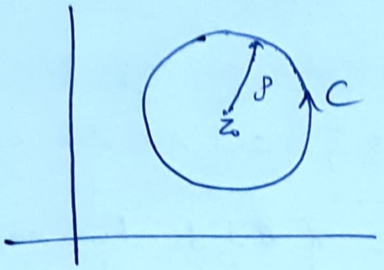
Recall $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = \left. 2e^{z/2} \right|_{8+\pi i}^{8-3\pi i} = 0$

__Aside__ $\displaystyle\int_{-\pi}^{\pi} e^{mit}\,dt = \begin{cases} \left. \dfrac{e^{mit}}{mi} \right|_{-\pi}^{\pi} & \text{if } m \neq 0 \\ 2\pi & \text{if } m = 0 \end{cases} = \begin{cases} \dfrac{e^{\pi i m} - e^{-\pi i m}}{mi} = (-1)^m - (-1)^m = 0 & \text{if } m \neq 0 \\ 2\pi & \text{if } m = 0 \end{cases}$

where $m \in \mathbb{Z}$

Pr 2) Let $f(z) = (z-z_0)^m$ where $m \in \mathbb{Z}$ & z_0 is a constant.

Integrate ccw around the circle C of radius ρ with center at z_0.



By CIT, $f(z)$ is analytic on any SCD that contains C as long as $m \geq 0$. Hence, clearly,

$$\oint_C (z-z_0)^m \, dz = 0 \quad \forall \; m \geq 0.$$

Case :- $m = -1$

$$\oint_C \frac{1}{z-z_0} \, dz$$

Let $z(t) = z_0 + \rho e^{it}$, $-\pi \leq t \leq \pi$

$$\frac{dz}{dt} = i\rho e^{it}$$

$$\int_{-\pi}^{\pi} \frac{1}{\rho e^{it}} \times i\rho e^{it} \, dt = 2\pi i$$

Case : $m \leq -2$: Let $n = |m|$

$$I = \oint_C \frac{1}{(z-z_0)^n} \, dz = \int_{-\pi}^{\pi} \frac{1}{(\rho e^{it})^n} i\rho e^{it} \, dt = i \int_{-\pi}^{\pi} (\rho e^{it})^{-n+1} \, dt$$

Recall that $\int_{-\pi}^{\pi} e^{imt} \, dt = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$

Hence, $I = i\rho^{1-n} \int_{-\pi}^{\pi} e^{i(1-n)t} \, dt$ where $n \neq 1$

Hence, $= 0$

We will not discuss the proof of CIT here. Cauchy proved this assuming $f'(z)$ is cts (~~which is known to us~~ which we know to be true now). This proof uses Green's theorem & interested people can read it in Kreyszig. (Pg 654).

(Pg 659 Kreyszig) (Problem Set 14.2)

Consider $\oint_C f(z)\,dz$ where C is unit circle oriented CCW.

In which of the cases is CIT applicable?

$f(z) = e^{z^2}$ (9), ~~$\tan\frac{z}{4}$~~ $\frac{1}{4z-1}$ (11), $\frac{1}{z^4-1.2}$ (13), $\frac{1}{\bar{z}}$ (14)

(12) (20)

5) $\operatorname{Re} z$ (15), $\frac{1}{\pi z-1}$ (16), $\frac{1}{|z|^2}$ (17), $\frac{1}{5z-1}$ (18), $(\bar{z})^3$, $\operatorname{Log}(1-z)$ with C: the boundary of the parallelogram with vertices $\pm i, \pm(1+i)$.

(9) 1) ✓

(11) 2) No as $z_0 = \frac{1}{4}$ is not a pt of analyticity of $f(z)$ & z_0 is in the interior of C.

(13) 3) ✓

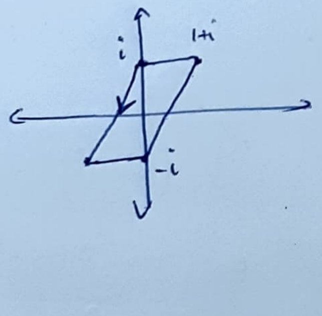
(14) 4) No, at $z=0$, $f(z)$ is not analytic

(15) 5) No   (12) 9) No

16 6) No   (20) 10) $\operatorname{Log}(1-z)$ is not analytic if $1-z \le 0$ $\iff z \ge 1$

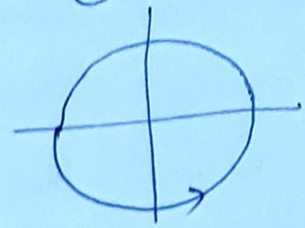
17 7) No

18 8) No   However, C is as given below



Since $\operatorname{Log}(1-z)$ is analytic on the interior of C & on C, $\oint_C \operatorname{Log}(1-z)\,dz = 0$

$I = \oint_C \frac{1}{4z-1}\, dz$ $\quad$ C : unit circle



$z(t) = e^{it} \quad -\pi \le t \le \pi$

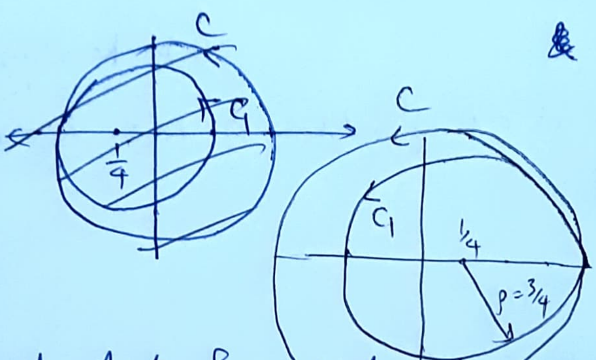
$$\int_{-\pi}^{\pi} \frac{1}{4e^{it}-1}\, i e^{it}\, dt = i\int_{-\pi}^{\pi} \frac{1}{4-e^{-it}}\, dt$$

$$= i\int_{-\pi}^{\pi} \left(\frac{1}{4-\cos t + i\sin t}\right) dt$$

One may multiply by conjugate & divide by conjugate, but it is lengthy.

Instead, write $I = \oint_C \frac{1/4}{z-1/4}\, dz$

Now,



Consider C_1

$|z-1/4| = 3/4$

$z(t) = \frac{1}{4} + \frac{3}{4}e^{it}$

$-\pi \le t \le \pi$

C_1 also starts from -1 & ends at $+1$.

But $\int_{C_1} \frac{1}{z-1/4}\, dz = \oint_C \frac{1}{z-1/4}\, dz = \int_C \frac{4}{4z-1}\, dz$

$= 2\pi i$ from prev. problem. by principle of deformation of path

Hence, $I = \frac{1}{4} \times (2\pi i) = \frac{\pi i}{2}$.

$I = \oint_C \frac{1}{z}\, dz$; Let $C : z(t) = e^{it}$, $-\pi \le t \le \pi$

$$= \int_{-\pi}^{\pi} e^{+it}\, i e^{it}\, dt = i\frac{e^{2it}}{2i}\Big|_{-\pi}^{\pi} = \frac{1}{2}\times\left[e^{2\pi i} - (e^{-2\pi i})\right] = 0$$

27/3/25

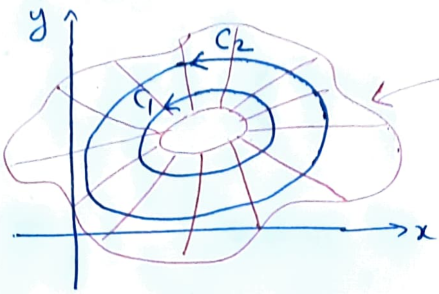
Thm (Principle of deformation of path) <u>Lec : 30</u>

7

Let C_1 & C_2 denote positively oriented simple closed contours where C_1 is interior to C_2. If a function f is analytic ~~in the closed region~~ on a doubly connected domain D consisting of both these contours & all points between them, then

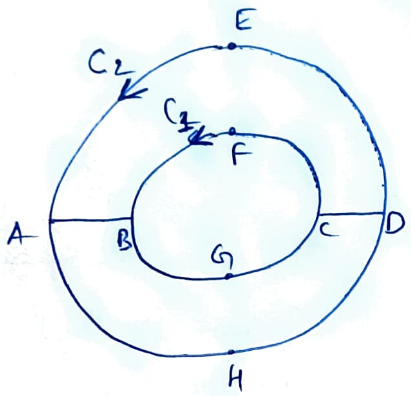
(Pg 159 Churchill) 8th edition

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz$$



Note that red lines are <u>dashed</u> red lines meaning the ~~boundary~~ these lines are not part of the doubly connected domain.

<u>Proof</u> :-



Consider the simple closed contours

$$\Gamma_1 = \widehat{AB} + \widehat{BFC} + \widehat{CD} + \widehat{DEA}$$

&

$$\Gamma_2 = \widehat{AB} + \widehat{BGC} + \widehat{CD} + \widehat{DHA}$$

$$\int_{C_2} f(z)dz = \left(\int_{\widehat{DEA}} + \int_{\widehat{AHD}}\right) f(z)dz$$

$$\int_{C_1} f(z)dz = \left(\int_{\widehat{CFB}} + \int_{\widehat{BGC}}\right) f(z)dz$$

Note
$$\int_{\Gamma_1+\Gamma_2} = \int_{\widehat{DEA}} + \int_{\widehat{AB}} + \int_{\widehat{BFC}} + \int_{\widehat{CD}} + \int_{\widehat{AB}} + \int_{\widehat{BGC}} + \int_{\widehat{CD}} + \int_{\widehat{DHA}}$$

$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$ as f is analytic on D & Γ_1 & Γ_2 are simple closed contours contained in D. And, I can find a SCD $D_1 \subseteq D$ which contains Γ_1. & another SCD $D_2 \subseteq D$ which contains Γ_2, such that f is analytic on D_1 & D_2

Thus,

$$\oint_{\Gamma_1} f(z)dz = \oint_{\Gamma_2} f(z)dz$$

Recall, $\cancel{\int_{C_2} f(z)dz = \left(\int_{\widehat{DEA}} + \int_{\widehat{AHD}}\right) f(z)dz \;\&\; \int_{C_1} f(z)dz =}$

$$\cancel{0 = \int_{\Gamma_1+\Gamma_2} f(z)dz =}\; 0 = \int_{AB} + \int_{BFC} + \int_{CD} + \int_{DEA} = \int_{AB} + \int_{BGC} + \int_{CD} + \int_{DHA}$$

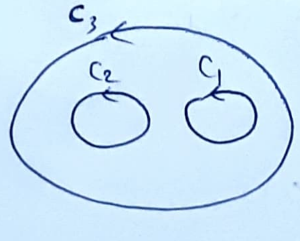
ie,
$$\left(\int_{DEA} - \int_{DHA}\right)(\ ) = \left(\int_{BGC} - \int_{BFC}\right)(\ )$$

$$\int_{\widehat{DEA}+\widehat{AHD}} (f(z)dz) = \int_{BGC+CFB}$$

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz$$

This is also the Cauchy's Integral Theorem for Doubly connected domain

Remark :- CIT for triply connected domain :-



$$\oint_{C_3} f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

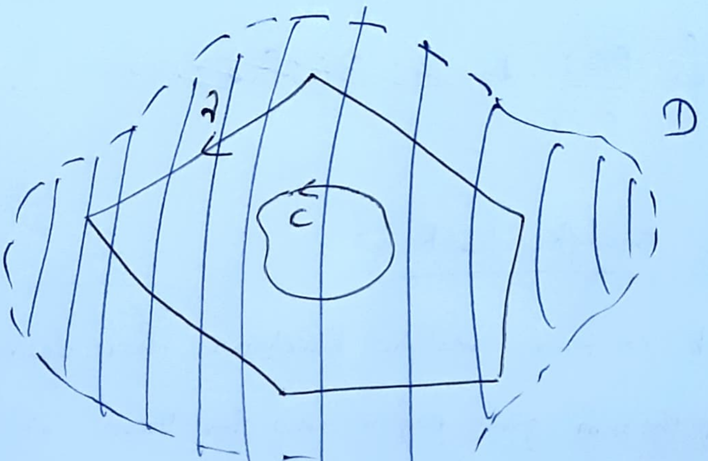
Conclusion :-

By principle of deformation of path, it follows that for $m \in \mathbb{Z}$ & $z_0 \in \mathbb{C}$

$$\oint_{\gamma} (z-z_0)^m \, dz = \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1 \end{cases}$$

is true for any γ that is a simple closed contour enclosing z_0. We had seen the special case when γ is the circle $|z-z_0| = \rho$.

Take away :- S'pz $\oint_C f(z)\,dz = \alpha$ for some C contained in a SCD D where f is analytic. As long as γ is a simple closed contour inside D, I can replace C by γ without any change in the integral.

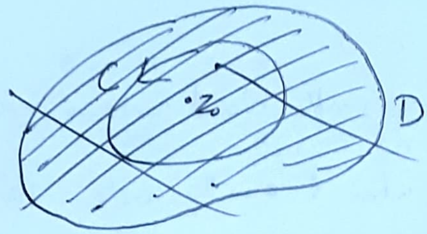


This ~~corollary~~ is known as principle of deformation of paths since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of the integral never changes.

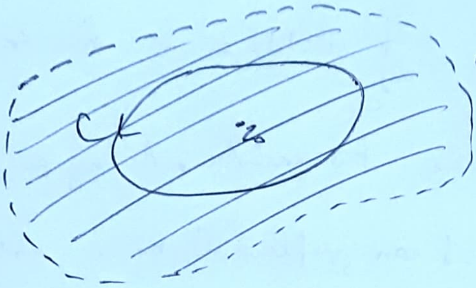
Cauchy's Integral Formula (Kreyszig Pg 660)

Theorem

Let f be analytic ~~everywhere inside~~ in a SCD D. Then for any point $z_0 \in D$ & any simple closed contour C in D that encloses z_0,
(CCW orientation)



D — dotted boundary



D

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \cancel{= 2\pi i f(z_0)}$$

Derivatives of analytic functions

Recall that complex analytic functions have derivatives of all orders. The following theorem gives expressions for them.

Theorem :-

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are also analytic functions in D. The values of these derivatives at a point z_0 in D is given by

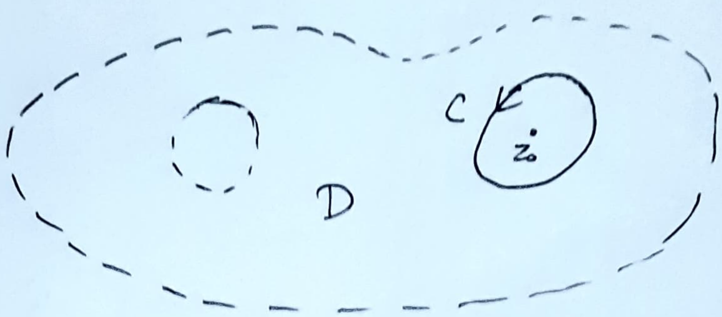
$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$. More

More generally,

$\forall n \geq 1$,
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

(6) Here, C is any simple closed contour that encloses z_0, lies in D & whose full interior belongs to D, oriented CCW.

No need for simply connected domain!



Eg 1) $$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz$$ where C is any simple closed contour enclosing πi (CCW)

$$= 2\pi i \times f'(\pi i) \quad \text{where } f(z) = \cos z$$

$$\text{Ans} = 2\pi i \times -\sin z\Big|_{z=\pi i} = 2\pi i \times -(\sin \pi i)$$

Eg. 2) $$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz$$ where C encloses $-i$ (CCW)

$$f(z) = z^4 - 3z^2 + 6 \quad \& \quad f''(z) = \frac{d}{dz}(4z^3 - 6z) = 12z^2 - 6$$

$$f''(-i) = 12(-1) - 6 = -18$$

$$\Rightarrow \frac{2!}{2\pi i} \times \oint_C \frac{f(z)}{[z-(-i)]^3}\,dz = f''(-i) = -18$$

$$\text{Ans} = \frac{-18 \times 2\pi i}{2} = -18\pi i$$

~~Cauchy's Inequality (SDE)~~.