Solve by PLU decomposition :

$$x + y + z = 1$$

$$2x + 2y + 5z = 0$$

$$4x + 6y + 8z = 0$$

Soln :

$$\underbrace{\begin{bmatrix} \boxed{1} & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 4R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & \boxed{0} & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$A = IA = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$P_{132} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \underbrace{P_{132} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}}_{\text{exchange columns}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

called as "pivoting"

$$= P_{132} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}}_{U} = P_{132} LU$$

exchange rows

We have $AX = b$

$$\Rightarrow (P_{132} L U) X = b$$

$$\Rightarrow (LU)X = P_{132}^{-1} b$$

$$= P_{132} b$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b$$

Let $UX = y$ ——— ①

Then $LY = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad y_1 = 1 \quad ; \quad 4y_1 + y_2 = 0$

$\Rightarrow \quad y_2 = -4$

$2y_1 + y_3 = 0 \quad \Rightarrow \quad y_3 = -2.$

$\therefore \; Y = \begin{matrix} 1 \\ -4 \\ -2 \end{matrix}$

(1) becomes,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}$$

$\Rightarrow \quad 3z = -2 \quad \Rightarrow \quad z = -2/3;$

$2y + 4z = -4 \quad \Rightarrow \quad 2y - 8/3 = -4$

$\Rightarrow y = -2/3;$

$x + y + z = 1 \quad \Rightarrow \quad x = 7/3.$

$$\therefore \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7/3 \\ -2/3 \\ -2/3 \end{bmatrix}.$$

Q. Values of X for $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

$b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$:

$$(LU)X = P_{(32)} b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $UX = Y$. Then $LY = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \quad y_1 = 0 \quad ; \quad 4y_1 + y_2 = 0$$

$$\Rightarrow y_2 = 0$$

$$2y_1 + y_3 = 1 \quad \Rightarrow \quad y_3 = 1.$$

$\therefore \quad UX = Y$ becomes $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$3z = 1 \Rightarrow z = 1/3$;

$2y + 4z = 0 \Rightarrow y = -2/3$

$x + y + z = 0 \Rightarrow x = 1/3$.

$$\therefore X = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

For $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: $(LU)X = P_{132}\, b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Let $UX = Y$ — (1).

$$LY = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$y_1 = 0$; $4y_1 + y_2 = 1 \Rightarrow y_2 = 1$;

$2y_1 + y_3 = 0 \Rightarrow y_3 = 0.$

① becomes,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$3z = 0 \Rightarrow z = 0$; $2y + 4z = 1$

$\Rightarrow y = 1/2$;

$x + y + z = 0 \Rightarrow x = -1/2$

$$\therefore X = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$AX = I$; $I = \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}$

$\Rightarrow X = A^{-1}$

$$= \begin{bmatrix} 7/3 & 1/3 & -1/2 \\ -2/3 & -2/3 & 1/2 \\ -2/3 & 1/3 & 0 \end{bmatrix}.$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} -7/3 & 1/3 & 1/2 \\ -2/3 & -2/3 & 1/2 \\ -2/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ Best method to find inverse of a large matrix.

↳ also, used for finding determinant of a large matrix.

$$\det A = \det(P_{132} LU)$$

Then $\det A = 1 \cdot \det(LU)$

$= \det L \cdot \det U$

$= 1 \cdot 6 = \underline{\underline{6}}.$

Permutation matrices.

$$P_{132} P_{132}^T = I = P_{132}^T P_{132}$$

"orthogonal matrix".

A matrix 'A' is an orthogonal matrix

if $AA^T = A^TA = I$.

Eg: Permutation matrix

Rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Orthogonal maps in 2D.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{—— (I)}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{—— (II)}$$

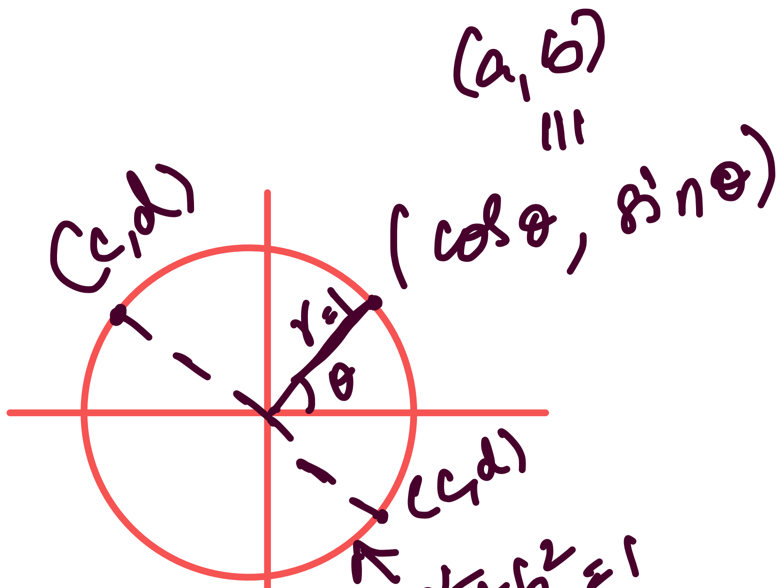
Solve (I) & (II)

(I) :

$$a^2 + b^2 = 1$$

$$ac + bd = 0$$

$$c^2 + d^2 = 1.$$



$a^2 + b^2 = 1$

for some $\theta \in [0, 2\pi]$.

(c,d) is on the circle.

$$ac + bd = 0 \Rightarrow \langle (a,b), (c,d) \rangle = 0$$

↳ dot product

$$\Rightarrow (a,b) \perp (c,d)$$

$$\Rightarrow \theta = \pi/2 \text{ or } -\pi/2.$$

$$(c,d) = \left(\cos\left(\theta + \pi/2\right), \sin\left(\theta + \pi/2\right)\right) \text{ or}$$

$$\left(\cos\left(\theta - \pi/2\right), \sin\left(\theta - \pi/2\right)\right)$$

ie, $(c, d) = (-\sin\theta, \cos\theta)$ or $(\sin\theta, -\cos\theta)$

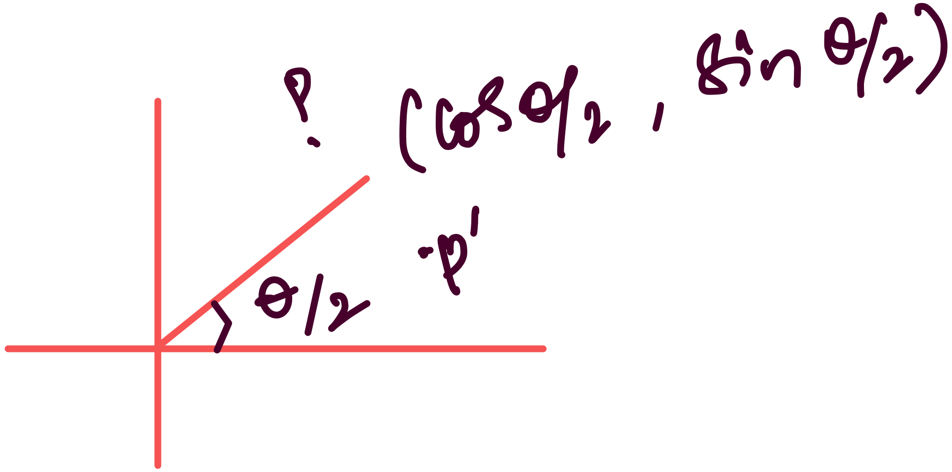
$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Rotation matrix clockwise angle θ

Reflection matrix about the line joining $(0,0)$ with $(\cos\theta/_2, \sin\theta/_2)$

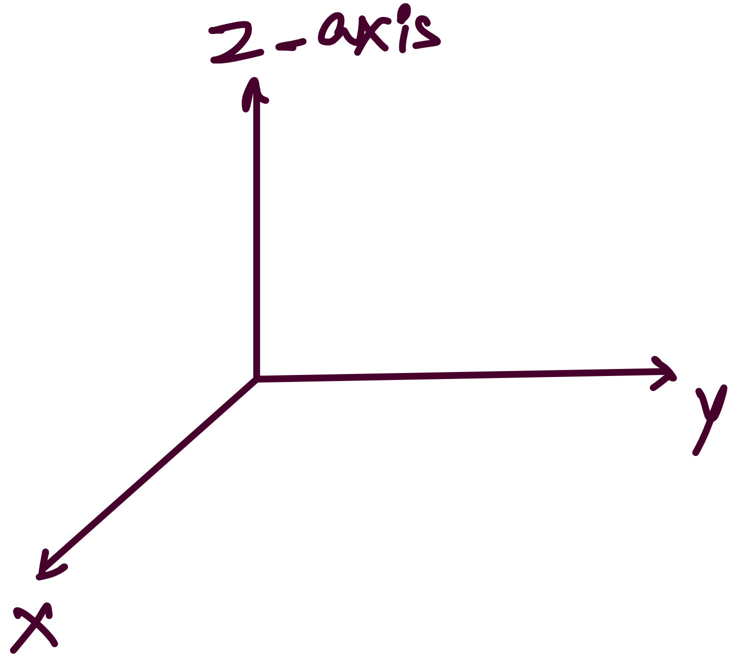
* $\det A = 1 \Rightarrow A$ is a rotation matrix.

$\det A = -1 \Rightarrow A$ is a reflection matrix.



(II) : $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = Reflection matrix or a Rotation matrix.

<u>Q</u>. In 3D ?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