Solve $x = 0$

$y = 0$

$x + y = 1.$

<u>Soln:</u>

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b}$$

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2/3} \\ 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{bmatrix}}_{R}$$

$$AX = b$$

$$QRX = b$$

$$\Rightarrow Q^TQRX = Q^Tb$$

$$\Rightarrow IRX = Q^Tb \qquad (Q^TQ = I)$$

$$\Rightarrow RX = Q^Tb$$

$$\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & \sqrt{1/6} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{6} \end{bmatrix}.$$

$$\Rightarrow \sqrt{2}\,x + \frac{1}{\sqrt{2}}\,y = 1/\sqrt{2}$$

$$\sqrt{3/2}\,y = 1/\sqrt{6}$$

$$\Rightarrow \left.\begin{array}{l} y = 1/3 \\ x = 1/3 \end{array}\right\} \text{least square solutions.}$$

QR decomposition :-

* useful for inconsistent systems.
* gives solution without matrix inverses.

Q. Find QR decomposition of

$$A = \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -3 & 5 \end{bmatrix} \end{array}$$

x

y

sparce matrices

Gram-Schmidt process $\rightarrow$ produces numerical errors.

$\downarrow$ solution

Length invariant operations are useful to handle these numerical errors.

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Length invariant operations.

1) Rotations (Given's rotation)

2) Reflections (Householder reflection method or Householder transformation)

Rotation matrix : $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\rightarrow$ orthogonal matrix.

Notations : $\cos\theta = c$

$\sin\theta = s$

(As θ need not be calculated explicitly)

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}; \quad r = \sqrt{x^2+y^2}$$

$$\Rightarrow \quad cx - sy = r$$

$$sx + cy = 0$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}^{-1}\begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c & -s \\ s & c \end{bmatrix}^{T}\begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} cr \\ -sr \end{bmatrix}.$$

$$\Rightarrow \boxed{\begin{aligned} c &= \frac{x}{r} \\ s &= \frac{-y}{r} \end{aligned}}$$

Take $x = 4$; $y = -3$

$$r = \sqrt{4^2 + (-3)^2} = 5$$

$$c = \frac{x}{r} = \frac{4}{5}$$

$$s = \frac{-y}{r} = \frac{3}{5}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -3 & 5 \end{bmatrix}$$

Given's matrix

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & -3 \\ 0 & 0 & 4 \end{bmatrix} = R \text{ (upper triangular)}$$

$$G_1 A = R$$

$$\Rightarrow A = G_1^{-1} R$$

$$= G_1^T R \quad \text{(as } G_1 \text{ is orthogonal)}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 3/5 \\ 0 & -3/5 & \frac{4}{5} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & -3 \\ 0 & 0 & 4 \end{bmatrix}}_{R}$$

$$= QR.$$

obtained QR decomposition
(without matrix inverses &
without Gram-Schmidt process)

Given's matrices :

yz - plane :
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
(rotation in yz-plane)

xy - plane :
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
(rotation in xy-plane)

XZ-plane : $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

(rotation ; but does not respect the right hand rule; Given's rotation not usual rotation)

Q. By Given's rotation (QR decomposition), solve

$$3x - y + 2z = 5$$

$$0x + 0y + 9z = 9$$

$$4x + 7y + 11z = 40.$$

Soln:

$$\underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 9 \\ 4 & 7 & 11 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 40 \end{bmatrix}$$

$$P_{132} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 9 \\ 4 & 7 & 11 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 3 & -1 & 2 \\ 4 & 7 & 11 \\ 0 & 0 & 9 \end{bmatrix} \end{array} = B.$$

x

y

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5.$$

$$c = \frac{x}{r} = \frac{3}{5}$$

$$s = \frac{-y}{r} = -\frac{4}{5}$$

Using Givens's rotation on B,

$$G_1 B = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 4 & 7 & 11 \\ 0 & 0 & 9 \end{bmatrix}$$

$$G_1 B = \begin{bmatrix} 5 & 5 & 10 \\ 0 & 5 & 5 \\ 0 & 0 & 9 \end{bmatrix} = R.$$

$$B = G_1^T R$$

$$= \underbrace{\begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 5 & 5 & 10 \\ 0 & 5 & 5 \\ 0 & 0 & 9 \end{bmatrix}}_{R}$$

$$= QR$$

$$AX = b$$

$$P_{132} A X = P_{132}\, b$$

$$\Rightarrow BX = P_{132} \begin{bmatrix} 5 \\ 9 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 40 \\ 9 \end{bmatrix}$$

$$\Rightarrow QRX = \begin{bmatrix} 5 \\ 40 \\ 9 \end{bmatrix}$$

$$\Rightarrow \underbrace{Q^T Q}_{I} R X = Q^T \begin{bmatrix} 5 \\ 40 \\ 9 \end{bmatrix}$$

$$\Rightarrow RX = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 40 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 35 \\ 20 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 & 10 \\ 0 & 5 & 5 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ 20 \\ 9 \end{bmatrix}.$$

$\Rightarrow 9z = 9 \qquad \Rightarrow z = 1$

$5y + 5z = 20 \qquad \Rightarrow y = 3$

$5x + 5y + 10z = 35 \qquad \Rightarrow x = 2.$