Linear Transformation

$T : V \to V$

(i) $T(X+Y) = T(X) + T(Y), \quad \forall X, Y \in V$

(ii) $T(\alpha X) = \alpha T(X), \quad \forall X \in V$, for any scalar α.

Eg: Suppose that $T(X) = AX$

Then $T(X+Y) = A(X+Y) = AX + AY = T(X) + T(Y)$

$T(\alpha X) = A(\alpha X) = \alpha(AX) = \alpha T(X)$

Thus T is a linear transformation.

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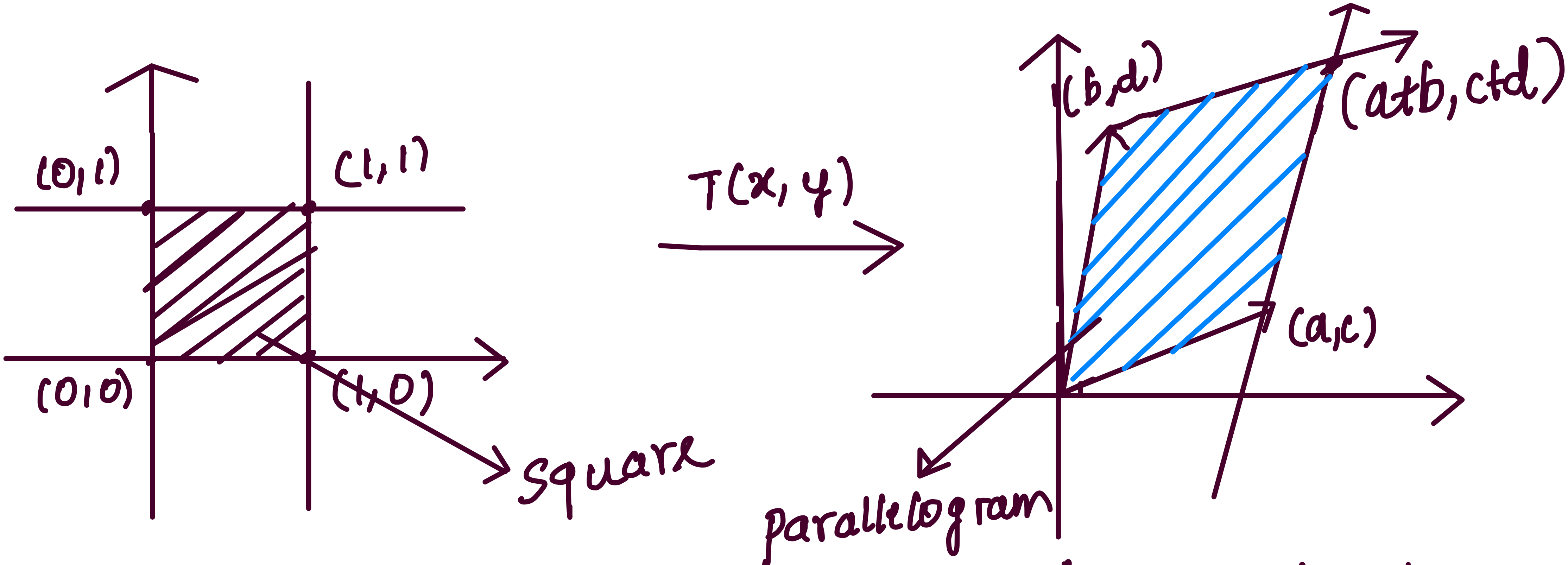
$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$T(x,y) = (ax+by,\ cx+dy)$$



Area = 1 $\qquad\qquad$ Area = |det A|

$$T(0,0) = (0,0)$$

$$T(1,0) = (a,c)$$

$$T(0,1) = (b,d)$$

$$T(1,1) = (a+b,\ c+d)$$

Q Find the matrix of linear transformation

$T(x,y) = (7x+2y, -4x+y)$.

(ii) Find T^{-1}

(iii) Verify your answer.

(iv) Find $T^2(x,y)$

(v) Verify your answer.

Soln:

(i) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$T(1,0) = (7,-4)$$

$$T(0,1) = (2,1)$$

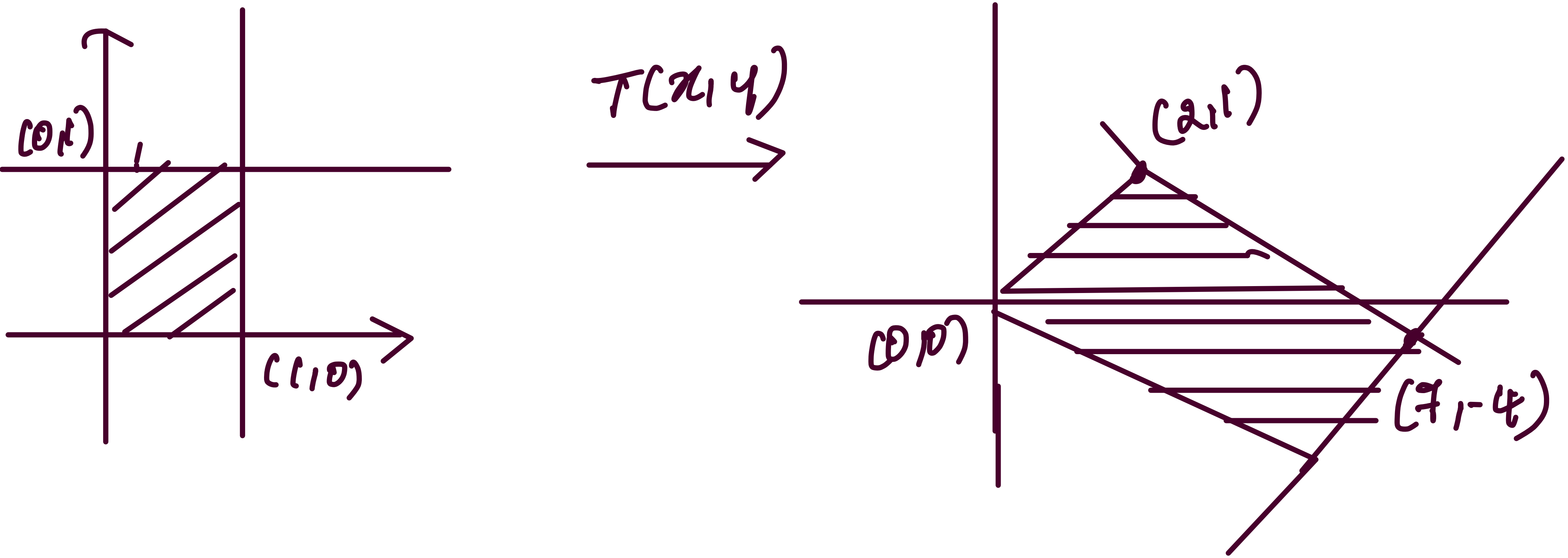
$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7x+2y \\ -4x+y \end{bmatrix}$$

$$T(1,0) = (7,-4) = 7(1,0) + (-4)(0,1)$$

$$T(0,1) = (2,1) = 2(1,0) + 1(0,1)$$

$$m(T) = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$



(ii) $M(T)$ Matrix $\longleftrightarrow$ Linear Transf. T

$A^{-1} \longleftrightarrow T^{-1}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{15} \begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}$$

$$T^{-1}(x, y) = \frac{1}{15}\begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} x-2y \\ 4x+7y \end{bmatrix}$$

$$T^{-1}(x, y) = \frac{1}{15}(x-2y, 4x+7y)$$

Sample Verification-

$$T^{-1}(0,0) = (0,0)$$

$$T^{-1}(7,-4) = \frac{1}{15}\left(7+8,\ 28-28\right) = (1,0)$$

$$T^{-1}(2,1) = \frac{1}{15}\left(2-2,\ 8+7\right) = (0,1)$$

$$T^{-1}(9,-3) = \frac{1}{15}\left(9+6,\ 36-21\right) = (1,1)$$

(iii) $T^{-1} \circ T = T \circ T^{-1} = I$ (Identity function)

$$(T^{-1} \circ T)(x, y) = T^{-1}(T(x, y))$$

$$= T^{-1}(\underbrace{7x+2y}_{X}, \underbrace{-4x+y}_{Y})$$

$$= T^{-1}(X, Y)$$

$$= \frac{1}{15}(X-2Y, 4X+7Y)$$

$$= \frac{1}{15}\left(7x+2y-2(-4x+y),\ 4(7x+2y) + 7(-4x+y)\right)$$

$$= \frac{1}{15}\left(7x+2y+8x-2y,\ 28x+8y-28x + 7y\right)$$

$$= \frac{1}{15}\left(15x,\ 15y\right)$$

$$= (x, y)$$

$$= I(x, y)$$

Therefore $T_0^{-1} T = I$.

Similarly, we can show that

$$T_0 T^{-1} = I$$

Therefore $\underline{\underline{T_0 T^{-1} = T_0^{-1} T = I}}$

(iv) $T^2(x,y) = A^2 \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 16 \\ -32 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= (41x + 16y, \quad -32x - 7y)$$

(v) $T^2(x,y)$

$= T \circ T(x,y)$

$= T(T(x,y))$

$= T(\underbrace{7x+2y}_{X}, \underbrace{-4x+y}_{Y})$

$= T(X, Y)$

$= (7X+2Y, -4X+Y)$

$$= \left(7(7x+2y) + 2(-4x+y),\ -4(7x+2y) + (-4x+y)\right)$$

$$= \left(49x+14y-8x+2y,\ -28x-8y-4x+y\right)$$

$$= \underline{\underline{\left(41x+16x,\ -32x-7y\right)}}$$

$$T^n(x,y) = A^n \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

$\lambda_1 + \lambda_2 = \text{trace } A = 8$

$\lambda_1 \lambda_2 = \det A = 15$

$\lambda_1 = 5$, $\lambda_2 = 3$ are eigen values of A.