A function $f: X \to Y$ is called onto if for every $y \in Y$, there exists $x \in X$ such that

$$f(x) = y.$$

(In other words, for every image $y \in Y$ there exists preimage $x \in X$ s.t. $f(x) = y$.).

Result : $T : V \to W$ be a L.T. Then

T is onto iff $\mathrm{Im}(T) = W$.

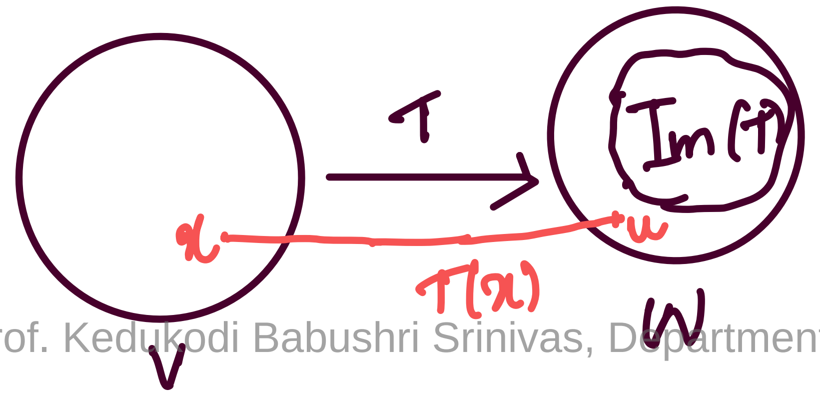
(A) $\Longleftrightarrow$ (B)

Proof : $(A \Rightarrow B)$: Assume T is onto.

To prove : $\mathrm{Im}(T) = W$.

$$\mathrm{Im}(T) = \{T(x) \mid x \in V\} \subseteq W \quad \text{--- (1)}$$

($\mathrm{Im}(T)$ is a subspace of W.)



We need to prove $W \subseteq Im(T)$.

Take $u \in W$.

As T is onto, there exists $x \in V$ s.t.

$T(x) = u$.

$$\Rightarrow u = T(x) \in Im(T)$$

$$\Rightarrow W \subseteq Im(T) \quad \text{——} \quad (2)$$

By (1) and (2), $\underline{\underline{W = Im(T)}}$.

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$(B \Rightarrow A)$: Assume $W = \text{Im}(T)$.

$$\Rightarrow W = \{T(x) \mid x \in V\}$$

Take $y \in W$.

$\Rightarrow y = T(x)$ for some $x \in V$. (Given)

$\Rightarrow T$ is onto.

* T is one-one $\Longleftrightarrow \text{Ker}\, T = \{0\}$

T is onto $\Longleftrightarrow \text{Im}(T) = W$.

Result :

$T : V \to W$ be a L.T. and T is one-one, T is onto. Then $\dim W = \dim V$.

Proof:

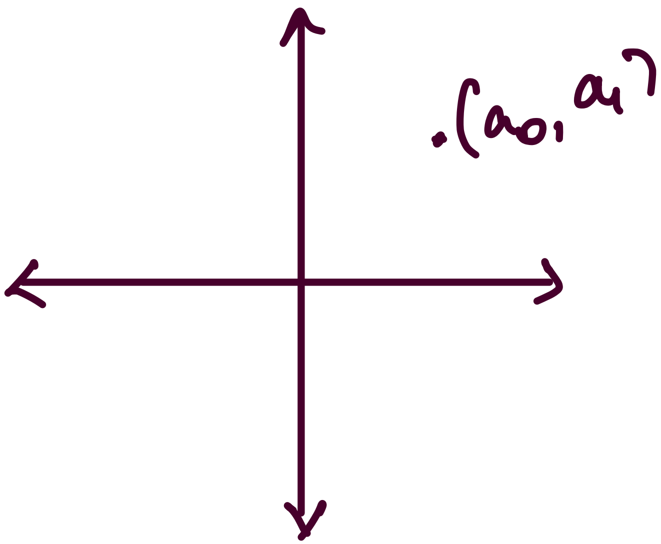
$\dim \operatorname{Ker} T + \dim \operatorname{Im}(T) = \dim V.$ (Rank-nullity theorem)

$$\Rightarrow \underbrace{\dim(\{0\})}_{\text{(as } T \text{ is one-one)}} + \underbrace{\dim W}_{\text{(as } T \text{ is onto)}} = \dim V.$$

$$\Rightarrow 0 + \dim W = \dim V. \Rightarrow \boxed{\dim W = \dim V}$$

Def^n: $T: V \to W$ be a L.T. If T is one-one and onto, then the vector space V is isomorphic to the vector space W; denoted by $V \cong W$.

Eg:



$$\mathbb{R}^2 = \{(a_0, a_1) \mid a_0, a_1 \in \mathbb{R}\}$$

$$P_1 = \{ a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \}$$

Q. Prove that as vector spaces, $\mathbb{R}^2 \cong P_1$.

Soln: Take $T : \mathbb{R}^2 \longrightarrow P_1$ given by

$$T(a_0, a_1) = a_0 + a_1 x .$$

Let $(a_0, a_1), (b_0, b_1) \in \mathbb{R}^2$.

(1). $T(\underset{\underset{v_1}{\uparrow}}{(a_0, a_1)} + \underset{\underset{v_2}{\uparrow}}{(b_0, b_1)}) = T((a_0 + b_0, a_1 + b_1))$

$= (a_0 + b_0) + (a_1 + b_1) x$

$$= (a_0 + a_1 x) + (b_0 + b_1 x)$$

$$= T(a_0, a_1) + T(b_0, b_1)$$

$$= T(v_1) + T(v_2)$$

(2) Let $\alpha \in \mathbb{R}$. $T(\alpha v_1) = T(\alpha(a_0, a_1))$

$$= T(\alpha a_0, \alpha a_1)$$

$$= \alpha a_0 + (\alpha a_1) x$$

$$= \alpha(a_0 + a_1 x)$$

$$= T(a_0, a_1) = \alpha T(v_1)$$

$\therefore$ T is a L.T. (from (1) & (2))

Take $T(v_1) = T(v_2)$

$\Rightarrow T(a_0, a_1) = T(b_0, b_1)$

$\Rightarrow a_0 + a_1 x = b_0 + b_1 x$

$\Rightarrow a_0 = b_0 \; ; \; a_1 = b_1$.

$\Rightarrow (a_0, a_1) = (b_0, b_1)$

$\Rightarrow v_1 = v_2$

$\Rightarrow$ T is one - one.

To show T is onto, take $v \in P_1$

$$\Rightarrow v = a_0 + a_1 x \quad ; \quad a_i \in \mathbb{R}.$$

$$\Rightarrow v = a_0 + a_1 x = T(a_0, a_1)$$

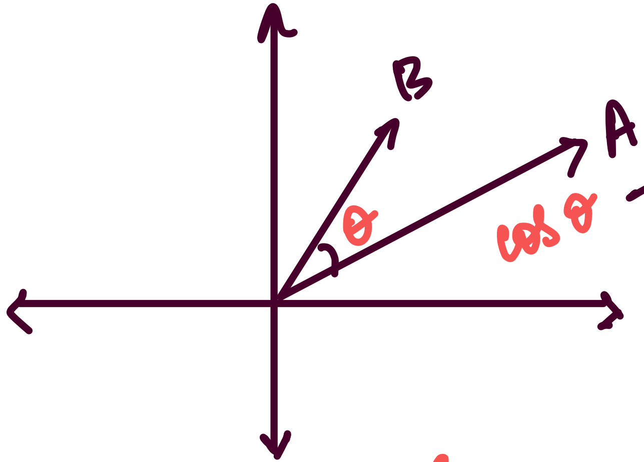
image (polynomial)

preimage (point in xy-plane)

$\Rightarrow T$ is onto.

$\therefore$ $\boxed{\mathbb{R}^2 \cong P_1.}$

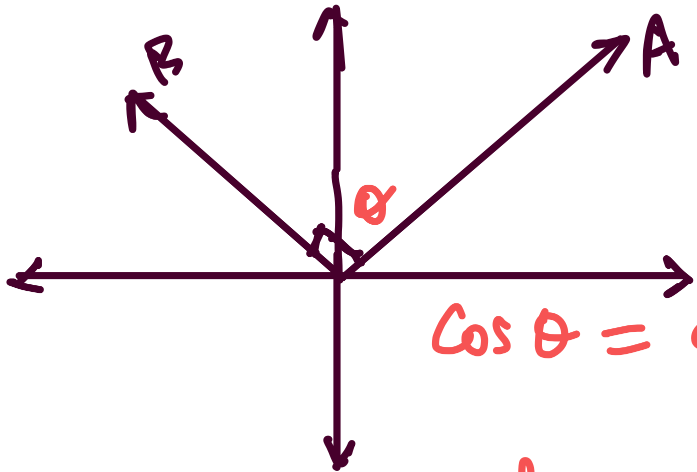
(isomorphic)



$\cos\theta \to +ve$

Similar vectors

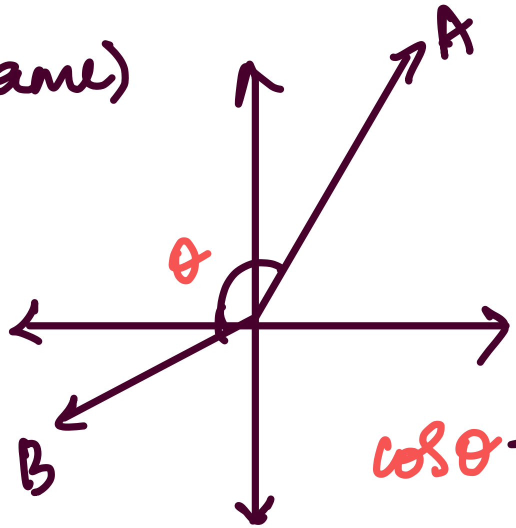
(directions are same)



$\cos\theta = 0$.

orthogonal vectors

(unrelated vectors)



$\cos\theta \to -ve$

Opposite vectors (directions are opposite)

Dot product :

$$A = (a_1, a_2, \dots, a_n)$$

$$B = (b_1, b_2, \dots, b_n)$$

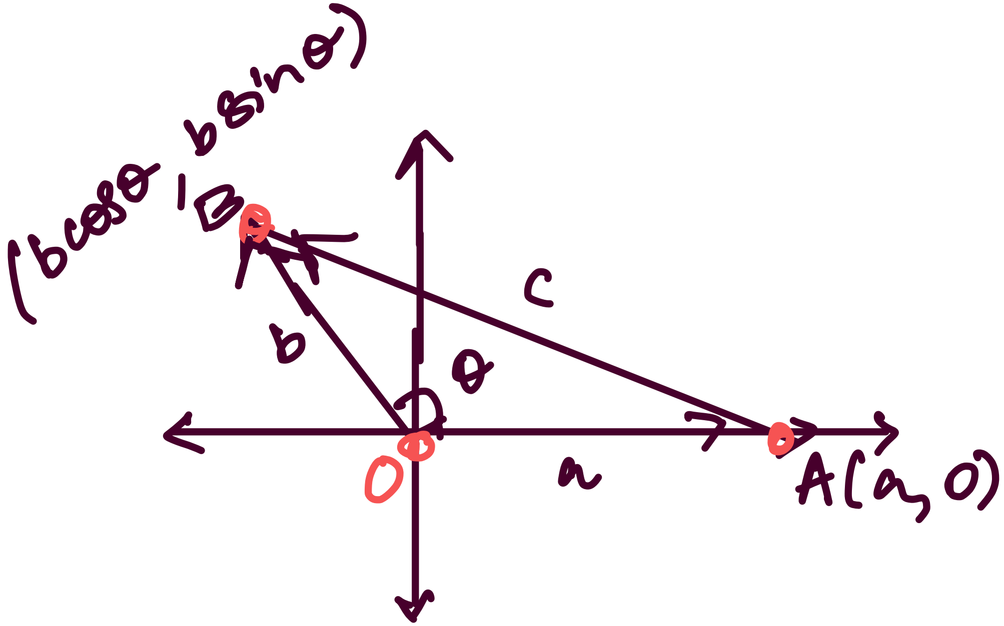
$$A \cdot B = \langle A, B \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n .$$

$$A \cdot A = a_1^2 + a_2^2 + \dots + a_n^2$$

$$= \|A\|^2$$

$$\Rightarrow \|A\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

distance from origin.



$$c = \| A - B \|$$

$$c = \sqrt{(b\cos\theta - a)^2 + (b\sin\theta - 0)^2}$$

$$c^2 = b^2\cos^2\theta + a^2 - 2ab\cos\theta + b^2\sin^2\theta$$

$$= a^2 + b^2(\cos^2\theta + \sin^2\theta) - 2ab\cos\theta$$

$$= a^2 + b^2 - 2ab\cos\theta .$$ (Law of cosines)

$\|A\| = a \quad ; \quad \|B\| = b$

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\,\|B\|\cos\theta$$

(Law of cosines) — (1)

$$\|A - B\|^2 = (A - B)\cdot(A - B)$$

$$= A\cdot A - A\cdot B - B\cdot A + B\cdot B$$

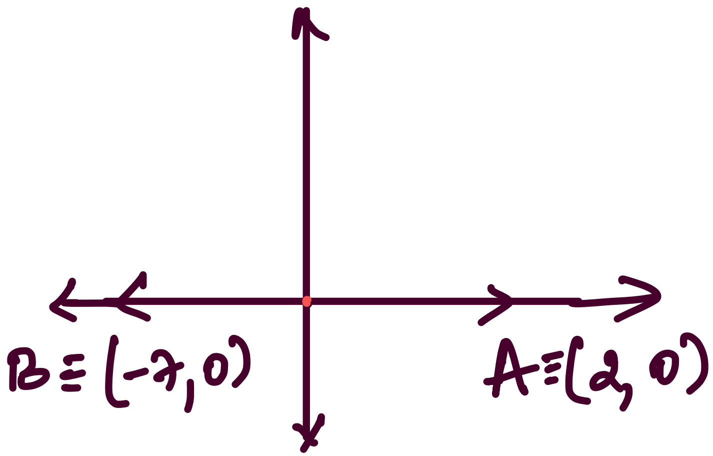
$$= \|A\|^2 - 2A\cdot B + \|B\|^2 \quad \text{— (2)}$$

By (1) & (2),

$$A \cdot B = \|A\| \, \|B\| \cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{A \cdot B}{\|A\| \, \|B\|}} \rightarrow \text{"cosine similarity formula"}$$

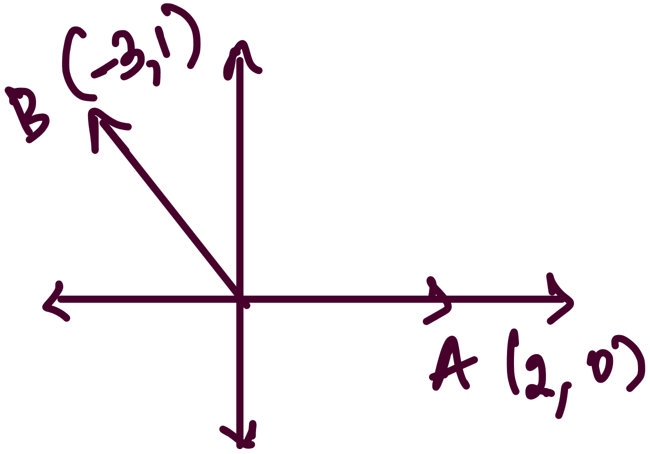
$$= \operatorname{Sim}(A, B)$$



$$\text{Sim}(A, B) = \frac{A \cdot B}{\|A\| \, \|B\|}$$

$$= \frac{2(-7) + 0 \cdot 0}{\sqrt{2^2 + 0^2} \cdot \sqrt{(-7)^2 + 0^2}} = \frac{-14}{14} = -1$$

exactly opposite direction.



$$Sim(A,B) = \frac{A \cdot B}{\|A\| \, \|B\|}$$

$$= \frac{2(-3) + 0 \cdot 1}{\sqrt{2^2+0^2} \cdot \sqrt{(-3)^2+1^2}} = \frac{-3}{\sqrt{10}}$$