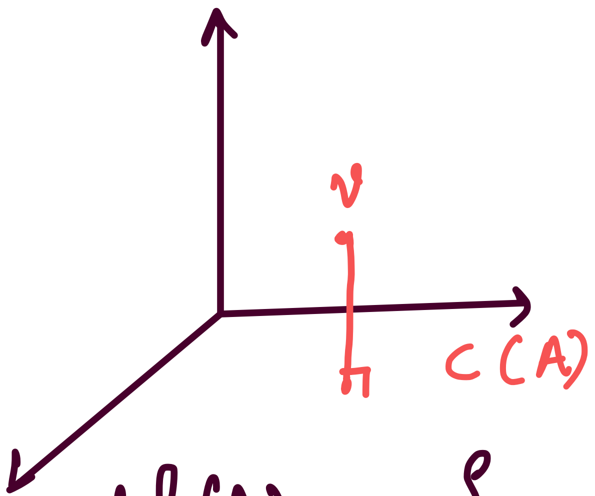
Result :



$$\text{col}(A) = \{\alpha_1 c_1 + \dots + \alpha_n c_n \mid c_i \to \text{columns of } A\} = C(A)$$

$\equiv$ set of all linear combinations of columns of A.

$$v \perp (\alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_n c_n) \iff \langle v, c_1 \rangle = 0;$$

$$\langle v, c_2 \rangle = 0;$$

$$\vdots$$

$$\langle v, c_n \rangle = 0$$

Proof : We know that $v \perp u \iff \langle v, u \rangle = 0$.

linear combination

Consider $v \perp (\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_n c_n)$

$$\iff \langle \alpha_1 c_1 + \cdots + \alpha_n c_n , v \rangle = 0$$

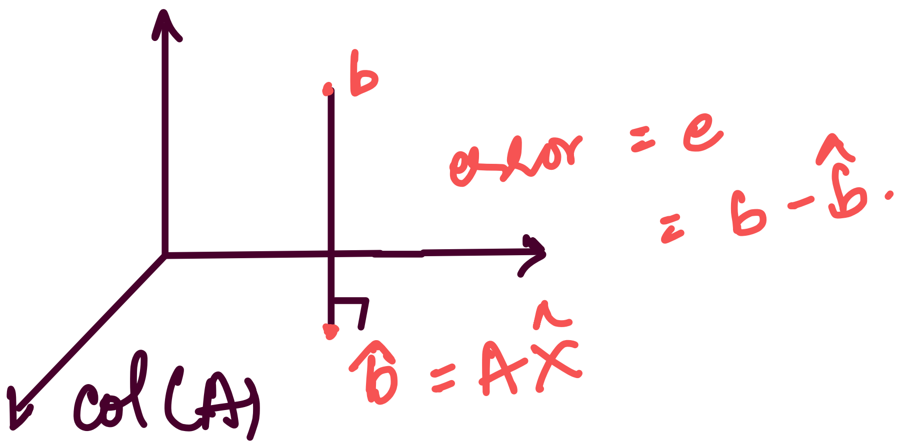
$$\iff \alpha_1 \langle c_1, v \rangle + \alpha_2 \langle c_2, v \rangle + \cdots + \alpha_n \langle c_n, v \rangle = 0$$

(This should hold for all linear combinations).

$$\iff \langle c_1, v \rangle = 0 \quad \text{and} \quad \langle c_2, v \rangle = 0 \quad \text{and} \quad \ldots$$

$$\langle c_n, v \rangle = 0.$$

Least squares solution $Ax = b$.



$Ax = b$

$col(A) = C(A)$

$= \{\alpha_1 C_1' + \alpha_2 C_2' + \dots + \alpha_n C_n' \mid \alpha_i \in \mathbb{R},$

$C_i' \rightarrow$ columns of $A\}$

$\operatorname{col}(A) = \operatorname{Span}\{c_1, c_2, \dots, c_r\}$

basis for $C(A)$

case 1: If $b \in AX = C(A)$ then we have exact solution.

case 2: Let $b \notin C(A)$.

Take $\hat{b}$ as the projection of b on $C(A)$.

As $\hat{b} \in C(A)$, there exists $\hat{X}$ s.t $A\hat{X} = \hat{b}$ (1)

The error vector $b-\hat{b} \perp C(A)$.

$$\Rightarrow b-\hat{b} \perp C(A) = \text{span}\{c_1, c_2, \dots, c_r\}$$

$$\Rightarrow b-\hat{b} \perp (\alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_r c_r)$$

$$\Rightarrow b-\hat{b} \perp c_1,$$

$$b-\hat{b} \perp c_2;$$

$$\vdots$$

$$b-\hat{b} \perp c_r.$$

$$\Rightarrow \langle b-\hat{b}, c_i \rangle = 0 \text{ for all } i, \quad i = 1, 2, \dots, r. \quad \text{--- (2)}$$

$$\Rightarrow A^T(b-\hat{b}) = 0$$

(vectorization or matrix product of ②)

$$\Rightarrow A^Tb - A^T\hat{b} = 0$$

$$\Rightarrow A^T\hat{b} = A^Tb$$

$$\Rightarrow A^T(A\hat{x}) = A^Tb \quad \text{(by (1))}$$

$$\Rightarrow \boxed{\hat{x} = (A^TA)^{-1}A^Tb}$$

called least squares solution if $(A^TA)^{-1}$ exists (approximate soln.)

$A\hat{x} = \hat{b}$ is projection of b on $C(A)$.

$$\boxed{\begin{aligned}\hat{b} &= A\hat{x} \\ &= A\,(A^TA)^{-1}A^Tb\end{aligned}}$$

$\rightarrow$ projection formula.

Eg: Solve
$x = 0$
$y = 0$
$x + y = 1$
by least squares.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b}$$

Approximate soln. $\hat{x} = (A^T A)^{-1} A^T b$

$$= \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} A^T b$$

$$= \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}} A^T b$$

$$= \frac{1}{3} \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \underset{2\times 3}{\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow x = 1/3 \; ; \; y = 1/3 .$$

error vector $e = b - \hat{b}$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - A\hat{X}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$\text{error} = \|e\|$$

$$= \sqrt{(-1/3)^2 + (-1/3)^2 + (1/3)^2}$$

$$= \frac{1}{\sqrt{3}}$$

* $B_{n\times m} A_{m\times n} = I_{n\times n}$

$\uparrow$

left inverse (pseudo inverse) of A

$$\underbrace{(A^TA)^{-1}A^T}_{B}A = I \quad \Rightarrow \quad B = (A^TA)^{-1}A^T$$

Eg: left inverse of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = (A^T A)^{-1} A^T$$

$$= \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \end{bmatrix}.$$

Q. Find left inverse of $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Soln: left inverse of $A = B$

$$= (A^T A)^{-1} A^T$$

$$= \left(\underset{2\times3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \right)^{-1} \underset{2\times3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}.$$

$$\underset{2\times 3}{B}\,A_{3\times 2} = I_{2\times 2}.$$

$$x = 0$$
$$y = 0 \quad \longleftrightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$x + y = 1$$

skinny matrix

$$\updownarrow$$

$$(Ax)^T = b^T$$

$$\Rightarrow \quad x^T A^T = b^T$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Q. Find right inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = B.$

wide matrix.

Soln: $B_{m \times n} A_{n \times m} = I_{m \times m}$

right inverse of B.

$$\underbrace{B B^T (BB^T)^{-1}}_{A} = I \quad \Rightarrow \quad A = B^T (BB^T)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1}.$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

↳ right inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Left inverse of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$.

↳ transpose of right inverse.