CHAPTER

2

ANALYTIC FUNCTIONS

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

12. FUNCTIONS OF A COMPLEX VARIABLE

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the *domain of definition* of f.*

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes between a given function and its values.

EXAMPLE 1. If f is defined on the set $z \neq 0$ by means of the equation $w = 1/z$, it may be referred to only as the function $w = 1/z$, or simply the function $1/z$.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

*Although the domain of definition is often a domain as defined in Sec. 11, it need not be.

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \tag{1}$$

If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{2}$$

EXAMPLE 2. If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

When polar coordinates are used,

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequently,

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

If, in either of equations (1) and (2), the function v always has value zero, then the value of f is always real. That is, f is a *real-valued function* of a complex variable.

EXAMPLE 3. A real-valued function that is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is zero or a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a *polynomial* of degree n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z plane. Quotients $P(z)/Q(z)$ of polynomials are called *rational functions* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of a real variable. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

EXAMPLE 4. Let z denote any nonzero complex number. We know from Sec. 9 that $z^{1/2}$ has the two values

$$z^{1/2} = \pm\sqrt{r}\exp\left(i\frac{\Theta}{2}\right),$$

where $r = |z|$ and Θ $(-\pi < \Theta \leq \pi)$ is the *principal value* of $\arg z$. But, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$f(z) = \sqrt{r}\exp\left(i\frac{\Theta}{2}\right) \qquad (r > 0, -\pi < \Theta \leq \pi), \tag{3}$$

the (single-valued) function (3) is well defined on the set of nonzero numbers in the z plane. Since zero is the only square root of zero, we also write $f(0) = 0$. The function f is then well defined on the entire plane.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$; *(b)* $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\overline{z}}$; *(d)* $f(z) = \dfrac{1}{1-|z|^2}$.

Ans. (a) $z \neq \pm i$; *(c)* $\operatorname{Re} z \neq 0$.

2. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

Ans. $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z-\overline{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

Ans. $f(z) = \overline{z}^2 + 2iz$.

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

$$\textit{Ans. } f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

13. MAPPINGS

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where z and w are complex, no such convenient graphical representation of the function f is available because each of the numbers z and w is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. To do this, it is generally simpler to draw the z and w planes separately.

When a function f is thought of in this way, it is often referred to as a *mapping*, or transformation. The *image* of a point z in the domain of definition S is the point $w = f(z)$, and the set of images of all points in a set T that is contained in S is called the image of T. The image of the entire domain of definition S is called the *range* of f. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f.

Terms such as *translation, rotation,* and *reflection* are used to convey dominant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider the z and w planes to be the same. For example, the mapping

$$w = z + 1 = (x + 1) + iy,$$

where $z = x + iy$, can be thought of as a translation of each point z one unit to the right. Since $i = e^{i\pi/2}$, the mapping

$$w = iz = r\exp\left[i\left(\theta + \frac{\pi}{2}\right)\right],$$

where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \overline{z} = x - iy$$

transforms each point $z = x + iy$ into its reflection in the real axis.

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following three examples, we illustrate this with the transformation $w = z^2$. We begin by finding the images of some *curves* in the z plane.

EXAMPLE 1. According to Example 2 in Sec. 12, the mapping $w = z^2$ can be thought of as the transformation

$$u = x^2 - y^2, \quad v = 2xy \tag{1}$$

from the xy plane into the uv plane. This form of the mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola

$$x^2 - y^2 = c_1 \qquad (c_1 > 0) \tag{2}$$

is mapped in a one to one manner onto the vertical line $u = c_1$. We start by noting from the first of equations (1) that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equations (1) tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

$$u = c_1, \quad v = 2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty);$$

and it is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction (Fig. 17). Likewise, since the pair of equations

$$u = c_1, \quad v = -2y\sqrt{y^2 + c_1} \qquad (-\infty < y < \infty)$$

furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going *downward* along the entire left-hand branch is seen to move up the entire line $u = c_1$.

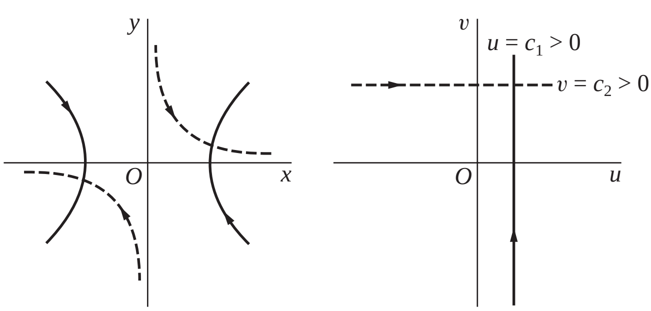

FIGURE 17
$w = z^2$.

On the other hand, each branch of a hyperbola

$$2xy = c_2 \qquad (c_2 > 0) \tag{3}$$

is transformed into the line $v = c_2$, as indicated in Fig. 17. To verify this, we note from the second of equations (1) that $v = c_2$ when (x, y) is a point on either branch. Suppose that (x, y) is on the branch lying in the first quadrant. Then, since $y = c_2/(2x)$, the first of equations (1) reveals that the branch's image has parametric representation

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2 \qquad (0 < x < \infty).$$

Observe that

$$\lim_{\substack{x \to 0 \\ x > 0}} u = -\infty \quad \text{and} \quad \lim_{x \to \infty} u = \infty.$$

Since u depends continuously on x, then, it is clear that as (x, y) travels down the entire upper branch of hyperbola (3), its image moves to the right along the entire horizontal line $v = c_2$. Inasmuch as the image of the lower branch has parametric representation

$$u = \frac{c_2^2}{4y^2} - y^2, \quad v = c_2 \qquad (-\infty < y < 0)$$

and since

$$\lim_{y \to -\infty} u = -\infty \quad \text{and} \quad \lim_{\substack{y \to 0 \\ y < 0}} u = \infty,$$

it follows that the image of a point moving *upward* along the entire lower branch also travels to the right along the entire line $v = c_2$ (see Fig. 17).

We shall now use Example 1 to find the image of a certain *region.*

EXAMPLE 2. The domain $x > 0, y > 0, xy < 1$ consists of all points lying on the upper branches of hyperbolas from the family $2xy = c$, where $0 < c < 2$ (Fig. 18). We know from Example 1 that as a point travels downward along the entirety of such a branch, its image under the transformation $w = z^2$ moves to the right along the entire line $v = c$. Since, for all values of c between 0 and 2, these upper branches fill out the domain $x > 0, y > 0, xy < 1$, that domain is mapped onto the horizontal strip $0 < v < 2$.

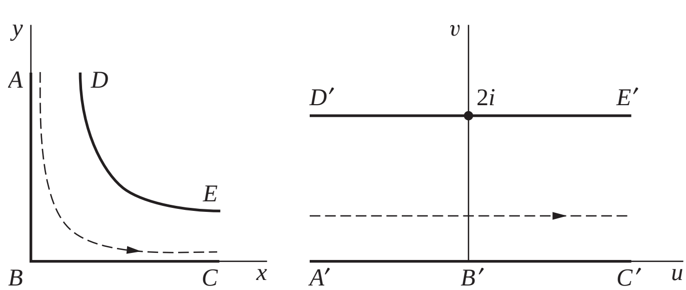

FIGURE 18
$w = z^2$.

In view of equations (1), the image of a point $(0, y)$ in the z plane is $(-y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, then, the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is mapped onto the closed strip $0 \leq v \leq 2$, as indicated in Fig. 18.

Our last example here illustrates how polar coordinates can be useful in analyzing certain mappings.

EXAMPLE 3. The mapping $w = z^2$ becomes

$$w = r^2 e^{i2\theta} \tag{4}$$

when $z = re^{i\theta}$. Evidently, then, the image $w = \rho e^{i\phi}$ of any nonzero point z is found by squaring the modulus $r = |z|$ and doubling the value θ of $\arg z$ that is used:

$$\rho = r^2 \quad \text{and} \quad \phi = 2\theta. \tag{5}$$

Observe that points $z = r_0 e^{i\theta}$ on a circle $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis (see Fig. 19). So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z and w planes fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one mapping of the first quadrant $r \geq 0$, $0 \leq \theta \leq \pi/2$ in the z plane onto the upper half $\rho \geq 0, 0 \leq \phi \leq \pi$ of the w plane, as indicated in Fig. 19. The point $z = 0$ is, of course, mapped onto the point $w = 0$.

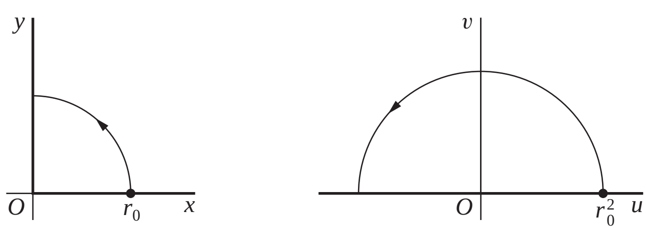

FIGURE 19
$w = z^2$.

The transformation $w = z^2$ also maps the upper half plane $r \geq 0, 0 \leq \theta \leq \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane.

When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $w = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \le r_0, 0 \le \theta \le 2\pi/n$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.

Other, but somewhat more involved, mappings by $w = z^2$ appear in Example 1, Sec. 97, and Exercises 1 through 4 of that section.

14. MAPPINGS BY THE EXPONENTIAL FUNCTION

In Chap. 3 we shall introduce and develop properties of a number of elementary functions which do not involve polynomials. That chapter will start with the exponential function

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

the two factors e^x and e^{iy} being well defined at this time (see Sec. 6). Note that definition (1), which can also be written

$$e^{x+iy} = e^x e^{iy},$$

is suggested by the familiar additive property

$$e^{x_1+x_2} = e^{x_1} e^{x_2}$$

of the exponential function in calculus.

The object of this section is to use the function e^z to provide the reader with additional examples of mappings that continue to be reasonably simple. We begin by examining the images of vertical and horizontal lines.

EXAMPLE 1. The transformation

$$w = e^z \tag{2}$$

can be written $w = e^x e^{iy}$, where $z = x + iy$, according to equation (1). Thus, if $w = \rho e^{i\phi}$, transformation (2) can be expressed in the form

$$\rho = e^x, \quad \phi = y. \tag{3}$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\phi = y$ in the w plane. That image moves counterclockwise around the circle shown in Fig. 20 as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

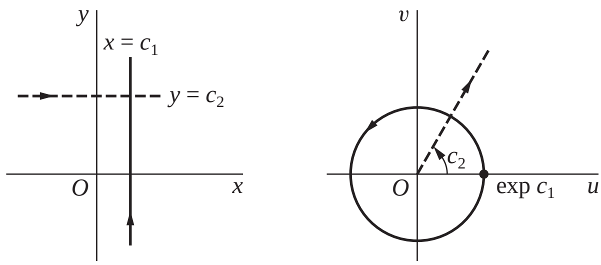

FIGURE 20
$w = \exp z$.

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\phi = c_2$. To see that this is so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\phi = c_2$. Consequently, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\phi = c_2$, as indicated in Fig. 20.

Vertical and horizontal line *segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example 1. This is illustrated in the following example.

EXAMPLE 2. Let us show that the transformation $w = e^z$ maps the rectangular region $a \leq x \leq b, c \leq y \leq d$ onto the region $e^a \leq \rho \leq e^b, c \leq \phi \leq d$. The two regions and corresponding parts of their boundaries are indicated in Fig. 21. The vertical line segment AD is mapped onto the arc $\rho = e^a, c \leq \phi \leq d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b, c \leq \phi \leq d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \leq \phi \leq \pi$; and the rectangular region is mapped onto half of a circular ring, as shown in Fig. 8, Appendix 2.

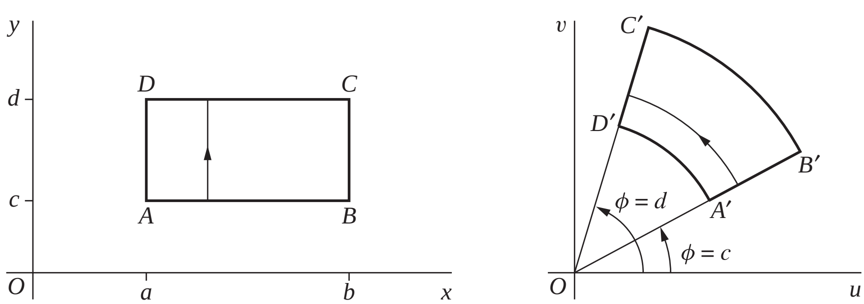

FIGURE 21
$w = \exp z$.

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

EXAMPLE 3. When $w = e^z$, the image of the infinite strip $0 \le y \le \pi$ is the upper half $v \ge 0$ of the w plane (Fig. 22). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$. This mapping is also shown in Fig. 6 of Appendix 2, where corresponding points on the boundaries of the two regions are indicated.

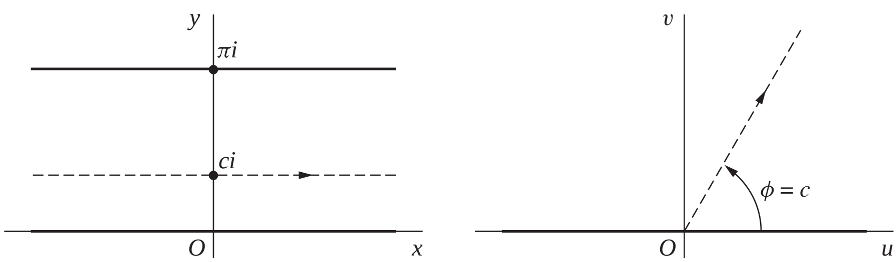

FIGURE 22
$w = \exp z$.

EXERCISES

1. By referring to Example 1 in Sec. 13, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1, u = 2, v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)
2. Find and sketch, showing corresponding orientations, the images of the hyperbolas
$$x^2 - y^2 = c_1 \ (c_1 < 0) \quad \text{and} \quad 2xy = c_2 \ (c_2 < 0)$$
under the transformation $w = z^2$.
3. Sketch the region onto which the sector $r \le 1, 0 \le \theta \le \pi/4$ is mapped by the transformation *(a)* $w = z^2$; *(b)* $w = z^3$; *(c)* $w = z^4$.
4. Show that the lines $ay = x$ $(a \ne 0)$ are mapped onto the spirals $\rho = \exp(a\phi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$.
5. By considering the images of *horizontal* line segments, verify that the image of the rectangular region $a \le x \le b, c \le y \le d$ under the transformation $w = \exp z$ is the region $e^a \le \rho \le e^b, c \le \phi \le d$, as shown in Fig. 21 (Sec. 14).
6. Verify the mapping of the region and boundary shown in Fig. 7 of Appendix 2, where the transformation is $w = \exp z$.
7. Find the image of the semi-infinite strip $x \ge 0, 0 \le y \le \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries.

8. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a *vector field* in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by *(a)* $w = iz$; *(b)* $w = z/|z|$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood (Sec. 11) of z_0. The statement that the *limit* of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 23). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

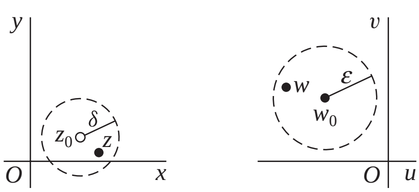

FIGURE 23

It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique.* To do this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1.$$

So if $0 < |z - z_0| < \delta$, where δ is any positive number that is smaller than δ_0 and δ_1, we find that

$$|w_1 - w_0| = |[f(z) - w_0] - [f(z) - w_1]| \le |f(z) - w_0| + |f(z) - w_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

But $|w_1 - w_0|$ is a nonnegative constant, and ε can be chosen arbitrarily small. Hence

$$w_1 - w_0 = 0, \quad \text{or} \quad w_1 = w_0.$$

Definition (2) requires that f be defined at all points in some deleted neighborhood of z_0. Such a deleted neighborhood, of course, always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that the first of inequalities (2) need be satisfied by only those points z that lie in both the region *and* the deleted neighborhood.

EXAMPLE 1. Let us show that if $f(z) = i\overline{z}/2$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}, \tag{3}$$

the point 1 being on the boundary of the domain of definition of f. Observe that when z is in the disk $|z| < 1$,

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\overline{z}}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2}.$$

Hence, for any such z and each positive number ε (see Fig. 24),

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 1| < 2\varepsilon.$$

Thus condition (2) is satisfied by points in the region $|z| < 1$ when δ is equal to 2ε or any smaller positive number.

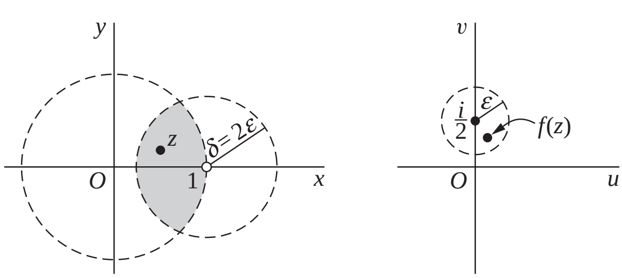

FIGURE 24

If limit (1) exists, the symbol $z \to z_0$ implies that z is allowed to approach z_0 in an arbitrary manner, not just from some particular direction. The next example emphasizes this.

EXAMPLE 2. If

$$f(z) = \frac{z}{\overline{z}}, \tag{4}$$

the limit

$$\lim_{z \to 0} f(z) \tag{5}$$

does not exist. For, if it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner. But when $z = (x, 0)$ is a nonzero point on the real axis (Fig. 25),

$$f(z) = \frac{x + i0}{x - i0} = 1;$$

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

$$f(z) = \frac{0 + iy}{0 - iy} = -1.$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit (5) does not exist.

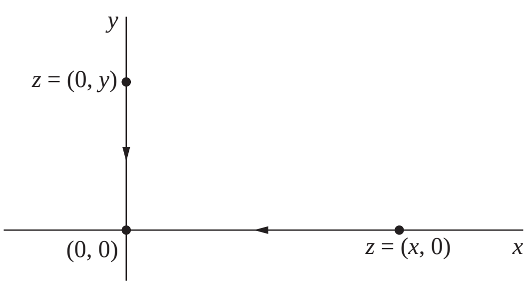

FIGURE 25

While definition (2) provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit. Theorems on limits, presented in the next section, will enable us to actually find many limits.

16. THEOREMS ON LIMITS

We can expedite our treatment of limits by establishing a connection between limits of functions of a complex variable and limits of real-valued functions of two real variables. Since limits of the latter type are studied in calculus, we use their definition and properties freely.

Theorem 1. *Suppose that*

$$f(z) = u(x, y) + iv(x, y) \quad (z = x + iy)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0.$$

Then

$$\lim_{z \to z_0} f(z) = w_0 \tag{1}$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0. \tag{2}$$

To prove the theorem, we first assume that limits (2) hold and obtain limit (1). Limits (2) tell us that for each positive number ε, there exist positive numbers δ_1 and δ_2 such that

$$|u - u_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \tag{3}$$

and

$$|v - v_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2. \tag{4}$$

Let δ be any positive number smaller than δ_1 and δ_2. Since

$$|(u + iv) - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| \le |u - u_0| + |v - v_0|$$

and

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)| = |(x + iy) - (x_0 + iy_0)|,$$

it follows from statements (3) and (4) that

$$|(u + iv) - (u_0 + iv_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta.$$

That is, limit (1) holds.

Let us now start with the assumption that limit (1) holds. With that assumption, we know that for each positive number ε, there is a positive number δ such that

$$|(u+iv)-(u_0+iv_0)|<\varepsilon \tag{5}$$

whenever

$$0<|(x+iy)-(x_0+iy_0)|<\delta. \tag{6}$$

But

$$|u-u_0| \le |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|,$$
$$|v-v_0| \le |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|,$$

and

$$|(x+iy)-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Hence it follows from inequalities (5) and (6) that

$$|u-u_0|<\varepsilon \quad \text{and} \quad |v-v_0|<\varepsilon$$

whenever

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta.$$

This establishes limits (2), and the proof of the theorem is complete.

Theorem 2. *Suppose that*

$$\lim_{z\to z_0} f(z) = w_0 \quad \textit{and} \quad \lim_{z\to z_0} F(z) = W_0. \tag{7}$$

Then

$$\lim_{z\to z_0}[f(z)+F(z)] = w_0+W_0, \tag{8}$$

$$\lim_{z\to z_0}[f(z)F(z)] = w_0W_0\,; \tag{9}$$

and, if $W_0 \neq 0$,

$$\lim_{z\to z_0}\frac{f(z)}{F(z)} = \frac{w_0}{W_0}. \tag{10}$$

This important theorem can be proved directly by using the definition of the limit of a function of a complex variable. But, with the aid of Theorem 1, it follows almost immediately from theorems on limits of real-valued functions of two real variables.

To verify property (9), for example, we write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y),$$

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0, \quad W_0 = U_0 + iV_0.$$

Then, according to hypotheses (7) and Theorem 1, the limits as (x, y) approaches (x_0, y_0) of the functions u, v, U, and V exist and have the values u_0, v_0, U_0, and V_0, respectively. So the real and imaginary components of the product

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$

have the limits $u_0U_0 - v_0V_0$ and $v_0U_0 + u_0V_0$, respectively, as (x, y) approaches (x_0, y_0). Hence, by Theorem 1 again, $f(z)F(z)$ has the limit

$$(u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0)$$

as z approaches z_0; and this is equal to w_0W_0. Property (9) is thus established. Corresponding verifications of properties (8) and (10) can be given.

It is easy to see from definition (2), Sec. 15, of limit that

$$\lim_{z \to z_0} c = c \quad \text{and} \quad \lim_{z \to z_0} z = z_0,$$

where z_0 and c are any complex numbers; and, by property (9) and mathematical induction, it follows that

$$\lim_{z \to z_0} z^n = z_0^n \qquad (n = 1, 2, \ldots).$$

So, in view of properties (8) and (9), the limit of a polynomial

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$

as z approaches a point z_0 is the value of the polynomial at that point:

$$\lim_{z \to z_0} P(z) = P(z_0). \tag{11}$$

17. LIMITS INVOLVING THE POINT AT INFINITY

It is sometimes convenient to include with the complex plane the *point at infinity,* denoted by ∞, and to use limits involving it. The complex plane together with this point is called the *extended* complex plane. To visualize the point at infinity, one can think of the complex plane as passing through the equator of a unit sphere centered at the origin (Fig. 26). To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is the point where the line through z and the north pole N intersects the sphere. In like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one

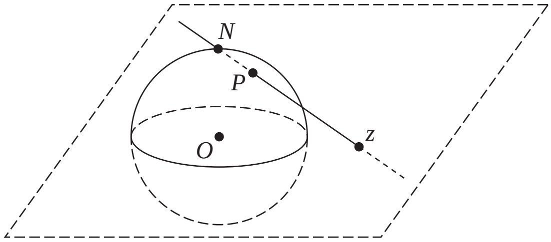

FIGURE 26

point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a one to one correspondence between the points of the sphere and the points of the extended complex plane. The sphere is known as the *Riemann sphere*, and the correspondence is called a *stereographic projection.*

Observe that the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted. Moreover, for each small positive number ε, those points in the complex plane exterior to the circle $|z| = 1/\varepsilon$ correspond to points on the sphere close to N. We thus call the set $|z| > 1/\varepsilon$ an ε *neighborhood,* or neighborhood, of ∞.

Let us agree that in referring to a point z, we mean a point in the *finite* plane. Hereafter, when the point at infinity is to be considered, it will be specifically mentioned.

A meaning is now readily given to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when either z_0 or w_0, or possibly each of these numbers, is replaced by the point at infinity. In the definition of limit in Sec. 15, we simply replace the appropriate neighborhoods of z_0 and w_0 by neighborhoods of ∞. The proof of the following theorem illustrates how this is done.

Theorem. *If z_0 and w_0 are points in the z and w planes, respectively, then*

(1) $$\lim_{z \to z_0} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

and

(2) $$\lim_{z \to \infty} f(z) = w_0 \qquad \textit{if and only if} \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0.$$

Moreover,

(3) $$\lim_{z \to \infty} f(z) = \infty \qquad \textit{if and only if} \quad \lim_{z \to 0} \frac{1}{f(1/z)} = 0.$$

We start the proof by noting that the first of limits (1) means that for each positive number ε, there is a positive number δ such that

$$(4) \qquad |f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

That is, the point $w = f(z)$ lies in the ε neighborhood $|w| > 1/\varepsilon$ of ∞ whenever z lies in the deleted neighborhood $0 < |z - z_0| < \delta$ of z_0. Since statement (4) can be written

$$\left|\frac{1}{f(z)} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta,$$

the second of limits (1) follows.

The first of limits (2) means that for each positive number ε, a positive number δ exists such that

$$(5) \qquad |f(z) - w_0| < \varepsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}.$$

Replacing z by $1/z$ in statement (5) and then writing the result as

$$\left|f\left(\frac{1}{z}\right) - w_0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta,$$

we arrive at the second of limits (2).

Finally, the first of limits (3) is to be interpreted as saying that for each positive number ε, there is a positive number δ such that

$$(6) \qquad |f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad |z| > \frac{1}{\delta}.$$

When z is replaced by $1/z$, this statement can be put in the form

$$\left|\frac{1}{f(1/z)} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta;$$

and this gives us the second of limits (3).

EXAMPLES. Observe that

$$\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty \quad \text{since} \quad \lim_{z \to -1} \frac{z + 1}{iz + 3} = 0$$

and

$$\lim_{z \to \infty} \frac{2z + i}{z + 1} = 2 \quad \text{since} \quad \lim_{z \to 0} \frac{(2/z) + i}{(1/z) + 1} = \lim_{z \to 0} \frac{2 + iz}{1 + z} = 2.$$

Furthermore,

$$\lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty \quad \text{since} \quad \lim_{z \to 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} = \lim_{z \to 0} \frac{z + z^3}{2 - z^3} = 0.$$

18. CONTINUITY

A function f is *continuous* at a point z_0 if all three of the following conditions are satisfied:

(1) $$\lim_{z \to z_0} f(z) \text{ exists,}$$

(2) $$f(z_0) \text{ exists,}$$

(3) $$\lim_{z \to z_0} f(z) = f(z_0).$$

Observe that statement (3) actually contains statements (1) and (2), since the existence of the quantity on each side of the equation there is needed. Statement (3) says, of course, that for each positive number ε, there is a positive number δ such that

(4) $$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R.

If two functions are continuous at a point, their sum and product are also continuous at that point; their quotient is continuous at any such point if the denominator is not zero there. These observations are direct consequences of Theorem 2, Sec. 16. Note, too, that a polynomial is continuous in the entire plane because of limit (11) in Sec. 16.

We turn now to two expected properties of continuous functions whose verifications are not so immediate. Our proofs depend on definition (4) of continuity, and we present the results as theorems.

Theorem 1. *A composition of continuous functions is itself continuous.*

A precise statement of this theorem is contained in the proof to follow. We let $w = f(z)$ be a function that is defined for all z in a neighborhood $|z - z_0| < \delta$ of a point z_0, and we let $W = g(w)$ be a function whose domain of definition contains the image (Sec. 13) of that neighborhood under f. The composition $W = g[f(z)]$ is, then, defined for all z in the neighborhood $|z - z_0| < \delta$. Suppose now that f is continuous at z_0 and that g is continuous at the point $f(z_0)$ in the w plane. In view of the continuity of g at $f(z_0)$, there is, for each positive number ε, a positive number γ such that

$$|g[f(z)] - g[f(z_0)]| < \varepsilon \quad \text{whenever} \quad |f(z) - f(z_0)| < \gamma.$$

(See Fig. 27.) But the continuity of f at z_0 ensures that the neighborhood $|z - z_0| < \delta$ can be made small enough that the second of these inequalities holds. The continuity of the composition $g[f(z)]$ is, therefore, established.

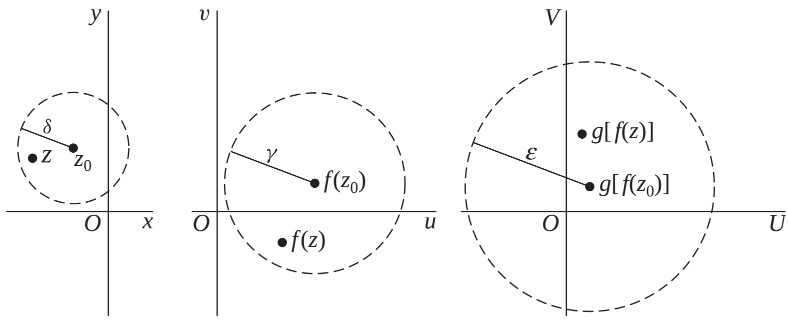

FIGURE 27

Theorem 2. *If a function* $f(z)$ *is continuous and nonzero at a point* z_0, *then* $f(z) \neq 0$ *throughout some neighborhood of that point.*

Assuming that $f(z)$ is, in fact, continuous and nonzero at z_0, we can prove Theorem 2 by assigning the positive value $|f(z_0)|/2$ to the number ε in statement (4). This tells us that there is a positive number δ such that

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever} \quad |z - z_0| < \delta.$$

So if there is a point z in the neighborhood $|z - z_0| < \delta$ at which $f(z) = 0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2};$$

and the theorem is proved.

The continuity of a function

$$f(z) = u(x, y) + iv(x, y) \tag{5}$$

is closely related to the continuity of its component functions $u(x, y)$ and $v(x, y)$. We note, for instance, how it follows from Theorem 1 in Sec. 16 that *the function* (5) *is continuous at a point* $z_0 = (x_0, y_0)$ *if and only if its component functions are continuous there.* Our proof of the next theorem illustrates the use of this statement. The theorem is extremely important and will be used often in later chapters, especially in applications. Before stating the theorem, we recall from Sec. 11 that a region R is *closed* if it contains all of its boundary points and that it is *bounded* if it lies inside some circle centered at the origin.

Theorem 3. *If a function* f *is continuous throughout a region* R *that is both closed and bounded, there exists a nonnegative real number* M *such that*

$$|f(z)| \leq M \qquad \textit{for all points } z \textit{ in } R, \tag{6}$$

where equality holds for at least one such z.

To prove this, we assume that the function f in equation (5) is continuous and note how it follows that the function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is continuous throughout R and thus reaches a maximum value M somewhere in R.* Inequality (6) thus holds, and we say that *f is bounded on R.*

EXERCISES

1. Use definition (2), Sec. 15, of limit to prove that

(a) $\lim_{z\to z_0} \operatorname{Re} z = \operatorname{Re} z_0\,$; *(b)* $\lim_{z\to z_0} \overline{z} = \overline{z_0}\,$; *(c)* $\lim_{z\to 0} \dfrac{\overline{z}^2}{z} = 0$.

2. Let a, b, and c denote complex constants. Then use definition (2), Sec. 15, of limit to show that

(a) $\lim_{z\to z_0} (az + b) = az_0 + b$; *(b)* $\lim_{z\to z_0} (z^2 + c) = z_0^2 + c$;

(c) $\lim_{z\to 1-i} [x + i(2x + y)] = 1 + i \quad (z = x + iy)$.

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(a) $\lim_{z\to z_0} \dfrac{1}{z^n} \ (z_0 \neq 0)$; *(b)* $\lim_{z\to i} \dfrac{iz^3 - 1}{z + i}$; *(c)* $\lim_{z\to z_0} \dfrac{P(z)}{Q(z)}$.

Ans. (a) $1/z_0^n$; *(b)* 0; *(c)* $P(z_0)/Q(z_0)$.

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z\to z_0} z^n = z_0^n$$

when n is a positive integer $(n = 1, 2, \ldots)$.

5. Show that the limit of the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

6. Prove statement (8) in Theorem 2 of Sec. 16 using

(a) Theorem 1 in Sec. 16 and properties of limits of real-valued functions of two real variables;

(b) definition (2), Sec. 15, of limit.

*See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 125–126 and p. 529, 1983.

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z \to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write

$$||f(z)| - |w_0|| \leq |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z \to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z \to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \leq M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 17 to show that

(a) $\displaystyle\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4;$ *(b)* $\displaystyle\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty;$ *(c)* $\displaystyle\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty.$

11. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0),$$

(a) $\displaystyle\lim_{z \to \infty} T(z) = \infty \quad \text{if } c = 0;$

(b) $\displaystyle\lim_{z \to \infty} T(z) = \frac{a}{c} \text{ and } \lim_{z \to -d/c} T(z) = \infty \quad \text{if } c \neq 0.$

12. State why limits involving the point at infinity are unique.

13. Show that a set S is unbounded (Sec. 11) if and only if every neighborhood of the point at infinity contains at least one point in S.

19. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be *differentiable* at z_0 when $f'(z_0)$ exists.

By expressing the variable z in definition (1) in terms of the new complex variable

$$\Delta z = z - z_0 \quad (z \neq z_0),$$

one can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2}$$

Because f is defined throughout a neighborhood of z_0, the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small (Fig. 28).

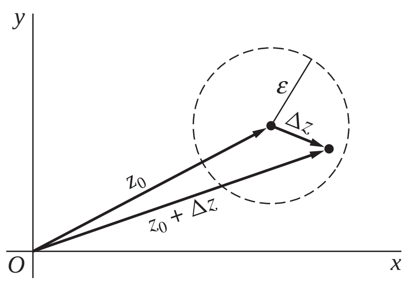

FIGURE 28

When taking form (2) of the definition of derivative, we often drop the subscript on z_0 and introduce the number

$$\Delta w = f(z + \Delta z) - f(z),$$

which denotes the change in the value $w = f(z)$ of f corresponding to a change Δz in the point at which f is evaluated. Then, if we write dw/dz for $f'(z)$, equation (2) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{3}$$

EXAMPLE 1. Suppose that $f(z) = z^2$. At any point z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

since $2z + \Delta z$ is a polynomial in Δz. Hence $dw/dz = 2z$, or $f'(z) = 2z$.

EXAMPLE 2. If $f(z) = \overline{z}$, then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}. \tag{4}$$

If the limit of $\Delta w/\Delta z$ exists, it can be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approach the origin $(0,0)$ in the Δz plane in any manner. In particular, as Δz approaches $(0,0)$ horizontally through the points $(\Delta x, 0)$ on the real axis (Fig. 29),

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z.$$

In that case, expression (4) tells us that

$$\frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = 1.$$

Hence if the limit of $\Delta w/\Delta z$ exists, its value must be unity. However, when Δz approaches $(0,0)$ vertically through the points $(0, \Delta y)$ on the imaginary axis, so that

$$\overline{\Delta z} = \overline{0 + i\Delta y} = 0 - i\Delta y = -(0 + i\Delta y) = -\Delta z,$$

we find from expression (4) that

$$\frac{\Delta w}{\Delta z} = \frac{-\Delta z}{\Delta z} = -1.$$

Hence the limit must be -1 if it exists. Since limits are unique (Sec. 15), it follows that dw/dz does not exist anywhere.

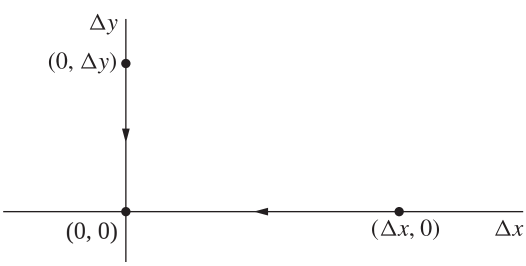

FIGURE 29

EXAMPLE 3. Consider the real-valued function $f(z) = |z|^2$. Here

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}. \tag{5}$$

Proceeding as in Example 2, where horizontal and vertical approaches of Δz toward the origin gave us

$$\overline{\Delta z} = \Delta z \quad \text{and} \quad \overline{\Delta z} = -\Delta z,$$

respectively, we have the expressions

$$\frac{\Delta w}{\Delta z} = \overline{z} + \overline{\Delta z} + z \quad \text{when} \quad \Delta z = (\Delta x, 0)$$

and

$$\frac{\Delta w}{\Delta z} = \overline{z} - \overline{\Delta z} - z \quad \text{when} \quad \Delta z = (0, \Delta y).$$

Hence if the limit of $\Delta w/\Delta z$ exists as Δz tends to zero, the uniqueness of limits, used in Example 2, tells us that

$$\overline{z} + z = \overline{z} - z,$$

or $z = 0$. Evidently, then dw/dz cannot exist when $z \neq 0$.

To show that dw/dz does, in fact, exist at $z = 0$, we need only observe that expression (5) reduces to

$$\frac{\Delta w}{\Delta z} = \overline{\Delta z}$$

when $z = 0$. We conclude, therefore, that dw/dz exists *only* at $z = 0$, its value there being 0.

Example 3 shows that a function $f(z) = u(x, y) + iv(x, y)$ can be differentiable at a point $z = (x, y)$ but nowhere else in any neighborhood of that point. Since

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0 \tag{6}$$

when $f(z) = |z|^2$, it also shows that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point $z = (x, y)$ and yet the function may not be differentiable there.

The function $f(z) = |z|^2$ is continuous at each point in the plane since its components (6) are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there. It is, however, true that *the existence of the derivative of a function at a point implies the continuity of the function at that point.* To see this, we assume that $f'(z_0)$ exists and write

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0,$$

from which it follows that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This is the statement of continuity of f at z_0 (Sec. 18).

Geometric interpretations of derivatives of functions of a complex variable are not as immediate as they are for derivatives of functions of a real variable. We defer the development of such interpretations until Chap. 9.

20. DIFFERENTIATION FORMULAS

The definition of derivative in Sec. 19 is identical in form to that of the derivative of a real-valued function of a real variable. In fact, the basic differentiation formulas given below can be derived from the definition in Sec. 19 by essentially the same steps as the ones used in calculus. In these formulas, the derivative of a function f at a point z is denoted by either

$$\frac{d}{dz} f(z) \quad \text{or} \quad f'(z),$$

depending on which notation is more convenient.

Let c be a complex constant, and let f be a function whose derivative exists at a point z. It is easy to show that

$$\frac{d}{dz}c = 0, \quad \frac{d}{dz}z = 1, \quad \frac{d}{dz}[cf(z)] = cf'(z). \tag{1}$$

Also, if n is a positive integer,

$$\frac{d}{dz}z^n = nz^{n-1}. \tag{2}$$

This formula remains valid when n is a negative integer, provided that $z \neq 0$.

If the derivatives of two functions f and g exist at a point z, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z), \tag{3}$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)\,; \tag{4}$$

and, when $g(z) \neq 0$,

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}. \tag{5}$$

Let us derive formula (4). To do this, we write the following expression for the change in the product $w = f(z)g(z)$:

$$\begin{aligned} \Delta w &= f(z + \Delta z)g(z + \Delta z) - f(z)g(z) \\ &= f(z)[g(z + \Delta z) - g(z)] + [f(z + \Delta z) - f(z)]g(z + \Delta z). \end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\,\frac{g(z + \Delta z) - g(z)}{\Delta z} + \frac{f(z + \Delta z) - f(z)}{\Delta z}\,g(z + \Delta z)\,;$$

and, letting Δz tend to zero, we arrive at the desired formula for the derivative of $f(z)g(z)$. Here we have used the fact that g is continuous at the point z, since

$g'(z)$ exists; thus $g(z + \Delta z)$ tends to $g(z)$ as Δz tends to zero (see Exercise 8, Sec. 18).

There is also a chain rule for differentiating composite functions. Suppose that f has a derivative at z_0 and that g has a derivative at the point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]f'(z_0). \tag{6}$$

If we write $w = f(z)$ and $W = g(w)$, so that $W = F(z)$, the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}.$$

EXAMPLE. To find the derivative of $(2z^2 + i)^5$, write $w = 2z^2 + i$ and $W = w^5$. Then

$$\frac{d}{dz}(2z^2 + i)^5 = 5w^4 4z = 20z(2z^2 + i)^4.$$

To start the derivation of formula (6), choose a specific point z_0 at which $f'(z_0)$ exists. Write $w_0 = f(z_0)$ and also assume that $g'(w_0)$ exists. There is, then, some ε neighborhood $|w - w_0| < \varepsilon$ of w_0 such that for all points w in that neighborhood, we can define a function Φ having the values $\Phi(w_0) = 0$ and

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad \text{when} \quad w \neq w_0. \tag{7}$$

Note that in view of the definition of derivative,

$$\lim_{w \to w_0} \Phi(w) = 0. \tag{8}$$

Hence Φ is continuous at w_0.

Now expression (7) can be put in the form

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \qquad (|w - w_0| < \varepsilon), \tag{9}$$

which is valid even when $w = w_0$; and since $f'(z_0)$ exists and f is therefore continuous at z_0, we can choose a positive number δ such that the point $f(z)$ lies in the ε neighborhood $|w - w_0| < \varepsilon$ of w_0 if z lies in the δ neighborhood $|z - z_0| < \delta$ of z_0. Thus it is legitimate to replace the variable w in equation (9) by $f(z)$ when z is any point in the neighborhood $|z - z_0| < \delta$. With that substitution, and with $w_0 = f(z_0)$, equation (9) becomes

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\}\frac{f(z) - f(z_0)}{z - z_0} \tag{10}$$

$$(0 < |z - z_0| < \delta),$$

where we must stipulate that $z \neq z_0$ so that we are not dividing by zero. As already noted, f is continuous at z_0 and Φ is continuous at the point $w_0 = f(z_0)$. Hence the composition $\Phi[f(z)]$ is continuous at z_0; and since $\Phi(w_0) = 0$,

$$\lim_{z \to z_0} \Phi[f(z)] = 0.$$

So equation (10) becomes equation (6) in the limit as z approaches z_0.

EXERCISES

1. Use results in Sec. 20 to find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$; *(b)* $f(z) = (1 - 4z^2)^3$;

(c) $f(z) = \dfrac{z - 1}{2z + 1} \quad (z \neq -1/2)$; *(d)* $f(z) = \dfrac{(1 + z^2)^4}{z^2} \quad (z \neq 0)$.

2. Using results in Sec. 20, show that

(a) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$ is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1};$$

(b) the coefficients in the polynomial $P(z)$ in part *(a)* can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

3. Apply definition (3), Sec. 19, of derivative to give a direct proof that

$$\frac{dw}{dz} = -\frac{1}{z^2} \quad \text{when} \quad w = \frac{1}{z} \qquad (z \neq 0).$$

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

5. Derive formula (3), Sec. 20, for the derivative of the sum of two functions.

6. Derive expression (2), Sec. 20, for the derivative of z^n when n is a positive integer by using

(a) mathematical induction and formula (4), Sec. 20, for the derivative of the product of two functions;

(b) definition (3), Sec. 19, of derivative and the binomial formula (Sec. 3).

7. Prove that expression (2), Sec. 20, for the derivative of z^n remains valid when n is a negative integer $(n = -1, -2, \ldots)$, provided that $z \neq 0$.

Suggestion: Write $m = -n$ and use the formula for the derivative of a quotient of two functions.

8. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when

(a) $f(z) = \operatorname{Re} z$; *(b)* $f(z) = \operatorname{Im} z$.

9. Let f denote the function whose values are

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x\ \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Example 2, Sec. 19.)

21. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

We start by writing

$$z_0 = x_0 + iy_0, \qquad \Delta z = \Delta x + i\Delta y,$$

and

$$\begin{aligned} \Delta w &= f(z_0 + \Delta z) - f(z_0) \\ &= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]. \end{aligned}$$

Assuming that the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{2}$$

exists, we know from Theorem 1 in Sec. 16 that

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Re} \frac{\Delta w}{\Delta z} \right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\operatorname{Im} \frac{\Delta w}{\Delta z} \right). \tag{3}$$

Now it is important to keep in mind that expression (3) is valid as $(\Delta x, \Delta y)$ tends to $(0, 0)$ in any manner that we may choose. In particular, we let $(\Delta x, \Delta y)$ tend to $(0, 0)$ horizontally through the points $(\Delta x, 0)$, as indicated in Fig. 29 (Sec. 19). Inasmuch as $\Delta y = 0$, the quotient $\Delta w/\Delta z$ becomes

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\,\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

Thus

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left(\operatorname{Re}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left(\operatorname{Im}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = v_x(x_0, y_0),$$

where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the first-order partial derivatives with respect to x of the functions u and v, respectively, at (x_0, y_0). Substitution of these limits into expression (3) tells us that

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0). \tag{4}$$

We might have let Δz tend to zero vertically through the points $(0, \Delta y)$. In that case, $\Delta x = 0$ and

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}\\ &= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\,\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}.\end{aligned}$$

Evidently, then,

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left(\operatorname{Re}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} = v_y(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left(\operatorname{Im}\frac{\Delta w}{\Delta z}\right) = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} = -u_y(x_0, y_0).$$

Hence it follows from expression (3) that

$$f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0), \tag{5}$$

where the partial derivatives of u and v are, this time, with respect to y. Note that equation (5) can also be written in the form

$$f'(z_0) = -i[u_y(x_0, y_0) + i v_y(x_0, y_0)].$$

Equations (4) and (5) not only give $f'(z_0)$ in terms of partial derivatives of the component functions u and v, but they also provide necessary conditions for the existence of $f'(z_0)$. To obtain those conditions, we need only equate the real parts and then the imaginary parts on the right-hand sides of equations (4) and (5) to see that the existence of $f'(z_0)$ requires that

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{6}$$

Equations (6) are the *Cauchy–Riemann equations*, so named in honor of the French mathematician A. L. Cauchy (1789–1857), who discovered and used them, and in honor of the German mathematician G. F. B. Riemann (1826–1866), who made them fundamental in his development of the theory of functions of a complex variable.

We summarize the above results as follows.

Theorem. *Suppose that*

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{7}$$

there. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x, \tag{8}$$

where these partial derivatives are to be evaluated at (x_0, y_0).

EXAMPLE 1. In Example 1, Sec. 19, we showed that the function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

is differentiable everywhere and that $f'(z) = 2z$. To verify that the Cauchy–Riemann equations are satisfied everywhere, write

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Thus

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x.$$

Moreover, according to equation (8),

$$f'(z) = 2x + i2y = 2(x + iy) = 2z.$$

Since the Cauchy–Riemann equations are necessary conditions for the existence of the derivative of a function f at a point z_0, they can often be used to locate points at which f does *not* have a derivative.

EXAMPLE 2. When $f(z) = |z|^2$, we have

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0.$$

If the Cauchy–Riemann equations are to hold at a point (x, y), it follows that $2x = 0$ and $2y = 0$, or that $x = y = 0$. Consequently, $f'(z)$ does not exist at any nonzero point, as we already know from Example 3 in Sec. 19. Note that the theorem just proved does not ensure the existence of $f'(0)$. The theorem in the next section will, however, do this.

22. SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

Satisfaction of the Cauchy–Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. (See Exercise 6, Sec. 23.) But, with certain continuity conditions, we have the following useful theorem.

Theorem. *Let the function*

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some ε *neighborhood of a point* $z_0 = x_0 + iy_0$, *and suppose that*

(a) the first-order partial derivatives of the functions u *and* v *with respect to* x *and* y *exist everywhere in the neighborhood;*

(b) those partial derivatives are continuous at (x_0, y_0) *and satisfy the Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0).

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = u_x + iv_x$$

where the right-hand side is to be evaluated at (x_0, y_0).

To prove the theorem, we assume that conditions *(a)* and *(b)* in its hypothesis are satisfied and write $\Delta z = \Delta x + i\Delta y$, where $0 < |\Delta z| < \varepsilon$, as well as

$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$

Thus

$$\Delta w = \Delta u + i\Delta v, \tag{1}$$

where

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

and

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

The assumption that the first-order partial derivatives of u and v are continuous at the point (x_0, y_0) enables us to write*

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \tag{2}$$

and

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y, \tag{3}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 tend to zero as $(\Delta x, \Delta y)$ approaches (0, 0) in the Δz plane. Substitution of expressions (2) and (3) into equation (1) now tells us that

$$\begin{aligned}\Delta w = {} & u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\ & + i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y].\end{aligned} \tag{4}$$

Because the Cauchy–Riemann equations are assumed to be satisfied at (x_0, y_0), one can replace $u_y(x_0, y_0)$ by $-v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in equation (4) and then divide through by the quantity $\Delta z = \Delta x + i\Delta y$ to get

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}. \tag{5}$$

But $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$, according to inequalities (3) in Sec. 4, and so

$$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \le 1.$$

Consequently,

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_1 + i\varepsilon_3| \le |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right| \le |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_2| + |\varepsilon_4|;$$

*See, for instance, W. Kaplan, "Advanced Calculus," 5th ed., pp. 86ff, 2003.

and this means that the last two terms on the right in equation (5) tend to zero as the variable $\Delta z = \Delta x + i\Delta y$ approaches zero. The expression for $f'(z_0)$ in the statement of the theorem is now established.

EXAMPLE 1. Consider the exponential function

$$f(z) = e^z = e^x e^{iy} \quad (z = x + iy),$$

some of whose mapping properties were discussed in Sec. 14. In view of Euler's formula (Sec. 6), this function can, of course, be written

$$f(z) = e^x \cos y + ie^x \sin y,$$

where y is to be taken in radians when $\cos y$ and $\sin y$ are evaluated. Then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere and since these derivatives are everywhere continuous, the conditions in the above theorem are satisfied at all points in the complex plane. Thus $f'(z)$ exists everywhere, and

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y.$$

Note that $f'(z) = f(z)$ for all z.

EXAMPLE 2. It also follows from our theorem that the function $f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0,$$

has a derivative at $z = 0$. In fact, $f'(0) = 0 + i0 = 0$. We saw in Example 2, Sec. 21, that this function *cannot* have a derivative at any nonzero point since the Cauchy–Riemann equations are not satisfied at such points. (See also Example 3, Sec. 19.)

23. POLAR COORDINATES

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

to restate the theorem in Sec. 22 in polar coordinates.

Depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta} \qquad (z \neq 0)$$

when $w = f(z)$, the real and imaginary components of $w = u + iv$ are expressed in terms of either the variables x and y or r and θ. Suppose that the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighborhood of a given nonzero point z_0 and are continuous at z_0. The first-order partial derivatives of u and v with respect to r and θ also have those properties, and the chain rule for differentiating real-valued functions of two real variables can be used to write them in terms of the ones with respect to x and y. More precisely, since

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$

one can write

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x\, r\sin\theta + u_y\, r\cos\theta. \tag{2}$$

Likewise,

$$v_r = v_x \cos\theta + v_y \sin\theta, \quad v_\theta = -v_x\, r\sin\theta + v_y\, r\cos\theta. \tag{3}$$

If the partial derivatives of u and v with respect to x and y also satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{4}$$

at z_0, equations (3) become

$$v_r = -u_y \cos\theta + u_x \sin\theta, \quad v_\theta = u_y\, r\sin\theta + u_x\, r\cos\theta \tag{5}$$

at that point. It is then clear from equations (2) and (5) that

$$ru_r = v_\theta, \quad u_\theta = -rv_r \tag{6}$$

at z_0.

If, on the other hand, equations (6) are known to hold at z_0, it is straightforward to show (Exercise 7) that equations (4) must hold there. Equations (6) are, therefore, an alternative form of the Cauchy–Riemann equations (4).

In view of equations (6) and the expression for $f'(z_0)$ that is found in Exercise 8, we are now able to restate the theorem in Sec. 22 using r and θ.

Theorem. *Let the function*

$$f(z) = u(r, \theta) + iv(r, \theta)$$

be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at (r_0, θ_0) *and satisfy the polar form*

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations at (r_0, θ_0).

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where the right-hand side is to be evaluated at (r_0, θ_0).

EXAMPLE 1. Consider the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) \quad (z \neq 0).$$

Since

$$u(r,\theta) = \frac{\cos\theta}{r} \quad \text{and} \quad v(r,\theta) = -\frac{\sin\theta}{r},$$

the conditions in this theorem are satisfied at every nonzero point $z = re^{i\theta}$ in the plane. In particular, the Cauchy–Riemann equations

$$ru_r = -\frac{\cos\theta}{r} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sin\theta}{r} = -rv_r$$

are satisfied. Hence the derivative of f exists when $z \neq 0$; and, according to the theorem,

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -e^{-i\theta}\frac{e^{-i\theta}}{r^2} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}.$$

EXAMPLE 2. The theorem can be used to show that when α is a fixed real number, the function

$$f(z) = \sqrt[3]{r}e^{i\theta/3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

has a derivative everywhere in its domain of definition. Here

$$u(r,\theta) = \sqrt[3]{r}\cos\frac{\theta}{3} \quad \text{and} \quad v(r,\theta) = \sqrt[3]{r}\sin\frac{\theta}{3}.$$

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$$ru_r = \frac{\sqrt[3]{r}}{3}\cos\frac{\theta}{3} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt[3]{r}}{3}\sin\frac{\theta}{3} = -rv_r$$

and since the other conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. The theorem tells us, moreover, that

$$f'(z) = e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}\cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}\sin\frac{\theta}{3}\right],$$

or

$$f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}.$$

Note that when a specific point z is taken in the domain of definition of f, the value $f(z)$ is one value of $z^{1/3}$ (see Sec. 9). Hence this last expression for $f'(z)$ can be put in the form

$$\frac{d}{dz}z^{1/3} = \frac{1}{3(z^{1/3})^2}$$

when that value is taken. Derivatives of such power functions will be elaborated on in Chap. 3 (Sec. 33).

EXERCISES

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$; *(b)* $f(z) = z - \overline{z}$;
(c) $f(z) = 2x + ixy^2$; *(d)* $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 22 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$; *(b)* $f(z) = e^{-x}e^{-iy}$;
(c) $f(z) = z^3$; *(d)* $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. (b) $f''(z) = f(z)$; *(d)* $f''(z) = -f(z)$.

3. From results obtained in Secs. 21 and 22, determine where $f'(z)$ exists and find its value when

(a) $f(z) = 1/z$; *(b)* $f(z) = x^2 + iy^2$; *(c)* $f(z) = z \operatorname{Im} z$.

Ans. (a) $f'(z) = -1/z^2$ $(z \neq 0)$; *(b)* $f'(x + ix) = 2x$; *(c)* $f'(0) = 0$.

4. Use the theorem in Sec. 23 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(a) $f(z) = 1/z^4$ $\quad (z \neq 0)$;

(b) $f(z) = \sqrt{r}e^{i\theta/2}$ $\quad (r > 0, \alpha < \theta < \alpha + 2\pi)$;

(c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $\quad (r > 0, 0 < \theta < 2\pi)$.

Ans. (b) $f'(z) = \dfrac{1}{2f(z)}$; *(c)* $f'(z) = i\dfrac{f(z)}{z}$.

5. Show that when $f(z) = x^3 + i(1 - y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

6. Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$. [Compare with Exercise 9, Sec. 20, where it is shown that $f'(0)$ nevertheless fails to exist.]

7. Solve equations (2), Sec. 23 for u_x and u_y to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that in Sec. 23 equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 23, are the Cauchy–Riemann equations in polar form.

8. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form (6), Sec. 23, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 22 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

9. *(a)* With the aid of the polar form (6), Sec. 23, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 8.

(b) Use the expression for $f'(z_0)$ in part *(a)* to show that the derivative of the function $f(z) = 1/z$ $(z \neq 0)$ in Example 1, Sec. 23, is $f'(z) = -1/z^2$.

10. *(a)* Recall (Sec. 5) that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part *(a)*, to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the *complex form* $\partial f/\partial \overline{z} = 0$ *of the Cauchy–Riemann equations.*

24. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0.* It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is *analytic in an open set* if it has a derivative everywhere in that set. If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Note that the function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.)

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a *singular point*, or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic.

A necessary, but by no means sufficient, condition for a function f to be analytic in a domain D is clearly the continuity of f throughout D. Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 22 and 23.

Other useful sufficient conditions are obtained from the differentiation formulas in Sec. 20. The derivatives of the sum and product of two functions exist wherever

*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.

the functions themselves have derivatives. Thus, *if two functions are analytic in a domain D, their sum and their product are both analytic in D*. Similarly, *their quotient is analytic in D provided the function in the denominator does not vanish at any point in D*. In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

From the chain rule for the derivative of a composite function, we find that *a composition of two analytic functions is analytic*. More precisely, suppose that a function $f(z)$ is analytic in a domain D and that the image (Sec. 13) of D under the transformation $w = f(z)$ is contained in the domain of definition of a function $g(w)$. Then the composition $g[f(z)]$ is analytic in D, with derivative

$$\frac{d}{dz}g[f(z)] = g'[f(z)]f'(z).$$

The following property of analytic functions is especially useful, in addition to being expected.

Theorem. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

We start the proof by writing $f(z) = u(x, y) + iv(x, y)$. Assuming that $f'(z) = 0$ in D, we note that $u_x + iv_x = 0$; and, in view of the Cauchy–Riemann equations, $v_y - iu_y = 0$. Consequently,

$$u_x = u_y = 0 \quad \text{and} \quad v_x = v_y = 0$$

at each point in D.

Next, we show that $u(x, y)$ is constant along any line segment L extending from a point P to a point P' and lying entirely in D. We let s denote the distance along L from the point P and let $\mathbf{U}$ denote the unit vector along L in the direction of increasing s (see Fig. 30). We know from calculus that the directional derivative du/ds can be written as the dot product

(1) $$\frac{du}{ds} = (\text{grad } u) \cdot \mathbf{U},$$

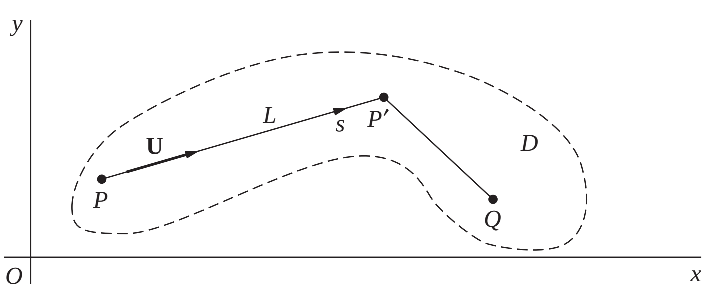

FIGURE 30

where grad u is the gradient vector

$$\text{grad}\, u = u_x \mathbf{i} + u_y \mathbf{j}. \tag{2}$$

Because u_x and u_y are zero everywhere in D, grad u is evidently the zero vector at all points on L. Hence it follows from equation (1) that the derivative du/ds is zero along L; and this means that u is constant on L.

Finally, since there is always a finite number of such line segments, joined end to end, connecting any two points P and Q in D (Sec. 11), the values of u at P and Q must be the same. We may conclude, then, that there is a real constant a such that $u(x, y) = a$ throughout D. Similarly, $v(x, y) = b$; and we find that $f(z) = a + bi$ at each point in D.

25. EXAMPLES

As pointed out in Sec. 24, it is often possible to determine where a given function is analytic by simply recalling various differentiation formulas in Sec. 20.

EXAMPLE 1. The quotient

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is evidently analytic throughout the z plane except for the singular points $z = \pm\sqrt{3}$ and $z = \pm i$. The analyticity is due to the existence of familiar differentiation formulas, which need to be applied only if the expression for $f'(z)$ is wanted.

When a function is given in terms of its component functions $u(x, y)$ and $v(x, y)$, its analyticity can be demonstrated by direct application of the Cauchy–Riemann equations.

EXAMPLE 2. If

$$f(z) = \cosh x \cos y + i \sinh x \sin y,$$

the component functions are

$$u(x, y) = \cosh x \cos y \quad \text{and} \quad v(x, y) = \sinh x \sin y.$$

Because

$$u_x = \sinh x \cos y = v_y \quad \text{and} \quad u_y = -\cosh x \sin y = -v_x$$

everywhere, it is clear from the theorem in Sec. 22 that f is entire.

Finally, we illustrate how the theorem in Sec. 24 can be used to obtain other properties of analytic functions.

EXAMPLE 3. Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are *both* analytic in a given domain D. It is now easy to show that $f(z)$ must be constant throughout D.

To do this, we write $\overline{f(z)}$ as

$$\overline{f(z)} = U(x, y) + iV(x, y)$$

where

$$U(x, y) = u(x, y) \quad \text{and} \quad V(x, y) = -v(x, y). \tag{1}$$

Because of the analyticity of $f(z)$, the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{2}$$

hold in D; and the analyticity of $\overline{f(z)}$ in D tells us that

$$U_x = V_y, \quad U_y = -V_x. \tag{3}$$

In view of relations (1), equations (3) can also be written

$$u_x = -v_y, \quad u_y = v_x. \tag{4}$$

By adding corresponding sides of the first of equations (2) and (4), we find that $u_x = 0$ in D. Similarly, subtraction involving corresponding sides of the second of equations (2) and (4) reveals that $v_x = 0$. According to expression (8) in Sec. 21, then,

$$f'(z) = u_x + iv_x = 0 + i0 = 0;$$

and it follows from the theorem in Sec. 24 that $f(z)$ is constant throughout D.

EXAMPLE 4. As in Example 3, we consider a function f that is analytic throughout a given domain D. Assuming further that the modulus $|f(z)|$ is constant throughout D, one can prove that $f(z)$ must be constant there too. This result is needed to obtain an important result later on in Chap. 4 (Sec. 54).

The proof is accomplished by writing

$$|f(z)| = c \quad \text{for all } z \text{ in } D, \tag{5}$$

where c is a real constant. If $c = 0$, it follows that $f(z) = 0$ everywhere in D. If $c \neq 0$, the fact that (see Sec. 5)

$$f(z)\overline{f(z)} = c^2$$

tells us that $f(z)$ is never zero in D. Hence

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{for all } z \text{ in } D,$$

and it follows from this that $\overline{f(z)}$ is analytic everywhere in D. The main result in Example 3 just above thus ensures that $f(z)$ is constant throughout D.

EXERCISES

1. Apply the theorem in Sec. 22 to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$; *(b)* $f(z) = \sin x \cosh y + i \cos x \sinh y$;

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; *(d)* $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$; *(b)* $f(z) = 2xy + i(x^2 - y^2)$; *(c)* $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

(a) $f(z) = \dfrac{2z+1}{z(z^2+1)}$; *(b)* $f(z) = \dfrac{z^3+i}{z^2-3z+2}$; *(c)* $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$.

Ans. (a) $z = 0, \pm i$; *(b)* $z = 1, 2$; *(c)* $z = -2, -1 \pm i$.

5. According to Exercise 4*(b)*, Sec. 23, the function

$$g(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is analytic in its domain of definition, with derivative

$$g'(z) = \frac{1}{2\,g(z)}.$$

Show that the composite function $G(z) = g(2z - 2 + i)$ is analytic in the half plane $x > 1$, with derivative

$$G'(z) = \frac{1}{g(2z - 2 + i)}.$$

Suggestion: Observe that $\mathrm{Re}(2z - 2 + i) > 0$ when $x > 1$.

6. Use results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0, y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0, y > 0$.

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughtout D.

26. HARMONIC FUNCTIONS

A real-valued function H of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as *Laplace's equation.*

Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

EXAMPLE 1. It is easy to verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi$, $y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig. 31. More precisely, it satisfies all of the conditions

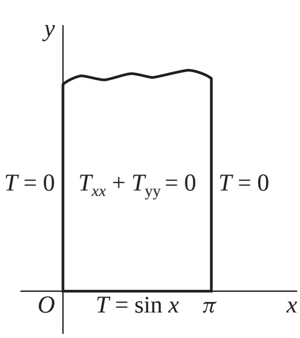

FIGURE 31

$$T_{xx}(x, y) + T_{yy}(x, y) = 0,$$

$$T(0, y) = 0, \quad T(\pi, y) = 0,$$

$$T(x, 0) = \sin x, \quad \lim_{y \to \infty} T(x, y) = 0,$$

which describe steady temperatures $T(x, y)$ in a thin homogeneous plate in the xy plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

The use of the theory of functions of a complex variable in discovering solutions, such as the one in Example 1, of temperature and other problems is described in considerable detail later on in Chap. 10 and in parts of chapters following it.* That theory is based on the theorem below, which provides a source of harmonic functions.

Theorem 1. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

To show this, we need a result that is to be proved in Chap. 4 (Sec. 52). Namely, if a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all orders at that point.

Assuming that f is analytic in D, we start with the observation that the first-order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations throughout D:

(2) $$u_x = v_y, \quad u_y = -v_x.$$

Differentiating both sides of these equations with respect to x, we have

(3) $$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}.$$

Likewise, differentiation with respect to y yields

(4) $$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}.$$

Now, by a theorem in advanced calculus,† the continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. It then follows from equations (3) and (4) that

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

That is, u and v are harmonic in D.

*Another important method is developed in the authors' "Fourier Series and Boundary Value Problems," 7th ed., 2008.

†See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 199–201, 1983.

EXAMPLE 2. The function $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire, as is shown in Exercise 1 *(c)*, Sec. 25. Hence its real component, which is the temperature function $T(x, y) = e^{-y}\sin x$ in Example 1, must be harmonic in every domain of the xy plane.

EXAMPLE 3. Since the function $f(z) = i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2} = \frac{i}{z^2} \cdot \frac{\overline{z}^2}{\overline{z}^2} = \frac{i\overline{z}^2}{(z\overline{z})^2} = \frac{i\overline{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2},$$

the two functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad \text{and} \quad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout D, then v is said to be a *harmonic conjugate* of u. The meaning of the word conjugate here is, of course, different from that in Sec. 5, where $\overline{z}$ is defined.

Theorem 2. *A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.*

The proof is easy. If v is a harmonic conjugate of u in D, the theorem in Sec. 22 tells us that f is analytic in D. Conversely, if f is analytic in D, we know from Theorem 1 that u and v are harmonic in D; furthermore, in view of the theorem in Sec. 21, the Cauchy–Riemann equations are satisfied.

The following example shows that if v is a harmonic conjugate of u in some domain, it is *not*, in general, true that u is a harmonic conjugate of v there. (See also Exercises 3 and 4.)

EXAMPLE 4. Suppose that

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Since these are the real and imaginary components, respectively, of the entire function $f(z) = z^2$, we know that v is a harmonic conjugate of u throughout the plane. But u cannot be a harmonic conjugate of v since, as verified in Exercise 2*(b)*, Sec. 25, the function $2xy + i(x^2 - y^2)$ is not analytic anywhere.

In Chap. 9 (Sec. 104) we shall show that a function u which is harmonic in a domain of a certain type always has a harmonic conjugate. Thus, in such domains, every harmonic function is the real part of an analytic function. It is also true (Exercise 2) that a harmonic conjugate, when it exists, is unique except for an additive constant.

EXAMPLE 5. We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x, y) = y^3 - 3x^2 y \tag{5}$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is related to $u(x, y)$ by means of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{6}$$

the first of these equations tells us that

$$v_y(x, y) = -6xy.$$

Holding x fixed and integrating each side here with respect to y, we find that

$$v(x, y) = -3xy^2 + \phi(x) \tag{7}$$

where ϕ is, at present, an arbitrary function of x. Using the second of equations (6), we have

$$3y^2 - 3x^2 = 3y^2 - \phi'(x),$$

or $\phi'(x) = 3x^2$. Thus $\phi(x) = x^3 + C$, where C is an arbitrary real number. According to equation (7), then, the function

$$v(x, y) = -3xy^2 + x^3 + C \tag{8}$$

is a harmonic conjugate of $u(x, y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2 y) + i(-3xy^2 + x^3 + C). \tag{9}$$

The form $f(z) = i(z^3 + C)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (9) becomes $f(x) = i(x^3 + C)$.

EXERCISES

1. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

(a) $u(x, y) = 2x(1 - y)$; *(b)* $u(x, y) = 2x - x^3 + 3xy^2$;
(c) $u(x, y) = \sinh x \sin y$; *(d)* $u(x, y) = y/(x^2 + y^2)$.

Ans. (a) $v(x, y) = x^2 - y^2 + 2y$; *(b)* $v(x, y) = 2y - 3x^2 y + y^3$;
(c) $v(x, y) = -\cosh x \cos y$; *(d)* $v(x, y) = x/(x^2 + y^2)$.

2. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

3. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

4. Use Theorem 2 in Sec. 26 to show that v is a harmonic conjugate of u in a domain D if and only if $-u$ is a harmonic conjugate of v in D. (Compare with the result obtained in Exercise 3.)

Suggestion: Observe that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in D if and only if $-if(z)$ is analytic there.

5. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (Sec. 23) and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the *polar form of Laplace's equation*. Show that the same is true of the function $v(r, \theta)$.

6. Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation, obtained in Exercise 5. Then use the technique in Example 5, Sec. 26, but involving the Cauchy–Riemann equations in polar form (Sec. 23), to derive the harmonic conjugate $v(r, \theta) = \theta$. (Compare with Exercise 6, Sec. 25.)

7. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of *level curves* $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the pair of equations $u(x, y) = c_1$ and $v(x, y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

8. Show that when $f(z) = z^2$, the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the component functions are the hyperbolas indicated in Fig. 32. Note the orthogonality of the two families, described in Exercise 7. Observe that the curves $u(x, y) = 0$ and $v(x, y) = 0$ intersect at the origin but are not, however, orthogonal to each other. Why is this fact in agreement with the result in Exercise 7?

9. Sketch the families of level curves of the component functions u and v when $f(z) = 1/z$, and note the orthogonality described in Exercise 7.

10. Do Exercise 9 using polar coordinates.

11. Sketch the families of level curves of the component functions u and v when

$$f(z) = \frac{z-1}{z+1},$$

and note how the result in Exercise 7 is illustrated here.

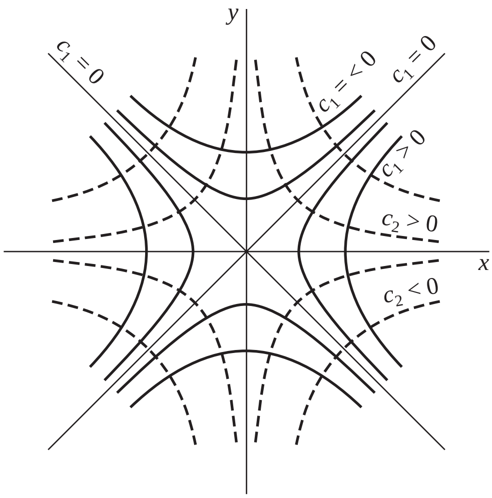

FIGURE 32

27. UNIQUELY DETERMINED ANALYTIC FUNCTIONS

We conclude this chapter with two sections dealing with how the values of an analytic function in a domain D are affected by its values in a subdomain of D or on a line segment lying in D. While these sections are of considerable theoretical interest, they are not central to our development of analytic functions in later chapters. The reader may pass directly to Chap. 3 at this time and refer back when necessary.

Lemma. *Suppose that*

(a) a function f is analytic throughout a domain D;

(b) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Then $f(z) \equiv 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

To prove this lemma, we let f be as stated in its hypothesis and let z_0 be any point of the subdomain or line segment where $f(z) = 0$. Since D is a *connected* open set (Sec. 11), there is a polygonal line L, consisting of a finite number of line segments joined end to end and lying entirely in D, that extends from z_0 to any other point P in D. We let d be the shortest distance from points on L to the boundary of D, unless D is the entire plane; in that case, d may be any positive number. We then form a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L, where the point z_n coincides with P (Fig. 33) and where each point is sufficiently close to adjacent ones that

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

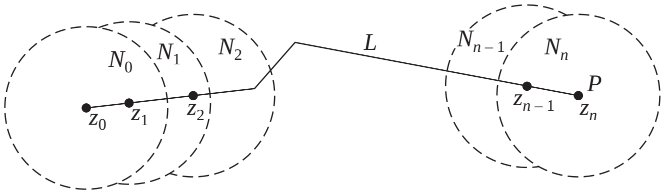

FIGURE 33

Finally, we construct a finite sequence of neighborhoods

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n,$$

where each neighborhood N_k is centered at z_k and has radius d. Note that these neighborhoods are all contained in D and that the center z_k of any neighborhood N_k $(k = 1, 2, \ldots, n)$ lies in the preceding neighborhood N_{k-1}.

At this point, we need to use a result that is proved later on in Chap. 6. Namely, Theorem 3 in Sec. 75 tells us that since f is analytic in N_0 and since $f(z) = 0$ in a domain or on a line segment containing z_0, then $f(z) \equiv 0$ in N_0. But the point z_1 lies in N_0. Hence a second application of the same theorem reveals that $f(z) \equiv 0$ in N_1; and, by continuing in this manner, we arrive at the fact that $f(z) \equiv 0$ in N_n. Since N_n is centered at the point P and since P was arbitrarily selected in D, we may conclude that $f(z) \equiv 0$ in D. This completes the proof of the lemma.

Suppose now that two functions f and g are analytic in the same domain D and that $f(z) = g(z)$ at each point z of some domain or line segment contained in D. The difference

$$h(z) = f(z) - g(z)$$

is also analytic in D, and $h(z) = 0$ throughout the subdomain or along the line segment. According to the lemma, then, $h(z) \equiv 0$ throught D; that is, $f(z) = g(z)$ at each point z in D. We thus arrive at the following important theorem.

Theorem. *A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.*

This theorem is useful in studying the question of extending the domain of definition of an analytic function. More precisely, given two domains D_1 and D_2, consider the *intersection* $D_1 \cap D_2$, consisting of all points that lie in both D_1 and D_2. If D_1 and D_2 have points in common (see Fig. 34) and a function f_1 is analytic in D_1, there *may* exist a function f_2, which is analytic in D_2, such that $f_2(z) = f_1(z)$ for each z in the intersection $D_1 \cap D_2$. If so, we call f_2 an *analytic continuation* of f_1 into the second domain D_2.

Whenever that analytic continuation exists, it is unique, according to the theorem just proved. That is, not more than one function can be analytic in D_2 and assume the value $f_1(z)$ at each point z of the domain $D_1 \cap D_2$ interior to D_2. However, if there is an analytic continuation f_3 of f_2 from D_2 into a domain D_3 which

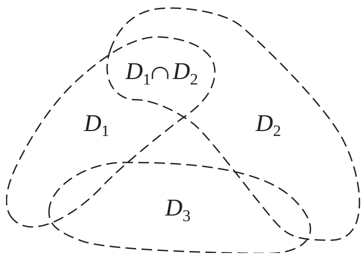

FIGURE 34

intersects D_1, as indicated in Fig. 34, it is not necessarily true that $f_3(z) = f_1(z)$ for each z in $D_1 \cap D_3$. Exercise 2, Sec. 28, illustrates this.

If f_2 is the analytic continuation of f_1 from a domain D_1 into a domain D_2, then the function F defined by means of the equations

$$F(z) = \begin{cases} f_1(z) & \text{when } z \text{ is in } D_1, \\ f_2(z) & \text{when } z \text{ is in } D_2 \end{cases}$$

is analytic in the *union* $D_1 \cup D_2$, which is the domain consisting of all points that lie in either D_1 or D_2. The function F is the analytic continuation into $D_1 \cup D_2$ of either f_1 or f_2; and f_1 and f_2 are called *elements* of F.

28. REFLECTION PRINCIPLE

The theorem in this section concerns the fact that some analytic functions possess the property that $\overline{f(z)} = f(\overline{z})$ for all points z in certain domains, while others do not. We note, for example, that the functions $z + 1$ and z^2 have that property when D is the entire finite plane; but the same is not true of $z + i$ and iz^2. The theorem here, which is known as the *reflection principle*, provides a way of predicting when $\overline{f(z)} = f(\overline{z})$.

Theorem. *Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then*

$$\overline{f(z)} = f(\overline{z}) \tag{1}$$

for each point z in the domain if and only if $f(x)$ is real for each point x on the segment.

We start the proof by assuming that $f(x)$ is real at each point x on the segment. Once we show that the function

$$F(z) = \overline{f(\overline{z})} \tag{2}$$

is analytic in D, we shall use it to obtain equation (1). To establish the analyticity of $F(z)$, we write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y)$$

and observe how it follows from equation (2) that since

$$\overline{f(\overline{z})} = u(x, -y) - iv(x, -y), \tag{3}$$

the components of $F(z)$ and $f(z)$ are related by the equations

$$U(x, y) = u(x, t) \quad \text{and} \quad V(x, y) = -v(x, t), \tag{4}$$

where $t = -y$. Now, because $f(x + it)$ is an analytic function of $x + it$, the first-order partial derivatives of the functions $u(x, t)$ and $v(x, t)$ are continuous throughout D and satisfy the Cauchy–Riemann equations*

$$u_x = v_t, \quad u_t = -v_x. \tag{5}$$

Furthermore, in view of equations (4),

$$U_x = u_x, \quad V_y = -v_t \frac{dt}{dy} = v_t;$$

and it follows from these and the first of equations (5) that $U_x = V_y$. Similarly,

$$U_y = u_t \frac{dt}{dy} = -u_t, \quad V_x = -v_x;$$

and the second of equations (5) tells us that $U_y = -V_x$. Inasmuch as the first-order partial derivatives of $U(x, y)$ and $V(x, y)$ are now shown to satisfy the Cauchy–Riemann equations and since those derivatives are continuous, we find that the function $F(z)$ is analytic in D. Moreover, since $f(x)$ is real on the segment of the real axis lying in D, we know that $v(x, 0) = 0$ on the segment; and, in view of equations (4), this means that

$$F(x) = U(x, 0) + iV(x, 0) = u(x, 0) - iv(x, 0) = u(x, 0).$$

That is,

$$F(z) = f(z) \tag{6}$$

at each point on the segment. According to the theorem in Sec. 27, which tells us that an analytic function defined on a domain D is uniquely determined by its

*See the paragraph immediately following Theorem 1 in Sec. 26.

values along any line segment lying in D, it follows that equation (6) actually holds throughout D. Because of definition (2) of the function $F(z)$, then,

$$\overline{f(\overline{z})} = f(z)\,; \tag{7}$$

and this is the same as equation (1).

To prove the converse in the theorem, we assume that equation (1) holds and note that in view of expression (3), the form (7) of equation (1) can be written

$$u(x, -y) - iv(x, -y) = u(x, y) + iv(x, y).$$

In particular, if $(x, 0)$ is a point on the segment of the real axis that lies in D,

$$u(x, 0) - iv(x, 0) = u(x, 0) + iv(x, 0);$$

and, by equating imaginary parts here, we see that $v(x, 0) = 0$. Hence $f(x)$ is real on the segment of the real axis lying in D.

EXAMPLES. Just prior to the statement of the theorem, we noted that

$$\overline{z+1} = \overline{z} + 1 \quad \text{and} \quad \overline{z^2} = \overline{z}^2$$

for all z in the finite plane. The theorem tells us, of course, that this is true, since $x + 1$ and x^2 are real when x is real. We also noted that $z + i$ and iz^2 do not have the reflection property throughout the plane, and we now know that this is because $x + i$ and ix^2 are *not* real when x is real.

EXERCISES

1. Use the theorem in Sec. 27 to show that if $f(z)$ is analytic and not constant throughout a domain D, then it cannot be constant throughout any neighborhood lying in D.

Suggestion: Suppose that $f(z)$ does have a constant value w_0 throughout some neighborhood in D.

2. Starting with the function

$$f_1(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, 0 < \theta < \pi)$$

and referring to Exercise 4*(b)*, Sec. 23, point out why

$$f_2(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < 2\pi\right)$$

is an analytic continuation of f_1 across the negative real axis into the lower half plane. Then show that the function

$$f_3(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \pi < \theta < \frac{5\pi}{2}\right)$$

is an analytic continuation of f_2 across the positive real axis into the first quadrant but that $f_3(z) = -f_1(z)$ there.

3. State why the function

$$f_4(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is the analytic continuation of the function $f_1(z)$ in Exercise 2 across the positive real axis into the lower half plane.

4. We know from Example 1, Sec. 22, that the function

$$f(z) = e^x e^{iy}$$

has a derivative everywhere in the finite plane. Point out how it follows from the reflection principle (Sec. 28) that

$$\overline{f(z)} = f(\overline{z})$$

for each z. Then verify this directly.

5. Show that if the condition that $f(x)$ is real in the reflection principle (Sec. 28) is replaced by the condition that $f(x)$ is pure imaginary, then equation (1) in the statement of the principle is changed to

$$\overline{f(z)} = -f(\overline{z}).$$

CHAPTER

3

ELEMENTARY FUNCTIONS

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

29. THE EXPONENTIAL FUNCTION

As anticipated earlier (Sec. 14), we define here the exponential function e^z by writing

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

where Euler's formula (see Sec. 6)

$$e^{iy} = \cos y + i \sin y \tag{2}$$

is used and y is to be taken in radians. We see from this definition that e^z reduces to the usual exponential function in calculus when $y = 0$; and, following the convention used in calculus, we often write $\exp z$ for e^z.

Note that since the *positive* nth root $\sqrt[n]{e}$ of e is assigned to e^x when $x = 1/n$ $(n = 2, 3, \ldots)$, expression (1) tells us that the complex exponential function e^z is also $\sqrt[n]{e}$ when $z = 1/n$ $(n = 2, 3, \ldots)$. This is an exception to the convention (Sec. 9) that would ordinarily require us to interpret $e^{1/n}$ as the set of nth roots of e.

According to definition (1), $e^x e^{iy} = e^{x+iy}$; and, as already pointed out in Sec. 14, the definition is suggested by the additive property

$$e^{x_1} e^{x_2} = e^{x_1+x_2}$$

of e^x in calculus. That property's extension,

$$e^{z_1} e^{z_2} = e^{z_1+z_2}, \tag{3}$$

to complex analysis is easy to verify. To do this, we write

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Then

$$e^{z_1} e^{z_2} = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}) = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}).$$

But x_1 and x_2 are both real, and we know from Sec. 7 that

$$e^{iy_1} e^{iy_2} = e^{i(y_1+y_2)}.$$

Hence

$$e^{z_1} e^{z_2} = e^{(x_1+x_2)} e^{i(y_1+y_2)};$$

and, since

$$(x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2,$$

the right-hand side of this last equation becomes $e^{z_1+z_2}$. Property (3) is now established.

Observe how property (3) enables us to write $e^{z_1-z_2} e^{z_2} = e^{z_1}$, or

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}. \tag{4}$$

From this and the fact that $e^0 = 1$, it follows that $1/e^z = e^{-z}$.

There are a number of other important properties of e^z that are expected. According to Example 1 in Sec. 22, for instance,

$$\frac{d}{dz} e^z = e^z \tag{5}$$

everywhere in the z plane. Note that the differentiability of e^z for all z tells us that e^z *is entire* (Sec. 24). It is also true that

$$e^z \neq 0 \quad \text{for any complex number } z. \tag{6}$$

This is evident upon writing definition (1) in the form

$$e^z = \rho e^{i\phi} \quad \text{where} \quad \rho = e^x \text{ and } \phi = y,$$

which tells us that

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \;\; (n = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

Statement (6) then follows from the observation that $|e^z|$ is always positive.

Some properties of e^z are, however, *not* expected. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z *is periodic,* with a pure imaginary period of $2\pi i$:

$$e^{z+2\pi i} = e^z. \tag{8}$$

For another property of e^z that e^x does not have, we note that while e^x is always positive, e^z can be negative. We recall (Sec. 6), for instance, that $e^{i\pi} = -1$. In fact,

$$e^{i(2n+1)\pi} = e^{i2n\pi + i\pi} = e^{i2n\pi} e^{i\pi} = (1)(-1) = -1 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

There are, moreover, values of z such that e^z is any given nonzero complex number. This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

EXAMPLE. In order to find numbers $z = x + iy$ such that

$$e^z = 1 + i, \tag{9}$$

we write equation (9) as

$$e^x e^{iy} = \sqrt{2}\, e^{i\pi/4}.$$

Then, in view of the statment in italics at the beginning of Sec. 9 regarding the equality of two nonzero complex numbers in exponential form,

$$e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Because $\ln(e^x) = x$, it follows that

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{4}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots);$$

and so

$$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{10}$$

EXERCISES

1. Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$; *(b)* $\exp\left(\dfrac{2+\pi i}{4}\right) = \sqrt{\dfrac{e}{2}}\,(1+i)$;

(c) $\exp(z + \pi i) = -\exp z$.

2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \overline{z}$ is not analytic anywhere.

4. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

Ans. $f'(z) = 2z\exp(z^2)$.

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

(a) $e^z = -2$; *(b)* $e^z = 1 + \sqrt{3}i$; *(c)* $\exp(2z - 1) = 1$.

Ans. *(a)* $z = \ln 2 + (2n+1)\pi i \;\; (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \ln 2 + \left(2n + \dfrac{1}{3}\right)\pi i \;\; (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $z = \dfrac{1}{2} + n\pi i \;\; (n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\overline{\exp(iz)} = \exp(i\overline{z})$ if and only if $z = n\pi \;\; (n = 0, \pm 1, \pm 2, \ldots)$. (Compare with Exercise 4, Sec. 28.)

10. *(a)* Show that if e^z is real, then $\operatorname{Im} z = n\pi \;\; (n = 0, \pm 1, \pm 2, \ldots)$.

(b) If e^z is pure imaginary, what restriction is placed on z?

11. Describe the behavior of $e^z = e^x e^{iy}$ as *(a)* x tends to $-\infty$; *(b)* y tends to ∞.

12. Write $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \quad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

14. Establish the identity

$$(e^z)^n = e^{nz} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

in the following way.

(a) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.
(b) Verify it for negative integers n by first recalling from Sec. 7 that

$$z^n = (z^{-1})^m \qquad (m = -n = 1, 2, \ldots)$$

when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part *(a)*, together with the property $1/e^z = e^{-z}$ (Sec. 29) of the exponential function.

30. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

for w, where z is any *nonzero* complex number. To do this, we note that when z and w are written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$ and $w = u + iv$, equation (1) becomes

$$e^u e^{iv} = re^{i\Theta}.$$

According to the statement in italics at the beginning of Sec. 9 about the equality of two complex numbers expressed in exponential form, this tells us that

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi$$

where n is any integer. Since the equation $e^u = r$ is the same as $u = \ln r$, it follows that equation (1) is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus, if we write

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{2}$$

equation (1) tells us that

$$e^{\log z} = z \qquad (z \neq 0), \tag{3}$$

which serves to motivate expression (2) as the *definition* of the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$.

EXAMPLE 1. If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -2\pi/3$. Hence

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i$$
$$(n = 0, \pm 1, \pm 2, \ldots).$$

It should be emphasized that it is *not* true that the left-hand side of equation (3) with the order of the exponential and logarithmic functions reversed reduces to just z. More precisely, since expression (2) can be written

$$\log z = \ln |z| + i \arg z$$

and since (Sec. 29)

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \ \ (n = 0, \pm 1, \pm 2, \ldots)$$

when $z = x + iy$, we know that

$$\log(e^z) = \ln |e^z| + i \arg(e^z) = \ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$\log(e^z) = z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{4}$$

The *principal value* of $\log z$ is the value obtained from equation (2) when $n = 0$ there and is denoted by Log z. Thus

$$\operatorname{Log} z = \ln r + i\Theta. \tag{5}$$

Note that Log z is well defined and single-valued when $z \neq 0$ and that

$$\log z = \operatorname{Log} z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

It reduces to the usual logarithm in calculus when z is a positive real number $z = r$. To see this, one need only write $z = re^{i0}$, in which case equation (5) becomes $\operatorname{Log} z = \ln r$. That is, $\operatorname{Log} r = \ln r$.

EXAMPLE 2. From expression (2), we find that

$$\log 1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

As anticipated, Log $1 = 0$.

Our final example here reminds us that although we were unable to find logarithms of *negative* real numbers in calculus, we can now do so.

EXAMPLE 3. Observe that

$$\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that $\operatorname{Log}(-1) = \pi i$.

31. BRANCHES AND DERIVATIVES OF LOGARITHMS

If $z = re^{i\theta}$ is a nonzero complex number, the argument θ has any one of the values $\theta = \Theta + 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\Theta = \operatorname{Arg} z$. Hence the definition

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

of the multiple-valued logarithmic function in Sec. 30 can be written

$$\log z = \ln r + i\theta. \tag{1}$$

If we let α denote any real number and restrict the value of θ in expression (1) so that $\alpha < \theta < \alpha + 2\pi$, the function

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi), \tag{2}$$

with components

$$u(r, \theta) = \ln r \quad \text{and} \quad v(r, \theta) = \theta, \tag{3}$$

is *single-valued* and continuous in the stated domain (Fig. 35). Note that if the function (2) were to be defined on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v are near α and also points such that the values of v are near $\alpha + 2\pi$.

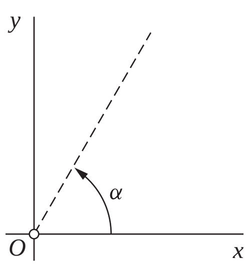

FIGURE 35

The function (2) is not only continuous but also analytic throughout the domain $r > 0$, $\alpha < \theta < \alpha + 2\pi$ since the first-order partial derivatives of u and v are continuous there and satisfy the polar form (Sec. 23)

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations. Furthermore, according to Sec. 23,

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}};$$

that is,

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi). \tag{4}$$

In particular,

$$\frac{d}{dz}\,\text{Log}\, z = \frac{1}{z} \qquad (|z| > 0, -\pi < \text{Arg}\ z < \pi). \tag{5}$$

A *branch* of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f. The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f. Observe that for each fixed α, the single-valued function (2) is a branch of the multiple-valued function (1). The function

$$\text{Log}\, z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi) \tag{6}$$

is called the *principal branch.*

A *branch cut* is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points (Sec. 24) of F, and any point that is common to all branch cuts of f is called a *branch point.* The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The branch cut for the principal branch (6) consists of the origin and the ray $\Theta = \pi$. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Special care must be taken in using branches of the logarithmic function, especially since expected identities involving logarithms do not always carry over from calculus.

EXAMPLE. When the principal branch (6) is used, one can see that

$$\text{Log}(i^3) = \text{Log}(-i) = \ln 1 - i\frac{\pi}{2} = -\frac{\pi}{2}i$$

and

$$3\,\text{Log}\, i = 3\left(\ln 1 + i\frac{\pi}{2}\right) = \frac{3\pi}{2}i.$$

Hence

$$\text{Log}(i^3) \neq 3\,\text{Log}\, i.$$

(See also Exercises 3 and 4.)

In Sec. 32, we shall derive some identities involving logarithms that *do* carry over from calculus, sometimes with qualifications as to how they are to be interpreted. A reader who wishes to pass to Sec. 33 can simply refer to results in Sec. 32 when needed.

EXERCISES

1. Show that

(a) $\mathrm{Log}(-ei) = 1 - \dfrac{\pi}{2}i$; *(b)* $\mathrm{Log}(1-i) = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i$.

2. Show that

(a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\log i = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that

(a) $\mathrm{Log}(1+i)^2 = 2\,\mathrm{Log}(1+i)$; *(b)* $\mathrm{Log}(-1+i)^2 \neq 2\,\mathrm{Log}(-1+i)$.

4. Show that

(a) $\log(i^2) = 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4}\right)$;

(b) $\log(i^2) \neq 2\log i$ when $\log z = \ln r + i\theta \left(r > 0, \dfrac{3\pi}{4} < \theta < \dfrac{11\pi}{4}\right)$.

5. Show that

(a) the set of values of $\log(i^{1/2})$ is

$$\left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that the same is true of $(1/2)\log i$;

(b) the set of values of $\log(i^2)$ is *not* the same as the set of values of $2\log i$.

6. Given that the branch $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 30)

$$e^{\log z} = z \qquad (z \neq 0)$$

and using the chain rule.

7. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

8. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (4), Sec. 30.]

9. Show that

(a) the function $f(z) = \mathrm{Log}(z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\mathrm{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

10. Show in two ways that the function $\ln(x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

11. Show that

$$\operatorname{Re}[\log(z-1)] = \frac{1}{2}\ln[(x-1)^2 + y^2] \qquad (z \neq 1).$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

32. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2}$$

was in Sec. 8. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \tag{3}$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 30 that

$$\log 1 = 2n\pi i \quad \text{and} \quad \log(-1) = (2n+1)\pi i,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Noting that $z_1 z_2 = 1$ and using the values

$$\log(z_1 z_2) = 0 \quad \text{and} \quad \log z_1 = \pi i,$$

we find that equations (1) is satisfied when the value $\log z_2 = -\pi i$ is chosen.

If, on the other hand, the principal values

$$\text{Log}\, 1 = 0 \quad \text{and} \quad \text{Log}(-1) = \pi i$$

are used,

$$\text{Log}(z_1 z_2) = 0 \quad \text{and} \quad \text{Log}\, z_1 + \log z_2 = 2\pi i$$

for the same numbers z_1 and z_2. Thus statement (1), which is sometimes true when *log* is replaced by *Log* (see Exercise 1), is not always true when principal values are used in all three of its terms.

Verification of the statement

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2, \tag{4}$$

which is to be interpreted in the same way as statement (1), is left to the exercises.

We include here two other properties of $\log z$ that will be of special interest in Sec. 33. If z is a nonzero complex number, then

$$z^n = e^{n \log z} \qquad (n = 0 \pm 1, \pm 2, \ldots) \tag{5}$$

for any value of $\log z$ that is taken. When $n = 1$, this reduces, of course, to relation (3), Sec. 30. Equation (5) is readily verified by writing $z = re^{i\theta}$ and noting that each side becomes $r^n e^{in\theta}$.

It is also true that when $z \neq 0$,

$$z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \qquad (n = 1, 2, \ldots). \tag{6}$$

That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write $z = r \exp(i\Theta)$, where Θ is the principal value of $\arg z$. Then, in view of definition (2), Sec. 30, of $\log z$,

$$\exp\left(\frac{1}{n} \log z\right) = \exp\left[\frac{1}{n} \ln r + \frac{i(\Theta + 2k\pi)}{n}\right]$$

where $k = 0, \pm 1, \pm 2, \ldots$. Thus

$$\exp\left(\frac{1}{n} \log z\right) = \sqrt[n]{r} \exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

Because $\exp(i2k\pi/n)$ has distinct values only when $k = 0, 1, \ldots, n-1$, the right-hand side of equation (7) has only n values. That right-hand side is, in fact, an expression for the nth roots of z (Sec. 9), and so it can be written $z^{1/n}$. This establishes property (6), which is actually valid when n is a negative integer too (see Exercise 5).

EXERCISES

1. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

Suggestion: Write $\Theta_1 = \operatorname{Arg} z_1$ and $\Theta_2 = \operatorname{Arg} z_2$. Then observe how it follows from the stated restrictions on z_1 and z_2 that $-\pi < \Theta_1 + \Theta_2 < \pi$.

2. Show that for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Exercise 1.)

3. Verify expression (4), Sec. 32, for $\log(z_1/z_2)$ by

(a) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 8);

(b) showing that $\log(1/z) = -\log z \ (z \neq 0)$, in the sense that $\log(1/z)$ and $-\log z$ have the same set of values, and then referring to expression (1), Sec. 32, for $\log(z_1 z_2)$.

4. By choosing specific nonzero values of z_1 and z_2, show that expression (4), Sec. 32, for $\log(z_1/z_2)$ is not always valid when *log* is replaced by *Log.*

5. Show that property (6), Sec. 32, also holds when n is a negative integer. Do this by writing $z^{1/n} = (z^{1/m})^{-1}$ $(m = -n)$, where n has any one of the negative values $n = -1, -2, \ldots$ (see Exercise 9, Sec. 10), and using the fact that the property is already known to be valid for positive integers.

6. Let z denote any nonzero complex number, written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \ldots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n} \ln r + i \frac{\Theta + 2(pn + k)\pi}{n},$$

where $p = 0, \pm 1, \pm 2, \ldots$ and $k = 0, 1, 2, \ldots, n - 1$. Then, after writing

$$\frac{1}{n} \log z = \frac{1}{n} \ln r + i \frac{\Theta + 2q\pi}{n},$$

where $q = 0, \pm 1, \pm 2, \ldots$, show that the set of values of $\log(z^{1/n})$ is the same as the set of values of $(1/n) \log z$. Thus show that $\log(z^{1/n}) = (1/n) \log z$ where, corresponding to a value of $\log(z^{1/n})$ taken on the left, the appropriate value of $\log z$ is to be selected on the right, and conversely. [The result in Exercise 5*(a)*, Sec. 31, is a special case of this one.]

Suggestion: Use the fact that the remainder upon dividing an integer by a positive integer n is always an integer between 0 and $n - 1$, inclusive; that is, when a positive integer n is specified, any integer q can be written $q = pn + k$, where p is an integer and k has one of the values $k = 0, 1, 2, \ldots, n - 1$.

33. COMPLEX EXPONENTS

When $z \neq 0$ and *the exponent c is any complex number,* the function z^c is defined by means of the equation

$$z^c = e^{c \log z}, \tag{1}$$

where $\log z$ denotes the multiple-valued logarithmic function. Equation (1) provides a consistent definition of z^c in the sense that it is already known to be valid (see Sec. 32) when $c = n$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $c = 1/n$ $(n = \pm 1, \pm 2, \ldots)$. Definition (1) is, in fact, suggested by those particular choices of c.

EXAMPLE 1. Powers of z are, in general, multiple-valued, as illustrated by writing

$$i^{-2i} = \exp(-2i \log i)$$

and then

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = \left(2n + \frac{1}{2}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

This shows that

$$i^{-2i} = \exp[(4n + 1)\pi] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Note that these values of i^{-2i} are all *real numbers.*

Since the exponential function has the property $1/e^z = e^{-z}$ (Sec. 29), one can see that

$$\frac{1}{z^c} = \frac{1}{\exp(c \log z)} = \exp(-c \log z) = z^{-c}$$

and, in particular, that $1/i^{2i} = i^{-2i}$. According to expression (2), then,

$$\frac{1}{i^{2i}} = \exp[(4n + 1)\pi] \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{3}$$

If $z = re^{i\theta}$ and α is any real number, the branch

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function is single-valued and analytic in the indicated domain (Sec. 31). When that branch is used, it follows that the function $z^c = \exp(c \log z)$ is single-valued and analytic in the same domain. The derivative of such a *branch of* z^c is found by first using the chain rule to write

$$\frac{d}{dz} z^c = \frac{d}{dz} \exp(c \log z) = \frac{c}{z} \exp(c \log z)$$

and then recalling (Sec. 30) the identity $z = \exp(\log z)$. That yields the result

$$\frac{d}{dz} z^c = c\frac{\exp(c \log z)}{\exp(\log z)} = c \exp[(c-1)\log z],$$

or

$$\frac{d}{dz} z^c = cz^{c-1} \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi). \tag{4}$$

The *principal value* of z^c occurs when $\log z$ is replaced by Log z in definition (1):

$$\text{P.V. } z^c = e^{c \operatorname{Log} z}. \tag{5}$$

Equation (5) also serves to define the *principal branch* of the function z^c on the domain $|z| > 0, -\pi < \operatorname{Arg} z < \pi$.

EXAMPLE 2. The principal value of $(-i)^i$ is

$$\exp[i \operatorname{Log}(-i)] = \exp\left[i\left(\ln 1 - i\frac{\pi}{2}\right)\right] = \exp\frac{\pi}{2}.$$

That is,

$$\text{P.V. } (-i)^i = \exp\frac{\pi}{2}. \tag{6}$$

EXAMPLE 3. The principal branch of $z^{2/3}$ can be written

$$\exp\left(\frac{2}{3}\operatorname{Log} z\right) = \exp\left(\frac{2}{3}\ln r + \frac{2}{3}i\Theta\right) = \sqrt[3]{r^2}\exp\left(i\frac{2\Theta}{3}\right).$$

Thus

$$\text{P.V. } z^{2/3} = \sqrt[3]{r^2}\cos\frac{2\Theta}{3} + i\sqrt[3]{r^2}\sin\frac{2\Theta}{3}. \tag{7}$$

This function is analytic in the domain $r > 0, -\pi < \Theta < \pi$, as one can see directly from the theorem in Sec. 23.

While familiar laws of exponents used in calculus often carry over to complex analysis, there are exceptions when certain numbers are involved.

EXAMPLE 4. Consider the nonzero complex numbers

$$z_1 = 1 + i, \quad z_2 = 1 - i, \quad \text{and} \quad z_3 = -1 - i.$$

When principal values of the powers are taken,

$$(z_1z_2)^i = 2^i = e^{i\operatorname{Log} 2} = e^{i(\ln 2 + i0)} = e^{i\ln 2}$$

and

$$z_1^i = e^{i\operatorname{Log}(1+i)} = e^{i(\ln\sqrt{2} + i\pi/4)} = e^{-\pi/4}e^{i(\ln 2)/2},$$
$$z_2^i = e^{i\operatorname{Log}(1-i)} = e^{i(\ln\sqrt{2} - i\pi/4)} = e^{\pi/4}e^{i(\ln 2)/2}.$$

Thus

$$(z_1z_2)^i = z_1^i z_2^i, \tag{8}$$

as might be expected.

On the other hand, continuing to use principal values, we see that

$$(z_2z_3)^i = (-2)^i = e^{i\operatorname{Log}(-2)} = e^{i(\ln 2 + i\pi)} = e^{-\pi}e^{i\ln 2}$$

and

$$z_3^i = e^{i\operatorname{Log}(-1-i)} = e^{i(\ln\sqrt{2} - i3\pi/4)} = e^{3\pi/4}e^{i(\ln 2)/2}.$$

Hence

$$(z_2z_3)^i = \left[e^{\pi/4}e^{i(\ln 2)/2}\right]\left[e^{3\pi/4}e^{i(\ln 2)/2}\right]e^{-2\pi},$$

or

$$(z_2z_3)^i = z_2^i z_3^i\, e^{-2\pi}. \tag{9}$$

According to definition (1), *the exponential function with base* c, where c is any nonzero complex constant, is written

$$c^z = e^{z\log c}. \tag{10}$$

Note that although e^z is, in general, multiple-valued according to definition (10), the usual interpretation of e^z occurs when the principal value of the logarithm is taken. This is because the principal value of $\log e$ is unity.

When a value of $\log c$ is specified, c^z is an entire function of z. In fact,

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log c} = e^{z\log c}\log c\,;$$

and this shows that

$$\frac{d}{dz}c^z = c^z\log c. \tag{11}$$

EXERCISES

1. Show that

(a) $(1+i)^i = \exp\left(-\dfrac{\pi}{4}+2n\pi\right)\exp\left(i\dfrac{\ln 2}{2}\right) \quad (n=0,\pm 1,\pm 2,\ldots)$;

(b) $(-1)^{1/\pi} = e^{(2n+1)i} \quad (n=0,\pm 1,\pm 2,\ldots)$.

2. Find the principal value of

(a) i^i; *(b)* $\left[\dfrac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; *(c)* $(1-i)^{4i}$.

Ans. (a) $\exp(-\pi/2)$; *(b)* $-\exp(2\pi^2)$; *(c)* $e^{\pi}[\cos(2\ln 2)+i\sin(2\ln 2)]$.

3. Use definition (1), Sec. 33, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing

(a) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;

(b) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 9 is the same as the principal value of $z_0^{1/n}$ defined by equation (5), Sec. 33.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a\ln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a+bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Ans. c is real.

8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then

(a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$; *(b)* $\dfrac{z^{c_1}}{z^{c_2}} = z^{c_1-c_2}$; *(c)* $(z^c)^n = z^{cn} \quad (n=1,2,\ldots)$.

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

34. TRIGONOMETRIC FUNCTIONS

Euler's formula (Sec. 6) tells us that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

for every real number x. Hence

$$e^{ix} - e^{-ix} = 2i\sin x \quad \text{and} \quad e^{ix}+e^{-ix} = 2\cos x.$$

That is,

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}.$$

It is, therefore, natural to *define* the sine and cosine functions of a complex variable z as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{1}$$

These functions are entire since they are linear combinations (Exercise 3, Sec. 25) of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives

$$\frac{d}{dz} e^{iz} = ie^{iz} \quad \text{and} \quad \frac{d}{dz} e^{-iz} = -ie^{-iz}$$

of those exponential functions, we find from equations (1) that

$$\frac{d}{dz} \sin z = \cos z \quad \text{and} \quad \frac{d}{dz} \cos z = -\sin z. \tag{2}$$

It is easy to see from definitions (1) that the sine and cosine functions remain odd and even, respectively:

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z. \tag{3}$$

Also,

$$e^{iz} = \cos z + i \sin z. \tag{4}$$

This is, of course, Euler's formula (Sec. 6) when z is real.

A variety of identities carry over from trigonometry. For instance (see Exercises 2 and 3),

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{5}$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \tag{6}$$

From these, it follows readily that

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z, \tag{7}$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z, \tag{8}$$

and [Exercise 4*(a)*]

$$\sin^2 z + \cos^2 z = 1. \tag{9}$$

The periodic character of $\sin z$ and $\cos z$ is also evident:

$$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z, \tag{10}$$

$$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z. \tag{11}$$

When y is any real number, definitions (1) and the hyperbolic functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

from calculus can be used to write

$$\sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y. \tag{12}$$

Also, the real and imaginary components of sin z and cos z can be displayed in terms of those hyperbolic functions:

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{13}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y, \tag{14}$$

where $z = x + iy$. To obtain expressions (13) and (14), we write

$$z_1 = x \quad \text{and} \quad z_2 = iy$$

in identities (5) and (6) and then refer to relations (12). Observe that once expression (13) is obtained, relation (14) also follows from the fact (Sec. 21) that if the derivative of a function

$$f(z) = u(x, y) + iv(x, y)$$

exists at a point $z = (x, y)$, then

$$f'(z) = u_x(x, y) + iv_x(x, y).$$

Expressions (13) and (14) can be used (Exercise 7) to show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{15}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{16}$$

Inasmuch as $\sinh y$ tends to infinity as y tends to infinity, it is clear from these two equations that $\sin z$ and $\cos z$ are *not bounded* on the complex plane, whereas the absolute values of $\sin x$ and $\cos x$ are less than or equal to unity for all values of x. (See the definition of a bounded function at the end of Sec. 18.)

A *zero* of a given function $f(z)$ is a number z_0 such that $f(z_0) = 0$. Since $\sin z$ becomes the usual sine function in calculus when z is real, we know that the real numbers $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ are all zeros of $\sin z$. To show that *there are no other zeros,* we assume that $\sin z = 0$ and note how it follows from equation (15) that

$$\sin^2 x + \sinh^2 y = 0.$$

This sum of two squares reveals that

$$\sin x = 0 \quad \text{and} \quad \sinh y = 0.$$

Evidently, then, $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $y = 0$; that is,

(17) $$\sin z = 0 \quad \text{if and only if} \quad z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots).$$

Since
$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$
according to the second of identities (8),

(18) $$\cos z = 0 \quad \text{if and only if} \quad z = \frac{\pi}{2} + n\pi \ (n = 0, \pm 1, \pm 2, \ldots).$$

So, as was the case with $\sin z$, the zeros of $\cos z$ are all real.

The other four trigonometric functions are defined in terms of the sine and cosine functions by the expected relations:

(19) $$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$

(20) $$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Observe that the quotients $\tan z$ and $\sec z$ are analytic everywhere except at the singularities (Sec. 24)
$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots),$$
which are the zeros of $\cos z$. Likewise, $\cot z$ and $\csc z$ have singularities at the zeros of $\sin z$, namely
$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By differentiating the right-hand sides of equations (19) and (20), we obtain the anticipated differentiation formulas

(21) $$\frac{d}{dz}\tan z = \sec^2 z, \qquad \frac{d}{dz}\cot z = -\csc^2 z,$$

(22) $$\frac{d}{dz}\sec z = \sec z \tan z, \quad \frac{d}{dz}\csc z = -\csc z \cot z.$$

The periodicity of each of the trigonometric functions defined by equations (19) and (20) follows readily from equations (10) and (11). For example,

(23) $$\tan(z + \pi) = \tan z.$$

Mapping properties of the transformation $w = \sin z$ are especially important in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared to read Sec. 96 (Chap. 8), where they are discussed.

EXERCISES

1. Give details in the derivation of expressions (2), Sec. 34, for the derivatives of $\sin z$ and $\cos z$.

2. *(a)* With the aid of expression (4), Sec. 34, show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 34, to show how it follows that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(b) Use the results in part *(a)* and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 34.

3. According to the final result in Exercise 2*(b)*,

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 34.

4. Verify identity (9) in Sec. 34 using

(a) identity (6) and relations (3) in that section;

(b) the lemma in Sec. 27 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

5. Use identity (9) in Sec. 34 to show that

(a) $1 + \tan^2 z = \sec^2 z$; *(b)* $1 + \cot^2 z = \csc^2 z$.

6. Establish differentiation formulas (21) and (22) in Sec. 34.

7. In Sec. 34, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

8. Point out how it follows from expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$ that

(a) $|\sin z| \geq |\sin x|$; *(b)* $|\cos z| \geq |\cos x|$.

9. With the aid of expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$, show that

(a) $|\sinh y| \le |\sin z| \le \cosh y$; *(b)* $|\sinh y| \le |\cos z| \le \cosh y$.

10. *(a)* Use definitions (1), Sec. 34, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

(b) With the aid of the identity obtained in part *(a)*, show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \overline{z}$ nor $\cos \overline{z}$ is an analytic function of z anywhere.

12. Use the reflection principle (Sec. 28) to show that for all z,

(a) $\overline{\sin z} = \sin \overline{z}$; *(b)* $\overline{\cos z} = \cos \overline{z}$.

13. With the aid of expressions (13) and (14) in Sec. 34, give direct verifications of the relations obtained in Exercise 12.

14. Show that

(a) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z;

(b) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

15. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and then the imaginary parts of $\sin z$ and $\cosh 4$.

Ans. $\left(\dfrac{\pi}{2} + 2n\pi\right) \pm 4i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

16. With the aid of expression (14), Sec. 34, show that the roots of the equaion $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2 + \sqrt{3}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

35. HYPERBOLIC FUNCTIONS

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore,

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z. \tag{2}$$

Because of the way in which the exponential function appears in definitions (1) and in the definitions (Sec. 34)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

of $\sin z$ and $\cos z$, the hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$$-i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z, \tag{3}$$

$$-i \sin(iz) = \sinh z, \quad \cos(iz) = \cosh z. \tag{4}$$

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \tag{5}$$

$$\cosh^2 z - \sinh^2 z = 1, \tag{6}$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \tag{7}$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \tag{8}$$

and

$$\sinh z = \sinh x \cos y + i \cosh x \sin y, \tag{9}$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y, \tag{10}$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \tag{11}$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y, \tag{12}$$

where $z = x + iy$. While these identities follow directly from definitions (1), they are often more easily obtained from related trigonometric identities, with the aid of relations (3) and (4).

EXAMPLE. To illustrate the method of proof just suggested, let us verify identity (11). According to the first of relations (4), $|\sinh z|^2 = |\sin(iz)|^2$. That is,

$$|\sinh z|^2 = |\sin(-y + ix)|^2, \tag{13}$$

where $z = x + iy$. But from equation (15), Sec. 34, we know that

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y\,;$$

and this enables us to write equation (13) in the desired form (11).

In view of the periodicity of $\sin z$ and $\cos z$, it follows immediately from relaions (4) that $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$. Relations (4), together with statements (17) and (18) in Sec. 34, also tell us that

$$\sinh z = 0 \quad \text{if and only if} \quad z = n\pi i \;\; (n = 0, \pm 1, \pm 2, \ldots) \tag{14}$$

and

$$\cosh z = 0 \quad \text{if and only if} \quad z = \left(\frac{\pi}{2} + n\pi\right) i \;\; (n = 0, \pm 1, \pm 2, \ldots). \tag{15}$$

The hyperbolic tangent of z is defined by means of the equation

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{16}$$

and is analytic in every domain in which $\cosh z \neq 0$. The functions $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are the reciprocals of $\tanh z$, $\cosh z$, and $\sinh z$, respectively. It is straightforward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable:

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z, \tag{17}$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z. \tag{18}$$

EXERCISES

1. Verify that the derivatives of $\sinh z$ and $\cosh z$ are as stated in equations (2), Sec. 35.
2. Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with
   (a) definitions (1), Sec. 35, of $\sinh z$ and $\cosh z$;
   (b) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 34) and using relations (3) in Sec. 35.
3. Show how identities (6) and (8) in Sec. 35 follow from identities (9) and (6), respectively, in Sec. 34.
4. Write $\sinh z = \sinh(x + iy)$ and $\cosh z = \cosh(x + iy)$, and then show how expressions (9) and (10) in Sec. 35 follow from identities (7) and (8), respectively, in that section.
5. Verify expression (12), Sec. 35, for $|\cosh z|^2$.
6. Show that $|\sinh x| \le |\cosh z| \le \cosh x$ by using
   (a) identity (12), Sec. 35;
   (b) the inequalities $|\sinh y| \le |\cos z| \le \cosh y$, obtained in Exercise 9*(b)*, Sec. 34.
7. Show that
   (a) $\sinh(z + \pi i) = -\sinh z$; *(b)* $\cosh(z + \pi i) = \cosh z$;
   (c) $\tanh(z + \pi i) = \tanh z$.

8. Give details showing that the zeros of $\sinh z$ and $\cosh z$ are as in statements (14) and (l5), Sec. 35.

9. Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.

10. Derive differentiation formulas (17), Sec. 35.

11. Use the reflection principle (Sec. 28) to show that for all z,

(a) $\overline{\sinh z} = \sinh \overline{z}$; *(b)* $\overline{\cosh z} = \cosh \overline{z}$.

12. Use the results in Exercise 11 to show that $\overline{\tanh z} = \tanh \overline{z}$ at points where $\cosh z \neq 0$.

13. By accepting that the stated identity is valid when z is replaced by the real variable x and using the lemma in Sec. 27, verify that

(a) $\cosh^2 z - \sinh^2 z = 1$; *(b)* $\sinh z + \cosh z = e^z$.

[Compare with Exercise 4*(b)*, Sec. 34.]

14. Why is the function $\sinh(e^z)$ entire? Write its real component as a function of x and y, and state why that function must be harmonic everywhere.

15. By using one of the identities (9) and (10) in Sec. 35 and then proceeding as in Exercise 15, Sec. 34, find all roots of the equation

(a) $\sinh z = i$; *(b)* $\cosh z = \dfrac{1}{2}$.

Ans. *(a)* $z = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \left(2n \pm \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

16. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 34.)

Ans. $z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

36. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function $\sin^{-1} z$, we write

$$w = \sin^{-1} z \quad \text{when} \quad z = \sin w.$$

That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is quadratic in e^{iw}, and solve for e^{iw} [see Exercise 8*(a)*, Sec. 10], we find that

$$e^{iw} = iz + (1 - z^2)^{1/2} \tag{1}$$

where $(1 - z^2)^{1/2}$ is, of course, a double-valued function of z. Taking logarithms of each side of equation (1) and recalling that $w = \sin^{-1} z$, we arrive at the expression

$$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]. \tag{2}$$

The following example emphasizes the fact that $\sin^{-1} z$ is a multiple-valued function, with infinitely many values at each point z.

EXAMPLE. Expression (2) tells us that

$$\sin^{-1}(-i) = -i \log(1 \pm \sqrt{2}).$$

But

$$\log(1 + \sqrt{2}) = \ln(1 + \sqrt{2}) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(1 - \sqrt{2}) = \ln(\sqrt{2} - 1) + (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since

$$\ln(\sqrt{2} - 1) = \ln \frac{1}{1 + \sqrt{2}} = -\ln(1 + \sqrt{2}),$$

then, the numbers

$$(-1)^n \ln(1 + \sqrt{2}) + n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

constitute the set of values of $\log(1 \pm \sqrt{2})$. Thus, in rectangular form,

$$\sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1 + \sqrt{2}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

One can apply the technique used to derive expression (2) for $\sin^{-1} z$ to show that

$$\cos^{-1} z = -i \log\left[z + i(1 - z^2)^{1/2}\right] \tag{3}$$

and that

$$\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}. \tag{4}$$

The functions $\cos^{-1} z$ and $\tan^{-1} z$ are also multiple-valued. When specific branches of the square root and logarithmic functions are used, all three inverse functions become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \tag{5}$$

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}. \tag{6}$$

The derivative of the last one,

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}, \tag{7}$$

does not, however, depend on the manner in which the function is made single-valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$$\sinh^{-1} z = \log\left[z + (z^2+1)^{1/2}\right], \tag{8}$$

$$\cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right], \tag{9}$$

and

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}. \tag{10}$$

Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

EXERCISES

1. Find all the values of

(a) $\tan^{-1}(2i)$; *(b)* $\tan^{-1}(1+i)$; *(c)* $\cosh^{-1}(-1)$; *(d)* $\tanh^{-1} 0$.

Ans. *(a)* $\left(n+\dfrac{1}{2}\right)\pi + \dfrac{i}{2}\ln 3 \ (n = 0, \pm 1, \pm 2, \ldots)$;

(d) $n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 36, for $\sin^{-1} z$.

Ans. $z = \left(2n+\dfrac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Derive formula (5), Sec. 36, for the derivative of $\sin^{-1} z$.

5. Derive expression (4), Sec. 36, for $\tan^{-1} z$.

6. Derive formula (7), Sec. 36, for the derivative of $\tan^{-1} z$.

7. Derive expression (9), Sec. 36, for $\cosh^{-1} z$.

CHAPTER

4

INTEGRALS

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally concise and powerful, and many of the proofs are short.

37. DERIVATIVES OF FUNCTIONS $w(t)$

In order to introduce integrals of $f(z)$ in a fairly simple way, we need to first consider derivatives of complex-valued functions w of a *real* variable t. We write

$$w(t) = u(t) + iv(t), \tag{1}$$

where the functions u and v are *real-valued* functions of t. The derivative

$$w'(t), \text{ or } \frac{d}{dt}w(t),$$

of the function (1) at a point t is defined as

$$w'(t) = u'(t) + iv'(t), \tag{2}$$

provided each of the derivatives u' and v' exists at t.

From definition (2), it follows that for every complex constant $z_0 = x_0 + iy_0$,

$$\begin{aligned}\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u + iv)]' = [(x_0 u - y_0 v) + i(y_0 u + x_0 v)]' \\ &= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' = (x_0 u' - y_0 v') + i(y_0 u' + x_0 v').\end{aligned}$$

But

$$(x_0u' - y_0v') + i(y_0u' + x_0v') = (x_0 + iy_0)(u' + iv') = z_0w'(t),$$

and so

(3) $$\frac{d}{dt}[z_0w(t)] = z_0w'(t).$$

Another expected rule that we shall often use is

(4) $$\frac{d}{dt}e^{z_0t} = z_0e^{z_0t},$$

where $z_0 = x_0 + iy_0$. To verify this, we write

$$e^{z_0t} = e^{x_0t}e^{iy_0t} = e^{x_0t}\cos y_0t + ie^{x_0t}\sin y_0t$$

and refer to definition (2) to see that

$$\frac{d}{dt}e^{z_0t} = (e^{x_0t}\cos y_0t)' + i(e^{x_0t}\sin y_0t)'.$$

Familiar rules from calculus and some simple algebra then lead us to the expression

$$\frac{d}{dt}e^{z_0t} = (x_0 + iy_0)(e^{x_0t}\cos y_0t + ie^{x_0t}\sin y_0t),$$

or

$$\frac{d}{dt}e^{z_0t} = (x_0 + iy_0)e^{x_0t}e^{iy_0t}.$$

This is, of course, the same as equation (4).

Various other rules learned in calculus, such as the ones for differentiating sums and products, apply just as they do for real-valued functions of t. As was the case with property (3) and formula (4), verifications may be based on corresponding rules in calculus. It should be pointed out, however, that not every such rule carries over to functions of type (1). The following example illustrates this.

EXAMPLE. Suppose that $w(t)$ is continuous on an interval $a \le t \le b$; that is, its component functions $u(t)$ and $v(t)$ are continuous there. Even if $w'(t)$ exists when $a < t < b$, the mean value theorem for derivatives no longer applies. To be precise, it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

To see this, consider the function $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$. When that function is used, $|w'(t)| = |ie^{it}| = 1$; and this means that the derivative $w'(t)$ is never zero, while $w(2\pi) - w(0) = 0$.

38. DEFINITE INTEGRALS OF FUNCTIONS $w(t)$

When $w(t)$ is a complex-valued function of a real variable t and is written

$$w(t) = u(t) + iv(t), \tag{1}$$

where u and v are real-valued, the definite integral of $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{2}$$

provided the individual integrals on the right exist. Thus

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re}[w(t)]\,dt \quad \text{and} \quad \operatorname{Im}\int_a^b w(t)\,dt = \int_a^b \operatorname{Im}[w(t)]\,dt. \tag{3}$$

EXAMPLE 1. For an illustration of definition (2),

$$\int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt = \frac{2}{3} + i.$$

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way.

The existence of the integrals of u and v in definition (2) is ensured if those functions are *piecewise continuous* on the interval $a \le t \le b$. Such a function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits. Of course, only the right-hand limit is required at a; and only the left-hand limit is required at b. When both u and v are piecewise continuous, the function w is said to have that property.

Anticipated rules for integrating a complex constant times a function $w(t)$, for integrating sums of such functions, and for interchanging limits of integration are all valid. Those rules, as well as the property

$$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt,$$

are easy to verify by recalling corresponding results in calculus.

The *fundamental theorem of calculus*, involving antiderivatives, can, moreover, be extended so as to apply to integrals of the type (2). To be specific, suppose that the functions

$$w(t) = u(t) + iv(t) \quad \text{and} \quad W(t) = U(t) + iV(t)$$

are continuous on the interval $a \le t \le b$. If $W'(t) = w(t)$ when $a \le t \le b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$. Hence, in view of definition (2),

$$\int_a^b w(t)\,dt = U(t)\Big]_a^b + iV(t)\Big]_a^b = [U(b) + iV(b)] - [U(a) + iV(a)].$$

That is,

$$\int_a^b w(t)\,dt = W(b) - W(a) = W(t)\Big]_a^b. \tag{4}$$

EXAMPLE 2. Since (see Sec. 37)

$$\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}e^{it} = \frac{1}{i}\,ie^{it} = e^{it},$$

one can see that

$$\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{\pi/4} = \frac{e^{i\pi/4}}{i} - \frac{1}{i} = \frac{1}{i}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - 1\right)$$
$$= \frac{1}{i}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}} + \frac{1}{i}\left(\frac{1}{\sqrt{2}} - 1\right).$$

Then, because $1/i = -i$,

$$\int_0^{\pi/4} e^{it}\,dt = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

We recall from the example in Sec. 37 how the mean value theorem for derivatives in calculus does not carry over to complex-valued functions $w(t)$. Our final example here shows that the mean value theorem for *integrals* does not carry over either. Thus special care must continue to be used in applying rules from calculus.

EXAMPLE 3. Let $w(t)$ be a continuous complex-valued function of t defined on an interval $a \le t \le b$. In order to show that it is not necessarily true that there is a number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\,dt = w(c)(b-a),$$

we write $a = 0$, $b = 2\pi$ and use the same function $w(t) = e^{it}\,(0 \le t \le 2\pi)$ as in the example in Sec. 37. It is easy to see that

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{2\pi} = 0.$$

But, for any number c such that $0 < c < 2\pi$,

$$|w(c)(b-a)| = |e^{ic}|\,2\pi = 2\pi;$$

and this means that $w(c)(b-a)$ is *not* zero.

EXERCISES

1. Use rules in calculus to establish the following rules when

$$w(t) = u(t) + iv(t)$$

is a complex-valued function of a real variable t and $w'(t)$ exists:

(a) $\dfrac{d}{dt}w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to t, evaluated at $-t$;

(b) $\dfrac{d}{dt}[w(t)]^2 = 2\,w(t)w'(t)$.

2. Evaluate the following integrals:

(a) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt\,;$ *(b)* $\displaystyle\int_0^{\pi/6} e^{i2t}\,dt\,;$ *(c)* $\displaystyle\int_0^{\infty} e^{-zt}\,dt$ $\quad(\operatorname{Re} z > 0)$.

Ans. (a) $-\dfrac{1}{2} - i\ln 4\,;$ *(b)* $\dfrac{\sqrt{3}}{4} + \dfrac{i}{4}\,;$ *(c)* $\dfrac{1}{z}$.

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

4. According to definition (2), Sec. 38, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\,dx = \int_0^{\pi} e^x \cos x\,dx + i\int_0^{\pi} e^x \sin x\,dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1+e^{\pi})/2, \quad (1+e^{\pi})/2$.

5. Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \leq t \leq a$.

(a) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_0^{a} w(t)\,dt.$$

(b) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)\,dt = 0.$$

Suggestion: In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of t, which is graphically evident.

39. CONTOURS

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

A set of points $z = (x, y)$ in the complex plane is said to be an *arc* if

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b), \tag{1}$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the xy, or z, plane; and the image points are ordered according to increasing values of t. It is convenient to describe the points of C by means of the equation

$$z = z(t) \qquad (a \le t \le b), \tag{2}$$

where

$$z(t) = x(t) + iy(t). \tag{3}$$

The arc C is a *simple arc*, or a Jordan arc,* if it does not cross itself; that is, C is simple if $z(t_1) \ne z(t_2)$ when $t_1 \ne t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a *simple closed curve*, or a Jordan curve. Such a curve is *positively oriented* when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter t in equation (2). This is, in fact, the case in the following examples.

EXAMPLE 1. The polygonal line (Sec. 11) defined by means of the equations

$$z = \begin{cases} x + ix & \text{when } 0 \le x \le 1, \\ x + i & \text{when } 1 \le x \le 2 \end{cases} \tag{4}$$

and consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$ (Fig. 36) is a simple arc.

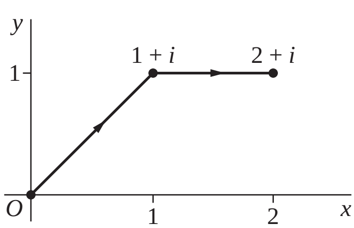

FIGURE 36

*Named for C. Jordan (1838–1922), pronounced *jor-don′*.

EXAMPLE 2. The unit circle

$$z = e^{i\theta} \qquad (0 \le \theta \le 2\pi) \tag{5}$$

about the origin is a simple closed curve, oriented in the counterclockwise direction. So is the circle

$$z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi), \tag{6}$$

centered at the point z_0 and with radius R (see Sec. 6).

The same set of points can make up different arcs.

EXAMPLE 3. The arc

$$z = e^{-i\theta} \qquad (0 \le \theta \le 2\pi) \tag{7}$$

is not the same as the arc described by equation (5). The set of points is the same, but now the circle is traversed in the *clockwise* direction.

EXAMPLE 4. The points on the arc

$$z = e^{i2\theta} \qquad (0 \le \theta \le 2\pi) \tag{8}$$

are the same as those making up the arcs (5) and (7). The arc here differs, however, from each of those arcs since the circle is traversed *twice* in the counterclockwise direction.

The parametric representation used for any given arc C is, of course, not unique. It is, in fact, possible to change the interval over which the parameter ranges to any other interval. To be specific, suppose that

$$t = \phi(\tau) \qquad (\alpha \le \tau \le \beta), \tag{9}$$

where ϕ is a real-valued function mapping an interval $\alpha \le \tau \le \beta$ onto the interval $a \le t \le b$ in representation (2). (See Fig. 37.) We assume that ϕ is continuous with

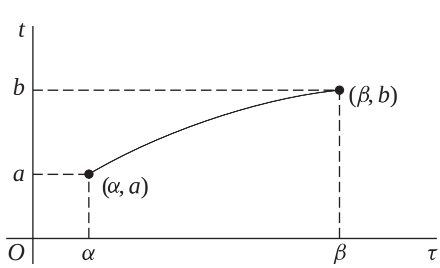

FIGURE 37
$t = \phi\ (\tau)$

a continuous derivative. We also assume that $\phi'(\tau) > 0$ for each τ; this ensures that t increases with τ. Representation (2) is then transformed by equation (9) into

$$z = Z(\tau) \qquad (\alpha \le \tau \le \beta), \tag{10}$$

where

$$Z(\tau) = z[\phi(\tau)]. \tag{11}$$

This is illustrated in Exercise 3.

Suppose now that the components $x'(t)$ and $y'(t)$ of the derivative (Sec. 37)

$$z'(t) = x'(t) + iy'(t) \tag{12}$$

of the function (3), used to represent C, are continuous on the entire interval $a \le t \le b$. The arc is then called a *differentiable arc*, and the real-valued function

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over the interval $a \le t \le b$. In fact, according to the definition of arc length in calculus, the length of C is the number

$$L = \int_a^b |z'(t)|\,dt. \tag{13}$$

The value of L is invariant under certain changes in the representation for C that is used, as one would expect. More precisely, with the change of variable indicated in equation (9), expression (13) takes the form [see Exercise 1*(b)*]

$$L = \int_\alpha^\beta |z'[\phi(\tau)]|\phi'(\tau)\,d\tau.$$

So, if representation (10) is used for C, the derivative (Exercise 4)

$$Z'(\tau) = z'[\phi(\tau)]\phi'(\tau) \tag{14}$$

enables us to write expression (13) as

$$L = \int_\alpha^\beta |Z'(\tau)|\,d\tau.$$

Thus the same length of C would be obtained if representation (10) were to be used.

If equation (2) represents a differentiable arc and if $z'(t) \ne 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t in that open interval, with angle of inclination $\arg z'(t)$. Also, when $\mathbf{T}$ turns, it does so continuously as the parameter t varies over the entire interval

$a < t < b$. This expression for $\mathbf{T}$ is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be *smooth*. In referring to a smooth arc $z = z(t)$ $(a \le t \le b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and nonzero throughout the open interval $a < t < b$.

A *contour*, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a *simple closed contour*. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept this statement, known as the *Jordan curve theorem*, as geometrically evident; the proof is not easy.*

EXERCISES

1. Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \le t \le b$, then

(a) $\displaystyle\int_{-b}^{-a} w(-t)\,dt = \int_a^b w(\tau)\,d\tau;$

(b) $\displaystyle\int_a^b w(t)\,dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)\,d\tau,$ where $\phi(\tau)$ is the function in equation (9), Sec. 39.

Suggestion: These identities can be obtained by noting that they are valid for *real-valued* functions of t.

2. Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which C is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 39.

3. Derive the equation of the line through the points (α, a) and (β, b) in the τt plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 39, to transform representation (2) in that section into representation (10) there.

Ans. $\phi(\tau) = \dfrac{b-a}{\beta-\alpha}\,\tau + \dfrac{a\beta - b\alpha}{\beta-\alpha}$.

4. Verify expression (14), Sec. 39, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

Suggestion: Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part *(a)* is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

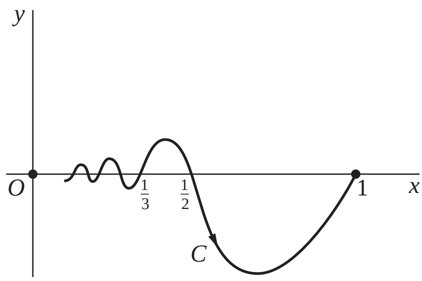

FIGURE 38

40. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour C as well as on the function f. It is written

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec. 38.

Suppose that the equation

$$z = z(t) \quad (a \le t \le b) \tag{1}$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 38) on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or *contour integral*, of f along C in terms of the parameter t:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt. \tag{2}$$

Note that since C is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 39. This can be seen by following the same general procedure that was used in Sec. 39 to show the invariance of arc length.

It follows immediately from definition (2) and properties of integrals of complex-valued functions $w(t)$ mentioned in Sec. 38 that

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz, \tag{3}$$

for any complex constant z_0, and

$$\int_C \left[f(z) + g(z)\right] dz = \int_C f(z)\,dz + \int_C g(z)\,dz. \tag{4}$$

Associated with the contour C used in integral (2) is the contour $-C$, consisting of the same set of points but with the order reversed so that the new contour extends from the point z_2 to the point z_1 (Fig. 39). The contour $-C$ has parametric representation

$$z = z(-t) \qquad (-b \le t \le -a).$$

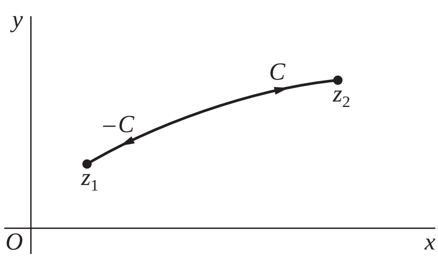

FIGURE 39

Hence, in view of Exercise 1*(a)*, Sec. 38,

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\,[z(-t)]\,\frac{d}{dt}z(-t)\,dt = -\int_{-b}^{-a} f\,[z(-t)]\,z'(-t)\,dt$$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to t, evaluated at $-t$. Making the substitution $\tau = -t$ in this last integral and referring to Exercise 1*(a)*, Sec. 39, we obtain the expression

$$\int_{-C} f(z)\,dz = -\int_a^b f[z(\tau)]z'(\tau)\,d\tau,$$

which is the same as

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz. \tag{5}$$

Consider now a path C, with representation (1), that consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3, the initial point of C_2 being the final point of C_1 (Fig. 40). There is a value c of t, where $a < c < b$, such that $z(c) = z_2$. Consequently, C_1 is represented by

$$z = z(t) \qquad (a \le t \le c)$$

and C_2 is represented by

$$z = z(t) \qquad (c \le t \le b).$$

Also, by a rule for integrals of functions $w(t)$ that was noted in Sec. 38,

$$\int_a^b f[z(t)]z'(t)\,dt = \int_a^c f[z(t)]z'(t)\,dt + \int_c^b f[z(t)]z'(t)\,dt.$$

Evidently, then,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{6}$$

Sometimes the contour C is called the *sum* of its legs C_1 and C_2 and is denoted by $C_1 + C_2$. The sum of two contours C_1 and $-C_2$ is well defined when C_1 and C_2 have the same final points, and it is written $C_1 - C_2$.

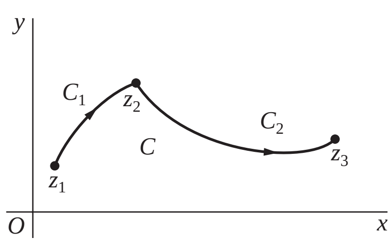

FIGURE 40
$C = C_1 + C_2$

Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.

41. SOME EXAMPLES

The purpose of this and the next section is to provide examples of the definition in Sec. 40 of contour integrals and to illustrate various properties that were mentioned there. We defer development of the concept of antiderivatives of the integrands $f(z)$ of contour integrals until Sec. 44.

EXAMPLE 1. Let us find the value of the integral

$$I = \int_C \overline{z}\,dz \tag{1}$$

when C is the right-hand half

$$z = 2\,e^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$ (Fig. 41). According to definition (2), Sec. 40,

$$I = \int_{-\pi/2}^{\pi/2} \overline{2\,e^{i\theta}}(2\,e^{i\theta})'\,d\theta = 4\int_{-\pi/2}^{\pi/2} \overline{e^{i\theta}}(e^{i\theta})'\,d\theta;$$

and, since

$$\overline{e^{i\theta}} = e^{-i\theta} \quad \text{and} \quad (e^{i\theta})' = i\,e^{i\theta},$$

this means that

$$I = 4\int_{-\pi/2}^{\pi/2} e^{-i\theta} i\,e^{i\theta}\,d\theta = 4i\int_{-\pi/2}^{\pi/2} d\theta = 4\pi i.$$

Note that $z\overline{z} = |z|^2 = 4$ when z is a point on the semicircle C. Hence the result

(2) $$\int_C \overline{z}\,dz = 4\pi i$$

can also be written

$$\int_C \frac{dz}{z} = \pi i.$$

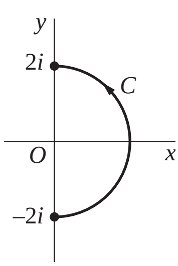

FIGURE 41

If $f(z)$ is given in the form $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, one can sometimes apply definition (2), Sec. 40, using one of the variables x and y as the parameter.

EXAMPLE 2. Here we first let C_1 denote the polygonal line OAB shown in Fig. 42 and evaluate the integral

(3) $$I_1 = \int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz,$$

where

$$f(z) = y - x - i3x^2 \quad (z = x + iy).$$

The leg *OA* may be represented parametrically as $z = 0 + iy$ $(0 \le y \le 1)$; and, since $x = 0$ at points on that line segment, the values of f there vary with the parameter y according to the equation $f(z) = y$ $(0 \le y \le 1)$. Consequently,

$$\int_{OA} f(z)\,dz = \int_0^1 yi\,dy = i\int_0^1 y\,dy = \frac{i}{2}.$$

On the leg AB, the points are $z = x + i$ $(0 \le x \le 1)$; and, since $y = 1$ on this segment,

$$\int_{AB} f(z)\,dz = \int_0^1 (1 - x - i3x^2)\cdot 1\,dx = \int_0^1 (1-x)\,dx - 3i\int_0^1 x^2\,dx = \frac{1}{2} - i.$$

In view of equation (3), we now see that

$$I_1 = \frac{1-i}{2}. \tag{4}$$

If C_2 denotes the segment OB of the line $y = x$ in Fig. 42, with parametric representation $z = x + ix$ $(0 \le x \le 1)$, the fact that $y = x$ on *OB* enables us to write

$$I_2 = \int_{C_2} f(z)\,dz = \int_0^1 -i3x^2(1+i)\,dx = 3(1-i)\int_0^1 x^2\,dx = 1 - i.$$

Evidently, then, the integrals of $f(z)$ along the two paths C_1 and C_2 have *different values* even though those paths have the same initial and the same final points.

Observe how it follows that the integral of $f(z)$ over the simple closed contour *OABO*, or $C_1 - C_2$, has the *nonzero value*

$$I_1 - I_2 = \frac{-1+i}{2}.$$

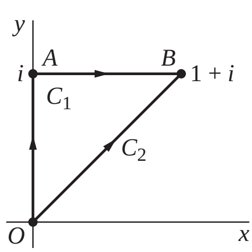

FIGURE 42

EXAMPLE 3. We begin here by letting C denote an arbitrary *smooth* arc (Sec. 39)

$$z = z(t) \quad (a \le t \le b)$$

from a fixed point z_1 to a fixed point z_2 (Fig. 43). In order to evaluate the integral

$$\int_C z\,dz = \int_a^b z(t)z'(t)\,dt,$$

we note that according to Exercise 1*(b)*, Sec. 38,

$$\frac{d}{dt}\frac{[z(t)]^2}{2} = z(t)z'(t).$$

Then, because $z(a) = z_1$ and $z(b) = z_2$, we have

$$\int_C z\,dz = \left.\frac{[z(t)]^2}{2}\right]_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

Inasmuch as the value of this integral depends only on the end points of C and is otherwise independent of the arc that is taken, we may write

$$\int_{z_1}^{z_2} z\,dz = \frac{z_2^2 - z_1^2}{2}. \tag{5}$$

(Compare with Example 2, where the value of an integral from one fixed point to another depended on the path that was taken.)

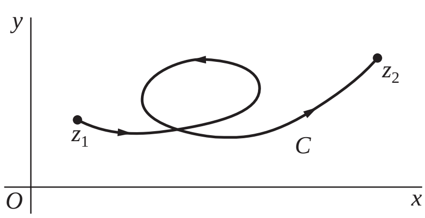

FIGURE 43

Expression (5) is also valid when C is a contour that is not necessarily smooth since a contour consists of a finite number of smooth arcs C_k $(k = 1, 2, \ldots, n)$, joined end to end. More precisely, suppose that each C_k extends from z_k to z_{k+1}. Then

$$\int_C z\,dz = \sum_{k=1}^{n}\int_{C_k} z\,dz = \sum_{k=1}^{n}\int_{z_k}^{z_{k+1}} z\,dz = \sum_{k=1}^{n}\frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}, \tag{6}$$

where this last summation has telescoped and z_1 is the initial point of C and z_{n+1} is its final point.

It follows from expression (6) that the integral of the function $f(z) = z$ around each closed contour in the plane has value zero. (Once again, compare with Example 2, where the value of the integral of a given function around a closed contour was *not* zero.) The question of predicting when an integral around a closed contour has value zero will be discussed in Secs. 44, 46, and 48.

42. EXAMPLES WITH BRANCH CUTS

The path in a contour integral can contain a point on a branch cut of the integrand involved. The next two examples illustrate this.

EXAMPLE 1. Let C denote the semicircular path

$$z = 3\,e^{i\theta} \quad (0 \le \theta \le \pi)$$

from the point $z = 3$ to the point $z = -3$ (Fig. 44). Although the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the multiple-valued function $z^{1/2}$ is not defined at the initial point $z = 3$ of the contour C, the integral

$$I = \int_C z^{1/2}\,dz \tag{1}$$

nevertheless exists. For the integrand is piecewise continuous on C. To see that this is so, we first observe that when $z(\theta) = 3\,e^{i\theta}$,

$$f[z(\theta)] = \exp\left[\frac{1}{2}(\ln 3 + i\theta)\right] = \sqrt{3}\,e^{i\theta/2}.$$

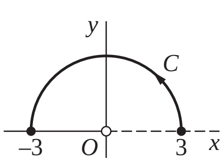

FIGURE 44

Hence the right-hand limits of the real and imaginary components of the function

$$f[z(\theta)]z'(\theta) = \sqrt{3}\,e^{i\theta/2}3ie^{i\theta} = 3\sqrt{3}ie^{i3\theta/2} = -3\sqrt{3}\sin\frac{3\theta}{2} + i3\sqrt{3}\cos\frac{3\theta}{2} \quad (0 < \theta \le \pi)$$

at $\theta = 0$ exist, those limits being 0 and $i3\sqrt{3}$, respectively. This means that $f[z(\theta)]z'(\theta)$ is continuous on the closed interval $0 \le \theta \le \pi$ when its value at $\theta = 0$ is defined as $i3\sqrt{3}$. Consequently,

$$I = 3\sqrt{3}i\int_0^{\pi} e^{i3\theta/2}\,d\theta.$$

Since

$$\int_0^\pi e^{i3\theta/2}\,d\theta = \frac{2}{3i}e^{i3\theta/2}\Big]_0^\pi = -\frac{2}{3i}(1+i),$$

we now have the value

$$I = -2\sqrt{3}(1+i) \tag{2}$$

of integral (1).

EXAMPLE 2. Suppose that C is the positively oriented circle (Fig. 45)

$$z = Re^{i\theta} \quad (-\pi \le \theta \le \pi)$$

about the origin, and left a denote any nonzero real number. Using the principal branch

$$f(z) = z^{a-1} = \exp[(a-1)\text{Log } z] \quad (|z| > 0, -\pi < \text{Arg } z < \pi)$$

of the power function z^{a-1}, let us evaluate the integral

$$I = \int_C z^{a-1}\,dz. \tag{3}$$

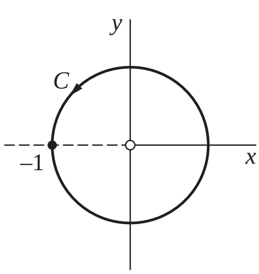

FIGURE 45

When $z(\theta) = Re^{i\theta}$, it is easy to see that

$$f[z(\theta)]z'(\theta) = iR^a e^{ia\theta} = -R^a \sin a\theta + iR^a \cos a\theta,$$

where the positive value of R^a is to be taken. Inasmuch as this function is piecewise continuous on $-\pi < \theta < \pi$, integral (3) exists. In fact,

$$I = iR^a\int_{-\pi}^{\pi} e^{ia\theta}\,d\theta = iR^a\left[\frac{e^{ia\theta}}{ia}\right]_{-\pi}^{\pi} = i\frac{2R^a}{a}\cdot\frac{e^{ia\pi}-e^{-ia\pi}}{2i} = i\frac{2R^a}{a}\sin a\pi. \tag{4}$$

Note that if a is a nonzero integer n, this result tells us that

$$\int_C z^{n-1}\,dz = 0 \quad (n = \pm 1, \pm 2, \ldots). \tag{5}$$

If a is allowed to be zero, we have

$$\int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} d\theta = 2\pi i. \tag{6}$$

EXERCISES

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\,dz.$$

1. $f(z) = (z+2)/z$ and C is

(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;
(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. (a) $-4 + 2\pi i$; *(b)* $4 + 2\pi i$; *(c)* $4\pi i$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

Ans. (a) 0; *(b)* 0.

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1+i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans. $2 + 3i$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.

Ans. $z_2 - z_1$.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. $i(1 - e^{-2\pi})$.

7. $f(z)$ is the principal branch

$$z^i = \exp(i\operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1+e^{-\pi}}{2}(1-i)$.

8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \overline{z}^n \, dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

9. Evaluate the integral I in Example 1, Sec. 41, using this representation for C:

$$z = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

(See Exercise 2, Sec. 39.)

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta} \ (-\pi \le \theta \le \pi) \quad \text{and} \quad z = Re^{i\theta} \ (-\pi \le \theta \le \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$$

when f is piecewise continuous on C.

(b) Apply the result in part *(a)* to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \ldots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

11. *(a)* Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 39, then

$$\int_a^b f[z(t)]z'(t)\, dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\, d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(b) Point out how it follows that the identity obtained in part *(a)* remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

Suggestion: In part *(a)*, use the result in Exercise 1*(b)*, Sec. 39, and then refer to expression (14) in that section.

43. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

We turn now to an inequality involving contour integrals that is extremely important in various applications. We present the result as a theorem but preface it with a needed lemma involving functions $w(t)$ of the type encountered in Secs. 37 and 38.

Lemma. *If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \le t \le b$, then*

$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt. \tag{1}$$

This inequality clearly holds when the value of the integral on the left is zero. Thus, in the verification we may assume that its value is a *nonzero* complex number and write

$$\int_a^b w(t)\,dt = r_0\, e^{i\theta_0}.$$

Solving for r_0, we have

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt. \tag{2}$$

Now the left-hand side of this equation is a real number, and so the right-hand side is too. Thus, using the fact that the real part of a real number is the number itself, we find that

$$r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t)\,dt,$$

or

$$r_0 = \int_a^b \text{Re}[e^{-i\theta_0} w(t)]\,dt. \tag{3}$$

But

$$\text{Re}[e^{-i\theta_0} w(t)] \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\,|w(t)| = |w(t)|,$$

and it follows from equation (3) that

$$r_0 \le \int_a^b |w(t)|\,dt.$$

Because r_0 is, in fact, the left-hand side of inequality (1), the verification of the lemma is complete.

Theorem. *Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that*

$$|f(z)| \le M \tag{4}$$

for all points z on C at which $f(z)$ is defined, then

$$\left|\int_C f(z)\,dz\right| \le ML. \tag{5}$$

To prove this, let $z = z(t)$ $(a \le t \le b)$ be a parametric representation of C. According to the above lemma,

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f[z(t)]z'(t)\,dt\right| \le \int_a^b |f[z(t)]z'(t)|\,dt.$$

Inasmuch as

$$|f[z(t)]z'(t)| = |f[z(t)]|\,|z'(t)| \le M\,|z'(t)|$$

when $a \le t \le b$, it follows that

$$\left|\int_C f(z)\,dz\right| \le M\int_a^b |z'(t)|\,dt.$$

Since the integral on the right here represents the length L of C (see Sec. 39), inequality (5) is established. It is, of course, a strict inequality if inequality (4) is strict.

Note that since C is a contour and f is piecewise continuous on C, a number M such as the one appearing in inequality (4) will always exist. This is because the real-valued function $|f[z(t)]|$ is continuous on the closed bounded interval $a \le t \le b$ when f is continuous on C; and such a function always reaches a maximum value M on that interval.* Hence $|f(z)|$ has a maximum value on C when f is continuous on it. The same is, then, true when f is *piecewise* continuous on C.

EXAMPLE 1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 46). Inequality (5) can be used to show that

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}. \tag{6}$$

This is done by noting first that if z is a point on C, so that $|z| = 2$, then

$$|z+4| \le |z| + 4 = 6$$

*See, for instance A. E. Taylor and W. R. Mann, “Advanced Calculus,” 3d ed., pp. 86–90, 1983.

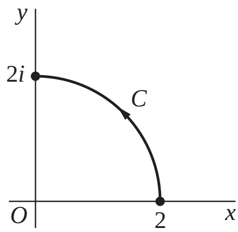

FIGURE 46

and

$$|z^3 - 1| \geq ||z|^3 - 1| = 7.$$

Thus, when z lies on C,

$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \leq \frac{6}{7}.$$

Writing $M = 6/7$ and observing that $L = \pi$ is the length of C, we may now use inequality (5) to obtain inequality (6).

EXAMPLE 2. Here C_R is the semicircular path

$$z = Re^{i\theta} \qquad (0 \leq \theta \leq \pi),$$

and $z^{1/2}$ denotes the branch

$$z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the square root function. (See Fig. 47.) Without actually finding the value of the integral, one can easily show that

$$\lim_{R\to\infty}\int_{C_R}\frac{z^{1/2}}{z^2+1}\,dz = 0. \tag{7}$$

For, when $|z| = R > 1$,

$$|z^{1/2}| = |\sqrt{R}e^{i\theta/2}| = \sqrt{R}$$

and

$$|z^2 + 1| \geq ||z^2| - 1| = R^2 - 1.$$

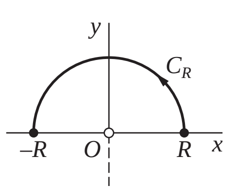

FIGURE 47

Consequently, at points on C_R,

$$\left|\frac{z^{1/2}}{z^2+1}\right| \le M_R \quad \text{where} \quad M_R = \frac{\sqrt{R}}{R^2-1}.$$

Since the length of C_R is the number $L = \pi R$, it follows from inequality (5) that

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \le M_R L.$$

But

$$M_R L = \frac{\pi R\sqrt{R}}{R^2-1} \cdot \frac{1/R^2}{1/R^2} = \frac{\pi/\sqrt{R}}{1-(1/R^2)},$$

and it is clear that the term on the far right here tends to zero as R tends to infinity. Limit (7) is, therefore, established.

EXERCISES

1. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$$

when C is the same arc as the one in Example 1, Sec. 43.

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction (see Fig. 48), then

$$\left|\int_C (e^z - \bar{z})\,dz\right| \le 60.$$

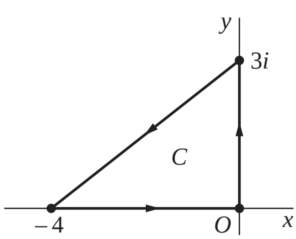

FIGURE 48

4. Let C_R denote the upper half of the circle $|z| = R\ (R > 2)$, taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz\right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

5. Let C_R be the circle $|z| = R\ (R > 1)$, described in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{\text{Log}\, z}{z^2}\, dz\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho\ (0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left|\int_{C_\rho} z^{-1/2} f(z)\, dz\right| \le 2\pi M\sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

7. Apply inequality (1), Sec. 43, to show that for all values of x in the interval $-1 \le x \le 1$, the functions*

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x + i\sqrt{1 - x^2}\cos\theta)^n\, d\theta \qquad (n = 0, 1, 2, \ldots)$$

satisfy the inequality $|P_n(x)| \le 1$.

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \ge |\sin x| \quad \text{and} \quad |\sin z| \ge |\sinh y|,$$

obtained in Exercises 8*(a)* and 9*(a)* of Sec. 34, show that $|\sin z| \ge 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \ge A$ for all points z lying on the contour C_N.

*These functions are actually polynomials in x. They are known as *Legendre polynomials* and are important in applied mathematics. See, for example, Chap. 4 of the book by Lebedev that is listed in Appendix 1.

(b) Using the final result in part *(a)*, show that

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

44. ANTIDERIVATIVES

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are *independent of path*. (Recall Examples 2 and 3 in Sec. 41.) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

The theorem contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. The extension involves the concept on an antiderivative of a continuous function $f(z)$ on a domain D, or a function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Note that an antiderivative is, of necessity, an analytic function. Note, too, that *an antiderivative of a given function $f(z)$ is unique except for an additive constant*. This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero; and, according to the theorem in Sec. 24, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem. *Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:*

(a) *$f(z)$ has an antiderivative $F(z)$ throughout D;*

(b) *the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely*

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) *the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.*

It should be emphasized that the theorem does *not* claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. The next section is devoted to the proof of the theorem and can be easily skipped by a reader who wishes to get on with other important aspects of integration theory. But we include here a number of examples illustrating how the theorem can be used.

EXAMPLE 1. The continuous function $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout the plane. Hence

$$\int_0^{1+i} z^2\,dz = \frac{z^3}{3}\Big]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$$

for every contour from $z = 0$ to $z = 1 + i$.

EXAMPLE 2. The function $f(z) = 1/z^2$, which is continuous everywhere except at the origin, has an antiderivative $F(z) = -1/z$ in the domain $|z| > 0$, consisting of the entire plane with the origin deleted. Consequently,

$$\int_C \frac{dz}{z^2} = 0$$

when C is the positively oriented circle (Fig. 49)

(1) $$z = 2\,e^{i\theta} \qquad (-\pi \le \theta \le \pi)$$

about the origin.

Note that the integral of the function $f(z) = 1/z$ around the same circle *cannot* be evaluated in a similar way. For, although the derivative of any branch $F(z)$ of $\log z$ is $1/z$ (Sec. 31), $F(z)$ is not differentiable, or even defined, along its branch cut. In particular, if a ray $\theta = \alpha$ from the origin is used to form the branch cut, $F'(z)$ fails to exist at the point where that ray intersects the circle C (see Fig. 49). So C does not lie in any domain throughout which $F'(z) = 1/z$, and one cannot make direct use of an antiderivative. Example 3, just below, illustrates how a combination of *two* different antiderivatives can be used to evaluate $f(z) = 1/z$ around C.

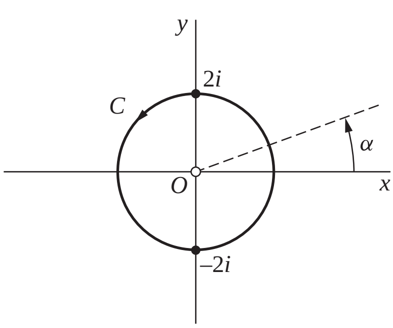

FIGURE 49

EXAMPLE 3. Let C_1 denote the right half

(2) $$z = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

of the circle C in Example 2. The principal branch

$$\operatorname{Log} z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi)$$

of the logarithmic function serves as an antiderivative of the function $1/z$ in the evaluation of the integral of $1/z$ along C_1 (Fig. 50):

$$\begin{aligned}\int_{C_1} \frac{dz}{z} &= \int_{-2i}^{2i} \frac{dz}{z} = \operatorname{Log} z \Big]_{-2i}^{2i} = \operatorname{Log}(2i) - \operatorname{Log}(-2i)\\ &= \left(\ln 2 + i\frac{\pi}{2}\right) - \left(\ln 2 - i\frac{\pi}{2}\right) = \pi i.\end{aligned}$$

This integral was evaluated in another way in Example 1, Sec. 41, where representation (2) for the semicircle was used.

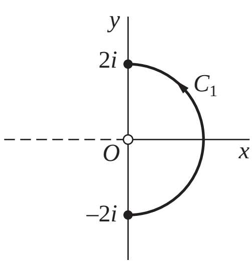

FIGURE 50

Next, let C_2 denote the *left* half

(3) $$z = 2\,e^{i\theta} \qquad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$$

of the same circle C and consider the branch

$$\log z = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function (Fig. 51). One can write

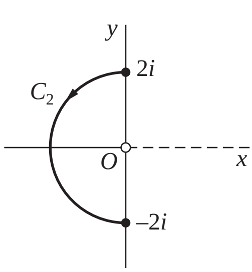

FIGURE 51

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = \log z\Big]_{2i}^{-2i} = \log(-2i) - \log(2i)$$
$$= \left(\ln 2 + i\frac{3\pi}{2}\right) - \left(\ln 2 + i\frac{\pi}{2}\right) = \pi i.$$

The value of the integral of $1/z$ around the entire circle $C = C_1 + C_2$ is thus obtained:

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

EXAMPLE 4. Let us use an antiderivative to evaluate the integral

$$\int_{C_1} z^{1/2}\, dz, \tag{4}$$

where the integrand is the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, 0 < \theta < 2\pi) \tag{5}$$

of the square root function and where C_1 is any contour from $z = -3$ to $z = 3$ that, except for its end points, lies above the x axis (Fig. 52). Although the integrand is piecewise continuous on C_1, and the integral therefore exists, the branch (5) of $z^{1/2}$ is not defined on the ray $\theta = 0$, in particular at the point $z = 3$. But another branch,

$$f_1(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

is defined and continuous everywhere on C_1. The values of $f_1(z)$ at all points on C_1 except $z = 3$ coincide with those of our integrand (5); so the integrand can be replaced by $f_1(z)$. Since an antiderivative of $f_1(z)$ is the function

$$F_1(z) = \frac{2}{3}z^{3/2} = \frac{2}{3}r\sqrt{r}e^{i3\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

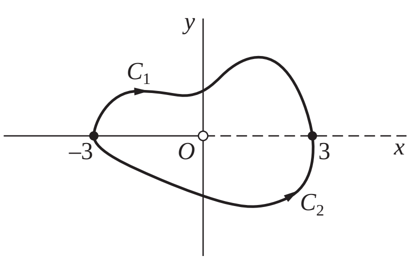

FIGURE 52

we can now write

$$\int_{C_1} z^{1/2}\,dz = \int_{-3}^{3} f_1(z)\,dz = F_1(z)\Big]_{-3}^{3} = 2\sqrt{3}(e^{i0} - e^{i3\pi/2}) = 2\sqrt{3}(1+i).$$

(Compare with Example 1 in Sec. 42.)

The integral

$$\int_{C_2} z^{1/2}\,dz \tag{6}$$

of the function (5) over any contour C_2 that extends from $z = -3$ to $z = 3$ *below* the real axis can be evaluated in a similar way. In this case, we can replace the integrand by the branch

$$f_2(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right),$$

whose values coincide with those of the integrand at $z = -3$ and at all points on C_2 below the real axis. This enables us to use an antiderivative of $f_2(z)$ to evaluate integral (6). Details are left to the exercises.

45. PROOF OF THE THEOREM

To prove the theorem in the previous section, it is sufficient to show that statement *(a)* implies statement *(b)*, that statement *(b)* implies statement *(c)*, and finally that statement *(c)* implies statement *(a)*.

Let us assume that statement *(a)* is true, or that $f(z)$ has an antiderivative $F(z)$ on the domain D being considered. To show how statement *(b)* follows, we need to show that integration in independent of path in D and that the fundamental theorem of calculus can be extended using $F(z)$. If a contour C from z_1 to z_2 is a *smooth* arc lying in D, with parametric representation $z = z(t)$ $(a \le t \le b)$, we know from Exercise 5, Sec. 39, that

$$\frac{d}{dt}F[z(t)] = F'[z(t)]z'(t) = f[z(t)]z'(t) \quad (a \le t \le b).$$

Because the fundamental theorem of calculus can be extended so as to apply to complex-valued functions of a real variable (Sec. 38), it follows that

$$\int_C f(z)\,dz = \int_a^b f\big[z(t)\big]z'(t)\,dt = F[z(t)]\Big]_a^b = F[z(b)] - F[z(a)].$$

Since $z(b) = z_2$ and $z(a) = z_1$, the value of this contour integral is then

$$F(z_2) - F(z_1);$$

and that value is evidently independent of the contour C as long as C extends from z_1 to z_2 and lies entirely in D. That is,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) = F(z)\Big]_{z_1}^{z_2} \tag{1}$$

when C is smooth. Expression (1) is also valid when C is *any* contour, not necessarily a smooth one, that lies in D. For, if C consists of a finite number of smooth arcs C_k $(k = 1, 2, \ldots, n)$, each C_k extending from a point z_k to a point z_{k+1}, then

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(z)\,dz = \sum_{k=1}^{n} [F(z_{k+1}) - F(z_k)].$$

Because the last sum here telescopes to $F(z_{n+1}) - F(z_1)$, we arrive at the expression

$$\int_C f(z)\,dz = F(z_{n+1}) - F(z_1).$$

(Compare with Example 3, Sec. 41.) The fact that statement *(b)* follows from statement *(a)* is now established.

To see that statement *(b)* implies statement *(c)*, we now show that the value of any integral around a closed contour in D is zero when integration is independent of path there. To do this, we let z_1 and z_2 denote two points on any closed contour C lying in D and form two paths C_1 and C_2, each with initial point z_1 and final point z_2, such that $C = C_1 - C_2$ (Fig. 53). Assuming that integration is independent of path in D, one can write

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz, \tag{2}$$

or

$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0. \tag{3}$$

That is, the integral of $f(z)$ around the closed contour $C = C_1 - C_2$ has value zero.

It remains to show statement *(c)* implies statement *(a)*. That is, we need to show that if integrals of $f(z)$ around closed contours in D always have value zero,

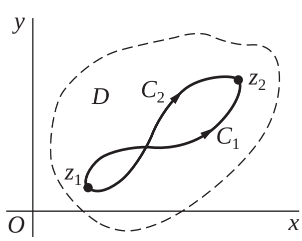

FIGURE 53

then $f(z)$ has an antiderivative on D. Assuming that the values of such integrals are in fact zero, we start by showing that integration is independent of path in D. We let C_1 and C_2 denote any two contours, lying in D, from a point z_1 to a point z_2 and observe that since integrals around closed paths lying in D have value zero, equation (3) holds (see Fig. 53). Thus equation (2) holds. Integration is, therefore, independent of path in D; and we can define the function

$$F(z) = \int_{z_0}^{z} f(s)\,ds$$

on D. The proof of the theorem is complete once we show that $F'(z) = f(z)$ everywhere in D. We do this by letting $z + \Delta z$ be any point distinct from z and lying in some neighborhood of z that is small enough to be contained in D. Then

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds = \int_{z}^{z+\Delta z} f(s)\,ds,$$

where the path of integration may be selected as a line segment (Fig. 54). Since

$$\int_{z}^{z+\Delta z} ds = \Delta z$$

(see Exercise 5, Sec. 42), one can write

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,ds;$$

and it follows that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(s) - f(z)]\,ds.$$

But f is continuous at the point z. Hence, for each positive number ε, a positive number δ exists such that

$$|f(s) - f(z)| < \varepsilon \quad \text{whenever} \quad |s - z| < \delta.$$

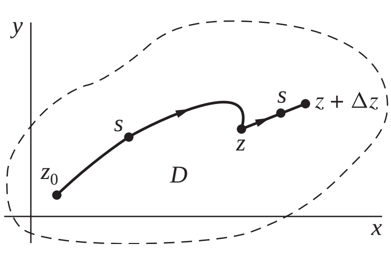

FIGURE 54

Consequently, if the point $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon;$$

that is,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z),$$

or $F'(z) = f(z)$.

EXERCISES

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n\, dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) $\displaystyle\int_i^{i/2} e^{\pi z}\, dz;$ *(b)* $\displaystyle\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz;$ *(c)* $\displaystyle\int_1^3 (z-2)^3\, dz.$

Ans. (a) $(1+i)/\pi$; *(b)* $e + (1/e)$; *(c)* 0.

3. Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \qquad (n = \pm 1, \pm 2, \ldots)$$

when C_0 is any closed contour which does not pass through the point z_0. [Compare with Exercise 10*(b)*, Sec. 42.]

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 44, to show that integral (6) there has value $2\sqrt{3}(-1+i)$. Note that the value of the integral of the function (5) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

5. Show that

$$\int_{-1}^{1} z^i\, dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 7, Sec. 42.)

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

46. CAUCHY–GOURSAT THEOREM

In Sec. 44, we saw that when a continuous function f has an antiderivative in a domain D, the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero. In this section, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a *simple* closed contour (Sec. 39) is zero. The theorem is central to the theory of functions of a complex variable; and some modifications of it, involving certain special types of domains, will be given in Secs. 48 and 49.

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, described in the *positive sense* (counterclockwise), and we assume that f is analytic at each point interior to and on C. According to Sec. 40,

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt; \tag{1}$$

and if

$$f(z) = u(x, y) + iv(x, y) \quad \text{and} \quad z(t) = x(t) + iy(t),$$

the integrand $f[z(t)]z'(t)$ in expression (1) is the product of the functions

$$u[x(t), y(t)] + iv[x(t), y(t)], \qquad x'(t) + iy'(t)$$

of the real variable t. Thus

$$\int_C f(z)\;dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt. \tag{2}$$

In terms of line integrals of real-valued functions of two real variables, then,

$$\int_C f(z)\;dz = \int_C u\;dx - v\;dy + i\int_C v\;dx + u\;dy. \tag{3}$$

Observe that expression (3) can be obtained formally by replacing $f(z)$ and dz on the left with the binomials

$$u + iv \quad \text{and} \quad dx + i\,dy,$$

respectively, and expanding their product. Expression (3) is, of course, also valid when C is any contour, not necessarily a simple closed one, and when $f[z(t)]$ is only piecewise continuous on it.

We next recall a result from calculus that enables us to express the line integrals on the right in equation (3) as double integrals. Suppose that two real-valued functions $P(x, y)$ and $Q(x, y)$, together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. According to *Green's theorem*,

$$\int_C P\,dx + Q\,dy = \int\!\!\int_R (Q_x - P_y)\,dA.$$

Now f is continuous in R, since it is analytic there. Hence the functions u and v are also continuous in R. Likewise, if the derivative f' of f is continuous in R, so are the first-order partial derivatives of u and v. Green's theorem then enables us to rewrite equation (3) as

$$\int_C f(z)\,dz = \int\!\!\int_R (-v_x - u_y)\,dA + i\int\!\!\int_R (u_x - v_y)\,dA. \tag{4}$$

But, in view of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

the integrands of these two double integrals are zero throughout R. So *when f is analytic in R and f' is continuous there,*

$$\int_C f(z)\,dz = 0. \tag{5}$$

This result was obtained by Cauchy in the early part of the nineteenth century.

Note that once it has been established that the value of this integral is zero, the orientation of C is immaterial. That is, statement (5) is also true if C is taken in the clockwise direction, since then

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz = 0.$$

EXAMPLE. If C is any simple closed contour, in either direction, then

$$\int_C \exp(z^3)\,dz = 0.$$

This is because the composite function $f(z) = \exp(z^3)$ is analytic everywhere and its derivative $f'(z) = 3z^2\exp(z^3)$ is continuous everywhere.

Goursat* was the first to prove that *the condition of continuity on f' can be omitted*. Its removal is important and will allow us to show, for example, that the derivative f' of an analytic function f is analytic without having to assume the continuity of f', which follows as a consequence. We now state the revised form of Cauchy's result, known as the *Cauchy–Goursat theorem*.

Theorem. *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\,dz = 0.$$

*E. Goursat (1858–1936), pronounced *gour-sah′*.

The proof is presented in the next section, where, to be specific, we assume that C is positively oriented. The reader who wishes to accept this theorem without proof may pass directly to Sec. 48.

47. PROOF OF THE THEOREM

We preface the proof of the Cauchy–Goursat theorem with a lemma. We start by forming subsets of the region R which consists of the points on a positively oriented simple closed contour C together with the points interior to C. To do this, we draw equally spaced lines parallel to the real and imaginary axes such that the distance between adjacent vertical lines is the same as that between adjacent horizontal lines. We thus form a finite number of closed square subregions, where each point of R lies in at least one such subregion and each subregion contains points of R. We refer to these square subregions simply as *squares*, always keeping in mind that by a square we mean a boundary together with the points interior to it. If a particular square contains points that are not in R, we remove those points and call what remains a *partial square*. We thus *cover* the region R with a finite number of squares and partial squares (Fig. 55), and our proof of the following lemma starts with this covering.

Lemma. *Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C together with the points on C itself. For any positive number ε, the region R can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \ldots, n$, such that in each one there is a fixed point z_j for which the inequality*

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \tag{1}$$

is satisfied by all points other than z_j in that square or partial square.

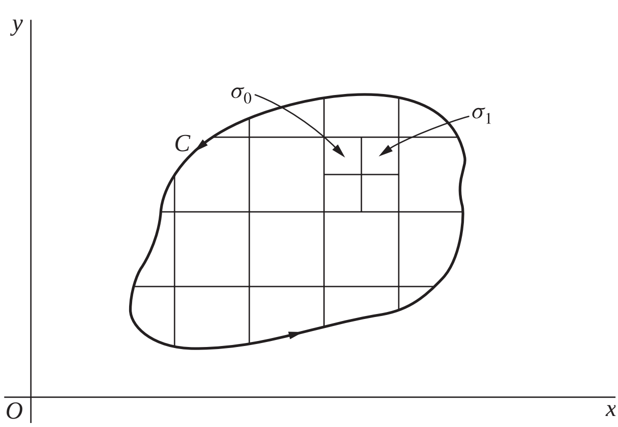

FIGURE 55

To start the proof, we consider the possibility that in the covering constructed just prior to the statement of the lemma, there is some square or partial square in which no point z_j exists such that inequality (1) holds for all other points z in it. If that subregion is a square, we construct four smaller squares by drawing line segments joining the midpoints of its opposite sides (Fig. 55). If the subregion is a partial square, we treat the whole square in the same manner and then let the portions that lie outside of R be discarded. If in any one of these smaller subregions, no point z_j exists such that inequality (1) holds for all other points z in it, we construct still smaller squares and partial squares, etc. When this is done to each of the original subregions that requires it, we find that *after a finite number of steps*, the region R can be covered with a finite number of squares and partial squares such that the lemma is true.

To verify this, we suppose that the needed points z_j do *not* exist after subdividing one of the original subregions a finite number of times and reach a contradiction. We let σ_0 denote that subregion if it is a square; if it is a partial square, we let σ_0 denote the entire square of which it is a part. After we subdivide σ_0, at least one of the four smaller squares, denoted by σ_1, must contain points of R but no appropriate point z_j. We then subdivide σ_1 and continue in this manner. It may be that after a square σ_{k-1} $(k = 1, 2, \ldots)$ has been subdivided, more than one of the four smaller squares constructed from it can be chosen. To make a specific choice, we take σ_k to be the one lowest and then furthest to the left.

In view of the manner in which the nested infinite sequence

$$\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \ldots \tag{2}$$

of squares is constructed, it is easily shown (Exercise 9, Sec. 49) that there is a point z_0 common to each σ_k; also, each of these squares contains points of R other than possibly z_0. Recall how the sizes of the squares in the sequence are decreasing, and note that any δ neighborhood $|z - z_0| < \delta$ of z_0 contains such squares when their diagonals have lengths less than δ. Every δ neighborhood $|z - z_0| < \delta$ therefore contains points of R distinct from z_0, and this means that z_0 is an accumulation point of R. Since the region R is a closed set, it follows that z_0 is a point in R. (See Sec. 11.)

Now the function f is analytic throughout R and, in particular, at z_0. Consequently, $f'(z_0)$ exists, According to the definition of derivative (Sec. 19), there is, for each positive number ε, a δ neighborhood $|z - z_0| < \delta$ such that the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

is satisfied by all points distinct from z_0 in that neighborhood. But the neighborhood $|z - z_0| < \delta$ contains a square σ_K when the integer K is large enough that the length of a diagonal of that square is less than δ (Fig. 56). Consequently, z_0 serves as the point z_j in inequality (1) for the subregion consisting of the square σ_K or a part of σ_K. Contrary to the way in which the sequence (2) was formed, then, it is not necessary to subdivide σ_K. We thus arrive at a contradiction, and the proof of the lemma is complete.

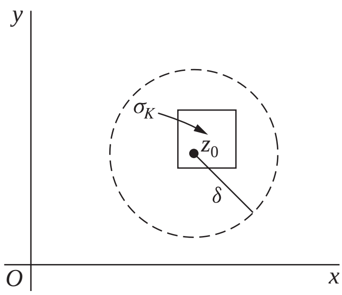

FIGURE 56

Continuing with a function f which is analytic throughout a region R consisting of a positively oriented simple closed contour C and points interior to it, we are now ready to prove the Cauchy–Goursat theorem, namely that

$$\int_C f(z)\,dz = 0. \tag{3}$$

Given an arbitrary positive number ε, we consider the covering of R in the statement of the lemma. We then define on the jth square or partial square a function $\delta_j(z)$ whose values are $\delta_j(z_j) = 0$, where z_j is the fixed point in inequality (1), and

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \quad \text{when } z \neq z_j. \tag{4}$$

According to inequality (1),

$$|\delta_j(z)| < \varepsilon \tag{5}$$

at all points z in the subregion on which $\delta_j(z)$ is defined. Also, the function $\delta_j(z)$ is continuous throughout the subregion since $f(z)$ is continuous there and

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0.$$

Next, we let C_j $(j = 1, 2, \ldots, n)$ denote the positively oriented boundaries of the above squares or partial squares covering R. In view of our definition of $\delta_j(z)$, the value of f at a point z on any particular C_j can be written

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z);$$

and this means that

$$\int_{C_j} f(z)\,dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz + f'(z_j) \int_{C_j} z\,dz + \int_{C_j} (z - z_j)\delta_j(z)\,dz. \tag{6}$$

But

$$\int_{C_j} dz = 0 \quad \text{and} \quad \int_{C_j} z\, dz = 0$$

since the functions 1 and z possess antiderivatives everywhere in the finite plane. So equation (6) reduces to

$$(7) \qquad \int_{C_j} f(z)\, dz = \int_{C_j} (z - z_j)\delta_j(z)\, dz \qquad (j = 1, 2, \ldots, n).$$

The sum of all n integrals on the left in equations (7) can be written

$$\sum_{j=1}^{n} \int_{C_j} f(z)\, dz = \int_C f(z)\, dz$$

since the two integrals along the common boundary of every pair of adjacent subregions cancel each other, the integral being taken in one sense along that line segment in one subregion and in the opposite sense in the other (Fig. 57). Only the integrals along the arcs that are parts of C remain. Thus, in view of equations (7),

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)\, dz\,;$$

and so

$$(8) \qquad \left| \int_C f(z)\, dz \right| \le \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)\, dz \right|.$$

We now use the theorem in Sec. 43 to find an upper bound for each modulus on the right in inequality (8). To do this, we first recall that each C_j coincides either

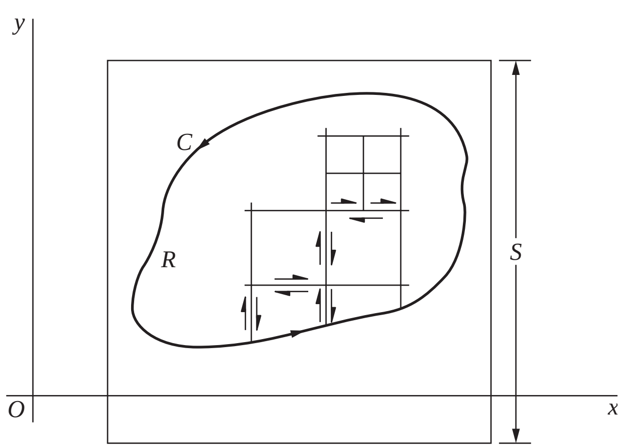

FIGURE 57

entirely or partially with the boundary of a square. In either case, we let s_j denote the length of a side of the square. Since, in the jth integral, both the variable z and the point z_j lie in that square,

$$|z - z_j| \leq \sqrt{2}s_j.$$

In view of inequality (5), then, we know that each integrand on the right in inequality (8) satisfies the condition

$$|(z - z_j)\delta_j(z)| = |z - z_j|\,|\delta_j(z)| < \sqrt{2}s_j\varepsilon. \tag{9}$$

As for the length of the path C_j, it is $4s_j$ if C_j is the boundary of a square. In that case, we let A_j denote the area of the square and observe that

$$\left|\int_{C_j} (z - z_j)\delta_j(z)\, dz\right| < \sqrt{2}s_j\varepsilon 4s_j = 4\sqrt{2}A_j\varepsilon. \tag{10}$$

If C_j is the boundary of a partial square, its length does not exceed $4s_j + L_j$, where L_j is the length of that part of C_j which is also a part of C. Again letting A_j denote the area of the full square, we find that

$$\left|\int_{C_j} (z - z_j)\delta_j(z)\, dz\right| < \sqrt{2}s_j\varepsilon(4s_j + L_j) < 4\sqrt{2}A_j\varepsilon + \sqrt{2}SL_j\varepsilon, \tag{11}$$

where S is the length of a side of some square that encloses the entire contour C as well as all of the squares originally used in covering R (Fig. 57). Note that the sum of all the A_j's does not exceed S^2.

If L denotes the length of C, it now follows from inequalities (8), (10), and (11) that

$$\left|\int_C f(z)\, dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon.$$

Since the value of the positive number ε is arbitrary, we can choose it so that the right-hand side of this last inequality is as small as we please. The left-hand side, which is independent of ε, must therefore be equal to zero; and statement (3) follows. This completes the proof of the Cauchy–Goursat theorem.

48. SIMPLY CONNECTED DOMAINS

A *simply connected* domain D is a domain such that every simple closed contour within it encloses only points of D. The set of points interior to a simple closed contour is an example. The annular domain between two concentric circles is, however, not simply connected. Domains that are not simply connected are discussed in the next section.

The closed contour in the Cauchy–Goursat theorem (Sec. 46) need not be simple when the theorem is adapted to simply connected domains. More precisely, the contour can actually cross itself. The following theorem allows for this possibility.

Theorem. *If a function f is analytic throughout a simply connected domain D, then*

$$\int_C f(z)\,dz = 0 \tag{1}$$

for every closed contour C lying in D.

The proof is easy if C is a *simple* closed contour or if it is a closed contour that intersects itself a *finite* number of times. For if C is simple and lies in D, the function f is analytic at each point interior to and on C; and the Cauchy–Goursat theorem ensures that equation (1) holds. Furthermore, if C is closed but intersects itself a finite number of times, it consists of a finite number of simple closed contours. This is illustrated in Fig. 58, where the simple closed contours C_k $(k = 1, 2, 3, 4)$ make up C. Since the value of the integral around each C_k is zero, according to the Cauchy–Goursat theorem, it follows that

$$\int_C f(z)\,dz = \sum_{k=1}^{4} \int_{C_k} f(z)\,dz = 0.$$

Subtleties arise if the closed contour has an *infinite* number of self-intersection points. One method that can sometimes be used to show that the theorem still applies is illustrated in Exercise 5, Sec. 49.*

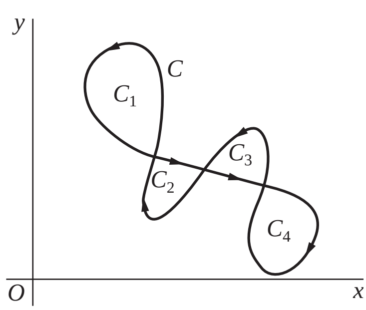

FIGURE 58

EXAMPLE. If C denotes any closed contour lying in the open disk $|z| < 2$ (Fig. 59), then

$$\int_C \frac{z\,e^z}{(z^2+9)^5}\,dz = 0.$$

*For a proof of the theorem involving more general paths of finite length, see, for example, Secs. 63–65 in Vol. I of the book by Markushevich that is cited in Appendix 1.

This is because the disk is a simply connected domain and the two singularities $z = \pm 3i$ of the integrand are exterior to the disk.

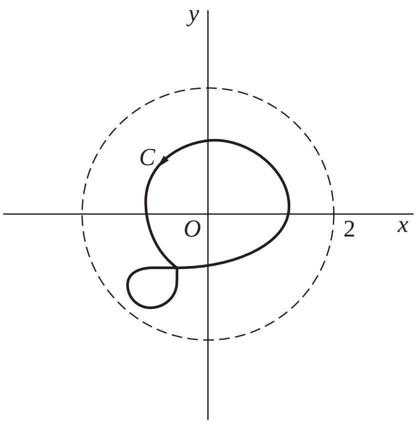

FIGURE 59

Corollary. *A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.*

We begin the proof of this corollary with the observation that a function f is continuous on a domain D when it is analytic there. Consequently, since equation (1) holds for the function in the hypothesis of this corollary and for each closed contour C in D, f has an antiderivative throughout D, according to the theorem in Sec. 44. Note that since the finite plane is simply connected, the corollary tells us that *entire functions always possess antiderivatives*.

49. MULTIPLY CONNECTED DOMAINS

A domain that is not simply connected (Sec. 48) is said to be *multiply connected*. The following theorem is an adaptation of the Cauchy–Goursat theorem to multiply connected domains.

Theorem. *Suppose that*

(a) C is a simple closed contour, described in the counterclockwise direction;

(b) C_k $(k = 1, 2, \ldots, n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common (Fig. 60).

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0. \tag{1}$$

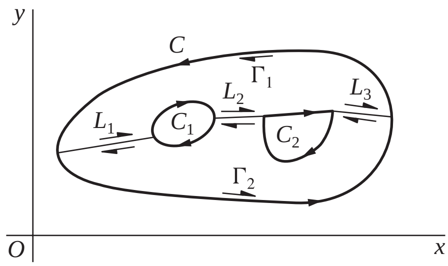

FIGURE 60

Note that in equation (1), the direction of each path of integration is such that the multiply connected domain lies to the *left* of that path.

To prove the theorem, we introduce a polygonal path L_1, consisting of a finite number of line segments joined end to end, to connect the outer contour C to the inner contour C_1. We introduce another polygonal path L_2 which connects C_1 to C_2; and we continue in this manner, with L_{n+1} connecting C_n to C. As indicated by the single-barbed arrows in Fig. 60, two simple closed contours Γ_1 and Γ_2 can be formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left. The Cauchy–Goursat theorem can now be applied to f on Γ_1 and Γ_2, and the sum of the values of the integrals over those contours is found to be zero. Since the integrals in opposite directions along each path L_k cancel, only the integrals along C and the C_k remain; and we arrive at statement (1).

Corollary. *Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2* (Fig. 61). *If a function f is analytic in the closed region consisting of those contours and all points between them, then*

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz. \tag{2}$$

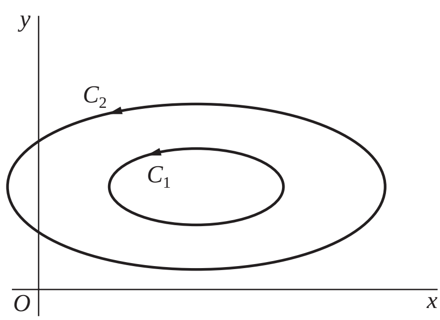

FIGURE 61

This corollary is known as the *principle of deformation of paths* since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes. To verify the corollary, we need only write equation (2) as

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

and apply the theroem.

EXAMPLE. When C is any positively oriented simple closed contour surrounding the origin, the corollary can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i.$$

This is done by constructing a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C (Fig. 62). Since (see Example 2, Sec. 42)

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

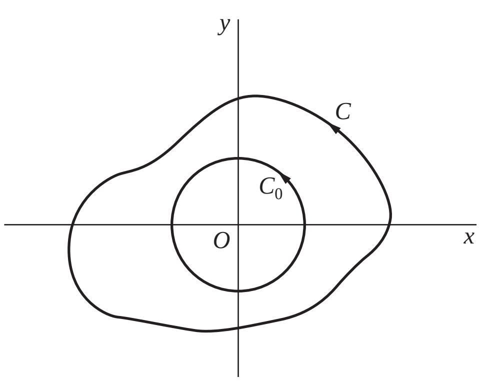

FIGURE 62

EXERCISES

1. Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$; *(b)* $f(z) = z\,e^{-z}$; *(c)* $f(z) = \dfrac{1}{z^2+2z+2}$;

(d) $f(z) = \text{sech}\, z$; *(e)* $f(z) = \tan z$; *(f)* $f(z) = \text{Log}\,(z+2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 63). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2 + 1}$; *(b)* $f(z) = \dfrac{z+2}{\sin(z/2)}$; *(c)* $f(z) = \dfrac{z}{1 - e^z}$.

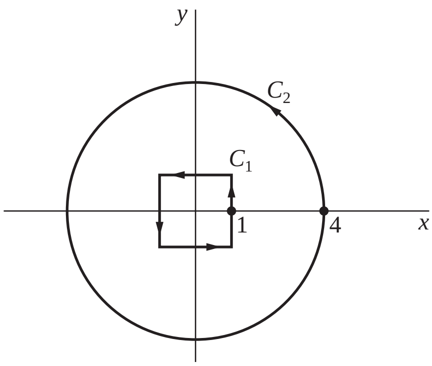

FIGURE 63

3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z - z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 10*(b)*, Sec. 42. Use that result and the corollary in Sec. 49 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$, described in the positive sense, then

$$\int_{C} (z - 2 - i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 64 can be written

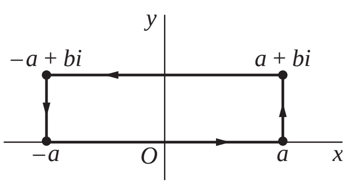

FIGURE 64

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\ dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\ dy.$$

(b) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part *(a)*.

5. According to Exercise 6, Sec. 39, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 65). Apply the Cauchy–Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)\ dz = \int_{C_3} f(z)\ dz \quad \text{and} \quad \int_{C_2} f(z)\ dz = -\int_{C_3} f(z)\ dz.$$

Conclude that even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)\ dz = 0.$$

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, “Advanced Calculus,” 3d ed., pp. 680–681, 1983.

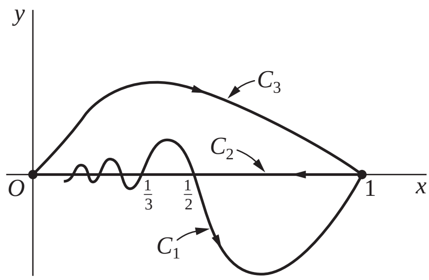

FIGURE 65

6. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)\, dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

7. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \overline{z}\, dz.$$

Suggestion: Note that expression (4), Sec. 46, can be used here even though the function $f(z) = \overline{z}$ is not analytic anywhere [see Example 2, Sec. 19].

8. *Nested Intervals.* An infinite sequence of closed intervals $a_n \le x \le b_n$ $(n = 0, 1, 2, \ldots)$ is formed in the following way. The interval $a_1 \le x \le b_1$ is either the left-hand or right-hand half of the first interval $a_0 \le x \le b_0$, and the interval $a_2 \le x \le b_2$ is then one of the two halves of $a_1 \le x \le b_1$, etc. Prove that there is a point x_0 which belongs to every one of the closed intervals $a_n \le x \le b_n$.

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \le a_n \le a_{n+1} < b_0$; hence they have a limit A as n tends to infinity. Show that the end points b_n also have a limit B. Then show that $A = B$, and write $x_0 = A = B$.

9. *Nested Squares.* A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 47). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the result in Exercise 8 to each of the sequences of closed intervals $a_n \le x \le b_n$ and $c_n \le y \le d_n$ $(n = 0, 1, 2, \ldots)$.

50. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}. \tag{1}$$

Formula (1) is called the *Cauchy integral formula*. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written as

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0), \tag{2}$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (2) tells us that

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)}\,dz = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 66). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 49) that

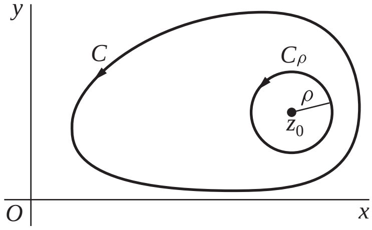

FIGURE 66

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}.$$

This enables us to write

$$\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{3}$$

But [see Exercise 10*(b)*, Sec. 42]

$$\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i;$$

and so equation (3) becomes

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{4}$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

$$|f(z)-f(z_0)| < \varepsilon \quad \text{whenever} \quad |z-z_0| < \delta. \tag{5}$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z-z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (5) holds when z is such a point; and the theorem in Sec. 43, giving upper bounds for the moduli of contour integrals, tells us that

$$\left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon.$$

In view of equation (4), then,

$$\left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it must be equal to zero. Hence equation (2) is valid, and the theorem is proved.

51. AN EXTENSION OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula in the theorem in Sec. 50 can be extended so as to provide an integral representation for derivatives of f at z_0. To obtain the extension, we consider a function f that is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. We then write the Cauchy integral formula as

$$(1) \qquad f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{s-z},$$

where z is interior to C and where s denotes points on C. Differentiating *formally* with respect to z under the integral sign here, without rigorous justification, we find that

$$(2) \qquad f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z)^2}.$$

To verify that $f'(z)$ exists and that expression (2) is in fact valid, we led d denote the smallest distance from z to points s on C and use expression (1) to write

$$\begin{aligned} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i} \int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z} \right) \frac{f(s)}{\Delta z}\, ds \\ &= \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)}, \end{aligned}$$

where $0 < |\Delta z| < d$ (see Fig. 67). Evidently, then,

$$(3) \qquad \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i} \int_C \frac{\Delta z\, f(s)\,ds}{(s-z-\Delta z)(s-z)^2}.$$

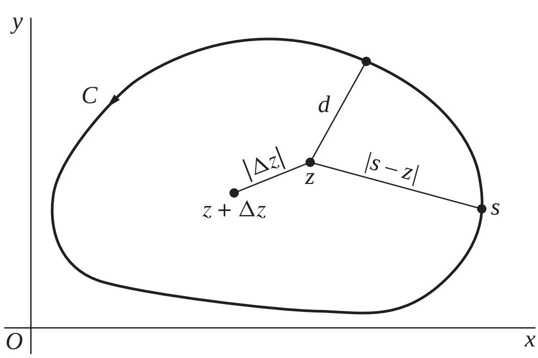

FIGURE 67

Next, we let M denote the maximum value of $|f(s)|$ on C and observe that since $|s-z| \geq d$ and $|\Delta z| < d$,

$$|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \geq d - |\Delta z| > 0.$$

Thus

$$\left| \int_C \frac{\Delta z\, f(s)\, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|)d^2} L,$$

where L is the length of C. Upon letting Δz tend to zero, we find from this inequality that the right-hand side of equation (3) also tends to zero. Consequently,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2} = 0\,;$$

and the desired expression for $f'(z)$ is established.

The same technique can be used to suggest and verify the expression

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)\, ds}{(s - z)^3}. \tag{4}$$

The details, which are outlined in Exercise 9, Sec. 52, are left to the reader. Mathematical induction can, moreover, be used to obtain the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{5}$$

The verification is considerably more involved than for just $n = 1$ and $n = 2$, and we refer the interested reader to other texts for it.* Note that with the agreement that

$$f^{(0)}(z) = f(z) \quad \text{and} \quad 0! = 1,$$

expression (5) is also valid when $n = 0$, in which case it becomes the Cauchy integral formula (1).

When written in the form

$$\int_C \frac{f(z)\; dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \ldots), \tag{6}$$

expressions (1) and (5) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 50 when $n = 0$.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

---

*See, for example, pp. 299–301 in Vol. I of the book by Markushevich, cited in Appendix 1.

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (6) shows that

$$\int_C \frac{dz}{z - z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = 0 \qquad (n = 1, 2, \ldots).$$

(Compare with Exercise 10*(b)*, Sec. 42.)

52. SOME CONSEQUENCES OF THE EXTENSION

We turn now to some important consequences of the extension of the Cauchy integral formula in the previous section.

Theorem 1. *If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.*

To prove this remarkable theorem, we assume that a function f is analytic at a point z_0. There must, then, be a neighborhood $|z - z_0| < \varepsilon$ of z_0 throughout which f is analytic (see Sec. 24). Consequently, there is a positively oriented circle C_0, centered at z_0 and with radius $\varepsilon/2$, such that f is analytic inside and on C_0 (Fig. 68). From expression (4), Sec. 51, we know that

$$f''(z) = \frac{1}{\pi i} \int_{C_0} \frac{f(s)\,ds}{(s - z)^3}$$

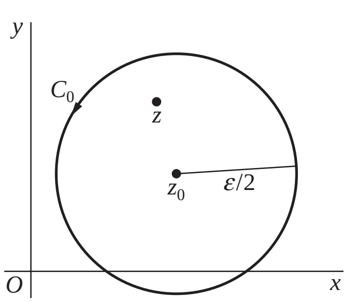

FIGURE 68

at each point z interior to C_0, and the existence of $f''(z)$ throughout the neighborhood $|z - z_0| < \varepsilon/2$ means that f' is analytic at z_0. One can apply the same argument to the analytic function f' to conclude that its derivative f'' is analytic, etc. Theorem 1 is now established.

As a consequence, when a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic at a point $z = (x, y)$, the differentiability of f' ensures the continuity of f' there (Sec. 19). Then, since (Sec. 21)

$$f'(z) = u_x + iv_x = v_y - iu_y,$$

we may conclude that the first-order partial derivatives of u and v are continuous at that point. Furthermore, since f'' is analytic and continuous at z and since

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx},$$

etc., we arrive at a corollary that was anticipated in Sec. 26, where harmonic functions were introduced.

Corollary. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.*

The proof of the next theorem, due to E. Morera (1856–1909), depends on the fact that the derivative of an analytic function is itself analytic, as stated in Theorem 1.

Theorem 2. *Let f be continuous on a domain D. If*

$$\int_C f(z)\, dz = 0 \tag{1}$$

for every closed contour C in D, then f is analytic throughout D.

In particular, when D is *simply connected*, we have for the class of continuous functions defined on D the converse of the theorem in Sec. 48, which is the adaptation of the Cauchy–Goursat theorem to such domains.

To prove the theorem here, we observe that when its hypothesis is satisfied, the theorem in Sec. 44 ensures that f has an antiderivative in D; that is, there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of F, it then follows from Theorem 1 that f is analytic in D.

Our final theorem here will be essential in the next section.

Theorem 3. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R* (Fig. 69). *If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

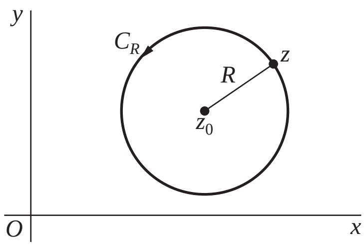

FIGURE 69

Inequality (2) is called *Cauchy's inequality* and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R}\frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

which is a slightly different form of equation (6), Sec. 51, when n is a positive integer. We need only apply the theorem in Sec. 43, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)}$; *(b)* $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$; *(c)* $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; *(e)* $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; *(b)* $\pi i/4$; *(c)* $-\pi i/2$; *(d)* 0; *(e)* $i\pi\sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2+4}$; *(b)* $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans. (a) $\pi/2$; *(b)* $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \qquad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following a procedure used in Sec. 51, prove that the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}\,(-\pi \leq \theta \leq \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

8. *(a)* With the aid of the binomial formula (Sec. 3), show that for each value of n, the function

$$P_n(z) = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n}(z^2 - 1)^n \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n.*

*These are Legendre polynomials, which appear in Exercise 7, Sec. 43, when $z = x$. See the footnote to that exercise.

(b) Let C denote any positively oriented simple closed contour surrounding a fixed point z. With the aid of the integral representation (5), Sec. 51, for the nth derivative of a function, show that the polynomials in part *(a)* can be expressed in the form

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds \qquad (n = 0, 1, 2, \ldots).$$

(c) Point out how the integrand in the representation for $P_n(z)$ in part *(b)* can be written $(s+1)^n/(s-1)$ if $z = 1$. Then apply the Cauchy integral formula to show that

$$P_n(1) = 1 \qquad (n = 0, 1, 2, \ldots).$$

Similarly, show that

$$P_n(-1) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

9. Follow these steps below to verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}$$

in Sec. 51.

(a) Use expression (2) in Sec. 51 for $f'(z)$ to show that

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3} = \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)\,ds.$$

(b) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 51 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part *(a)* is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(c) Use the results in parts *(a)* and *(b)* to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

53. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 52 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as *Liouville's theorem*, states this result in a somewhat different way.

Theorem 1. *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant througout the plane.*

To start the proof, we assume that f is as stated and note that since f is entire, Theorem 3 in Sec. 52 can be applied with any choice of z_0 and R. In particular, Cauchy's inequality (2) in that theorem tells us that when $n = 1$,

$$|f'(z_0)| \le \frac{M_R}{R}. \tag{1}$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that $|f(z)| \le M$ for all z; and, because the constant M_R in inequality (1) is always less than or equal to M, it follows that

$$\left|f'(z_0)\right| \le \frac{M}{R}, \tag{2}$$

where R can be arbitrarily large. Now the number M in inequality (2) is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function, according to the theorem in Sec. 24.

The following theorem, called the *fundamental theorem of algebra*, follows readily from Liouville's theorem.

Theorem 2. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \ne 0)$$

of degree n $(n \ge 1)$ has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that $P(z)$ is *not* zero for any value of z. Then the reciprocal

$$f(z) = \frac{1}{P(z)}$$

is clearly entire, and it is also bounded in the complex plane.

To show that its is bounded, we first write

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}, \tag{3}$$

so that

$$P(z) = (a_n + w)z^n. \tag{4}$$

Next, we observe that a sufficiently large positive number R can be found such that the modulus of each of the quotients in expression (3) is less than the number $|a_n|/(2n)$ when $|z| > R$. The generalized triangle inequality (10), Sec. 4, which applies to n complex numbers, thus shows that

$$|w| < \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

Consequently,

$$|a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

This inequality and expression (4) enable us to write

$$(5) \qquad |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| > R.$$

Evidently, then,

$$|f(z)| = \frac{1}{|P(z)|} < \frac{2}{|a_n|R^n} \quad \text{whenever} \quad |z| > R.$$

So f is bounded in the region *exterior to* the disk $|z| \leq R$. But f is continuous in that closed disk, and this means that f is bounded there too (Sec. 18). Hence f is bounded in the entire plane.

It now follows from Liouville's theorem that $f(z)$, and consequently $P(z)$, is constant. But $P(z)$ is not constant, and we have reached a contradiction.*

The fundamental theorem tells us that any polynomial $P(z)$ of degree n ($n \geq 1$) can be expressed as a product of linear factors:

$$(6) \qquad P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n),$$

where c and z_k ($k = 1, 2, \ldots, n$) are complex constants. More precisely, the theorem ensures that $P(z)$ has a zero z_1. Then, according to Exercise 9, Sec. 54,

$$P(z) = (z - z_1)Q_1(z),$$

where $Q_1(z)$ is a polynomial of degree $n - 1$. The same argument, applied to $Q_1(z)$, reveals that there is a number z_2 such that

$$P(z) = (z - z_1)(z - z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we arrive at expression (6). Some of the constants z_k in expression (6) may, of course, appear more than once, and it is clear that $P(z)$ can have no more than n *distinct* zeros.

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*For an interesting proof of the fundamental theorem using the Cauchy–Goursat theorem, see R. P. Boas, Jr., *Amer. Math. Monthly*, Vol. 71, No. 2, p. 180, 1964.

54. MAXIMUM MODULUS PRINCIPLE

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

Lemma. *Suppose that $|f(z)| \le |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.*

To prove this, we assume that f satisfies the stated conditions and let z_1 be any point other than z_0 in the given neighborhood. We then let ρ be the distance between z_1 and z_0. If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, centered at z_0 and passing through z_1 (Fig. 70), the Cauchy integral formula tells us that

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)\,dz}{z - z_0}; \tag{1}$$

and the parametric representation

$$z = z_0 + \rho e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

for C_ρ enables us to write equation (1) as

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta. \tag{2}$$

We note from expression (2) that when a function is analytic within and on a given circle, its value at the center is the arithmetic mean of its values on the circle. This result is called *Gauss's mean value theorem*.

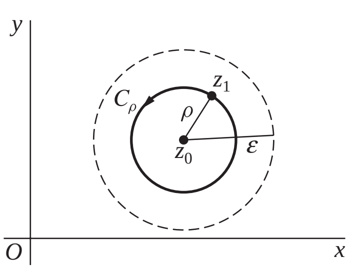

FIGURE 70

From equation (2), we obtain the inequality

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta. \tag{3}$$

On the other hand, since

$$|f(z_0 + \rho e^{i\theta})| \le |f(z_0)| \qquad (0 \le \theta \le 2\pi), \tag{4}$$

we find that

$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \le \int_0^{2\pi} |f(z_0)|\, d\theta = 2\pi |f(z_0)|.$$

Thus

$$|f(z_0)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta. \tag{5}$$

It is now evident from inequalities (3) and (5) that

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta,$$

or

$$\int_0^{2\pi} [|f(z_0)| - |f(z_0 + \rho e^{i\theta})|]\, d\theta = 0.$$

The integrand in this last integral is continuous in the variable θ; and, in view of condition (4), it is greater than or equal to zero on the entire interval $0 \le \theta \le 2\pi$. Because the value of the integral is zero, then, the integrand must be identically equal to zero. That is,

$$|f(z_0 + \rho e^{i\theta})| = |f(z_0)| \qquad (0 \le \theta \le 2\pi). \tag{6}$$

This shows that $|f(z)| = |f(z_0)|$ *for all points* z *on the circle* $|z - z_0| = \rho$.

Finally, since z_1 is any point in the deleted neighborhood $0 < |z - z_0| < \varepsilon$, we see that the equation $|f(z)| = |f(z_0)|$ is, in fact, satisfied by all points z lying on any circle $|z - z_0| = \rho$, where $0 < \rho < \varepsilon$. Consequently, $|f(z)| = |f(z_0)|$ everywhere in the neighborhood $|z - z_0| < \varepsilon$. But we know from Example 4. Sec. 25, that when the modulus of an analytic function is constant in a domain, the function itself is constant there. Thus $f(z) = f(z_0)$ for each point z in the neighborhood, and the proof of the lemma is complete.

This lemma can be used to prove the following theorem, which is known as the *maximum modulus principle*.

Theorem. *If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it.*

Given that f is analytic in D, we shall prove the theorem by assuming that $|f(z)|$ *does* have a maximum value at some point z_0 in D and then showing that $f(z)$ must be constant throughout D.

The general approach here is similar to that taken in the proof of the lemma in Sec. 27. We draw a polygonal line L lying in D and extending from z_0 to any other point P in D. Also, d represents the shortest distance from points on L to the boundary of D. When D is the entire plane, d may have any positive value. Next, we observe that there is a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L such that z_n coincides with the point P and

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

In forming a finite sequence of neighborhoods (Fig. 71)

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n$$

where each N_k has center z_k and radius d, we see that f is analytic in each of these neighborhoods, which are all contained in D, and that the center of each neighborhood N_k $(k = 1, 2, \ldots, n)$ lies in the neighborhood N_{k-1}.

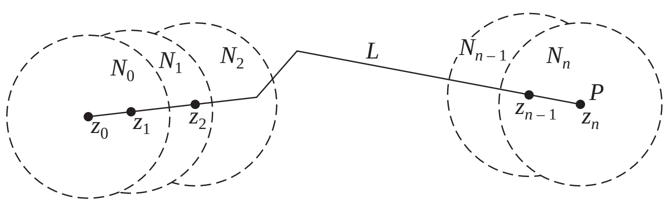

FIGURE 71

Since $|f(z)|$ was assumed to have a maximum value in D at z_0, it also has a maximum value in N_0 at that point. Hence, according to the preceding lemma, $f(z)$ has the constant value $f(z_0)$ throughout N_0. In particular, $f(z_1) = f(z_0)$. This means that $|f(z)| \leq |f(z_1)|$ for each point z in N_1; and the lemma can be applied again, this time telling us that

$$f(z) = f(z_1) = f(z_0)$$

when z is in N_1. Since z_2 is in N_1, then, $f(z_2) = f(z_0)$. Hence $|f(z)| \leq |f(z_2)|$ when z is in N_2; and the lemma is once again applicable, showing that

$$f(z) = f(z_2) = f(z_0)$$

when z is in N_2. Continuing in this manner, we eventually reach the neighborhood N_n and arrive at the fact that $f(z_n) = f(z_0)$.

Recalling that z_n coincides with the point P, which is any point other than z_0 in D, we may conclude that $f(z) = f(z_0)$ for *every* point z in D. Inasmuch as $f(z)$ has now been shown to be constant throughout D, the theorem is proved.

If a function f that is analytic at each point in the interior of a closed bounded region R is also continuous throughout R, then the modulus $|f(z)|$ has a maximum value somewhere in R (Sec. 18). That is, there exists a nonnegative constant M such that $|f(z)| \leq M$ for all points z in R, and equality holds for at least one such point.

If f is a constant function, then $|f(z)| = M$ for all z in R. If, however, $f(z)$ is not constant, then, according to the theorem just proved, $|f(z)| \neq M$ for any point z in the interior of R. We thus arrive at an important corollary.

Corollary. *Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.*

EXAMPLE. Let R denote the rectangular region $0 \leq x \leq \pi$, $0 \leq y \leq 1$. The corollary tells us that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R that occurs somewhere on the boundary of R and not in its interior. This can be verified directly by writing (see Sec. 34)

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

and noting that the term $\sin^2 x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh^2 y$ is greatest when $y = 1$. Thus the maximum value of $|f(z)|$ in R occurs at the boundary point $z = (\pi/2, 1)$ and at no other point in R (Fig. 72).

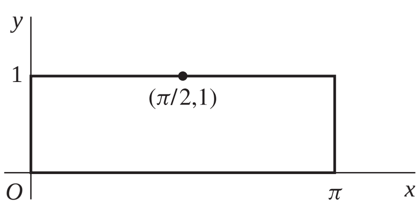

FIGURE 72

When the function f in the corollary is written $f(z) = u(x, y) + iv(x, y)$, *the component function $u(x, y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior*, where it is harmonic (Sec. 26). This is because the composite function $g(z) = \exp[f(z)]$ is continuous in R and analytic and not constant in the interior. Hence its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in R, must assume its maximum value in R on the boundary. In view of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.

Properties of *minimum* values of $|f(z)|$ and $u(x, y)$ are treated in the exercises.

EXERCISES

1. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is, $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

Suggestion: Apply Liouville's theorem (Sec. 53) to the function $g(z) = \exp[f(z)]$.

2. Show that for R sufficiently large, the polynomial $P(z)$ in Theorem 2, Sec. 53, satisfies the inequality

$$|P(z)| < 2|a_n||z|^n \quad \text{whenever} \quad |z| \geq R.$$

[Compare with the first of inequalities (5), Sec. 53.]

Suggestion: Observe that there is a positive number R such that the modulus of each quotient in expression (3), Sec. 53, is less than $|a_n|/n$ when $|z| > R$.

3. Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 54) to the function $g(z) = 1/f(z)$.

4. Use the function $f(z) = z$ to show that in Exercise 3 the condition $f(z) \neq 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ *can* reach its minimum value at an interior point when the minimum value is zero.

5. Consider the function $f(z) = (z+1)^2$ and the closed triangular region R with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values, thus illustrating results in Sec. 54 and Exercise 3.

Suggestion: Interpret $|f(z)|$ as the square of the distance between z and -1.

Ans. $z = 2, z = 0$.

6. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 3.)

7. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1, 0 \leq y \leq \pi$. Illustrate results in Sec. 54 and Exercise 6 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

Ans. $z = 1, z = 1 + \pi i$.

8. Let the function $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed bounded region R, and suppose that it is analytic and not constant in the interior of R. Show that the component function $v(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never in the interior, where it is harmonic.

Suggestion: Apply results in Sec. 54 and Exercise 6 to the function $g(z) = -if(z)$.

9. Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$. Show in the following way that

$$P(z) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$.

(a) Verify that

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z\,z_0^{k-2} + z_0^{k-1}) \qquad (k = 2, 3, \ldots).$$

(b) Use the factorization in part *(a)* to show that

$$P(z) - P(z_0) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$, and deduce the desired result from this.