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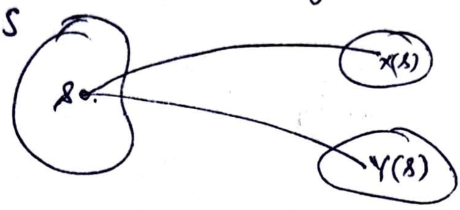
Two - dimensional Random variables:

Def: Let S be the sample space associated with a random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. ~~to~~ then we call the pair (X, Y) is a $\boxed{\text{2-dimensional random variable.}}$ (or a random vector)

Ex: The height and weight of a randomly chosen person.



Note: If $X_1 = X_1(s)$, $X_2 = X_2(s)$, ---, $X_n = X_n(s)$ are n-functions each assigning a real number to every out come $s \in S$, we call $(X_1, X_2, ---, X_n)$ an n- dimensional random variable.

Def: We say that a two dimensional random variable (X, Y) is $\boxed{\text{discrete}}$ if the possible values of (X, Y) are finite or countably infinite.

That is, the possible values of (X, Y) may be represented as (x_i, y_j) for $i = 1, 2, ---, n, ..$ and $j = 1, 2, ---, m, ...$

Def: A two - dim r.v. (X, Y) is $\boxed{\text{continuous}}$ if (X, Y) assumes all values in some uncountable set of the Euclidean plane.

EX: (X, Y) assume all values in the rectangle

$$\{(x, y) / a \le x \le b, c \le y \le d\}.$$

or all the values ~~of~~ in the circle $\{(x, y) / x^2 + y^2 \le 1\}$.

Def: Let (X, Y) be a two dimensional discrete r.v. with each possible outcome (x_i, y_j), we associate a number $p(x_i, y_j)$ representing $P(X = x_i, Y = y_j)$ and satisfying the following conditions.

(i) $p(x_i, y_j) \geq 0$ for all (x_i, y_j)

(ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p(x_i, y_j) = 1$.

The function p defined for all (x_i, y_j) in the range space of (X, Y) is called the $\boxed{\text{probability function}}$ of (X,Y).

Note: The set of all triples $(x_i, y_j, p(x_i, y_j))$ $i,j = 1, 2, \ldots$ is called the probability distribution of (X,Y).

Def: Let (X,Y) be a continuous random variable assuming all values in some region R of the Euclidean plane. The $\boxed{\text{joint probability density function}}$ f is a function satisfying

(i) $f(x,y) \geq 0$ for all $(x,y) \in R$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$.

Def: Cumulative distribution function: (CDF).

let (X, Y) be a two dimensional random variable

The CDF is defined by.

$$F(x,y) = P(X \leq x, Y \leq y).$$

If (X,Y) is a d.r.v., then $F(x,y) = \sum_{x_i \leq x} \sum_{y_j \leq y} p(x_i, y_j)$

If (X,Y) is a c.r.v., then

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dy\, dx.$$

Note: If $F(x,y)$ is the cdf of (X,Y) then the joint PDF is given by.

$$f(x,y) = \frac{d^2 F(x,y)}{dx\, dy} \quad \text{if } F \text{ is differentiable.}$$

Two production lines manufacture a certain type of item. Suppose that the capacity is 5 items for line I and three items for line II. Assume that the no. of items actually produced by either production line is a random variable.

(X, Y) = two-dim r.v. yielding number of items produced by line I and line II resply.

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Consider the table.

Y \ X	0	1	2	3	4	5	Sum marginal Y
0	0	0.01	0.03	0.05	0.07	0.09	0.25
1	0.01	0.02	0.04	0.05	0.06	0.08	0.26
2	0.01	0.03	0.05	0.05	0.05	0.06	0.25
3	0.01	0.02	0.04	0.06	0.06	0.05	0.24
marginal X	0.03	0.08	0.16	0.21	0.24	0.28	1

$$p(x_i, y_j) = p(X = x_i, Y = y_j)$$

$=0, y=1,2,3$

$$p(2,3) = P(X=2, Y=3) = 0.04.$$

Define $B = \{(X,Y) \,/\, X$ produced more than $Y\}$.

we have

$$P(B) = p(1,0) + p(2,1) + p(2,0) + p(3,0) + p(3,1) + p(3,2) + p(4,0) + p(4,1) + p(4,2) + p(4,3) + p(5,0) + p(5,1) + p(5,2) + p(5,3)$$

$$= 0.01 + 0.04 + 0.03 + 0.05 + 0.05 + 0.05 + 0.07 + 0.06 + 0.05 + 0.06 + 0.09 + 0.08 + 0.06 + 0.05 = 0.75.$$

Ex: Suppose that the two-dimensional continuous random variable (X,Y) has joint pdf given by.

$$f(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 \le x \le 1, \ 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(B)$ where $B = \{X + Y \geq 1\}$.

To verify $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \left(x^2 + \frac{xy}{3}\right) dx\,dy.$$

$$= \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_{x=0}^{x=1} dy$$

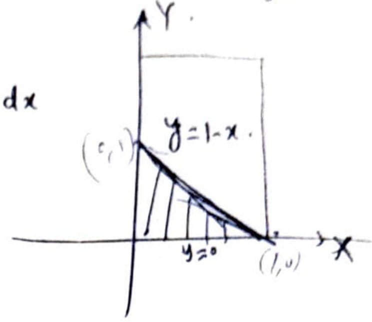
$$= \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{1}{3}y + \frac{y^2}{12}\Big]_0^2 = \frac{2}{3} + \frac{4}{12} = 1.$$

clearly $\bar{B} = \{X+Y<1\}$ and $P(B) = 1 - P(\bar{B})$.

$$= 1 - \int_0^1 \int_{y=0}^{1-x} \left(x^2 + \frac{xy}{3}\right) dy\, dx \qquad \left[\text{here } x+y<1 \Rightarrow y<1-x\right]$$

$$= 1 - \int_0^1 \left[x^2(1-x) + \frac{x(1-x)^2}{6}\right] dx$$

$$= 1 - \frac{7}{72} = \frac{65}{72}.$$



Marginals:

Def: Let X be a discrete random variable. Since $X = x_i$ must occur with $Y = y_j$ for some j and can occur $Y = y_j$ for only one j, we have

$$P(x_i) = P(X=x_i) = P(X = x_i, Y = y_1 \text{ or } X = x_i, Y = y_2, \text{ or } \dots)$$

$$= \sum_{j=1}^{\infty} p(x_i, y_j)$$

The function p defined for $x_1, x_2, \dots$, represents the $\boxed{\text{marginal probability distribution}}$ of X.

Similarly, we can define $q(y_j) = P(Y = y_j) = \sum_{i=1}^{\infty} p(x_i, y_j)$ as the marginal probability distribution of Y.

Def: Let (X, Y) be a continuous two-dim. r.v. with joint pdf $f(x,y)$, then the $\boxed{\text{marginal pdfs}}$ of X and Y defined by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy, \qquad h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

Conditional pdf:

Discrete case: let (X, Y) be a d.r.v. with joint pdf

$p(x_i, y_j)$ $\quad i = 1, 2, \ldots n, \ldots$
$\quad j = 1, 2, \ldots, m, \ldots$

and the marginal pdf's are $p(x_i)$ and $q(y_j)$, then

$$p(x_i | y_j) = P(X = x_i | Y = y_j)$$

$$= \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{p(x_i, y_j)}{q(y_j)} \text{ if } q(y_j) > 0$$

$$= \cancel{P(X = x_i, Y = y_j)}$$

and

$$q(y_j | x_i) = P(Y = y_j | X = x_i)$$

$$= \frac{p(x_i, y_j)}{p(x_i)} \quad \text{if } p(x_i) > 0.$$

Continuous case: Let (X, Y) be a continuous two dimensional random variable with joint pdf f. let g and h be the marginal pdf's of X and Y respectively.

The conditional pdf of X for given $Y = y$ is defined by

$$g(x|y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0.$$

The cond. pdf. of Y, given $X = x$ is defined by.

$$h(y|x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0.$$

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Note: In above table $P(X=3) = 0.21$, $P(Y=1) = 0.26$, ---etc.

Problem: Suppose that, a two dim. c.r.v. has joint pdf

II test

$$f(x,y) = \begin{cases} kx(x-y), & 0 < x < 2,\ -x < y < x. \\ 0 & \text{else where} \end{cases}$$

(a) Evaluate k

(b) Find marginal pdf's of X and Y.

Sol: $f(x,y)$ is a pdf iff $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$.

$$\Rightarrow \int_0^2 \int_{-x}^{x} kx(x-y)\,dy\,dx = 1.$$

$$\Rightarrow k\int_0^2 \int_{-x}^{x} (x^2 - xy)\,dy\,dx = 1$$

$$\Rightarrow k\int_0^2 \left[\int_{-x}^{x} x^2\,dy - \int_{-x}^{x} xy\,dy\right] dx = 1$$

$$\Rightarrow k\int_0^2 \left[x^2(x+x) - \left[x\cdot\frac{x^2}{2} - x\cdot\frac{x^2}{2}\right]\right] dx = 1$$

$$\Rightarrow k\int_0^2 2x^3\,dx = 1 \Rightarrow k\cdot\left[\frac{2x^4}{4}\right]_0^2 = 1$$

$$\Rightarrow k\cdot 8 = 1 \Rightarrow k = 1/8.$$

Therefore $f(x,y) = \begin{cases} \frac{1}{8}x(x-y), & 0<x<2,\ -x<y<x \\ 0 & \text{elsewhere.} \end{cases}$

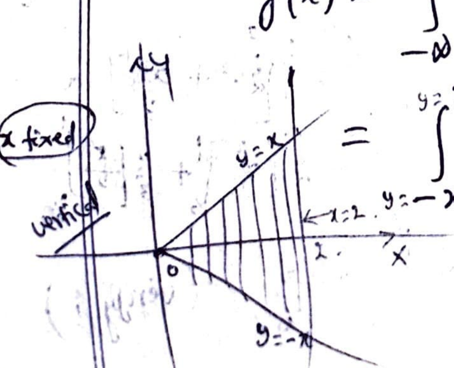
Marginal pdf of X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{y=-x}^{y=x} \frac{1}{8}x(x-y)\,dy \begin{cases} = \frac{x^3}{4} & 0 \le x < 2. \\ = 0 & \text{elsewhere.} \end{cases}$$

$$= \frac{1}{8}x^2\int_{-x}^{x} dy - \frac{1}{8}x\int_{-x}^{x} y\,dy$$

$$= \frac{x^3}{4} - \frac{1}{8}x\underbrace{\left[\frac{x^2}{2} - \frac{x^2}{2}\right]}_{0} = \frac{x^3}{4}.$$



Marginal pdf of Y :

(i) $-x<y \Rightarrow -y<x<2$.

$$h(y) = \int_{x=-y}^{2} \frac{1}{8}\, x(x-y)\, dx$$

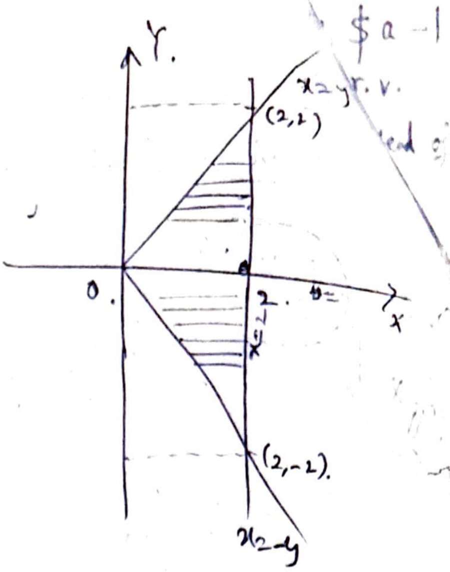
$$= \frac{1}{8}\int_{x=-y}^{2} (x^2 - xy)\, dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2}{2}y\right]_{-y}^{2}$$

$$= \frac{1}{8}\left\{\left(\frac{8}{3} - 2y\right) - \left(\frac{-y^3}{3} - \frac{y^3}{2}\right)\right\}$$

$$= \frac{1}{3} + \frac{5y^3}{48} - \frac{y}{4}, \quad 0 < -y < 2$$

ie, $-2 \le y \le 0$.



(ii) If $y<x$, $0<x<2 \Rightarrow 0<y<x<2$.

$$h(y) = \int_{x=y}^{2} \frac{1}{8}x(x-y)\, dx = \frac{1}{3} - \frac{y}{4} + \frac{y^3}{48} \quad \text{for } 0<y<$$

Therefore
$$h(y) = \begin{cases} \frac{1}{3} + \frac{5y^3}{48} - \frac{y}{4} & \text{if } -2 \le y \le 0. \\ \frac{1}{3} + \frac{y^3}{48} - \frac{y}{4} & \text{if } 0 \le y \le 2. \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that joint pdf of a two dim. random variable is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & , 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere.} \end{cases}$$

Compute the following.

(a) $P(X > \frac{1}{2})$ (b) $P(Y<X)$ (c) $P(Y<\frac{1}{2} \mid X<\frac{1}{2})$

$$P(X \le x) = P(-\infty \le X \le x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx.$$

$$P(X > \tfrac{1}{2}) = P(\tfrac{1}{2} \le X < \infty)$$

$$= \int_{1/2}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx.$$

$$= \int_{x=1/2}^{1}\int_{y=0}^{2} \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_{1/2}^{1}\left[x^2 y + x\frac{y^2}{6}\right]_0^2 dx$$

$$= \int_{1/2}^{1}\left[2x^2 + \frac{x \cdot 2}{3}\right] dx$$

$$= 2\left[\frac{x^3}{3}\right]_{1/2}^{1} + \frac{2}{3}\left[\frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \frac{5}{6} \quad \text{(verify!)}$$

$P(Y < X) = P(-\infty < Y < X < \infty)$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x,y)\, dy\, dx.$

$= \int_{x=0}^{1} \int_{y=0}^{x} \left(x^2 + \frac{xy}{3}\right) dy\, dx. = \frac{7}{24}$

Q7

Q8

$P\left(Y < \frac{1}{2} \middle| X < \frac{1}{2}\right) = \frac{P\left(X < \frac{1}{2},\ Y < \frac{1}{2}\right)}{P\left(X < \frac{1}{2}\right)}$

$= \frac{\int_{0}^{1/2} \int_{0}^{1/2} f(x,y)\, dy\, dx}{\int_{0}^{1/2} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx.}$

$= \frac{\int_{0}^{1/2} \int_{0}^{1/2} \left(x^2 + \frac{xy}{3}\right) dy\, dx.}{\int_{0}^{1/2} \int_{0}^{2} \left(x^2 + \frac{xy}{3}\right) dy\, dx.}$

$= \frac{5}{32}.$

Pb: If $f(x,y) = \begin{cases} 2/a^2 & 0 \le x \le y \le a \\ 0 & \text{elsewhere.} \end{cases}$

find ~~g(y/x)~~, and ~~h(x/y)~~. $g(x|y)$ and $h(y|x)$

Sol.

$g(x|y) = \frac{f(x,y)}{h(y)} = \frac{2/a^2}{\int_{-\infty}^{\infty} f(x,y)\, dx.} = \frac{2/a^2}{\int_{0}^{y} \frac{2}{a^2}\, dx} = 1/y.$

$h(y/x) = \frac{f(x,y)}{g(x)} = \frac{2/a^2}{\int_{x}^{a} 2/a^2\, dy} = \frac{1}{a-x}.$

II Test

Pb: Suppose that the joint pdf of (X,Y) is given by.

$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{elsewhere} \end{cases}$

(a) find marginal pdf of X (b) find marginal pdf of Y.

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<u>Sol:</u> The marginal pdf of $X = g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

$$= \int_{x}^{\infty} f(x,y)\,dy = \int_{x}^{\infty} e^{-y}\,dy = -e^{-y}\Big]_{x}^{\infty} = +e^{-x}\ ;\ 0<x<\infty$$

The marginal pdf of $Y = h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

$$= \int_{0}^{y} e^{-y}\,dx = ye^{-y},\ 0<y<\infty.$$

The two dim. r. v. (X,Y) has the joint density function given by

$$f(x,y) = \begin{cases} 6\left(e^{-2x-3y}\right), & x,y \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Find (i) $P(1<X<2,\ 2<Y<3)$

(ii) $P(0<X<2,\ Y>2)$

(iii) marginal and conditional densities.

(i) $P(1<X<2,\ 2<Y<3)$

$$= \int_{1}^{2}\int_{2}^{3} f(x,y)\,dy\,dx = \int_{1}^{2}\int_{2}^{3} 6e^{-2x-3y}\,dy\,dx = 6\left[\int_{2}^{3} e^{-3y}\,dy\right]\left[\int_{1}^{2} e^{-2x}\,dx\right]$$

$$= 6\left[\frac{e^{-3y}}{-3}\right]_{2}^{3}\left[\frac{e^{-2x}}{-2}\right]_{1}^{2} = \frac{6}{-3\times-2}\left[e^{-9}-e^{-6}\right]\left[e^{-4}-e^{-2}\right]$$

$$= e^{-8}\left(1+e^{-5}-e^{-2}-e^{-3}\right).$$

(ii) $P(0<X<2,\ Y>2)$

$$= \int_{0}^{2}\int_{2}^{\infty} f(x,y)\,dy\,dx = \int_{0}^{2}\int_{2}^{\infty} 6e^{-2x}\cdot e^{-3y}\,dy\,dx.$$

$$= \int_{0}^{2} 6e^{-2x}\cdot\left[\frac{e^{-3y}}{-3}\right]_{2}^{\infty} dx.$$

$$= +2\cdot\int_{0}^{2} e^{-2x}e^{-6}\,dx. = +2e^{-6}\left[\frac{e^{-2x}}{-2}\right]_{0}^{2} = -e^{-6}\left(e^{-4}-1\right).$$

iii) Marginal pdf of X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = 6\int_{0}^{\infty} e^{-2x-3y}\,dy.$$

$$= 6\,e^{-2x}\cdot\left[\frac{e^{-3y}}{-3}\right]_{0}^{\infty} = -2\,e^{-2x}(-1) = 2e^{-2x}, \quad x>0.$$

Marginal pdf of Y: $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{\infty} 6\,e^{-2x}\cdot e^{-3y}\,dx$

$$= 6e^{-3y}\left[\frac{e^{-2x}}{-2}\right]_{0}^{\infty} = 3e^{-3y}, \quad y>0$$

iv) Conditional probability of X/Y:

$$g(x/y) = \frac{f(x,y)}{h(y)} = \frac{6\,e^{-2x}\,e^{-3y}}{3e^{-3y}} = 2e^{-2x}, \quad x\geq 0.$$

Conditional probability of Y/X:

$$h(y/x) = \frac{f(x,y)}{g(x)} = \frac{6\,e^{-2x}\cdot e^{-3y}}{2e^{-2x}} = 3e^{-3y}, \quad y\geq 0.$$

Independent Random variables:

Def: Let (X,Y) be a two dimensional d. random variable. we say that X and Y are $\boxed{\text{independent}}$ if and only if

$$p(x_i, y_j) = p(x_i)\cdot q(y_j) \text{ for all } i \text{ and } j.$$

That is, $P(X=x_i, Y=y_j) = P(X=x_i)\cdot P(Y=y_j)$ for all i and j

Def: Let (X,Y) be a two dimensional continuous random variable. we say that X and Y are $\boxed{\text{independent}}$ random variables if and only if $f(x,y) = g(x)\cdot h(y)$ for all (x,y) where f is the joint pdf and g, h are marginal pdf's of X and Y respectively.

The two dimensional random variable (X,Y) have the joint pdf $f(x,y) = 2e^{x-y}$, $0<x<y<\infty$

prove that X and Y are dependent.

We have to prove that $f(x,y) \neq g(x)\,h(y)$.

given by variable (X,Y) has a joint density

$$f(x,y) = x+y \quad , \; 0 \le x \le 1, \quad 0 \le y \le 1.$$

Compute the correlation coefficient between X and Y.

Sol:

$$\rho_{xy} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)\,V(Y)}}$$

Consider $E(xy) = \int_0^1 \int_0^1 xy \, f(x,y)\,dy\,dx = \int_0^1 \int_0^1 xy\,(x+y)\,dy\,dx = 1/3.$

$E(X) = \int_0^1 \int_0^1 x \, f(x,y)\,dy\,dx = \int_0^1 \int_0^1 x\cdot(x+y)\,dy\,dx = 7/12$

$E(Y) = \int_0^1 \int_0^1 y f(x,y)\,dy\,dx = 7/12$

$E(X^2) = \int_0^1 \int_0^1 x^2 \, f(x,y)\,dy\,dx = 5/12$

$E(Y^2) = \int_0^1 \int_0^1 y^2 f(x,y)\,dy\,dx = 5/12$

Therefore $V(X) = E(X^2) - [E(X)]^2 = 11/144.$

$V(Y) = 11/144.$

Therefore

$$\rho_{xy} = \frac{\frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12}}{\sqrt{11/144 \times 11/144}} = \frac{\frac{-1}{144}}{11/144} = \frac{-1}{11}.$$

Pb: prove that $V(aX+bY) = a^2 V(X) + b^2 V(Y) + 2ab\,Cov(X,Y)$

Pf: $V(aX+bY) = E[(aX+bY) - E(aX+bY)]^2$

$= E[aX+bY - aE(X) - bE(Y)]^2 = E[a(X-E(X)) + b(Y-E(Y))]^2$

$= E\{a^2[X-E(X)]^2 + b^2[Y-E(Y)]^2 + 2ab[X-E(X)][Y-E(Y)]\}$

$= a^2 E[X-E(X)]^2 + b^2[Y-E(Y)]^2 + 2ab\,Cov(X,Y)$

$= a^2 V(X) + b^2 V(Y) + 2ab\,Cov(X,Y).$

Note: If X and Y are independent, then $Cov(X,Y)=0$,

and hence $V(aX+bY) = a^2 V(X) + b^2 V(Y).$

$= k^2 \sum (x_i - E|X|)\; p(x_i)$