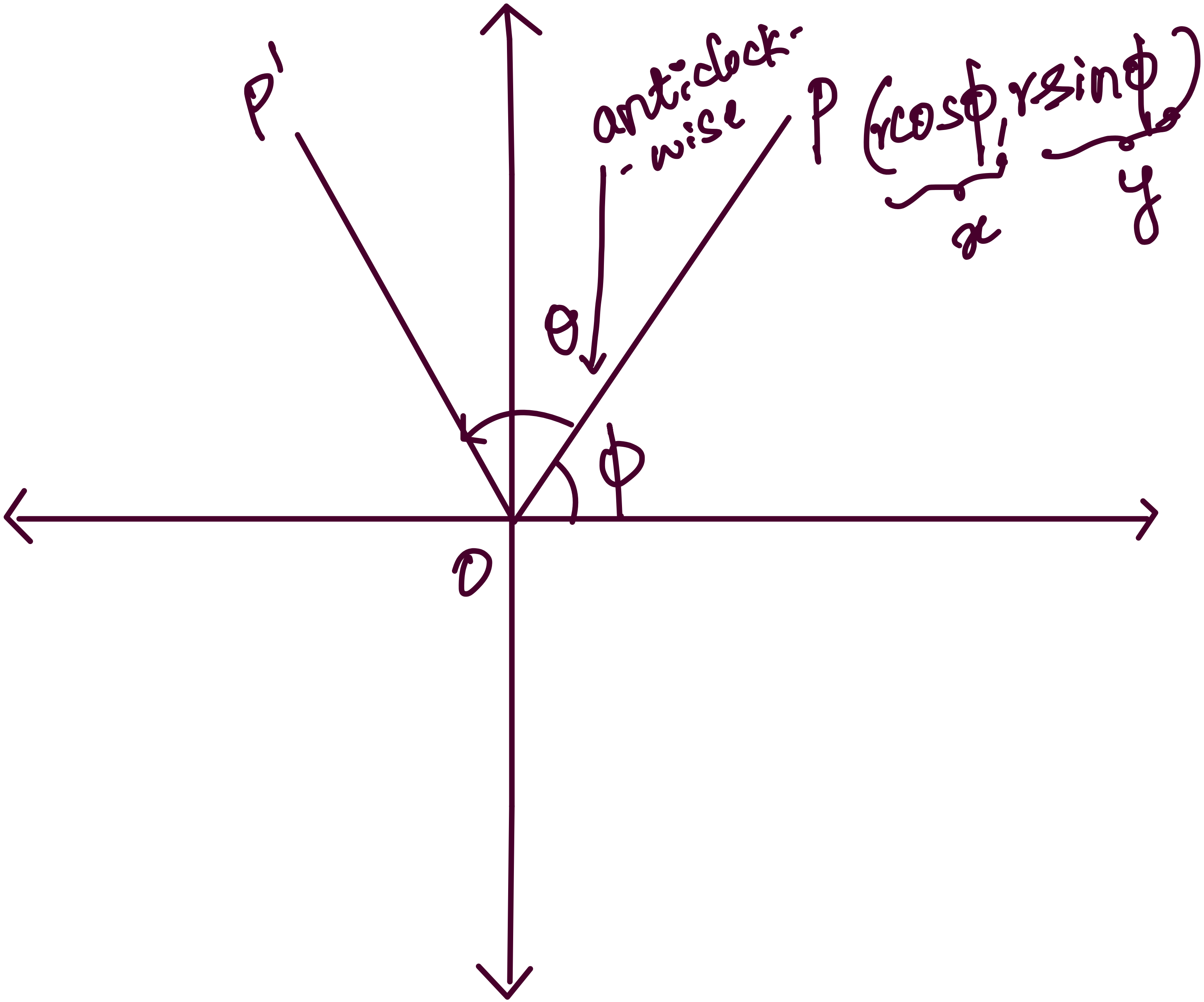

P'
anticlock-
-wise
P
$(r\cos\phi, r\sin\phi)$
x
y
θ
φ
O

$$p' = (r\cos(\phi+\theta),\ r\sin(\phi+\theta))$$

$$= \Big(r(\cos\phi\cos\theta - \sin\phi\sin\theta),$$

$$r(\sin\phi\cos\theta + \cos\phi\sin\theta)\Big)$$

$$= (x\cos\theta - y\sin\theta,\ \ y\cos\theta + x\sin\theta)$$

$$\equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix which rotates a vector (x, y) by an angle θ in anticlockwise sense is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{(anticlockwise)}$$

$$R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{(clockwise)}$$

$$R_{\pi/2} = \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \equiv i$$

$$R_{-\pi/2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -i$$

$$a+bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\downarrow$

Multiplication by $e^{i\theta}$ rotates a vector by angle θ.

$$R_\theta R_\theta^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv I$$

$$R_\theta^T R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus $R_\theta R_\theta^T = R_\theta^T R_\theta = I$

$$\boxed{(R_\theta)^{-1} = R_\theta^T}$$ → easy to compute

Challenging to compute for large matrices.

Some std linear Transformations

1. $T(x, y) = (x, y)$ (Identity)

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\downarrow$

Identity Matrix.

② Scaling (Stretching) (X-direction)

$$T(x, y) = (kx, y)$$

$$= \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

3) Scaling (stretching) (Y-direction

$$T(x, y) = (x, ky)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

4) Shear about X-axis

$$T(x, y) = (x+ky, y)$$

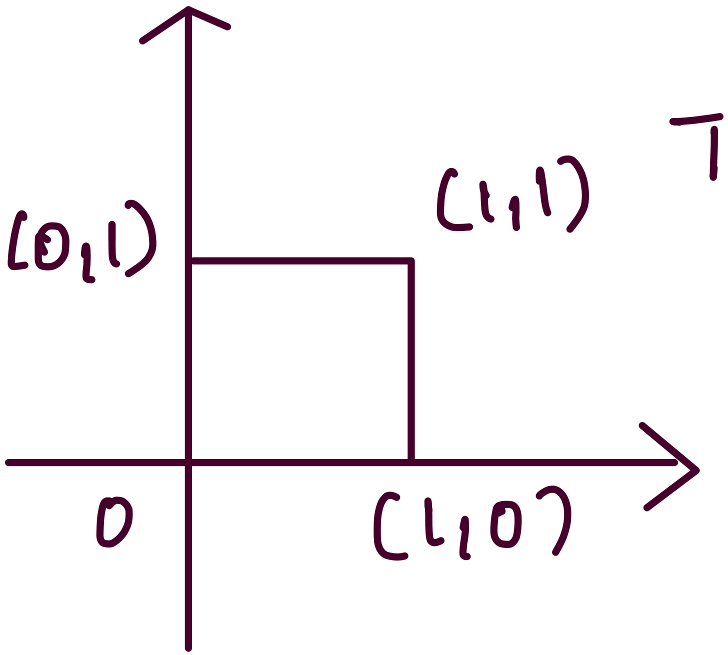
$$= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$T(0,0) = (0,0)$

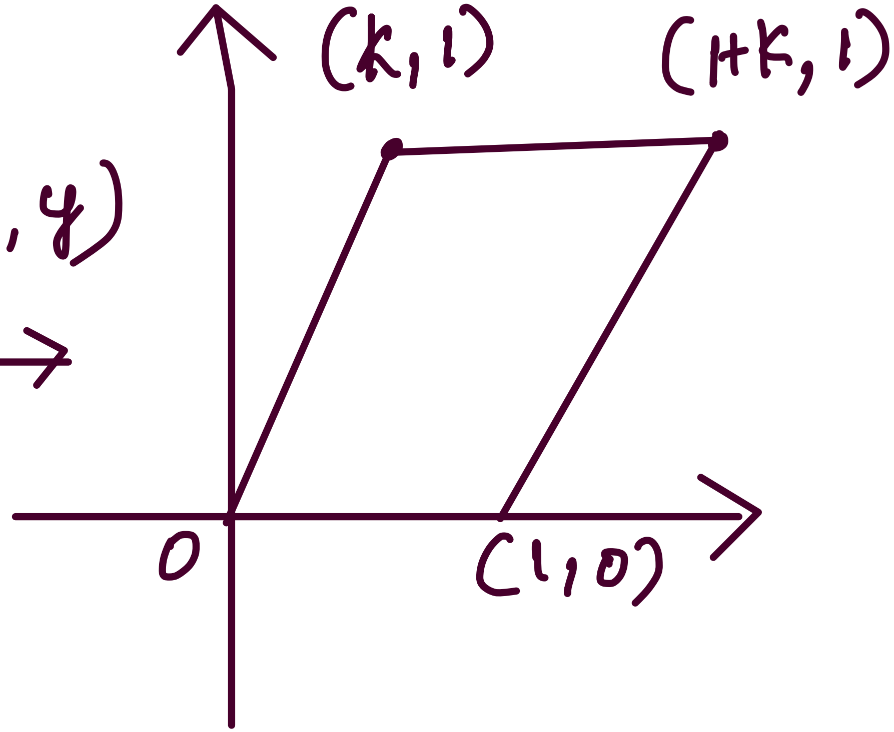
$T(1,0) = (1,0)$

$T(0,1) = (k,1)$

$T(1,1) = (1+k,1)$



$T(x,y) = (x+ky,y)$



Shear about y-axis

$$T(x, y) = (x, kx + y)$$

$$= \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$T(0,0) = (0,0)$

$T(1,0) = (1,k)$

$T(0,1) = (0,1)$

$T(1,1) = (1,k+1)$

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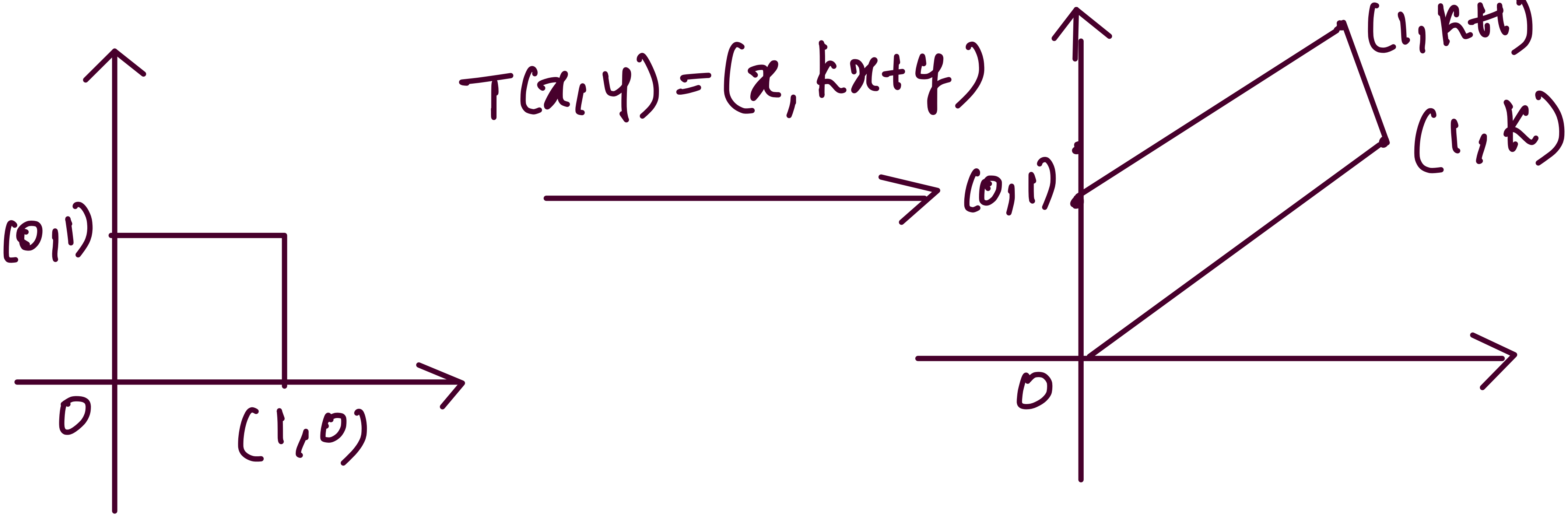
$T(x, y) = (x, kx + y)$

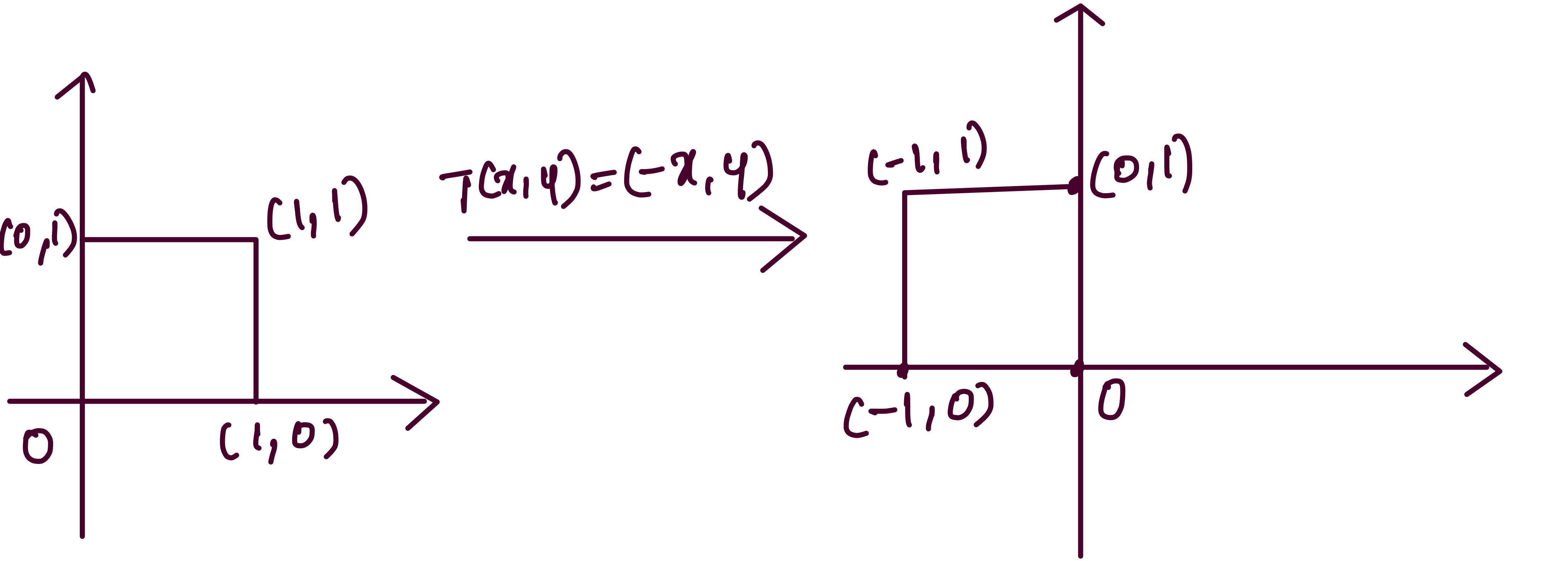


6) Reflection wrt y-axis

$$T(x, y) = (-x, y)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T(0,0) = (0,0)$$
$$T(0,1) = (0,1)$$
$$T(1,0) = (-1,0)$$
$$T(1,1) = (-1,1)$$



7) Reflection wrt X-axis

$$T(x, y) = (x, -y)$$

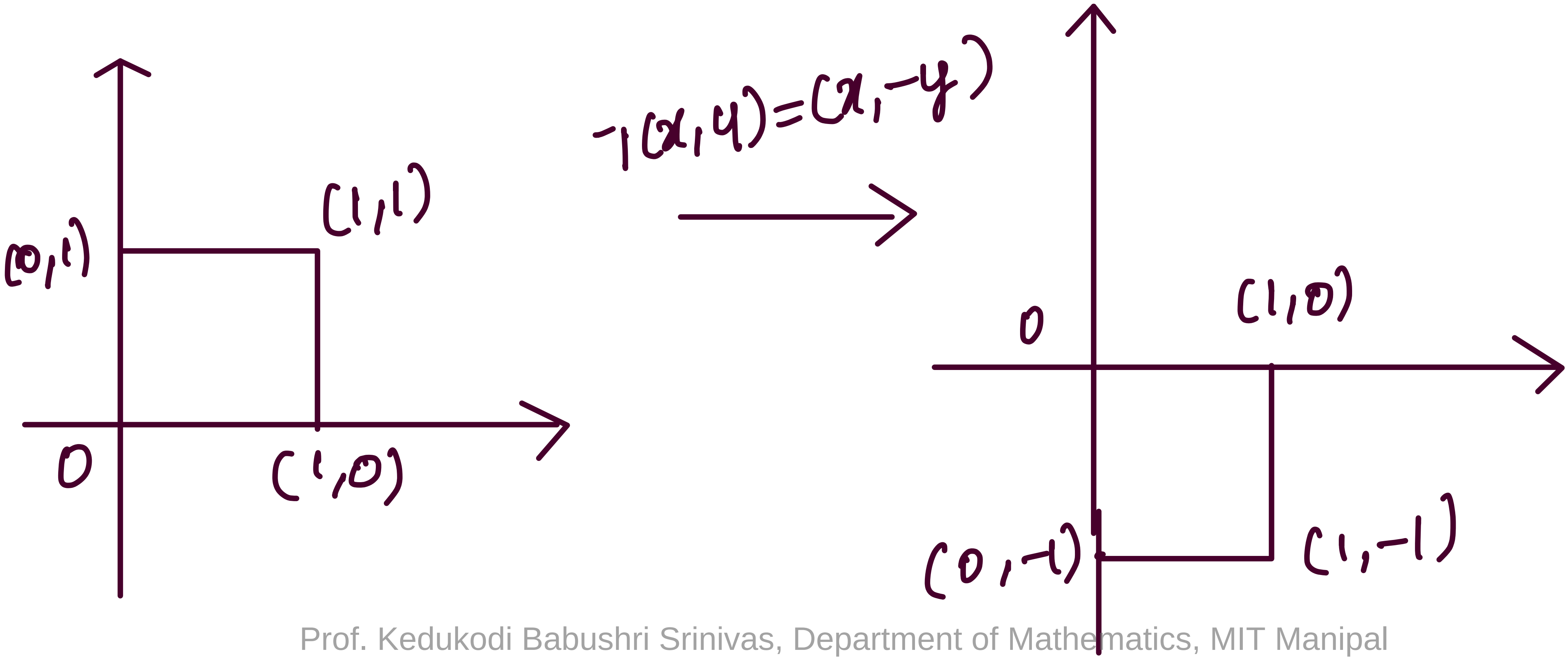
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$T(0,0) = (0,0)$

$T(1,0) = (1,0)$

$T(0,1) = (0,-1)$

$T(1,1) = (1,-1)$

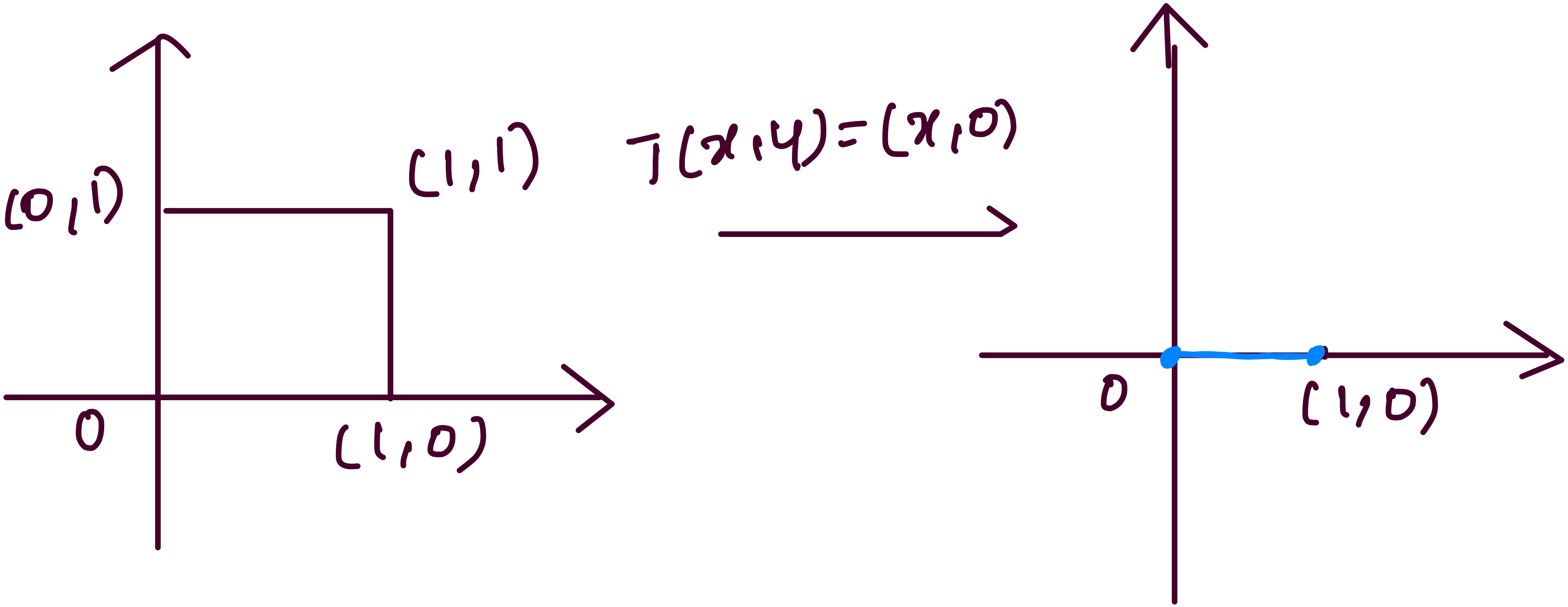


8) Projection about X-axis

$$T(x, y) = (x, 0)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

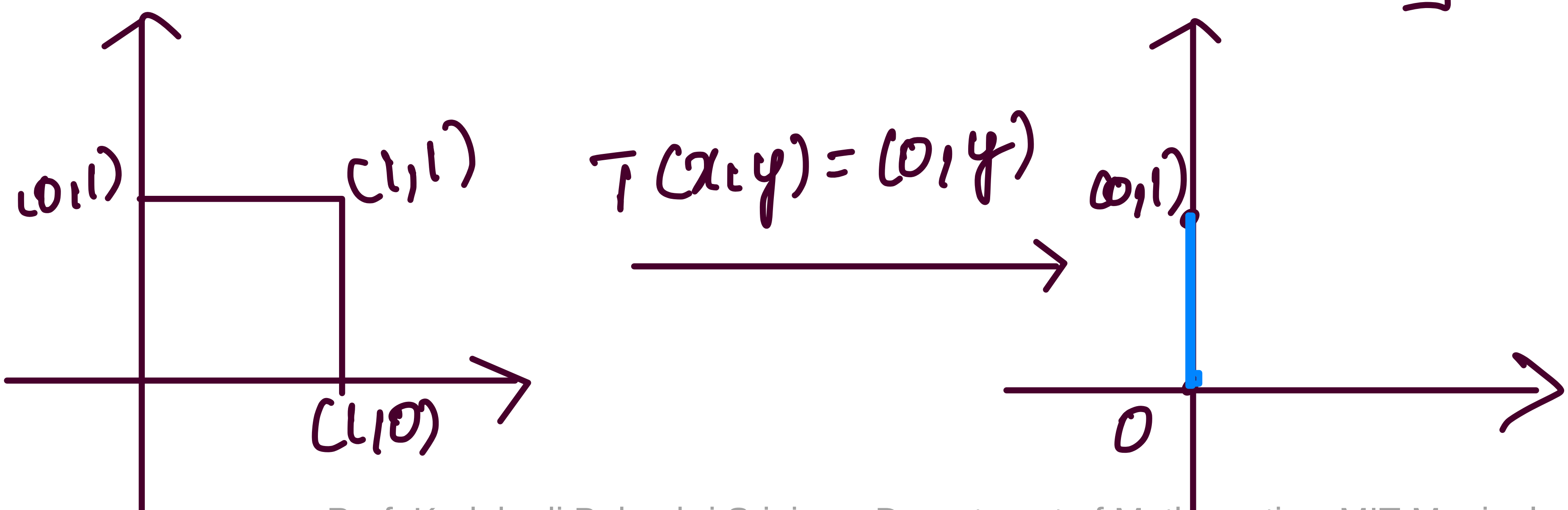
$$T(0,0) = (0,0)$$
$$T(0,1) = (0,0)$$
$$T(1,0) = (1,0)$$
$$T(1,1) = (1,0)$$



9) Projection about y-axis

$$T(x,y) = (0,y)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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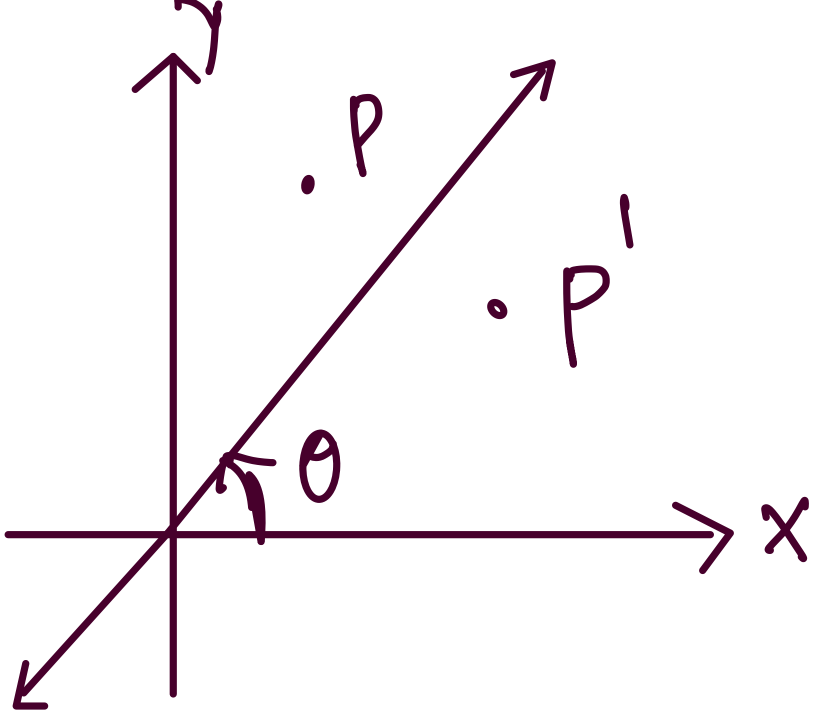
10) <u>Rotation</u> <u>by angle θ</u>

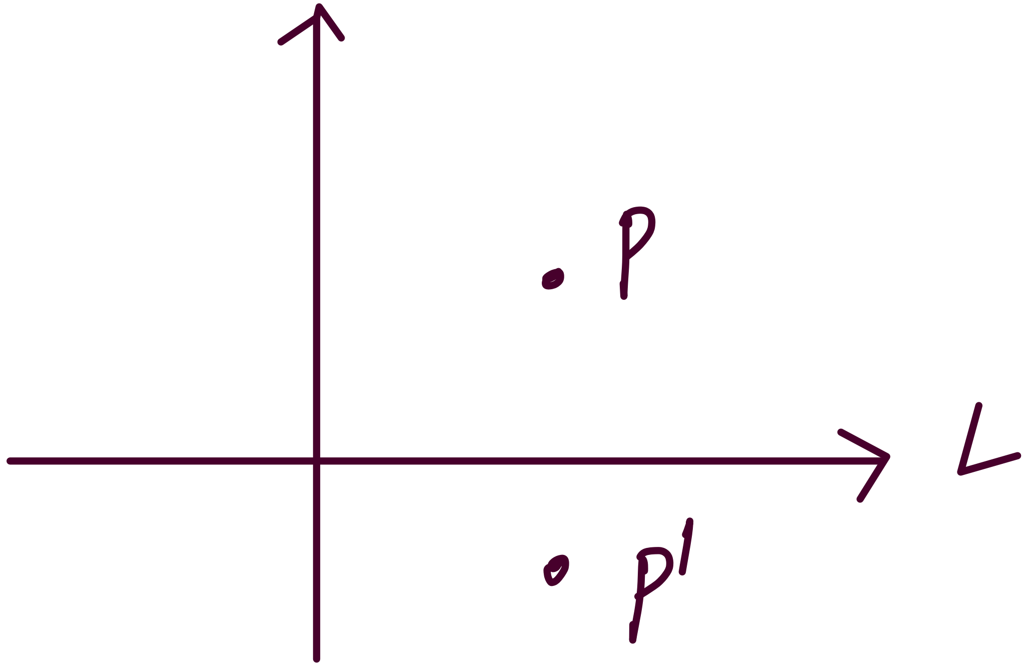
$$T(x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

11) <u>Reflection</u> <u>about a line L</u>



Step I : Make the line L as X-axis by rotating L by an angle $-\theta$

matrix $= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$



Step 2: Reflect about x-axis

matrix $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Step 3 : Rotate back by an angle θ

$$\text{matrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Thus the required linear transformation is the composition of above 3 linear transformations.

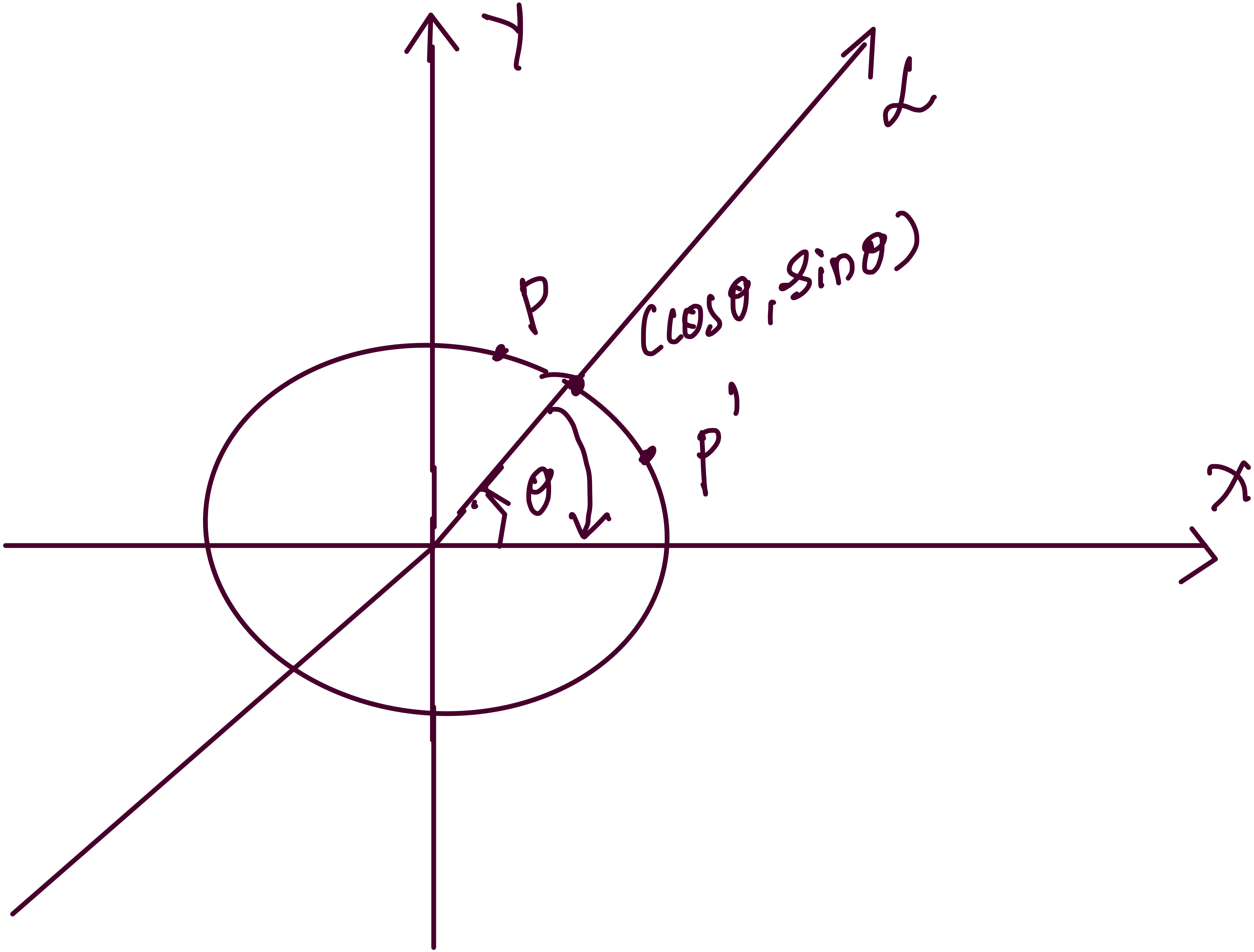
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ \sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$


Y
L
P
(cosθ, sinθ)
P'
θ
X