

1

Sequence and Series of Real Numbers

1.1 Sequence of Real Numbers

Suppose for each positive integer n , we are given a real number a_n . Then, the list of numbers,

$$a_1, a_2, \dots, a_n, \dots,$$

is called a *sequence*, and this *ordered list* is usually written as

$$(a_1, a_2, \dots, \dots) \quad \text{or} \quad (a_n) \quad \text{or} \quad \{a_n\}.$$

More precisely, we define a sequence as follows:

Definition 1.1 A **sequence** of real numbers is a function from the set \mathbb{N} of natural numbers to the set \mathbb{R} of real numbers. If $f : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence, and if $a_n = f(n)$ for $n \in \mathbb{N}$, then we write the sequence f as (a_n) or (a_1, a_2, \dots) .

A sequence of real numbers is also called a *real sequence*. □

Remark 1.1 (a) It is to be born in mind that a sequence (a_1, a_2, \dots, \dots) is different from the set $\{a_n : n \in \mathbb{N}\}$. For instance, a number may be repeated in a sequence (a_n) , but it need not be written repeatedly in the set $\{a_n : n \in \mathbb{N}\}$. As an example, $(1, 1/2, 1, 1/3, \dots, 1, 1/n, \dots)$ is a sequence (a_n) with $a_{2n-1} = 1$ and $a_{2n} = 1/(n+1)$ for each $n \in \mathbb{N}$, where as the set $\{a_n : n \in \mathbb{N}\}$ is same as the set $\{1/n : n \in \mathbb{N}\}$.

(b) Instead of sequence of real numbers, we can also talk about a sequence of elements from any nonempty set S , such as sequence of sets, sequence of functions and so on. Thus, given a nonempty set S , a sequence in S is a function $f : \mathbb{N} \rightarrow S$. For example, for each $n \in \mathbb{N}$, consider the set $A_n = \{j \in \mathbb{N} : j \leq n\}$. Then we obtain a sequence of subsets of \mathbb{N} , namely, (A_1, A_2, \dots) .

In this chapter, we shall consider only sequence of real numbers. In some of the later chapters we shall consider sequences of functions as well. ◆

EXAMPLE 1.1 (i) (a_n) with $a_n = 1$ for all $n \in \mathbb{N}$ – a constant sequence with value 1 throughout.

(ii) (a_n) with $a_n = n$ for all $n \in \mathbb{N}$.

(iii) (a_n) with $a_n = 1/n$ for all $n \in \mathbb{N}$.

(iv) (a_n) with $a_n = n/(n+1)$ for all $n \in \mathbb{N}$.

(v) (a_n) with $a_n = (-1)^n$ for all $n \in \mathbb{N}$ – the sequence takes values 1 and -1 alternately. \square

Question: Consider a sequence (a_1, a_2, \dots) . Is (a_2, a_3, \dots) also a sequence? Why?

1.1.1 Convergence and divergence

A fundamental concept in mathematics is that of *convergence*. We consider convergence of sequences.

Consider the sequences listed in Example 1.1 and observe the way a_n vary as n becomes larger and larger:

(i) $a_n = 1$: every term of the sequence is same.

(ii) $a_n = n$: the terms becomes larger and larger.

(iii) $a_n = 1/n$: the terms come closer to 0 as n becomes larger and larger.

(iv) $a_n = n/(n+1)$: the terms come closer to 1 as n becomes larger and larger.

(v) $a_n = (-1)^n$: the terms of the sequence oscillates with values -1 and 1 , and does not come closer to any number as n becomes larger and larger.

Now, we make precise the statement “ a_n comes closer to a number a ” as n becomes larger and larger.

Definition 1.2 A sequence (a_n) in \mathbb{R} is said to **converge** to a real number a if for every $\varepsilon > 0$, there exists positive integer N (in general depending on ε) such that

$$|a_n - a| < \varepsilon \quad \forall n \geq N,$$

and in that case, the number a is called a **limit** of the sequence (a_n) , and (a_n) is called a **convergent sequence**. \square

Remark 1.2 (a) Note that, different ε can result in different N , i.e., the number N may vary as ε varies. We shall illustrate this in Example 1.2.

(b) In Definition 1.2, the relation $|a_n - a| < \varepsilon$ can be replaced by $|a_n - a| < c_0 \varepsilon$ for any $c_0 > 0$ which is independent of n (Why?). Thus, given a (a_n) in \mathbb{R} , if we are able to identify a number $a \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ satisfying $|a_n - a| < c_0 \varepsilon$ for all $n \geq N$ for some constant $c_0 > 0$, then (a_n) converges to a . \blacklozenge

NOTATION: If (a_n) converges to a , then we write

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{or} \quad a_n \rightarrow a \quad \text{as} \quad n \rightarrow \infty$$

or simply as $a_n \rightarrow a$.

Note that

$$|a_n - a| < \varepsilon \quad \forall n \geq N$$

if and only if

$$a - \varepsilon < a_n < a + \varepsilon \quad \forall n \geq N.$$

Thus, $\lim_{n \rightarrow \infty} a_n = a$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n \in (a - \varepsilon, a + \varepsilon) \quad \forall n \geq N.$$

Thus, $a_n \rightarrow a$ if and only if for every $\varepsilon > 0$, a_n belongs to the open interval $(a - \varepsilon, a + \varepsilon)$ for all n after some finite stage, and this finite stage may vary according as ε varies.

Remark 1.3 Suppose (a_n) is a sequence and $a \in \mathbb{R}$. Then to show that (a_n) does not converge to a , we should be able to find an $\varepsilon > 0$ such that infinitely many a_n 's are outside the interval $(a - \varepsilon, a + \varepsilon)$. \blacklozenge

Exercise 1.1 Show that a sequence (a_n) converges to a if and only if for every open interval I containing a , there exists $N \in \mathbb{N}$ such that $a_n \in I$ for all $n \geq N$. \blacktriangleleft

Before further discussion, let us observe an important property.

Theorem 1.1 *Limit of a convergent sequence is unique. That, is if $a_n \rightarrow a$ and $a_n \rightarrow a'$ as $n \rightarrow \infty$, then $a = a'$.*

Proof. Suppose $a_n \rightarrow a$ and $a_n \rightarrow a'$ as $n \rightarrow \infty$, and suppose that $a' \neq a$. Now, for $\varepsilon > 0$, suppose $N_1, N_2 \in \mathbb{N}$ be such that

$$a_n \in (a - \varepsilon, a + \varepsilon) \quad \forall n \geq N_1, \quad a_n \in (a' - \varepsilon, a' + \varepsilon) \quad \forall n \geq N_2.$$

In particular,

$$a_n \in (a - \varepsilon, a + \varepsilon), \quad a_n \in (a' - \varepsilon, a' + \varepsilon) \quad \forall n \geq N := \max\{N_1, N_2\}.$$

If we take $\varepsilon < |a - a'|/2$, then we see that $(a - \varepsilon, a + \varepsilon)$ and $(a' - \varepsilon, a' + \varepsilon)$ are disjoint intervals. Thus the above observation leads to a contradiction. \blacksquare

An alternate proof. Note that

$$|a - a'| = |(a - a_n) + (a_n - a')| \leq |a - a_n| + |a_n - a'|.$$

Now, for $\varepsilon > 0$, let $N_1, N_2 \in \mathbb{N}$ be such that

$$|a - a_n| < \varepsilon/2 \quad \text{for all } n \geq N_1, \quad |a' - a_n| < \varepsilon/2 \quad \forall n \geq N_2.$$

Then it follows that

$$|a - a'| \leq |a - a_n| + |a_n - a'| < \varepsilon \quad \forall n \geq N := \max\{N_1, N_2\}.$$

Since this is true for all $\varepsilon > 0$, it follows that $a' = a$. \blacksquare

EXAMPLE 1.2 The sequences $(1/n)$, $((-1)^n/n)$, $(1 - \frac{1}{n})$ are convergent with limit 0, 0, 1 respectively:

For the sake of illustrating how to use the definition to justify the above statement, let us provide the details of the proofs:

(i) Let $a_n = 1/n$ for all $n \in \mathbb{N}$, and let $\varepsilon > 0$ be given. We have to identify an $N \in \mathbb{N}$ such that $1/n < \varepsilon$ for all $n \geq N$. Note that

$$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}.$$

Thus, if we take $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$, then we have

$$|a_n - 0| = \frac{1}{n} < \varepsilon \quad \forall n \geq N.$$

Hence, $(1/n)$ converges to 0.

Here $\lceil x \rceil$ denotes the integer part of x .

(ii) Next, let $a_n = (-1)^n/n$ for all $n \in \mathbb{N}$. Since $|a_n| = 1/n$ for all $n \in \mathbb{N}$, in this case also, we see that

$$|a_n - 0| < \varepsilon \quad \forall n \geq N := \left\lceil \frac{1}{\varepsilon} \right\rceil + 1.$$

Hence, $((-1)^n/n)$ converges to 0.

(iii) Now, let $a_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Since, $|a_n - 1| = 1/n$ for all $n \in \mathbb{N}$, we have

$$|a_n - 1| < \varepsilon \quad \forall n \geq N := \left\lceil \frac{1}{\varepsilon} \right\rceil + 1.$$

Hence, $(1 - 1/n)$ converges to 1. □

EXAMPLE 1.3 Every constant sequence is convergent to the constant term in the sequence.

To see this, let $a_n = a$ for all $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, we have

$$|a_n - a| = 0 < \varepsilon \quad \forall n \geq N := 1.$$

Thus, (a_n) converges to a . □

EXAMPLE 1.4 For a given $k \in \mathbb{N}$, let $a_n = 1/n^{1/k}$ for all $n \in \mathbb{N}$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

To see this, first let $\varepsilon > 0$ be given. Note that

$$\frac{1}{n^{1/k}} < \varepsilon \iff n > \frac{1}{\varepsilon^k}.$$

Hence,

$$\frac{1}{n^{1/k}} < \varepsilon \quad \forall n \geq N := \left\lceil \frac{1}{\varepsilon^k} \right\rceil + 1.$$

Thus, $1/n^{1/k} \rightarrow 0$. □

Exercise 1.2 Corresponding to a sequence (a_n) and $k \in \mathbb{N}$, let (b_n) be defined by $b_n = a_{k+n}$ for all $n \in \mathbb{N}$. Show that, for $a \in \mathbb{R}$, $a_n \rightarrow a$ if and only if $b_n \rightarrow a$. ◀

Definition 1.3 A sequence (a_n) is said to be **eventually constant** if there exists $k \in \mathbb{N}$ such that $a_{k+n} = a_k$ for all $n \geq 1$. □

Exercise 1.3 Show that every eventually constant sequence converges. ◀

Exercise 1.4 Prove the following.

1. Let $b \geq 0$ such that $b < \varepsilon$ for all $\varepsilon > 0$. Then $b = 0$.
2. Let $a_n \geq 0$ for all $n \in \mathbb{N}$ such that $a_n \rightarrow a$. Then $a \geq 0$.

◀

The following theorem can be easily proved.

Theorem 1.2 Suppose $a_n \rightarrow a$, $b_n \rightarrow b$ as $n \rightarrow \infty$. Then we have the following :

- (i) $a_n + b_n \rightarrow a + b$ as $n \rightarrow \infty$,
- (ii) For every real number c , $ca_n \rightarrow ca$ as $n \rightarrow \infty$
- (iii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
- (iv) **(Sandwich theorem)** If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$, and if $a = b$, then $c_n \rightarrow a$ as $n \rightarrow \infty$.

Exercise 1.5 Supply proof for Theorem 1.2. ◀

Exercise 1.6 If $a_n \rightarrow a$ and there exists $b \in \mathbb{R}$ such that $a_n \geq b$ for all $n \in \mathbb{N}$, then show that $a \geq b$. ◀

Exercise 1.7 If $a_n \rightarrow a$ and $a \neq 0$, then show that there exists $k \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq k$. ◀

Exercise 1.8 Consider the sequence (a_n) with $a_n = \left(1 + \frac{1}{n}\right)^{1/n}$, $n \in \mathbb{N}$. Then show that $\lim_{n \rightarrow \infty} a_n = 1$.

[Hint: Observe that $1 \leq a_n \leq (1 + 1/n)$ for all $n \in \mathbb{N}$.] ◀

Exercise 1.9 Consider the sequence (a_n) with $a_n = \frac{1}{n^k}$, $n \in \mathbb{N}$. Then show that for any given $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n = 0$.

[Hint: Observe that $1 \leq a_n \leq 1/n$ for all $n \in \mathbb{N}$.] ◀

Remark 1.4 In Theorem 1.2 (c) and (d), instead of assuming the inequalities for all $n \in \mathbb{N}$, we can assume them to hold for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ (Why?). ◆

Definition 1.4 A sequence which does not converge is called a **divergent sequence**. □

Definition 1.5 (i) If (a_n) is such that for every $M > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n > M \quad \forall n \geq N,$$

then we say that (a_n) **diverges to** $+\infty$.

(ii) If (a_n) is such that for every $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n < -M$ for all $n \geq N$, then we say that (a_n) **diverges to** $-\infty$. □

Definition 1.6 If (a_n) is such that $a_n a_{n+1} < 0$ for every $n \in \mathbb{N}$, that is a_n changes sign alternately, then we say that (a_n) is an **alternating sequence**. □

An alternating sequence converge or diverge. For example, (*Verify that*) the sequence $((-1)^n)$ diverges, whereas $((-1)^n/n)$ converges to 0.

Definition 1.7 A sequence (a_n) is said to be **bounded above** if there exists a real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$; and the sequence (a_n) is said to be **bounded below** if there exists a real number M' such that $a_n \geq M'$ for all $n \in \mathbb{N}$. A sequence which is bound above and bounded below is said to be a **bounded sequence**. □

Exercise 1.10 Show that a sequence (a_n) is bounded if and only if there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. ◀

Exercise 1.11 Prove the following.

1. If (a_n) is not bounded above, then there exists a strictly increasing sequence (k_n) of natural numbers such that $a_{k_n} \rightarrow +\infty$ as $n \rightarrow \infty$.
2. If (a_n) is not bounded below, then there exists a strictly increasing sequence (k_n) of natural numbers such that $a_{k_n} \rightarrow -\infty$ as $n \rightarrow \infty$.

◀

Theorem 1.3 Every convergent sequence is bounded. The converse is not true.

Proof. Suppose $a_n \rightarrow a$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| \leq 1$ for all $n \geq N$. Hence

$$|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| \leq 1 + |a| \quad \forall n \geq N.$$

Thus, if take $K = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|\}$, then we have

$$|a_n| \leq \max\{1 + |a|, K\} \quad \forall n \in \mathbb{N}.$$

To see that the converse of the theorem is not true, consider the sequence $((-1)^n)$. It is a bounded sequence, but not convergent. ■

The above theorem can be used to show that certain sequence is not convergent, as in the following example.

EXAMPLE 1.5 For $n \in \mathbb{N}$, let

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

Then (a_n) diverges: To see this, observe that

$$\begin{aligned} a_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \\ &\quad \dots + \left(\frac{1}{2^{n-1} + 1} + \dots + \frac{1}{2^n}\right) \\ &\geq 1 + \frac{n}{2}. \end{aligned}$$

Hence, (a_n) is not a bounded sequence, so that it diverges. □

Using Theorem 1.3, the following result can be deduced

Theorem 1.4 Suppose $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$. Then we have the following.

- (i) $a_n b_n \rightarrow ab$ as $n \rightarrow \infty$.
- (ii) If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then $a_n/b_n \rightarrow a/b$ as $n \rightarrow \infty$.

Exercise 1.12 Prove Theorem 1.4. ◀

Exercise 1.13 If (a_n) converges to a and $a \neq 0$, then show that there exists $k \in \mathbb{N}$ such that $|a_n| \geq |a|/2$ for all $n \geq k$, and $(1/a_{n+k})$ converges to $1/a$. ▶

1.1.2 Monotonic sequences

We can infer the convergence or divergence of a sequence in certain cases by observing the way the terms of the sequence varies.

Definition 1.8 Consider a sequence (a_n) .

- (i) (a_n) is said to be **monotonically increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
- (ii) (a_n) is said to be **monotonically decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

If strict inequality occur in (i) and (ii), then we say that the sequence is **strictly increasing** and **strictly decreasing**, respectively.

Also, a monotonically increasing (respectively, a monotonically decreasing) sequence is also called an increasing (respectively, a decreasing) sequence. \square

We shall make use of some important properties of the set \mathbb{R} of real numbers.

Definition 1.9 Let S be a subset of \mathbb{R} . Then

- (i) S is said to be **bounded above** if there exists $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in S$, and in that case b is called an **upper bound** of S ;
- (ii) S is said to be **bounded below** if there exists $a \in \mathbb{R}$ such that $a \leq x$ for all $x \in S$, and in that case a is called a **lower bound** of S . \square

Definition 1.10 Let S be a subset of \mathbb{R} .

- (i) A number $b_0 \in \mathbb{R}$ is called a **least upper bound** (lub) or **supremum** of $S \subseteq \mathbb{R}$ if b_0 is an upper bound of S and for any upper bound b of S , $b_0 \leq b$.
- (ii) A number $a_0 \in \mathbb{R}$ is called a **greatest lower bound** (glb) or **infimum** of $S \subseteq \mathbb{R}$ if a_0 is a lower bound of S and for any lower bound a of S , $a \leq a_0$. \square

Thus, we have the following:

- $b_0 \in \mathbb{R}$ is supremum of $S \subseteq \mathbb{R}$ if and only if b_0 is an upper bound of S and if $\beta < b_0$, then β is not an upper bound of S , i.e., there exists $x \in S$ such that $\beta < x$.
- $a_0 \in \mathbb{R}$ is infimum of $S \subseteq \mathbb{R}$ if and only if a_0 is a lower bound of S and if $\alpha > a_0$, then α is not a lower bound of S , i.e., there exists $x \in S$ such that $x < \alpha$.

The above two statements can be rephrased as follows:

- $b_0 \in \mathbb{R}$ is supremum of $S \subseteq \mathbb{R}$ if and only if b_0 is an upper bound of S and for every $\varepsilon > 0$, there exists $x \in S$ such that $b_0 - \varepsilon < x \leq b_0$.
- $a_0 \in \mathbb{R}$ is infimum of $S \subseteq \mathbb{R}$ if and only if a_0 is a lower bound of S and for every $\varepsilon > 0$, there exists $x \in S$ such that $a_0 \leq x < a_0 + \varepsilon$.

Exercise 1.14 Show that supremum (respectively, infimum) of a set $S \subseteq \mathbb{R}$, if exists, is unique. \blacktriangleleft

Exercise 1.15 Prove the following:

- (i) If b_0 is the supremum of S , then there exists a sequence (x_n) in S which converges to b_0 .

(ii) If a_0 is the infimum of S , then there exists a sequence (x_n) in S which converges to a_0 . ◀

EXAMPLE 1.6 (i) If S is any of the intervals $(0, 1)$, $[0, 1)$, $(0, 1]$, $[0, 1]$, then 1 is the supremum of S and 0 is the infimum of S .

(ii) If $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then 1 is the supremum of S and 0 is the infimum of S .

(iii) For $k \in \mathbb{N}$, if $S_k = \{n \in \mathbb{N} : n \geq k\}$, then k is the infimum of S_k , and S_k has no supremum.

(iv) For $k \in \mathbb{N}$, if $S_k = \{n \in \mathbb{N} : n \leq k\}$, then k is the supremum of S_k , and S_k has no infimum. ◻

The above examples show the following:

- The supremum and/or infimum of a set S , if exists, need not belong to S .
- If S is not bounded above, then S does not have supremum.
- If S is not bounded below, then S does not have infimum.

However, we have the following properties for \mathbb{R} :

Least upper bound property: If $S \subseteq \mathbb{R}$ is bounded above, then S has a least upper bound. We may write this least upper bound as $\text{lub}(S)$ or $\sup(S)$.

Greatest lower bound property: If $S \subseteq \mathbb{R}$ is bounded below, then S has a greatest lower bound, and we write it as $\text{glb}(S)$ or $\inf(S)$.

Theorem 1.5 (i) Every sequence which is monotonically increasing and bounded above is convergent.

(ii) Every sequence which is monotonically decreasing and bounded below is convergent.

Proof. Suppose (a_n) is a monotonically increasing sequence of real numbers which is bounded above. Then the set $S := \{a_n : n \in \mathbb{N}\}$ is bounded above. Hence, by the least upper bound property of \mathbb{R} , S has a least upper bound, say b . Now, let $\varepsilon > 0$ be given. Then, by the definition of the least upper bound, there exists $N \in \mathbb{N}$ such that $a_N > b - \varepsilon$. Since $a_n \geq a_N$ for every $n \geq N$, we get

$$b - \varepsilon < a_N \leq a_n \leq b < b + \varepsilon \quad \forall n \geq N.$$

Thus we have proved that $a_n \rightarrow b$ as $n \rightarrow \infty$.

To see the last part, suppose that (b_n) is a monotonically decreasing sequence which is bounded below. Then, it is seen that the sequence (a_n) defined by $a_n = -b_n$ for all $n \in \mathbb{N}$ is monotonically increasing and bounded above. Hence, by the first part of the theorem, $a_n \rightarrow a$ for some $a \in \mathbb{R}$. Then, $b_n \rightarrow b := -a$. ■

Note that a convergent sequence need not be monotonically increasing or monotonically decreasing. For example, look at the sequence $((-1)^n/n)$.

1.1.3 Subsequences

Definition 1.11 A sequence (b_n) is called a **subsequence** of a sequence (a_n) if there is a strictly increasing sequence (k_n) of natural numbers such that $b_n = a_{k_n}$ for all $n \in \mathbb{N}$. \square

Thus, subsequences of a real sequence (a_n) are of the form (a_{k_n}) , where (k_n) is a strictly increasing sequence natural numbers.

For example, given a sequence (a_n) , the sequences (a_{2n}) , (a_{2n+1}) , (a_{n^2}) , (a_{2^n}) are some of its subsequences. As concrete examples, $(1/2n)$, and $(1/(2n+1))$, $(1/2^n)$ are subsequences of $(1/n)$.

A sequence may not converge, but it can have convergent subsequences. For example, we know that the sequence $((-1)^n)$ diverges, but the subsequences (a_n) and (b_n) defined by $a_n = 1, b_n = -1$ for all $n \in \mathbb{N}$ are convergent subsequences of $((-1)^n)$.

However, we have the following result.

Theorem 1.6 *If a sequence (a_n) converges to x , then all its subsequences converge to the same limit x .*

Exercise 1.16 Prove Theorem 1.6. \blacktriangleleft

What about the converse of the above theorem? Obviously, if all subsequences of a sequence (a_n) converge to the same limit x , then (a_n) also has to converge to x , as (a_n) is a subsequence of itself.

Suppose every subsequence of (a_n) has at least one subsequence which converges to x . Does the sequence (a_n) converges to x ? The answer is affirmative, as the following theorem shows.

Theorem 1.7 *If every subsequence of (a_n) has at least one subsequence which converges to x , then (a_n) also converges to x .*

Proof. Proof is left as an exercise. \blacksquare

We have seen in Theorem 1.3 that every convergent sequence is bounded, but a bounded sequence need not be convergent.

Question: For every bounded sequence, can we have a convergent subsequence?

The answer is in affirmative:

Theorem 1.8 (Bolzano-Weirstrass theorem). *Every bounded sequence of real numbers has a convergent subsequence.*

We shall relegate its proof to the appendix (See Section 1.4).

Remark 1.5 We may observe that the proof of Theorem 1.8 can be slightly modified so that we, in fact, have the following: *Every sequence in \mathbb{R} has a monotonic subsequence.* ♦

Exercise 1.17 Prove the statement in italics in Remark 1.5. ◀

1.1.4 Further examples

EXAMPLE 1.7 Let a sequence (a_n) be defined as follows :

$$a_1 = 1, \quad a_{n+1} = \frac{2a_n + 3}{4}, \quad n = 1, 2, \dots$$

We show that (a_n) is monotonically increasing and bounded above.

Note that

$$a_{n+1} = \frac{2a_n + 3}{4} = \frac{a_n}{2} + \frac{3}{4} \geq a_n \quad \Longleftrightarrow \quad a_n \leq \frac{3}{2}.$$

Thus it is enough to show that $a_n \leq 3/2$ for all $n \in \mathbb{N}$.

Clearly, $a_1 \leq 3/2$. If $a_n \leq 3/2$, then $a_{n+1} = a_n/2 + 3/4 < 3/4 + 3/4 = 3/2$. Thus, we have proved that $a_n \leq 3/2$ for all $n \in \mathbb{N}$. Hence, by Theorem 1.5, (a_n) converges. Let its limit be a . Then taking limit on both sides of $a_{n+1} = \frac{2a_n+3}{4}$ we have

$$a = \frac{2a + 3}{4} \quad \text{i.e.,} \quad 4a = 2a + 3 \quad \text{so that} \quad a = \frac{3}{2}.$$

□

EXAMPLE 1.8 Let a sequence (a_n) be defined as follows :

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right), \quad n = 1, 2, \dots$$

It is seen that, if the sequence converges, then its limit would be $\sqrt{2}$.

Since $a_1 = 2$, one may try to show that (a_n) is monotonically decreasing and bounded below.

Note that

$$a_{n+1} := \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \leq a_n \quad \Longleftrightarrow \quad a_n^2 \geq 2, \quad \text{i.e.,} \quad a_n \geq \sqrt{2}.$$

Clearly $a_1 \geq \sqrt{2}$. Now,

$$a_{n+1} := \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \geq \sqrt{2} \iff a_n^2 - 2\sqrt{2}a_n + 2 \geq 0,$$

i.e., if and only if $(a_n - \sqrt{2})^2 \geq 0$. This is true for every $n \in \mathbb{N}$. Thus, (a_n) is monotonically decreasing and bounded below, so that by Theorem 1.5, (a_n) converges. \square

EXAMPLE 1.9 Consider the sequence (a_n) with $a_n = (1 + 1/n)^n$ for all $n \in \mathbb{N}$. We show that (a_n) is monotonically increasing and bounded above. Hence, by Theorem 1.5, (a_n) converges.

Note that

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n} \right)^n \\ &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdots 2 \cdot 1}{n!} \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n} \right) + \frac{1}{3!} \left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) + \\ &\quad \cdots + \frac{1}{n!} \left(1 - \frac{1}{n} \right) \cdots \left(1 - \frac{n-1}{n} \right) \\ &\leq a_{n+1}. \end{aligned}$$

Also

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2!} \frac{n(n-1)}{n^2} + \cdots + \frac{1}{n!} \frac{n(n-1) \cdots 2 \cdot 1}{n^n} \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &< 3. \end{aligned}$$

Thus, (a_n) is monotonically increasing bounded above. \square

Exercise 1.18 Show that $(n^{1/n})_{n=3}^\infty$ is a monotonically decreasing sequence. \blacktriangleleft

In the four examples that follow, we shall be making use of Theorem 1.2 without mentioning it explicitly.

EXAMPLE 1.10 Consider the sequence (b_n) with

$$b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \quad \forall n \in \mathbb{N}.$$

Clearly, (b_n) is monotonically increasing, and we have noticed in the last example that it is bounded above by 3. Hence, by Theorem 1.5, it converges.

Let (a_n) and (b_n) be as in Examples 1.9 and 1.10 respectively, and let a and b their limits. We show that $a = b$.

We have observed in last example that $2 \leq a_n \leq b_n \leq 3$. Hence, taking limits, it follows that $a \leq b$. Notice that

$$\begin{aligned} a_n = & 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \\ & \dots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Hence, for m, n with $m \leq n$, we have

$$\begin{aligned} a_n \geq & 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \\ & \dots + \frac{1}{m!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{m-1}{n}\right). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get (cf. Theorem 1.2 (c))

$$a \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{m!} = b_m.$$

Now, taking limit as $m \rightarrow \infty$, we get $a \geq b$. Thus we have proved $a = b$.

The common limit in the above two examples is denoted by the letter e . □

EXAMPLE 1.11 Let $a > 0$. We show that, if $0 < a < 1$, then the sequence (a^n) converges to 0, and if $a > 1$, then (a^n) diverges to infinity.

(i) Suppose $0 < a < 1$. Then we can write $a = 1/(1+r)$, $r > 0$, and we have

$$a^n = 1/(1+r)^n = 1/(1+nr + \dots + r^n) < 1/(1+nr) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $a^n \rightarrow 0$ as $n \rightarrow \infty$.

An alternate way: Let $x_n = a^n$. Then (x_n) is monotonically decreasing and bounded below by 0. Hence (x_n) converges, to say x . Then $x_{n+1} = a^{n+1} = ax_n \rightarrow ax$. Hence, $x = ax$. This shows that $x = 0$.

(ii) Suppose $a > 1$. Then, since $0 < 1/a < 1$, the sequence $(1/a^n)$ converges to 0, so that (a^n) diverges to infinity. (*Why ?*) □

EXAMPLE 1.12 The sequence $(n^{1/n})$ converges and the limit is 1.

Note that $n^{1/n} = 1 + r_n$ for some sequence (r_n) of positive reals. Then we have

$$n = (1 + r_n)^n \geq \frac{n(n-1)}{2} r_n^2,$$

so that $r_n^2 \leq 2/(n-1)$ for all $n \geq 2$. Since $2/(n-1) \rightarrow 0$, by Theorem 1.2(c), that $r_n \rightarrow 0$, and hence by Theorem 1.2(c), $n^{1/n} = 1 + r_n \rightarrow 1$. □

Remark 1.6 In Example 1.12 for showing $n^{1/n} \rightarrow 1$ we did not require the fact that the limit is 1. But, if one is asked to show that $n^{1/n} \rightarrow 1$, then we can give another proof of the same by using only the definition, as follows:

Another proof without using Theorem 1.2. Let $\varepsilon > 0$ be given. To find $n_0 \in \mathbb{N}$ such that $n^{1/n} - 1 < \varepsilon$ for all $n \geq n_0$. Note that

$$\begin{aligned} n^{1/n} - 1 < \varepsilon &\iff n^{1/n} < 1 + \varepsilon \\ &\iff n < (1 + \varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2}\varepsilon^2 + \dots + \varepsilon^n. \end{aligned}$$

Hence

$$n^{1/n} - 1 < \varepsilon \quad \text{if} \quad n > 1 + 2/\varepsilon^2.$$

So, we may take any $n_0 \in \mathbb{N}$ which satisfies $n_0 \geq 2(1 + 1/\varepsilon^2)$. ◆

EXAMPLE 1.13 For any $a > 0$, $(a^{1/n})$ converges to 1.

If $a > 1$, then we can write $a^{1/n} = 1 + r_n$ for some sequence (r_n) of positive reals. Then we have

$$a = (1 + r_n)^n \geq nr_n \quad \text{so that} \quad r_n \leq a/n.$$

Since $a/n \rightarrow 0$, by Theorem 1.2(c), $r_n \rightarrow 0$, and hence by Theorem 1.2(c), $a^{1/n} = 1 + r_n \rightarrow 1$.

In case $0 < a < 1$, then $1/a > 1$. Hence, by the first part, $1/a^{1/n} = (1/a)^{1/n} \rightarrow 1$, so that $a^n \rightarrow 1$. □

Exercise 1.19 Give another proof for the result in Example 1.13 without using Theorem 1.2. ◀

EXAMPLE 1.14 Let (a_n) be a bounded sequence of non-negative real numbers. Then $(1 + a_n)^{1/n} \rightarrow 1$ as $n \rightarrow \infty$:

This is seen as follows: Let $M > 0$ be such that $0 \leq a_n \leq M$ for all $n \in \mathbb{N}$. Then,

$$1 \leq (1 + a_n)^{1/n} \leq (1 + M)^{1/n} \quad \forall n \in \mathbb{N}.$$

By Example 1.13, $(1 + M)^{1/n} \rightarrow 1$. Hence the result follows by making use of part (d) of Theorem 1.2. □

EXAMPLE 1.15 As an application of some of the results discussed above, consider the sequence (a_n) with $a_n = (1 + 1/n)^{1/n}$, $n \in \mathbb{N}$. We already know that $\lim_{n \rightarrow \infty} a_n = 1$. Now, another proof for the same:

Note that $1 \leq a_n \leq 2^{1/n}$, and $2^{1/n} \rightarrow 1$ as $n \rightarrow \infty$. □

EXAMPLE 1.16 Consider the sequence (a_n) with $a_n = (1 + n)^{1/n}$. Then $a_n \rightarrow 1$ as $n \rightarrow \infty$. We give two proofs for this result.

(i) Observe that $a_n = n^{1/n} (1 + 1/n)^{1/n}$. We already know that $n^{1/n} \rightarrow 1$, and $(1 + 1/n)^{1/n} \rightarrow 1$ as $n \rightarrow \infty$.

(ii) Observe that $n^{1/n} \leq (1+n)^{1/n} \leq (2n)^{1/n} = 2^{1/n}n^{1/n}$, where $n^{1/n} \rightarrow 1$ and $2^{1/n} \rightarrow 1$ as $n \rightarrow \infty$. \square

EXAMPLE 1.17 Suppose $a_n > 0$ for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \ell < 1$. We show that $a_n \rightarrow 0$.

Since $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \ell < 1$, there exists q such that $\ell < q < 1$ and $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \leq q$ for all $n \geq N$. Hence,

$$0 \leq a_n \leq q^{n-N} a_N \quad \forall n \geq N.$$

Now, since $q^{n-N} \rightarrow 0$ as $n \rightarrow \infty$, it follows that $a_n \rightarrow 0$ as $n \rightarrow \infty$. \square

Exercise 1.20 Suppose $a_n > 0$ for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \ell > 1$. Show that (a_n) diverges to ∞ . \blacktriangleleft

EXAMPLE 1.18 Let $0 < a < 1$. Then $na^n \rightarrow 0$ as $n \rightarrow \infty$.

To see this let $a_n := na^n$ for $n \in \mathbb{N}$. Then we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)a^{n+1}}{na^n} = \frac{(n+1)a}{n} \quad \forall n \in \mathbb{N}.$$

Hence, $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = a < 1$. Thus, the result follows from the last example. \square

Exercise 1.21 Obtain the result in Example 1.18 by using the arguments in Example 1.11. \blacktriangleleft

Remark 1.7 Suppose for each $k \in \mathbb{N}$, $a_n^{(k)} \rightarrow 0$, $b_n^{(k)} \rightarrow 1$ as $n \rightarrow \infty$, and also $a_n^{(n)} \rightarrow 0$, $b_n^{(n)} \rightarrow 1$ as $n \rightarrow \infty$. In view of Theorems 1.2 and 1.4, one may think that

$$a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$b_n^{(1)} b_n^{(2)} \cdots b_n^{(n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Unfortunately, that is not the case. To see this consider

$$a_n^{(k)} = \frac{k}{n^2}, \quad b_n^{(k)} = k^{1/n} \quad k, n \in \mathbb{N}.$$

Then, for each $k \in \mathbb{N}$, $a_n^{(k)} \rightarrow 0$, $b_n^{(k)} \rightarrow 1$ as $n \rightarrow \infty$, and also $a_n^{(n)} \rightarrow 0$, $b_n^{(n)} \rightarrow 1$ as $n \rightarrow \infty$. But,

$$a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(n)} = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} = \frac{n+1}{2n} \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty$$

and

$$b_n^{(1)} b_n^{(2)} \cdots b_n^{(n)} = 1^{1/n} 2^{1/n} \cdots n^{1/n} = (n!)^{1/n} \not\rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

In fact, for any $k \in \mathbb{N}$, if $n \geq k$, then

$$(n!)^{1/n} \geq (k!)^{1/n} (k^{n-k})^{1/n} = (k!)^{1/n} k^{1-k/n} = k \left(\frac{k!}{k^k} \right)^{1/n}.$$

Since for any given $k \in \mathbb{N}$, $k \left(\frac{k!}{k^k} \right)^{1/n} \rightarrow k$ as $n \rightarrow \infty$, it follows that $(n!)^{1/n} \not\rightarrow 1$. In fact, there exists $n_0 \in \mathbb{N}$ such that

$$(n!)^{1/n} \geq \frac{k}{2} \quad \forall n \geq \max\{n_0, k\}.$$

Thus, the sequence $((n!)^{1/n})$ is unbounded. ♦

EXAMPLE 1.19 Let $a_n = (n!)^{1/n^2}$, $n \in \mathbb{N}$. Then $a_n \rightarrow 1$ as $n \rightarrow \infty$. We give two proofs for this.

(i) Note that, for every $n \in \mathbb{N}$,

$$1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}.$$

Now, since $n^{1/n} \rightarrow 1$, we have $(n!)^{1/n^2} \rightarrow 1$.

(ii) By GM-AM inequality, for $n \in \mathbb{N}$,

$$(n!)^{1/n} = (1 \cdot 2 \cdot \dots \cdot n)^{1/n} \leq \frac{1 + 2 + \dots + n}{n} = \frac{n+1}{2} \leq n.$$

Thus,

$$1 \leq (n!)^{1/n^2} \leq n^{1/n}.$$

Since $n^{1/n} \rightarrow 1$ we have $(n!)^{1/n^2} \rightarrow 1$. □

1.1.5 Cauchy sequence

Theorem 1.9 If a real sequence (a_n) converges, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall n, m \geq N.$$

Proof. Suppose $a_n \rightarrow a$ as $n \rightarrow \infty$, and let $\varepsilon > 0$ be given. Then we know that there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ for all $n \geq N$. Hence, we have

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon \quad \forall n, m \geq N.$$

This completes the proof. ■

Definition 1.12 A sequence (a_n) is said to be a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall n, m \geq N.$$

□

Theorem 1.9 show that every convergent sequence is a Cauchy sequence. In particular, if (a_n) is not a Cauchy sequence, then (a_n) does not converge to any $a \in \mathbb{R}$. Thus, Theorem 1.9 may help us to show that certain sequence is not convergent. For example, see the following exercise.

Exercise 1.22 Let $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ for $n \in \mathbb{N}$. Then show that (s_n) is not a Cauchy sequence. [*Hint*: For any $n \in \mathbb{N}$, note that $s_{2n} - s_n \geq \frac{1}{2}$.] ◀

Let us observe certain properties of Cauchy sequences.

Theorem 1.10 Let (a_n) be a Cauchy sequence. Then we have the following.

- (i) (a_n) is a bounded sequence.
- (ii) If (a_n) has a convergent subsequence with limit a , then the sequence (a_n) itself will converge to a .

Proof. Let (a_n) be Cauchy sequence and $\varepsilon > 0$ be given.

- (i) Let $N \in \mathbb{N}$ be such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$. In particular,

$$|a_n| \leq |a_n - a_N| + |a_N| < \varepsilon + |a_N| \quad \forall n \geq N.$$

This proves (i).

- (ii) Suppose (a_{n_k}) is a convergent subsequence of (a_n) , and let $a = \lim_{k \rightarrow \infty} a_{n_k}$. Let $n_0 \in \mathbb{N}$ be such that $|a_{n_k} - a| < \varepsilon$ for all $k \geq n_0$. Hence, we have

$$|a_k - a| \leq |a_k - a_{n_k}| + |a_{n_k} - a| < 2\varepsilon \quad \forall k \geq \max\{N, n_0\}.$$

Thus, $a_n \rightarrow a$ as $n \rightarrow \infty$. ■

In fact, the converse of Theorem 1.9 also holds:

Theorem 1.11 Every Cauchy sequence of real numbers converges.

Proof. Let (a_n) be a Cauchy sequence. By Theorem 1.10(i), (a_n) is bounded. Then, by Bolzano-Weierstrass theorem (Theorem 1.8), (a_n) has a convergent subsequence (a_{n_k}) . Let $a = \lim_{k \rightarrow \infty} a_{n_k}$. Now, by Theorem 1.10(ii), $a_n \rightarrow a$ as $n \rightarrow \infty$. ■

Remark 1.8 For an alternate proof of Theorem 1.11, without using Bolzano-Weierstrass theorem, see Section 1.4. ♦

Here is an example of a general nature.

EXAMPLE 1.20 Let (a_n) be a sequence of real numbers. Suppose there exists a positive real number $\rho < 1$ such that

$$|a_{n+1} - a_n| \leq \rho |a_n - a_{n-1}| \quad \forall n \in \mathbb{N}, n \geq 2.$$

Then (a_n) is a Cauchy sequence. To see this first we observe that

$$|a_{n+1} - a_n| \leq \rho^{n-1} |a_2 - a_1| \quad \forall n \in \mathbb{N}, n \geq 2.$$

Hence, for $n > m$,

$$\begin{aligned} |a_n - a_m| &\leq |a_n - a_{n-1}| + \dots + |a_{m+1} - a_m| \\ &\leq (\rho^{n-2} + \dots + \rho^{m-1}) |a_2 - a_1| \\ &\leq \rho^{m-1} (1 + \rho + \dots + \rho^{n-m-3}) |a_2 - a_1| \\ &\leq \frac{\rho^{m-1}}{1 - \rho} |a_2 - a_1|. \end{aligned}$$

Since $\rho^{m-1} \rightarrow 0$ as $m \rightarrow \infty$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$. \square

Exercise 1.23 Given $a, b \in \mathbb{R}$ and $0 < \lambda < 1$, let (a_n) be a sequence of real numbers defined by $a_1 = a, a_2 = b$ and

$$a_{n+1} = (1 + \lambda)a_n - \lambda a_{n-1} \quad \forall n \in \mathbb{N}, n \geq 2.$$

Show that (a_n) is a Cauchy sequence and its limit is $(b + \lambda a)/(1 - \lambda)$. \blacktriangleleft

Exercise 1.24 Suppose f is a function defined on an interval J . If there exists $0 < \rho < 1$ such that

$$|f(x) - f(y)| \leq \rho |x - y| \quad \forall x, y \in J,$$

then for any $a \in J$, the sequence (a_n) defined by

$$a_1 = f(a), \quad a_{n+1} := f(a_n) \quad \forall n \in \mathbb{N},$$

is a Cauchy sequence. Show also that the limit of the sequence (a_n) is independent of the choice of a . \blacktriangleleft

1.2 Series of Real Numbers

Definition 1.13 A **series** of real numbers is an expression of the form

$$a_1 + a_2 + a_3 + \dots,$$

or more compactly as $\sum_{n=1}^{\infty} a_n$, where (a_n) is a sequence of real numbers.

The number a_n is called the n -th term of the series and the sequence $s_n := \sum_{i=1}^n a_i$ is called the n -th partial sum of the series $\sum_{n=1}^{\infty} a_n$. \square