$$P_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$$

$$\dim P_n = n+1$$

Basis $B = \{1, x, \ldots, x^n\}$

$$T : P_3 \to P_2$$

$$T(f) = f'$$

Then $m(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ → corresponds to derivative.

$$\ker T = \{ f \in P_3 \mid T(f) = 0 \}$$

$$= \{ f \in P_3 \mid f' = 0 \}$$

$$f \in P_3 \Rightarrow f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\Rightarrow f' = a_1 + 2a_2 x + 3a_3 x^2 = 0$$

$$\Rightarrow a_1 = 0 \; ; \; a_2 = 0 \; ; \; a_3 = 0$$

$$\therefore \ker T = \{ a_0 \mid a_0 \in \mathbb{R} \} = \{ a_0 \cdot 1 \mid a_0 \in \mathbb{R} \}$$

dim Ker $T = 1$; Basis for Ker $T = \{1\}$.

Eg: $f = 2 + 4x + 7x^2 + 21x^3$; $T(f) = f'$.

lost information

$\Rightarrow f' = 4 + 14x + 3(21)x^2$

Integrating,

$$f = 4x + 14\frac{x^2}{2} + 3(21)\frac{x^3}{3} + c$$

$$= 4x + 7x^2 + 21x^3 + c$$;

c = constant.

Q. Prove that $T : P_2 \to P_3$

$T(f) = \int f\,dx$ is a L.T.

Find $m(T)$.

Soln: Let $f, g \in P_2$.

$$T(f+g) = \int (f+g)\,dx$$

$$= \int f\,dx + \int g\,dx$$

$$= T(f) + T(g).$$

Let α be a scalar.

$$T(\alpha f) = \int (\alpha f)\,dx$$

$$= \alpha \int f\,dx$$

$$= \alpha \cdot T(f)$$

$\therefore$ T is a L.T.

Basis for P_2 = $\{1, x, x^2\}$

Basis for $P_3 = \{1, x, x^2, x^3\}$

$$T(1) = \int 1\,dx = x + c_1 = c_1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$

$$T(x) = \int x\,dx = \frac{x^2}{2} + c_2 = c_2 \cdot 1 + 0 \cdot x + \frac{1}{2} \cdot x^2 + 0 \cdot x^3$$

$$T(x^2) = \int x^2\,dx = \frac{x^3}{3} + c_3 = c_3 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \frac{1}{3} \cdot x^3$$

where c_1, c_2, c_3 are constants.

$$\therefore \quad m(T) = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3} \end{array}$$

$\leftarrow$ corresponds to Anti derivative.

$$\text{Ker } T = \{ f \in P_2 \mid T(f) = 0 \}$$

$$= \{ f \in P_2 \mid \int f\,dx = 0 \}$$

$f \in P_2 \Rightarrow f = a_0 + a_1 x + a_2 x^2 \quad ; \quad a_i \in \mathbb{R}$

$$\int f\,dx = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + c \quad ;$$

c = constant.

$$\int f\,dx = 0 \Rightarrow a_0 = 0 \ ; \ a_1 = 0 \ ; \ a_2 = 0 \ ; \ c = 0.$$

$$\Rightarrow f = 0$$

$$\therefore \operatorname{Ker} T = \{0\} \Rightarrow \dim \operatorname{Ker} T = 0.$$

Function f is called one - one if

$$f(x) = f(y) \Rightarrow x = y.$$

(Images coincide) (preimages coincide)

Result : $T : V \to W$ be a L.T. Then

(A) T is one - one $\Leftrightarrow$ (iff) (B) $\ker T = \{0\}$.

Proof : $(A \Rightarrow B)$: Assume T is one - one.

To prove : $\ker T = \{0\}$.

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

ie, to prove $\ker T \subseteq \{0\}$ and $\{0\} \subseteq \ker T$.

$$0 + \cancel{T(0)} = T(0+0)$$

$$= \cancel{T(0)} + T(0) \quad \Rightarrow T(0) = 0.$$

or, (T is a L.T)

$$T(0) = T(0 \cdot x) = 0 \cdot T(x) = 0 \quad \Rightarrow T(0) = 0.$$

(T is a L.T)

$$\therefore T(0) = 0 \quad \text{——— (1)}$$

$$\ker T = \{ x \in V \mid T(x) = 0 \} \quad \text{——— (2)}$$

By (1) and (2), $0 \in \operatorname{Ker} T$

$$\Rightarrow \{0\} \subseteq \operatorname{Ker} T \quad \text{——} (\mathrm{I}).$$

To prove : $\operatorname{Ker} T \subseteq \{0\}$.

Take $t \in \operatorname{Ker} T$

$$\Rightarrow T(t) = 0$$

$$= T(0) \quad (\text{by } (1))$$

As T is one-one, $t = 0$

$$\Rightarrow \operatorname{Ker} T \subseteq \{0\} \quad \text{——} (\mathrm{II})$$

From (I) and (II),

$$\underline{\underline{\operatorname{Ker} T = \{0\}}}.$$

$(B \Rightarrow A)$: Assume $\operatorname{Ker} T = \{0\}$.

To prove : T is one-one.

Take $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$ (T is a L.T).

$\Rightarrow x-y \in \operatorname{Ker} T$ (defn. of $\operatorname{Ker} T$)

$\Rightarrow x-y \in \operatorname{Ker} T = \{0\}$. (Given)

$\Rightarrow x-y = 0$

$\Rightarrow x = y$

$\Rightarrow T$ is one-one.

Result: $T : V \to W$ be a L.T.

T is one-one (A) $\iff$ $T(x) = 0$ implies $x = 0$ (B)

Proof: $(A \Rightarrow B)$: Assume T is one-one.

Take $T(x) = 0$

$= T(0)$

$\Rightarrow \underline{\underline{x = 0}}.$ (given : T is one-one)

$(B \Rightarrow A)$: Given $T(x) = 0$ implies $x = 0$.

To prove : T is one-one.

Take $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$ (T is a L.T.)

$\Rightarrow \quad x - y = 0$ (given)

$\Rightarrow \quad x = y$

$\Rightarrow$ T is one-one.

Result: T is one-one $\Leftrightarrow$ $\ker T = \{0\}$

$\Leftrightarrow$ $T(x) = 0$ implies $x = 0$.

characterization of one-one linear transformation.
(All three statements are equivalent.)