Suppose $A = \begin{bmatrix} 1 & 10 \\ 4 & 5 \end{bmatrix}$

$$R_2 \to R_2 - 4R_1 \sim \begin{bmatrix} 1 & 10 \\ 0 & -35 \end{bmatrix} = \text{Ans}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1 \sim \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \underbrace{\begin{array}{c} C_1 \qquad C_2 \\ \begin{bmatrix} 1 & 10 \\ 4 & 5 \end{bmatrix} \end{array}}_{= A} = \begin{bmatrix} 1 & 10 \\ 0 & -35 \end{bmatrix} = \text{Ans}$$

Shear of column vectors C_1 & C_2.

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} 1 & 10 \\ 0 & -35 \end{bmatrix}}_{\text{Ans}} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & -35 \end{bmatrix}$$

undoing shear

$$= \begin{bmatrix} 1 & 10 \\ 4 & 5 \end{bmatrix} = A.$$

Q. $A = LU$?

write $A = L_0 U_0 = \underbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \, 1 \end{bmatrix}}_{L_0} U_0$

Solve by LU decomposition :

$$12y + 4z + 12u - 5w = -1$$

$$3y + z + 2u - 3w = -3$$

$$-9y - 3z - 4u + 10w = 12$$

$$2x + 4y - z + 5u - 2w = 0.$$

Soln:

pivot

$$A = \begin{bmatrix} \boxed{0} & 12 & 4 & 12 & -5 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 2 & 4 & -1 & 5 & -2 \end{bmatrix}$$

$\downarrow$

coefficient matrix

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$b = \begin{bmatrix} -1 \\ -3 \\ 12 \\ 0 \end{bmatrix} , \quad X = \begin{bmatrix} x \\ y \\ z \\ u \\ w \end{bmatrix}$$

System is $AX = b$. → LU decomposition does not exist.

So, exchange rows of A.

<u>Eg</u>: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \leftrightarrow R_1$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P_{12}$

$$P_{12}\, B = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}_{B} = \begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix} \rightarrow \text{exchange of rows of } B$$

left multiplication

$$BP_{12} = \overset{B}{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix} \rightarrow \text{exchange of columns of } B$$

right multiplication

Such matrix P_{12} is a permutation matrix.

Practically, $\underbrace{\text{PLU}}_{\text{Permuted LU}}$ decomposition is useful.

$$P_{4231} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$P_{4231} A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}_{4\times 5} = B$$

Finding LU decomposition of B.

$R_3 \to R_3 + 3R_2$; $R_4 \to R_4 - 4R_2$

$$B = IB = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}}_{4\times 4} \underbrace{\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}}_{4\times 5}$$

$$R_4 \to R_4 - 2R_3.$$

$$B = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 4 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}}_{U}$$

3rd pivot $\updownarrow$ 3rd column.

$$B = LU$$

We have $AX = b$ and $B = P_{4231}A$.

$$\Rightarrow P_{4231}(AX) = P_{4231}\, b$$

$$(LU)X = BX = P_{4231}\begin{bmatrix}-1\\-3\\12\\0\end{bmatrix} = \begin{bmatrix}0\\-3\\12\\-1\end{bmatrix} = b_1$$

same as exchange of equations.

Let $UX = Y$ —— (1).

Then $LY = b_1$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 12 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = 0$$

$$y_2 = -3$$

$$-3y_2 + y_3 = 12 \qquad \Rightarrow \quad y_3 = 3$$

$$4y_2 + 2y_3 + y_4 = 1 \qquad y_4 = 5$$

Put in ①.

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 5 \end{bmatrix}$$

Then $5w = 5 \Rightarrow w = 1$

$2u + w = 3 \Rightarrow 2u + 1 = 3$

$\Rightarrow u = 1.$

$$3y + z + 2u - 3w = -3$$

$$\Rightarrow 3y + z - 1 = -3 \quad \Rightarrow 3y + z = -2$$

$$2x + 4y - z + 5u - 2w = 0$$

$$\Rightarrow 2x + 4y - z = -3.$$

Choose $y = k.$

Then $z = -2 - 3k$;

$$x = \frac{1}{2}\left(-3 - 4y + z\right)$$

$$\Rightarrow x = \frac{1}{2}\left(-3-4k-2-3k\right)$$

$$= \frac{1}{2}\left(-5-7k\right)$$

Put $k=0$. Then $u=w=1$;

$y=0$; $z=-2$; $x=-5/2$.

Put $k=1$. Then $u=w=1$;

$y=1$; $z=-5$; $x=-6$.

* Solves shortage of pivots as well.

Solve by LU decomposition:

$$x + y + z = 1$$

$$2x + 2y + 5z = 0$$

$$4x + 6y + 8z = 0$$

Soln: $Ax = b$

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \to R_2 - 2R_1 \quad ; \quad R_3 \to R_3 - 4R_1$

$$A = IA \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \boxed{0} & 3 \\ 0 & 2 & 4 \end{bmatrix}$$