Lec 11: Divergence & Curl of a vector field

Till introduction of total derivative, we will assume that "differentiability" of a function of more than one variable at a point P means that all its partial derivatives exist & are continuous at P.

Let $\vec{v}(x,y,z)$ be a differentiable vector function & let v_1, v_2, v_3 be the components of $\vec{v}$. Then, the function

$$\text{div } \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad -(1)$$

is called the divergence of $\vec{v}$.

Symbolical vector : ∇ can be thought of as the symbolic vector $\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ so that $\nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$
$= (1)$.

Eg 1) Suppose $\vec{v} = x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$. Find div $\vec{v}$ at $(1,-1,1)$.

$$\nabla \cdot \vec{v} = 2xz^2 + (-4yz^2) + xy^2 \Big|_{(1,-1,1)} = 2 + -4\times -1 + 1 = 7.$$

Eg. 2) Given $\phi(x,y,z) = 6x^3y^2z$, a scalar function,

a) find grad ϕ

b) find div(grad ϕ).

Soln :- a) $\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \nabla\phi = (18x^2y^2z, 12x^3yz, 6x^3y^2)$

b) $\nabla \cdot (\nabla \phi) = \text{div}(\text{grad } \phi) = \cancel{\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y}}$

$= (36xy^2z, 12x^3z, 0)$

$\cancel{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}}$ Note that we had computed

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

This operator $\cdot \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

You can define it on any twice differentiable scalar function.

Defn:- If a vector function $\vec{v}(P)$ ~~is~~ can be written as gradient of a scalar function f say $\vec{v}(P) = \nabla f(P)$, then the function $f(P)$ is called a potential of $\vec{v}(P)$. Then ~~Eg 3) (Gravitational field)~~ ~~Further,~~ $\vec{v}(P)$ defines a conservative vector field.

Lec 11 :- Eg 3) (Gravitational field) ~~force of~~ Laplace's eqn

a) The force of attraction between 2 particles at points $P_0 = (x_0, y_0, z_0)$

& $P = (x, y, z)$ given by $\vec{p} = -\frac{c}{r^3}\vec{r}$

where $\vec{r} = \overrightarrow{P_0P}$ can be written as

$$\vec{p} = \nabla f$$

where $f(x,y,z) = \frac{c}{r}$ where $r = \|\vec{r}\|$. That is, $\vec{p} = \text{grad}\left(\frac{c}{r}\right)$.

b) This potential f is a soln of Laplace eqn, ie,

$$\nabla^2 f = 0.$$

Equivalently $\text{div}\ \vec{p} = 0$.

Soln :- $\nabla\left(\frac{c}{r}\right) = \nabla(f) = \left(c\frac{\partial}{\partial x}\left(\frac{1}{r}\right), c\frac{\partial}{\partial y}\left(\frac{1}{r}\right), c\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right)$.

$$r = \|\vec{r}\| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.$$

~~$\frac{\partial}{\partial x}\left(\frac{1}{x}\right) =$~~ Let $g(r) = \frac{1}{r}$. Applying Chain Rule, we get

$$\Rightarrow \frac{\partial g}{\partial x} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} = -\frac{1}{r^2} \times \frac{1 \times 2(x-x_0)}{2\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

ie, $\frac{\partial r}{\partial x} = \frac{x-x_0}{r}$

$$= -\frac{(x-x_0)}{r^3}.$$

$|||^{ly}$, we get $\frac{\partial g}{\partial y} = -\left(\frac{y-y_0}{r^3}\right)$ & $\frac{\partial g}{\partial z} = -\frac{(z-z_0)}{r^3}$.

$$\Rightarrow \quad \nabla g = -\left(\frac{x-x_0}{r^3}, \frac{y-y_0}{r^3}, \frac{z-z_0}{r^3}\right) \quad \text{wrt } x, y \,\&\, z.$$

$$\therefore \nabla\left(\frac{c}{r}\right) = -\frac{c}{r^3}(x-x_0, y-y_0, z-z_0)$$

$$= -\frac{c\vec{r}}{r^3} = \vec{p}.$$

Thus, the ~~gr~~ force of attraction $\vec{p} = \nabla\left(\frac{c}{r}\right)$, ~~ie~~ is ~~a~~ the gradient of a scalar function $f(x,y,z) = \frac{c}{r}$. Thus, f is the potential function of $\vec{p}$.

Thus, vector field defined by the force of attraction $\vec{p}$, called the <u>gravitational field</u> is a conservative field.

No energy is lost (or gained) in displacing a body from a pt P to P_0 ~~&~~ in the field & back to P.

b) In addition, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla\cdot(\nabla f) = \nabla\cdot\vec{p}$

To compute this, see that

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(-\frac{(x-x_0)}{r^3}\right) = -(x-x_0)\times -\frac{3}{r^4}\frac{\partial r}{\partial x} + \frac{1}{r^3}\times(-1)$$

$$= \frac{3(x-x_0)}{r^4} \times \frac{x-x_0}{r} - \frac{1}{r^3}$$

$$= \frac{3(x-x_0)^2}{r^5} - \frac{1}{r^3}.$$

$|||^{ly}$, $\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = \frac{3(y-y_0)^2}{r^5} - \frac{1}{r^3}$ & $\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \frac{3(z-z_0)^2}{r^5} - \frac{1}{r^3}$

$$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^5}\times r^2 - \left(\frac{1}{r^3}\times 3\right) = 0.$$

ie, $\nabla^2\left(\frac{c}{r}\right) = c\times\nabla^2\left(\frac{1}{r}\right) = 0 = \nabla^2 f$

ie, f satisfies Laplace eqn & $\operatorname{div}(\vec{p}) = \operatorname{div}(\nabla f) = 0$.

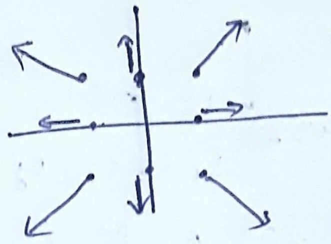
Divergence of $\vec{F}$ tells you locally how much it is sourcing out or sinking in or neither ($\mathrm{div}\, f) = 0$ (3)

$\mathrm{div}(f+g) = \mathrm{div}\, f + \mathrm{div}\, g$ & $\mathrm{div}(cf) = c\, \mathrm{div}\, f$.

Eg 2) $f(x,y) = x\hat{i} + y\hat{j}$

$f(1,0) = \hat{i}$; $f(0,1) = \hat{j}$

$f(1,1) = \hat{i} + \hat{j}$



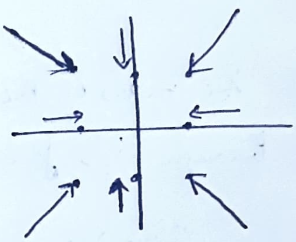
From the diagram, it seems that in the field, things are diverging out. So we feel intuitively that $\mathrm{div}\, f > 0$.

Check it out (verify)! $\mathrm{div}.\, \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1 + 1 = 2 > 0.$

We say this flow has +ve divergence

Eg 3) $f(x,y) = -x\hat{i} - y\hat{j}$

is an example of a flow with negative divergence



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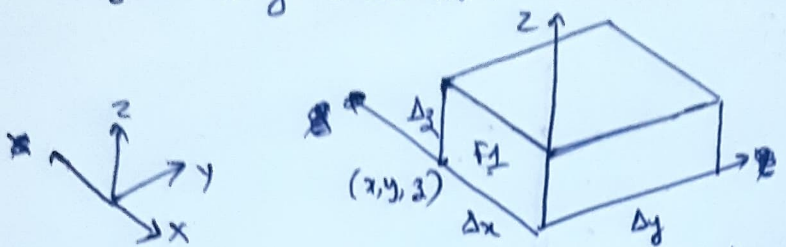
Remark: If $\operatorname{div} \vec{v} = 0$, $\vec{v}$ is called a solenoidal vector.

Physical meaning of divergence

Consider a (compressible) fluid flowing through a rectangular box B of small edges $\Delta x, \Delta y$ & Δz ||r to the co-ordinate axes.



Let $\vec{v}(x,y,z,t) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ denote the velocity ~~velo~~ vector of the motion.
& $\rho = \rho(x,y,z,t)$ denote the density of the fluid. (~~depends~~ density changes as fluid travels & hence, there is change in mass)

(Steady flow if ρ is indep't of time $\frac{\partial \rho}{\partial t} = 0$)

Let $\vec{u} = \rho\vec{v} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ (Assume $\vec{u}, \vec{v}$ are C^1 wrt variables x, y, z & t)

Consider the mass of the fluid flowing in through F1 & leaving the box B through opp. face F2 during a short time interval Δt.

$(\rho v_2)(y)\,\Delta x\,\Delta z$ denotes approximately the mass of fluid passing through F1 in unit time. $\Rightarrow$ In time interval Δt,

$(\rho v_2)(y)(\Delta x\,\Delta z)\,\Delta t$. III^{ly}, the mass of fluid leaving B through F2 in that interval Δt is $(\rho v_2)(y+\Delta y)\,\Delta x\,\Delta z\,\Delta t$

$\therefore$ total change in mass along Y in time Δt is

$$u_2(y+\Delta y) - u_2(y) \times \Delta x\,\Delta z\,\Delta t$$

$$= \frac{\Delta u_2}{\Delta y}\,\Delta V\,\Delta t$$

. Same way for X & Z dim

$\therefore$, total change in mass of the fluid in B

$$= \left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\Delta V\,\Delta t$$

From physics, we ~~understand~~ assume that this is caused by the rate of change of density wrt time & hence,

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\Delta V \Delta t = -\frac{\partial \rho}{\partial t}\Delta V \Delta t$$

Dividing by $\Delta V \Delta t$, we see that as Δx, Δy & $\Delta z \to 0$,

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = -\frac{\partial \rho}{\partial t}$$

or $$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \vec{v} = 0}$$

Lec 12

Continuity eqn of a compressible fluid flow.

Lec 12 :-

Special cases : a) Steady flow ie, $\frac{\partial \rho}{\partial t} = 0$ (density indpt of time)

$$\operatorname{div}(\rho \vec{v}) = 0$$

a1) Steady flow & incompressible fluid

$\operatorname{div} \vec{v} = 0.$ $\boxed{\text{Condition for incompressibility}}$

Says: ~~Balance~~ Difference of outflow & inflow for a given volume element is zero at any time. ~~Thus,~~ $\boxed{\text{divergence measures outflow minus inflow}}$

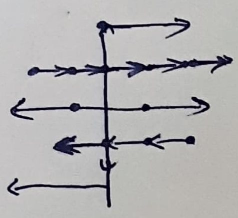
Eg 1) ~~Solenoidal~~ ~~For~~ Show that the flow with velocity vector

$\vec{v} = y\hat{i}$

is incompressible

$v(1,0) = 0$
$v(0,1) = 1$; $v(0,2) = 2$
$v(-1,0) = 0$
$v(0,-1) =$

$(-1,1) =$



$\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 \Rightarrow \vec{v}.$

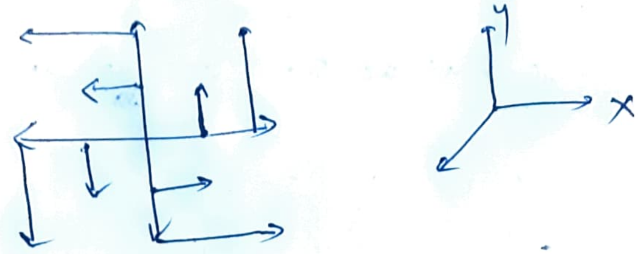
~~Eg 2)~~ A vector $\vec{v}$ whose divergence is zero is called solenoidal.

Eg 2) Determine the constant a so that the following vector is solenoidal.

$$\vec{v} = (-4x-6y+3z)\hat{i} + (-2x+y-5z)\hat{j} + (5x+6y+az)\hat{k}$$

$$0 = \nabla \cdot \vec{v} = -4+1+a = 0 \Rightarrow a = 3.$$

Eg 3) Find the divergence & curl of the vector field $\vec{f}(x,y) = -y\hat{i} + x\hat{j}$.



$$\text{div}(\vec{f}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

$$\text{curl}(\vec{f}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(1-(-1)) = 2\hat{k}$$

Curl(f) given by cross product is along the axis of rotation of the vector field ~~& clockwise direc~~ such that rotation appear clockwise if one looks ~~from~~ in the direction of cross product (by right hand screw rule).

Eg 4) (Exercise) Find curl $\vec{F}$ at $(1,-1,1)$ if $\vec{F} = x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$

~~(Eg 5)~~ ~~Eg 5)~~ A vector $\vec{v}$ is called <u>irrotational</u> if $\nabla \times \vec{v} = 0$.

Eg 5) i) Find a, b, c so that $\vec{v} = (-4x-3y+az)\hat{i} + (bx+3y+5z)\hat{j} + (4x+cy+3z)\hat{k}$ is irrotational.

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4x-3y+az & bx+3y+5z & 4x+cy+3z \end{vmatrix} = \hat{i}(c-5) - \hat{j}(4-a) + \hat{k}(b+3) = 0$$

$\Rightarrow a = 4$

$b = -3$

$c = 5$.

~~Now~~ (ii) Show that $\vec{v}$ is a conservative vector field.

Ans:- If $\exists$ scalar field $f: \mathbb{R}^3 \to \mathbb{R}$ st $\nabla f = \vec{v}$, then

$$\frac{\partial f}{\partial x} = -4x - 3y + 4z;$$

$$\frac{\partial f}{\partial y} = -3x + 3y + 5z$$

$$\frac{\partial f}{\partial z} = 4x + 5y + 3z$$

$$\Rightarrow f = -2x^2 - 3xy + \cancel{2}4zx + c_1(y,z)$$

$$= -3xy + \frac{3}{2}y^2 + 5yz + c_2(x,z)$$

$$= +4xz + 5yz + \frac{3}{2}z^2 + c_3(x,y)$$

$\Rightarrow$ By taking common terms from each, we get

$$f(x,y,z) = -2x^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 - 3xy + 5yz + 4xz + c$$

~~ie, if curl $\vec{v} = 0$, then & $\exists$ f s.t $\nabla f = \vec{v}$~~

~~$\vec{v}$ is irrotational $\Rightarrow$ $\vec{v}$ is conservative~~

~~A continuously differentiable vector function~~

Remark In general, if $\vec{v}$ is an irrotational (differentiable) vector field, then $\exists$ a scalar field f such that $\nabla f = \vec{v}$. (To be checked)

Thm 1) Let $\vec{v}$ be a ~~C^1 vector func~~ continuously differentiable vector function, ~~then~~ that can be written as gradient of a scalar function (f), then its curl is the zero vector.

ie, $\text{curl}(\nabla f) = 0$

b) If $\vec{v}$ is twice-~~differentiable~~ continuously differentiable (C^2 smooth) vector function, then

$$\text{div}(\text{curl}\,\vec{v}) = 0.$$

Proof :-

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

Note that $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ if all partial derivatives exist & are cts ~~on the~~ at that point.

$\because \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}(v_3)$ is cont. by assumption

Thus, by C^1 ness of $\vec{v}$, we see that curl (grad f) $= \vec{0}$.

Thus, gravitational field is irrotational too.

Summary : gravitational field is conservative ($\vec{F} = \nabla f$)
irrotational (curl $(\vec{F}) = 0$)
solenoidal (div $\vec{F} = 0$)

b) $\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$$

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

Since v_1, v_2 & v_3 have 2nd order partial derivatives cts at every pt of the domain,

$$\nabla \cdot (\nabla \times \vec{v}) = 0$$