Then (a_n) is a cauchy sequence. To see this first we observe that

$$|a_{n+1} - a_n| \leq \rho^{n-1}|a_2 - a_1| \quad \forall\, n \in \mathbb{N}, n \geq 2.$$

Hence, for $n > m$,

$$\begin{aligned} |a_n - a_m| &\leq |a_n - a_{n-1}| + \ldots + |a_{m+1} - a_m| \\ &\leq (\rho^{n-2} + \ldots + \rho^{m-1})|a_2 - a_1| \\ &\leq \rho^{m-1}(1 + \rho + \ldots + \rho^{n-m-3})|a_2 - a_1| \\ &\leq \frac{\rho^{m-1}}{1-\rho}|a_2 - a_1|. \end{aligned}$$

Since $\rho^{m-1} \to 0$ as $m \to \infty$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$. $\square$

Exercise 1.23 Given $a, b \in \mathbb{R}$ and $0 < \lambda < 1$, let (a_n) be a sequence of real numbers defined by $a_1 = a, a_2 = b$ and

$$a_{n+1} = (1 + \lambda)a_n - \lambda a_{n-1} \quad \forall\, n \in \mathbb{N},\, n \geq 2.$$

Show that (a_n) is a Cauchy sequence and its limit is $(b + \lambda a)/(1 - \lambda)$. ◀

Exercise 1.24 Suppose f is a function defined on an interval J. If there exists $0 < \rho < 1$ such that

$$|f(x) - f(y)| \leq \rho|x - y| \quad \forall\, x, y \in J,$$

then for any $a \in J$, the sequence (a_n) defined by

$$a_1 = f(a), \quad a_{n+1} := f(a_n) \quad \forall\, n \in \mathbb{N},$$

is a Cauchy sequence. Show also that the limit of the sequence (a_n) is independent of the choice of a. ◀

1.2 Series of Real Numbers

Definition 1.13 A **series** of real numbers is an expression of the form

$$a_1 + a_2 + a_3 + \ldots,$$

or more compactly as $\sum_{n=1}^{\infty} a_n$, where (a_n) is a sequence of real numbers.

The number a_n is called the n-th term of the series and the sequence $s_n := \displaystyle\sum_{i=1}^{n} a_i$ is called the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$.

$\square$

1.2.1 Convergence and divergence of series

Definition 1.14 A series $\sum_{n=1}^{\infty} a_n$ is said to *converge* (to $s \in \mathbb{R}$) if the sequence $\{s_n\}$ of partial sums of the series converge (to $s \in \mathbb{R}$).

If $\sum_{n=1}^{\infty} a_n$ converges to s, then we write $\sum_{n=1}^{\infty} a_n = s$.

A series which does not converge is called a *divergent series.* □

A necessary condition

Theorem 1.12 *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$. Converse does not hold.*

Proof. Clearly, if s_n is the n-th partial sum of the convergent series $\sum_{n=1}^{\infty} a_n$, then

$$a_n = s_n - s_{n-1} \to 0 \quad \text{as} \quad n \to \infty.$$

To see that the converse does not hold it is enough to observe that the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ diverges whereas $a_n \to 0$. ∎

The proof of the following corollary is immediate from the above theorem.

Corollary 1.13 *Suppose (a_n) is a sequence of positive terms such that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

The above theorem and corollary shows, for example, that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

EXAMPLE 1.21 We have seen that the sequence (s_n) with $s_n = \sum_{k=1}^{n} \frac{1}{k!}$ converges. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. Also, we have seen that the sequence (σ_n) with $\sigma_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges. Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

EXAMPLE 1.22 Consider the geometric series $\sum_{n=1}^{\infty} aq^{n-1}$, where $a, q \in \mathbb{R}$. Note that $s_n = a + aq + \ldots + aq^{n-1}$ for $n \in \mathbb{N}$. Clearly, if $a = 0$, then $s_n = 0$ for all $n \in \mathbb{N}$. Hence, assume that $a \neq 0$. Then we have

$$s_n = \begin{cases} na & \text{if} \quad q = 1, \\ \frac{a(1-q^n)}{1-q} & \text{if} \quad q \neq 1. \end{cases}$$

Thus, if $q = 1$, then (s_n) is not bounded; hence not convergent. If $q = -1$, then we have

$$s_n = \begin{cases} a & \text{if} \quad n \text{ odd}, \\ 0 & \text{if} \quad n \text{ even}. \end{cases}$$

Thus, (s_n) diverges for $q = -1$ as well. Now, assume that $|q| \neq 1$. In this case, we have

$$\left| s_n - \frac{a}{1-q} \right| = \frac{|a|}{|1-q|} |q|^n.$$

This shows that, if $|q| < 1$, then (s_n) converges to $\frac{a}{1-q}$, and if $|q| > 1$, then (s_n) is not bounded, hence diverges. $\square$

Theorem 1.14 *Suppose (a_n) and (b_n) are sequences such that for some $k \in \mathbb{N}$, $a_n = b_n$ for all $n \geq k$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.*

Proof. Suppose s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Let $\alpha = \sum_{i=1}^{k} a_i$ and $\beta = \sum_{i=1}^{k} b_i$. Then we have

$$s_n - \alpha = \sum_{i=k+1}^{n} a_i = \sum_{i=k+1}^{n} b_i = \sigma_n - \beta \quad \forall\, n \geq k.$$

From this it follows that the sequence (s_n) converges if and only if (σ_n) converges. ∎

From the above theorem it follows if $\sum_{n=1}^{\infty} b_n$ is obtained from $\sum_{n=1}^{\infty} a_n$ by omitting or adding a finite number of terms, then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \sum_{n=1}^{\infty} b_n \quad \text{converges.}$$

In particular,

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \sum_{n=1}^{\infty} a_{n+k} \quad \text{converges}$$

for any $k \in \mathbb{N}$.

The proof of the following theorem is left as an exercise.

Theorem 1.15 *Suppose $\sum_{n=1}^{\infty} a_n$ converges to s and $\sum_{n=1}^{\infty} b_n$ converges to σ. Then for every $\alpha, \beta \in \mathbb{R}$, $\sum_{n=1}^{\infty}(\alpha\, a_n + \beta\, b_n)$ converges to $\alpha\, s + \beta\, \sigma$.*

1.2.2 Some tests for convergence

Theorem 1.16 (**Comparison test**) *Suppose (a_n) and (b_n) are sequences of* **non-negative terms**, *and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then,*

(i) $\sum_{n=1}^{\infty} b_n$ *converges* $\Longrightarrow \sum_{n=1}^{\infty} a_n$ *converges,*

(ii) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Longrightarrow \sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. Suppose s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. By the assumption, we get $0 \leq s_n \leq \sigma_n$ for all $n \in \mathbb{N}$, and both (s_n) and (σ_n) are monotonically increasing.

(i) Since (σ_n) converges, it is bounded. Let $M > 0$ be such that $\sigma_n \leq M$ for all $n \in \mathbb{N}$. Then we have $s_n \leq M$ for all $n \in \mathbb{N}$. Since (s_n) are monotonically increasing, it follows that (s_n) converges.

(ii) Proof of this part follows from (i) (*How?*). ∎

Corollary 1.17 *Suppose* (a_n) *and* (b_n) *are sequences of* **positive terms**.

(a) Suppose $\ell := \lim_{n\to\infty} \frac{a_n}{b_n}$ *exists. Then we have the following:*

(i) If $\ell > 0$, *then* $\sum_{n=1}^{\infty} b_n$ *converges* $\iff$ $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) If $\ell = 0$, *then* $\sum_{n=1}^{\infty} b_n$ *converges* $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ *converges.*

(b) Suppose $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. *Then* $\sum_{n=1}^{\infty} a_n$ *converges* $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ *converges.*

Proof. (a) Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$.

(i) Let $\ell > 0$. Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\ell - \varepsilon < \frac{a_n}{b_n} < \ell + \varepsilon$ for all $n \geq N$. Equivalently, $(\ell - \varepsilon)b_n < a_n < (\ell + \varepsilon)b_n$ for all $n \geq N$. Had we taken $\varepsilon = \ell/2$, we would get $\frac{\ell}{2} b_n < a_n < \frac{3\ell}{2} b_n$ for all $n \geq N$. Hence, the result follows by comparison test.

(ii) Suppose $\ell = 0$. Then for $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $-\varepsilon < \frac{a_n}{b_n} < \varepsilon$ for all $n \geq N$. In particular, $a_n < \varepsilon b_n$ for all $n \geq N$. Hence, we get the result by using comparison test.

(b) By assumption, there exists $N \in \mathbb{N}$ such that $\frac{a_n}{b_n} \geq 1$ for all $n \geq N$. Hence the result follows by comparison test. ∎

EXAMPLE 1.23 We have already seen that the sequence (s_n) with $s_n = \sum_{k=1}^{n} \frac{1}{k!}$ converges. Here is another proof for the same fact: Note that $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges, it follows from the above theorem that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. □

EXAMPLE 1.24 Since $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ for all $n \in \mathbb{N}$, and since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows from the above theorem that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges. □

Theorem 1.18 (**De'Alembert's ratio test**) *Suppose* (a_n) *is a sequence of positive terms such that* $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$ *exists. Then we have the following:*

(i) If $\ell < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) If $\ell > 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. (i) Suppose $\ell < q < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} < q \quad \forall\, n \geq N.$$

In particular,

$$a_{n+1} < q\, a_n < q^2 a_{n-1} < \ldots < q^n\, a_1, \forall\, n \geq N.$$

Since $\sum_{n=1}^{\infty} q^n$ converges, by comparison test, $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) Let $1 < p < \ell$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} > p > 1 \, n \geq N.$$

From this it follows that (a_n) does not converge to 0. Hence $\sum_{n=1}^{\infty} a_n$ diverges. ∎

Theorem 1.19 (**Cauchy's root test**) *Suppose (a_n) is a sequence of positive terms such that $\lim_{n\to\infty} {a_n}^{1/n} = \ell$ exists. Then we have the following:*

(i) If $\ell < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\ell > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Suppose $\ell < q < 1$. Then there exists $N \in \mathbb{N}$ such that

$${a_n}^{1/n} < q \quad \forall\, n \geq N.$$

Hence, $a_n < q^n$ for all $n \geq N$. Since the $\sum_{n=1}^{\infty} q^n$ converges, by comparison test, $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) Let $1 < p < \ell$. Then there exists $N \in \mathbb{N}$ such that

$${a_n}^{1/n} > p > 1 \quad \forall\, n \geq N.$$

Hence, $a_n \geq 1$ for all $n \geq N$. Thus, (a_n) does not converge to 0. Hence, $\sum_{n=1}^{\infty} a_n$ also diverges. ∎

Remark 1.9 We remark that both d'Alembert's test and Cauchy test are silent for the case $\ell = 1$. But, for such case, we may be able to infer the convergence or divergence by some other means. ♦

EXAMPLE 1.25 For every $x \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges:

Here, $a_n = \frac{x^n}{n!}$. Hence

$$\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \quad \forall\, n \in \mathbb{N}.$$

Hence, it follows that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, so that by d'Alemberts test, the series converges. □

EXAMPLE 1.26 The series $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ converges: Here

$${a_n}^{1/n} = \frac{n}{2n+1} \to \frac{1}{2} < 1.$$

Hence, by Cauchy's test, the series converges. □

EXAMPLE 1.27 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. In this series, we see that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 = \lim_{n\to\infty} {a_n}^{1/n}$. However, the n-th partial sum s_n is given by

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}.$$

Hence $\{s_n\}$ converges to 1. $\square$

EXAMPLE 1.28 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. In this case, we see that

$$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \quad \forall\, n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, by comparison test, the given series also converges. $\square$

EXAMPLE 1.29 Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by comparison test, we see that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p \geq 2$ and diverges for $p \leq 1$. $\square$

EXAMPLE 1.30 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. To discuss this example, consider the function $f(x) := 1/x^p$, $x \geq 1$. Then, using teh fact that indefinite integral of x^q for $q \neq -1$ is $x^{q+1}/(q+1)$, we see that for each $k \in \mathbb{N}$,

$$k - 1 \leq x \leq k \Longrightarrow \frac{1}{k^p} \leq \frac{1}{x^p} \Longrightarrow \frac{1}{k^p} \leq \int_{k-1}^{k} \frac{dx}{x^p}.$$

Hence,

$$\sum_{k=2}^{n} \frac{1}{k^p} \leq \sum_{k=2}^{n} \int_{k-1}^{k} \frac{dx}{x^p} = \int_{1}^{n} \frac{dx}{x^p} = \frac{n^{1-p} - 1}{1-p} \leq \frac{1}{p-1}.$$

Thus,

$$s_n := \sum_{k=1}^{n} \frac{1}{k^p} \leq \frac{1}{p-1} + 1.$$

Hence, (s_n) is monotonically increasing and bounded above. Therefore, (s_n) converges. $\square$

A more general result on convergence of series in terms of integrals will be proved in Chapter 3.

1.2.3 Alternating series

Definition 1.15 A series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ where (u_n) is a sequence of positive terms is called an **alternating series**. $\square$

Theorem 1.20 (**Leibniz's theorem**) *Suppose (u_n) is a sequence of positive terms such that $u_n \geq u_{n+1}$ for all $n \in \mathbb{N}$, and $u_n \to 0$ as $n \to \infty$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.*

Proof. Let s_n be the n-th partial sum of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$. We observe that

$$s_{2n+1} = s_{2n} + u_{2n+1} \quad \forall\, n \in \mathbb{N}.$$

Since $u_n \to 0$ as $n \to \infty$, it is enough to show that (s_{2n}) converges (Why?). Note that

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}),$$

$$s_{2n} = u_1 - (u_2 - u_3) - \ldots (u_{2n-2} - u_{2n-1}) - u_{2n}$$

for all $n \in \mathbb{N}$. Since $u_i - u_{i+1} \geq 0$ for each $i \in \mathbb{N}$, (s_{2n}) is monotonically increasing and bounded above. Therefore (s_{2n}) converges. In fact, if $s_{2n} \to s$, then we have $s_{2n+1} = s_{2n} + u_{2n+1} \to s$, and hence $s_n \to s$ as $n \to \infty$. ∎

By the above theorem the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Likewise, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$ also converge[1]

Suppose (u_n) is as in Leibniz's theorem (Theorem 1.20), and let $s \in \mathbb{R}$ be such that $s_n \to s$, where s_n is the n^{th} partial sum of $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}u_n$.

How fast (s_n) converges to s?

In the proof of Theorem 1.20, we have shown that $\{s_{2n}\}$ is a monotonically increasing sequence. Similarly, it can be shown that $\{s_{2n-1}\}$ is a monotonically decreasing sequence.

Since (s_{2n-1}) is monotonically decreasing and (s_{2n}) is monotonically increasing, we have

$$s_{2n-1} = s_{2n} + u_{2n} \leq s + u_{2n}, \qquad s \leq s_{2n+1} = s_{2n} + u_{2n+1}.$$

Thus,

$$s_{2n-1} - s \leq u_{2n}, \qquad s - s_{2n} \leq u_{2n+1}.$$

Consequently,

$$|s - s_n| \leq u_{n+1} \quad \forall\, n \in \mathbb{N}.$$

1.2.4 Absolute convergence

Definition 1.16 A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely**, if $\sum_{n=1}^{\infty} |a_n|$ converges. □

Theorem 1.21 *Every absolutely convergent series converges.*

[1]The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ appear in the work of a Kerala mathematician *Madhava* did around 1425 which wa presented later in the year around 1550 by another Kerala mathematician *Nilakantha*. The discovery of the above series is normally attributed to Leibniz and James Gregory after nearly 300 years of its discovery.

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respectively. Then, for $n > m$, we have

$$|s_n - s_m| = \Big| \sum_{j=m+1}^{n} a_n \Big| \le \sum_{j=m+1}^{n} |a_n| = |\sigma_n - \sigma_m|.$$

Since, $\{\sigma_n\}$ converges, it is a Cauchy sequence. Hence, form the above relation it follows that $\{s_n\}$ is also a Cauchy sequence. Therefore, by the *Cauchy criterion*, it converges. ▮

Another proof without using Cauchy criterion. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respectively. Then it follows that

$$s_n + \sigma_n = 2p_n,$$

where p_n is the sum of all positive terms from $\{a_1, \ldots, a_n\}$. Since $\{\sigma_n\}$ converges, it is bounded, and since $p_n \le \sigma_n$ for all $n \in \mathbb{N}$, the sequence $\{p_n\}$ is also bounded. Moreover, $\{p_n\}$ is monotonically increasing. Hence $\{p_n\}$ converge as well. Thus, both $\{\sigma_n\}$, $\{p_n\}$ converge. Now, since $s_n = 2p_n - \sigma_n$ for all $n \in \mathbb{N}$, the sequence $\{s_n\}$ also converges. ▮

Definition 1.17 A series $\sum_{n=1}^{\infty} a_n$ is said to **converge conditionally** if $\sum_{n=1}^{\infty} a_n$ converges, but not absolutely. □

EXAMPLE 1.31 We observe the following:

(i) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

(ii) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

(iii) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent. □

EXAMPLE 1.32 For any $\alpha \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^2}$ is absolutely convergent: Note that

$$\left| \frac{\sin(n\alpha)}{n^2} \right| \le \frac{1}{n^2} \quad \forall\, n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, $\sum_{n=1}^{\infty} \left| \frac{\sin(n\alpha)}{n^2} \right|$ also converges. □

EXAMPLE 1.33 The series $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \log n}{n \log(\log n)}$ is conditionally convergent. To see this, let $u_n = \dfrac{\log n}{n \log(\log n)}$. Since $n \ge \log n \ge \log(\log n)$ we have

$$\frac{1}{n} \le \frac{\log n}{n \log(\log n)} \le \frac{1}{\log(\log n)} \qquad (*)$$

so that $u_n \to 0$. It can be easily seen that $u_{n+1} \le u_n$. Hence, by Leibnitz theorem, the given series converges. Inequality $(*)$ also shows that the series $\sum_{n=3}^{\infty} u_n$ does not converge. $\square$

Here are two more results whose proofs are based on some advanced topics in *analysis*

Theorem 1.22 *Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series and (b_n) is a sequence obtained by rearranging the terms of (a_n). Then $\sum_{n=1}^{\infty} b_n$ is also absolutely convergent, and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.*

Theorem 1.23 *Suppose $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series. The for every $\alpha \in \mathbb{R}$, there exists a sequence (b_n) whose terms are obtained by rearranging the terms of (a_n) such that $\sum_{n=1}^{\infty} b_n = \alpha$.*

To illustrate the last theorem consider the conditionally convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Consider the following rearrangement of this series:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} + \cdots .$$

Thus, if $a_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$, the rearranged series is $\sum_{n=1}^{\infty} b_n$, where

$$b_{3k-2} = \frac{1}{2k-1}, \qquad b_{3k-1} = \frac{1}{4k-2}, \qquad b_{3k} = \frac{1}{4k}$$

for $k = 1, 2, \ldots$. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Then we see that

$$\begin{aligned}
\sigma_{3k} &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \\
&= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}\right) \\
&= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{4k-2} - \frac{1}{4k}\right) \\
&= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right)\right] \\
&= \frac{1}{2} s_{2k}.
\end{aligned}$$

Also, we have

$$\sigma_{3k+1} = \sigma_{3k} + \frac{1}{2k+1}, \qquad \sigma_{3k+2} = \sigma_{3k} + \frac{1}{2k+1} - \frac{1}{4k+2}.$$

We know that $\{s_n\}$ converge. Let $\lim_{n\to\infty} s_n = s$. Since, $a_n \to 0$ as $n \to \infty$, it then follows that

$$\lim_{k\to\infty} \sigma_{3k} = \frac{s}{2}, \qquad \lim_{k\to\infty} \sigma_{3k+1} = \frac{s}{2}, \qquad \lim_{k\to\infty} \sigma_{3k+2} = \frac{s}{2}.$$

Hence, we can infer that $\sigma_n \to s/2$ as $n \to \infty$.