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Complex integration

Recall that an analytic function has derivatives of all orders. This is due to Cauchy's integral formula.

Applications of complex integration : problems in physics
special functions like gamma fn & error function.

In real analysis / calculus, we study ~~real~~

indefinite integrals $\int_{x_0}^{x} f(t)\,dt$ Integrating over

or

definite integrals $\int_{a}^{b} f(t)\,dt$

Here, in complex analysis, there is not one path between x_0 & x. We study complex integrals along a given contour (between a & b or x_0 & x) & these are called contour integrals or line integrals.

Review of definitions

1) Arc/path/curve : A set of points $z = x+iy$ in the complex plane is said to be a curve if $x = x(t)$ & $y = y(t)$ are functions of t where $a \le t \le b$ & $x(t)$ & $y(t)$ are continuous w.r.t t.

If we call this curve as C, then $z(t) = x(t) + iy(t)$ is called a parametric representation of C.

2) ~~Eg 1) The unit circle~~ Simple closed curve : If a curve is such that only the initial & final values of $z(t)$ are the same, then such curves are called simple closed curves.

Eg 1) The unit circle $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ about the origin is a simple closed curve with ~~then~~ a parametrisation

$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

It also has another parametrisation

~~3). A curve C or z(t)~~ $z = e^{-i\theta}$, $\quad 0 \le \theta \le 2\pi$

3) A curve C or $z(t)$, $a \le t \le b$ is said to be a smooth curve (C^1 smooth) if

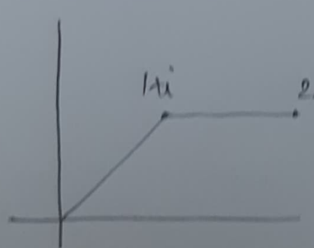
a) $\frac{dz}{dt}$ (exists &) is continuous on $[a,b]$, &

b) $\frac{dz}{dt} \neq 0$ throughout (a,b)

4) A contour or piecewise smooth curve is a curve consisting of a finite number of smooth curves joined end to end.

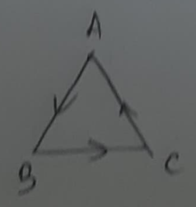
Eg. The polygonal line

$$z = \begin{cases} x + ix & 0 \le x \le 1 \\ x + i & 1 \le x \le 2 \end{cases}$$



$\therefore$ if $z(t)$, $a \le t \le b$ represents a contour, then $z(t)$ is continuous on $[a,b]$ & $z'(t)$ is piecewise continuous.

Simple closed curves



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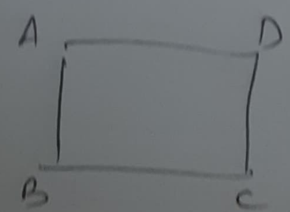
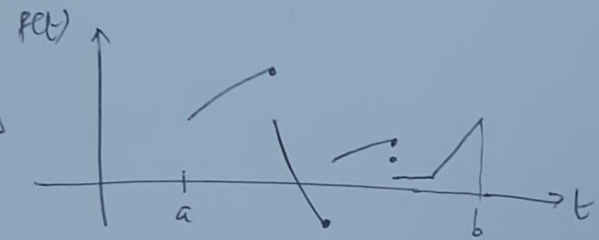


Piecewise cts on [a,b]

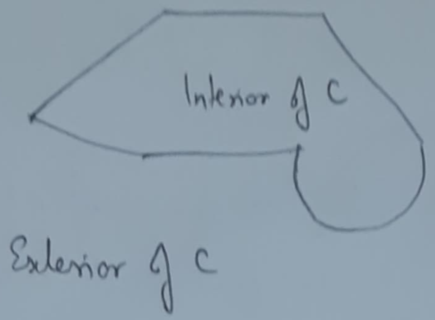
f is said to be piecewise cts on [a,b] if it is continuous everywhere on [a,b] except possibly for a finitely many points where although discontinuous, it has one sided limits.

Piecewise smooth on [a,b]

f is said to be piecewise smooth on [a,b] if it is C^1 smooth everywhere on [a,b] except possibly for finitely many pts where ~~although~~ although derivatives exist, they may not be continuous.



5) Interior & exterior of a simple closed contour C



Interior of C is bounded & exterior of C is unbounded (Jordan's curve theorem)

Points on C are the boundaries of both these domains.

Contour Integral

Given a contour C parametrised by $z : [a,b] \to \mathbb{C}$ & f a continuous function on C, we define the contour integral / line integral / integral of f along C as

$$\int_C f(z)\,dz := \int_a^b f(z(t))\,\frac{dz}{dt}(t)\,dt$$

Note that a contour may have more than one parametrisation, however, integral value is independent of the parametrisation.

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Properties of Contour Integrals

1) $\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$

(Integration is linear)

2) (Sense reversal) $\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$

3) Partitioning of path

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

A domain D is called simply connected if every simple closed curve within it encloses only points of D.

Eg Open circular disc $B(a,r)$ Non eg. Open Annulus $r_1 < |z-a| < r_2$

Evaluating Contour Integrals

There are two methods to evaluate contour integrals.

Method 1) Using "antiderivative." (will be explained later)

Eg 5) $\displaystyle\int_0^{1+i} z^2 dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{(1+i)^3}{3}$.

(Here, we used the fact that z^2 has an antiderivative $\frac{z^3}{3}$)

The following theorem guarantees the existence of "antiderivatives" in SCDs.

Theorem (Kreyszig Pg 647 Section 14.1)

Let $f(z)$ be analytic in a simply connected domain D. Then, $\exists$ an

a) analytical function $F(z)$ in D such that

$$F'(z) = f(z) \text{ on } D$$

(This will be called the indefinite integral of f in D) &

Let z_0 & z_1 be 2 pts on D & let C be any path with IP z_0 & EP z_1

b) ~~For any~~ path ~~C in D, joining two points~~ (~~z_0 & z_1 in D~~ (inside D)) ~~(oriented from z_0 to z_1)~~. Then, we have

$$\int_C f(z)\,dz = F(z_1) - F(z_0)$$

<u>Remark</u> :- Since integral is indpt of the path chosen from z_0 to z_1, we can as well write it uniformly as $\int_{z_0}^{z_1} f(z)\,dz$.

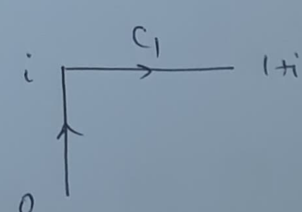
Eg 1) $$\int_{-\pi i}^{\pi i} \cos z\,dz = 2\sin(\pi i)$$

2) $$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 0$$

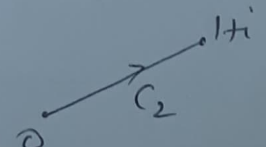
<u>Method 2</u>: <u>Using parametrisation</u>

Eg Find the contour integral a) $\int_{C_1} f(z)\,dz$ where (Eg 2 : Churchill)

$f(z) = y - x - i3x^2$



b) $\int_{C_2} f(z)\,dz$



a)

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Note that if we do integration on simple closed curves, there is an ambiguity in the orientation. In such cases, we often assume (if not explicitly given otherwise) that orientation is in the direction along which parameter t increases.

For example if

2) $\oint_C \frac{dz}{z}$ where C is the unit circle ~~oriented in the counter clockwise direction~~ parametrised as $z(t) = \cos t + i \sin t$, $-\pi \leq t \leq \pi$

If nothing is specified, it is assumed that it is counter clockwise as ~~t~~ as t increases, $z(t)$ traverses CCW.

2) b) $\oint_C \frac{dz}{z}$. Let $z = e^{it}$, $-\pi \leq t \leq \pi$

$$\int_{-\pi}^{\pi} \frac{1}{e^{it}} i e^{it} \, dt = 2\pi i$$

a) $\oint_C z^2 \, dz = \int_{-\pi}^{\pi} e^{2it} i e^{it} \, dt = i \left. \frac{e^{3it}}{3i} \right|_{-\pi}^{\pi} = \frac{1}{3} \times \left[e^{3\pi i} - e^{-3\pi i} \right]$

$$= \frac{1}{3} \left[e^{\pi i} - e^{\pi i} \right] = 0.$$

Conclusion: $\frac{1}{z}$ is not analytic on any SCD which contains C.

c) $\int_{-\pi}^{\pi} e^{mxi} \, dx$

d) $\oint_C (z - z_0)^m \, dz$, $m \in \mathbb{Z}$ & z_0 is a constant.

Integrate counter clockwise around the circle C of radius ρ with center at z_0.

$z(t) - z_0 = \rho e^{it}$ $\quad -\pi \leq t \leq \pi$ or $0 \leq t \leq 2\pi$

$C : z(t) = z_0 + \rho e^{it}$ $\qquad \frac{dz}{dt} = \rho \times i e^{it}$

$$\int_{-\pi}^{\pi} (\rho e^{it})^m \, i \rho e^{it} \, dt = i \rho^{m+1} \int_{-\pi}^{\pi} e^{i(m+1)t} \, dt$$

$$\int_{-\pi}^{\pi} e^{mxi}\,dx = \frac{e^{mxi}}{mi}\Big|_{-\pi}^{\pi} \qquad \text{if } m \neq 0$$

$$= 2\pi \qquad \text{if } m = 0$$

& $e^{m\pi i} = \cancel{\ } = \left(e^{\pi i}\right)^m = (-1)^m$

& $e^{-m\pi i} = \left(e^{\pi i}\right)^{-m} = (-1)^{-m}$

both are same. Hence,

$$\int_{-\pi}^{\pi} e^{mxi}\,dx = \begin{cases} 0 & \text{if } m \neq 0 \\ 2\pi & \text{if } m = 0. \end{cases}$$

Exercise: If $m, n \in \mathbb{Z}$,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \neq n \\ 2\pi & \text{when } m = n \end{cases}$$

<u>Back to part (c)</u> :-

$$= \begin{cases} 0 & \text{if } m \neq -1 \\ 2\pi i & \text{if } m = -1 \end{cases} = \begin{cases} 0 & \text{if } m \neq -1 \\ 2\pi i & \text{if } m = -1. \end{cases}$$

~~Suggests f(z) is analytic~~ could be ~~on some SCD containing~~

Since $f(z) = (z-z_0)^m$ is analytic on any SCD containing C, this was expected, as integral depends on end pts only.

~~ML inequality~~

Let C denote a ~~contour of~~ smooth C^1 smooth curve in $\mathbb{C}$.

Say $C : z(t)$, $a \le t \le b$

ie, $\frac{dz}{dt}(t)$ is cts on $[a,b]$ & ~~differentiable~~ non zero on (a,b).

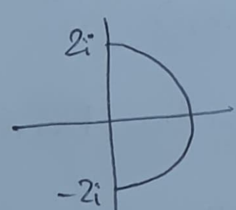
Recall length of a smooth curve (arclength) was defined as

$$len(C) := \int_a^b \left| \frac{dz}{dt}(t) \right| dt$$

Now if C is a piecewise smooth curve or contour, then ~~say C~~ C can be written as union of n many smooth curves say $C = \bigcup_{i=1}^{n} C_i$, then

$$len(C) = \sum_{i=1}^{n} len(C_i)$$

Eg Perimeter of a ~~circle~~ half circle (of radius 2)



$z(t) = 2e^{it}$ $\quad -\pi/2 \le t \le \pi/2$

$$len(C) = \int_{-\pi/2}^{\pi/2} |2ie^{it}|\, dt = 2\pi$$

Bounds for contour integral

Many a times, there will be need for estimating the absolute value of contour integrals. For this, we have the following theorem :-

Theorem ~~called~~ (M-L inequality)

Let C denote a contour of length L & suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defn. at the end defined, then

$$\left| \int_C f(z)dz \right| \leq ML.$$

Pr 1) Find an upper bound for the absolute value of the integral

$\int_C z^2 dz$ where C is the straight line segment from 0 to $1+i$.

We had done this : $\frac{(1+i)^3}{3}$

Find M s.t $|f(z)| \leq M$

$$|z^2| = |(x+iy)^2|$$

Parametrisation $z(t) : t+it$

$0 \leq t \leq 1$.

[You may also write $z(x) : x+ix$, $0 \leq x \leq 1$.]

$\therefore \; |(t+it)^2| = t^2 \times |(1+i)^2| = 2t^2 \leq 2$

$M = 2$ is an option (Any $M \geq 2$ works)

So, now we find $\text{len}(C)$ (obviously we know it is $\sqrt{2}$)

$$\text{len}(C) = \int_0^1 \left| \frac{dz(t)}{dt} \right| dt = \int_0^1 |(1+i)| dt = \sqrt{2}.$$

$$\therefore \left| \int_C f(z)\, dz \right| \leq 2\sqrt{2}.$$

Pr 2) Let C be the arc of the circle $|z| = 2$ from $z=2$ to $z=2i$, that lies in the first quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}.$$

f satisfies the conditions of ML inequality theorem

What is the M

$$\text{st } |f(z)| \le M ? \quad \forall z \in C.$$

$$|f(z)| = \left|\frac{z+4}{z^3-1}\right| \le \frac{|z|+4}{|z^3-1|} = \frac{6}{|z^3-1|}$$

Lower bound for $|z^3-1|$ is given by reverse triangle inequality

$$|a-b| \geq \big|\, |a| - |b| \,\big| .$$

Proof:- Let $a, b \in \mathbb{C}$.

$$|a| = |a+b-b| \le |a+b| + |-b|$$

$$\Rightarrow |a+b| \geq |a| - |b|$$

$$|||^{ly}, \quad |b| = |b+a-a| \le |a+b| + |a|$$

$$\Rightarrow |a+b| \geq |b| - |a|.$$

Put together

$$|a+b| \geq \big|\, |a| - |b| \,\big|$$

Hence,

$$|z^3-1| \geq \big(\, |z|^3 - 1 \,\big) = 8-1 = 7$$

$$\therefore \frac{1}{|z^3-1|} \le \frac{1}{7}$$

$$\Rightarrow |f(z)| \le \frac{6}{7}$$

$$\therefore \left|\int_C f(z)dz\right| \le \frac{6}{7} \times L \quad \text{where}$$

L is $\frac{1}{4}$ (perimeter of circle of radius 2) $= \frac{2\pi \times 2}{4} = \pi$

$\therefore$ Ans : $\frac{6\pi}{7}$.

$$z(t) = 2e^{it}, \quad 0 \le t \le \pi/2$$

$$z'(t) = 2ie^{it}$$

$$\text{len}(C) = \int_0^{\pi/2} 2\, dt = \pi.$$