The Rank-Nullity Theorem:

Let $T: V \to W$ be a linear transformation.

$$\operatorname{Ker} T = \{x \in V \mid T(x) = 0\}$$

$$\operatorname{Im} T = \{T(x) \mid x \in V\}$$

Denote $\operatorname{rank}(T) = \dim \operatorname{Ker} T$

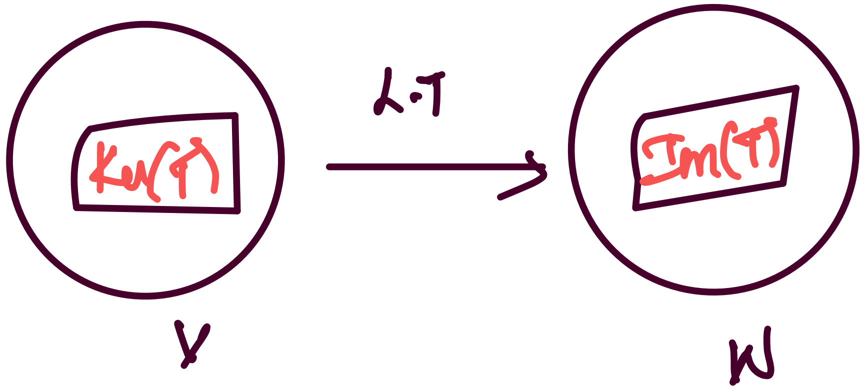
$\operatorname{nullity}(T) = \dim \operatorname{Im} T$

Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V.$$

Proof:

$T :$

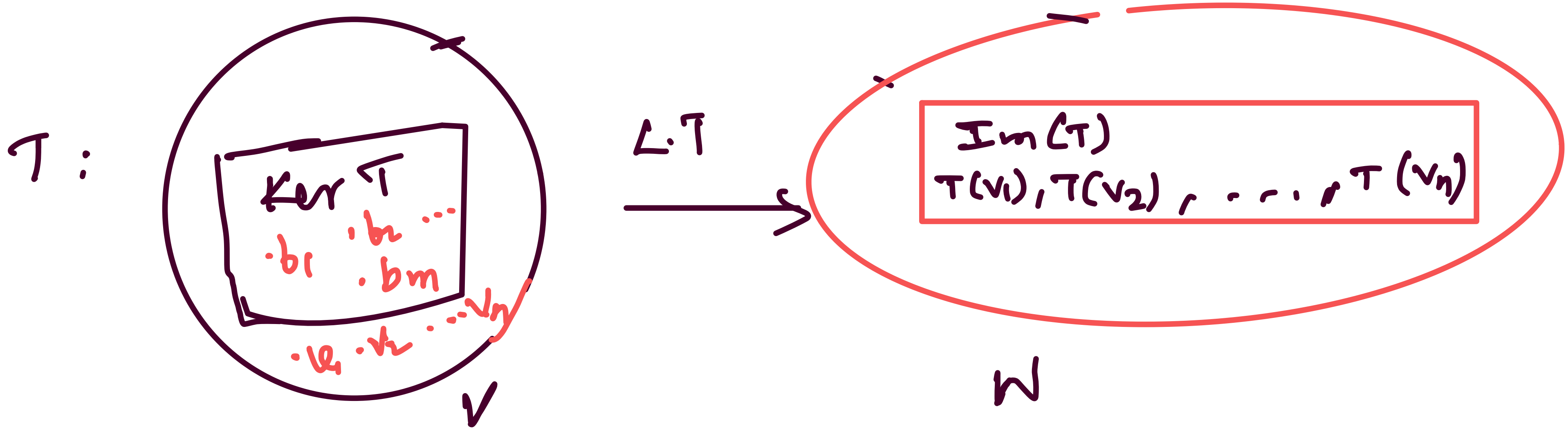


$\ker(T) = \{ x \in V \mid T(x) = 0 \}$: subspace of V

$\mathrm{Im}(T) = \{ T(x) \mid x \in V \}$: subspace of W.

Proofs already done.

Take $U = \{b_1, b_2, \dots, b_m\}$ be basis of Ker T.



As Ker T is a subspace of V, we can extend it to basis

$$B = \{b_1, b_2, \dots, b_m, v_1, v_2, \dots v_n\}$$

of whole space V.

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$\dim V = n(B) = m + n$$

claim: The basis for Im(T) is

$$\{T(v_1), T(v_2), \dots, T(v_n)\}$$

To prove Basis

L.I. set — ①

spanning set for Im(T) — ②

Need to prove 1 and 2.

1 Take $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0$

for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \cdots + T(\alpha_n v_n) = 0$$

(because T is a L.T)

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0$$

(because T is a L.T)

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in \ker T.$$

(by definition of $\ker T$)

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_m b_m$$

(because U is a basis of $\ker T$)

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n - \beta_1 b_1 - \beta_2 b_2 - \dots - \beta_m b_m = 0$$

$\Rightarrow$ $\boxed{\alpha_i = 0 \text{ for all } i,\ 1 \leq i \leq n}$ and $\beta_i = 0$ for all i, $1 \leq i \leq m$.

(because B is linearly independent (L.I), each scalar is zero).

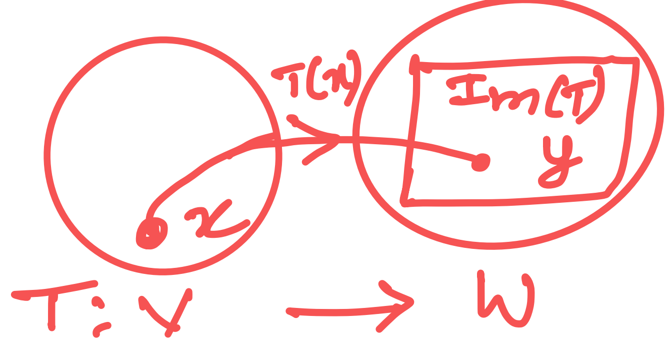
$\uparrow$ basis of V

$$\Rightarrow \alpha_1 = 0, \ \alpha_2 = 0, \ - - - , \ \alpha_n = 0.$$

$\therefore \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is L.I.

(① proved)

To prove ②:



Take $y \in Im(T) = \{T(x) \mid x \in V\}$

$\Rightarrow y = T(x)$ for some $x \in V$.

As B is a basis of V,

$$x = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_m b_m + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n .$$

Now, $y = T(x)$

$$= T(\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_m b_m + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n)$$

$$\Rightarrow y = \alpha_1 \underset{\downarrow\, 0}{T(b_1)} + \alpha_2 \underset{\downarrow\, 0}{T(b_2)} + \cdots + \alpha_m \underset{\downarrow\, 0}{T(b_m)} +$$

$$\beta_1 T(v_1) + \beta_2 T(v_2) + \cdots + \beta_n T(v_n)$$

(because T is a L.T.)

$$= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \cdots + \alpha_m \cdot 0 + \beta_1 T(v_1) + \beta_2 T(v_2) + \cdots + \beta_n T(v_n)$$

(because $b_i \in \operatorname{Ker} T$, for all i, $1 \le i \le m$)

$$= 0 + \beta_1 T(v_1) + \beta_2 T(v_2) + \cdots + \beta_n T(v_n)$$

$\Rightarrow \quad y = \beta_1 T(v_1) + \beta_2 T(v_2) + \cdots + \beta_n T(v_n)$

$\therefore \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a spanning set for $Im(T)$.

By ① and ②

$\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for $Im(T)$.

$\dim Im(T) = n$

$\dim Ker T = m$

The Rank - Nullity theorem

$$\boxed{\dim V = m + n = \dim Ker T + \dim Im(T)}$$

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$\boxed{\text{dim of domain} = \text{Nullity}(T) + \text{Rank}(T)}$$

Q:

Verify Rank-Nullity theorem for $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T(x, y, z) = (x+y,\ x+2y+z,\ 2x+3y+z).$$

Soln: dim of domain $= \dim(\mathbb{R}^3)$ $\hookrightarrow$ XYZ-space

$= 3.$

$$\text{Ker } T = \{\bar{x} \in \mathbb{R}^3 \mid T(\bar{x}) = \bar{0}\}$$

$$= \{\bar{x} \in \mathbb{R}^3 \mid T(x,y,z) = (x+y,\ x+2y+z,\ 2x+3y+z) = (0,0,0)\}$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x+y=0; \\ x+2y+z=0; \\ 2x+3y+z=0 \end{array}\right\}$$

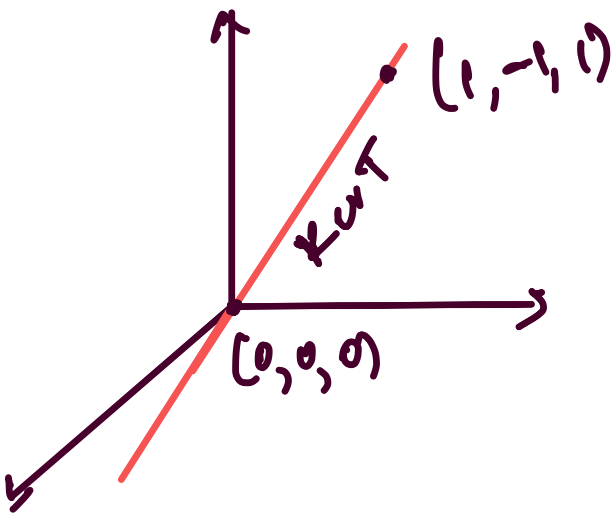
$$= \{(x,-x,x) \mid x \in \mathbb{R}\}$$

as $y = -x$;

$$z = -x - 2y$$
$$= -x + 2x$$
$$= x.$$

$$= \{x(1,-1,1) \mid x \in \mathbb{R}\}$$

Basis of Ker T $= \{(1,-1,1)\} = B$



$\Rightarrow$ $\dim \operatorname{Ker} T = n(B) = 1.$

$$\operatorname{Im}(T) = \{ T(\bar{x}) \mid \bar{x} \in \mathbb{R}^3 \}$$

$$= \{ T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3 \}$$

$$\operatorname{Im}(T) = \{ (x+y,\ x+2y+z,\ 2x+3y+z) \mid x, y, z \in \mathbb{R} \}$$

Suppose $T(x,y,z) = (a,b,c)$; $a, b, c \in \mathbb{R}$

$$\Rightarrow \quad x+y = a$$
$$x+2y+z = b$$
$$2x+3y+z = c$$

Then $\operatorname{Im}(T) = \left\{ \begin{array}{l} x+y=a \\ x+2y+z=b \\ 2x+3y+z=c \end{array} \;\middle|\; a, b, c \in \mathbb{R} \right\}$