2.12 Neighbourhood of a point in $\mathbb{R}$

We will later need precise language to discuss the notion of one real number being "close to" another. If a is a given real number, then saying that a real number x is "close to" a should mean that the distance $|x-a|$ between them is "small". A context in which this idea can be discussed is provided by the terminology of neighborhoods, which we now define.

Definition 2.8: Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then the *ϵ-neighbourhood* of a is the set

$$N(a,\epsilon) = \{x \in \mathbb{R} : |x-a| < \epsilon\}.$$

For $a \in \mathbb{R}$, the statement that $x \in N(a,\epsilon)$ is equivalent to either of the statements

$$-\epsilon < x - a < \epsilon \qquad \Leftrightarrow \qquad a - \epsilon < x < a + \epsilon.$$

$N(a,\epsilon) \setminus \{a\}$ is called the *deleted ϵ-neighbourhood* of a and is denoted by $N'(a,\epsilon)$. $N(a) \setminus \{a\}$ is called the *deleted neighbourhood* of a and is denoted by $N'(a)$.

Theorem 2.22: Let $a \in \mathbb{R}$. If $x \in N(a,\epsilon)$ for every $\epsilon > 0$, then $x = a$.

Proof: Exercise!

Theorem 2.23: Let $c \in \mathbb{R}$. Then
(a) The union of a finite number of neighbourhoods of c is a neighbourhood of c.
(b) The intersection of a finite number of neighbourhoods of c is a neighbourhood of c.

Proof: Exercise!

Note: The intersection of an infinite number of neighbourhoods of a point $c \in \mathbb{R}$ may not be a neighbourhood of c. For example, for every $n \in \mathbb{N}$, $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is a neighbourhood of 0. But, $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, which is not a neighbourhood of 0.

Example 2.17: Let $U = \{x : 0 < x < 1\}$. If $a \in U$, then let ϵ be the smaller of the two numbers a and $1-a$. Then, $N(a,\epsilon)$ is contained in U (**Proof!**). Thus each element of U has some ϵ-neighborhood of it contained in U.

Example 2.18: If $I = \{x : 0 \le x \le 1\}$, then for any $\epsilon > 0$, $N(0,\epsilon)$

contains points not in I, and so $N(0, \epsilon)$ is not contained in I. For example, the number $x = -\frac{\epsilon}{2}$ is in $N(0, \epsilon)$ but not in I.

Example 2.19: If $|x - a| < \epsilon$ and $|y - b| < \epsilon$, then the Triangle Inequality implies that

$$\begin{aligned} |(x+y) - (a+b)| &= |(x-a) + (y-b)| \\ &\leq |x-a| + |y-b| \\ &< 2\epsilon. \end{aligned}$$

Thus if x, y belong to the ϵ-neighborhoods of a, b respectively, then $x + y$ belongs to the 2ϵ-neighborhood of $a + b$ (but not necessarily to the ϵ-neighborhood of $a + b$).

2.13 Interior Point, Limit Point, Isolated Point, and Interior of a Set

Definition 2.9: Let S be a subset of $\mathbb{R}$. A point $x \in S$ is said to be an *interior point* of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subseteq S$.

Definition 2.10: The set of all interior points of S is said to be the *interior* of S and is denoted by int S (or by S°).

Example 2.20: (a) Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So, x cannot be an interior of S. Therefore, int $S = \phi$.

(b) Let $S = \mathbb{N}$. Every neighbourhood of x contains points not belonging to S. So, x cannot be an interior of S. Therefore, int $S = \phi$.

(c) Let $S = \mathbb{Q}$. Every neighbourhood of x contains rational as well as irrational points. So, x cannot be an interior of S. Therefore, int $S = \phi$.

(d) Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Every point of S is an interior point of S. Therefore, int $S = S$.

(e) Let $S = \mathbb{R}$. Every point of S is an interior point of S. Therefore, int $S = S$.

(f) Let $S = \phi$. S has no interior point. Therefore, int $S = \phi$.

Definition 2.11: Let S be a subset of $\mathbb{R}$. A point $x \in S$ is said to be a *limit point* (or an *accumulation point* or a *cluster point*) of S if every neighbourhood of x contains a point of S other than x.

Therefore, x is a limit point of S if for each positive ϵ,

$$N'(x, \epsilon) \cap S \neq \phi,$$

i.e., every deleted neighbourhood of x contains a point of S.

Note that a limit point of a set S may or may not belong to S. When we say that a set $S \subseteq \mathbb{R}$ has a limit point, we mean that some real number x is a limit point of S and no assertion is made as to whether x belongs to S or not.

Example 2.21: Prove that a finite set has no limit points.

Solution: Let S be a finite set and $S = \{x_1, x_2, \ldots, x_m\}$. Let $p \in \mathbb{R}$. p cannot be a limit point of S because if p be a limit point of S, then every neighbourhood of p must contain infinitely many elements of S, which is an impossibility since S contains only a finite number of elements. Therefore, the finite set S has no limit points.

Example 2.22: Prove that $\mathbb{N}$ has no limit points.

Solution: Let $p \in \mathbb{R}$. Let $\epsilon = \frac{1}{2}$. Then the ϵ-neighbourhood $N\left(p, \frac{1}{2}\right)$ of p contains at most one natural number and p cannot be a limit point of $\mathbb{N}$, beacuse, in order that p may be a limit point of $\mathbb{N}$, each neighbourhood of p must contain infinitely many elements of $\mathbb{N}$. Therefore, $\mathbb{N}$ has no limit points.

Example 2.23: Let S be a subset of $\mathbb{R}$. Prove that an interior point of S is a limit point of S.

Solution: Let x be an interior point of S. Then there exists a positive δ such that the neighbourhood $N(x, \delta)$ of x is entirely contained in S. Let us choose $\epsilon > 0$.

Case I: $0 < \epsilon < \delta$: Then $N(x, \epsilon) \subseteq N(x, \delta) \subseteq S$ and therefore $N'(x, \epsilon) \cap S \neq \phi$.

Case II: $\epsilon \geq \delta$: Then $N(x.\delta) \subseteq N(x, \epsilon)$. $N(x, \delta) \subseteq S$ and $N(x, \delta) \subseteq N(x, \epsilon) \implies N(x, \delta) \subseteq N(x, \epsilon) \cap S$. Then clearly, $N'(x, \epsilon) \cap S \neq \phi$.

Definition 2.12: Let S be a subset of $\mathbb{R}$. A point $y \in S$ is said to be an *isolated point* of S if y is not a limit point of S.

Since y is not a limit point of S, there exists a neighbourhood $N(y)$ of x such that $N'(y) \cap S \neq \phi$. Since $y \in S$, $N(y) \cap S = \{y\}$. Therefore, y is an isolated point of S if for some positive ϵ, $N(y, \epsilon)$ contains no point of S other than y.

Example 2.24: (a) Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Every point of S is an isolated point of S. We prove that 0 is a limit point of S. Let $\epsilon > 0$. By Archimedian Property of $\mathbb{R}$, there exists a natural number m such that $0 < \frac{1}{m} < \epsilon$. Now, $\frac{1}{m} \in S$ and $\frac{1}{m} \in N'(0, \epsilon)$. Thus, the deleted ϵ-neighbourhood of 0 contains a point of S and this holds for each positive ϵ. Hence, 0 is a limit point of S.

(b) Let $S = \mathbb{Z}$. Every point of $\mathbb{Z}$ is an isolated point of $\mathbb{Z}$. Therefore, no point of $\mathbb{Z}$ is a limit point of $\mathbb{Z}$. Let $x \in \mathbb{R} \setminus \mathbb{Z}$. Then there exists an integer m such that $m - 1 < x < m$. Let $\epsilon = \min\{|x - m|, |x - m \bar{-} 1|\}$. Then the neighbourhood $N(x, \epsilon)$ of x contains no point of $\mathbb{Z}$ and therefore x cannot be a limit point of $\mathbb{Z}$.

(c) Let $S = \mathbb{Q}$. No point of S is an isolated point of S. Every point $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, since each deleted neighbourhood of x contains a point of $\mathbb{Q}$.

(d) Let $S = \mathbb{R}$. No point of S is an isolated point of S. Every point $x \in \mathbb{R}$ is a limit point of $\mathbb{R}$, since each deleted neighbourhood of x contains a point of $\mathbb{R}$.

Theorem 2.24: Let $S \subseteq \mathbb{R}$ and x be a limit point of S. Then every neighbourhood of x contains infinitely many elements of S.

Proof: Let $\epsilon > 0$. Since x is a limit point of S, the deleted neighbourhood $N'(x, \epsilon)$ contains a point of S, i.e., $N'(x, \epsilon) \cap S \neq \phi$. Let $A = N'(x, \epsilon) \cap S$. We prove that A is an infinite set. If not, let A contain only a finite number of elements of S, say $a_1, a_2, \ldots, a_m$. Let $\epsilon_1 = |x - a_1|$, $\epsilon_2 = |x - a_2|$, ..., $\epsilon_m = |x - a_m|$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$. Then $\epsilon > 0$ and $a_i \notin N(x, \epsilon)$, $i = 1, 2, \ldots, m$. It follows that $N'(x, \epsilon) \cap S \neq \phi$ and this diswallows x to be a limit point of S. Thus A is an infinite set. In other words, $N(x, \epsilon)$ contains infinitely many elements of S.

We shall often use the above theorem to determine that a given set has no limit points.

Theorem 2.25 (Bolzano-Weierstrass Theorem for sets): Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof: Let S be a bounded infinite subset of $\mathbb{R}$. Since S is non-empty, both sup S and inf S exist. Let $s^* = \sup S$ and $s_* = \inf S$. Then $x \in S \implies s_* \leq x \leq s^*$. Let H be a subset of $\mathbb{R}$ defined by

$$H = \{x \in \mathbb{R} : x \text{ is greater than infinitely many elements of } S\}.$$

Then $s^* \in H$ and so H is a non-empty subset of $\mathbb{R}$. Let $h \in H$. Then h is greater than infinitely many elements of S and therefore $h > s_*$, because no element less or equal to s_* exceeds infinitely many elements of S. Thus, H is a non-empty subset of $\mathbb{R}$, bounded below, s_* being a lower bound. So inf H exists. Let $\inf H = \xi$. We now show that ξ is a limit point of S. Let us choose $\epsilon > 0$. Since $\inf H = \xi$, there exists an element $y \in H$ such that $\xi \leq y < \xi + \epsilon$. Since $y \in H$, y exceeds infinitely many elements of S and consequently $\xi + \epsilon$ exceeds infinitely many elements of S. Since ξ is the infimum of H, $\xi - \epsilon$ does not belong to H and so $\xi - \epsilon$ can exceed at most a finite number of elements of S. Thus, the neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ contains infinitely many elements of S. This holds for each $\epsilon > 0$. Therefore, ξ is a limit point of S.

This completes the proof.

Definition 2.13: Let $S \subseteq \mathbb{R}$. The set of all limit points of S is said to be the *derived set* of S and is denoted by S'.

Example 2.25: (a) If S is either a finite set or $S = \phi$, then $S' = \phi$.

(b) If $S = \mathbb{N}$ or $\mathbb{Z}$, then $S' = \phi$.

(c) If $S = \mathbb{Q}$ or $\mathbb{R}$, then $S' = \mathbb{R}$.

Example 2.26: Let S be a bounded subset of $\mathbb{R}$. Prove that the derived set S' is bounded.

Solution: Case I: Let S be a finite subset of $\mathbb{R}$. Then $S' = \phi$ and it is bounded.

Case II: Let S be an infinite subset of $\mathbb{R}$. By Bolzano-Weierstrass theorem, S' is a non-empty subset of $\mathbb{R}$.

Let $\sup S = m^*$. Then $x \in S \implies x \leq m^*$. Let $c > m^*$. Let us choose $\epsilon = \frac{c - m^*}{2}$. Then $m^* + \epsilon = c - \epsilon$ and the ϵ-neighbourhood $(c - \epsilon, c + \epsilon)$ of c contains no point of S. Therefore, c cannot be a limit point of S, i.e., $c \notin S'$. Thus, $c > m^* \implies c \notin S'$. Contrapositively, $c \in S' \implies c \leq m^*$. This shows that m^* is an upper bound of S', i.e., S' is bounded above.

Let $\inf S = m_*$ and let $d < m_*$. By similar arguments, d cannot be a limit point of S, i.e., $d \notin S'$. Thus, $d < m_* \implies d \notin S'$. Contrapositively,

$d \in S' \implies d \geq m_*$. This shows that m_* is a lower bound of S', i.e., S is bounded below.

Therefore, S' is a bounded subset of $\mathbb{R}$.

Example 2.27: Let S be a non-empty subset of $\mathbb{R}$ bounded above and $s^* = \sup S$. If s^* does not belong to S. prove that s^* is a limit point of S and s^* is the greatest element of S'.

Solution: Let $\epsilon > 0$. Since $s^* = \sup S$,
(i) $x \in S \implies x < s^*$ (since $s^* \notin S$) and
(ii) there is an element $y \in S$ such that $s^* - \epsilon < y < s^*$. We have

$$s^* - \epsilon < y < s^* < s^* + \epsilon.$$

Thus, the ϵ-neighbourhood $(s^* - \epsilon, s^* + \epsilon)$ of s^* contains a point y of S other than s^*. Since ϵ is arbitrary, s^* is a limit point of S.

Let $t > s^*$ and $\epsilon = \frac{t - s^*}{2}$. Then $\epsilon > 0$ and $s^* + \epsilon = t - \epsilon$. Since $s^* = \sup S$, no point of S is greater than S^*. Therefore, the neighbourhood $(t - \epsilon, t + \epsilon)$ of t contains no point of S and so t is not a limit point of S. Consequently, s^* is the greatest element of S'.

Example 2.28: Let $S = (a, b)$ be an open bounded interval. Prove that $S' = [a, b]$.

Solution: **Case I:** Let $x \in (a, b)$. Then x is an interior point of S and therefore x is a limit point of S, since an interior point of a set is a limit point of the set.

Case II: Let $x = a$. Let us choose $\epsilon > 0$. Let $\delta = \min\{\epsilon, b - a\}$. Then $\delta > 0$ and

$$a < a + \frac{\delta}{2} < a + \delta \leq a + \epsilon \ , \quad a < a + \frac{\delta}{2} < a + \delta \leq b.$$

Now,

$$a < a + \frac{\delta}{2} < a + \epsilon \implies a + \frac{\delta}{2} \in N'(a, \epsilon) \text{ and } a < a + \frac{\delta}{2} < b \implies a + \frac{\delta}{2} \in S.$$

Therefore, $a + \frac{\delta}{2} \in N'(a, \epsilon) \cap S$. As $N'(a, \epsilon) \cap S \neq \phi$, a is a limit point of S.

Case III: Let $x = b$. The proof is similar to **Case II** above.

Case IV: Let $x > b$. Let us choose $\epsilon = \frac{x - b}{2}$. Then $\epsilon > 0$ and $b + \epsilon = x - \epsilon$. The neighbourhood $(x - \epsilon, x + \epsilon)$ contains no point of S and this proves that

x is not a limit point of S.

Case V: Let $x < a$. Let us choose $\epsilon = \frac{a-x}{2}$. The $\epsilon > 0$ and $x + \epsilon = a - \epsilon$. The neighbourhood $(x - \epsilon, x + \epsilon)$ contains no point of S and this proves that x is not a limit point of S.

From the above cases, we conclude that $S' = [a, b]$.

Example 2.29: Let $S = [a, b]$ be a closed bounded interval. Prove that $S' = S = [a, b]$.

Solution: Exercise!

Example 2.30: Find the derived set of the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Solution: $\cdots$

Example 2.31: Let $S = \{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
(i) Show that 0 is a limit point of S.
(ii) If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of S.

Solution: $\cdots$

Example 2.32: Let $S = \{(-1)^m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$. Show that 1 and -1 are limit points of S.

Solution: $\cdots$

Theorem 2.26: (a) Let A, B be subsets of $\mathbb{R}$ and $A \subseteq B$. Then $A' \subseteq B'$.
(b) Let $A \subseteq \mathbb{R}$. Then $(A')' \subseteq A'$.
(c) Let A, B be subsets of $\mathbb{R}$. Then $(A \cap B)' \subseteq A' \cap B'$.
(d) Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cap A_2 \cap \cdots \cap A_m)' \subseteq A_1^{'} \cap A_2^{'} \cap \cdots \cap A_m^{'}$.
(e) Let A, B be subsets of $\mathbb{R}$. Then $(A \cup B)' = A' \cup B'$.
(f) Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cup A_2 \cup \cdots \cup A_m)' = A_1^{'} \cup A_2^{'} \cap \cdots \cup A_m^{'}$.

Proof: Exercise!

2.14 Open & Closed Sets

Definition 2.13: A set $S \subseteq \mathbb{R}$ is said to be an *open set* if each point of S is an interior point of S.

Example 2.33: $\cdots$

Theorem 2.27: Let $S \subseteq \mathbb{R}$. Then S is an open set if and only if $S = \operatorname{int} S$.

Proof: $\cdots$

Theorem 2.28: (a) The union of a finite number of open sets in $\mathbb{R}$ is an open set.
(b) The union of an arbitrary collection of open sets in $\mathbb{R}$ is an open set.
(c) The intersection of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof: Exercise!

Note: The intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set. Let us consider the sets G_i where

$$\begin{array}{rcl} G_1 & = & \{x \in \mathbb{R} : -1 < x < 1\} \\ G_2 & = & \{x \in \mathbb{R} : -\dfrac{1}{2} < x < \dfrac{1}{2}\} \\ \cdots & \cdots & \cdots \\ G_n & = & \{x \in \mathbb{R} : -\dfrac{1}{n} < x < \dfrac{1}{n}\} \\ \cdots & \cdots & \cdots \end{array}$$

Each G_i is an open set but $\bigcap_{i=1}^{\infty} G_i = \{0\}$ is not an open set.

Again, let us consider the sets G_i where

$$\begin{array}{rcl} G_1 & = & \{x \in \mathbb{R} : -1 < x < 1\} \\ G_2 & = & \{x \in \mathbb{R} : -2 < x < 2\} \\ \cdots & \cdots & \cdots \\ G_n & = & \{x \in \mathbb{R} : -n < x < n\} \\ \cdots & \cdots & \cdots \end{array}$$

Each G_i is an open set and $\bigcap_{i=1}^{\infty} G_i = G_1$ is also an open set.

Definition 2.14: A set $S \subseteq \mathbb{R}$ is said to be a *closed set* if it contains all its limit points, i.e., $S' \subseteq S$.

Theorem 2.29: Let S be a subset of $\mathbb{R}$. Then int S is an open set.

Proof: $\cdots$

Theorem 2.30: Let S be a subset of $\mathbb{R}$. Then int S is the largest open set contained in S.

Proof: $\cdots$

Theorem 2.31: An open interval is an open set.

Proof: $\cdots$

Theorem 2.32: A non-empty bounded open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

Example 2.34: Let $S = (0, 1]$ and $T = \{\frac{1}{n} : n = 1, 2, 3, \cdots\}$. Show that $S \setminus T$ is an open set.

Solution: Observe that $S \setminus T = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \cdots$. Thus, $S \setminus T$ is the union of an infinite number of open intervals. Since an open interval is an open set, $S \setminus T$ is the union of an infinite number of open sets and hence is an open set.

Definition 2.15: A set $S \subseteq \mathbb{R}$ is said to be a *closed set* if $\bar{S} \subseteq S$. Alternatively, a set $S \subseteq \mathbb{R}$ is said to be a *closed set* if $\mathbb{R} \setminus S$ is an open set.

Example 2.35: $\cdots$

Theorem 2.33: Let $S \subseteq \mathbb{R}$. Then S is a closed set if and only if $S' \subseteq S$.

Proof: $\cdots$

Theorem 2.34: (a) The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.
(b) The intersection of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof: Exercise!

Note: Since $\mathbb{R}$ is an open set, ϕ being the complement of $\mathbb{R}$, is a closed set. Since ϕ is an open set, $\mathbb{R}$ being the complement of ϕ, is a closed set. Therefore, the set $\mathbb{R}$ is both open and closed; the set ϕ is both open and closed in $\mathbb{R}$. The next theorem shows that no other subset of $\mathbb{R}$ has this property.

Theorem 2.35: No non-empty proper subset of $\mathbb{R}$ is both open and closed in $\mathbb{R}$.

Proof: $\cdots$

2.15 Adherent Point, Exterior Point, and Closure of a Set

Definition 2.14: Let S be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an *adherent point* of S if every neighbourhood of x contains a point of S.

It follows that x is an adherent point of S if $N(x, \epsilon) \cap S \neq \phi$ for every $\epsilon > 0$.

Definition 2.15: The set of all adherent points of S is said to be the *closure* of S and is denoted by $\bar{S}$.

From definition, it follows that $S \subseteq \bar{S}$ for any set $S \subseteq \mathbb{R}$.