The exponential function

Let $z = x+iy$. We define $e^z = e^x e^{iy}$ where the symbol e^{iy} stands for $\cos y + i\sin y$. (y in radians).

Properties of exponential fn $\exp(z)$ or e^z

1) $e^{z_1} e^{z_2} = e^{z_1+z_2} \Rightarrow \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

~~$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$~~

~~2) $e^{2\pi i} = 1 \Rightarrow e^{z+2\pi i} = e^z$~~

2) $\frac{d(e^z)}{dz} = e^z$ 3) $|e^z| = e^x$ & $\arg(e^z) = \{y + 2n\pi \mid n \in \mathbb{Z}\}$

4) $e^z \neq 0$ for any complex no z.

If $e^{z_0} = 0$ for some $z_0 \in \mathbb{C}$, then, $e^{x_0} = 0$ ⫫

Properties not possessed by $f(x) = e^x$, $x \in \mathbb{R}$.

5) (Periodicity) $e^{2\pi i + z} = e^z$

e^z is periodic with a purely imaginary period of $2\pi i$

6) e^z can be negative even though e^x is always +ve

(Eg $z = \pi i$)

7) $\mathbb{C} \twoheadrightarrow \mathbb{C} \setminus \{0\}$ is onto $\mathbb{R}$ —

$z \longrightarrow e^z$

ie, $\forall\, w_0 \neq 0$ in $\mathbb{C}$, $\exists\, z_0 \in \mathbb{C}$ st $e^{z_0} = w_0$.

Recall $\mathbb{R} \to \mathbb{R}\setminus\{0\}$ is not onto under e^x

but $\mathbb{R} \overset{\exp}{\twoheadrightarrow} \mathbb{R}_{>0}$ is onto

$x \longrightarrow e^x$

$\forall\, y_0 > 0$, $\exists\, x_0 \in \mathbb{R}$ st $e^{x_0} = y_0$. x_0 is what we call as $\ln y_0$.

Pr 1) Find all $z = x+iy$ for which $\cancel{e^z = 1+i}$. $e^z = 1+\sqrt{3}i$

Soln : $e^{x+iy} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2e^{i\pi/3}$

$\therefore$ $e^x = 2$ & $y \in \{\pi/3 + 2n\pi \mid n \in \mathbb{Z}\}$

ie, $x = \ln 2$ & y can be any no in the set above

$\therefore z = \ln 2 + i(\pi/3 + 2n\pi)$ for any $n \in \mathbb{Z}$

satisfies $e^z = 1+\sqrt{3}i$

Problems in exponential function

Churchill Pg 91 Example

Pg 92 : 1, 3, 5, 7, 8a, 8c, 10

Logarithmic function

Recall that $\ln x$ (only applicable if $x>0$) is that unique real no. y such that $e^y = x$

Similarly, here we seek for all $\omega \in \mathbb{C}$ (omega) such that
$$e^{\omega} = z \quad — \text{①}$$
& would like to define these ω as candidates for $\boxed{\log z.}$

Let us write $z = re^{i\theta}$ where $-\pi < \theta \le \pi$, $z \ne 0$.
Note that if ω_0 is a soln for ①, then so is $\omega_0 + 2\pi n i$ for any $n \in \mathbb{Z}$ as
$$e^{\omega_0 + 2\pi n i} = e^{\omega_0} = z.$$
Hence, $\log z$ if exists is a multi-valued function.
Now, if $\omega = u + iv$, then ① is eqv.t to saying
$$e^{u+iv} = re^{i\theta}$$
That is, $e^u = r$ & v is of the form $\theta + 2\pi n$ for any $n \in \mathbb{Z}$.
ie, $u = \ln r$ & $v \in \{\theta + 2\pi n, n \in \mathbb{Z}\}$.
Hence, $e^{\omega} = z$ iff
$$\omega = \ln r + i(\theta + 2\pi n), \quad n \in \mathbb{Z}.$$
$$\boxed{\omega = \ln r + i\theta + 2\pi n i, \quad n \in \mathbb{Z}} \longrightarrow \text{①}$$

But $\theta + 2\pi n$, $n \in \mathbb{Z}$ are the values of $\arg z$.

Definition :- Let $z \ne 0 \in \mathbb{C}$. Then,
$$\log z \overset{\text{defn}}{=} \ln r + i \arg z$$

Obsn 1: $e^{\log z} = e^{\ln r} e^{i\arg z} = re^{i\theta} = z$

Obsn 2 :- $e^z = e^x \cdot e^{iy} =$
Comparing the polar forms, we get $|e^z| = e^x$ & $\arg(e^z) = y + 2\pi n$, $n \in \mathbb{Z}$.
$$\Rightarrow \log(e^z) = \ln e^x + i\arg(e^z)$$
$$= x + i(y + 2\pi n)$$
$$= z + 2\pi i n, \quad n \in \mathbb{Z} \longrightarrow \text{②}$$

Comparison between real logarithm & complex logarithm

Real variable $\ln x$	Complex logarithm
$\log(e^x) = x$ ✓	$\log(e^z) = z$ is false (see ②)
$e^{\log x} = x$ ✓	$e^{\log z} = z$

exp(z) is single valued although $\log z$ isn't.

Pr 1) Find $\log(-1-\sqrt{3}i)$

Let $z_0 = -1-\sqrt{3}i = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{i(-2\pi/3)}$
$$\therefore \log z_0 = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right), \quad n \in \mathbb{Z}$$

Principal value of the logarithm

To avoid the ambiguities when multi-valued functions arise, we need to define a single valued logarithm function (like in case of argument function).

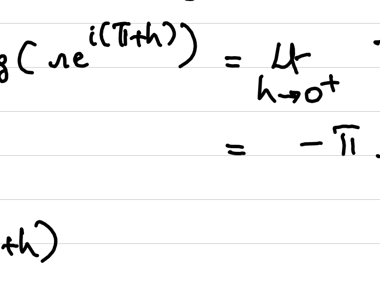
Define (capital L)
$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$
Further, if $z = re^{i\theta}$ where $-\pi < \theta \le \pi$, then
$$\mathrm{Log}\, z = \ln r + i\theta \quad (\text{ie, } n = 0 \text{ in ①})$$
$$\therefore \log z = \mathrm{Log}\, z + 2\pi n i$$
Further if $z \in \mathbb{R}_{>0}$, then $z = re^{i\theta}$ (where $\theta = 0$ & hence $z = r$) $\Rightarrow \mathrm{Log}\, z = \ln r$
Thus, $\mathrm{Log}\, z$, the principal value of the logarithm is the extension of the usual real (natural) logarithm.

Continuity of the Arg z (principal argument)

$\mathrm{Arg}\, z$ is defined on $\mathbb{C} \setminus \{0\}$.
It is continuous at every point in $\mathbb{C}$ except at zero & on the negative real axis.

To see this, consider the polar form of z ie, $re^{i\theta}$ & write $\mathrm{Arg}(z)$ as a function $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow (-\pi, \pi]$
$$f(r,\theta) = \mathrm{Arg}(re^{i\theta})$$
Let $S = \{(r,\theta) \in \mathbb{R}^+ \times \mathbb{R} \mid \theta = (2n+1)\pi, n \in \mathbb{Z}\}$
For simplicity, consider $(r, \pi) \in S$.

This can be generalised easily to any $(r, (2n+1)\pi) \in S$.



To show discontinuity at a point, it is enough to find two curves C_1 & C_2 that approach (r,π) but give different limits.
$$\lim_{\theta \to \pi^+} f(r,\theta) = \lim_{\substack{h \to 0 \\ h > 0}} \mathrm{Arg}(re^{i(\pi+h)}) = \lim_{h \to 0^+} \pi + h - 2\pi = -\pi.$$

(r,π) C_1 $(r, \theta = \pi + h)$

$$\lim_{\theta \to \pi^-} f(r,\theta) = \lim_{h \to 0^+} \mathrm{Arg}(re^{i(\pi-h)}) = \lim_{h \to 0^+} \pi - h = \pi$$

C_2 $(r, \pi - h)$ (r,π)

At other points, ie, $\mathbb{C} \setminus S$, it is easily seen that $f(r,\theta)$ is continuous.

Conclusion :- $\mathrm{Log}\, z$ is cont. at $z_0 \iff \mathrm{Arg}\, z$ is cont. at z_0.

Theorem (Sufficient condition for differentiability in terms of polar form)

Let the function $f(z) = u(r,\theta) + iv(r,\theta)$ be defined throughout some ε-nbhd of a non-zero point $z_0 = r_0 e^{i\theta_0}$ & suppose that

a) the first order partial derivatives of the functions u & v with respect to r & θ exist everywhere in the nbhd

b) those partial derivatives are continuous at (r_0, θ_0) & satisfy the polar form of the CR eqns
$$r u_r = v_\theta$$
$$u_\theta = -r v_r \quad \text{at } (r_0, \theta_0).$$
Then, $\left(\frac{df}{dz}\right)(z)$ exists at $z = z_0$ & its value is
$$f'(z_0) = e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)}.$$

Derivative of the Logarithm

Consider $r \in \mathbb{R}^+$ & $\theta \in \mathbb{R}$. Then, $\mathrm{Log}(re^{i\theta})$ is defined as
$$\left.\begin{aligned} &\mathrm{Log}(re^{i\theta}) = \ln r + i\,\mathrm{Arg}(re^{i\theta}) \\ &u(r,\theta) = \ln r \quad \& \quad v(r,\theta) = \mathrm{Arg}(re^{i\theta}). \end{aligned}\right\} \text{③}$$

Remark :- Note that if $\theta \in (-\pi, \pi]$, then
$$\mathrm{Log}(re^{i\theta}) = \ln r + i\theta$$
with $u(r,\theta) = \ln r$ & $v(r,\theta) = \theta$.

Going back to eqn ③, for any $(r_0, \theta_0) \in \mathbb{R}^+ \times \mathbb{R}$,
$$u_r(r_0,\theta_0) = \frac{1}{r_0}, \quad u_\theta(r_0,\theta_0) = 0$$
$$v_r(r_0,\theta_0) = 0$$
However, $\frac{\partial v}{\partial \theta}(r_0,\theta_0)$ doesn't exist if $(r_0,\theta_0) \in S$.
$$\lim_{\theta \to \pi^+} \frac{v(r_0,\theta) - v(r_0,\pi)}{\theta - \pi}. \quad \text{But } v(r_0,\theta) = \pi + h - 2\pi = -\pi + h$$
$$= \lim_{h \to 0^+} \frac{-\pi + h - \pi}{h} = \text{doesn't exist.}$$
On the other hand, the left hand derivative
$$\lim_{\theta \to \pi^-} \frac{v(r_0,\theta) - v(r_0,\pi)}{\theta - \pi} = \lim_{h \to 0^+} \frac{(\pi - h) - \pi}{-h} = 1$$
Hence, $\frac{\partial v}{\partial \theta}(r_0,\theta_0)$ doesn't exist if $\theta_0 \in \{(2n+1)\pi \mid n \in \mathbb{Z}\}$.

Check: $\frac{\partial v}{\partial \theta}(r_0,\theta_0) = 1$ if $(r_0,\theta_0) \in \mathbb{C} \setminus S$.

(Note that $\mathrm{Arg}(re^{i\theta})$ equals θ upto a difference factor which is a multiple of 2π)
$$\therefore \frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}(\theta - 2\pi n) = 1. \quad \text{if } \frac{\partial v}{\partial \theta} \text{ exists.}$$
One can now easily verify that all the conditions of Theorem above are met if we define $\mathrm{Log}\, z$ for $z \in \mathbb{C} \setminus S$. Thus, its derivative exists for every $z_0 \in \mathbb{C} \setminus S$ & given by
$$f'(z_0) = \left[e^{-i\theta}(u_r + iv_r)\right]\Big|_{(r_0,\theta_0)} \qquad z_0 = r_0 e^{i\theta_0}$$
$$u_r(r_0,\theta_0) = \frac{1}{r_0}$$
$$v_r(r_0,\theta_0) = 0$$
$$\Rightarrow f'(z_0) = e^{-i\theta_0} \times \frac{1}{r_0} = \frac{1}{r_0 e^{i\theta_0}} = \frac{1}{z_0}$$
$\therefore$ If $z_0 \in \mathbb{C} \setminus S$, then
$$\left[\frac{d}{dz}(\mathrm{Log}\, z)\right](z_0) = \frac{1}{z_0}.$$

Summary :-

1) $\mathrm{Log}\, z$ is defined on $\mathbb{C} \setminus \{0\}$.

2) $\mathrm{Log}\, z$ (Arg z) is not continuous on $\mathbb{R}_{\le 0}$.

3) $\mathrm{Log}\, z$ is analytic on $\mathbb{C} \setminus S$ & its derivative is $\frac{1}{z}$.

Corollary :- $\exists$ no open ball $B(0,\rho)$ centered around 0 with radius ρ where $\mathrm{Log}\, z$ is analytic
Hence, $\exists$ no simply connected domain containing origin where $\mathrm{Log}\, z$ is analytic.

Hence, $\frac{1}{z}$ will never have an antiderivative on any simply connected domain containing origin.

Recall: If C = unit circle with CCW orientation, then
$$\oint_C \left(\frac{1}{z}\right) dz = 2\pi i$$

This remark also tells you that $\frac{1}{z}$ doesn't have an antiderivative on any SCD that contains C inside it.

Identities involving log

(Note that for real $x_1 > 0$, $x_2 > 0$,
$$\log(x_1 x_2) = \log x_1 + \log x_2$$
where $\log x$ is single valued.)

1. $\log(z_1 z_2) = \log z_1 + \log z_2$

However, $\mathrm{Log}(z_1 z_2) \ne \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$.

Pr 1) ~~What is the value of Log z~~ Show that when

$z_1 = i$ & $z_2 = \frac{-1-i}{\sqrt{2}}$,

$\text{Log}(z_1 z_2) \neq \text{Log}\, z_1 + \text{Log}\, z_2$

$\text{Log}\, z_1 = \text{Log}(e^{i\pi/2}) = i\pi/2.$

$\text{Log}\, z_2 = \text{Log}(e^{i3\pi/4}) = i3\pi/4.$

$\text{Log}(z_1 z_2) = \text{Log}\left(i\left(\frac{-1-i}{\sqrt{2}}\right)\right) = \text{Log}\left(\frac{1-i}{\sqrt{2}}\right) = \text{Log}(e^{-i\pi/4})$

$= -i\pi/4.$

Hence, proved.

~~Ex~~ Pr 2) Show that if $\text{Re}(z_1) > 0$ & $\text{Re}\, z_2 > 0$, then

$\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2$

$\text{Re}(z_j) > 0 \Rightarrow \text{Log}\, z_j = \ln|z_j| + i\,\text{Arg}\, z_j$ where ~~Arg z~~ $-\frac{\pi}{2} < \text{Arg}\, z_j < \frac{\pi}{2}$

Hence, $z_1 z_2 = r_1 r_2 e^{i(\text{Arg}\, z_1 + \text{Arg}\, z_2)}$

Since $\text{Arg}\, z_1 + \text{Arg}\, z_2 \in (-\pi, \pi]$, $\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2$ &

hence, $\cancel{z_1 z_2 = r_1 r_2 e^{i\,\text{Arg}\, z_1 z_2}}$

$\text{Log}\, z_1 z_2 = \ln|z_1 z_2| + i(\text{Arg}\, z_1 + \text{Arg}\, z_2)$

$= \ln|z_1| + \ln|z_2| +$ "

$= \text{Log}\, z_1 + \text{Log}\, z_2.$