$A = LU$

Q. Find LU decomposition of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Soln:

$$A = IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

pivot

elementary matrix

$R_2 \to R_2 - 4R_1$; $R_3 \to R_3 - 5R_1$

elementary operations.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}}_{\text{undo elementary operations}}^{L_1} \overbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{-3} & -6 \\ 0 & -3 & -6 \end{bmatrix}}^{U_1}$$

$= A$.

$R_3 \to R_3 - R_2$.

$L_2 = L \qquad U_2 = U$ "Shortage of pivots"

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & +1 & 1 \end{bmatrix}}_{3\times 3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} = LU \quad ; \text{ 2 pivots.}$$

$\underbrace{}_{\text{undo}}$: $f \circ g(\ )$; f, g are functions.

some operation

Eg: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$

$$I \rightsquigarrow \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 + 4R_1]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

Eg: $f(x) = \sin x$

$g(x) = \cos x$

$$(f \circ g)(x) = f(g(x)) = f(\cos x) = \sin(\cos x)$$

* Shortage of pivots.
  ↳ add last column of identity matrix.

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Find LU decompositio[n] of A.

$\text{rank}(A) = 1 \implies$ 1-pivot.

$R_2 \to R_2 - 2R_1 \quad ; \quad R_3 \to R_3 - 3R_1$

$$A = IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of pivots $= 1$

Expected no. of pivots $= 3$

So, 2 columns are added from the identity matrix.

Q. Find LU decomposition of

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & 2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}_{5\times4}.$$

Soln: $A = IA = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{5\times5} \underbrace{\begin{bmatrix} \boxed{2} & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & 2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}}_{5\times4}$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - 2R_1$;

$R_5 \to R_5 + 3R_1$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & \boxed{3} & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 12 \end{bmatrix}$$

$R_3 \to R_3 + R_2$; $R_4 \to R_4 - 2R_2$; $R_5 \to R_5 + 3R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ -3 & -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & \boxed{5} \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$R_4 \to R_4 + R_3 \quad ; \quad R_5 \to R_5 - 2R_3$$

$$A = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{U}$$

↳ columns from identity matrix.