Partial Derivatives of scalar fields :-

Let $f: \mathbb{R}^2 \to \mathbb{R}$

$\frac{\partial f}{\partial x}(x_1, y_1)$: The partial deriv of f wrt x at (x_1, y_1) is
or
$$f_x(x_1, y_1) = \lim_{h \to 0} \frac{f(x_1+h, y_1) - f(x_1, y_1)}{h}$$

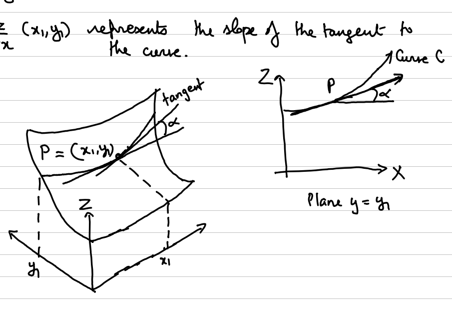
|||^{ly}, the partial deriv of f wrt y at (x_1, y_1) is
$$f_y(x_1, y_1) = \lim_{k \to 0} \frac{f(x_1, y_1+k) - f(x_1, y_1)}{k}$$

The partial derivatives f_x & f_y are also functions of x & y.

Geometric interpretation

Let $z = f(x,y)$ represent a surface in space. The plane $y = y_1$ (vertical plane) cuts the surface, intersecting it along a curve & the partial derivative

$\frac{\partial z}{\partial x}(x_1, y_1)$ represents the slope of the tangent to the curve.



$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \text{ may or may not be equal to}$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}.$$

Two mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ & $\frac{\partial^2 f}{\partial y \partial x}$ are equal at a point (x_0, y_0) if

1) All first order partial derivatives f_x, f_y; second order partial derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ exists (here, $n=2 \Rightarrow$ open disc) on an n-ball containing (x_0, y_0), say B

2) $\frac{\partial^2 f}{\partial x \partial y}$ & $\frac{\partial^2 f}{\partial y \partial x}$ are continuous at (x_0, y_0). OR

2') Either $\frac{\partial^2 f}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial y \partial x}$ is continuous on B.

Ex 1). $f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Soln :- $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$. From defn, both 1st order partial derivatives of f exist at pts other than origin.

$$\therefore f_x(x,y) = y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy \times \left[\frac{(x^2+y^2)(2x) - (x^2-y^2)\times 2x}{(x^2+y^2)^2}\right]$$

if $(x,y) \neq (0,0)$. Hence, $f_x(0,k) = -k$ for $k \neq 0$.

$$\therefore (f_x)_y(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{-k-0}{k} = -1.$$

Similarly,
$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$
$$f_y(x,y) = x\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy \times \left[\frac{(x^2+y^2)(-2y) - (x^2-y^2)\times 2y}{(x^2+y^2)^2}\right]$$

if $(x,y) \neq (0,0)$. Hence, $f_y(h,0) = h$ for $h \neq 0$.

$$\therefore (f_y)_x(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

GRADIENT OF A SCALAR FIELD :-

Let $f: S \to \mathbb{R}$ be a scalar field, where $S \subseteq \mathbb{R}^n, n \geq 1$. The gradient of f, written grad f or ∇f is defined as the vector function

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Useful in

a) finding the rate of change of $f(x,y,z)$ in any direction in space.

b) to obtain surface normal vector

c) in deriving vector fields from scalar fields.

We also introduce the differential operator ∇ as
$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

If you know Laplace eqn in PDE, then

by noting $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$,

we can rewrite it as
$$\nabla^2 f = 0.$$

DIRECTIONAL DERIVATIVE : Generalisation of partial derivatives

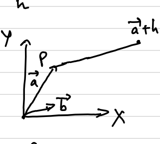
This addresses the question of rate of change of f (scalar function) in an arbitrary direction in space.

Let $f: S \to \mathbb{R}$ be a scalar function & $S \subseteq \mathbb{R}^n, n \geq 1$. The directional derivative of f at a point $P \in S$ along a non-zero vector $\vec{b}$ is defined by

$$\left(D_{\vec{b}} f\right)(P) := \lim_{h \to 0} \frac{f(\vec{a} + h\hat{b}) - f(\vec{a})}{h}$$

where $\vec{OP} = \vec{a}$, & $\hat{b} = \frac{\vec{b}}{\|\vec{b}\|}$.

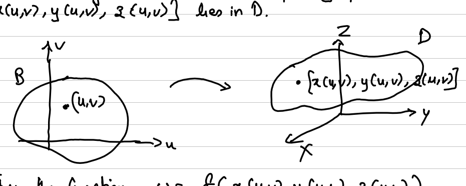
Illustration
$S = \mathbb{R}^2$



Now, let us recall the chain rule. (Statement only).

Theorem 1 (Chain Rule)

Let $w = f(x,y,z)$ be continuous and have continuous first order partial derivatives in a domain D in xyz space. Let
$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$
be functions that are continuous & have first order partial derivatives in a domain B in the uv plane, where B is such that for every $(u,v) \in B$, the corresponding point $[x(u,v), y(u,v), z(u,v)]$ lies in D.



Then, the function $w = f(x(u,v), y(u,v), z(u,v))$ is defined in B & has first order partial derivatives w.r.t u & v in B &

$$\left.\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}, \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}. \end{aligned}\right\} \quad (1)$$

Special cases :- If we drop the variable z, ie, if $w = f(x,y)$ then, the third term in the above eqn (1) will be dropped.

If we drop the variable v, we get
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. \quad (1a)$$

If we drop both z & v, we get
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad (1b)$$

Finally, if we drop y, z & v, we get the familiar one variable chain rule.
ie, if $z = f(x)$ & $x = g(t)$, then
$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt}. \quad (1c)$$

3) Chain rule illustration :-

If $w = x^2 - y^2$ & we define polar co-ordinates r, θ by $x = r\cos\theta$, $y = r\sin\theta$, then, what is $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial \theta}$ in terms of r & θ ?

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= 2x \times \cos\theta + (-2y)\times \sin\theta = 2r\cos^2\theta - 2r\sin^2\theta \\ &= 2r\cos 2\theta. \end{aligned}$$

|||^{ly}, $$\begin{aligned} \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} \\ &= 2x(-r\sin\theta) + (-2y)(r\cos\theta) \\ &= -2r(r\cos\theta \cdot \sin\theta + r\sin\theta\cos\theta) \\ &= -4r^2\sin\theta\cos\theta = -2r^2\sin 2\theta = -4xy \end{aligned}$$

Chain rule as matrix multiplication :-

Coming back to eqn (1), we see that

$$\frac{dw}{dt} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} \quad \text{in case of 1b}$$

OR

$$= \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \quad \text{in case of 1a}$$

More generally, (1) can be rewritten as

$$\underbrace{\underset{1\times 2}{\begin{bmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{bmatrix}}}_{\text{Gradient of } w \text{ w.r.t } (u,v)} = \underbrace{\underset{1\times 3}{\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}}}_{\text{Gradient of } w \text{ wrt } (x,y,z)} \underset{3\times 2}{\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}}$$

Interpretation of $D_{\vec{b}}(f)$ in terms of gradient

Let $S \subseteq \mathbb{R}^3$.

Lemma 2: If $f: S \to \mathbb{R}$ has continuous first order partial derivatives in S, then,
$$\left(D_{\hat{b}} f\right)(\vec{a}) = \nabla f(\vec{a}) \cdot \hat{b}$$

Proof :- Consider $g(t) = f(\vec{a} + t\vec{b})$
$$\Rightarrow g(0) = f(\vec{a})$$
$$\therefore \lim_{h \to 0} \frac{g(h) - g(0)}{h} = (D_{\hat{b}} f)(\vec{a}).$$
$$\Rightarrow g'(0). = \frac{dg}{dt}(0) = (D_{\hat{b}} f)(\vec{a}).$$

Let $\vec{r}(t) = \vec{a} + t\vec{b} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$
where, $x(t) = a_1 + tb_1$, $y(t) = a_2 + tb_2$
$z(t) = a_3 + tb_3$.

$$\therefore g(t) = f(\vec{a} + t\vec{b}) \Rightarrow \frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

ie, $g'(0) = \frac{\partial f}{\partial x}(\vec{a}) \cdot b_1 + \frac{\partial f}{\partial y}(\vec{a})\, b_2 + \frac{\partial f}{\partial z}(\vec{a})\, b_3$

ie, $(D_{\hat{b}} f)(\vec{a}) = \nabla f(\vec{a}) \cdot \hat{b}$

This shows that the directional derivative is simply the component of the gradient vector in the direction of $\vec{b}$.

Note: $D_{\vec{b}} f(\vec{a})$ & $D_{\hat{b}} f(\vec{a})$ are same by definition $\nabla f(\vec{a}) \cdot \hat{b}$.

$\hat{b}$ – unit vector along $\vec{b}$. Equivalently,
$$(D_{\vec{b}} f)(\vec{a}) = (D_{\hat{b}} f)(\vec{a}) = \frac{\nabla f(\vec{a}) \cdot \vec{b}}{\|\vec{b}\|}.$$

Q 1) Consider the scalar function
$$f(x,y,z) = x^2 + y^2 + xz.$$

a) Find grad f

b) Find grad f at the point $P = (2,-1,3)$.

c) Find the directional derivative of f in the direction of $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ at the point $P = (2,-1,3)$.

Soln :- a) $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

where $\frac{\partial f}{\partial x}(x,y,z) = 2x + z$
$\frac{\partial f}{\partial y}(x,y,z) = 2y$
$\frac{\partial f}{\partial z}(x,y,z) = x$
$\Rightarrow \nabla f(x,y,z) = [2x+z, 2y, x]$.

b) $\nabla f(P) = [2x+z, 2y, x]\big|_{(2,-1,3)}$
$= [7, -2, 2]$.

c) $$\begin{aligned} (D_{\vec{b}} f)(P) &= \nabla f(P) \cdot \left(\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}\right) \\ &= (7\hat{i} - 2\hat{j} + 2\hat{k}) \cdot \left(\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}\right) \\ &= \frac{7-4+2}{\sqrt{6}} = \frac{5}{\sqrt{6}}. \end{aligned}$$

The +ve sign indicates that f is increasing along $\vec{b}$.

2) Suppose $\phi(x,y,z) = xy^2z$ &
$\vec{A}(x,y,z) = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$.

Find $\frac{\partial^3}{\partial x^2 \partial z}(\phi A)$ at the point $(2,-1,1)$.

Ans : $4\hat{i} - 2\hat{j}$ (Exercise)

Next theorem tells you that gradient points in the direction of maximum increase of f. In other words, f undergoes its maximum rate of change in the direction of the gradient vector.

Let $S \subseteq \mathbb{R}^3$.

Theorem 3: Let $f(x,y,z)$ be a scalar function on S, having continuous first order partial derivatives in S. Then, ∇f exists in & if $\nabla f(P) \neq \vec{0}$ at some point P, then $\nabla f(P)$ has the direction of maximum increase of f at P.

Proof :- Recall that
$$D_{\hat{b}}(f)(P) = \nabla f(P) \cdot \hat{b} = \|\nabla f(P)\| \cos\theta$$

Thus, the directional derivative of f at P is maximum along the direction of the vector $\nabla f(P)$, provided $\nabla f(P) \neq \vec{0}$.

Pr 4) In which direction the directional derivative of $\phi(x,y,z) = x^2yz^3$ is maximum at $(2,1,-1)$ & find the magnitude of the maximum.

Soln :- $\nabla\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$

$\nabla\phi(x,y,z) = (2xyz^3, x^2z^3, 3x^2yz^2)$

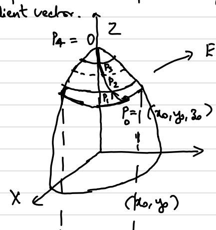
$\Rightarrow \nabla\phi(2,1,-1) = (-4,-4,12)$

The directional derivative of f at $P = (2,1,-1)$ is maximum when the direction is that of gradient at P

$\therefore \hat{b}$ can be chosen at $\frac{-4\hat{i} - 4\hat{j} + 12\hat{k}}{\sqrt{16+16+144}} = \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}$.

Magnitude of $D_{\overrightarrow{\nabla\phi(P)}} f(P) = \|\vec{\nabla}\phi(P)\| = 4\sqrt{11}$.

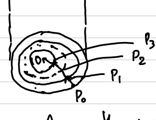
Illustration :- Consider a hill (surface) whose height above the sea level is a function of x & y, say $H(x,y)$. The gradient of H at a point P is a plane vector in the XY plane which points in the direction of steepest slope at P. How steep it is at that point is given by the magnitude of the gradient vector.



Eg. Paraboloid
$z = f(x,y)$

The direction of max ascent will be along $\vec{\nabla f}(P)$ &
(") max descent (") $-\vec{\nabla f}(P)$.

Top view



Projections of P_0, P_1, P_2 & P_3 in XY plane

On the above surface, the curve through which the steepest ascent happens say $P_0 - P_1 - P_2 - \dots - P_n = O$ (approximately) will have each segment's (say $P_i - P_{i+1}$) projection in the XY plane to be the gradient of f (at P_i), ie, normal to level curve passing through P_i.