Normal Distribution:

A Continuous random variable X is said to have a normal distribution with parameters μ and σ^2 if its density function is given by the probability law:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$
$$-\infty < \mu < \infty, \ \sigma > 0.$$

Note:

(i) μ and σ^2 are called the mean and Variance respectively of the normal distribution. It is represented as $X \sim N(\mu, \sigma^2)$

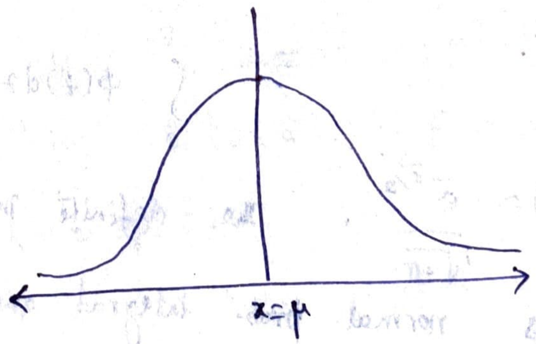
(ii) Let $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$.

Then $E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{E(X)-\mu}{\sigma} = 0.$

$Var(Z) = \frac{1}{\sigma^2} V(X) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1.$

Z is called a Standard normal variate and is written as $Z \sim N(0,1)$.

Properties: The graph of $y = f(x)$ is given below.



(i) The curve is bell shaped and symmetrical about the line $x = \mu$.

(ii) The max. prob. occurs at the point $x = \mu$ and is given by $[f(x)]_{max} = \frac{1}{\sigma\sqrt{2\pi}}$.

(iii) Since $f(x)$ being prob., it can never be negative and hence no portion of the curve lies below the x-axis.

(iv) X- axis is the asymptote of the curve.

(v) Area:

If $X \sim N(\mu, \sigma^2)$, then

$$P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$$

put $\frac{x-\mu}{\sigma} = z$, $x = \mu + \sigma z$. when $x = \mu$, $z = 0$

when $x = x_1$, $z = z_1 = \frac{x_1-\mu}{\sigma}$.

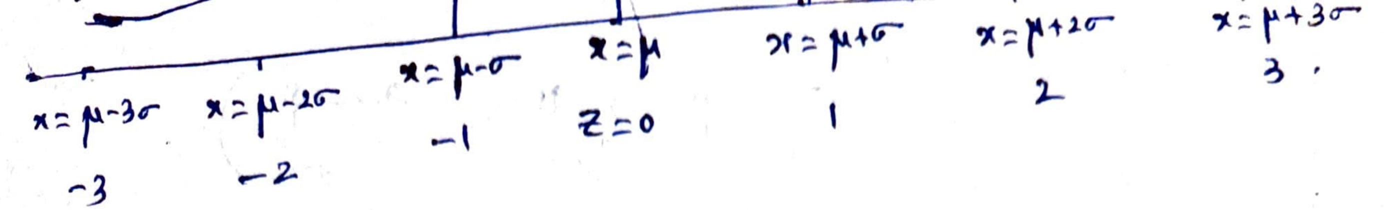
Therefore $P(\mu < X < x_1) = P(0 < Z < z_1)$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_0^{z_1} e^{-\frac{z^2}{2}}\,\cancel{\sigma}\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{z^2}{2}}\,dz.$$

$$= \int_0^{z_1} \phi(z)\,dz.$$

Where $\phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$. The definite integral $\int_0^{z_1} \phi(z)\,dz$ is known as normal prob. integral and it gives the area under ~~nor~~ standard normal curve between the ordinates $z = 0$ and $z = z_1$.



Conversion: (i) $P(X \geq a)$

$$= P\left(\frac{X-\mu}{\sigma} \geq \frac{a-\mu}{\sigma}\right)$$

$$= P\left(Z \geq \frac{a-\mu}{\sigma}\right)$$

(ii) $P(a \leq X \leq b)$

$$= P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right).$$

Note: The cdf of the standard normal distri. is denoted by $\Phi(z)$ ie; $\Phi(z = a) = P(z \leq a)$

i) $\Phi(a) = P(z \leq a)$

(ii) $\Phi(-a) = 1 - \Phi(a)$

(iii) $P(a < z < b) = \Phi(b) - \Phi(a)$

Since in tables we are given the areas under S.N. curve, in numerical problems, we shall deal with the S.N. variate Z rather than X.

Example: If $X \sim N(2, 0.16)$, then find

(i) $P(X \geq 2.3)$

(ii) $P(1.8 \leq X \leq 2.1)$

Sol: mean $\mu = 2$, $\sigma^2 = 0.16 \Rightarrow \sigma = 0.4$

$$Z = \frac{X-\mu}{\sigma} = \frac{x-2}{0.4} \quad \text{and} \quad Z \sim N(0,1)$$

$$P(X \geq 2.3) = P(x-\mu \geq 2.3-\mu)$$

$$= P\left(\frac{X-\mu}{\sigma} \geq \frac{2.3-\mu}{\sigma}\right)$$

$$= P\left(Z \geq \frac{2.3-2}{0.4}\right)$$

$$= P(Z \geq 0.75)$$

$$= 1 - P(Z \leq 0.75)$$

$$= 1 - \Phi(0.75)$$

$$= 1 - 0.7734 \quad \text{(by tables)}$$

$$= 0.2266.$$

(ii) $P(1.8 \leq X \leq 2.1)$

$$= P\left(\frac{1.8-2}{0.4} \leq Z \leq \frac{2.1-2}{0.4}\right)$$

$$= P(-0.5 \leq Z \leq 0.25)$$

$$= \Phi(0.25) - \Phi(-0.5)$$

$$= \Phi(0.25) - [1 - \Phi(0.5)]$$

$$= 0.5987 - [1 - 0.6915]$$

$$= 0.5987 + 0.6915 - 1 = 0.2902.$$

Problem: Suppose that the temperature X is normally distributed with expectation 50, variance 4. What is the probability that the temperature will lie between 48°c and 53°c.

Solution: $X \sim N(50, 4)$, $\mu = 50$, $\sigma = 2$.

$$P(48 < X < 53) = P\left(\frac{48-50}{2} < z < \frac{53-50}{2}\right)$$

$$= P(-1 < z < 1.5)$$

$$= \Phi(1.5) - \Phi(-1)$$

$$= 0.9332 - [1 - \Phi(1)]$$

$$= 0.9332 - [1 - 0.8413]$$

$$= 0.7745.$$

Problem: The diameter of an electric cable X is normally distributed with mean 0.8 and variance 0.0004. What is the probability that the diameter will exceed 0.81 inch?

Solution: $\mu = 0.8$, $\sigma = 0.02$

$$P(X \geq 0.81) = P\left(Z \geq \frac{0.81 - 0.8}{0.02}\right)$$

$$= P(Z \geq 0.5)$$

$$= 1 - P(Z \leq 0.5)$$

$$= 1 - \Phi(0.5) = 1 - 0.6915$$

$$= 0.3085$$

Problem: Suppose that the life lengths of two electronic divices say X_1 and X_2 have distributions $N(40, 36)$, and $N(45, 9)$ respectively. If an electronic device is to be used for a 45 hr. period, which device is preferred?

Solution: $X_1 \sim N(40, 36)$, $X_2 = N(45, 9)$

$$P(X_1 \geq 45) = P\left(Z_1 \geq \frac{45-40}{\sqrt{36}}\right)$$

$$= P\left(Z_1 \geq \frac{45-40}{6}\right)$$

$$= P\left(Z_1 \geq \frac{5}{6} = 0.8333\right)$$

$$= 1 - P(Z \leq 0.83)$$

$$= 1 - 0.7967 = 0.2033.$$

$$P(X_2 \geq 45) = P\left(Z_2 \geq \frac{45-45}{3}\right)$$

$$= P(Z_2 \geq 0) = 1 - P(Z_2 \leq 0) = 1 - 0.5 = 0.5.$$

$$\Rightarrow P(X_1 \geq 45) < P(X_2 \geq 45)$$

Therefore X_2 should be prefered.

Problem: Let X be a normal variate with mean 30 and S.D. 5. Find

i) $P(26 \leq X \leq 40)$

ii) $P(X \geq 45)$

iii) $P(|X-30| > 5)$

Solution: $P(26 \leq X \leq 40) = P\left(\frac{26-30}{5} \leq Z \leq \frac{40-30}{5}\right)$

$$= P(-0.8 \leq Z \leq 2)$$

$$= \Phi(2) - \Phi(-0.8)$$

$$= \Phi(2) - [1 - \Phi(0.8)]$$

$=$

$= 0.7563 (?)$

(ii) $P(X \geq 45) = P\left(Z \geq \frac{45-30}{5}\right)$

$= P(Z \geq 3)$

$= 1 - P(Z < 3)$

$= 1 - \Phi(3)$

$\vdots$

$= 0.0013.$

(iii) $P(|X-30| > 5) = 1 - P(|X-30| \leq 5)$

$P(|X-30| \leq 5) = P(25 \leq X \leq 35)$

$= P\left(\frac{25-30}{5} \leq Z \leq \frac{35-30}{5}\right)$

$= P(-1 \leq Z \leq 1)$

$= \Phi(1) - \Phi(-1)$

$= \Phi(1) - [1 - \Phi(1)]$

$= \quad \ldots \quad = 0.6826$

Therefore $P(|X-30| > 5) = 1 - 0.6826 = 0.3174.$

Problem: The standard deviation of a certain group of 1000 high school grades was 11% and the mean grade 78%. Assuming the distribution to be normal find (i) How many grades were above 90%? (ii) how many grades were below 60%? (iii) how many grades were between 75% and 85%. (iv) what was the highest grade of the lowest 10?

(ii) If Z denotes the number of men that have height greater than the mean $\mu = 167$, then the r.v. Z has the binomial distribution.

$$n = 4, \quad p = P(X > 167) = 0.5$$

$$P(Z = 2) = {}^4C_2 \, (0.5)^4 = 0.375 \quad \left(\begin{array}{l}\text{here } p = 0.5 \\ q = 0.5\end{array}\right)$$

Problem: (95/22) A machine produces bolts the length of which (in cms) obeys a normal probability law with mean 5 and standard deviation $\sigma = 0.2$. A bolt is called defective if its length falls outside the interval $(4.8, 5.2)$

(a) what is the proportion of defective bolts that this machine produces?

(b) What is the probability that among ten bolts none will be defective?

Solution: Let X be the length of a bolt

Then $X \sim N(5, (0.2)^2)$, $\mu = 5$, $\sigma = 0.2$

$$P(\text{bolt is defective}) = P(X \notin (4.8, 5.2))$$

$$= 1 - P[4.8 < X < 5.2]$$

$$= 1 - P\left(\frac{4.8 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{5.2 - \mu}{\sigma}\right)$$

$$= 1 - P(-1 < Z < 1)$$

$$= 1 - [\Phi(1) - \Phi(-1)]$$

$$= 1 - [\Phi(1) - (1 - \Phi(1))]$$

$$= 2(1 - \Phi(1)) = \ldots = 0.32$$