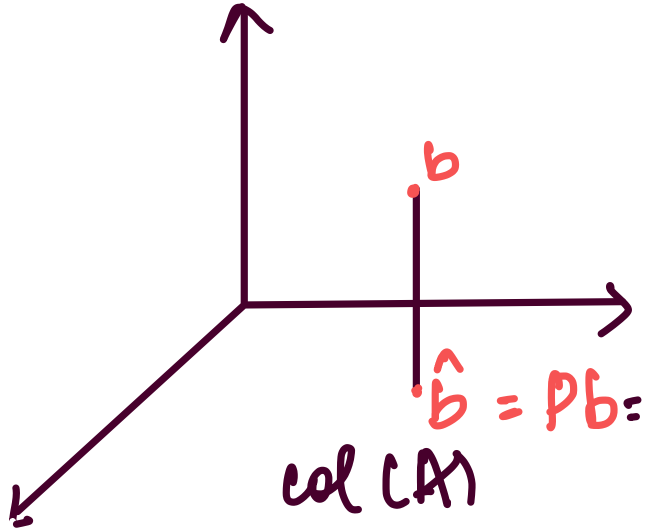
Projection matrix (Hat matrix)



$\hat{b} = Pb = A\hat{x}$; P = projection matrix.

$$Ax = b$$

$$A^T A\hat{x} = A^T b$$

$$\hat{b} = A\hat{x}$$

$$= \underbrace{A(A^TA)^{-1}A^T}_{P} b$$

$$\therefore \boxed{P = A(A^TA)^{-1}A^T} \quad \text{(projection matrix)}$$

↪ projects a vector b on the col(A).

Properties.

1). $P^2 = P$ (Idempotent)

Proof: $P^2 = P \cdot P$

$$= \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right)$$

$$= A(A^TA)^{-1}\underbrace{A^T A(A^TA)^{-1}}_{= I}A^T$$

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$$= A(A^TA)^{-1}A^T = P.$$ (associativity of matrix multiplication)

2). $P^T = P$ (Symmetry)

Proof: $P^T = \left(A(A^TA)^{-1}A^T\right)^T$

$= (A^T)^T\left[(A^TA)^{-1}\right]^T A^T$ (because $(AB)^T = B^TA^T$)

$= A\left[(A^TA)^T\right]^{-1} A^T.$

$= A\left[A^T(A^T)^T\right]^{-1} A^T$ (because $(B^T)^{-1} = (B^{-1})^T$)

$= A(A^TA)^{-1}A^T$

$= P.$

Used : $(B^T)^{-1} = (B^{-1})^T$.

Justification : (i). $B^T (B^{-1})^T = I$

(ii). $(B^{-1})^T B^T = I$

(i). $B^T (B^{-1})^T = (YX)^T$; where $Y = B^{-1}$
$X = B$.

$= (B^{-1} B)^T$

$= I^T$

$= I$.

(iii) $(B^{-1})^T B^T = Y^T X^T$; where $Y = B^{-1}$

$X = B$.

$= (XY)^T$

$= (BB^{-1})^T$

$= I^T$

$= I$.

Q. Given $A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 4 & 7 \end{bmatrix}$, find projection matrix P.

Soln: $P = A(A^T A)^{-1} A^T$

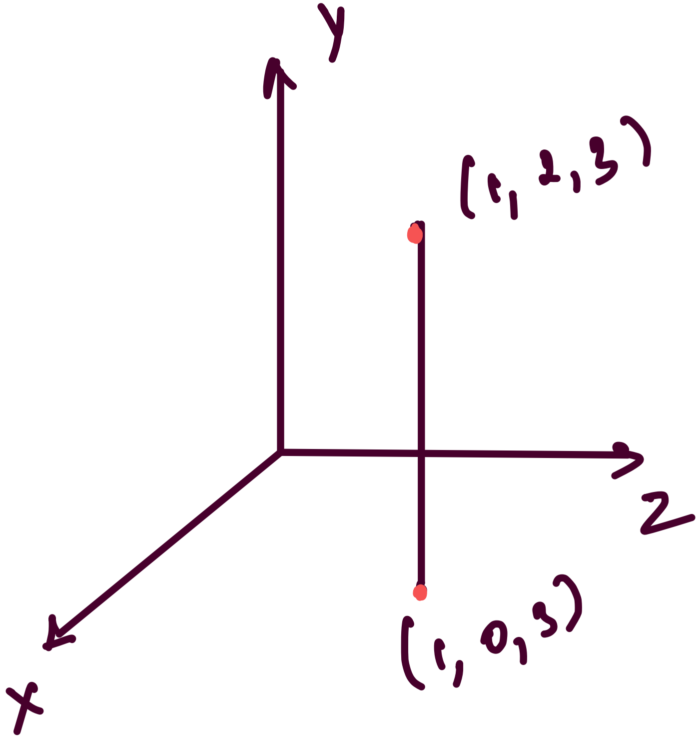
$$= A\left(\underbrace{\begin{bmatrix} 3 & 0 & 4 \\ -1 & 0 & 7 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 4 & 7 \end{bmatrix}}_{3\times 2} \right)^{-1} A^T$$

$$= A \underbrace{\begin{bmatrix} 25 & 25 \\ 25 & 50 \end{bmatrix}^{-1}}_{\text{symmetric}} A^T$$

$$= \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 4 & 7 \end{bmatrix} \frac{1}{(25)^2} \begin{bmatrix} 50 & -25 \\ -25 & 25 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ -1 & 0 & 7 \end{bmatrix}$$

$$= \frac{1}{25} \underset{3\times 2}{\begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 4 & 7 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 3 & 0 & 4 \\ -1 & 0 & 7 \end{bmatrix}}.$$

$$= \frac{1}{25} \underset{3\times 2}{\begin{bmatrix} 7 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 3 & 0 & 4 \\ -1 & 0 & 7 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

(tall matrix)

orthonormal vectors

$$U^T U = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = I_{2\times 2}.$$

$$UU^T = \frac{1}{2}\underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}}\underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}}$$

$$= \frac{1}{2}\underset{3\times 3}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}}^{I_{2\times 2}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}.$$

$$\neq I.$$

Algorithm to find projection matrix.

Given : $W = \text{col}(A)$; A is tall matrix.

$= \text{span}\{c_1, c_2, \dots, c_n\}$

Step 1 : Use Gram-Schmidt process to get orthonormal basis for W.

Step 2: Let projection of b on W given by $\hat{b}$.

$\Rightarrow \hat{b} = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ where

u_i's are given by step 1.

$\langle u_i, u_j \rangle = 0$ for $i \neq j$.

and $\| u_i \|^2 = \langle u_i , u_i \rangle = 1$.