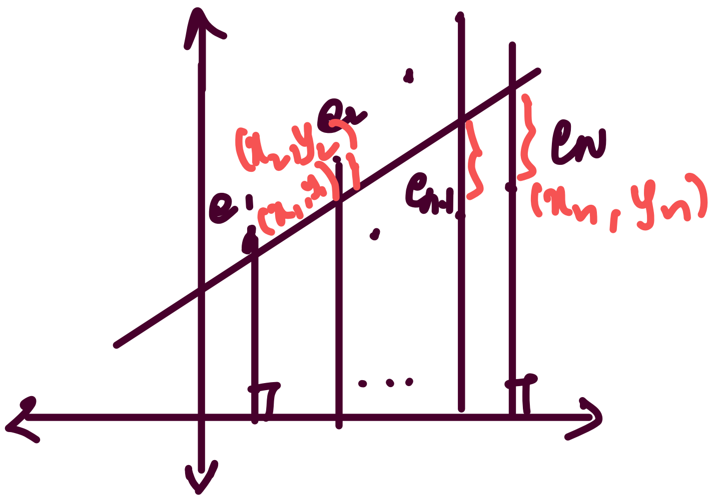
Regression:



Fit $y = ax + b$ with min. least squares error.
(ie, to find a, b)

$$e_1^2 + e_2^2 + \dots + e_n^2 = \text{least square error} = E.$$

minimize

$$e_1 = (a + bx_1 - y_1)^2$$

$$e_2 = (a + bx_2 - y_2)^2$$

$$\vdots$$

$$e_n = (a + bx_n - y_n)^2$$

$$E = \sum_{i=1}^{n} (a + bx_i - y_i)^2$$

$$\frac{\partial E}{\partial a} = 0 \Rightarrow \sum_{i=1}^{n} 2(a + bx_i - y_i)\cdot 1 = 0$$

$$\Rightarrow a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$\Rightarrow an + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{— (1)}$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} 2(a + bx_i - y_i)\cdot x_i = 0$$

$$\Rightarrow \quad a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \quad \text{—— (2)}$$

(1), (2) are called normal equations.

Solve (1), (2) to get a, b.

$y = ax + b \rightarrow$ regression equation of y on x.

Vectorize the set of equations :

$$a + bx_1 \cong y_1$$

$$a + bx_2 \cong y_2$$

$$\vdots$$

$$a + bx_n \cong y_n$$

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix}_{n\times 2}}_{X} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}_{2\times 1}}_{v} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}}_{y}$$

$$\Rightarrow X v = y$$

$$\Rightarrow X^T X v = X^T y$$

$$\Rightarrow v = (X^T X)^{-1} X^T y \longrightarrow \text{least squares solution.}$$

Consider $X^T X v = X^T y$.

i.e., $\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}}_{2\times n} \underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix}}_{n\times 2} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}}_{2\times n} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n\times 1}$

$$\begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}_{2\times 2} \begin{bmatrix} a \\ b \end{bmatrix}_{2\times 1} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}_{2\times 1}$$

↳ vectorized form of normal equations.

$$\Rightarrow \left.\begin{aligned} a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} y_i \\ a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} x_i y_i \end{aligned}\right\} \text{normal equations.}$$

To Fit : $z = a + bx + cy$ $\rightarrow$ plane

Find a, b, c.

$$a + bx_1 + cy_1 \cong z_1$$

$$a + bx_2 + cy_2 \cong z_2$$

$$\vdots$$

$$a + bx_n + cy_n \cong z_n$$

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & & \\ 1 & x_n & y_n \end{bmatrix}_{n\times 3}}_{X} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1}}_{v} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_{n\times 1}}_{z}$$

$$\Rightarrow \quad XV = Z$$

$$\Rightarrow \quad X^T X V = X^T Z$$

$$\underset{3\times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}} \underset{n\times 3}{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & y_n \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \underset{3\times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}$$

$$\underset{3\times 3}{\begin{bmatrix} \sum_{i=1}^n 1 & \sum_{i=1}^n x_i & \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i & \sum_{i=1}^n x_i y_i & \sum_{i=1}^n y_i^2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} \sum_{i=1}^n z_i \\ \sum_{i=1}^n x_i z_i \\ \sum_{i=1}^n y_i z_i \end{bmatrix}}.$$

(3 normal equations)

To Fit :

$$y = a + bx + cx^2 \rightarrow \text{parabola.}$$

Find a, b, c.

$$a + bx_1 + cx_1^2 \approx y_1$$

$$a + bx_2 + cx_2^2 \approx y_2$$

$$\vdots$$

$$a + bx_n + cx_n^2 \approx y_n$$

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & & \\ 1 & x_n & x_n^2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{Y}.$$

$$\Rightarrow XV = y$$

$$\Rightarrow X^T X V = X^T y$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & & \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} \sum_{i=1}^n 1 & \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i^3 \\ \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n x_i^2 y_i \end{bmatrix}.$$

$$a + bx_i + cx_i^2 = y_i$$

$$a\sum 1 + b\sum x_i + c\sum x_i^2 = \sum y_i$$

$$a\sum x_i + b\sum x_i + c\sum x_i^3 = \sum x_i y_i$$

$$a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 = \sum x_i^2 y_i \,.$$

3 normal equations to fit a parabola

Q. Fit $y = a + bx + cx^2$; given

x	y
0	2
1	3
2	12
3	29

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Soln.

x	y	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
0	2	0	0	0	0	0
1	3	1	1	1	3	3
2	12	4	8	16	24	48
3	29	9	27	81	87	261
$\sum x_i = 6$	$\sum y_i = 46$	$\sum x_i^2 = 14$	$\sum x_i^3 = 36$	$\sum x_i^4 = 98$	$\sum x_i y_i = 114$	$\sum x_i^2 y_i = 812$

$n = 4$ data points.

normal equations :

$$a(4) + b(6) + c(14) = 46$$

$$a(6) + b(14) + c(36) = 114$$

$$a(14) + b(36) + c(98) = 312$$

↳ solve by LU-decomposition or Gauss elimination.

$$\Rightarrow a = 2$$

$$b = -3$$

$$\underline{\underline{c = 4}}.$$

Q. Fit $z = a + bx + cy$; given

x	y	z
0	0	7
0	1	17
1	0	5
1	1	15

normal equations :