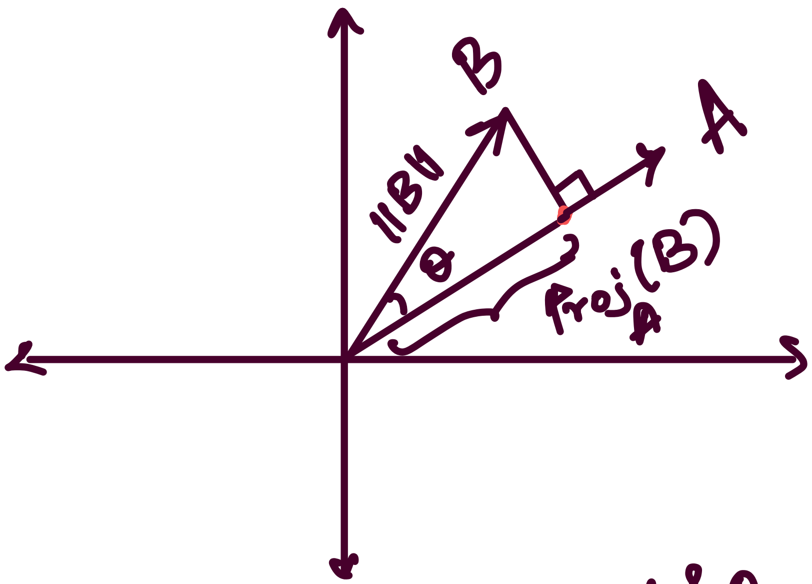
* $\langle A, B \rangle = \|A\| \, \|B\| \cos\theta$

↳ cosine formula.

Used in

* word to vector (word2vec in Natural Language Processing (NPL)).



projection of B on A.

$$\cos\theta = \frac{Proj_A(B)}{\|B\|}$$

$$\langle A, B\rangle = \|A\| \, \|B\| \cos\theta$$ → real number

$$= (\|B\| \cos\theta) \|A\|$$

$$= \left(\|B\| \frac{\operatorname{Proj}_A(B)}{\|B\|} \right) \|A\|$$

$$= \operatorname{Proj}_A(B) \, \|A\|$$

$$\Rightarrow \boxed{\operatorname{proj}_A(B) = \frac{\langle A, B\rangle}{\|A\|}}.$$ → scalar distance from (0, 0) to the foot of the perpendicular.

or, $\overrightarrow{\text{proj}_A (B)} = \dfrac{\langle A, B\rangle}{\|A\|} \cdot \dfrac{A}{\|A\|} = \dfrac{\langle A, B\rangle}{\|A\|^2} A$

$\hookrightarrow$ scalar

unit vector in direction of A denoted by u_A.

$\underline{Q}$. Project the point $B(7, 9)$ on the line $\underbrace{y = x}_{\text{same as } \alpha(1,1)}$.

same as $\alpha(1, 1)$; $\alpha \in \mathbb{R}$ $\Rightarrow$ $A \equiv (1, 1)$

$B \equiv (7, 9)$

$\underline{\text{Soln:}}$ $\text{proj}_A (B) = \dfrac{\langle A, B\rangle}{\|A\|}$

$$\text{proj}_A(B) = \frac{\langle (1,1), (7,9) \rangle}{\sqrt{1^2+1^2}}$$

$$= \frac{1\cdot 7 + 1\cdot 9}{\sqrt{2}} = \frac{16}{\sqrt{2}} = 8\sqrt{2}$$

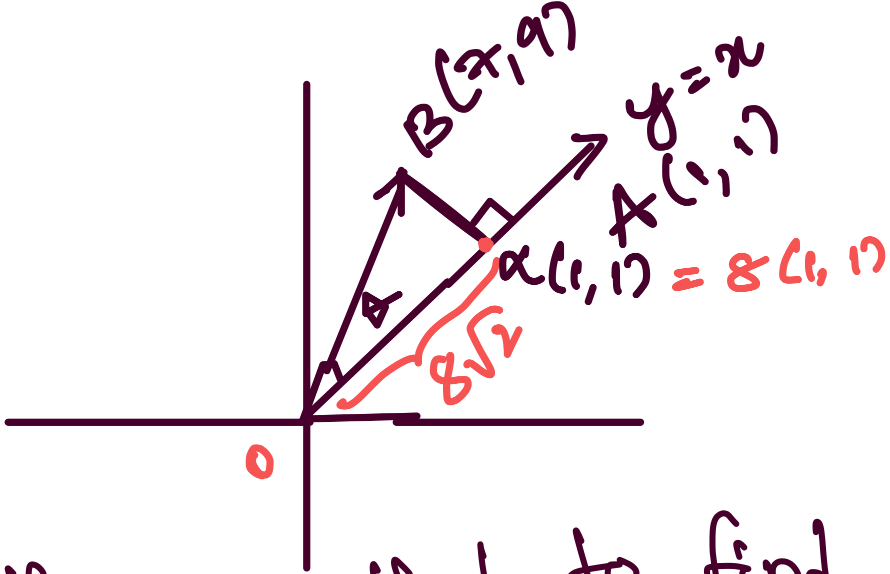
length

(distance from (0,0) to (8,8))

$$\overrightarrow{\text{proj}_A(B)} = (8\sqrt{2}) \frac{A}{\|A\|}$$

$$= 8\sqrt{2}\,\frac{(1,1)}{\sqrt{2}} = 8(1,1).$$

projection of B in the direction of A.



Another method to find α: OA ⊥ AB.

$$\langle \alpha(1,1), (7-\alpha, 9-\alpha)\rangle = 0$$

$$\Rightarrow \langle (\alpha,\alpha), (7-\alpha, 9-\alpha)\rangle = 0$$

$$\Rightarrow \alpha(7-\alpha) + \alpha(9-\alpha) = 0 \Rightarrow \alpha = 8.$$

$$\overrightarrow{\text{proj}_A(B)} = \frac{\langle A, B\rangle u_A}{\|A\|} = \frac{\langle A, B\rangle}{\|A\|} \cdot \frac{A}{\|A\|}.$$

projection of B in the direction of vector of A.

$$\Rightarrow \overrightarrow{\text{proj}_A(B)} = \frac{1\cdot 7 + 1\cdot 9}{\sqrt{2}\cdot\sqrt{2}} \cdot (1,1) = \underline{8\,(1,1)} = (8,8).$$

vector

$\text{Proj}_A(B) = 8\sqrt{2}.$

Scalar

$A_{m\times n} \longleftrightarrow T : \mathbb{R}^n \xrightarrow[\text{L.T.}]{} \mathbb{R}^m.$

matrix

$$T(X) = \underset{m\times n}{A}\,X_{n\times 1}$$

$$\begin{aligned}
\operatorname{Ker} T &= \{ X \in \mathbb{R}^n \mid T(X) = 0 \} \\
&= \{ X \in \mathbb{R}^n \mid AX = 0 \} \\
&= \operatorname{Null}(A) \\
&= N(A)
\end{aligned}$$

(Null space of A)

$$A_{m\times n} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

↳ n columns of A.

$$\operatorname{Im} T = \{ T(X) \mid X \in \mathbb{R}^n \}$$

$$= \{ AX \mid X \in \mathbb{R}^n \}$$

$$= \left\{ \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_i \in \mathbb{R} \right\}$$

$$\operatorname{Im} T = \{ x_1 c_1 + x_2 c_2 + \cdots + x_n c_n \mid x_i \in \mathbb{R} \}$$

set of all linear combinations
of columns of A

$= \text{column space of } A = C(A).$

Q Solve $x = 0$

$y = 0$

$x + y = 1.$

Soln:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{\substack{3\times 2 \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b}.$$

$$AX = b$$

$$A^T(AX) = A^T b.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

symmetric

$$\Rightarrow \quad 2x + y = 1$$
$$x + 2y = 1$$

$\Rightarrow x = 1/3,$

$y = 1/3.$

↳ Interpret this soln.

↳ Inconsistent

Q. Solve $\quad 2x - y = 0$

$$x - 3y = -20.$$

Soln :

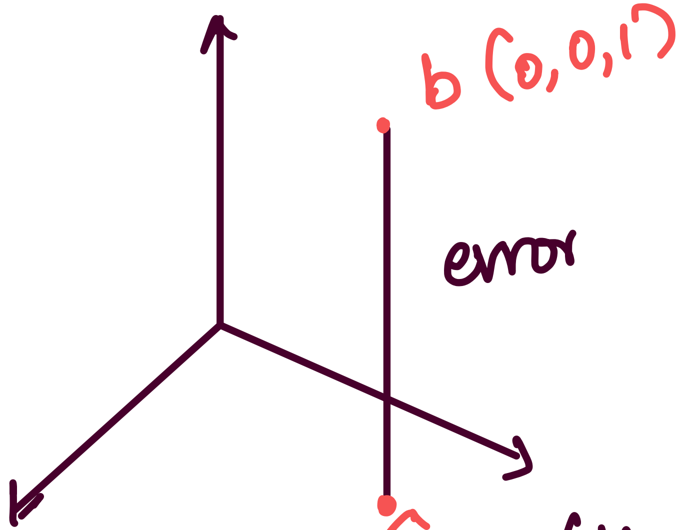
$$\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ -20 \end{bmatrix}}_{b}$$

$$AX = b$$

$$\Rightarrow A^T A X = A^T b$$

$$\Rightarrow x = 4 \;,\; y = 8$$ → precise soln.

↳ consistent system.



$C(A)$ $\quad \hat{b} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)$

$$Ax = b \quad ; \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$\Rightarrow A\hat{x} = \hat{b}$$

$$\Rightarrow \hat{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}.$$

error $: \| b - \hat{b} \|$

$$= \sqrt{(1/3 - 0)^2 + (1/3 - 0)^2 + (2/3 - 1)^2}$$

$$= 1/\sqrt{3}.$$

Q. $(b - \hat{b}) \perp C(A)$?

i.e, $(b - \hat{b}) \perp \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$?

$(b - \hat{b}) \perp \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

$$b - \hat{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$\left\langle b - \hat{b}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

$$= -1/3 \cdot 1 + (-1/3) \cdot 0 + 1/3 \cdot 1 = \underline{\underline{0}}$$

(1)

$$\left\langle b-\hat{b}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}-1/3\\-1/3\\1/3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\rangle$$

$$= (-1/3)\cdot 0 + (-1/3)\cdot 1 + 1/3\cdot 1 = 0$$

— (2)

$$\hat{x} = 1/3$$

$$\hat{y} = 1/3$$

} least squares solutions (unique).

Vectorizing ① & ②,

$$\left(\quad ? \quad \right)(b-\hat{b})_{3\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\substack{2\times 3 \\ A^T}} \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}.$$

vectorization of equations ①, ② : $\boxed{A^T(b-\hat{b}) = 0.}$