MANIPAL ACADEMY of HIGHER EDUCATION
(Institution of Eminence Deemed to be University)

Reg.No.									

III SEMESTER B. Tech. (Mathematics & Computing)
END SEMESTER EXAMINATION, November 2024
Computational Linear Algebra [MAT 2135]

Time: 09:30 AM to 12:30 PM — Date: 20 November 2024 — MAX. MARKS: 50

Note (i) Answer ALL questions
(ii) Draw diagrams, and write equations wherever necessary

Q.1A Show that $T(x, y) = (x+5y, 5x+y)$ is a linear transformation. Find $T^{-1}(x, y)$ and $T^{100}(x, y)$.

(3 Marks; CO: 1; BL: 3)

Q.1B Express the following matrix A as product of elementary matrices and then describe the geometric effect of multiplication of a vector by A.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(3 Marks; CO: 1; BL: 3)

Q.1C Define the kernel of a linear transformation $T : V \to W$. With explicit details, show that $dim(Ker(T)) + dim(Image(T)) = dim(V)$.

(4 Marks; CO: 2; BL: 4)

Q.2A Find the least squares solution to the inconsistent system of equations given by $AX = b$ given

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$$

Compute the error in the solution.

(3 Marks; CO: 2; BL: 3)

Q.2B Find QR decomposition of the following matrix:

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 7 & 11 \\ 0 & 0 & 9 \end{bmatrix}$$

(3 Marks; CO: 3; BL: 3)

Q.2C Derive the following statements:

(i) The non-zero eigenvalues of AA^T and A^TA are equal.

(ii) The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

(4 Marks; CO: 1; BL: 4)

Q.3A Compute the basis for the four fundamental subspaces given the matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Validate the direct sum conditions resulting from the four fundamental subspaces.

(3 Marks; CO: 3; BL: 3)

Q.3B Fit $y = a + b\frac{x}{log_e x}$ given the following data:

X	e	e^2	e^3
Y	1	4	7

Hence find $y(e^4)$.

(3 Marks; CO: 2; BL: 3)

Q.3C Let A be a symmetric matrix with all real entries. Then

(i) show that all eigenvalues of A are real.

(ii) show that the quadratic form $X^T AX$ is positive definite if and only if all the eigenvalues of A are positive.

(4 Marks; CO: 3; BL: 4)

Q.4A Let $m \geq n$ and suppose $A_{m\times n}$ has n independent columns. Then show that

(i) $A^T A$ is invertible.

(ii) left inverse of A exists.

(3 Marks; CO: 2; BL: 4)

Q.4B Find LU decomposition of the following matrix:

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

(3 Marks; CO: 3; BL: 3)

Q.4C Find the singular value decomposition (SVD) and the reduced SVD of the following matrix:

$$\begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

Give the geometrical interpretation of the obtained SVD.

(4 Marks; CO: 4; BL: 4)

Q.5A Using eigenvalues and eigenvectors, find the maximum and minimum values of the function

$$117x^2 + 162xy + 333y^2$$

subject to the constraint

$$x^2 + y^2 = 1.$$

Give a pictorial representation of your computations using standard axes and principal axes.

(3 Marks; CO: 4; BL: 4)

Q.5B The sequence of numbers $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is called the Fibonacci sequence. Let F_n denote the n^{th} term of Fibonacci sequence where $n = 0, 1, 2, 3, \ldots$. Test whether F_{n+2} gives the number of binary strings of length n without consecutive $1s$ by taking $n = 4$. Using eigenvalues and eigenvectors, find F_{n+2} and hence deduce the Binet's formula.

(3 Marks; CO: 4; BL: 3)

Q.5C Use Principal Component Analysis (PCA) to reduce the following 2D data to 1D data. Then give the geometrical interpretation of the computations involved and mention the benefits of PCA.

X	0	1	2	-1	-2
Y	0	-1	-2	1	2

(4 Marks; CO: 4; BL: 4)