Problem; (82/20): A machine normally makes items of which 4% are defective. Every hour the producer draws a sample of size 10 for inspection. If the sample contains no defective items he does not stop the machine. what is the probability that the machine will not be stopped when it has started producing items of which 10% are defective.

Solution: Since here there are only two cases defective or non-defective.

Define X: the number of defective items in a sample of 10.

X obeys a binomial law. $pr(\text{one is defective}) = \frac{1}{10}$

$pr(\text{non defective}) = \frac{9}{10}$.

$$P(X=k) = {}^{10}C_k \left(\frac{1}{10}\right)^{k} \left(\frac{9}{10}\right)^{10-k}, \quad k=0,1,2,\dots,10.$$

Therefore the required probability

$$pr(\text{no defective}) = P(X=0) = {}^{10}C_0 \left(\frac{1}{10}\right)^{0} \cdot \left(\frac{9}{10}\right)^{10-0}$$

$$= \left(\frac{9}{10}\right)^{10}.$$

Problem; (83/20): One per thousand of a population is subject to certain kinds of accident each year. Given that an insurance company has insured 5,000 persons from the population, find the probability that at most 2 persons will incur this accident.

Binomial
poisson

Solution: Define X: the number of persons that incur the accident in a year.

~~$P(X) =$~~ $P(\text{accident in year}) = \frac{1}{1000} = p$ (say) (very small)

$n = 5000$ (very large)

Since $m = np = 5000 \times \frac{1}{1000} = 5$, the poisson approximation to the binomial (for large n, small p)

applies.

Therefore the required probability;

P (at most 2 accidents will incur)

$$= P(X \le 2) = \sum_{x=0}^{2} \frac{e^{-m} m^{x}}{x!}, \quad (m=5)$$

$$= e^{-5}\left(\frac{5^0}{0!} + \frac{5}{1!} + \frac{5^2}{2!}\right) = \frac{37}{2e^5}.$$

Problem (84/20): A certain airline company, having observed that 5% of the persons making reservations on a flight do not show up for the flight, sells 100 seats on a plane that has 95 seats. What is the probability that there will be a seat available for every person who shows up for the flight?

Solution: Define

X: Number of persons making reservations on a flight and not showing up for the flight.

Then X has binomial distribution with $n = 100$ and $p = 0.05$.

Hence

$$P(\text{seat aviable}) = P(X \le 95) = 1 - P(X > 95)$$

$$= 1 - \sum_{96}^{100} \binom{100}{k} (0.05)^k (0.95)^{100-k}.$$

$=$

Problem (85/20): Workers in a factory incur accidents, at the rate of two accidents per week. Calculate the probability that there will be at most two accidents, (i) during 1 week, (ii) during 2 weeks (iii) in each of 2 weeks. (i.e. In one week at most 2 and other week at most 2)

Solution: Define X: number of accidents per week

X obeys a poisson distribution with $\lambda = 2$ (or m)

(i) P (at most 2 accidents in 1 week)

$$= P(X \leq 2) = \sum_{x=0}^{2} e^{-2} \frac{2^x}{x!} = e^{-2}\left(\frac{1}{0!} + \frac{2}{1!} + \frac{2^2}{2!}\right) = 5e^{-2}.$$

(ii) Define Y: number of accidents in 2 weeks, then Y has Poisson distribution with $m = 4$ $(2+2)$

$$P(Y \leq 2) = P(\text{at most 2 accidents in 2 weeks})$$

$$= \sum_{x=0}^{2} e^{-4} \frac{4^x}{x!} = e^{-4}\left(1 + \frac{4}{1!} + \frac{4^2}{2!}\right) = 13e^{-4}.$$

(iii) Therefore the required probability is

$$\left[P(X \leq 2)\right]^2 = 25e^{-4}.$$

Problem: (86/24) Suppose that the suicide rate in a certain State is four suicides per one million inhabitants per month. Find the probability that in a certain town of population 500,000 there will be at most four suicides in a month. Would you find it surprising that during 1 year there were at least 2 months in which more than four suicides occurred?

Therefore $g(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-x^2}, & |x| \le R. \\ 0 & \text{if } |x| > R \end{cases}$

By symmetry marginal pdf of Y is

$$h(y) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-y^2}, & |y| \le R \\ 0 & \text{if } |y| > R. \end{cases}$$

(iii) The prob. that the distance from the origin to the point selected is not greater than a is the probability that the second point lie within the circular region $x^2+y^2 \le a^2$ which is given by

$$P(x^2+y^2 \le a^2) = \iint_{x^2+y^2 \le a^2} f(x,y)\,dx\,dy$$

$$= \frac{1}{\pi R^2}\iint_{x^2+y^2\le a^2} dx\,dy$$

$$= \frac{1}{\pi R^2}\left[\text{Area of the circle } x^2+y^2=a^2\right]$$

$$= \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}.$$

Problem: A passenger arrives at a bus stop at 10 am knowing that the bus will arrive at some time uniformly distributed between 10 Am and 10.30 Am. What is the probability that he will have to wait longer than 10 min? If at 10.15 Am the bus has not yet arrived, what is the probability that he will have to wait atleast 10 additional minutes?

Let X denotes the waiting time ... the pdf of X is

$$f(x) = \begin{cases} \frac{1}{30} , & 0 < x < 30 \\ 0 & \text{otherwise} \end{cases}$$

P(he will have to wait longer than 10 min)

$$= P(X > 10)$$

$$= \int_{10}^{30} \frac{1}{30}\, dx = \frac{2}{3}.$$

P(he has to wait 25 min | he has already waited 15 min)

$$= P(X > 25 | X > 15)$$

$$= \frac{P(X > 25) \cap P(X > 15)}{P(X > 15)}$$

$$= \frac{P(X > 25)}{P(X > 15)} = \frac{\int_{25}^{30} \frac{dx}{30}}{\int_{15}^{30} \frac{dx}{30}} = \frac{1}{3}.$$

Problem: A random variable X has uniform distribution over $(-3, 3)$ compute

(i) $P(X = 2)$, $P(X < 2)$, $P(|X| < 2)$, and $P(|X-2| < 2)$

(ii) Find k for which $P(X > k) = \frac{1}{3}$

Solution: The pdf of X is $f(x) = \begin{cases} \frac{1}{3-(-3)} & \text{If } -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

(i) $P(X = 2) = 0$, since the prob. of a continuous r.v. at a particular point is zero.

$$P(X<2) = \int_{-3}^{2} \frac{1}{6}\, dx = \frac{5}{6}$$

$$P(|X|<2) = \int_{-2}^{2} \frac{1}{6}\, dx = \frac{2}{3}$$

$$P(|X-2|<2) = P(-2<X-2<2)$$

$$= P(0<X<4)$$

$$= \int_{0}^{3} \frac{1}{6}\, dx = \frac{1}{2}$$

(ii) $P(X>k) = \int_{k}^{3} \frac{1}{6}\, dx = \frac{1}{3}$ (given)

$$\Rightarrow \frac{3-k}{6} = \frac{1}{3} \Rightarrow k=1$$

Problem: In 100 sets of 10 tosses of an unbiased coin, in how many cases should we expect (i) 7 heads and 3 tails (ii) at least 7 heads?

Binomial

Solution: Given the number of sets $N=100$

Number of tosses $= n = 10$

p = prob. of getting head $= \frac{1}{2}$

$q = 1-p = \frac{1}{2}$.

The prob. of getting x heads in 10 tosses is given by

$$P(x) = \binom{10}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{10-x}$$

$$= \binom{10}{x} \left(\frac{1}{2}\right)^{10}, \quad x = 0, 1, \cdots, 10.$$

(i) The prob. of getting 7 heads (Consequently 3 tails is)

$$P(7) = \binom{10}{7} \left(\frac{1}{2}\right)^{10} = --$$

Problem: A box contains tags numbered $1, 2, \ldots, n$. Two tags are chosen at random successively

i) without replacement

ii) with replacement

In each case, find the prob. that the numbers on the tags are consecutive integers.

Sol: i) (Without replacement):

Exhaustive cases : $n(n-1)$

Define event $A = \{$ numbers on the tags are consecutive integers $\}$

$$= \left\{ \begin{matrix} (1,2), (2,3) & \cdots & (n-1, n) \\ (2,1)\ (3,2) & \cdots & (n, n-1) \end{matrix} \right\}$$

Favourable cases : $2(n-1)$

Required prob. $\dfrac{2(n-1)}{n(n-1)} = \dfrac{2}{n}$.

ii) (with replacement)

Exhaustive cases: n^2.

favourable cases : $2(n-1)$

Required prob: $\dfrac{2(n-1)}{n^2}$.

Problem: If the random variable K is uniformly distributed over $(0, 5)$. what is the prob. that the roots of the equation $4x^2 + 4xk + k + 2 = 0$ are real?

Sol: K is uniformly distributed over $(0, 5)$

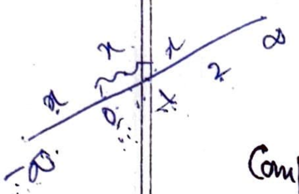
Pdf of K : $f(K) = \begin{cases} \frac{1}{5} & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$

The roots are real if $b^2 - 4ac \ge 0$.

$$P(2 \leq X \leq 3) = \int_{2}^{3} \frac{1}{18}(3+2x)\,dx = \frac{4}{9}.$$

Imp

Prob: The p.d.f. of a r.v. X is



$$f(x) = \begin{cases} x & \text{for } 0 < x \leq 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{for } \cancel{x \geq 2}\text{. otherwise} \end{cases}$$

Compute the ~~cumm~~ cumulative distribution function of X.

Sol:

(i) For any x, $x \leq 0$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\,dx = 0.$$

(ii) $0 < x < 1$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} x\,dx = \frac{1}{2}x^2.$$

(iii) for any x in $1 < x \leq 2$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\cdot dx + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= 2x - \frac{x^2}{2} - 2$$

$$= 2x - \frac{x^2}{2} - 1.$$

(iv) $x \geq 2$.

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx$$

$$+ \int_{2}^{\infty} 0\cdot dx = 1$$

Def: The pdf of the uniform distribution (continuous) is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & , \text{ if } a < x < b \text{ and} \\ 0 & \text{Otherwise.} \end{cases}$$

Problem: A random variable X has a uniform probability distribution given by the pdf.

$$f(x) = \begin{cases} \frac{1}{6} & , -3 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find (i) $P(X<0)$, (ii) $P(-1<x<1)$, (iii) $P(|X| > \frac{1}{2})$

Solution: (i) $P(X<0) = P(-3 < X < 0)$

$$= \int_{-3}^{0} f(x)\,dx = \int_{-3}^{0} \frac{1}{6}\,dx = \cdots = \frac{1}{2}.$$

(ii) $P(-1 < x < 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{1}{6}\,dx = \frac{1}{6}[x]_{-1}^{1} = \frac{1}{3}.$

(iii) $P(|x| > \frac{1}{2}) = 1 - P(|x| \leq \frac{1}{2}) = 1 - P(-\frac{1}{2} \leq x \leq \frac{1}{2})$

$$= 1 - \int_{-1/2}^{1/2} f(x)\,dx = 1 - \int_{-1/2}^{1/2} \frac{1}{6}\,dx = \cdots = \frac{5}{6}.$$

Problem: On a certain city transport route ~~buttes~~ buses play every ~~half~~ half an hour between 6 AM and 10 pm. If a person reaches a bus stop on this route at a random time during this period, what is the prob. that he will have to wait at least 20 min?

Sol: The waiting time X is a rv uniformly distributed over (0, 30), since the person has to wait any where from 0 min to 30 min.

Problem: A box contains tags numbered $1, 2, \dots, n$. Two tags are chosen at random successively

i) without replacement

ii) with replacement

in each case, find the prob. that the numbers on the tags are consecutive integers.

Sol: i) (Without replacement):

Exhaustive cases : $n(n-1)$

Define event $A = \{$ numbers on the tags are consecutive integers $\}$

$$= \left\{ \begin{matrix} (1,2), (2,3) \; \dots \; (n-1, n) \\ (2,1) \; (3,2) \; \dots \; (n, n-1) \end{matrix} \right\}$$

Favourable cases : $2(n-1)$

Required prob. $\dfrac{2(n-1)}{n(n-1)} = \dfrac{2}{n}$.

(ii) (with replacement)

Exhaustive cases : n^2

favourable cases : $2(n-1)$

Required prob : $\dfrac{2(n-1)}{n^2}$

Problem: If the random variable K is uniformly distributed over $(0,5)$. What is the prob. that the roots of the equation $4x^2 + 4xk + (k+2) = 0$ are real?

Sol: K is uniformly distributed over $(0,5)$

Pdf of K : $f(K) = \begin{cases} \frac{1}{5} & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$

The roots are real if $b^2 - 4ac \geq 0$.

ie, $(4k)^2 - 4\cdot 4\,(k+2) \geq 0$

$\Rightarrow 16k^2 - 16(k+2) \geq 0$

$\Rightarrow 16(k^2 - k - 2) \geq 0$

$\Rightarrow k^2 - k - 2 \geq 0$

$\Rightarrow (k+1)(k-2) \geq 0$

(i) $k+1 \geq 0,\ k-2 \geq 0$

$\Rightarrow k \geq -1,\ k \geq 2$

ie, $k \geq 2$.

(ii) $k+1 \leq 0,\ k-2 \leq 0$

$\Rightarrow k \leq -1,\ k \leq 2$

$\Rightarrow k \leq -1$

Therefore $pr(\text{roots are real}) = pr\{k \leq -1 \text{ or } k \geq 2\}$

$= pr\{k \leq -1\} + pr\{k \geq 2\}$

$= 0 + \int_2^5 f(k)\,dk.$

$= \int_2^5 \frac{1}{5}\,dk = \cdots = \frac{3}{5}.$

Problem: Given $P(A) = 3/4$, $P(B) = \frac{3}{8}$

S.T. (i) $\frac{3}{4} \leq P(A\cup B)$

(ii) $\frac{1}{8} \leq P(A\cap B) \leq \frac{3}{8}$

(iii) $\frac{3}{8} \leq P(A\cap\bar{B}) \leq \frac{5}{8}$

Sol: (i) $A \subseteq A\cup B$

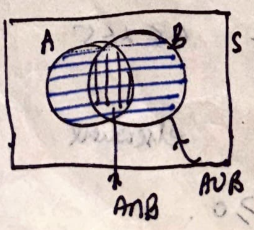
$\Rightarrow P(A) \leq P(A\cup B)$

$\Rightarrow \frac{3}{4} \leq P(A\cup B)$

(ii) $A\cap B \subseteq B$

$P(A\cap B) \leq P(B)$

$P(A\cap B) \leq \frac{3}{8}$



Also $P(A\cup B) \leq 1$

ie, $P(A) + P(B) - P(A\cap B) \leq 1$

$\Rightarrow P(A) + P(B) - 1 \leq P(A\cap B)$

$\Rightarrow \frac{3}{4} + \frac{3}{8} - 1 \leq P(A\cap B)$

$\Rightarrow \frac{1}{8} \leq P(A\cap B)$

Thus $\frac{1}{8} \leq P(A\cap B) \leq 3/8$.