$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x, y) = (7x + 2y \,,\, -4x + y)$

$$\equiv \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$T^n(x, y) = ?$

$A^n = ?$

$A = PDP^{-1}$ (eigen/spectral decomposition of A)

Then $A^n = PD^nP^{-1}$

Eigen values of A, $|A-\lambda I| = 0$ (To get non-zero X)

$$\left(\because \quad AX = \lambda X \right.$$

$$(A - \lambda I)X = 0$$

If $|A-\lambda I| \neq 0$ then $X = 0$)

$$\begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)(1-\lambda)+8=0$$

$$\Rightarrow 7-7\lambda-\lambda+\lambda^2+8=0$$

$$\Rightarrow \lambda^2 \underbrace{-8}_{(-\text{trace}A)}\lambda + \underbrace{15}_{(\det A)} = 0 \longleftrightarrow A^2-8A+15I = [0]$$

$$A^2 - 8A + 15I = [0]$$

$$A^{-1}(A^2 - 8A + 15I) = A^{-1}([0]) = [0]$$

$$\Rightarrow A - 8I + 15A^{-1} = [0]$$

$$\Rightarrow A^{-1} = \frac{1}{15}(8I - A)$$

$$(\lambda - 5)(\lambda - 3) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = 3$$

$$\lambda_1 + \lambda_2 = 8 = \text{trace}(A)$$

$$\lambda_1 \lambda_2 = 15 = \det(A)$$

Eigen vector corresponding to $\lambda_1 = 5$

$$AX = \lambda_1 X = 5X$$

$$(A - 5I)X = [0]$$

$$\begin{bmatrix} 7-5 & 2 \\ -4 & 1-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x + 2y = 0$$

$$-4x - 4y = 0$$

i.e. $x + y = 0$

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow \text{eigen vector}$$

Eigen Vector corresponding to $\lambda_2 = 3$.

$$AX = \lambda_2 X = 3X$$

$$(A - 3I)X = [0]$$

$$\begin{bmatrix} 7-3 & 2 \\ -4 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x + 2y = 0$$

$$-4x - 2y = 0$$

i.e. $2x + y = 0$

Eigen Vector $= \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = A$$

$$A^n = PD^nP^{-1}$$

$$PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}^n \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^n & 3^n \\ -5^n & (-2)3^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(5^n) - 1(3^n) & 5^n - 1(3^n) \\ -2(5^n) + 2(3^n) & -5^n + 2(3^n) \end{bmatrix}$$

$$T^n(x, y) = \begin{bmatrix} 2(5^n) - 1(3^n) & 5^n - 1(3^n) \\ -2(5^n) + 2(3^n) & -5^n + 2(3^n) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \Big(\big(2(5^n) - 3^n\big)x + \big(5^n - 3^n\big)y \, , \, \big(-2(5^n) + 2(3^n)\big)x + \big(-5^n - 2(3^n)\big)y \Big)$$

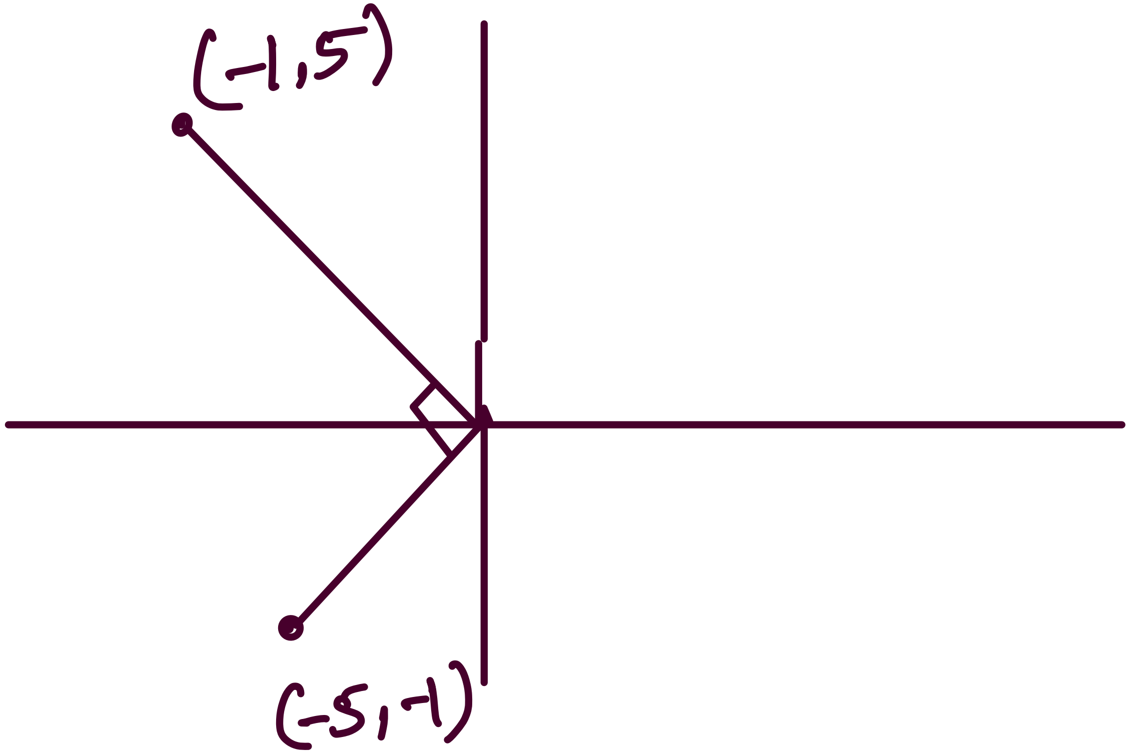
when $n=2$

$$T^2(x,y) = \underline{\underline{(41x+16y,\ -32x-7y)}}$$

If $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{5} \\ -1/\sqrt{2} & -2/\sqrt{5} \end{bmatrix}$

$$P^{-1} = -\sqrt{10}\begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ -\sqrt{5} & -\sqrt{5} \end{bmatrix}$$

Rotations



$(-1+5i)\,i = (-5-i)$

$(-1+5i)\,i = -5-i = (-5,-1).$

Multiplication by i rotates a vector by 90°.

we've $e^{i\theta} = \cos\theta + i\sin\theta$

when $\theta = \pi/2$

$$e^{i\pi/2} = \cos \pi/2 + i \sin \pi/2 = 0 + i(1) = i.$$

when $\theta = \pi$

$$e^{i\pi} = -1 + 0$$

$$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

$$(a+ib)(x+iy)$$

$$= (ax + aiy + ibx + i^2by)$$

$$= (ax + i(ay+bx) - by)$$

$$= (ax-by) + i(ay+bx)$$

$$\equiv (ax-by, \quad ay+bx)$$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus i is the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$i^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$