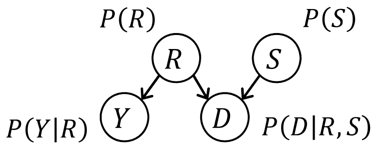
SOLUTIONS



Random variables in the graph are defined by:

$P(R) = [P(R = \text{'low'}) \quad P(R = \text{'high'})]^{\mathrm{T}} = [30\% \quad 70\%]^{\mathrm{T}}$

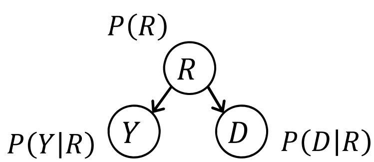
$P(S) = [P(S = \text{'low'}) \quad P(S = \text{'high'})]^{\mathrm{T}} = [70\% \quad 30\%]^{\mathrm{T}}$

$$P(D|S, R = \text{'low'}) = \begin{bmatrix} P(D = \text{'low'}|S = \text{'low'}, R = \text{'low'}) & P(D = \text{'low'}|S = \text{'high'}, R = \text{'low'}) \\ P(D = \text{'high'}|S = \text{'low'}, R = \text{'low'}) & P(D = \text{'high'}|S = \text{'high'}, R = \text{'low'}) \end{bmatrix} = \begin{bmatrix} 85\% & 50\% \\ 15\% & 50\% \end{bmatrix}$$

$$P(D|S, R = \text{'high'}) = \begin{bmatrix} P(D = \text{'low'}|S = \text{'low'}, R = \text{'high'}) & P(D = \text{'low'}|S = \text{'high'}, R = \text{'high'}) \\ P(D = \text{'high'}|S = \text{'low'}, R = \text{'high'}) & P(D = \text{'high'}|S = \text{'high'}, R = \text{'high'}) \end{bmatrix} = \begin{bmatrix} 100\% & 90\% \\ 0\% & 10\% \end{bmatrix}$$

$$P(Y|R) = \begin{bmatrix} P(Y = \text{'high'}|R = \text{'high'}) & P(Y = \text{'high'}|R = \text{'low'}) \\ P(Y = \text{'low'}|R = \text{'high'}) & P(Y = \text{'low'}|R = \text{'low'}) \end{bmatrix} = \begin{bmatrix} 95\% & 5\% \\ 5\% & 95\% \end{bmatrix}$$

We can eliminate variable S, finding conditional probability $P(D|R)$:



to do so, we compute $P(D|R = \text{'low'}) = [P(D = \text{'low'}|R = \text{'low'}) \quad P(D = \text{'high'}|R = \text{'low'})]^{\mathrm{T}} = \sum_S P(D|S, R = \text{'low'})P(S)$

by multiplying matrix $P(D|S, R = \text{'low'})$ and vector $P(S)$, we get

$P(D|R = \text{'low'}) = [74.5\% \quad 25.5\%]^{\mathrm{T}}$

and

$$P(D|R = \text{'high'}) = [P(D = \text{'low'}|R = \text{'high'}) \quad P(D = \text{'high'}|R = \text{'high'})]^{\mathrm{T}} = \sum_S P(D|S, R = \text{'high'})P(S)$$

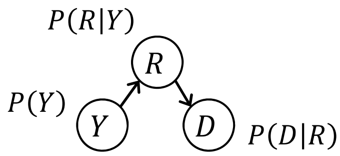
by multiplying matrix $P(D|S, R = '\text{high}')$ and vector $P(S)$, we get

$P(D|R = '\text{high}') = [97\% \quad 3\%]^{\mathrm{T}}$

By arranging the two vectors in a single matrix, we obtain:

So $P(D|R) = \begin{bmatrix} P(D = '\text{low}'|R = '\text{low}') & P(D = '\text{low}'|R = '\text{high}') \\ P(D = '\text{high}'|R = '\text{low}') & P(D = '\text{high}'|R = '\text{high}') \end{bmatrix} = \begin{bmatrix} 74.5\% & 97\% \\ 25.5\% & 3\% \end{bmatrix}$

We invert the link between Y and R:



Marginal of Y is:

$$P(Y) = [P(Y = '\text{low}') \quad P(Y = '\text{high}')]^{\mathrm{T}} = \sum_R P(Y|R)P(R) = [32\% \quad 68\%]^{\mathrm{T}}$$

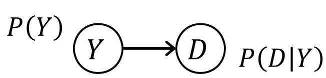
by multiplying matrix $P(Y|R)$ and vector $P(R)$.

Conditional of R given Y is

$$P(R|Y) = \begin{bmatrix} P(R = '\text{low}'|Y = '\text{low}') & P(R = '\text{low}'|Y = '\text{high}') \\ P(R = '\text{high}'|Y = '\text{low}') & P(R = '\text{high}'|Y = '\text{high}') \end{bmatrix} = \frac{P(Y|R)P(R)}{P(Y)} = \begin{bmatrix} 89.1\% & 2.2\% \\ 10.9\% & 97.8\% \end{bmatrix}$$

by multiplying matrix $P(Y|R)$ element by element by vector $P(R)$ and dividing by vector $P(Y)$.

Now we marginalize variable R:



Probability $P(D|Y)$ is given by:

$$P(D|Y) = \begin{bmatrix} P(D = '\text{low}'|Y = '\text{low}') & P(D = '\text{low}'|Y = '\text{high}') \\ P(D = '\text{high}'|Y = '\text{low}') & P(D = '\text{high}'|Y = '\text{high}') \end{bmatrix} = \sum_R P(D|R)P(R|Y)$$
$$= \begin{bmatrix} 77.0\% & 96.5\% \\ 23.0\% & 3.5\% \end{bmatrix}$$

where we get the matrix by matrix multiplication.

Marginal distribution of damage is:

$$P(D) = \sum_Y P(D|Y)P(Y) = [P(D = '\text{low}') \quad P(D = '\text{high}')]^{\mathrm{T}} = [90.2\% \quad 9.8\%]^{\mathrm{T}}$$

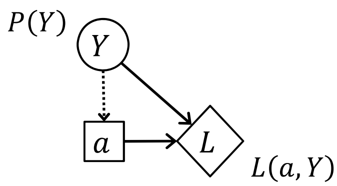
where we get the vector by matrix-vector multiplication.

So probability of failure is $P(D = \text{‘high’}) = 9.8\%$, it goes up to $P(D = \text{‘high’}|Y = \text{‘low’}) = 23.0\%$, or down to $P(D = \text{‘high’}|Y = \text{‘high’}) = 3.5\%$, depending to the observed value of Y.

The loss matrix is

$$L(a, D) = \begin{bmatrix} L(a = \text{‘do nothing’}, D = \text{‘low’}) & L(a = \text{‘do nothing’}, D = \text{‘high’}) \\ L(a = \text{‘repair’}, D = \text{‘low’}) & L(a = \text{‘repair’}, D = \text{‘high’}) \end{bmatrix} = \begin{bmatrix} 0 & 100 \\ 20 & 25 \end{bmatrix}$$

The corresponding loss matrix depending by Y is



We compute the matrix as:

$$L(a, Y) = \begin{bmatrix} L(a = \text{‘do nothing’}, Y = \text{‘low’}) & L(a = \text{‘do nothing’}, Y = \text{‘high’}) \\ L(a = \text{‘repair’}, Y = \text{‘low’}) & L(a = \text{‘repair’}, Y = \text{‘high’}) \end{bmatrix} = \sum_R L(a, D) P(D|Y)$$
$$= \begin{bmatrix} 23.0 & 3.5 \\ 21.1 & 20.1 \end{bmatrix}$$

The expected loss relating to action a is:

$$EL(a) = [L(a = \text{‘do nothing’}) \quad L(a = \text{‘repair’})]^{\mathrm{T}} = \sum_Y L(a, Y) P(Y) = [9.75 \quad 20.49]^{\mathrm{T}}$$

The optimal action is that minimizing the expected loss:

$$a^* = \operatorname*{argmin}_a EL(a) = \text{‘do nothing’}$$

and the corresponding loss is

$$EL^* = \min_a EL(a) = 9.75$$

If $Y = \text{‘low’}$, the optimal action becomes

$$a^*(Y = \text{‘low’}) = \operatorname*{argmin}_a L(a, Y = \text{‘low’}) = \text{‘repair’}$$

and the corresponding loss is

$$L^*(Y = \text{‘low’}) = \min_a L(a, Y = \text{‘low’}) = 21.1$$

If $Y = \text{‘high’}$, the optimal action is

$$a^*(Y = \text{‘high’}) = \operatorname*{argmin}_a L(a, Y = \text{‘high’}) = \text{‘do nothing’}$$

and the corresponding loss is

$$L^*(Y = \text{‘high’}) = \min_a L(a, Y = \text{‘high’}) = 3.5$$

The expected loss observing Y is:

$$EL^*_{\text{obs}} = \sum_Y L^*(Y)P(Y) = 21.1 \cdot 32\% + 3.5 \cdot 68\% = 9.14$$

and the Value of information is

$$VoI = EL^* - EL^*_{\text{obs}} = 0.60$$