Name: BAGAS TRI WIBOWO NIM: 1301194051 Class: IF-43-04

Assignment 3: Proof Methods and Mathematical Induction

Mathematical Logic - A (MSH1B3)

First Term 2019-2020

Instructions:

1. This assignment is due **Friday November 22, 2019 at 5:00 p.m.**. Please submit your work at School of Computing academic roster (*roster akademik Fakultas Informatika*), room A203A (building A room A203A). Do not forget to write your identity on the space provided. You may submit this assignment as of **Monday November 18, 2019 at 8:00 a.m.**.
2. In order to prevent any academic misconduct, you also need to submit a readable scan or photograph of this assignment to the provided submission slot in CeLoE. Please submit it in a **.pdf file**. Please contact your class instructor for more detailed information. The due date of this online submission is the same as the hardcopy. **Please make sure that your file size do not exceed the maximum file size allowed.**
3. Please upload your assignment to the CeLoE under the file name: `A3-<student ID>.pdf`, for example: `A3-1301198888.pdf`.
4. To save paper, you may print and reproduce this assignment on both sides of a paper.
5. Your answers should be handwritten. You may use: HB or 2B pencil, or pen with blue or black ink.
6. All problems in this assignment are adapted from the textbooks. **The problems are written in English.** If you are a student in a regular class, you may answer the problems in Bahasa Indonesia. However, if you are a student in international class, your answers must be written in English—otherwise your assignment will not be graded. You may ask your class instructor or teaching assistant for helping you understanding the problem, but you should not ask them to give the solution of any problem.
7. Write your solutions on the space provided. If you need more space, you may use additional A4 papers and attach them to your assignment.
8. Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
9. This assignment consists of **10** problems, each problem is worth **10 points**.
10. Please retain yourself from copying answers from elsewhere without understanding the steps. This assignment is an individual evaluation.
11. **Important:** late submission without reasonable explanation will not be graded.

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Problem 1 Suppose a is an even integer and b is an odd integer. Determine the parity (i.e., whether the integer is even or odd) of each of these following integers. Justify your answer using mathematical proofs. Each justification is worth **1.5 points.**

Part	Integer	Parity (i.e., even or odd) **[0.5 point]**
a. **[2 points]**	$(a-2)b$	genap
b. **[2 points]**	$(a-1)(b-1)$	genap
c. **[2 points]**	$3b-a^2$	ganjil
d. **[2 points]**	$2a-b^2$	ganjil
e. **[2 points]**	$2a^3-3b^2$	ganjil

PROOF:

a. Ambil bilangan bulat a dan b, karena a genap dan b ganjil, maka berlaku $a=2k$ dan $b=2k+1$, untuk suatu bilangan bulat k. Dari persamaan diatas didapat pula $(a-2)b = (2k-2)(2k+1) = 2(k-1)(2k+1)$. Kita misalkan $(k-1)(2k+1) = l$. Untuk suatu bilangan bulat l. Sehingga didapat $2(k-1)(2k+1) = 2l$ Sehingga $(a-2)b$ = bilangan genap.

b. Ambil bilangan bulat a dan b, karena a genap dan b ganjil, maka berlaku $a=2k$ dan $b=2k+1$, untuk suatu bilangan bulat k. Dari persamaan $(a-1)(b-1)$, diperoleh $(2k-1)(2k+1-1) = (2k-1)(2k) = 2(k^2-2k)$. Kita misalkan $(k^2-2k) = l$. untuk setiap bilangan bulat l. Sehingga diperoleh $2(k^2-2k) = 2l$. Sehingga $(a-1)(b-1)$ = bilangan genap

c. Ambil bilangan a dan b, karena a genap dan b ganjil, maka berlaku $a=2k$ dan $b=2k+1$, untuk suatu bilangan bulat k. Dari persamaan $3b-a^2$, diperoleh $3(2k+1)-(2k)^2 = 6k+3-2k^2 = 2(-k^2+3k+1)+1$. Kita misalkan $(-k^2+3k+1) = l$, kemudian diperoleh $2(-k^2+3k+1)+1 = 2l+1$. Sehingga $3b-a^2$ adalah bilangan ganjil

d. Ambil bilangan a dan b, karena a genap dan b ganjil, maka berlaku $a=2k$ dan $b=2k+1$, untuk suatu bilangan bulat k. Dari persamaan $2a-b^2$, diperoleh $2(2k)-(2k+1)^2 = 4k-(4k^2+4k+1) = -4k^2-1 = 2(-2k^2)-1$. Kita misalkan $(-2k^2) = l$. Kemudian diperoleh $2(-2k^2)-1 = 2l-1$. Sehingga $2a-b^2$ adalah bilangan ganjil.

e. Ambil sembarang a dan b, karena a genap dan b ganjil, maka berlaku $a=2k$ dan $b=2k+1$, untuk suatu bilangan bulat k. Dari persamaan $2a^3-3b^2$ diperoleh $2(2k)^3-3(2k+1)^2 = 16k^3-(12k^2+12k+3) = 2(8k^3-6k^2+6k+1)+1$. Kita misalkan $(8k^3-6k^2+6k+1) = l$. Kemudian diperoleh $2(8k^3-6k^2+6k+1)+1 = 2l+1$. Sehingga $2a^3-3b^2$ adalah bilangan ganjil.

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Problem 2 Prove or disprove of each of the following statements and justify your answer using mathematical proof or provide a relevant counterexample.

(a). **[2.5 points]** Prove or disprove: suppose a and b are integers, if a and b are perfect squares, then $a+b+2\sqrt{ab}$ is also a perfect square. (Definition: an integer a is a perfect square if there is an integer $s \geq 0$ such that $a = s^2$. Example: 0 and 9 are perfect squares because $0 = 0^2$ and $9 = 3^2$.)

(b). **[2.5 points]** Prove or disprove: suppose a and b are integers, if a and b are perfect squares, then $a^2 + b^2$ is also a perfect square.

(c). **[2.5 points]** Prove or disprove: for every integer a, b, and c, if $a \geq 2b$ and $b \geq 2c$, then $a \geq 2c$.

(d). **[2.5 points]** Prove or disprove: the sum of three consecutive even integers is divisible by 6.

PROOF/COUNTEREXAMPLE:

a. Ambil bilangan a dan b, karena a dan b kuadrat sempurna, maka berlaku $a = s^2$ dan $b = s^2$ untuk suatu bilangan bulat s. Dari persamaan $a+b+2\sqrt{ab}$ diperoleh $= s^2 + s^2 + 2\sqrt{s^2 . s^2} = 2s^2 + 2s^2 = 4s^2$. Karena $\sqrt{4s^2} = 2s$, maka terbukti bahwa $a+b+2\sqrt{ab}$ merupakan kuadrat sempurna.

b. Ambil bilangan a dan b, karena a dan b kuadrat sempurna, maka berlaku $a = s^2$ dan $b = s^2$. Untuk suatu bilangan bulat s. Dari persamaan a^2+b^2 diperoleh $(s^2)^2 + (s^2)^2 = 2s^4$. Karena $2s^4 = \sqrt{2}.s^2$. Maka a^2+b^2 bukan kuadrat sempurna.

Bukti lain : $a^2+b^2 = (4)^2+(4)^2$
$= 32 \longrightarrow \sqrt{32} = 2\sqrt{8}$. Bukan kuadrat sempurna.

C. Ambil sembarang bilangan bulat a, b dan c. misalkan $a \geq 2b$ dan $b \geq 2c$. Kita memiliki $a \geq 2b \geq 4c$. Sehingga $a \geq 2c$ bernilai benar

D. Ambil bilangan bulat a, b, c, dimana $a = 2k$, $b = 2k+2$, $c = 2k+4$. Dari persamaan $a+b+c$, diperoleh $2k + (2k+2) + (2k+4) = 6k+6$. Karena $\frac{6k+6}{6} = k+1$. Sehingga pernyataan diatas benar.

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Problem 3 Verify the truth value of the following statements and justify your answer.

(a). **[5 points]** For any integer n, if $3n+3$ is even, then $n+4$ is odd.

(b). **[5 points]** Let n, a, b, c be integers such that $n = abc$. If n is even, then one of a, b, or c must be even.

PROOF/COUNTEREXAMPLE:

a. kita ambil kontraposisi dari pernyataan, yaitu jika $n+4$ adalah genap, maka $3n+3$ adalah ganjil. kita peroleh $n+4 = 2k$, maka diperoleh $n = 2k-4$, diperoleh pula $3(2k-4)+3 = 6k-9 = 2(3k-5)+1$. kita misalkan $3k+5 = l$. kemudian diperoleh $2l+1$. sehingga didapat $3n+3$ ganjil. maka pernyataan tersebut benar.

b. kita memiliki n, a, b, c bilangan bulat, dan $n = a.b.c$. jika n genap maka salah satu a, b, atau c adalah genap. kita ambil kontraposisinya jika a, b, dan c ganjil, maka n ganjil. karena a, b, c ganjil maka berlaku $a = 2k+1$; $b = 2k+1$, $c = 2k+1$. dengan demikian $n = a.b.c = (2k+1)(2k+1) \times (2k+1) = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$. kita misalkan $4k^3+6k^2+3k = l$, kemudian kita peroleh $= 2l+1$, sehingga n ganjil, dan pernyataan tersebut terbukti.

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Problem 4 Answer each of the following problems clearly and justify its truth using mathematical proof.

(a). **[5 points]** Is it true that there is no smallest positive rational number? Justify your reason with mathematical proof. (Hint: a number x is positive if $x > 0$; observe the relationship between x and $\frac{x}{2}$ for any rational number x.)

(b). **[5 points]** Is it true that if q is a rational number and x is an irrational number, then $q + x$ is an irrational number? Justify your reason with mathematical proof. (Hint: a number q is a rational number if $q = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a, b) = 1$.)

PROOF/COUNTEREXAMPLE:

a. kita buat falsifikasinya. Apakah benar ada bilangan rasional positif terkecil? misalkan a adalah bilangan rasional positif terkecil, kemudian b adalah bilangan rasional positif dimana $a > b$. kemudian ada $M = a - b$, $M < a$ M bilangan rasional positif juga, tentu pernyataan ini kontradiksi dengan pernyataan awal. Sehingga pernyataan tidak ada bilangan rasional terkecil benar. □

b. kita buat falsifikasinya. ~~q adalah f~~

Andaikan q adalah bilangan rasional dan x adalah bilangan irasional dan jika q dijumlahkan dengan x, maka hasilnya adalah rasional $\frac{c}{d}$, misalkan $q = \frac{a}{b}$. maka $\frac{a}{b} + x = \frac{c}{d}$. untuk suatu bilangan bulat a, b, c, d. dengan $b, d \neq 0$. Maka $\frac{a}{b} + x = \frac{c}{d}$, $x = \frac{c}{d} - \frac{a}{b}$, $x = \frac{cb - ad}{bd}$. Dikarenakan a, b, c, d adalah bilangan bulat, maka cb, ad, bd adalah bilangan bulat juga. Akibatnya $\frac{cb - ad}{bd}$ adalah bilangan rasional. Hal ini bertentangan dengan pernyataan x adalah bilangan irrasional. Akibatnya, penjumlahan bilangan rasional dengan irasional adalah bilangan irrasional benar. □

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Problem 5 Prove or disprove that: for any integer n, $n+1$ is even if and only if n^2-1 is even.

PROOF/COUNTEREXAMPLE:

$p \leftrightarrow q$
kita harus buktikan
$p \rightarrow q \equiv$ True
$q \rightarrow p \equiv$ True

Bukti $p \rightarrow q \equiv$ True

: Jika $n+1$ genap, maka n^2-1 genap. karena $n+1$ genap, maka berlaku $n+1 = 2k$, ~~kemudian diperoleh pula n^2+~~ $n = 2k-1$, kemudian diperoleh pula $n^2-1 = (2k-1)^2-1 = 4k^2-4k+1-1 = 2(2k^2-2k)$. kita misalkan $2k^2-2k = l$. kita peroleh $2(2k^2-2k) = 2l$, maka n^2+1 adalah bilangan genap sehingga pernyataan $p \rightarrow q$ benar.

Bukti $q \rightarrow p \equiv$ True

: kita ambil kontraposisinya jika $n+1$ ganjil, maka n^2-1 ganjil, karena $n+1$ ganjil, maka berlaku $n+1 = 2k+1$, $n = 2k$. kemudian diperoleh $n^2-1 = (2k)^2-1 = 4k^2-1 = 2(2k^2)-1$. kita misalkan $2k^2 = l$, kemudian diperoleh $2(2k^2)-1 = 2l-1$. Maka terbukti n^2-1 ganjil, sehingga pernyataan

$q \rightarrow p \equiv$ True.

Berdasarkan 2 bukti diatas maka pernyataan untuk semua bilangan bulat n, $n+1$ genap jika dan hanya jika n^2-1 genap terbukti benar.

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Problem 6 Use mathematical induction to prove that $6^n - 1$ is divisible by 5 for every integer $n \geq 0$.

PROOF:

$P(n) = 6^n - 1$, habis dibagi 5 untuk setiap $n \geq 0$

* Langkah basis

$P(0) = 6^0 - 1$

$= 1 - 1 = 0 \leftarrow$ habis dibagi 5, maka basis benar

* Langkah Induksi

$P(k) = 6^k - 1$, benar habis dibagi 5

$P(k+1) = 6^{k+1} - 1$, harus benar habis dibagi 5

> Perhatikan

$6^{k+1} - 1 = 6 . 6^k - 1$

$= (5+1) . 6^k - 1$

$= 5 . 6^k + [6^k - 1]$

Karena $5 . 6^k$ dan $6^k - 1$ habis dibagi 5, maka dapat disimpulkan $6^n - 1$, habis dibagi 5 untuk setiap bilangan bulat $n \geq 0$.

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Problem 7 Determine all positive integers n that make the inequality $2n + 3 < 2^n$ holds! Justify your answer with mathematical induction!

PROOF:

$\boxed{\text{misal } n \geq 4}$

*Langkah basis

$P(4) = 2(4) + 3 < 2^4$

$= 11 < 16 \rightarrow$ Benar

* Langkah Induksi

$P(k) = 2k + 3 < 2^k \rightarrow$ Benar

$P(k+1) = 2(k+1) + 3 < 2^{k+1} \rightarrow$ harus benar, untuk $n \geq 4$

> Perhatikan

$2(k+1) + 3 = 2k + 5$

$= 2k + 3 + 2$

$< 2^k + 2$ (Induksi)

$< 2^{k+1}$

Sehingga berdasarkan 2 langkah diatas $n \geq 4$, ~~[illegible]~~ berlaku untuk $2n + 3 < 2^n$.

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Problem 8 One day Alvin, an informatics student, created the following Python program:

```
def sum_something(n):
    sum = 0
    for i in range(1,n+1):
        sum = i*(i+1) + sum
    return sum
```

Alvin use the above program to calculate following mathematical expression:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1). \quad (1)$$

One day, Alvin encountered Vishnu, a senior student who regularly participated at programming contest for the last two years. Vishnu laughed at Alvin and he said that Alvin's code is very inefficient. Vishnu said to Alvin that the expression (1) can be expressed in a simple mathematical expression.

(a). **[2 points]** Vishnu gave Alvin hints that expression (1) can be expressed in one of the following form, **choose one correct expression**:

(1) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = n^2 + n.$

(2) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$

(3) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{(n-1)(n)(n+1)}{3}.$

(Hint: substitute $n = 1$ and $n = 2$ to the above equations, the correct expression yields correct equations.)

ANSWER:

(2) $1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = \frac{n(n+1)(n+2)}{3}$

$* n = 1$

$1.2 = \frac{1(1+1)(1+2)}{3}$

$2 = 2$ ✓

$* n = 2$

$1.2 + 2.3 = \frac{2(2+1)(2+2)}{3}$

$8 = 8$ ✓

(b). **[1 point]** Use your choice in (a) to compute the following expression

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 99 \cdot 100,$$

express your result as an integer.

ANSWER: $\frac{n(n+1)(n+2)}{3} = \frac{\overset{33}{\cancel{99}}(100)(101)}{\cancel{3}} = 100 . 3333 = 333300$

(c). **[7 points]** Prove that your choice in (a) is correct using mathematical induction.

PROOF:

→ Langkah basis

$P(2) = 1.2 + 2.3 = \frac{2(2+1)(2+2)}{3}$

$8 = 8$ True

→ Langkah Induksi

$P(k) = 1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

$P(k+1) = 1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)[(k+1)+1]$

$= \frac{(k+1)[(k+1)+1][(k+1)+2]}{3}$

→ Perhatikan

$\underbrace{1.2 + 2.3 + 3.4 + k(k+1)}_{P(k)} + (k+1)[(k+1)+1]$

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$= (k+1)(k+2)\left(\frac{k}{3} + 1\right)$

$= (k+1)(k+2)\left(\frac{k+3}{3}\right)$

$= \frac{(k+1)[(k+1)+1][(k+1)+2]}{3}$

Sehingga terbukti bahwa $P(k+1)$ benar, sehingga langkah induksi terbukti.

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Problem 9 A sequence $\langle a_n \rangle$ is defined recursively as

$$a_0 = 4,\ a_1 = 12, \text{ and } a_n = a_{n-1} + a_{n-2}.$$

(a). **[1 point]** Use the recursive definition to determine the value of a_4.

ANSWER:

$$\begin{aligned} a_4 &= a_{4-1} + a_{4-2} \\ &= a_3 + a_2 \\ &= (a_2 + a_1) + (a_1 + a_0) \\ &= (16 + 12) + (12 + 4) \\ &= 28 + 16 \\ &= 44 \end{aligned}$$

(b). **[2 points]** Assuming that $P(n)$ is the predicate stating that "a_n is divisible by 4", write the predicate $P(k-1)$, $P(k)$, and $P(k+1)$.

ANSWER:

~~$P(k-1)$: a_{k-1} habis dibagi 4~~

~~$P(k)$: a_k habis dibagi 4~~

~~$P(k+1)$: a_{k+1} habis dibagi 4~~

$P(k+1)$: $a_k + a_{k-1}$, habis dibagi 4

$P(k)$: $a_{k-1} + a_{k-2}$, habis dibagi 4

$P(k-1)$: $a_{k-2} + a_{k-3}$, habis dibagi 4

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(c). **[7 points]** Use strong induction to prove that a_n is divisible by 4 for every integer $n \geq 0$.

PROOF:

$P_n = a_{n-1} + a_{n-2}$, habis dibagi 4, $n \in N$, $N \geq 0$

* Langkah basis

$P(0) = 4$, habis dibagi 4

$P(1) = 12$, habis dibagi 4

$P(2) = 16$, habis dibagi 4

* Langkah Induksi

Kita asumsikan $P(0), P(1), \ldots, P(k)$ benar, maka selanjutnya kita buktikan $P(k+1)$ benar

$P(k+1) = a_k + a_{k-1}$

$= (a_{k-1} + a_{k-2}) + (a_{k-2} + a_{k-3})$

$= 4x + 4y$ (berdasarkan pernyataan sebelumnya, bahwa $P(k)$ dan $P(k-1)$ habis dibagi 4)

$= 4(x+y) \rightarrow$ habis dibagi 4

Sehingga dari langkah induksi di atas, terbukti $P(n)$ habis dibagi 4, untuk setiap $n \geq 0$.

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Problem 10 Observe the following excerpt of a program in Python:

```
def a(n):
    if n == 0:   return 1
    if n == 1:   return 6
    else:   return 6*a(n-1)-9*a(n-2)
```

The above program computes the n-th term of the sequence $\langle a_n \rangle$ recursively defined as

$$a_0 = 1, a_1 = 6, \text{ and } a_n = 6a_{n-1} - 9a_{n-2} \text{ for all integers } n \geq 2. \quad (2)$$

(a). **[1 point]** Use the above recursive program (or the recursive definition $\langle a_n \rangle$) to compute a_3. (The value of a_3 is equal to a(3) in the above Python program.)

ANSWER:

$a_3 = 6a_2 - 9a_1$

$= 6 \cdot (27) - 9(6)$

$= 162 - 54$

$= 108$

(b). **[2 points]** The recursive sequence in (2) can be expressed using a simple mathematical formula, which one of the following formulas is correct? Choose **one correct expression!**

(1) $a_n = (1 - n) \cdot 3^n$.

(2) $a_n = (1 + n) \cdot 3^n$.

(Hint: substitute $n = 0$ and $n = 1$ to the above formulas, the correct expression yields correct equations.)

ANSWER:

(2) $a_n = (1+n).3^n$

$n = 0$

$a_0 = (1+0).3^0$

$= 1$ ✓

$n = 1$

$a_1 = (1+1).3^1$

$= 3$ ✓

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(c). **[7 points]** Prove that your choice in (b) is correct for all integers $n \geq 0$ using mathematical induction. (Define the predicate $P(n)$ first. Hint: $6 \cdot 3^k + 3 \cdot k \cdot 3^k = 2 \cdot 3 \cdot 3^k + k \cdot 3 \cdot 3^k$.)

PROOF: $P(n) : 6a_{n-1} - 9a_{n-2} = (1+n)3^n$

* langkah basis

$P(2) = 6a_1 - 9a_0 = (1+2)3^2 \iff 27 = 27 \rightarrow$ True
$P(3) = 6a_2 - 9a_1 = (1+3)3^3 \iff 108 = 108 \rightarrow$ True
$P(4) = 6a_3 - 9a_2 = (1+4)3^4 \iff 405 = 405 \rightarrow$ True

* Langkah Induksi

kita asumsikan $P(0), P(1), \ldots, P(k)$ adalah benar, selanjutnya akan kita buktikan $P(k+1)$ benar

$$P(k+1) = (1 + (k+1)).3^{k+1} = 6a_k - 9a_{k-1}$$
$$= 6(1+k).3^k - 9((k).3^{k-1})$$
$$= 3(2(3^k + 3^k.k) - 3.3^{k-1}.k)$$
$$= 3(2.3^k + 3^k.k)$$
$$= 3.3^k.(2+k)$$
$$= (k+2).3^{k+1}$$

Sehingga terbukti bahwa $P(n)$ berlaku $n \geq 0$, berdasarkan langkah induksi diatas / $P(k+1)$ terbukti.

□