

Probability Basic

Yanzhong (Eric) Huang.
2025

Ref: A first course in probability (ROSS).

1. Combinatorial Analysis

1.1 Permutations

→ distinct/different.

Suppose we have n objects, their are

$$n(n-1) \cdots 3 \cdot 2 \cdot 1 = n! \quad \text{permutations.}$$

When we have n_1, \dots, n_r are alike:

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

Example: PEPPER $\rightarrow n=6, n_1=3, n_2=2 \rightarrow \frac{6!}{3! 2!}$

1.2 Combinations

From n objects pick $r \rightarrow$ the first r permutations.

$$\binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r!} = \frac{n!}{(n-r)! r!}$$

$$* \binom{n}{0} = \binom{n}{n} = 1 = \frac{n!}{n! 0!} = 1.$$

$$\binom{n}{i} = 0, \text{ if } i < 0 \text{ or } i > n.$$

* Identities :

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\begin{aligned} \text{proof : } &= \frac{(n-1)!}{(r-1)!(n-1-r+1)!} + \frac{(n-1)!}{r!(n-1-r)!} \\ &= \frac{(n-1)! r}{(n-r)! r!} + \frac{(n-1)! (n-r)}{(n-r)! r!} \\ &= \frac{n!}{(n-r)! r!} = \binom{n}{r} \end{aligned}$$

* Binomial theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

* Multinomial theorem .

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$$

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{\substack{(n_1, \dots, n_r) : \\ n_1 + \dots + n_r = n}} \binom{n}{n_1, \dots, n_r} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

known as multinomial coefficient

2. Axioms of Probability

Sample space $\rightarrow \Omega$ or S . all possible outcome.

Event $\rightarrow E$ or A , subset of Ω .

Axioms:

1) $0 \leq P(A) \leq 1$

2) $P(\Omega) = 1$

3) For any sequence of mutually exclusive A_1, A_2, \dots
$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i).$$

2.1 conditional probability and independence

$$P(A \text{ given } B) = P(A|B) = \frac{P(AB)}{P(B)}$$

$$* P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Independence. $P(AB) = P(A)P(B).$

equivalent: $P(A|B) = P(A)$, $P(B|A) = P(B)$

2.2 Random Variables

The quantities of numerical experiment outcomes:

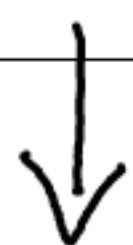
cdf: $F(x) = P\{X \leq x\} = P\{\text{outcome} \leq x\}.$

pmf (finite/discrete) \rightarrow countable values

$$p(x) = P\{X = x\} \rightarrow \sum_{i=1}^{\infty} p(x_i) = 1$$

continuous r.v. if exists a non-negative $f(x)$ (pdf).

$$P\{X \in C\} = \int_C f(x) dx.$$



$$F(x) = P\{X \in (-\infty, x]\} = \int_{-\infty}^x f(s) ds.$$

* Multivariable distribution.

$$F_{X,Y}(x,y) = P\{X \leq x, Y \leq y\} = \int_{-\infty}^x \int_{-\infty}^y f(u,v) du dv$$

Independence : $X \perp Y$

$$P(X \in C, Y \in D) = P(X \in C) P(Y \in D)$$

$$\int_{C,D} f_{X,Y}(x,y) dx dy = \int_C f_X(x) dx \int_D f_Y(y) dy.$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

2.3 Expectation

discrete : $E[X] = \sum_i x_i p(x_i) = \sum_i x_i P\{X = x_i\}.$

continuous : $E[X] = \int_{-\infty}^{+\infty} x f(x) dx$

$$E[g(X)] = \sum_i g(x_i) p(x_i).$$

or

$$= \int_{-\infty}^{+\infty} g(x) f(x) dx.$$

2.4 Variance

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}(X))^2] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2.\end{aligned}$$

$$\begin{aligned}\text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].\end{aligned}$$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{cov}(X, Y).$$

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$$

2.5