Probability Basic

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Ref: A first course in probability (ROSS).

1. Combinatorial Analysis

1.1 Permutations

distinct / different.

Suppose we have n objects, their are

$$n(n-1)\cdots 3\cdot 2\cdot 1 = n! \quad \text{permutations.}$$

When we have $n_1, \cdots, n_r$ are alike:

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Example: PEPPER $\rightarrow n=6,\ n_1=3,\ n_2=2 \rightarrow \frac{6!}{3!\,2!}$

1.2 Combinations

From n objects pick r $\rightarrow$ the first r pemutations.

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

$$* \binom{n}{0} = \binom{n}{n} = 1 = \frac{n!}{n!\,0!} = 1.$$

$$\binom{n}{i} = 0, \text{ if } i<0 \text{ or } i>n.$$

* Identities:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

proof: $= \dfrac{(n-1)!}{(r-1)!(n-1-r+1)!} + \dfrac{(n-1)!}{r!(n-1-r)!}$

$= \dfrac{(n-1)!\, r}{(n-r)!\, r!} + \dfrac{(n-1)!\,(n-r)}{(n-r)!\, r!}$

$= \dfrac{n!}{(n-r)!\, r!} = \binom{n}{r}$

* Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

* Multinomial theorem.

$$\binom{n}{n_1, n_2 \cdots n_r} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, \cdots, n_r): \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, \cdots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$

known as multinomial coefficient

2. Axioms of Probability

Sample space $\rightarrow \Omega$ or S. all possible outcome.

Event $\rightarrow E$ or A, subset of Ω.

Axioms:

1) $0 \leq \mathbb{P}(A) \leq 1$

2) $\mathbb{P}(\Omega) = 1$

3) For any sequence of mutually exclusive $A_1 A_2 \cdots$

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

2.1 conditional probability and independence

$$P(A \text{ given } B) = P(A|B) = \frac{P(AB)}{P(B)}$$

$\star$ $P(A) = P(A|B)P(B) + P(A|B^c)\,P(B^c)$.

Independence. $P(AB) = P(A)P(B)$.

equivelent: $P(A|B) = P(B)$, $P(B|A) = P(A)$

2.2 Random Variables

The quantites of numerical experiment outcomes:

cdf: $F(x) = \mathbb{P}\{X \le x\} = \mathbb{P}\{\text{outcome} \le x\}$.

pmf (finite/discrete) $\rightarrow$ countable values

$$p(x) = \mathbb{P}\{X = x\} \rightarrow \sum_{i=1}^{\infty} p(x_i) = 1$$

continuous r.v. if exists a non-negative $f(x)$ (pdf).

$$P\{X \in C\} = \int_C f(x)\,dx.$$

$$\downarrow$$

$$F(x) = P\{X \in (-\infty, x]\} = \int_{-\infty}^{x} f(s)\,ds.$$

* Multivariable distribution.

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\} = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

Independence : $X \perp Y$

$$P(X \in C, Y \in D) = P(X \in C)\, P(Y \in D)$$

$$\int_{C,D} f_{X,Y}(x,y)\, dx\, dy = \int_C f_X(x)\, dx \int_D f_Y(y)\, dy.$$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y).$$

2.3 Expectation

discrete : $\mathbb{E}[X] = \sum_i x_i\, p(x_i) = \sum_i x_i\, P\{X = x_i\}.$

continuous : $\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$

$$\mathbb{E}[g(X)] = \sum_i g(x_i)\, p(x_i).$$

or

$$= \int_{-\infty}^{+\infty} g(x) f(x)\, dx.$$

2.4 Variance

$$Var(X) = \mathbb{E}\left[(X-\mathbb{E}(X))^2\right]$$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

$$Cov(X,Y) = \mathbb{E}\left[(X-\mathbb{E}(X))\,(Y-\mathbb{E}(Y))\right].$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

$$Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y).$$

$$Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

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