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P1. Error Probability

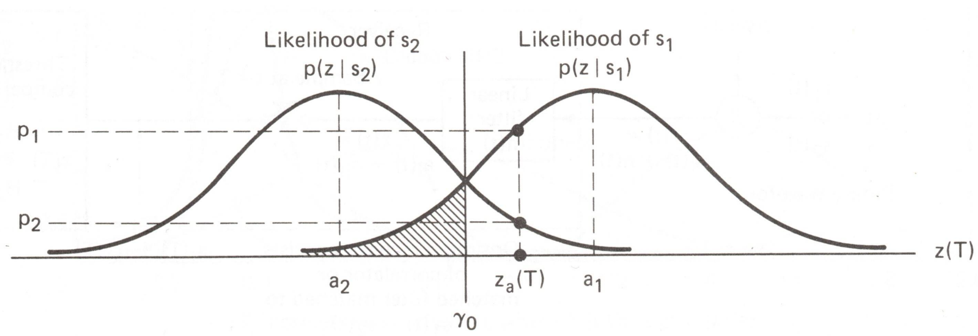

FIGURE 1. Conditional probability density functions: $p(z|s_1)$ and $p(z|s_2)$.

For the binary example in Fig. 1, there are two ways in which errors can occur. An error, e, will occur when $s_1(t)$ is sent, and channel noise results in the receiver output signal $z(T)$, being less than γ_0. The probability of such an occurence is:

$$P(e|s_1) = P(H_2|s_1) = \int_{-\infty}^{\gamma_0} p(z|s_1)dz \tag{1}$$

This is illustrated by the shaded area to the left of γ_0 in Fig. 1. Similarly an error occurs when $s_2(t)$ is sent and the channel noise results in $z(T)$, being greater than γ_0. The probability of this occurence is:

$$P(e|s_2) = P(H_1|s_2) = \int_{\gamma_0}^{\infty} p(z|s_2)dz \tag{2}$$

The probability of an error is the sum of the probabilities of all the ways that an error can occur. For the binary case, we can express the probability of bit error, P_B, as follows:

$$P_B = \sum_{i=1}^{2} P(e, s_i) \tag{3}$$

Combining the previous three equations, we can write:

$$P_B = P(H_2|s_1)P(s_1) + P(H_1|s_2)P(s_2) \tag{4}$$

That is, given that signal $s_1(t)$ was transmitted, an error results if hypothesis H_2 is chosen, or given that the signal $s_2(t)$ was transmitted, an error results if hypothesis H_1 is chosen. For the case where the a priori probabilities are equal, that is, $P(s_1) = P(s_2) = \frac{1}{2}$,

$$P_B = \frac{1}{2}P(H_2|s_1) + \frac{1}{2}P(H_1|s_2) \tag{5}$$

and because of the symmetry of the probability density functions

$$P_B = P(H_2|s_1) = P(H_1|s_2) \tag{6}$$

The probability of a bit error, P_B, is numerically equal to the area under the ”tail” of either likelihood function, $p(z|s_1)$ or $p(z|s_2)$, falling on the ”incorrect” side of the threshold. We can therefore compute P_B by integrating $p(z|s_1)$ between the limits $-\infty$ and γ_0, or as shown below, by integrating $p(z|s_2)$ between the limits γ_0 and ∞:

$$P_B = \int_{\gamma_0=(a_1+a_2)/2}^{\infty} p(z|s_2)dz \tag{7}$$

where $\gamma_0 = (a_1 + a_2)/2$ is the optimum threshold from Fig. 1. Replacing the likelihood $p(z|s_2)$ with its Gaussian equivalent, we have:

$$P_B = \int_{\gamma_0=(a_1+a_2)/2}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right] dz \tag{8}$$

where σ_0^2 is the variance of the noise out of the correlator.

Let $u = (z - a_2)/\sigma_0$. Then $\sigma_0 du = dz$ and:

$$P_B = \int_{u=(a_1-a_2)/(2\sigma_0)}^{u=\infty} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{u^2}{2}\right) du = Q\left(\frac{a_1-a_2}{2\sigma_0}\right) \tag{9}$$

where Q(x), called the complementary error function or co-error function, is a commonly used symbol for the probability under the tails of the Gaussian distribution. It is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} exp\left(-\frac{u^2}{2}\right) du \tag{10}$$

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