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P2. The Matched Filter

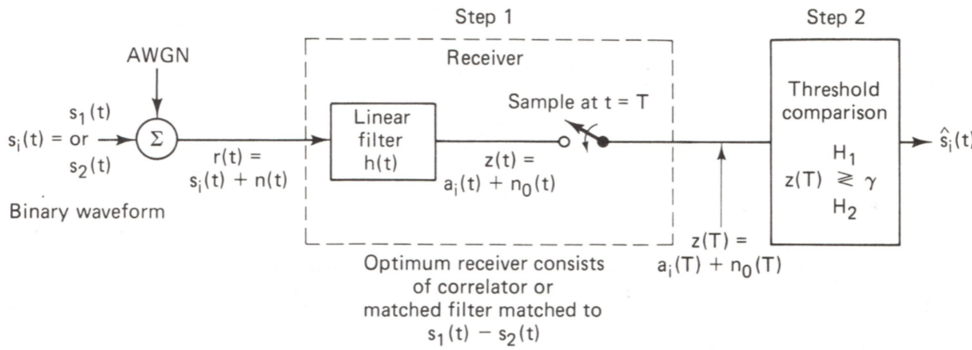

FIGURE 1. Steps in digital signal detection.

A matched filter is a linear filter designed to provide the maximum signal-to-noise power ratio at its output for a given transmitted symbol waveform. Consider that a known signal $s(t)$ plus AWGN, n(t), is the input to a linear, time-invariant filter followed by a sampler, as shown in Fig. 1. At time $t = T$, the receiver output z(T), consists of a signal component, a_i, and a noise component, n_0. The variance of the output noise (average noise power) is denoted by σ_0^2, so that the ratio of instantaneous signal power to average noise power, $(S/N)_T$, at time $t = T$, out of the receiver in block 1, is:

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} \tag{1}$$

We wish to find the filter transfer function $H_0(f)$ that maximizes the previous equation.

We can express the signal, a(t), at the filter output, in terms of the filter transfer function, $H(f)$ (before optimization), and the Fourier transform of the input signal as follows

$$a(t) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df \tag{2}$$

where $S(f)$ is the Fourier transform of the input signal $s(t)$. Assume the two-sided power spectral density of the input noise is $N_0/2$ watts/hertz. The input power spectral density, $G_X(f)$, and the output power spectral density, $G_Y(f)$, are related as follows:

$$G_Y(f) = G_X(f)|H(f)|^2 \tag{3}$$

We can express the output noise power, σ_0^2, as:

$$\sigma_0^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df \tag{4}$$

We then combine the previous equations to express $(S/N)_T$, as follows:

$$\left(\frac{S}{N}\right)_T = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi f_t} df\right|}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df} \tag{5}$$

We next find that value of $H(f) = H_0(f)$ for which the maximum $(S/N)_T$ is achieved, by using Schwarz's inequality. One form of the inequality can be stated as:

$$\left|\int_{-\infty}^{\infty} f_1(x) f_2(x)\right| \leq \int_{-\infty}^{\infty} |f_1(x)|^2 dx \int_{-\infty}^{\infty} |f_2(x)|^2 dx \tag{6}$$

The equality holds if $f_1(x) = kf_2^*$, where k is an arbitrary constant and * indicates complex conjugate. If we identify $H(f)$ with $f_1(x)$ and $S(f)e^{j2\pi fT}$ with $f_2(x)$ we can write

$$\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df\right| \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |S(f)|^2 df \tag{7}$$

Substituting in one of the previous equations yields:

$$\left(\frac{S}{N}\right)_T \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |S(f)|^2 df \tag{8}$$

or

$$max \left(\frac{S}{N}\right)_T = \frac{2E}{N_0} \tag{9}$$

where the energy, E, of the input signal s(t) is:

$$E = \int_{-\infty}^{\infty} |S(f)|^2 df \tag{10}$$

Thus the maximum output $(S/N)_T$ depends on the input signal energy and the power spectral density of the noise, not on the particular shape of the waveform that is used. The equality in the previous inequality holds only if the optimum filter transfer function, $H_0(f)$, is employed, such that:

$$H(f) = H_0(f) = kS^*(f)e^{-j2\pi fT} \tag{11}$$

or

$$h(t) = \mathcal{F}^{-1}\{kS^*(f)e^{-j2\pi fT}\} \tag{12}$$

Since $s(t)$ is a real valued signal we can write using Fourier transformations table:

$$h(t) = \begin{cases} ks(T-t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{13}$$

Thus the impulse response of a filter that produces the maximum output signal-to-noise ratio is the mirror image of the message signal s(t), *delayed* by the symbol time duration, T.Note that the delay of T seconds makes the previous equations causal; that is, the delay of T seconds makes $h(t)$ a function of positive time in the interval $0 \leq t \leq T$. Without the delay of T seconds, the response, $s(-t)$, is unrealizable because it describes a response as a function of negative time.

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