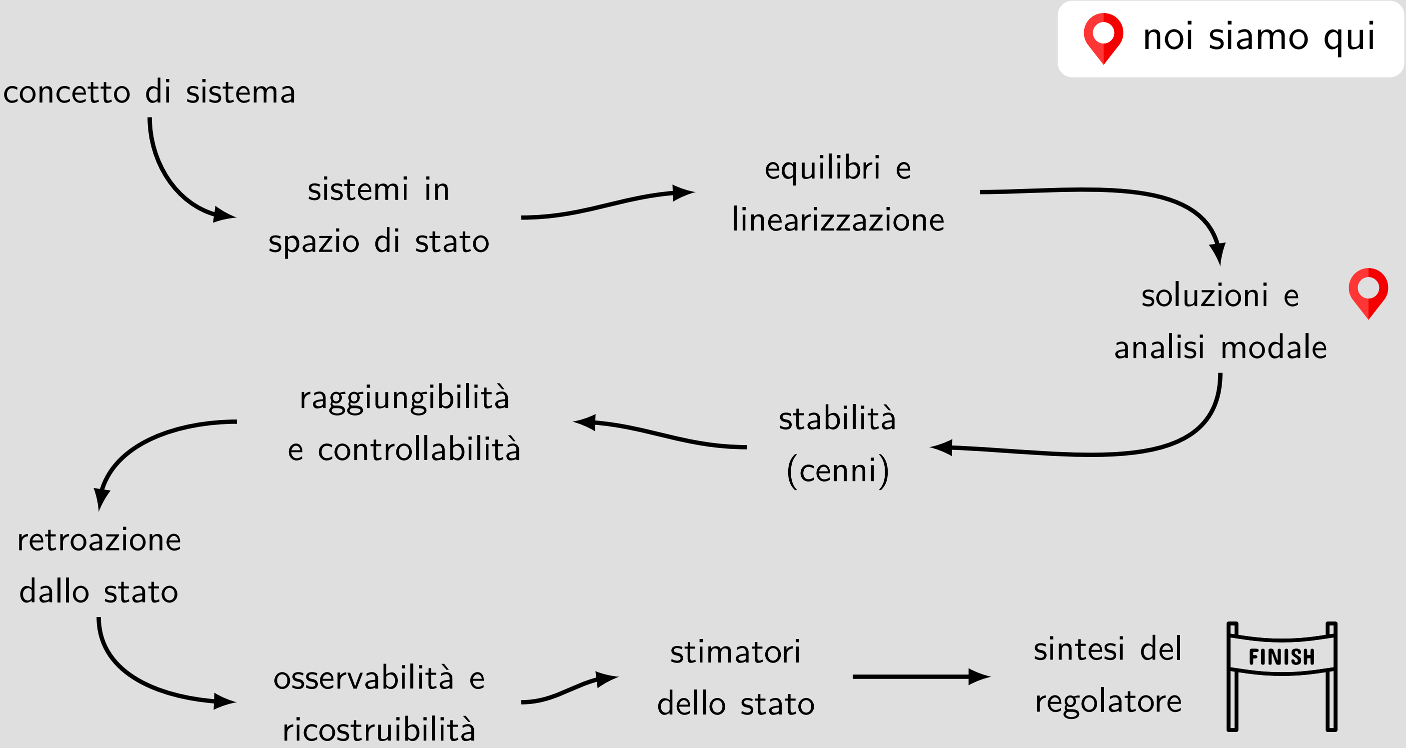
Teoria dei Sistemi e Controllo Ottimo e Adattativo (C. I.)
Teoria dei Sistemi (Mod. A)

Docente: Giacomo Baggio

Lez. 10: Modi di un sistema lineare, risposta libera e forzata (tempo discreto)

Corso di Laurea Magistrale in Ingegneria Meccatronica

A.A. 2021-2022


noi siamo qui
concetto di sistema
sistemi in spazio di stato
equilibri e linearizzazione
soluzioni e analisi modale
stabilità (cenni)
raggiungibilità e controllabilità
retroazione dallo stato
osservabilità e ricostruibilità
stimatori dello stato
sintesi del regolatore
FINISH

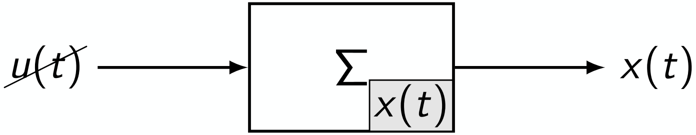
Nella scorsa lezione

- Analisi modale ed evoluzione libera di un sistema lineare a t.c.
- Evoluzione complessiva di un sistema lineare a t.c.
- Equivalenza algebrica e matrice di trasferimento

In questa lezione

- ▷ Analisi modale ed evoluzione libera di un sistema lineare a t.d.
- ▷ Evoluzione complessiva di un sistema lineare a t.d.

Soluzioni di un sistema lineare autonomo?



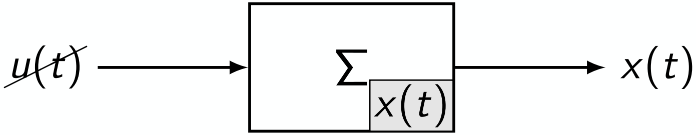
Caso vettoriale $\qquad x(t) = y(t) \in \mathbb{R}^n$

$$x(t+1) = Fx(t), \qquad x(0) = x_0$$

$$x(t) = ??$$

$x(1) = Fx(0)$
$x(2) = Fx(1) = F^2x(0)$
$x(3) = Fx(2) = F^3x(0)$
$\vdots$
$x(t) = F^t x(0)$

Soluzioni di un sistema lineare autonomo?



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$$x(t) = F^t x_0$$

Calcolo di F^t tramite Jordan

1. $F = TF_J T^{-1} \quad \Longrightarrow \quad F^t = TF_J^t T^{-1}$

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2. $$F_J = \left[\begin{array}{c|c|c|c} J_{\lambda_1} & 0 & \cdots & 0 \\ \hline 0 & J_{\lambda_2} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & J_{\lambda_k} \end{array}\right] \quad \Longrightarrow \quad F_J^t = \left[\begin{array}{c|c|c|c} J_{\lambda_1}^t & 0 & \cdots & 0 \\ \hline 0 & J_{\lambda_2}^t & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & J_{\lambda_k}^t \end{array}\right]$$

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3. $$J_{\lambda_i} = \left[\begin{array}{c|c|c|c} J_{\lambda_i,1} & 0 & \cdots & 0 \\ \hline 0 & J_{\lambda_i,2} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & J_{\lambda_i,g_i} \end{array}\right] \quad \Longrightarrow \quad J_{\lambda_i}^t = \left[\begin{array}{c|c|c|c} J_{\lambda_i,1}^t & 0 & \cdots & 0 \\ \hline 0 & J_{\lambda_i,2}^t & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & J_{\lambda_i,g_i}^t \end{array}\right]$$

Calcolo di F^t tramite Jordan

$$\lambda_i \neq 0$$

$$\textbf{4(i).}\ J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij} \times r_{ij}} \implies J_{\lambda_i,j}^t = (\lambda_i I + N)^t, \quad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

note

Calcolo di F^t tramite Jordan

$$\textbf{4(i).}\ J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij}\times r_{ij}} \overset{\lambda_i \neq 0}{\Longrightarrow} J_{\lambda_i,j}^t = (\lambda_i I + N)^t, \quad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow J_{\lambda_i,j}^t = \begin{bmatrix} \binom{t}{0}\lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \binom{t}{2}\lambda_i^{t-2} & \cdots & \binom{t}{r_{ij}-1}\lambda_i^{t-r_{ij}+1} \\ 0 & \binom{t}{0}\lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{t}{2}\lambda_i^{t-2} \\ \vdots & \ddots & \ddots & \ddots & \binom{t}{1}\lambda_i^{t-1} \\ 0 & \cdots & \cdots & 0 & \binom{t}{0}\lambda_i^t \end{bmatrix}$$

note

Calcolo di F^t tramite Jordan

$$\lambda_i = 0$$

4(ii). $J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij} \times r_{ij}} \implies J^t_{\lambda_i,j} = N^t, \quad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$

Calcolo di F^t tramite Jordan

$$\textbf{4(ii).}\ J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij}\times r_{ij}} \xRightarrow{\lambda_i = 0} J^t_{\lambda_i,j} = N^t, \qquad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

delta Kronecker
impulso discreto
$\leftarrow$ $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

$$\implies J^t_{\lambda_i,j} = \begin{bmatrix} \delta(t) & \delta(t-1) & \delta(t-2) & \cdots & \delta(t-r_{ij}+1) \\ 0 & \delta(t) & \delta(t-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \delta(t-2) \\ \vdots & \ddots & \ddots & \ddots & \delta(t-1) \\ 0 & \cdots & \cdots & 0 & \delta(t) \end{bmatrix}$$

Modi elementari

$\lambda_i^t \quad t\,\lambda_i^t \quad t^2\lambda_i^t \quad t^{r_{ij}-1}\lambda_i^t$

$$\binom{t}{0}\lambda_i^t,\ \binom{t}{1}\lambda_i^{t-1},\ \binom{t}{2}\lambda_i^{t-2},\ \dots,\ \binom{t}{r_{ij}-1}\lambda_i^{t-r_{ij}+1}$$

$$\delta(t),\ \delta(t-1),\ \delta(t-2),\ \dots,\ \delta(t-r_{ij}+1)$$

$=$ modi elementari del sistema

Modi elementari

$$\left.\begin{array}{l}\binom{t}{0}\lambda_i^t,\ \binom{t}{1}\lambda_i^{t-1},\ \binom{t}{2}\lambda_i^{t-2},\ \ldots,\ \binom{t}{r_{ij}-1}\lambda_i^{t-r_{ij}+1}\\ \delta(t),\ \delta(t-1),\ \delta(t-2),\ \ldots,\ \delta(t-r_{ij}+1)\end{array}\right. = \text{modi elementari del sistema}$$

$$t^k e^{\ln \lambda_i^t} = t^k e^{t \ln \lambda_i} \qquad \ln \lambda_i = \ln|\lambda_i| + i \underbrace{\arg(\lambda_i)}_{\in [0, 2\pi)}$$

1. $\lambda_i \neq 0$: $\binom{t}{k}\lambda_i^{t-k} \sim t^k \lambda_i^t = t^k e^{t(\ln \lambda_i)}$ (**N.B.** $\ln(\cdot)$ = logaritmo naturale complesso)

Modi elementari

$$\left.\begin{matrix}\binom{t}{0}\lambda_i^t,\ \binom{t}{1}\lambda_i^{t-1},\ \binom{t}{2}\lambda_i^{t-2},\ \dots,\ \binom{t}{r_{ij}-1}\lambda_i^{t-r_{ij}+1} \\ \delta(t),\ \delta(t-1),\ \delta(t-2),\ \dots,\ \delta(t-r_{ij}+1)\end{matrix}\right. = \text{modi elementari del sistema}$$

1. $\lambda_i \neq 0$: $\binom{t}{k}\lambda_i^{t-k} \sim t^k \lambda_i^t = t^k e^{t(\ln \lambda_i)}$ (**N.B.** $\ln(\cdot)$ = logaritmo naturale complesso)

2. $\lambda_i = 0$: modi elementari si annullano dopo un numero finito di passi !

Non esiste una controparte modale a tempo continuo !!

Evoluzione libera

$$x(t+1) = Fx(t) + \cancel{Gu(t)}, \qquad x(0) = x_0$$

$$y(t) = Hx(t) + \cancel{Ju(t)}$$

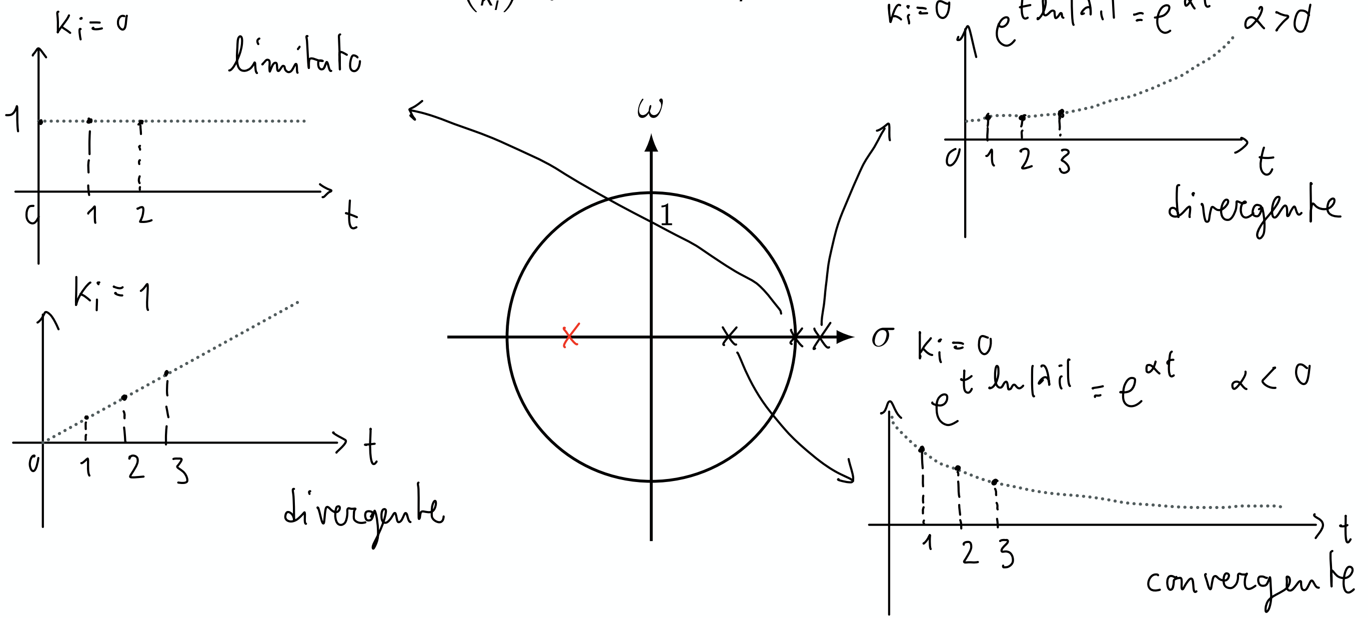
$$T^{-1} F_J^t \, T$$

$$y(t) = y_\ell(t) = HF^t x_0 = \sum_{i,j} t^j \lambda_i^t v_{ij} + \sum_j \delta(t-j) w_j$$

$$= \text{combinazione lineare dei modi elementari}$$

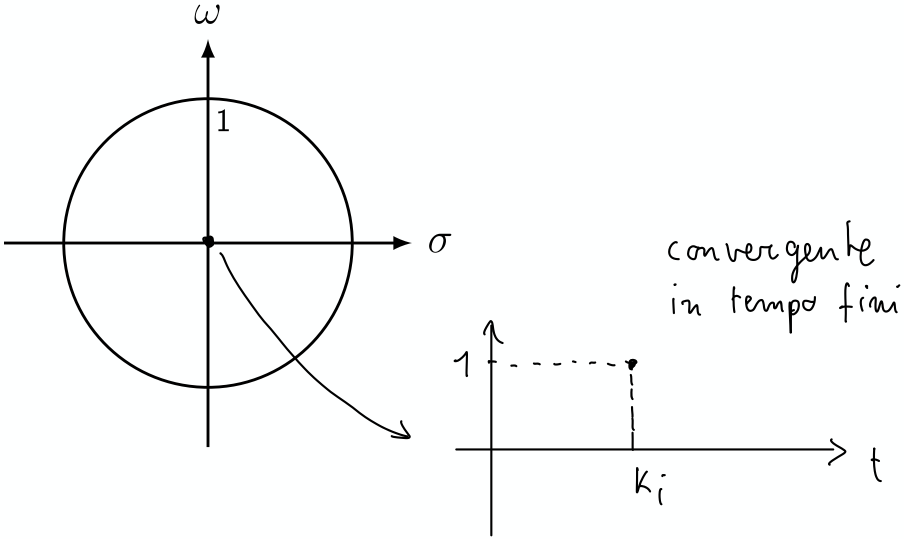
Carattere dei modi elementari

$$\lambda_i = \sigma_i + i\omega_i \in \mathbb{C}, \lambda_i \neq 0 : \binom{t}{k_i}\lambda_i^{t-k_i} \sim t^{k_i}\lambda_i^t = t^{k_i} e^{t(\ln\lambda_i)} = t^{k_i} e^{t(\ln|\lambda_i| + i\arg(\lambda_i))}$$



Carattere dei modi elementari

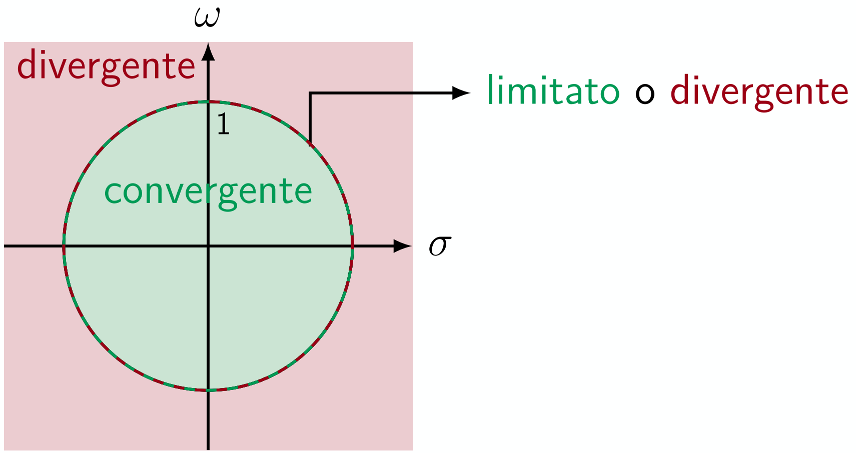
$$\lambda_i = 0\text{: } \delta(t - k_i)$$



convergente
in tempo finito

Carattere dei modi elementari

modo associato a $\lambda_i = \sigma_i + i\omega_i$



Comportamento asintotico

$$F \in \mathbb{R}^{n\times n} \text{ con autovalori } \{\lambda_i\}_{i=1}^{k}$$

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$$|\lambda_i| < 1, \forall i \qquad \Longleftrightarrow \qquad F^t \xrightarrow{t \to \infty} 0 \implies y(t) = HF^t x_0 \xrightarrow{t \to \infty} 0$$

$F^t = 0$ per t finito se $\lambda_i = 0$!!

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$$\begin{array}{c} |\lambda_i| \le 1,\ \forall i \text{ e} \\ \nu_i = g_i \text{ se } |\lambda_i| = 1 \end{array} \quad \iff \quad F^t \text{ limitata} \Rightarrow y(t) = HF^t x_0 \text{ limitata}$$

Comportamento asintotico

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$F^t = 0$ per t finito se $\lambda_i = 0$!!

$|\lambda_i| \leq 1, \ \forall i$ e $\nu_i = g_i$ se $|\lambda_i| = 1 \iff F^t$ limitata $\Rightarrow y(t) = HF^t x_0$ limitata $\quad \forall x_0, H$

$\exists\, \lambda_i$ tale che $|\lambda_i| > 1$ o $|\lambda_i| = 1$ e $\nu_i > g_i \iff F^t$ non limitata $\Rightarrow y(t) = HF^t x_0$?

$\downarrow$

dipende da x_0, H

In questa lezione

- ▷ Analisi modale ed evoluzione libera di un sistema lineare a t.d.
- ▷ Evoluzione complessiva di un sistema lineare a t.d.

Evoluzione complessiva (libera + forzata)

$$x(t+1) = Fx(t) + Gu(t), \qquad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = x_\ell(t) + x_f(t), \qquad x_\ell(t) = F^t x_0, \qquad x_f(t) \; ??$$

$$y(t) = y_\ell(t) + y_f(t), \qquad y_\ell(t) = HF^t x_0, \qquad y_f(t) \; ??$$

note

Evoluzione complessiva (libera + forzata)

$$x(t+1) = Fx(t) + Gu(t), \qquad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = \underbrace{F^t x_0}_{=x_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} F^{t-k-1} G u(k)}_{=x_f(t)}$$

$$y(t) = \underbrace{HF^t x_0}_{=y_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} HF^{t-k-1} G u(k) + Ju(t)}_{=y_f(t)}$$

$$w(t) = \text{ risposta impulsiva } = \begin{cases} J, & t = 0 \\ HF^{t} G, & t \geq 1 \end{cases}$$

$t-1$

Evoluzione complessiva (libera + forzata)

$$x(t+1) = Fx(t) + Gu(t), \qquad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = \underbrace{F^t x_0}_{=x_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} F^{t-k-1} Gu(k)}_{=x_f(t)} = \underbrace{F^t x_0}_{=x_\ell(t)} + \underbrace{\mathcal{R}_t u_t}_{=x_f(t)} \qquad u_t \triangleq \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(0) \end{bmatrix}$$

$$y(t) = \underbrace{HF^t x_0}_{=y_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} HF^{t-k-1} Gu(k) + Ju(t)}_{=y_f(t)} = \underbrace{HF^t x_0}_{=y_\ell(t)} + \underbrace{H\mathcal{R}_t u_t + Ju(t)}_{=y_f(t)}$$

$\mathcal{R}_t \triangleq \left[\ G \mid FG \mid F^2 G \mid \cdots \mid F^{t-1} G\ \right]$ = matrice di raggiungibilità in t passi

Evoluzione complessiva con Zeta

$$zX(z) - zx_0 = FX(z) + GU(z)$$

$$Y(z) = HX(z) + JU(z)$$

$$V(z) \triangleq \mathcal{Z}[v(t)] = \sum_{t=0}^{\infty} v(t) z^{-t}$$

note

Evoluzione complessiva con Zeta

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$$X(z) = \underbrace{z(zI - F)^{-1}x_0}_{=X_\ell(z)} + \underbrace{(zI - F)^{-1}GU(z)}_{=X_f(z)}$$

$$Y(z) = \underbrace{Hz(zI - F)^{-1}x_0}_{=Y_\ell(z)} + \underbrace{[H(zI - F)^{-1}G + J]U(z)}_{=Y_f(z)}$$

note

Equivalenze dominio temporale/Zeta

1. $W(z) = \mathcal{Z}[w(t)] = H(zI - F)^{-1}G + J =$ matrice di trasferimento

2. $\mathcal{Z}[F^t] = z(zI - F)^{-1} =$ metodo alternativo per calcolare F^t !!

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Calcolo di F^t tramite Jordan

4(i). $J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij}\times r_{ij}} \overset{\lambda_i \neq 0}{\Longrightarrow} J^t_{\lambda_i,j} = (\lambda_i I + N)^t, \quad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$

note

$J^t_{\lambda_{i,j}}$?

$J_{\lambda_{i,j}} = \lambda_i I + N$

$N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$ nilpotente con indice di nilpotenza r_{ij}

$$\binom{t}{k} = \frac{t!}{(t-k)!\,k!}$$

<u>Proprietà</u>: $A, B \in \mathbb{R}^{n\times n}$ con $AB = BA$ (A, B commutano)

$$(A+B)^t = \sum_{k=0}^{t} \binom{t}{k} A^{t-k} B^k \quad \text{(binomio di Newton)}$$

$$J^t_{\lambda_{i,j}} = (\lambda_i I + N)^t \underset{\lambda_i I,\ N \text{ commutano}}{=} \sum_{k=0}^{t} \binom{t}{k} (\lambda_i I)^{t-k} N^k \overset{t \geq r_{ij}}{=} \overbrace{\binom{t}{0}}^{1} \lambda_i^t + \overbrace{\binom{t}{1}}^{t} \lambda_i^{t-1} N + \binom{t}{2} \lambda_i^{t-2} N^2 + \cdots + \binom{t}{r_{ij}-1} \lambda_i^{t-(r_{ij}-1)} N^{r_{ij}-1}$$

$$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}, \ \ldots, \quad N^{r_{ij}-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad \underset{k \geq r_{ij}}{N^k = 0}$$

Evoluzione complessiva (libera + forzata)

$$x(t+1) = Fx(t) + Gu(t), \qquad x(0) = x_0$$
$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = x_\ell(t) + x_f(t), \qquad x_\ell(t) = F^t x_0, \qquad x_f(t)\ ??$$
$$y(t) = y_\ell(t) + y_f(t), \qquad y_\ell(t) = HF^t x_0, \qquad y_f(t)\ ??$$

note

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Ju(t) \end{cases} \qquad x(0) = x_0$$

$$x(1) = Fx(0) + Gu(0)$$

$$x(2) = Fx(1) + Gu(1) = F(Fx(0) + Gu(0)) + Gu(1) = F^2 x(0) + FGu(0) + Gu(1)$$

$$x(3) = Fx(2) + Gu(2) = F(Fx(1) + Gu(1)) + Gu(2)$$
$$= F(F^2 x(0) + FGu(0) + Gu(1)) + Gu(2)$$
$$= F^3 x(0) + F^2 G u(0) + FGu(1) + Gu(2)$$

$\vdots$

$$x(t) = F^t x(0) + \underbrace{\sum_{k=0}^{t-1} F^{t-1-k} G u(k)}$$

$$\underbrace{\begin{bmatrix} G & FG & F^2G & \cdots & F^{t-1}G \end{bmatrix}} \underbrace{\begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(0) \end{bmatrix}}_{u_t \in \mathbb{R}^{mt}}$$

R_t = matrice di raggiungibilità in t passi

$$R_t \in \mathbb{R}^{n \times (mt)}$$

$$x(t) = \underbrace{F^t x_0}_{\text{evoluzione libera}} + \underbrace{R_t u_t}_{\text{evoluzione forzata}}$$

$$y(t) = H x(t) + J u(t)$$

$$y(t) = H F^{t} x_0 + \sum_{k=0}^{t-1} H F^{t-1-k} G u(k) + J u(t)$$

$$\overset{!}{=} \underbrace{H F^{t} x_0}_{\text{evoluzione libera}} + \underbrace{H R_t u_t + J u(t)}_{\text{evoluzione forzata } y_f(t)}$$

$$y_f(t) = [w \underset{\substack{\uparrow \\ \text{convoluzione discreta}}}{*} u](t) = \sum_{k=-\infty}^{\infty} w(t-k)\, u(k) = \sum_{k=0}^{t} w(t-k)\, u(k)$$

$u(k) = 0 \quad k<0$

$w(k) = 0 \quad k<0$

$$w(t) = \begin{cases} J & t = 0 \\ H F^{t-1} G & t \geq 1 \end{cases} = \text{matrice delle risposte impulsive}$$

Evoluzione complessiva con Zeta

$$zX(z) - zx_0 = FX(z) + GU(z)$$
$$Y(z) = HX(z) + JU(z)$$

$$V(z) \triangleq \mathcal{Z}[v(t)] = \sum_{t=0}^{\infty} v(t) z^{-t}$$

$$\mathcal{Z}[v(t+1)] = z\,\mathcal{Z}[v(t)] - z\,v(0) = z\,V(z) - z\,v(0)$$

$$\begin{cases} x(t+1) = F x(t) + G u(t) \\ y(t) = H x(t) + J u(t) \end{cases} \xrightarrow{\mathcal{Z}} \begin{cases} \mathcal{Z}[x(t+1)] = F\,\mathcal{Z}[x(t)] + G\,\mathcal{Z}[u(t)] \\ \mathcal{Z}[y(t)] = H\,\mathcal{Z}[x(t)] + J\,\mathcal{Z}[u(t)] \end{cases}$$

$$\xrightarrow{\mathcal{Z}} \begin{cases} zX(z) - zx_0 = F X(z) + G U(z) \\ Y(z) = H X(z) + J U(z) \end{cases}$$

$$\longrightarrow \begin{cases} X(z) = (zI-F)^{-1} z x_0 + (zI-F)^{-1} G U(z) \\ Y(z) = H(zI-F)^{-1} z x_0 + H(zI-F)^{-1} G U(z) + J U(z) \end{cases}$$

$$X(z) = \underbrace{(zI-F)^{-1} z x_0}_{X_\ell(z)} + \underbrace{(zI-F)^{-1} G\, U(z)}_{X_f(z)}$$

$$Y(z) = \underbrace{H(zI-F)^{-1} z x_0}_{Y_\ell(z)} + \underbrace{\left[H(zI-F)^{-1} G + J\right]}_{W(z)} U(z)$$

$W(z)$ = matrice di trasferimento $= \mathcal{Z}[w(t)]$

$$X_\ell(z) = (zI - F)^{-1} z\, x_0 = \mathcal{Z}[x_\ell(t)] = \mathcal{Z}[F^t x_0] = \mathcal{Z}[F^t]\, x_0$$

$$\mathcal{Z}[F^t] = z(zI - F)^{-1}$$

$$\Big\downarrow \mathcal{Z}^{-1}$$

$$F^t$$