Esercizio 1

$$x(t+1) = Fx(t), \qquad F = \begin{bmatrix} 1 & 1 & \alpha - \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix}, \qquad \alpha \in \mathbb{R}$$

$$y(t) = Hx(t), \qquad H = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

1. Osservabilità, ricostruibilità e rivelabilità al variare di $\alpha \in \mathbb{R}$?
2. Spazi non osservabili $X_{NO}(t)$, $t \geq 1$, al variare di $\alpha \in \mathbb{R}$?

$$F = \begin{bmatrix} 1 & 1 & \alpha - 1/2 \\ 0 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

$$H = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

1) Osservabilità, ricostruibilità, rivelabilità per $\alpha \in \mathbb{R}$.

Autovalori di F: $1, \alpha$

Caso $\alpha = 1$: $\lambda_1 = 1$, $\nu_1 = 3$

Test PBH di osservabilità:

$$PBH(\lambda_1) = \begin{bmatrix} \lambda_1 I - F \\ H \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1/2 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathrm{rank}(PBH(\lambda_1)) = 3$$

$\Rightarrow \Sigma$ osservabile

$\Rightarrow \Sigma$ ricostr., rivelabile

Caso $\alpha \neq 1$: $\lambda_1 = 1$, $\nu_1 = 2$, $\lambda_2 = \alpha$, $\nu_1 = 1$

$$PBH(\lambda_1) = \begin{bmatrix} \lambda_1 I - F \\ H \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1/2 - \alpha \\ 0 & 0 & 0 \\ 0 & -1 & 1 - \alpha \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathrm{rank}(PBH(\lambda_1)) = 3 \quad \forall \alpha$$

$$PBH(\lambda_2) = \begin{bmatrix} \lambda_2 I - F \\ H \end{bmatrix} = \begin{bmatrix} \alpha - 1 & -1 & 1/2 - \alpha \\ 0 & \alpha - 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathrm{rank}(PBH(\lambda_2)) = \begin{cases} 2 & \text{se } \alpha = 1/2 \\ 3 & \text{se } \alpha \neq 1/2 \end{cases}$$

Σ osservabile se $\alpha \neq \frac{1}{2}$

Σ ricostruibile se $\alpha \neq \frac{1}{2}$

Σ rivelabile $\forall \alpha \in \mathbb{R}$ (perché: 1) Σ oss $\Rightarrow$ Σ rivelabile ($\alpha \neq 1/2$)

2) $\alpha = 1/2$ matrice PBH(λ_2) cade di rango, ma in questo caso $\lambda_2 = 1/2$ e $|\lambda_2| < 1$)

2) Spazi non osservabili $X_{NO}(t)$, $t \geq 1$

$$X_{NO}(1) = \ker \mathcal{O}_1 = \ker H = \ker \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \left\{ x \in \mathbb{R}^3 : Hx = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 = 0 \right\} = \left\{ \begin{bmatrix} \beta \\ -\beta \\ \gamma \end{bmatrix}, \beta, \gamma \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$X_{NO}(2) = \ker \mathcal{O}_2 = \ker \begin{bmatrix} H \\ HF \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix}$$

$$= \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$X_{no}(2) = \ker \mathcal{O}_2 = \ker \begin{bmatrix} H \\ HF \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix}$$

$$= \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha - 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 + (\alpha - 1/2) x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \begin{cases} \left\{ \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}, \gamma \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} & \alpha = 1/2 \\ \left\{ \begin{bmatrix} (\alpha - 1/2)\gamma \\ (1/2 - \alpha)\gamma \\ \gamma \end{bmatrix}, \gamma \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} \alpha - 1/2 \\ 1/2 - \alpha \\ 1 \end{bmatrix} \right\} & \alpha \neq 1/2 \end{cases}$$

$X_{NO}(3)$:

$\alpha = 1/2$: $X_{NO}(3) = \mathrm{Ker} \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \mathrm{Ker} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\begin{cases} x_1 = -x_2 \\ x_1 = -2x_2 \\ x_1 = -3x_2 \end{cases} \quad \begin{cases} x_1 = 0 \\ -x_2 = -2x_2 \Rightarrow x_2 = 0 \\ \quad " \end{cases}$

$= \left\{ \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix}, y \in \mathbb{R} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\alpha \neq 1/2$: $X_{NO}(3) = \{0\}$ perché Σ osservabile

$X_{NO}(t) = X_{NO}(3) \quad \forall t \geq 3$