Esercizio 1 [riadattato da Es. 2 tema d'esame 28 Gennaio 2010]

$$x(t+1) = Fx(t) + Gu(t), \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & \alpha & \alpha \end{bmatrix}, \; G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; \alpha \in \mathbb{R}$$

1. Raggiungibilità e controllabilità al variare di $\alpha \in \mathbb{R}$?
2. Spazio raggiungibile $X_R(t)$ e controllabile $X_C(t)$ al variare di $t \geq 1$ e $\alpha \in \mathbb{R}$?

$$F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & \alpha & \alpha \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

1)+2): Calcoliamo gli spazi ragg. e contr. e poi verifichiamo ragg. e contr. completa

N.B.: i) Per il primo i t.c. $X_R(i) = X_R(i+1) \implies X_R(j) = X_R(i) \; \forall j \geq i$

ii) Se $X_R(t) = \mathbb{R}^n \implies X_C(t) = \mathbb{R}^n$

Calcolo spazi ragg. e raggiungibilità:

$$X_R(1) = \operatorname{im} R_1 = \operatorname{im} G = \operatorname{im} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$X_R(2) = \operatorname{im} R_2 = \operatorname{im} [G \;\; FG] = \operatorname{im} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} = \begin{cases} \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} & \alpha = 0 \\ \mathbb{R}^3 & \alpha \neq 0 \end{cases}$$

Quindi:

- $\alpha = 0$: per i) $X_R(1) = X_R(2) \implies X_R(t) = X_R(1) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \; \forall t \geq 1$

  $\implies \Sigma$ non ragg.

- $\alpha \neq 0$: $X_R(1) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

  $X_R(t) = \mathbb{R}^3 \quad t \geq 2 \implies \Sigma$ ragg. (in 2 passi)

Calcolo spazi controllabili e controllabilità:

$$X_c(1) = \{x \in \mathbb{R}^3 : Fx \in X_R(1) = \underbrace{\operatorname{im} R_1 = \operatorname{im} G}_{\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}}\}$$

$$= \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 1\\0 & \alpha & \alpha\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\beta\\\gamma\\0\end{bmatrix},\ \beta,\gamma \in \mathbb{R}\right\}$$

$$= \left\{\quad '' \quad : \begin{bmatrix}x_1\\-x_1+x_2+x_3\\\alpha(x_2+x_3)\end{bmatrix} = \begin{bmatrix}\beta\\\gamma\\0\end{bmatrix},\ \beta,\gamma \in \mathbb{R}\right\}$$

$$= \begin{cases} \left\{\begin{bmatrix}\beta\\\delta\\\gamma+\beta-\delta\end{bmatrix},\ \beta,\gamma,\delta \in \mathbb{R}\right\} = \mathbb{R}^3 & \alpha = 0 \\ \left\{\begin{bmatrix}\beta\\\delta\\-\delta\end{bmatrix},\ \beta,\delta \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\} & \alpha \neq 0 \end{cases}$$

Quindi:

- $\alpha = 0$: $X_c(t) = \mathbb{R}^3 \quad \forall t \geq 1 \implies \Sigma$ controllabile (in 1 passo)
- $\alpha \neq 0$: $X_c(1) = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\} \ [\neq X_R(1)]$

per ii) $X_c(t) = \mathbb{R}^3 \quad \forall t \geq 2 \implies \Sigma$ controllabile (in 2 passi)

Esercizio 2

$$x(t+1) = Fx(t) + Gu(t), \qquad F = \begin{bmatrix} 2 & 4 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1. Forma di Kalman di raggiungibilità?
2. Ingresso che porta nel minor numero possibile di passi lo stato da

$$\text{da} \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{a} \quad x^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ?$$

note

$$F = \begin{bmatrix} 2 & 4 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1) Forma di Kalman:

$$X_R = \operatorname{im} R = \operatorname{im} [\, G \;\; FG \;\; F^2G \,] = \operatorname{im} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \operatorname{span} \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{v_2} \right\}$$

$$T = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & \tilde{v}_1 \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \end{array} \qquad T^{-1} = T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad (T \text{ matrice di permutazione})$$

$$F_K = T^{-1}FT = TFT = T \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix} = \left[\begin{array}{cc:c} 2 & 0 & 4 \\ 1 & 3 & 2 \\ \hdashline 0 & 0 & -1 \end{array}\right] = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}$$

$$G_K = T^{-1}G = TG = \left[\begin{array}{c} 1 \\ 0 \\ \hdashline 0 \end{array}\right] = \left[\begin{array}{c} G_1 \\ \hdashline 0 \end{array}\right]$$

2) Calcolare $u(t)$ t.c. $x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x(\bar{t}) = x^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ con $\bar{t}$ più piccolo possibile

- Esistenza di $u(t)$:

$$x^* \in X_R \,? \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{sì}$$

- Calcolo di u:

$t=1: \quad \underset{\begin{bmatrix}0\\0\\1\end{bmatrix}}{\overset{\shortparallel}{x^*}} \in X_R(1) = \operatorname{im} G = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$? No

$t=2: \quad x^* \in X_R(2) = \operatorname{im}[G \ \ FG] = X_R = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$? Sì

$$u_2 = \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$x^* = x(2) = R_2 u_2 = [G \ \ FG]\, u_2$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}u(1)\\u(0)\end{bmatrix} \qquad \begin{cases} u(1)+2u(0)=0 \\ 0=0 \\ 1 = u(0) \end{cases} \quad \begin{cases} u(1) = -2 \\ u(0) = 1 \end{cases}$$