

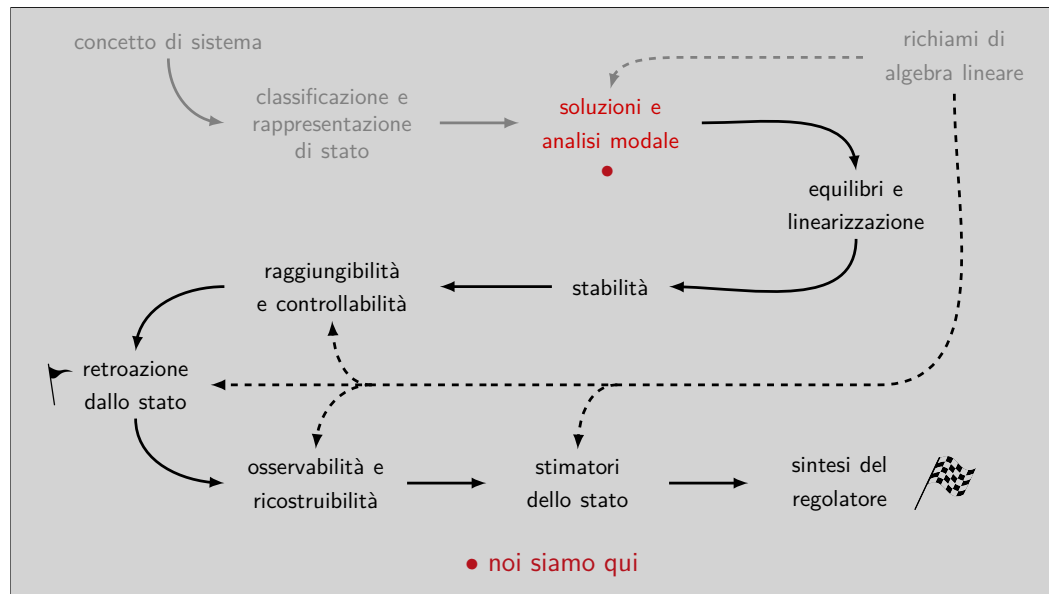
# Teoria dei Sistemi e Controllo Ottimo e Adattativo (C. I.)

## Teoria dei Sistemi (Mod. A)

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Lez. 7: Modi di un sistema lineare, risposta libera e forzata  
(tempo discreto)

Corso di Laurea Magistrale in Ingegneria Meccatronica  
A.A. 2019-2020



## Nella scorsa lezione

- ▷ Modi elementari e evoluzione libera di un sistema lineare a tempo continuo
  - ▷ Analisi modale di un sistema lineare a tempo continuo
    - ▷ Evoluzione forzata di un sistema lineare a tempo continuo
      - ▷ Matrice di trasferimento e equivalenza algebrica
        - ▷ Addendum: calcolo di  $e^{Ft}$  tramite Laplace

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## In questa lezione

- ▷ Modi elementari e evoluzione libera di un sistema lineare **a tempo discreto**
  - ▷ Analisi modale di un sistema lineare **a tempo discreto**
    - ▷ Evoluzione forzata di un sistema lineare **a tempo discreto**

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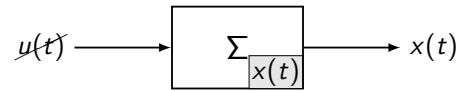
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## Soluzioni di un sistema lineare autonomo?



Caso vettoriale  $x(t) = y(t) \in \mathbb{R}^n$

$$x(t+1) = Fx(t), \quad x(0) = x_0$$

$$x(t) = F^t x_0$$

## Usiamo Jordan!

$$1. F = TF_J T^{-1} \implies F^t = TF_J^t T^{-1}$$

$$2. F_J = \begin{bmatrix} J_{\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_k} \end{bmatrix} \implies F_J^t = \begin{bmatrix} J_{\lambda_1}^t & 0 & \cdots & 0 \\ 0 & J_{\lambda_2}^t & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_k}^t \end{bmatrix}$$

$$3. J_{\lambda_i} = \begin{bmatrix} J_{\lambda_i,1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_i,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_i,\ell_i} \end{bmatrix} \implies J_{\lambda_i}^t = \begin{bmatrix} J_{\lambda_i,1}^t & 0 & \cdots & 0 \\ 0 & J_{\lambda_i,2}^t & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_i,\ell_i}^t \end{bmatrix}$$

## Usiamo Jordan!

$$4. J_{\lambda_i, j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij} \times r_{ij}} \xRightarrow{\lambda_i \neq 0} J_{\lambda_i, j}^t = (\lambda_i I + N)^t, \quad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\Rightarrow J_{\lambda_i, j}^t = \begin{bmatrix} \binom{t}{0} \lambda_i^t & \binom{t}{1} \lambda_i^{t-1} & \binom{t}{2} \lambda_i^{t-2} & \cdots & \binom{t}{r_{ij}-1} \lambda_i^{t-r_{ij}+1} \\ 0 & \binom{t}{0} \lambda_i^t & \binom{t}{1} \lambda_i^{t-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{t}{2} \lambda_i^{t-2} \\ \vdots & \ddots & \ddots & \ddots & \binom{t}{1} \lambda_i^{t-1} \\ 0 & \cdots & \cdots & 0 & \binom{t}{0} \lambda_i^t \end{bmatrix}$$

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## Usiamo Jordan!

$$4. J_{\lambda_i, j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{r_{ij} \times r_{ij}} \xRightarrow{\lambda_i = 0} J_{\lambda_i, j}^t = N^t, \quad N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\Rightarrow J_{\lambda_i, j}^t = \begin{bmatrix} \delta(t) & \delta(t-1) & \delta(t-2) & \cdots & \delta(t-r_{ij}+1) \\ 0 & \delta(t) & \delta(t-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \delta(t-2) \\ \vdots & \ddots & \ddots & \ddots & \delta(t-1) \\ 0 & \cdots & \cdots & 0 & \delta(t) \end{bmatrix}$$

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## Modi elementari

$$\binom{t}{0}\lambda_i^t, \binom{t}{1}\lambda_i^{t-1}, \binom{t}{2}\lambda_i^{t-2}, \dots, \binom{t}{r_{ij}-1}\lambda_i^{t-r_{ij}+1} \\ \delta(t), \delta(t-1), \delta(t-2), \dots, \delta(t-r_{ij}-1) = \text{Modi elementari del sistema}$$

1.  $\lambda_i \neq 0$ :  $\binom{t}{k}\lambda_i^{t-k} \sim t^k \lambda_i^t = t^k e^{t(\ln \lambda_i)}$  ( $\ln(\cdot)$  = logaritmo naturale complesso)

2.  $\lambda_i = 0$ : modo elementari si annullano dopo un numero finito di passi !

Non esiste una controparte modale a tempo continuo !!

## Evoluzione libera

$$x(t+1) = Fx(t) + Gu(t), \quad x(0) = x_0$$

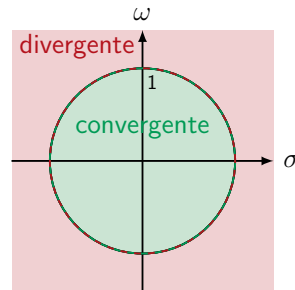
$$y(t) = Hx(t) + Ju(t)$$

$$y(t) = y_\ell(t) = HF^t x_0 = \sum_{i,j} t^j \lambda_i^t v_{ij}$$

= combinazione lineare dei modi elementari

## Carattere dei modi elementari

modo associato a  $\lambda_i = \sigma_i + i\omega_i$



## Comportamento asintotico

$F \in \mathbb{R}^{n \times n}$  con autovalori  $\{\lambda_i\}_{i=1}^k$

$$|\lambda_i| < 1, \forall i \quad \Rightarrow \quad F^t \xrightarrow{t \rightarrow \infty} 0 \Rightarrow y(t) = HF^t x_0 \xrightarrow{t \rightarrow \infty} 0$$

$F^t = 0$  per  $t$  finito se  $\lambda_i = 0$  !

$$|\lambda_i| \leq 1, \forall i \text{ e } \nu_i = g_i \text{ se } |\lambda_i| = 1 \quad \Rightarrow \quad F^t \text{ limitata} \Rightarrow y(t) = HF^t x_0 \text{ limitata}$$

$$\exists \lambda_i \text{ tale che } |\lambda_i| > 1 \text{ o } |\lambda_i| = 1 \text{ e } \nu_i > g_i \quad \Rightarrow \quad F^t \text{ non limitata} \Rightarrow y(t) = HF^t x_0 ?$$

## Evoluzione forzata

$$x(t+1) = Fx(t) + Gu(t), \quad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = x_\ell(t) + x_f(t), \quad x_\ell(t) = F^t x_0, \quad x_f(t) ??$$

$$y(t) = y_\ell(t) + y_f(t), \quad y_\ell(t) = HF^t x_0, \quad y_f(t) ??$$

## Evoluzione forzata

$$x(t+1) = Fx(t) + Gu(t), \quad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = \underbrace{F^t x_0}_{=x_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} F^{t-k-1} Gu(k)}_{=x_f(t)}$$

$$y(t) = \underbrace{HF^t x_0}_{=y_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} HF^{t-k-1} Gu(k)}_{=y_f(t)} + Ju(t)$$

$$w(t) = HF^t G + J\delta(t) = \text{risposta impulsiva}$$

## Evoluzione forzata

$$x(t+1) = Fx(t) + Gu(t), \quad x(0) = x_0$$

$$y(t) = Hx(t) + Ju(t)$$

$$x(t) = \underbrace{F^t x_0}_{=x_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} F^{t-k-1} Gu(k)}_{=x_r(t)} = \underbrace{F^t x_0}_{=x_\ell(t)} + \underbrace{\mathcal{R}_t u_t}_{=x_r(t)} \quad u_t \triangleq \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(0) \end{bmatrix}$$

$$y(t) = \underbrace{HF^t x_0}_{=y_\ell(t)} + \underbrace{\sum_{k=0}^{t-1} HF^{t-k-1} Gu(k)}_{=y_r(t)} + Ju(t) = \underbrace{HF^t x_0}_{=y_\ell(t)} + \underbrace{H\mathcal{R}_t u_t + Ju(t)}_{=y_r(t)}$$

$$\mathcal{R}_t \triangleq \begin{bmatrix} G & FG & F^2G & \dots & F^{t-1}G \end{bmatrix} = \text{matrice di raggiungibilità in } t \text{ passi}$$

## Evoluzione forzata (con trasformata Zeta)

$$zX(z) - zx_0 = FX(z) + GU(z)$$

$$Y(z) = HX(z) + JU(z)$$

$$V(z) \triangleq \mathcal{Z}[v(t)] = \sum_{t=0}^{\infty} v(t)z^{-t}$$

$$X(z) = \underbrace{z(zI - F)^{-1}x_0}_{=X_\ell(z)} + \underbrace{(zI - F)^{-1}G}_{=X_r(z)}$$

$$Y(z) = \underbrace{Hz(zI - F)^{-1}x_0}_{=Y_\ell(z)} + \underbrace{[H(zI - F)^{-1}G + J]U(z)}_{=Y_r(z)}$$



## Equivalenze dominio temporale/Zeta

1.  $W(z) = \mathcal{Z}[w(t)] = H(zI - F)^{-1}G + J =$  matrice di trasferimento

2.  $\mathcal{Z}[F^t] = z(zI - F)^{-1} =$  metodo alternativo per calcolare  $F^t$  !!

## Struttura della matrice di trasferimento

$T \in \mathbb{R}^{n \times n} =$  base di Jordan

$$(F, G, H, J) \xrightarrow{z=Tx} (F_J = T^{-1}FT, G_J = T^{-1}G, H_J = HT, J_J = J)$$

$$W(z) = W_J(z) = H_J(zI - F_J)^{-1}G_J + J_J$$

## Struttura della matrice di trasferimento

$$F_J = \begin{bmatrix} J_{\lambda_1,1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_1,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_k,\ell_k} \end{bmatrix}, \quad G_J = \begin{bmatrix} G_{\lambda_1,1} \\ G_{\lambda_1,2} \\ \vdots \\ G_{\lambda_k,\ell_k} \end{bmatrix}, \quad H_J = \begin{bmatrix} H_{\lambda_1,1} & H_{\lambda_1,2} & \cdots & H_{\lambda_k,\ell_k} \end{bmatrix}$$

## Struttura della matrice di trasferimento

$$F_J = \begin{bmatrix} J_{\lambda_1,1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_1,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_k,\ell_k} \end{bmatrix}, \quad G_J = \begin{bmatrix} G_{\lambda_1,1} \\ G_{\lambda_1,2} \\ \vdots \\ G_{\lambda_k,\ell_k} \end{bmatrix}, \quad H_J = \begin{bmatrix} H_{\lambda_1,1} & H_{\lambda_1,2} & \cdots & H_{\lambda_k,\ell_k} \end{bmatrix}$$

$$\begin{aligned} W(z) &= H_{\lambda_1,1}(zI - J_{\lambda_1,1})^{-1}G_{\lambda_1,1} + H_{\lambda_1,2}(zI - J_{\lambda_1,2})^{-1}G_{\lambda_1,2} + \cdots + H_{\lambda_k,\ell_k}(zI - J_{\lambda_k,\ell_k})^{-1}G_{\lambda_k,\ell_k} + J \\ &= W_{\lambda_1,1}(z) + W_{\lambda_1,2}(z) + \cdots + W_{\lambda_k,\ell_k}(z) + J \end{aligned}$$

## Struttura della matrice di trasferimento

$$\text{miniblocco } J_{\lambda_i,j} \in \mathbb{R}^{r_{ij} \times r_{ij}} \implies W_{\lambda_i,j}(z) = \frac{A_1}{z - \lambda_i} + \frac{A_2}{(z - \lambda_i)^2} + \dots + \frac{A_{r_{ij}}}{(z - \lambda_i)^{r_{ij}}}$$

$$y_f(t) = \mathcal{Z}^{-1} \left[ \sum_{i,j} W_{\lambda_i,j}(z) U(z) + JU(z) \right]$$