LAB3: INTERPOLATION
DUE: SUNDAY, NOVEMBER 15

1. Chebyshev polynomials

Chebyshev polynomials have many important applications in (numerical) analysis. The Chebyshev polynomial T_n of degree n on $[-1,1]$ is defined by

$$T_n(x) = \cos(n\cos^{-1}(x)), \quad x \in [-1,1], n = 0,1,2,....$$[1]

At the first glance T_n does not look like a polynomial, but, in fact, it is. For $n = 0$, we have $T_0(x) = 1$ and for $n = 1$ we have $T_1(x) = x$. If one uses the trigonometric identity[2]

$$\cos[(n+1)y] + \cos[(n-1)y] = 2\cos(y)\cos(ny)$$

and then uses the change of variables $y = \cos^{-1}(x)$ one sees that

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \quad n = 1,2,...$$

and hence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,... \tag{1}$$

As T_0 and T_1 are polynomials of degree 0 and 1, respectively, it follows by induction and (1) that T_n is a polynomial of degree n with leading term $2^{n-1}x^n$. It follows from their definition that T_n maps $[-1,1]$ to itself and that T_n is and even function for n even and T_n is an odd function for n odd (this latter also follows from (1) and induction). It is also straightforward to see that, for $n \geq 1$, the zeros of T_n are given by

$$x_j = \cos\left[\frac{(2j-1)\pi}{2n}\right], \quad j = 1,...,n.$$

Next we summarize some important features of Chebyshev polynomials:

(a) The polynomial p_n of degree n defined by

$$p_n(x) = x^{n+1} - 2^{-n}T_{n+1}(x), \quad x \in [-1,1],$$

is the minmax approximation of degree n to the function $f(x) = x^{n+1}$ on $[-1,1]$.

(b) Let $n \geq 0$. Among all the polynomials of degree $n+1$ with leading coefficient 1 the polynomials $\pm 2^{-n}T_{n+1}$ have the smallest ∞-norm on $[-1,1]$.

(c) If p_n denotes the Lagrange interpolation polynomial of degree n of a function $f \in C^1[-1,1]$, with interpolation points given by the zeros of T_{n+1}, then

$$\|p_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

That is, if one chooses the interpolation points cleverly for Lagrange interpolation, then Runge's phenomenon does not occur. This is true for more general intervals $[a,b]$: one first linearly maps the zeros x_j, $j = 1,2,...,n+1$,

[1] Some students would be more familiar with the notation $\arccos(x)$ instead of $\cos^{-1}(x)$.

[2] Prove it!

of T_{n+1} from $(-1,1) \to (a,b)$ using the mapping $t \to \frac{1}{2}(b-a)t + \frac{1}{2}(b+a)$ resulting in points ξ_j, $j = 1, 2, ..., n+1$. Then, for the Lagrange interpolation polynomial p_n of degree n with interpolation points $\xi_1, ..., \xi_{n+1}$ of a function $f \in C^1[a,b]$ one has

$$\|p_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

Exercise 1 (Runge's phenomenon)**.**
(a) For the function $f(x) = \frac{1}{1+x^2}$ on $[-5,5]$, consider the Lagrange interpolation polynomial p_n of degree $n = 2, 4, 6, ..., 24$, with $n+1$ equidistant interpolation points (always choose the first and the last interpolation points to be the endpoints of $[-5,5]$). Calculate, and put in a table, the values of $\|f - p_n\|_\infty$, $n = 2, 4, 6, ..., 24$.

(b) For the function $f(x) = \frac{1}{1+x^2}$ on $[-5,5]$, consider the Lagrange interpolation polynomial p_n of degree $n = 2, 4, 6, ..., 24$, but with interpolation points described above at item (c). Calculate, and put in a table, the values of $\|f - p_n\|_\infty$, $n = 2, 4, 6, ..., 24$. Compare the results with the equidistant case from part (a). What do you observe?

2. Spline interpolation

Unlike Lagrange and Hermite interpolation, spline interpolation is rather local in nature. Given an interval $[a,b]$, a continuous function $f : [a,b] \to \mathbb{R}$ and set of knots $K = \{a = x_0 < x_1, ... < x_m = b\}$ we look for a function s that interpolates f at the knots, where p is a polynomial (of typically low degree) on each interval $[x_{i-1}, x_i]$, $i = 1, .., m$ and has a certain number of continuous derivatives. Thus, in general, s is not a global polynomial on $[a,b]$.

2.1. Linear interpolating splines.

Given an interval $[a,b]$, a continuous function $f : [a,b] \to \mathbb{R}$ and set of knots $K = \{a = x_0 < x_1, ... < x_m = b\}$ the *linear interpolating spline* s_L of f at the knots x_i, $i = 0, 1, ..., m$ is given by

$$s_L(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_1), \quad x \in [x_{i-1}, x_i],\ i = 1, ..., m.$$

A linear spline can be represented as

$$s_L(x) = \sum_{k=0}^{m} \phi_k(x) f(x_i),$$

where ϕ_k, $k = 0, 1, ..., m$ are basis functions. The function ϕ_k is defined as the unique continuous function, that is linear (that is, affine) on each interval $[x_{i-1}, x_i]$, $i = 1, .., m$, and $\phi_k(x_i) = \delta_{ik}$[3], $i, k = 0, 1, ..., m$.

2.2. Cubic interpolating splines.

Given an interval $[a,b]$, a continuous function $f : [a,b] \to \mathbb{R}$ and set of knots $K = \{a = x_0 < x_1, ... < x_m = b\}$ we consider the set $\mathcal{S}$ of all functions $s : [a,b] \to \mathbb{R}$ such that

- $s \in C^2[a,b]$,
- $s(x_i) = f(x_i)$, $i = 0, 1, ..., m$,
- s is a cubic polynomial on each interval $[x_{i-1}, x_i]$, $i = 1, .., m$.

[3]The symbol δ_{ik} denotes the Kronecker delta.

An element of $\mathcal{S}$ is called an *interpolating cubic spline* for f. Counting the number of degrees of freedom one easily sees that there are more that one interpolating cubic spline for f. The interpolating cubic spline $s_2 \in \mathcal{S}$ for f which, in addition, satisfies that $s_2''(x_0) = s_2''(x_m) = 0$ is called the *natural cubic spline*. It exists and it is unique. There is no explicit formula for s_2, however, one may write a uniquely solvable system of linear equations for the coefficients of s_2. Let $h_i = x_i - x_{i-1}$, $i = 1, ..., m$. The spline s_2 has the form

$$s_2(x) = \frac{(x_i - x)^3}{6h_i}\sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h_i}\sigma_i + \alpha_i(x - x_{i-1}) + \beta_i(x_i - x),$$
$$x \in [x_{i-1}, x_i],\ i = 1, ..., m.$$

Here

$$\alpha_i = \frac{1}{h_i} f(x_i) - \frac{1}{6}\sigma_i h_i,\ i = 1, ..., m$$
$$\beta_i = \frac{1}{h_i} f(x_{i-1}) - \frac{1}{6}\sigma_{i-1} h_i,\ i = 1, ..., m,$$

while σ_i can be found solving the linear system of equations

$$h_i\sigma_{i-1} + 2(h_{i+1} + h_i)\sigma_i + h_{i+1}\sigma_{i+1} = 6\left(\frac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \frac{f(x_i) - f(x_{i-1})}{h_i}\right)$$
$$i = 1, ..., m-1, \quad (2)$$

together with $\sigma_0 = \sigma_m = 0$.

2.3. **Hermite cubic splines.** Given an interval $[a, b]$, a function $f \in C^1[a, b]$ and set of knots $K = \{a = x_0 < x_1, ... < x_m = b\}$ the function $s \in C^1[a, b]$ which is the Hermite interpolation polynomial of degree 3 of f on each interval $[x_{i-1}, x_i]$, $i = 1, .., m$, is called the *Hermite cubic spline*.

Exercise 2. Consider the function

$$f(x, t) = \sin(5\pi x)\cos(10\pi t) + 2\sin(7\pi x)\cos(14\pi t)$$

for $(x, t) \in [0, 1] \times [0, 1]$. Consider the set of equidistant knots $x_i = i/50$, $i = 0, 1, ..., 50$, and plot the (a) linear, (b) natural cubic, and (c) Hermite cubic spline interpolation polynomials for the function $x \to f(x, t)$ for each $t = t_j = j/50$, $j = 0, 1, ..., 50$, as a movie in t. For determining the natural cubic spline interpolation, solve the linear system (2) by standard Gaussian elimination.