a)

I. QR decomposition:

- Gram-Schmidt
- Householder:

$$A : \begin{bmatrix} \overset{v_1}{\times} & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} \|v_1\| & \times & \times \\ 0 & \times & \times \\ 0 & \underset{v_2}{\times} & \times \end{bmatrix} \xrightarrow{Q_2} \begin{bmatrix} \|v_1\| & \times & \times \\ 0 & \|v_2\| & \times \\ 0 & 0 & \underset{v_3}{\times} \end{bmatrix} \xrightarrow{Q_3}$$

$$\xrightarrow{Q_3} \underbrace{\begin{bmatrix} \|v_1\| & \times & \times \\ 0 & \|v_2\| & \times \\ 0 & 0 & \|v_3\| \end{bmatrix}}_{R} \Longrightarrow \underbrace{Q_3 Q_2 Q_1}_{Q} A = R \qquad \left(Q^{-1} = Q^T\right)$$

$$\boxed{A = Q^T R}$$

Calculation of Q_k:

$$Q_k = \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix} \qquad F = I - 2\frac{vv^T}{v^Tv}$$

$$v = \begin{bmatrix} \\ \\ \end{bmatrix} \qquad v^T = [\qquad]$$

This v is modified this way:

$$v := \operatorname{sign}(v[1])\,\|v\| \cdot e_1 + v$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

II.

1, $A^{(k)} - \mu_k I \underset{\uparrow}{=} Q^{(k)} R^{(k)}$

Run the QR decomposition

$$A^{(k+1)} = R^{(k)} Q^{(k)} + \mu_k I$$

Where $\mu_k = a^{(k)}_{n,n} + \varepsilon$ if $\mu_k = \lambda$, then the iteration stucks

2, Check if $a^{(k)}_{n,n-1} \approx 0$,

then remove the last column and row

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & \sim 0 & \underset{\big\downarrow \; \lambda_k}{\boxed{x}} \end{bmatrix} \longrightarrow \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

if size is 2, then the iteration stops, the eigenvalues of a 2×2 matrix is trivial

b) $\vartheta \approx \lambda$ Warning: $\vartheta \neq \lambda$

$\|v^{(0)}\| = 1$ $\left(v^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \text{ or anything} \right)$

$$\underbrace{(A - \vartheta I)\, \underset{\substack{\uparrow \\ \text{unknown}}}{w}^{(k)} = v^{(k)}}$$

Solve this to get $w^{(k)}$ (Thomas algorithm)

$$v^{(k+1)} = \frac{w^{(k)}}{\|w^{(k)}\|_2} \quad \text{(normalization)}$$

Stop the iteration, when $v^{(k)} \approx v^{(k+1)}$