Question

Estimate the percentage of your peers you are taller than?

Statistical Distributions: Normal Distribution

Data Science Immersive

FLATIRON SCHOOL

Agenda and Goals

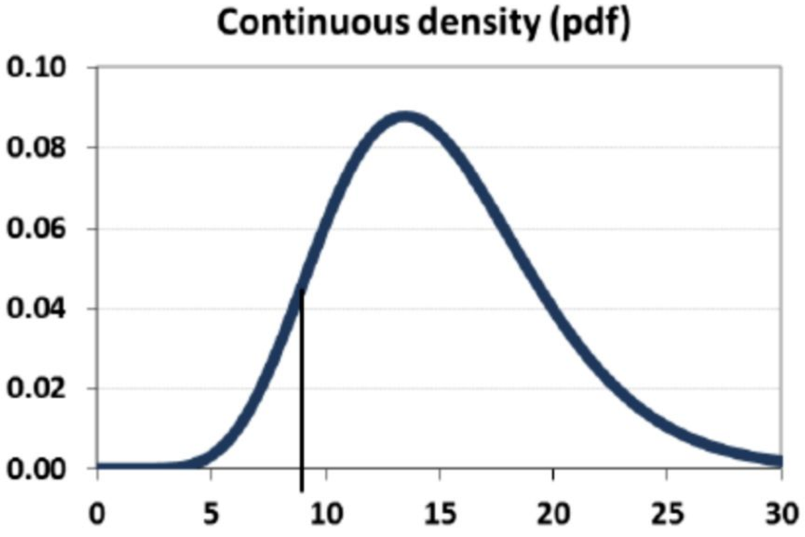
1. Normal Distribution
2. Standard Normal Distribution (the Z Distribution)
3. Exercises

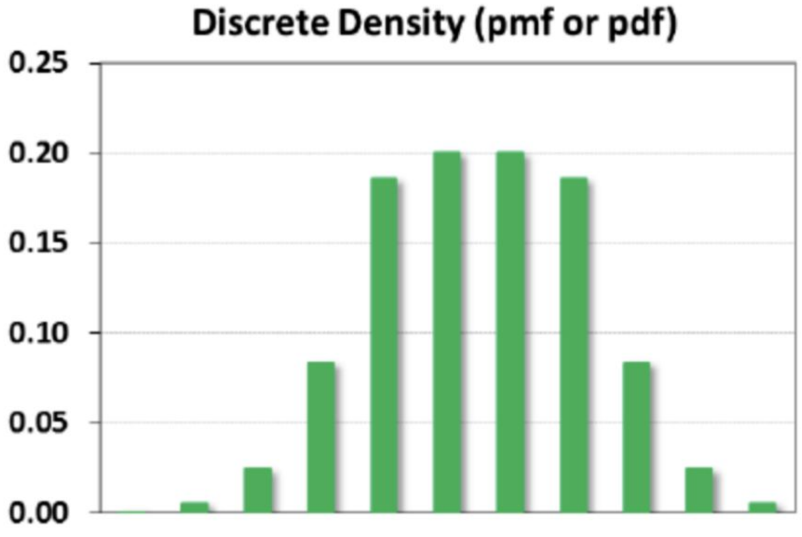
Goals

- Describe the properties of the normal distribution.
- Find probabilities and percentiles of any normal distribution.
- Apply the Empirical rule.

Continuous Probability Functions

If the data is continuous the distribution is modeled using a probability density function (PDF).

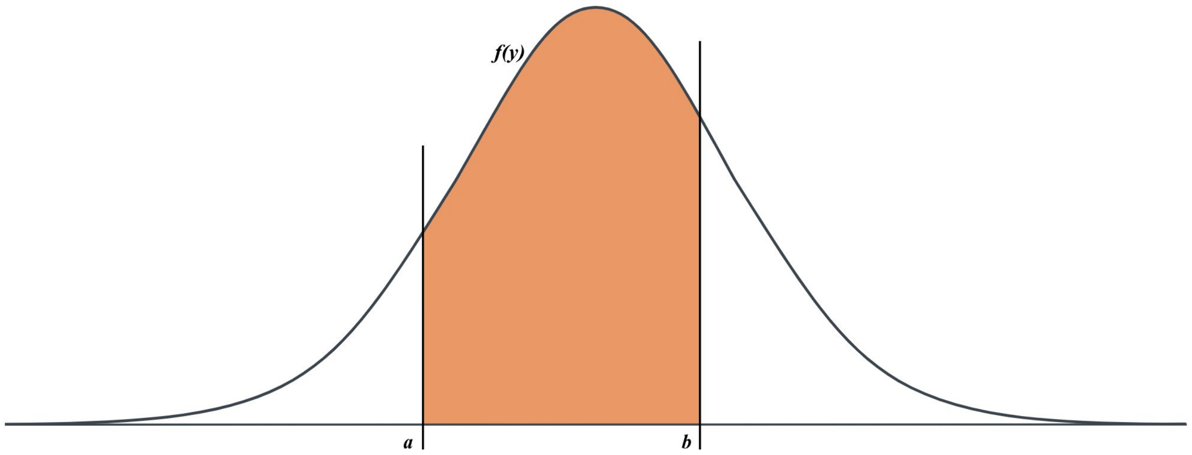




The probability of any given point in a PDF is 0!

Continuous Probability Functions

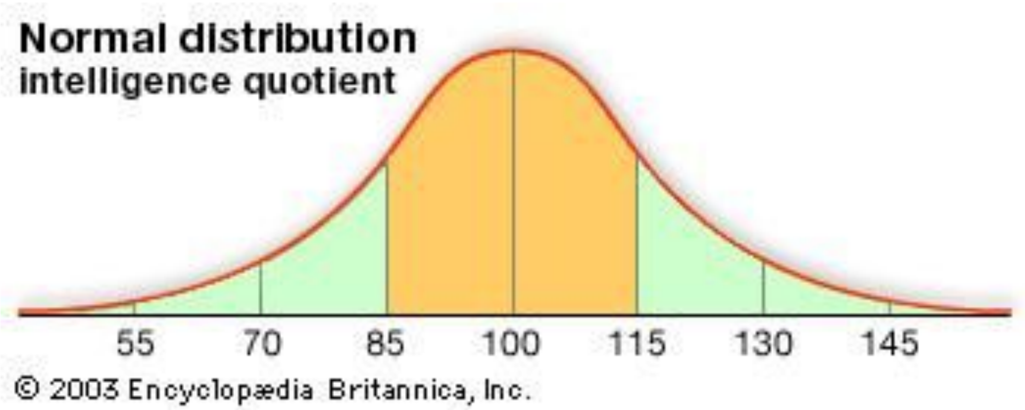
We define the probability distribution function (PDF) of *Y* as *f(y)* where: $P(a<Y<b)$ is the area under *f(y)* over the interval from *a* to *b*.



Gaussian/Normal Distribution

The most common continuous probability distribution is a normal curve. It has two has two parameters:

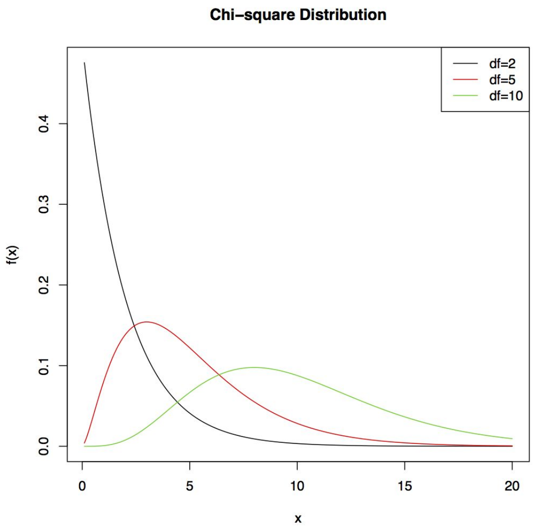
- mean μ (center of the curve)
- standard deviation σ(spread about the center) (..and variance σ2)

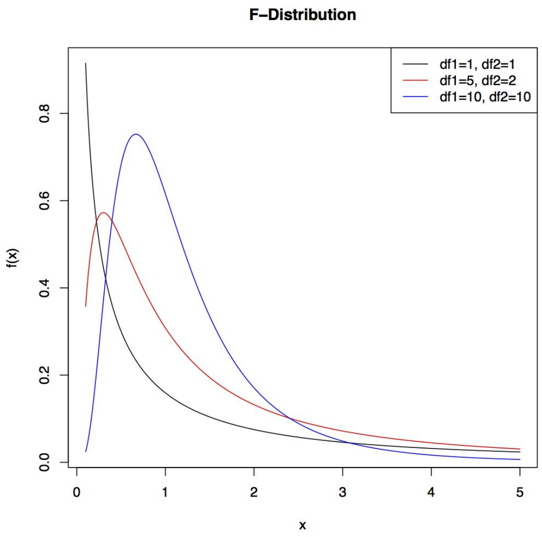


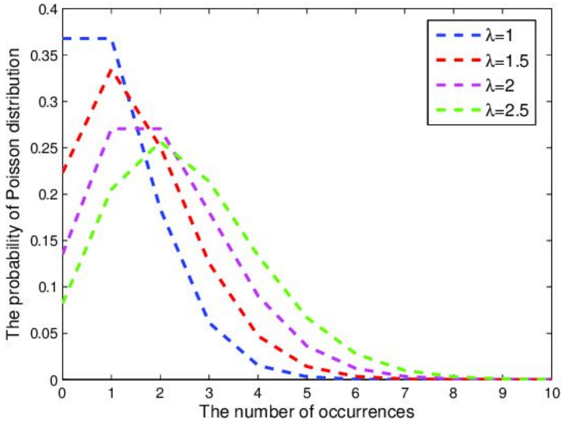
Review: What's the total area under the curve?

Continuous Distributions

The normal distributions is the most common and discussed distribution, there are many other important continuous distributions that are utilized in data analysis. Each has their own parameters that determine the shape of the distribution.







Gaussian/Normal Distribution

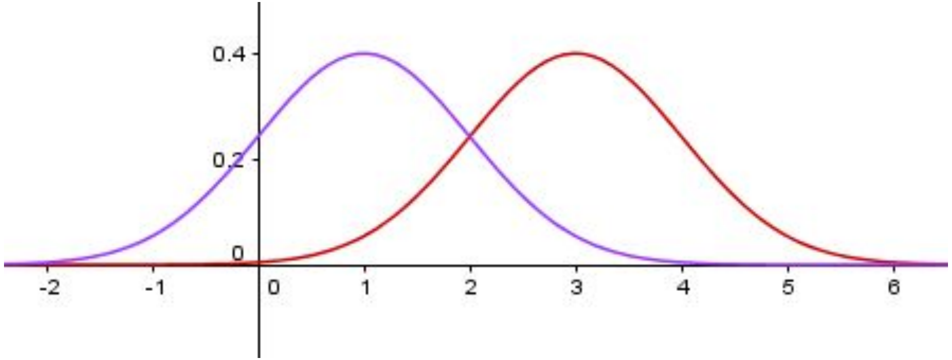
The most well-known distribution is the Gaussian distribution, often called the Normal distribution.

Characteristics:

1. Perfectly symmetrical around its center.
2. Only one mode, or peak, in a normal distribution.
3. Normal distributions are continuous and have tails that are asymptotic.
4. The center of a normal distribution is located at its peak, and 50% of the data lies above the mean, while 50% lies below.
5. The mean, median, and mode are all equal in a normal distribution.

Mean

The “location” of the distribution



$$\mu = E[X]$$

Discrete Probability Distributions

$$E(X) = \sum_{j=1}^{n} p(x_j)\, x_j = p(x_1)x_1 + p(x_2)x_2 + \ldots + p(x_n)x_n.$$

Continuous Probability Distributions

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

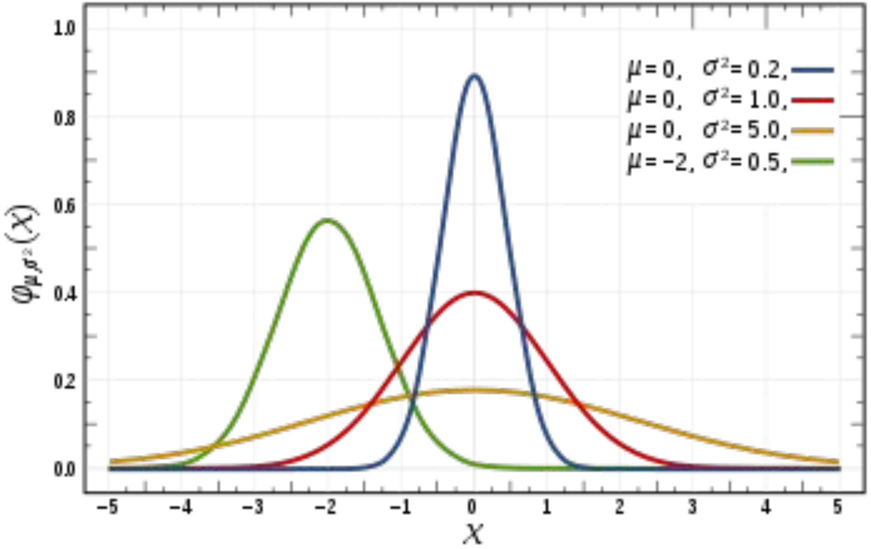
Variance

The "Spread" of the data

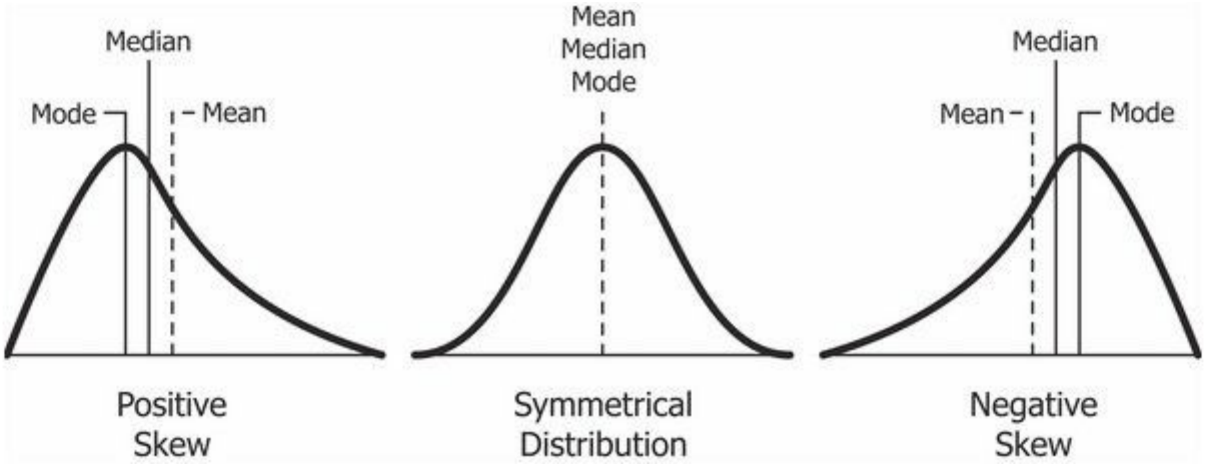
$$\sigma^2 = E[(X - E[X])^2]$$

Probability Density Function:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$



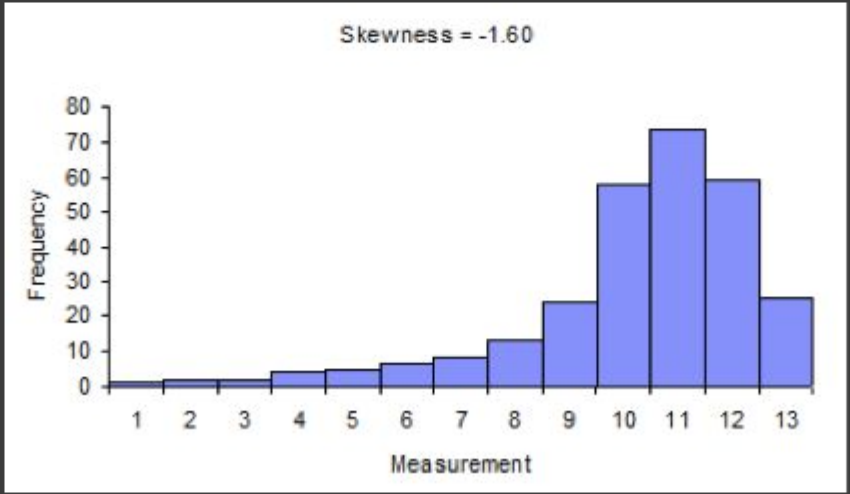
Skewness

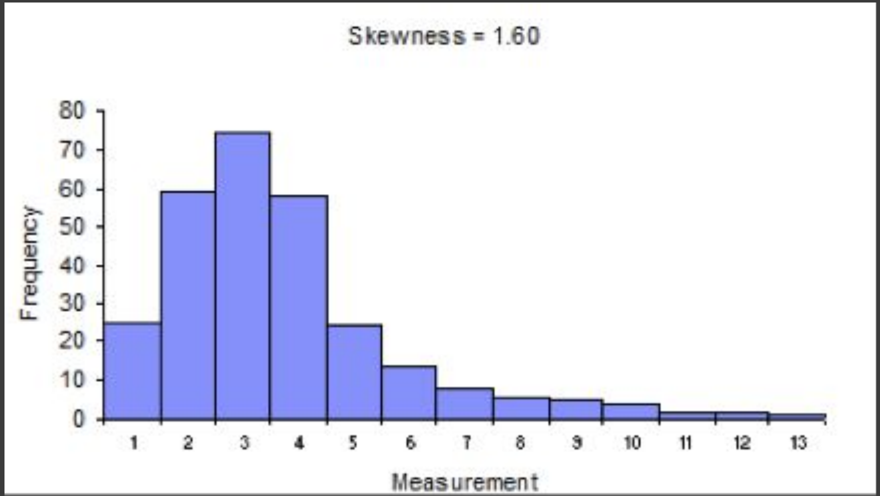


$$\frac{\sum_{i=1}^{N}(Y_i - \bar{Y})^3/N}{s^3}$$

As a general rule of thumb:

- If -1 < skewness > 1, the distribution is highly skewed.
- If skewness is between -1 and -0.5 or between 0.5 and 1, the distribution is moderately skewed.
- If skewness is between -0.5 and 0.5, the distribution is approximately symmetric.





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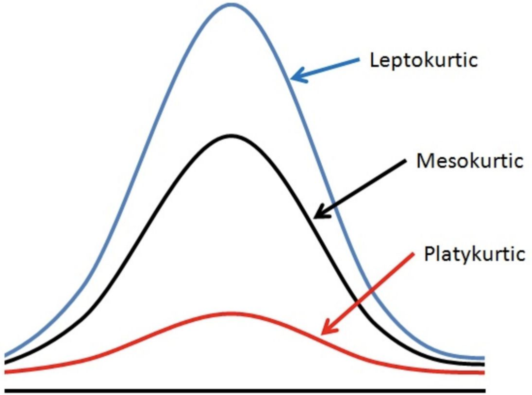
Kurtosis



$$\mathrm{Kurt}[X] = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma^4} = \frac{\mathrm{E}[(X-\mu)^4]}{(\mathrm{E}[(X-\mu)^2])^2},$$

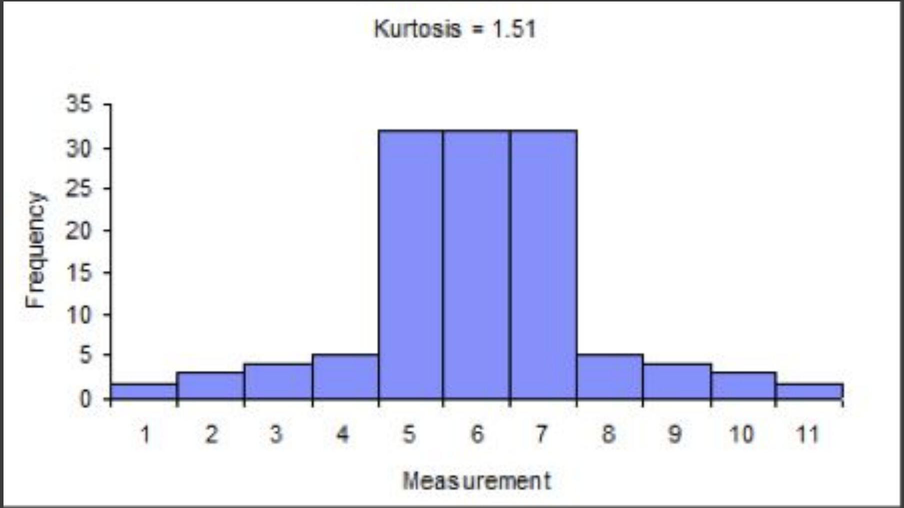
The kurtosis of a normal distribution is **3**

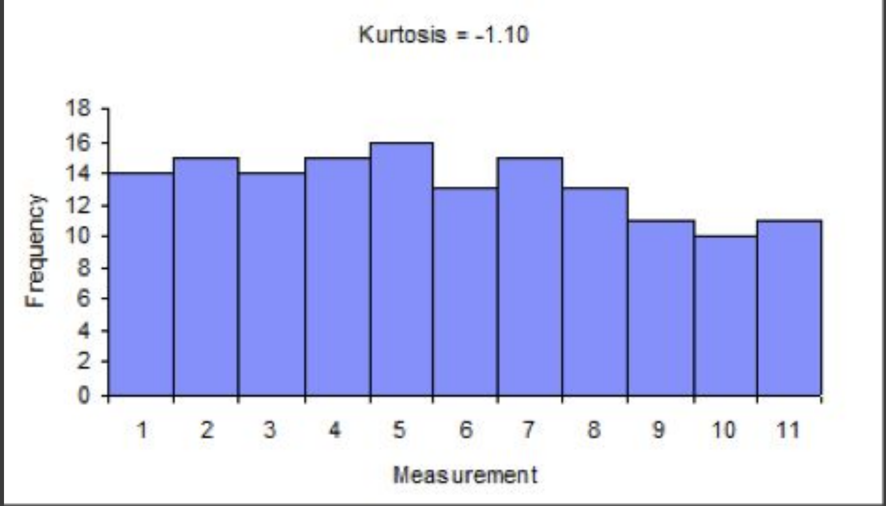
Excess Kurtosis = Kurtosis - 3

Range of kurtosis [1, positive infinity)

A measure of “fatness of tails.” It is a comparison of the combined weight of a distributions’ tails compared to its peak center. A higher number of outliers will lead to a larger kurtosis value.

Kurtosis tells you the height and sharpness of the central peak, relative to that of a standard bell curve.





Kurtosis Real World Application

Fund managers usually focus on risks and returns, kurtosis (in particular if an investment is lepto- or platykurtic).

According to stock trader and analyst Michael Harris, a leptokurtic return means that risks are coming from outlier events. This would be a stock for investors willing to take extreme risks. For example, real estate (with a kurt of 8.75) and High Yield US bonds (8.63) are high risk investments while Investment grade US bonds (1.06) and Small cap US stocks (1.08) would be considered safer investments.

https://www.statisticshowto.datasciencecentral.com/probability-and-statistics/statistics-definitions/kurtosis-leptokurtic-platykurtic/#platykurtic

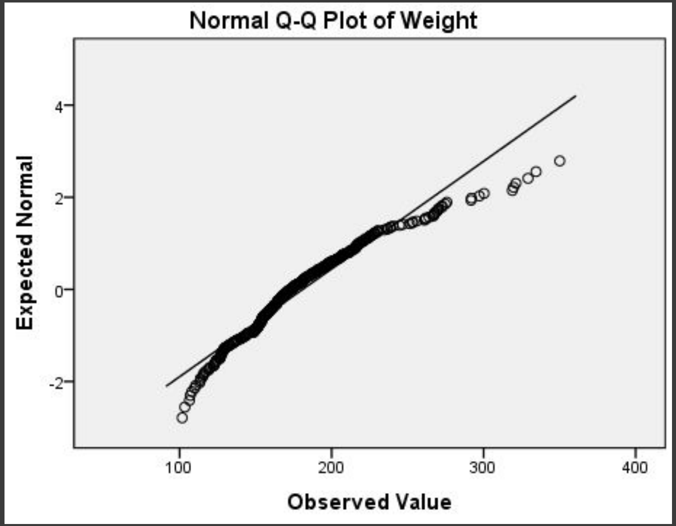
https://www.spcforexcel.com/knowledge/basic-statistics/are-skewness-and-kurtosis-useful-statistics

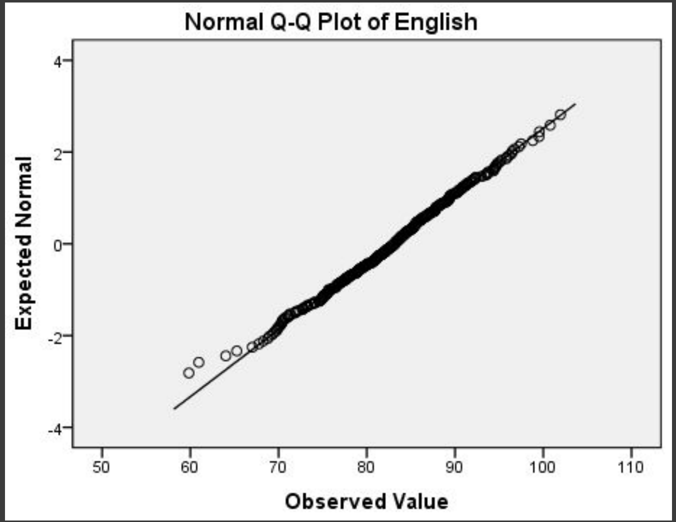
Test of Normality

Some of the traditional goodness-of-fit tests like ***Shapiro-Wilk*** and ***Kolmogorov-Smirnov*** are really designed for tiny data sets. If you have data beyond a few hundred observations, these tests may not be ideal.

The Shapiro-Wilk and Kolmogorov-Smirnov tests were created many decades ago at a time when statistics centered around tiny samples of data. So for statisticians back then, these tests were sufficient. But in this day and age of big data, they may be somewhat outdated.

“Normal Q-Q Plot” provides a graphical way to determine the level of normality.

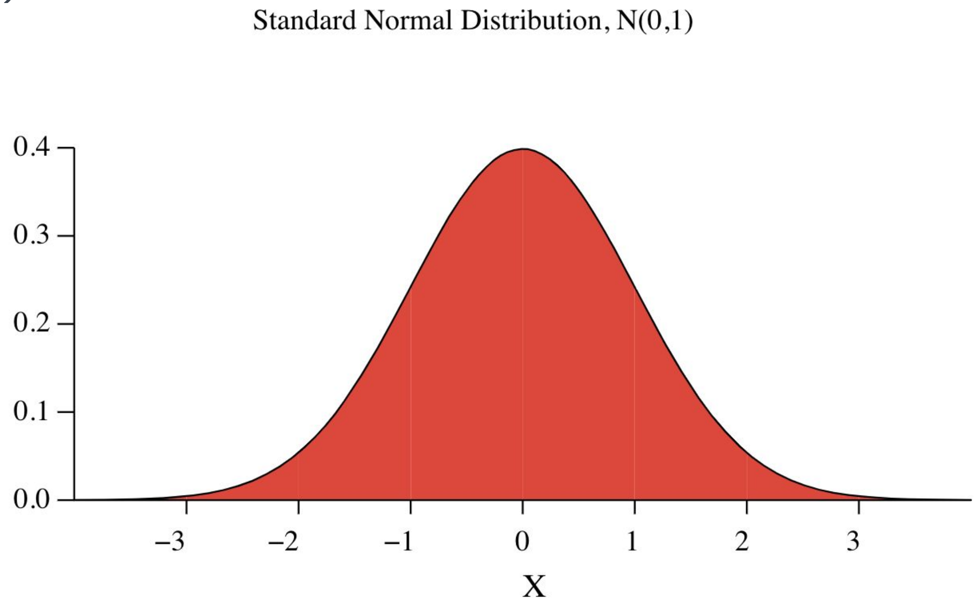




The black line indicates the values your sample should adhere to if the distribution was normal. If the dots fall exactly on the black line, then your data are normal. If they deviate from the black line, your data are non-normal.

Standard Normal Distribution

A standard normal distribution has a mean of 0 and a standard deviation of 1. This is also known as a z distribution. You may see the notation $N(\mu,\sigma 2)$ where N signifies that the distribution is normal, μ is the mean, and $\sigma 2$ is the variance. A Z distribution may be described as $N(0,1)$.

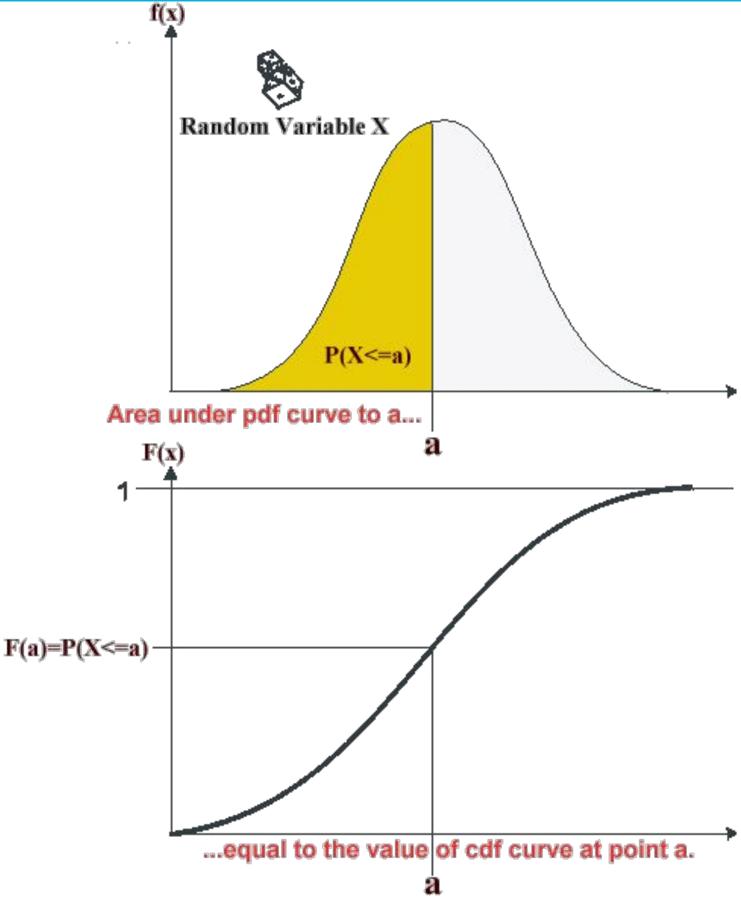


Calculating Probabilities

To calculate the probability of a random variable being less than or equal to a:

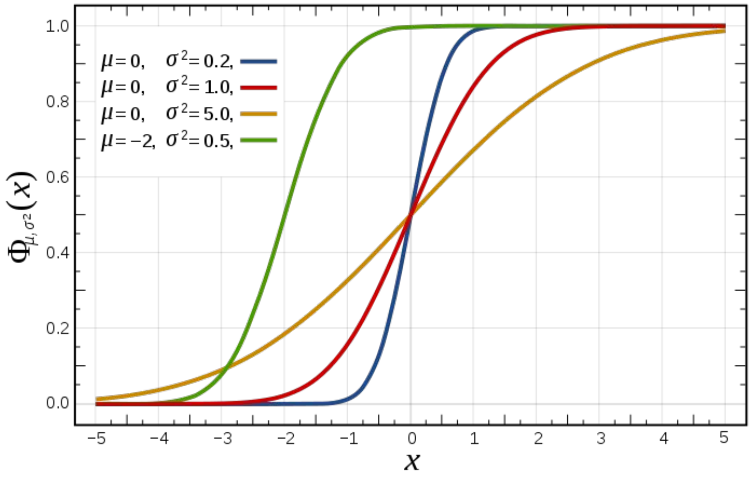
Calculate the area under the pdf curve up to the value of a….

Which is equal to the value of the cdf curve at point a



Normal Distribution

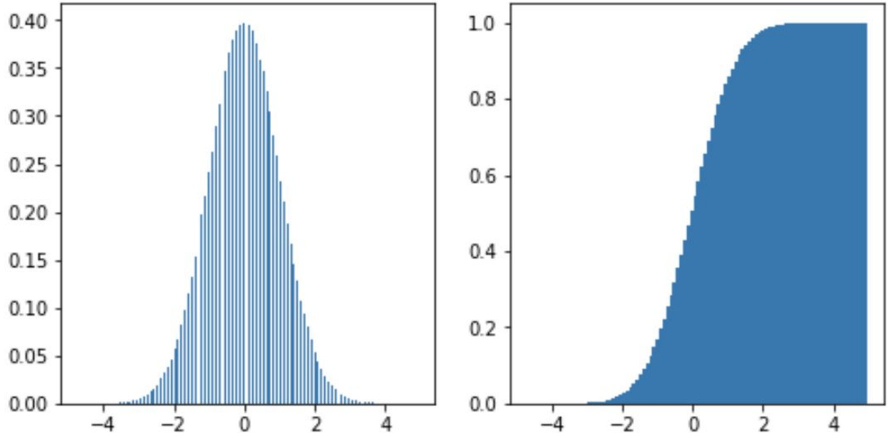
Cumulative Distribution Function



$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

Calculating Probabilities

```python
# Draw 1000000 samples from Normal distribution
mean = 0
std=1
samples = np.random.normal(mean, std, size=1000000)
# Make histograms and CDF
fig, ax = plt.subplots(1,2, figsize=(8,4))
ax[0].hist(samples, bins=100, density=True, rwidth=0.5)
ax[1].hist(samples, bins=100, density=True, cumulative=True)
```




```python
# Compute the fraction that are less than 2: prob
prob = len(samples[samples<2])/1000000

# Print the result
print('Probability of being less than 2:', prob)

```

```
Probability of being less than 2: 0.977387
```

Z-Score Tables

There is no closed form integral of the Normal Distribution, but people have calculated it for all different value of Z. (Thank you for that!)

Alternatively you can use scipy.stats.norm

Here's an example:

```
>>> from scipy.stats import norm
>>> norm.cdf(1.96)
0.9750021048517795
>>> norm.cdf(-1.96)
0.024997895148220435
```

Standard Normal Probabilities

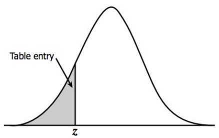


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard Normal Probabilities

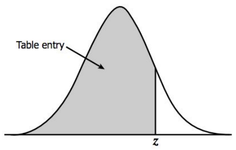
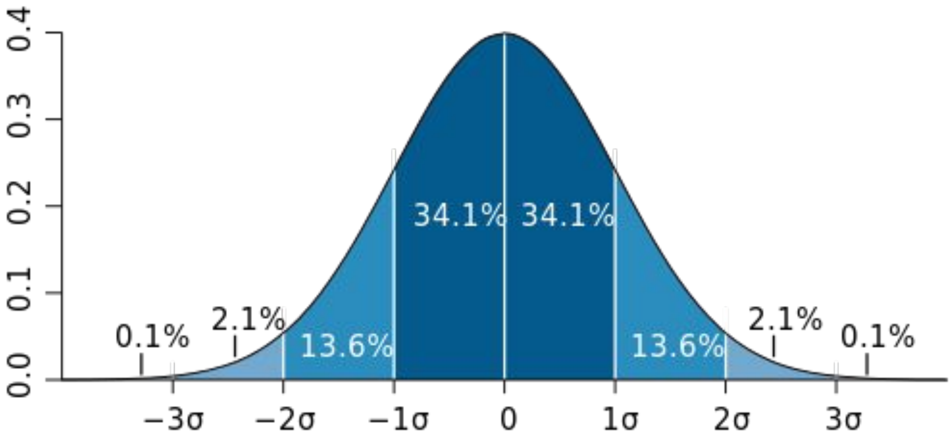


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Empirical Rule

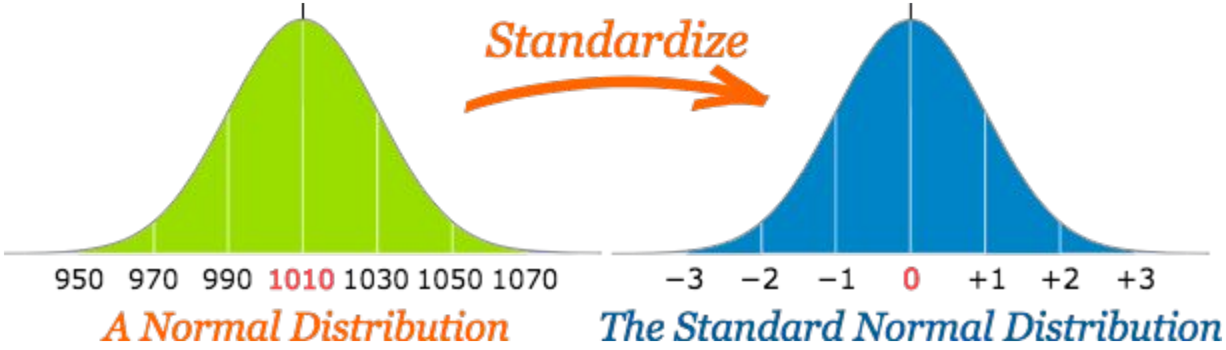
≈ 68% of the area lies between -1 and +1 standard deviations
≈ 95% of the area lies between -2 and +2 standard deviations
≈ 99% of the area lies between -3 and +3 standard deviation



What percentage of the area is less than +1 standard deviation from the mean?

Standard Normal Distribution

We can convert any normal distribution into the standard normal distribution in order to find probability and apply the properties of the standard normal. In order to do this, we use the z-value.



$$z = \frac{x - \mu}{\sigma}$$

$\mu =$ Mean

$\sigma =$ Standard Deviation

Why Standardizing?

- Gives us a good idea the relative location of raw values
- Allows us to compare different values in a more informative way
- Scaling for features if we conduct algorithms that rely on distance metrics

Grading on a curve

A statistics professor gives a really tough semester exam to 100 students, and the results were an average score of 45 with a standard deviation of 11.

- A student with a score of 56 did better than how many students?

The teacher wants to curve the scores so that the new average is 80 and the standard deviation is 9.

- Calculate the new grade for the students that score 23 and 56.

Example 1

Assume snowfall follows a normal distribution over time and the mean snowfall in New York City is 25 inches with a variance of 16 inches.

What is:

1) P(X < 25) = ?
2) P(17 < X < 32) = ?
3) P(X = 25) = ?

Recall:

$$z = \frac{x - \mu}{\sigma}$$

$\mu =$ Mean
$\sigma =$ Standard Deviation

Example 1

Assume snowfall follows a normal distribution over time and the mean snowfall in New York City is 25 inches with a variance of 16 inches.

What is:

1) P(X < 25) = 0.5
2) P(17 < X < 32) = 0.93
3) P(X = 25) = Not possible!!!!

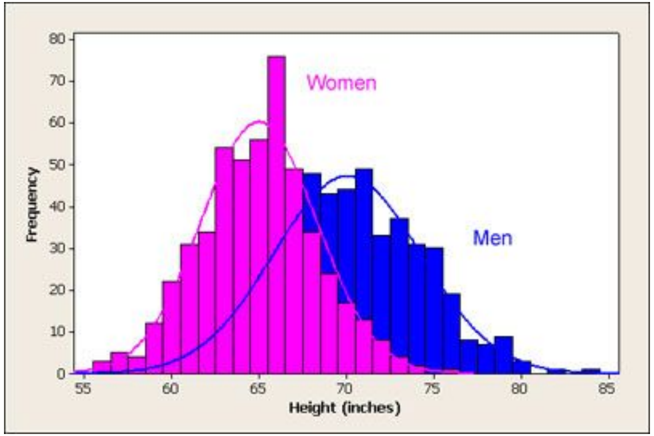

```
z_first = (17 - 25)/4
z_second = (32-25)/4
print('z score of 17 is : ',z_first)
print('z score of 33 is : ',z_second)
stats.norm.cdf(1.75) - stats.norm.cdf(-2)
```

```
z score of 17 is :  -2.0
z score of 33 is :  1.75
```

```
0.9371907111880037
```

Example 2

Adult male heights are on average 70 inches (5'10) with a standard deviation of 4 inches. Adult women are on average a bit shorter and less variable in height with a mean height of 65 inches (5'5) and standard deviation of 3.5 inches.



What is:

1) The probability that a random man you encounter will be taller than you?
2) The probability that a random woman you encounter will be taller than you?