Week 10
Signal Processing S2
Engineering Physics ITS

@btatmaja

Index DFT

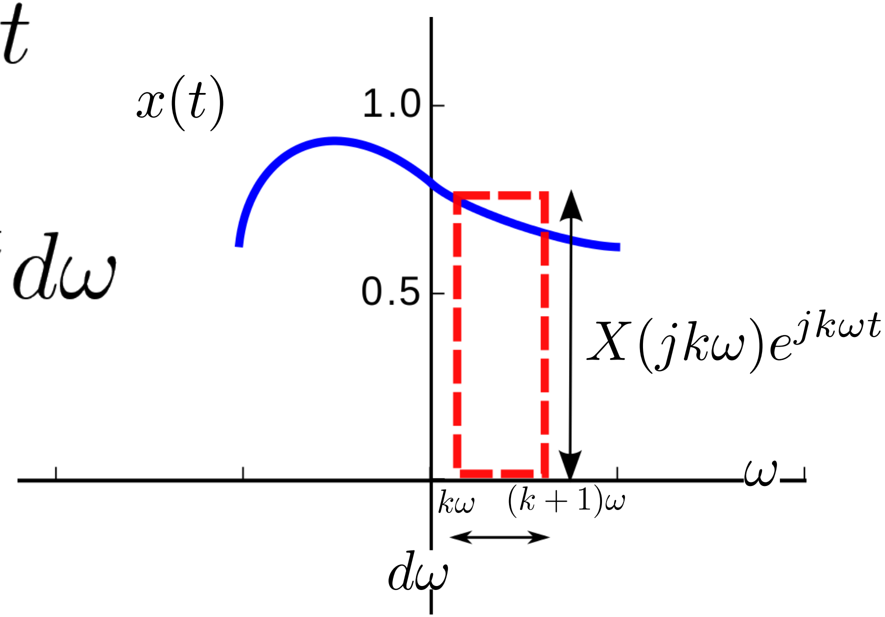
- Review CTFT dan DTFT
- DFT equation
- Complex exponentials in the DFT
- Scalar product in the DFT
- DFT of complex sinusoids
- DFT of real sinusoids
- Inverse-DFT

Review Fourier Transform

continuous

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$



discrete

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$\omega = 2\pi f$$

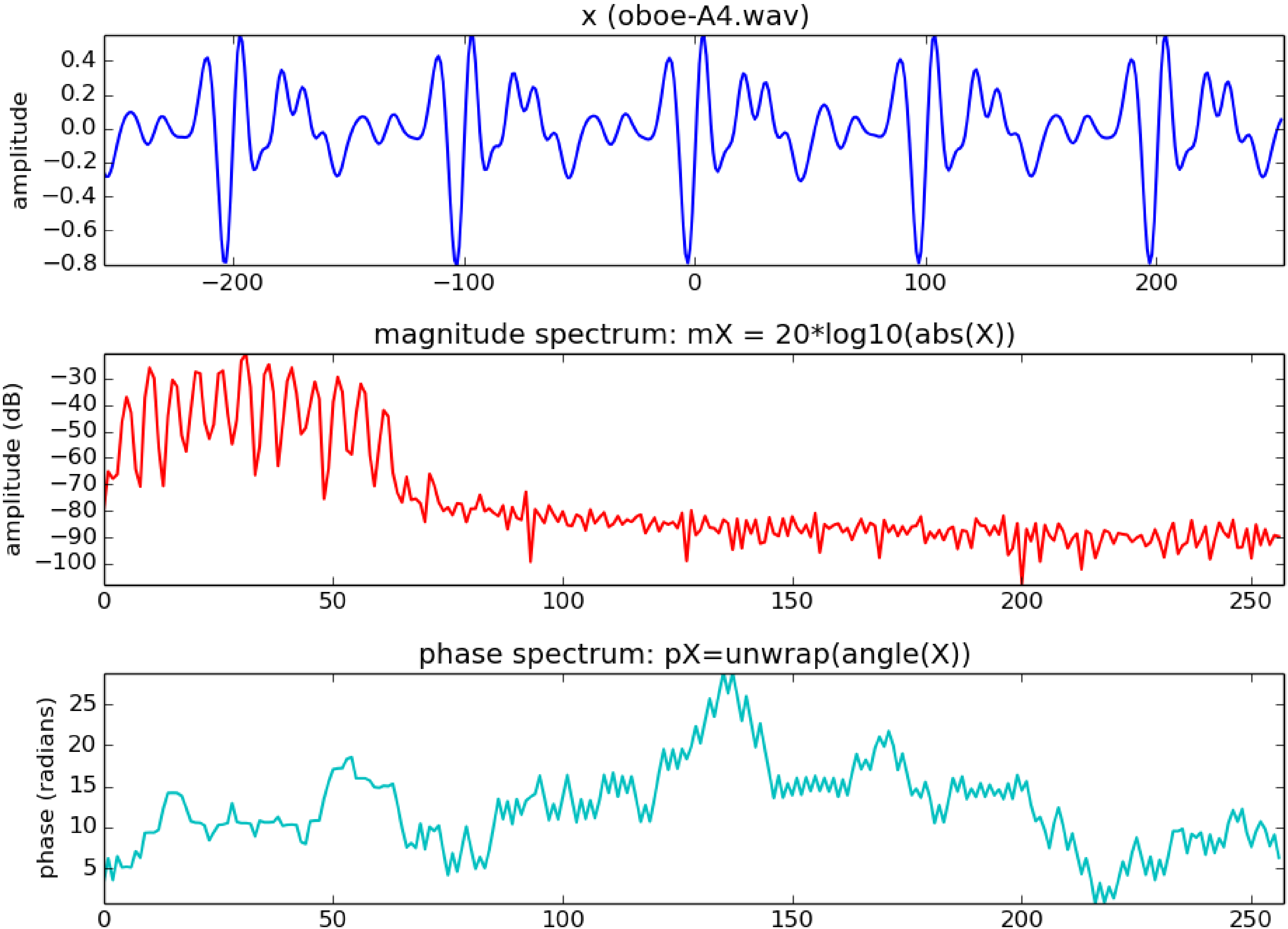
Discrete Fourier Transform

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad k=0,\dots,N-1$$

$$n:\text{discrete time index}\left(\text{normalized time, } \; T=1\right)$$
$$k:\text{discrete frequency index}$$
$$\omega_k=2\pi k/N:\text{frequency in radians per second}$$
$$f_k=f_s k/N:\text{frequency in Hz}\left(f_s:\text{sampling rate}\right)$$

Recall Euler formula

$$e^{-j\theta} = \cos - j\sin\theta$$



x (oboe-A4.wav)
amplitude
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−200
−100
0
100
200
magnitude spectrum: mX = 20*log10(abs(X))
amplitude (dB)
−30
−40
−50
−60
−70
−80
−90
−100
0
50
100
150
200
250
phase spectrum: pX=unwrap(angle(X))
phase (radians)
25
20
15
10
5
0
50
100
150
200
250

DFT: complex exponentials

real

imaginer

$$s_k^* = e^{-j2\pi kn/N} = \cos(2\pi kn/N) - j\sin(2\pi kn/N)$$

$$\text{for } N=4, \quad \text{thus for} \quad n=0,1,2,3\,; \quad k=0,1,2,3$$

$$k=0 \rightarrow \quad s_0^* = \cos(2\pi\times 0\times n/4) - j\sin(2\pi\times 0\times n/4) = [1,1,1,1]$$

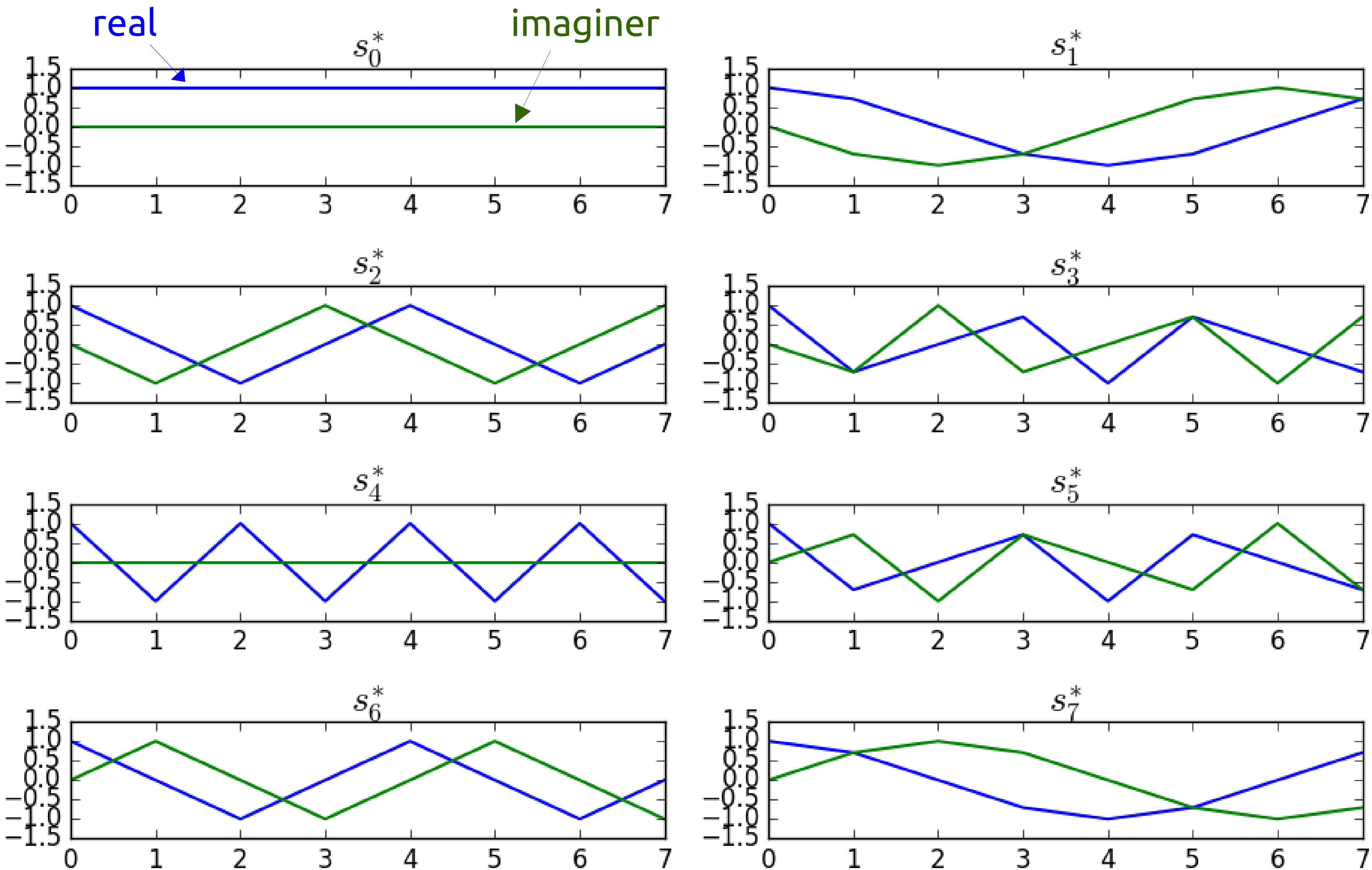
$$k=1 \rightarrow \quad s_1^* = \cos(2\pi\times 1\times n/4) - j\sin(2\pi\times 1\times n/4) = [1,-j,-1,j]$$

$$k=2 \rightarrow \quad s_2^* = \cos(2\pi\times 2\times n/4) - j\sin(2\pi\times 2\times n/4) = [1,-1,1,-1]$$

$$k=3 \rightarrow \quad s_3^* = \cos(2\pi\times 3\times n/4) - j\sin(2\pi\times 3\times n/4) = [1,j,-1,-j]$$

Note that * indicates complex conjugate, e.g., **a = 1 + 2j --> a* = 1 - 2j**

DFT: complex exponentials



DFT: scalar product

$$\langle x, s_k \rangle = \sum_{n=0}^{N-1} x[n] s_k^*[n] = \sum_{n=0}^{N-1} x[n] e^{-j 2\pi kn/N}$$

Example:

$$x[n] = [1, -1, 1, -1]; N = 4$$

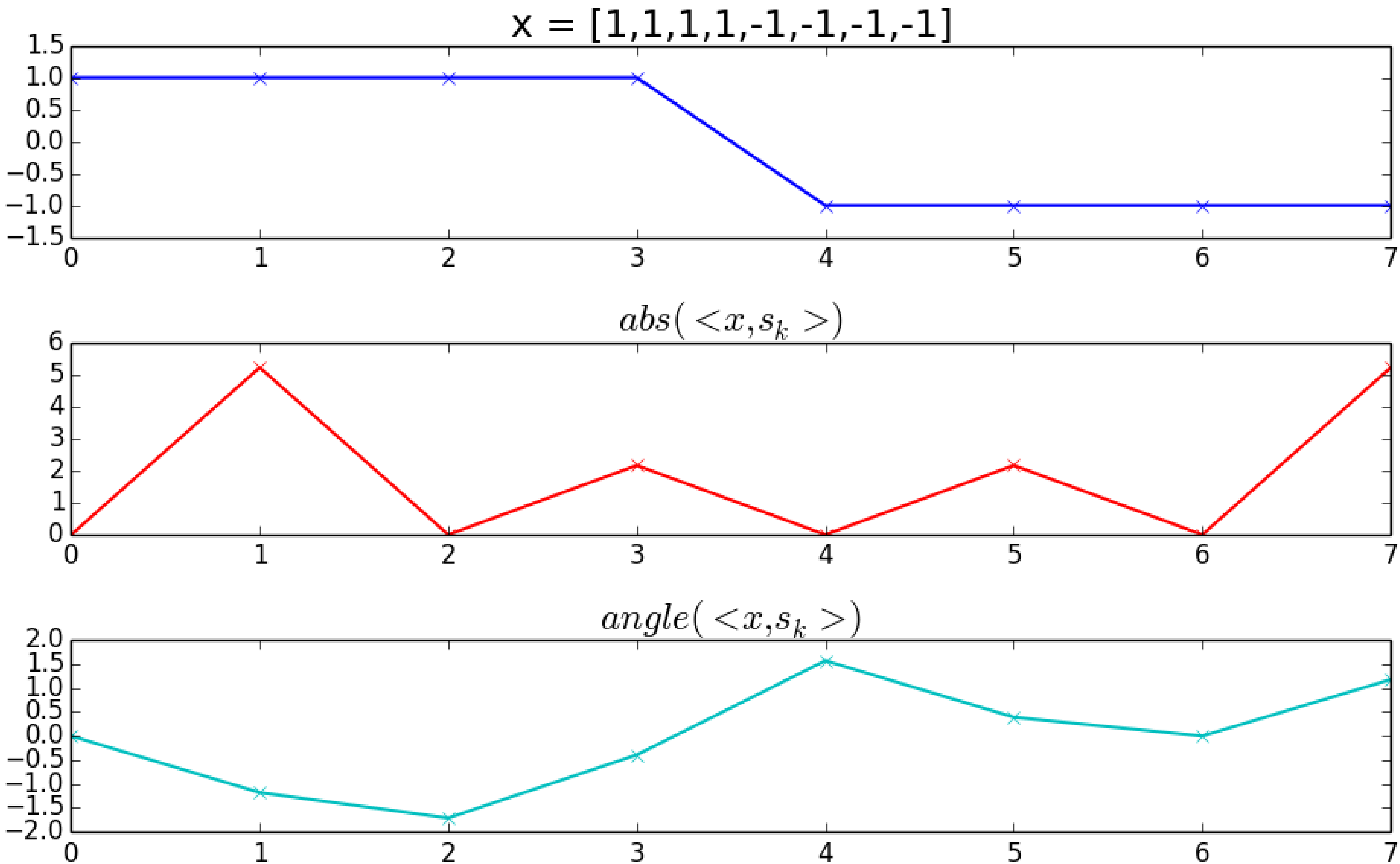
$$\langle x, s_0 \rangle = 1 \times 1 + (-1) \times 1 + 1 \times 1 + (-1) \times 1 = 0$$

$$\langle x, s_1 \rangle = 1 \times 1 + (-1) \times (-j) + 1 \times (-1) + (-1) \times j = 0$$

$$\langle x, s_2 \rangle = 1 \times 1 + (-1) \times (-1) + 1 \times 1 + (-1) \times (-1) = 4$$

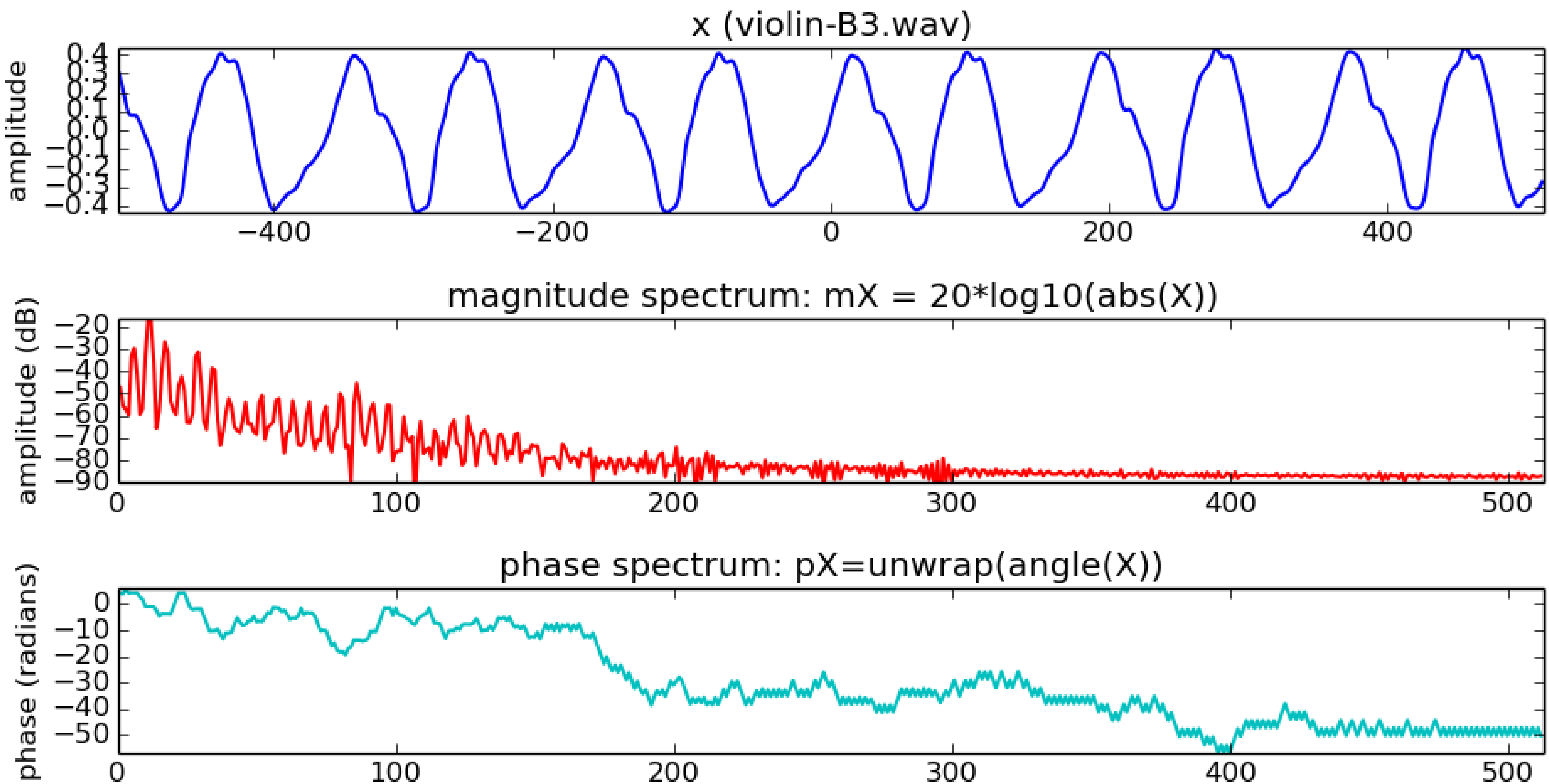
$$\langle x, s_3 \rangle = 1 \times 1 + (-1) \times j + 1 \times (-1) + (-1) \times (-j) = 0$$

DFT: scalar product



Discrete Fourier Transform

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad k=0,\dots,N-1$$



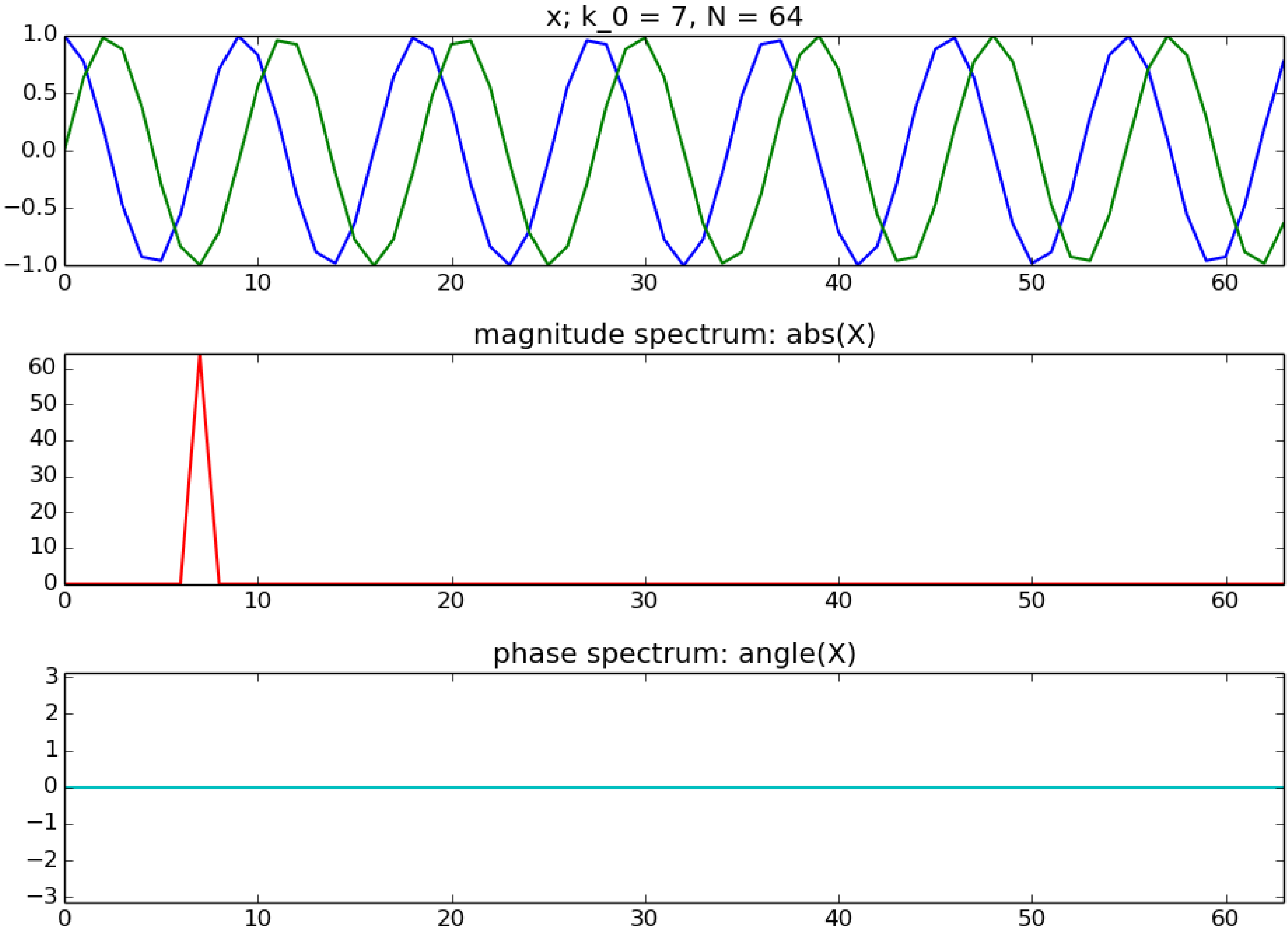
DFT of complex sinusoid

$$x_1[n] = e^{j2\pi k_0 n/N} \quad \text{for } n = 0, \ldots, N-1$$

$$\begin{aligned}
X_1[k] &= \sum_{n=0}^{N-1} x_1[n] e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} e^{-j2\pi k n/N} \\
&= \sum_{n=0}^{N-1} e^{-j2\pi(k-k_0)n/N} \\
&= \frac{1-e^{-j2\pi(k-k_0)}}{1-e^{-j2\pi(k-k_0)/N}} \qquad \text{(sum of a geometric series)}
\end{aligned}$$

if $k \neq k_0$, denominator $\neq 0$ and numerator $= 0$

thus $X_1[k] = N$ for $k = k_0$ and $X_1[k] = 0$ for $k \neq k_0$

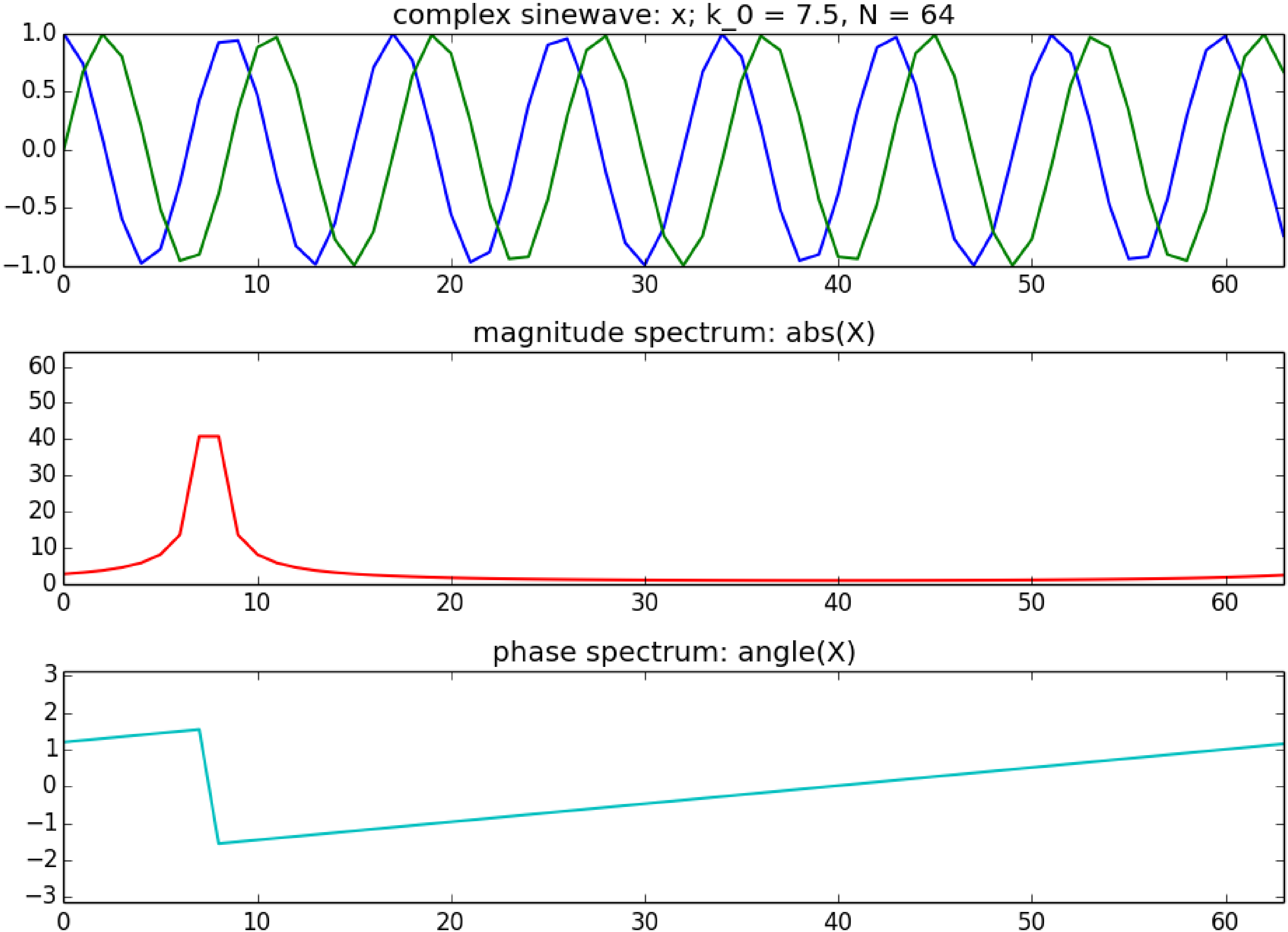


x; k_0 = 7, N = 64
magnitude spectrum: abs(X)
phase spectrum: angle(X)

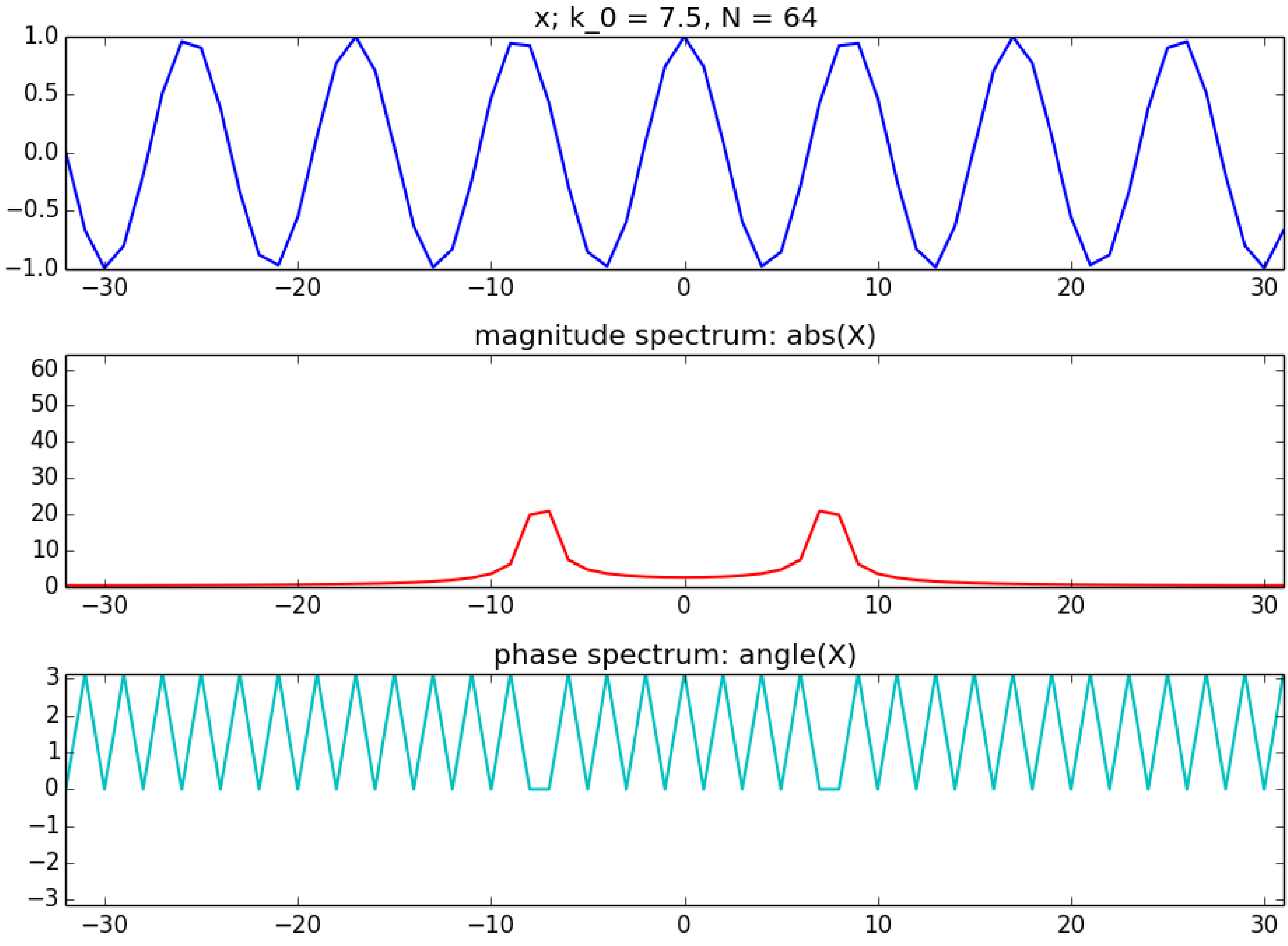
DFT of any complex sinusoid

$$x_2[n] = e^{j(2\pi f_0 n + \phi)} \quad \text{for } n = 0, \ldots, N-1$$

$$\begin{aligned}
X_2[k] &= \sum_{n=0}^{N-1} x_2[n] e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} e^{j(2\pi f_0 n + \phi)} e^{-j2\pi kn/N} \\
&= e^{j\phi} \sum_{n=0}^{N-1} e^{-j2\pi(k/N - f_0)n} \\
&= e^{j\phi} \frac{1 - e^{-j2\pi(k/N - f_0)N}}{1 - e^{-j2\pi(k/N - f_0)}}
\end{aligned}$$



complex sinewave: x; k_0 = 7.5, N = 64
magnitude spectrum: abs(X)
phase spectrum: angle(X)



x; k_0 = 7.5, N = 64
magnitude spectrum: abs(X)
phase spectrum: angle(X)

DFT of real sinusoids

$$x_3[n] = A_0 \cos(2\pi k_0 n/N) = \tfrac{A_0}{2} e^{j2\pi k_0 n/N} + \tfrac{A_0}{2} e^{-j2\pi k_0 n/N}$$

$$X_3[k] = \sum_{n=-N/2}^{N/2-1} x_3[n] e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \left(\frac{A_0}{2} e^{j2\pi k_0 n/N} + \frac{A_0}{2} e^{-j2\pi k_0 n/N} \right) e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{j2\pi k_0 n/N} e^{-j2\pi kn/N} + \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi k_0 n/N} e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi (k-k_0) n/N} + \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi (k+k_0) n/N}$$

$$= N \frac{A_0}{2} \text{ for } k = k_{0,} - k_0; 0 \text{ for rest of } k$$

Inverse DFT

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]s_k[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \quad n=0,1,\ldots,N-1$$

Example:

$$X[k]=[0,4,0,0];N=4$$

$$x[0]=\frac{1}{4}(X*s)[n=0]=\frac{1}{4}(0*1+4*1+0*1+0*1)=1$$

$$x[1]=\frac{1}{4}(X*s)[n=1]=\frac{1}{4}(0*1+4*j+0*(-1)+0*(-j))=j$$

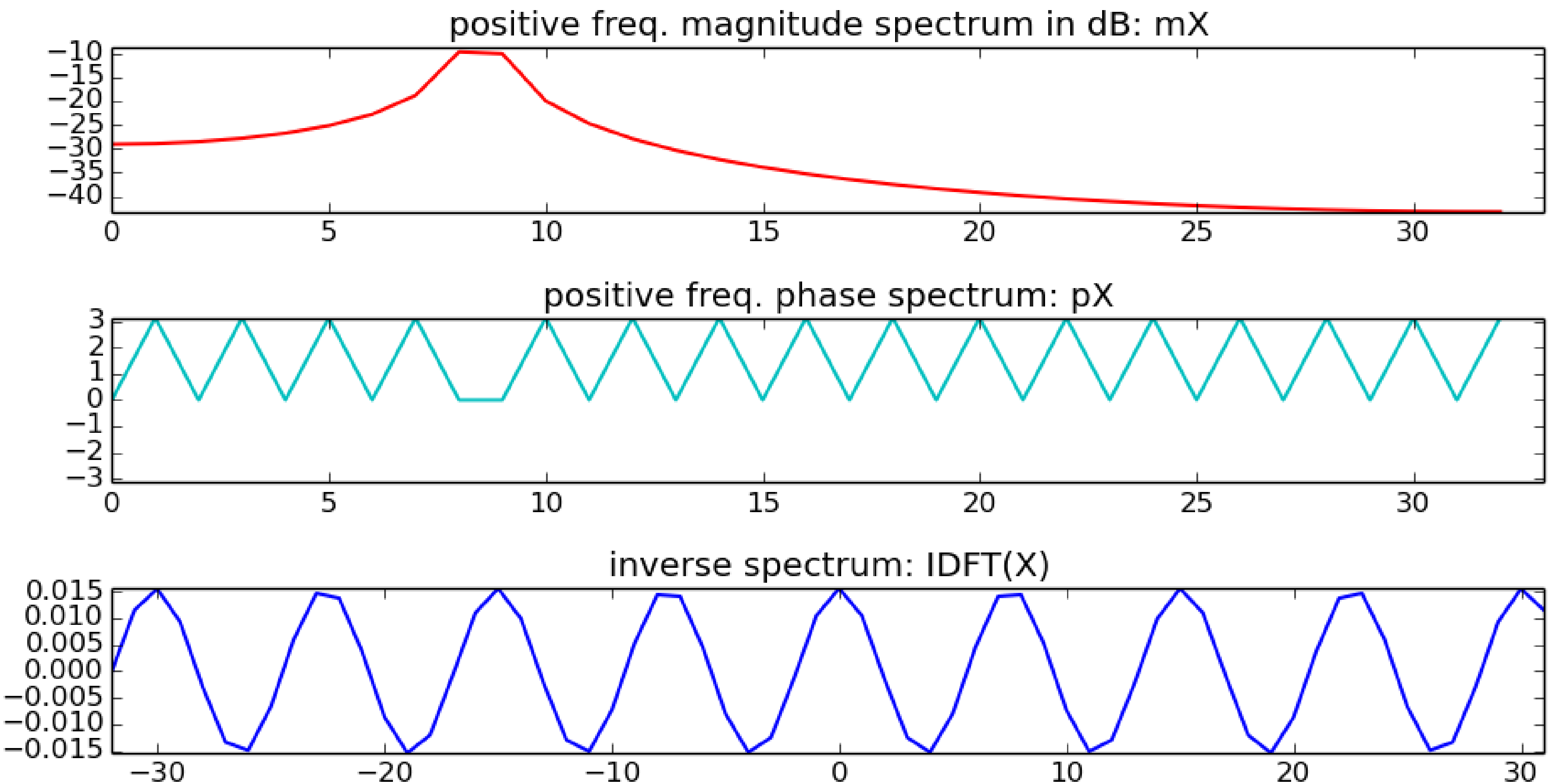
$$x[2]=\frac{1}{4}(X*s)[n=2]=\frac{1}{4}(0*1+4*(-1)+0*1+0*(-1))=-1$$

$$x[3]=\frac{1}{4}(X*s)[n=3]=\frac{1}{4}(0*1+4*(-j)+0*(-1)+0*j)=-j$$

Inverse DFT for real signals

$$X[k]=|X[k]|e^{j\angle X[k]} \quad \text{and} \quad X[-k]=|X[k]|e^{-j\angle X[k]}$$

for $k=0,1,\ldots,N/2$



Practice session with Python

- Download zip/git clone https://github.com/MTG/sms-tools
- Run all codes in sms-tools/lectures/02-DFT/plots-code
- Play with DFT

Additional resources

- https://github.com/bagustris/python-for-signal-processing/blob/master/notebook/Fourier_Transform.ipynb
- https://github.com/spatialaudio/signals-and-systems-lecture/tree/master/discrete_fourier_transform

Homework

- Find DFT of
  (a) x[n] = [1, 2, 3, 4]
  (b) x[n] = [1, 0, -1, -1, -1, -1, 0 1]
- Find IDFT of x[k] = [0, 0, 4, 0]

Lab

- github.com/bagustris/python-for-signal-processing > lab > lab1_sampling