Week 10
Computational Vibration

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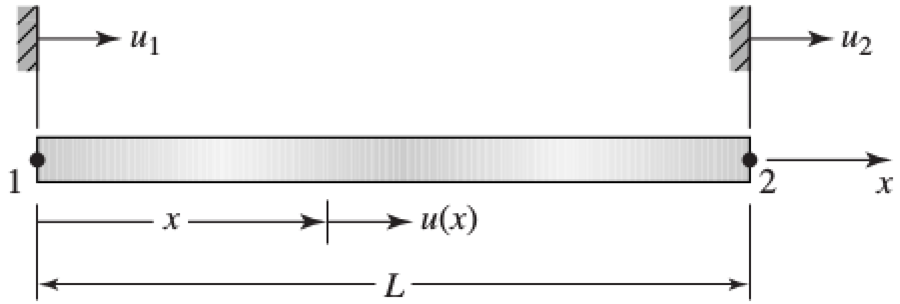
Engineering Physics ITS

Bar element

- Bar element –> subjected to axial forces only
- Assumption:
  - The bar is geometrically straight.
  - The material obeys Hooke’s law.
  - Forces are applied only at the ends of the bar.
  - The bar supports axial loading only; bending, torsion, and shear are not transmitted to the element via the nature of its connections to other elements.

Interpolation/shape function: equal to one at their respective nodes and 0 at the other nodes

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$



$$u(x=0) = u_1 \quad u(x=L) = u_2$$

$$N_1(0) = 1 \quad N_2(0) = 0$$

$$N_1(L) = 0 \quad N_2(L) = 1$$

$$N_1(x) = a_0 + a_1 x$$

$$N_2(x) = b_0 + b_1 x$$

$$a_0 = 1$$

$$b_0 = 0$$

$$a_1 = -\frac{1}{L}$$

$$b_1 = \frac{x}{L}$$

$$N_1(x) = 1 - \frac{x}{L}$$

$$N_2(x) = \frac{x}{L}$$

$$u(x) = (1 - \tfrac{x}{L})u_1 + (\tfrac{x}{L})u_2$$

$$\mathrm{u(x)} = [\mathrm{N}_1(x) \quad N_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{u\}$$

Matriks kekakuan elemen bar

→ relation between the nodal displacements and applied forces

deflection

$$\delta = \frac{PL}{AE}$$

Modulus elastisitas

Konstanta pegas

$$k = \frac{P}{\delta} = \frac{AE}{L}$$

$$\varepsilon_x = \frac{du}{dx} \qquad \varepsilon_x = \frac{u_2 - u_1}{L}$$

The axial stress, by Hooke's law

$$\sigma_x = E\varepsilon_x = E\frac{u_2 - u_1}{L}$$

The associated axial force

$$P = \sigma_x A = \frac{AE}{L}(u_2 - u_1)$$

Hubungan gaya f_1 dan f_2 terhadap nodal displacement u_1 dan u_2

Kebalikan f_2 ←

$$f_1 = -\frac{AE}{L}(u_2 - u_1)$$

$$f_2 = \frac{AE}{L}(u_2 - u_1)$$

→ positive

Dalam bentuk matriks

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Matriks kekakuan

$$[k_e] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Contoh soal

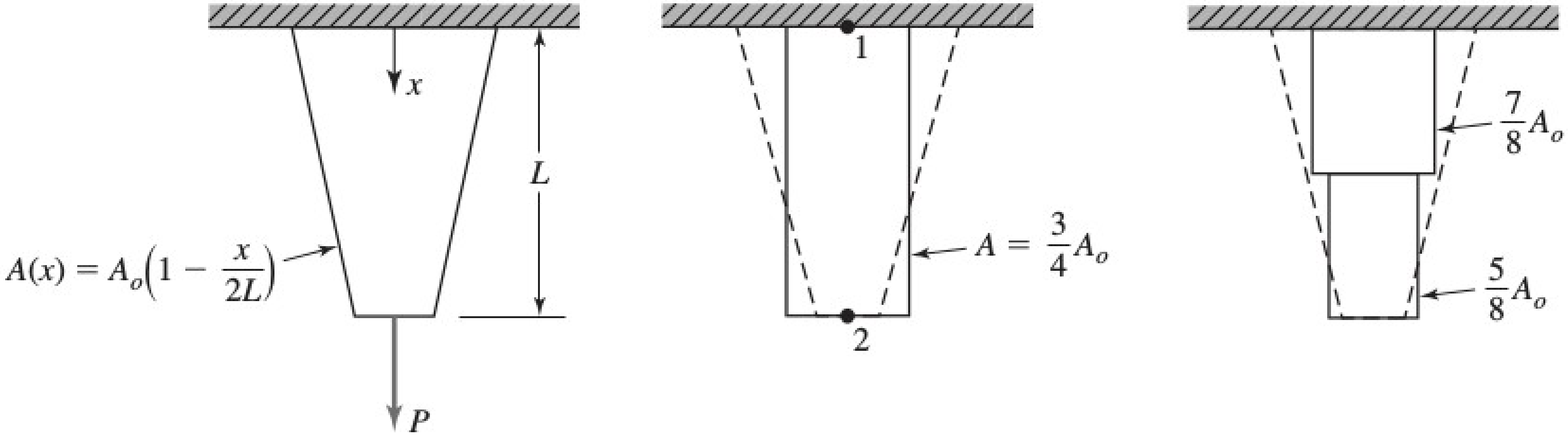


Fig. above depicts a tapered elastic bar subjected to an applied tensile load P at one end and attached to a fixed support at the other end. The cross-sectional area varies linearly from A 0 at the fixed support at x = 0 to A 0 /2 at x = L . Calculate the displacement of the end of the bar (a) by modeling the bar as a single element having cross-sectional area equal to the area of the actual bar at its midpoint along the length, (b) using two bar elements of equal length and similarly evaluating the area at the midpoint of each, and (c) using integration to obtain the exact solution.

Solusi

a) For a single element, the cross-sectional area is $3A_0/4$ and the element "spring constant" is

$$k = \frac{AE}{L} = \frac{3A_0E}{4L}$$

$$\frac{3A_0E}{4L}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \end{Bmatrix}$$

$$U_2 = \frac{4PL}{3A_0E} = 1.333\frac{PL}{A_0E}$$

Solusi (b): Two elements of equal length L /2,
for element 1, A1 = 7 A0 / 8:

$$k_1 = \frac{A_1 E}{L_1} = \frac{7 A_0 E}{8(L/2)} = \frac{7 A_0 E}{4L}$$

$$A_1 = \frac{5 A_0}{8} \quad \text{and} \quad k_2 = \frac{A_2 E}{L_2} = \frac{5 A_0 E}{8(L/2)} = \frac{5 A_0 E}{4L}$$

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \end{Bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

$$U_2 = \frac{P}{k_1} = \frac{4PL}{7 A_0 E} \quad U_3 = \frac{k_1 + k_2}{k_2} = \frac{48 PL}{35 A_0 E} = 1.371 \frac{PL}{A_0 E}$$

Solusi (c), at x = L:

$$\sigma_x A = P \quad \text{and since} \quad A = A(x) = A_0\left(1 - \frac{x}{2L}\right)$$

$$\sigma_x = \frac{P}{A_0\left(1 - \frac{x}{2L}\right)} \qquad \varepsilon_x = \frac{\sigma_x}{E} = \frac{P}{EA_0\left(1 - \frac{x}{2L}\right)}$$

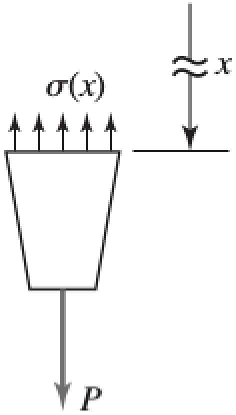
$$\delta = \int_0^L \varepsilon_x \, \mathrm{d}x = \frac{P}{EA_0}\int_0^L \frac{\mathrm{d}x}{\left(1 - \frac{x}{2L}\right)}$$

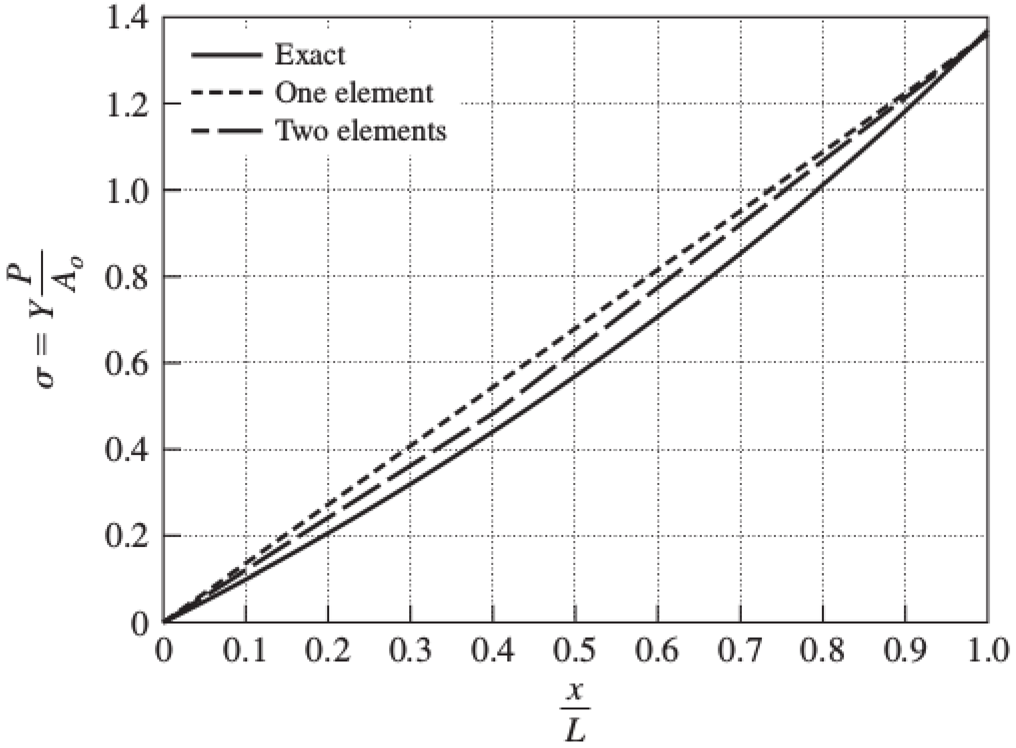
$$= \frac{2PL}{EA_0}[-\ln(2L - x)]\Big|_0^L = \frac{2PL}{EA_0}[\ln(2L) - \ln L] = \frac{2PL}{EA_0}\ln 2$$

$$= 1.386\frac{PL}{A_0 E}$$

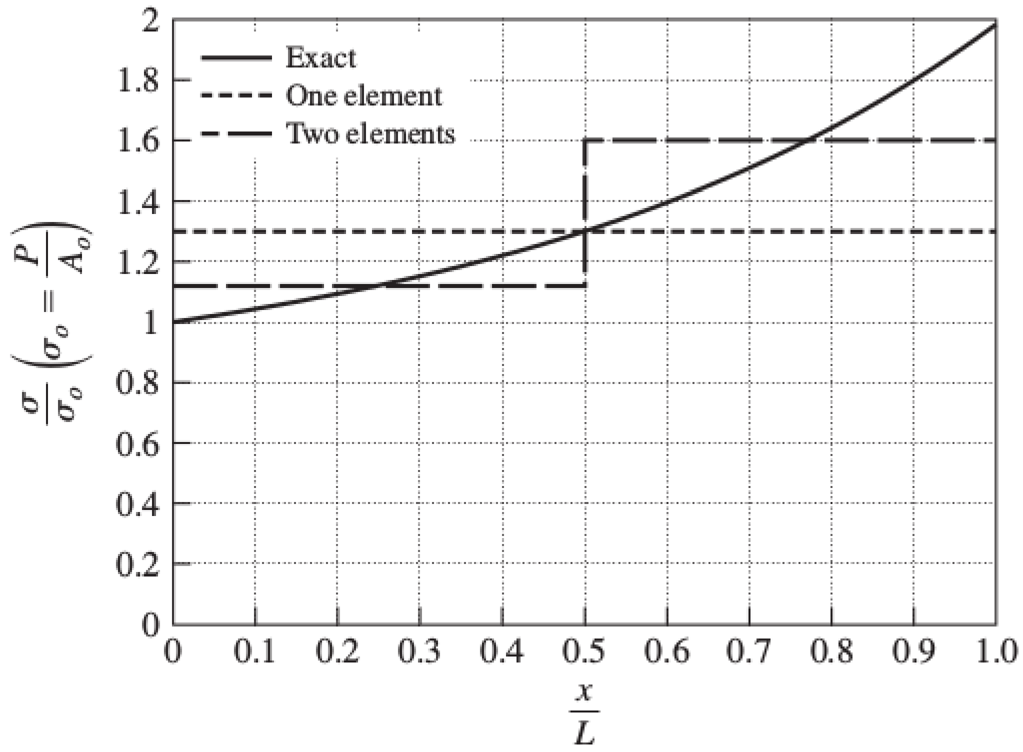
displacement solutions

free-body diagram
for an exact solution





stress solutions



Strain energy

When external forces are applied to a body, the mechanical work done by those forces ***is converted, in general, into a combination of kinetic and potential energies*****:** in an elastic body to prevent motion, all the work is stored in the body as elastic potential energy, which is also commonly referred to as strain energy.

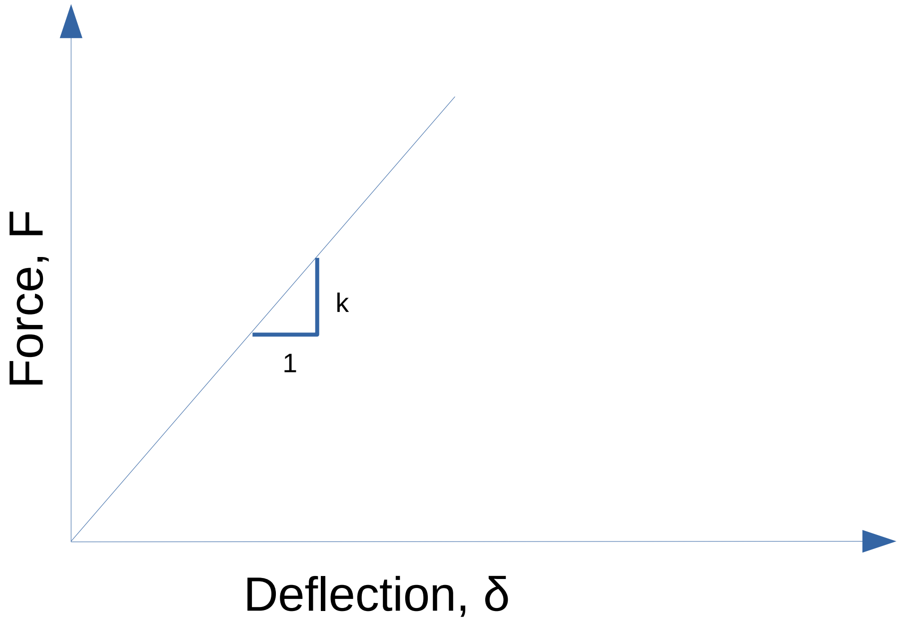
Strain Energy

- Dalam Fisika dasar kita mengenal W = F s

$$W = \int_1^2 \vec{F} \cdot \mathrm{d}\vec{r}$$
$$\mathrm{d}\vec{r} = \mathrm{d}x\vec{i} + \mathrm{d}y\vec{j} + \mathrm{d}z\vec{k}$$

$$W = \int_{x_1}^{x_2} F_x \; \mathrm{d}x + \int_{y_1}^{y_2} F_y \; \mathrm{d}y + \int_{z_1}^{22} F_z \; \mathrm{d}z$$

Hubungan force-deflection untuk pegas linear elastis



Usaha yang diperlukan untuk merubah pegas sebesar δ_0

$$W = \int_0^{\delta_0} F \, \mathrm{d}\delta = \int_0^{\delta_0} k\delta \mathrm{d}\delta = \frac{1}{2} k\delta_0^2 = U_e$$

$$U_e = \frac{1}{2}k\delta^2 = \frac{1}{2}\frac{AE}{L}\delta^2$$

$$U_e = \tfrac{1}{2}k\delta^2 = \tfrac{1}{2}\tfrac{AE}{L}\left(\tfrac{PL}{AE}\right)^2 = \tfrac{1}{2}\left(\tfrac{P}{A}\right)\left(\tfrac{P}{AE}\right)AL = \underbrace{\tfrac{1}{2}\sigma\varepsilon}_{\text{strain energy density}}V$$

$$U_e = \int_0^{\varepsilon} \sigma \, d\varepsilon$$

Castigliano’s First Theorem

For an elastic system in equilibrium, the partial derivative of total strain energy with respect to deflection at a point is equal to the applied force in the direction of the deflection at that point.

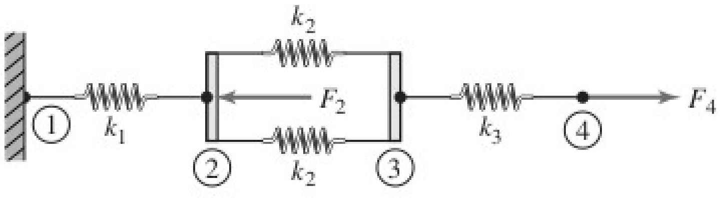
$$U_e = W = \sum_{j=1}^{N} \int_0^{\delta_j} F_j \, \mathrm{d}\delta_j$$

$$U_e = \Delta W = F_i \Delta\delta_i + \int_0^{\Delta\delta_i} \Delta F_i \, \mathrm{d}\delta_i$$

$$\frac{\Delta U_e}{\Delta\delta_i} = F_i \qquad \frac{\partial U_e}{\partial u_1} = \frac{AE}{L}(u_1 - u_2) = f_1$$

$$\frac{\partial U}{\partial\delta_i} = F_i \qquad \frac{\partial U_e}{\partial u_2} = \frac{AE}{L}(u_2 - u_1) = f_2$$

Contoh Soal



(a) Apply Castigliano's first theorem to the system of four spring elements depicted in Fig. above to obtain the system stiffness matrix. The vertical members at nodes 2 and 3 are to be considered rigid;

(b) Solve for the displacements and the reaction force at node 1 if
k_1 = 4 N/mm; k_2 = 6 N/mm; k_3 = 3 N/mm; F_2 = − 30 N; F_3 = 0; F_4 = 50 N

Total energi sistem empat pegas dinyatakan dalam nodal displacement dan konstanta pegas sebagai berikut:

$$U_e = \frac{1}{2}k_1\left(U_2 - U_1\right)^2 + 2\left[\frac{1}{2}k_2\left(U_3 - U_2\right)^2\right] + \frac{1}{2}k_3\left(U_4 - U_3\right)^2$$

$$\frac{\partial U_e}{\partial U_1} = F_1 = k_1\left(U_2 - U_1\right)\left(-1\right) = k_1\left(U_1 - U_2\right)$$

$$\frac{\partial U_e}{\partial U_2} = F_2 = k_1\left(U_2 - U_1\right) + 2k_2\left(U_3 - U_2\right)\left(-1\right) = -k_1U_1 + \left(k_1 + 2k_2\right)U_2 - 2k_2U$$

$$\frac{\partial U_e}{\partial U_3} = F_3 = 2k_2\left(U_3 - U_2\right) + k_3\left(U_4 - U_3\right)\left(-1\right) = -2k_2U_2 + \left(2k_2 + k_3\right)U_3 - k_3U$$

$$\frac{\partial U_e}{\partial U_4} = F_4 = k_3\left(U_4 - U_3\right) = -k_3U_3 + k_3U_4$$

Dalam bentuk matriks...

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 16 + -12 & 0 & \\ 0 & -12 & 15 + -3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30 \\ 0 \\ 50 \end{Bmatrix}$$

Eliminasi baris paling atas, -4 U_2=F_1, selesaikan sisanya dengan komputer.

Solusi dengan Python NumPy

```
bagus@bagus-IStDXi-M049:...        IPython: /tmp

In [3]: a = np.array([[16, -12, 0],
   ...:               [-12, 15, -3],
   ...:               [0, -3, 3]])

In [4]: c = np.array([-30, 0, 50])

In [5]: b = np.linalg.solve(a, c)

In [6]: b
Out[6]: array([ 5.        ,  9.16666667, 25.83333333])
```

U_2 U_3 U_4

$$\mathrm{F}_1 = -4U_2 = -4(5.0) = -20\ N$$

Practice Session with CALFEM