

Integral (tak tertentu)

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Integral tertentu dan tak tertentu

Integral:

- ▶ tak tertentu: Jika $f(x)$ ditentukan maka setiap fungsi $F(x)$ hingga $F'(x) = f(x)$ disebut Integral tak tertentu dari $f(x)$ dituliskan $\int f(x) = F(x) + C$.

Contoh: $x^3, x^3 - 2, x^3 + 1$ adalah integral tak tertentu dari $f(x) = 3x^2$, dituliskan $\int 3x^2 dx = x^3 + C$

- ▶ tertentu: Jika $f(x)$ fungsi kontinyu pada $[a, b]$ maka dikatakan bahwa $f(x)$ terintegral (integrable) pada $[a, b]$ dan disebut integral tertentu dan dinyatakan dengan $\int_a^b f(x) dx$.

Sifat-sifat integral tak tertentu

1. $\int kf(x)dx = k \int f(x)dx$ dimana k konstanta

Bukti: Untuk sisi kiri, $\frac{d\{\int kf(x)dx\}}{dx} = kf(x)$

Untuk sisi kanan,

$$\frac{d\{k \int f(x)dx\}}{dx} = \frac{kd\{\int f(x)dx\}}{dx} = kf(x)$$

2. $\int\{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$

Bukti: Untuk sisi kiri, $\frac{d[\int\{f(x) \pm g(x)\}dx]}{dx} = f(x) \pm g(x)$

Untuk sisi kanan,

$$\begin{aligned}\frac{d[\int f(x)dx \pm \int g(x)dx]}{dx} &= \frac{d\int f(x)dx}{dx} \pm \frac{d\int g(x)}{dx} \\ &= f(x) \pm g(x)\end{aligned}$$

Beberapa rumus integral tak tertentu[1]

1. $\int 0dx = C$
2. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C; \quad (n \neq -1)$
3. $\int \frac{1}{x} dx = \ln|x| + C$
4. $\int e^x dx = e^x + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C$
6. $\int \sin x dx = -\cos x + C$
7. $\int \cos x dx = \sin x + C$
8. $\int \tan x dx = \ln|\sec x| + C$
9. $\int \cot x dx = \ln|\sin x| + C$
10. $\int \sec x dx = \ln|\sec x + \tan x| + C$
11. $\int \csc x dx = \ln|\csc x - \cot x| + C$

Beberapa rumus integral tak tertentu[2]

$$12. \int \sec^2 x dx = \tan x + C$$

$$13. \int \csc^2 x dx = -\cot x + C$$

$$14. \int \sinh x dx = \cosh x + C$$

$$15. \int \cosh x dx = \sinh x + C$$

$$16. \int \tanh x dx = \ln |\cosh x| + C$$

$$17. \int \coth x dx = \ln |\sinh x| + C$$

$$18. \int \operatorname{sech}^2 x dx = \tanh x + C$$

$$19. \int \operatorname{csch}^2 x dx = -\coth x + C$$

$$20. \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$21. \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$$

$$22. \int \frac{dx}{1+x^2} = \arctan x + C$$

Beberapa rumus integral tak tertentu[3]

$$23. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$24. \int \ln x dx = x \ln x - x + C$$

$$25. \int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln |x + \sqrt{x^2 \pm 1}| + C$$

$$26. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C$$

$$27. \int \sqrt{1 - x^2} = \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \arcsin x + C$$

$$28. \int \sqrt{a^2 - x^2} = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

Integrasi parsial

Jika $u = f(x)$ dan $v = g(x)$ adalah fungsi-fungsi yang diferensiabel maka:

$$\boxed{\int u dv = uv - \int v du}$$

Cara menggunakan rumus diatas:

- ▶ dv dipilih sehingga v mudah dicari
- ▶ $\int v du$ harus menjadi lebih mudah daripada $\int u dv$

Contoh:

- ▶ $I = \int xe^x dx$
- ▶ $I = \int e^x \sin x dx$

Penerapan $1/D$ pada integral tak tertentu

$D = \frac{d}{dx}$ operator derivatif

$$\frac{1}{D} = D^{-1} = \int \dots dx \text{ operator integral}$$

$$\frac{1}{1-D} = 1 + D + D^2 + \dots; \Rightarrow \frac{1}{1+D} = 1 - D - D^2 - \dots$$

$$D \sin ax = a \cos ax; \boxed{D^2 \sin ax = -a^2 \sin ax}$$

$$D \cos ax = -a \sin ax; \boxed{D^2 \cos ax = -a^2 \cos ax}$$

$$(D^2 - b^2) \sin ax = (-a^2 - b^2) \sin ax$$

$(D^2 - b^2) \cos ax = (-a^2 - b^2) \cos ax$, sehingga:

$$\boxed{\frac{1}{D^2 - b^2} \sin ax = \frac{\sin ax}{-a^2 - b^2}}$$

$$\boxed{\frac{1}{D^2 - b^2} \cos ax = \frac{\cos ax}{-a^2 - b^2}}$$

Rumus: $\frac{1}{D} e^{ax} V = e^{ax} \frac{1}{(D + a)} V$

Bukti :

$$D(e^{ax} U) = ae^{ax} U + e^{ax} DU$$

$$D(e^{ax} U) = e^a x(D + a)U$$

$$D\left\{e^{ax} \frac{1}{(D + a)} V\right\} = ea^{ax} V$$

Misal:

$$(D + a)U = V \implies U = \frac{1}{D + a} V$$

$$\frac{1}{D}(e^{ax} V) = e^{ax} \frac{1}{(D + a)} V$$

Buktikan:

$$\int e^{ax} \cos bx dx = e^{ax} (D - a) \frac{\cos bx}{-b^2 - a^2}$$

$$\int e^{2x} (2x^2 - 4x + 8) dx = \frac{1}{D} e^{2x} (2x^2 - 4x + 8)$$

Rumus: $\frac{1}{D}x V = \left(x - \frac{1}{D}\right) \frac{1}{D} V$

Bukti:

$$D(x U) = x DU + U \text{ Misal : } DU = V \rightarrow U = \frac{1}{D}$$

$$\begin{aligned} D\left(x \frac{1}{D} x V\right) &= xV + \frac{1}{D} V \\ x \frac{1}{D} V &= \frac{1}{D}(xV) + \frac{1}{D^2} V \\ \frac{1}{D}(xV) &= x \frac{1}{D} V - \frac{1}{D^2} V \\ \frac{1}{D}(xV) &= \left(x - \frac{1}{D}\right) \frac{1}{D} V \end{aligned}$$

Contoh:

$$\int x e^x \cos 2x dx = \dots$$

Rumus:

$$\frac{1}{D}(UV) = U\frac{1}{D}V - DU\frac{1}{D^2}V + D^2U\frac{1}{D^3}V - D^3U\frac{1}{D^4}V + \dots$$

Bukti:

Jika $y = UV$ dimana $U = f(x)$ dan $v = g(x)$ maka turunan tingkat ke n ,

$y^{(n)} = D^{(n)}(UV)$ dirumuskan oleh Leibnitz sbb:

$$D^{(n)} = UD^n V + nDUD^{n-1}V + \frac{1}{D^2!}n(n-1)D^2UD^{n-2}V + \frac{1}{D^3!}n(n-1)(n-2)D^3UD^{n-3}V + \dots$$

Dengan memasang $n=-1$, maka:

$$\frac{1}{D}(UV) = U\frac{1}{D}V - DU\frac{1}{D^2}V + D^2U\frac{1}{D^3}V - D^3U\frac{1}{D^4}V + \dots$$

Integrasi fungsi pecah rasional

1. Jika $N(x) = (ax + b)(cx + d)$ maka dibawa ke bentuk

$$\frac{A}{ax + b} + \frac{B}{cx + d}.$$

2. Jika $N(x) = (ax + b)^2(cx + d)$ maka dibawa ke bentuk

$$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}.$$

3. Jika $N(x) = (ax^2 + bx + c)$ dengan $D < 0$ maka dibawa ke

bentuk $\frac{A}{cx + d} + \frac{Bx + C}{ax^2 + b}$.

Contoh:

1. $I = \int \frac{x - 1}{(x + 1)(x^2 + 1)} dx$

2. $I = \int \frac{3x + 1}{x^2 + 2x + 1} dx$

3. $I = \int \frac{23 - 2x}{x^2 + 9x - 5} dx$

Formula operasi fungsi trigonometri

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin 2x = 2 \sin x \cos x$
3. $\cos 2x = \cos^2 x - \sin^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin(-x) = -\sin x; \quad \cos -x = \cos x$
7. $\sin \alpha x \cos \beta x = \frac{1}{2} \{ \sin(\alpha + \beta)x + \sin(\alpha - \beta)x \}$
8. $\cos \alpha x \sin \beta x = \frac{1}{2} \{ \sin(\alpha + \beta)x - \sin(\alpha - \beta)x \}$
9. $\cos \alpha x \cos \beta x = \frac{1}{2} \{ \cos(\alpha + \beta)x + \cos(\alpha - \beta)x \}$
10. $-\sin \alpha x \sin \beta x = \frac{1}{2} \{ \cos(\alpha + \beta)x - \cos(\alpha - \beta)x \}$

Integrasi fungsi trigonometri[1]

Bentuk: $\int \alpha x \cos \beta x dx; \int \cos \alpha x \sin \beta x dx; \int \sin \alpha x \beta x dx$

$$\int \sin ax \sin bx dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)}, \quad (a^2 \neq b^2) \quad \int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos ax \cos bx dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)}, \quad (a^2 \neq b^2) \quad \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \sin ax \cos bx dx = -\frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, \quad (a^2 \neq b^2) \quad \int \sin ax \cos ax dx = \frac{\sin^2 ax}{2a}$$

$$\int x^2 \sin ax dx = \frac{2x}{a^2} \sin ax - \frac{a^2 x^2 - 2}{a^3} \cos ax \quad \int x \sin ax dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$$

$$\int x^2 \cos ax dx = \frac{2x}{a^2} \cos ax + \frac{a^2 x^2 - 2}{a^3} \sin ax \quad \int x \cos ax dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$$

Contoh:

$$\int \sin 5x \cos x dx$$

Integrasi fungsi trigonometri[2]

- Bentuk: $\int R(\sin x, \cos x)dx$; R= fungsi rasional

Subtitusi : $\tan \frac{x}{2} = t \implies \frac{x}{2} = \arctan t, \rightarrow x = 2 \arctan x$

$$dx = \frac{2dt}{1+t^2}; \sin x = \frac{2t}{1+t^2}; \cos x = \frac{1-t^2}{1+t^2}$$

$$\text{Maka: } \int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2dt}{1+t^2}$$

- Bentuk: $R(\sin x, \cos x)dx = \int R(-\sin x, -\cos x)dx$

Subtitusi: $\tan x = t \implies x = \arctan t \rightarrow \cos x = \frac{1}{\sqrt{1+x^2}}$

- Bentuk: $R(\tan x)dx$

Subtitusi: $\tan x = t \implies x = \arctan t \rightarrow dx = \frac{dt}{1+t^2}$

Contoh:

$$1. \ I = \int \frac{dx}{5 + 4 \cos x}$$

$$2. \ I = \int dx \sin^2 x - \sin x \cos x$$

$$3. \ I = \int \frac{1 + \tan x}{1 - \tan x} dx$$