Linear Time-Invariant Systems

Features :

1. Homogeneity
2. Additivity
3. Time-Invariant

Homogeneity

A system is homogeneous if it multiplies all input amplitudes of a certain frequency by a constant:

if:

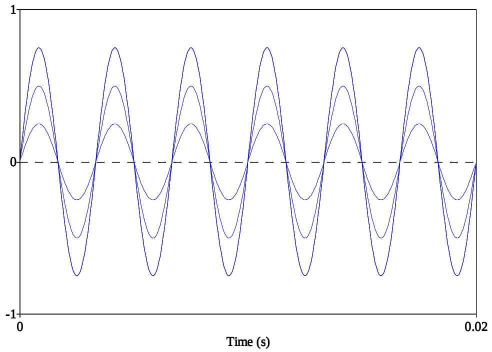
inp(t) → outp(t)

then:

k x inp(t) → k x outp(t)

Homogeneity

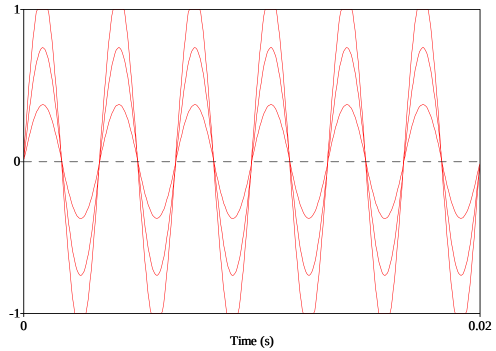
300 Hz 25/50/75 mV

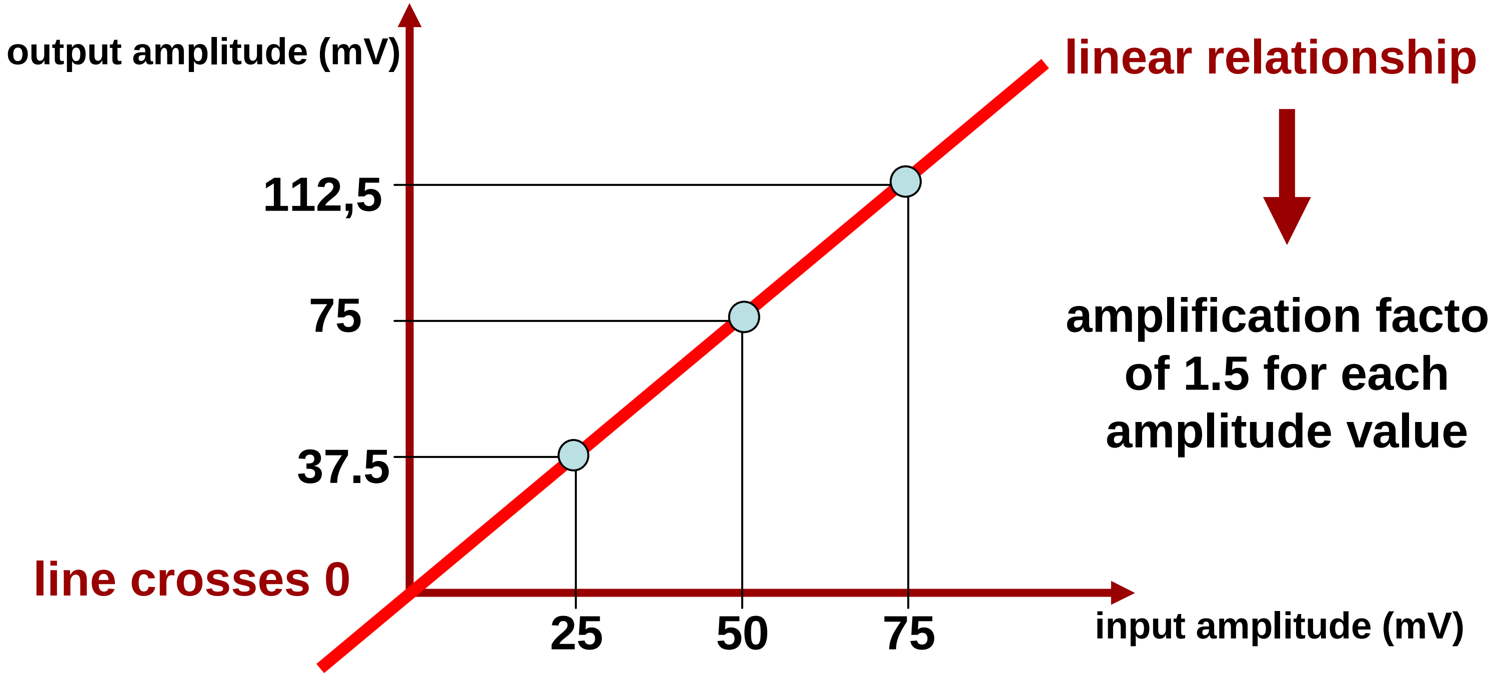


SYSTEM

amplifier

300 Hz 37.5/75/112,5 mV





amplification factor of 1.5 for each amplitude value

Additivity

A system is additive if the sum of individual output signals eaquals the sum of the same individual input signals after having passed through the system, i.e:

if:

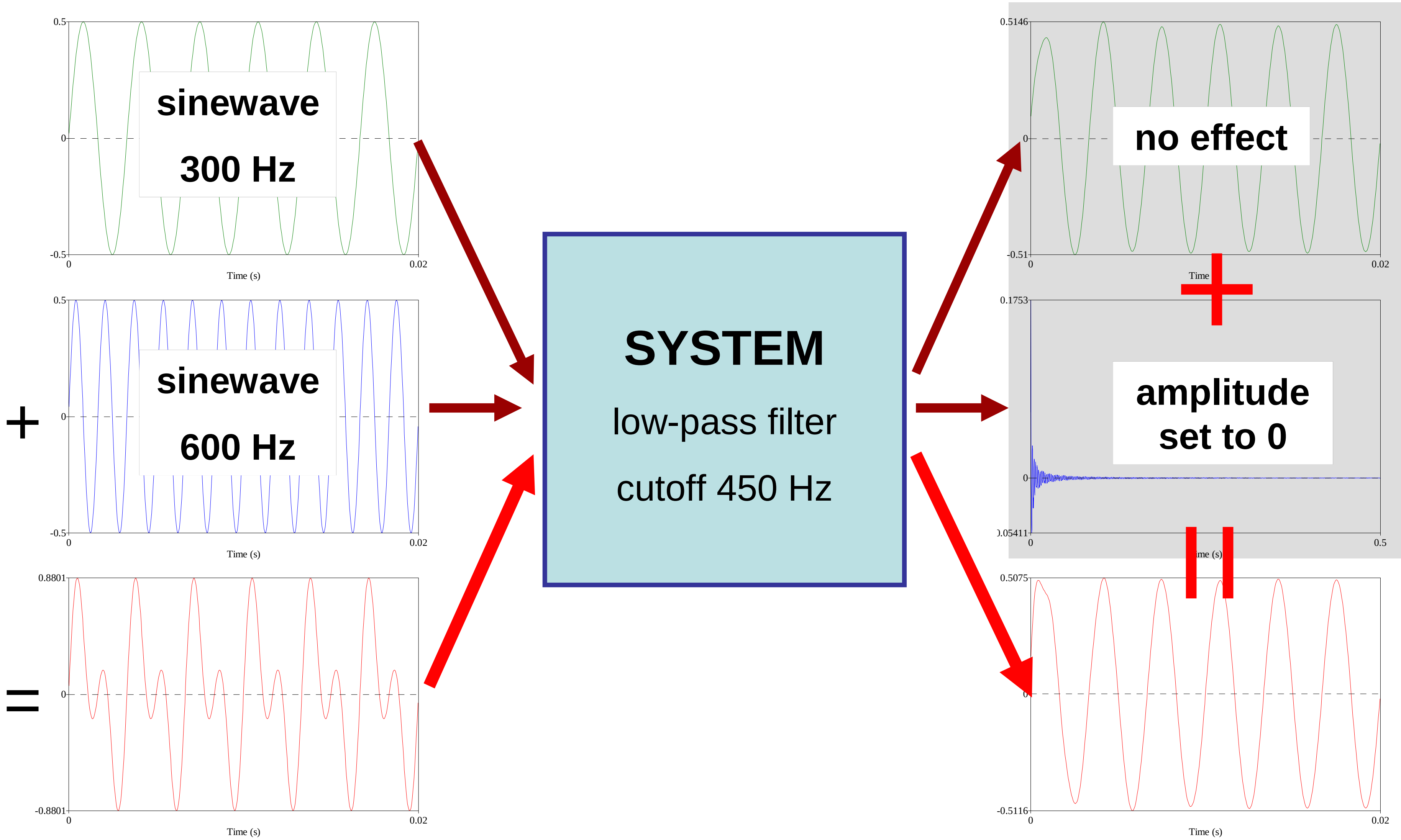
$inp1(t) \rightarrow outp1(t)$

$inp2(t) \rightarrow outp2(t)$

then:

$[inp1(t) + inp2(t)] \rightarrow [outp1(t) + outp2(t)]$

Additivity



sinewave
300 Hz
+
sinewave
600 Hz
=
SYSTEM
low-pass filter
cutoff 450 Hz
no effect
amplitude
set to 0
Time (s)

Time-Invariance

A system is time-invariant if it does not change the output over time, i.e.:

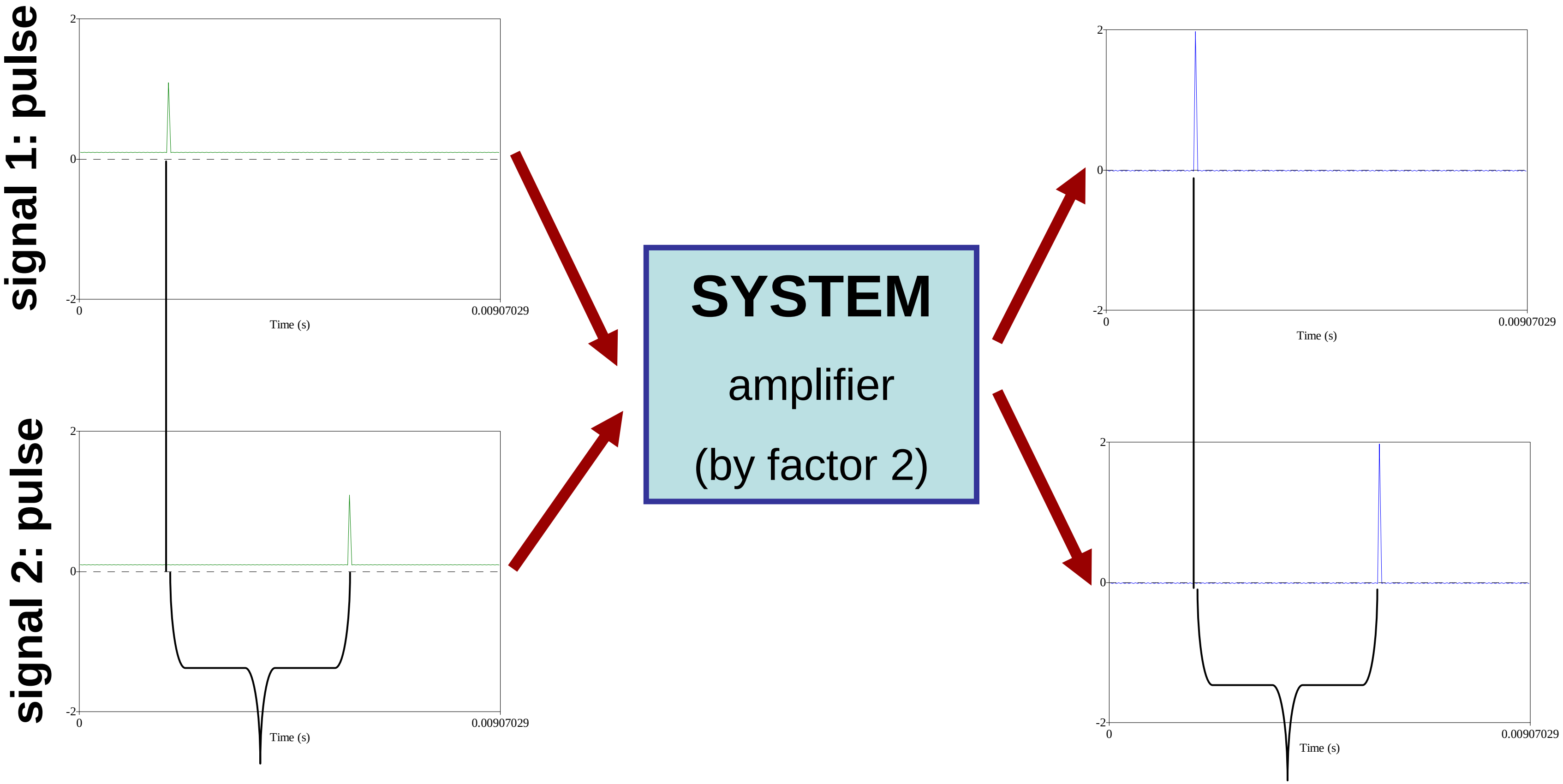
if:

inp(t) → outp(t)

then:

inp(t) + d seconds → outp(t) + d seconds

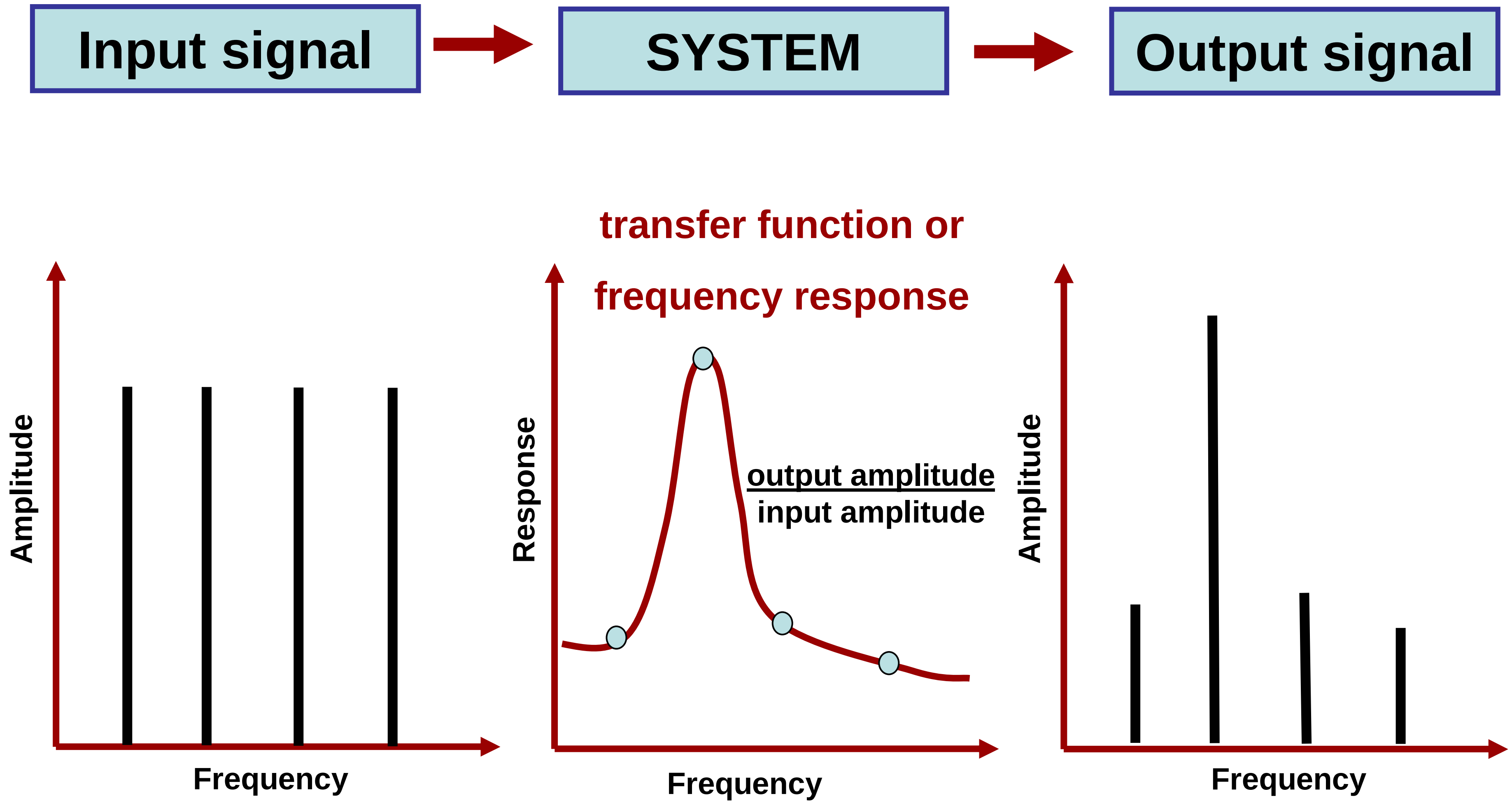
Time-Invariance



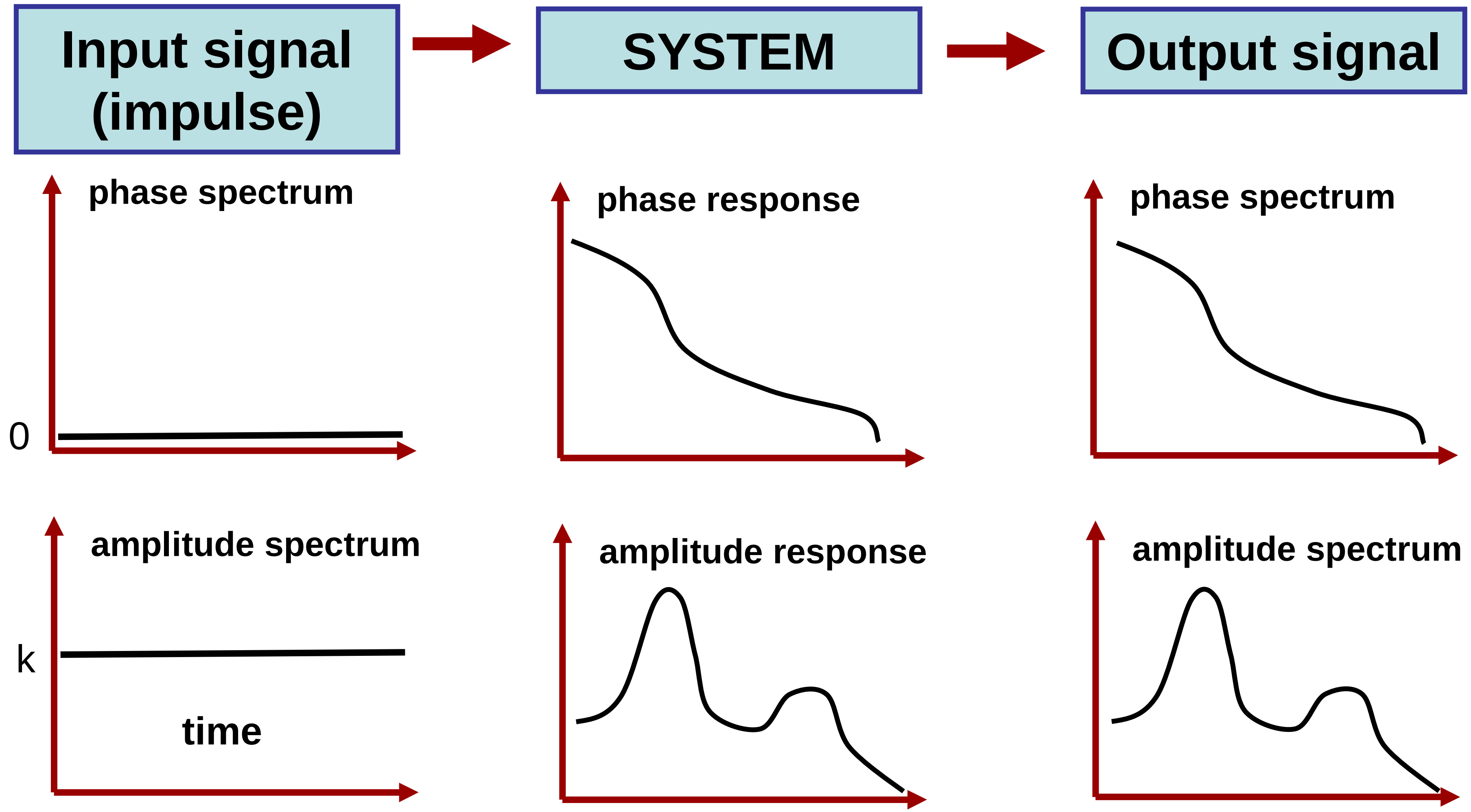
delayed by 3 ms

IMPORTANT: delayed of 3 ms stays constant

Frequency Domain Characterisation of LTIs

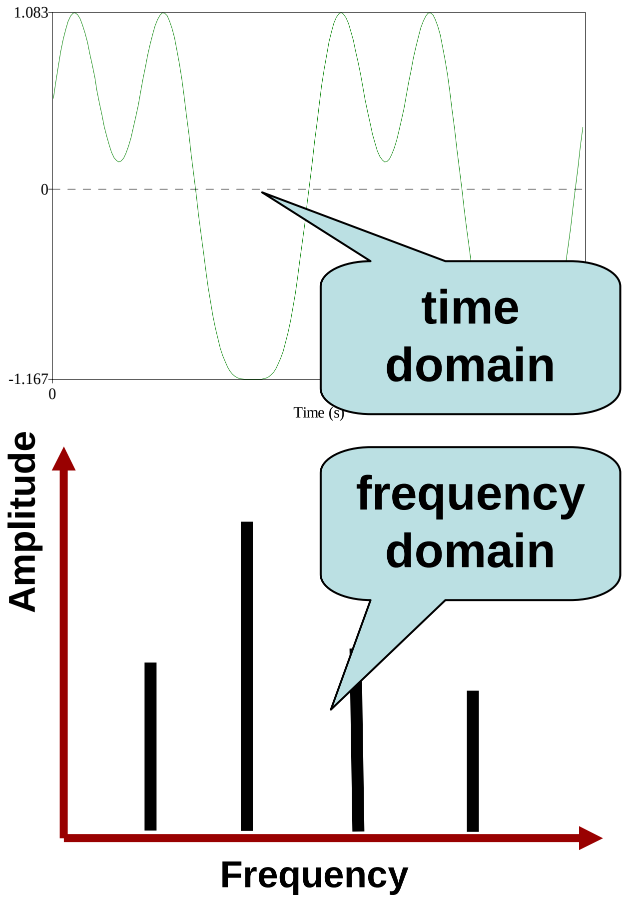


Time-Domain Characterisation of LTIs



Charaterisation of signals and systems

SIGNALS

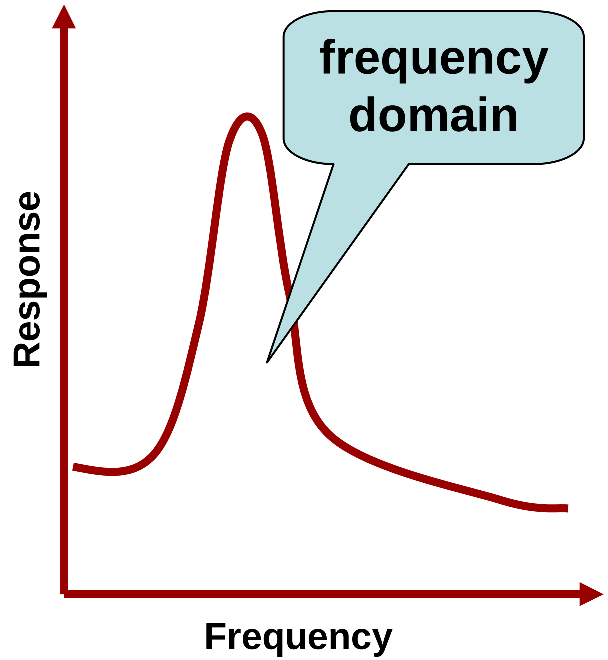


SYSTEM

transfer function

or

frequency response



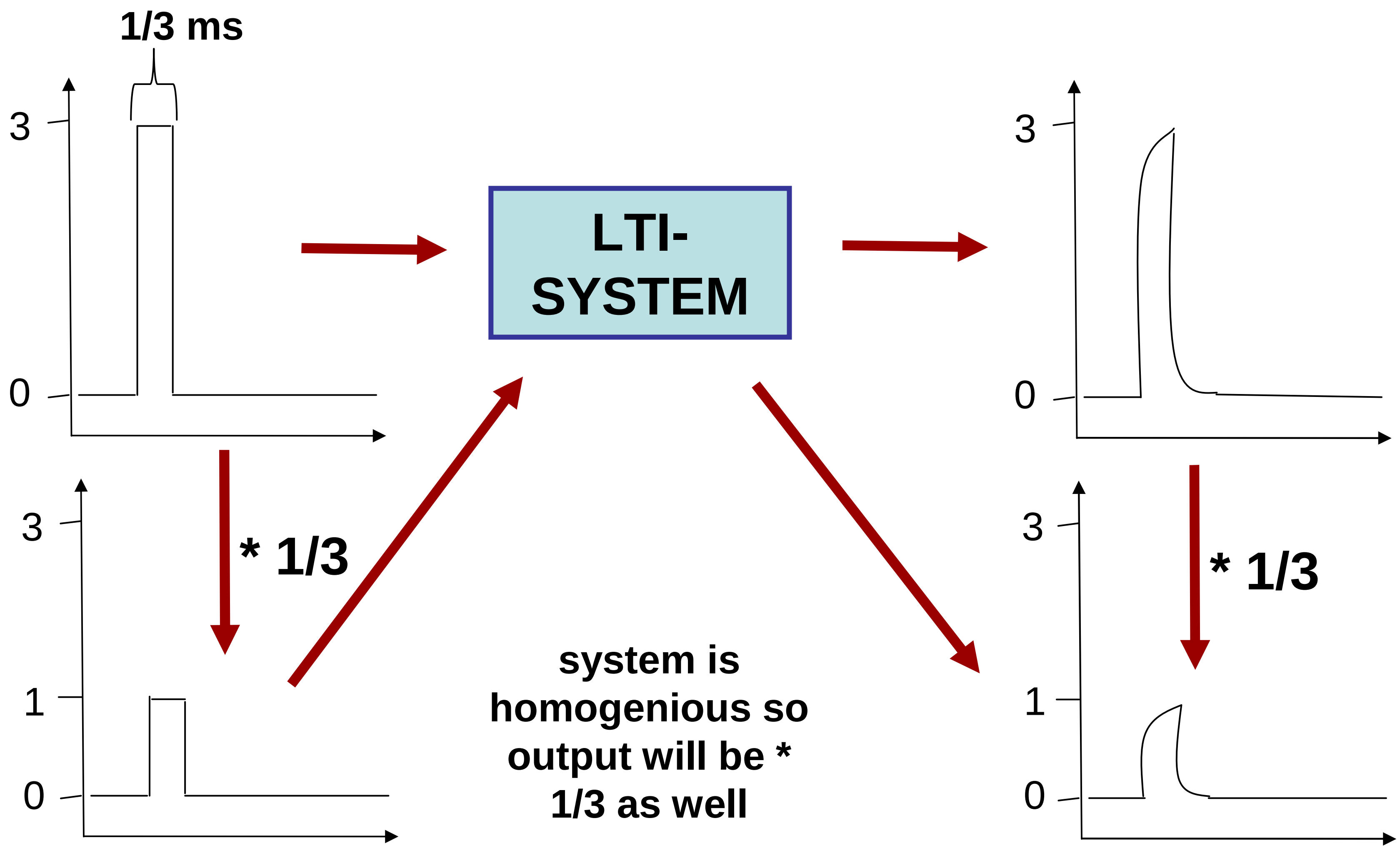
Question:

Can systems be characterised by a time domain?

Answer:

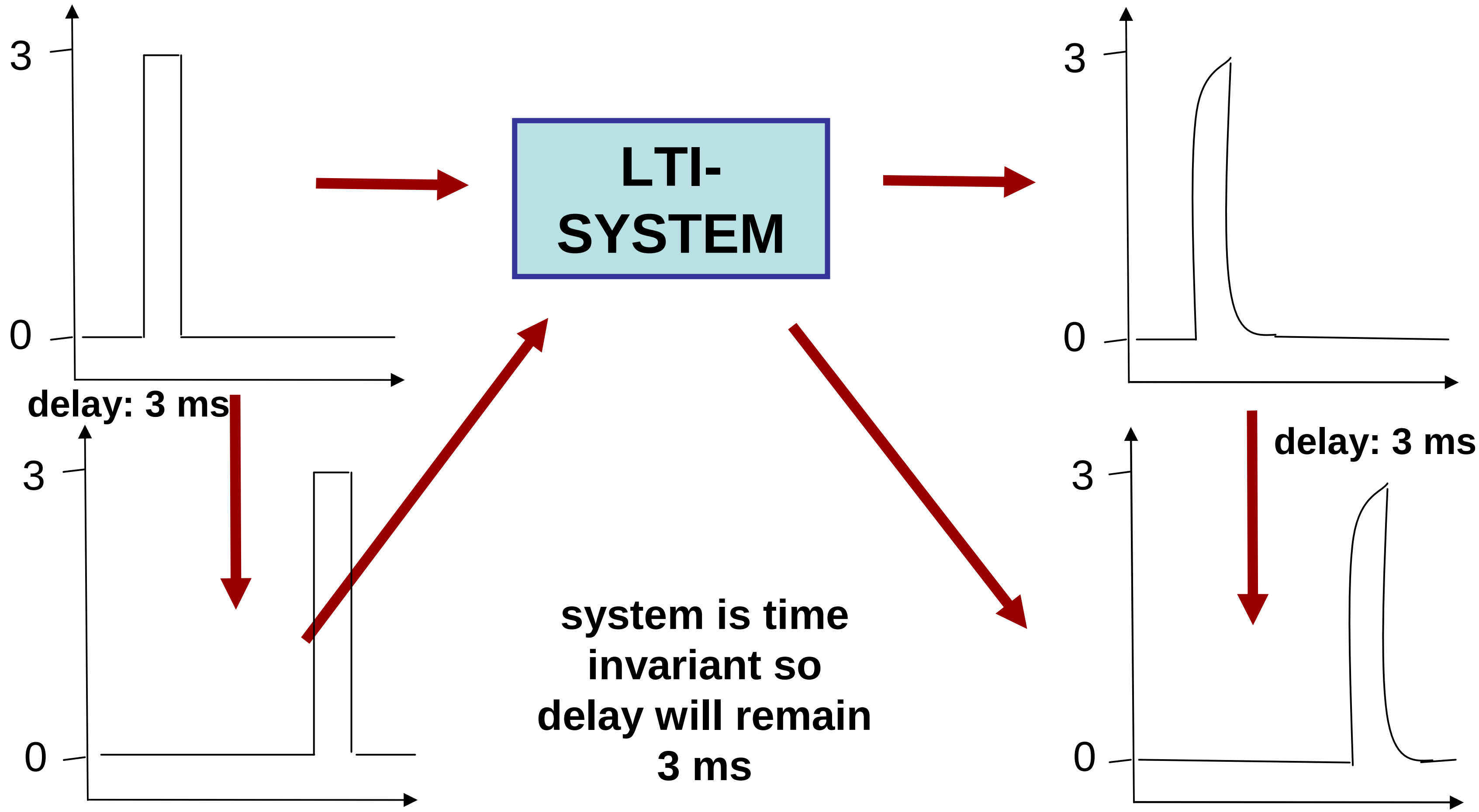
YES!

Time-domain characterisation of LTI systems

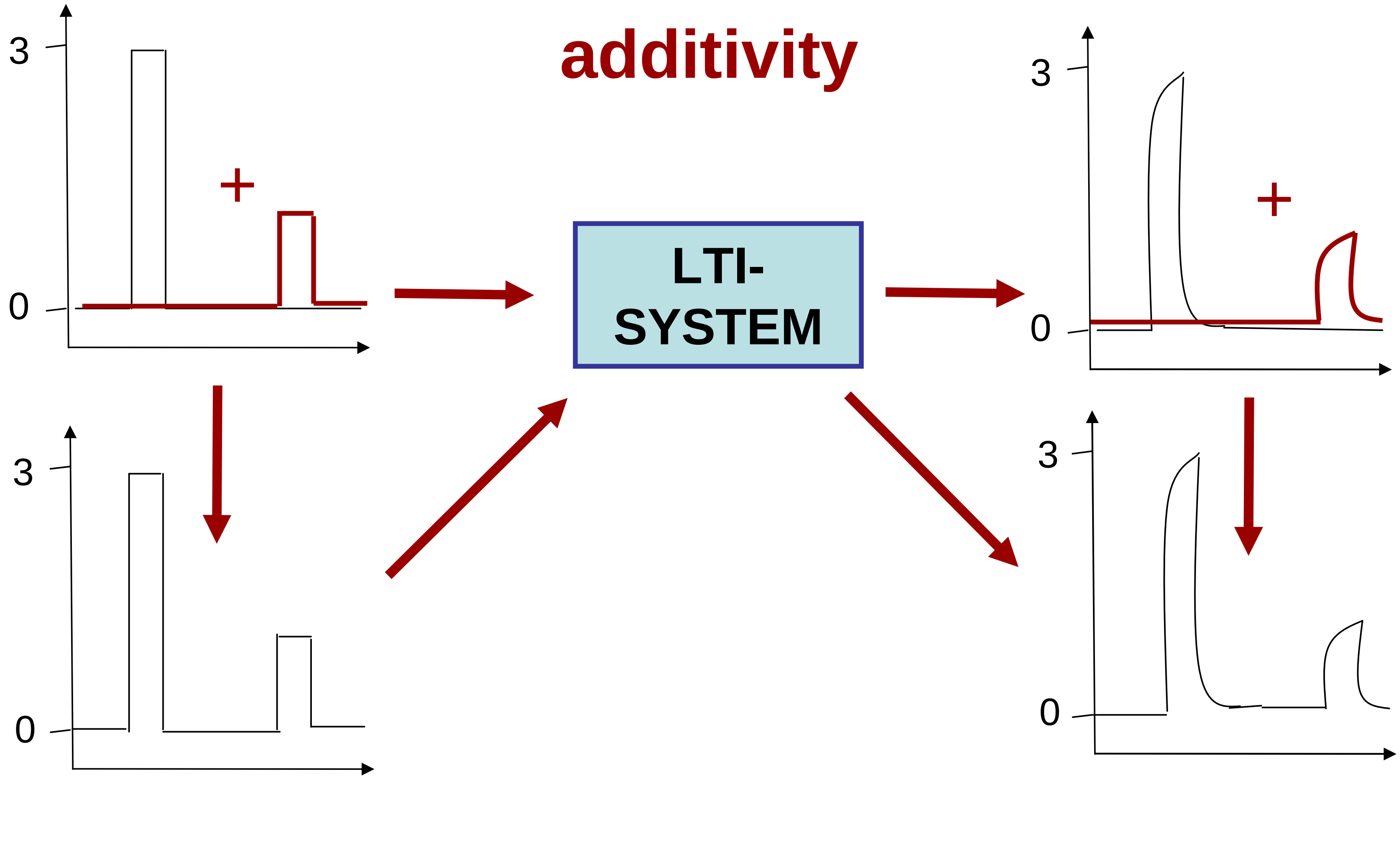


system is homogenious so output will be * 1/3 as well

Time-domain characterisation of LTI systems

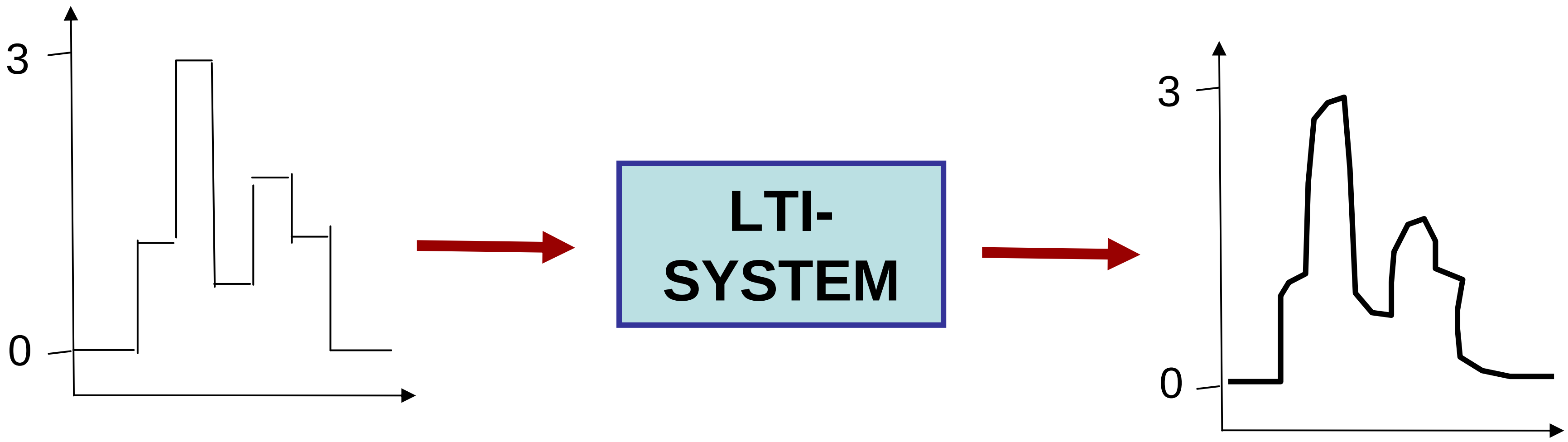


Time-domain characterisation of LTI systems



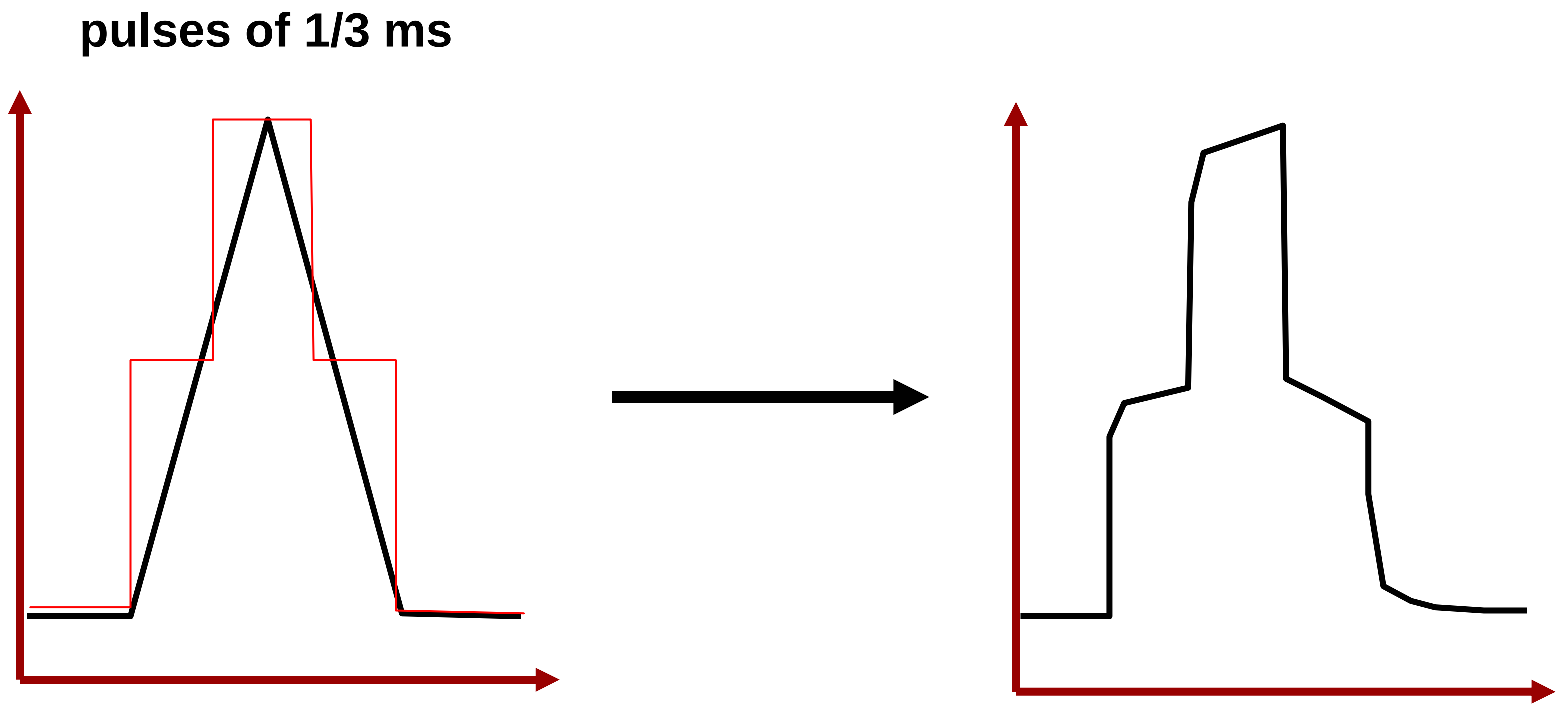
Time-domain characterisation of LTI systems

What we can do already:



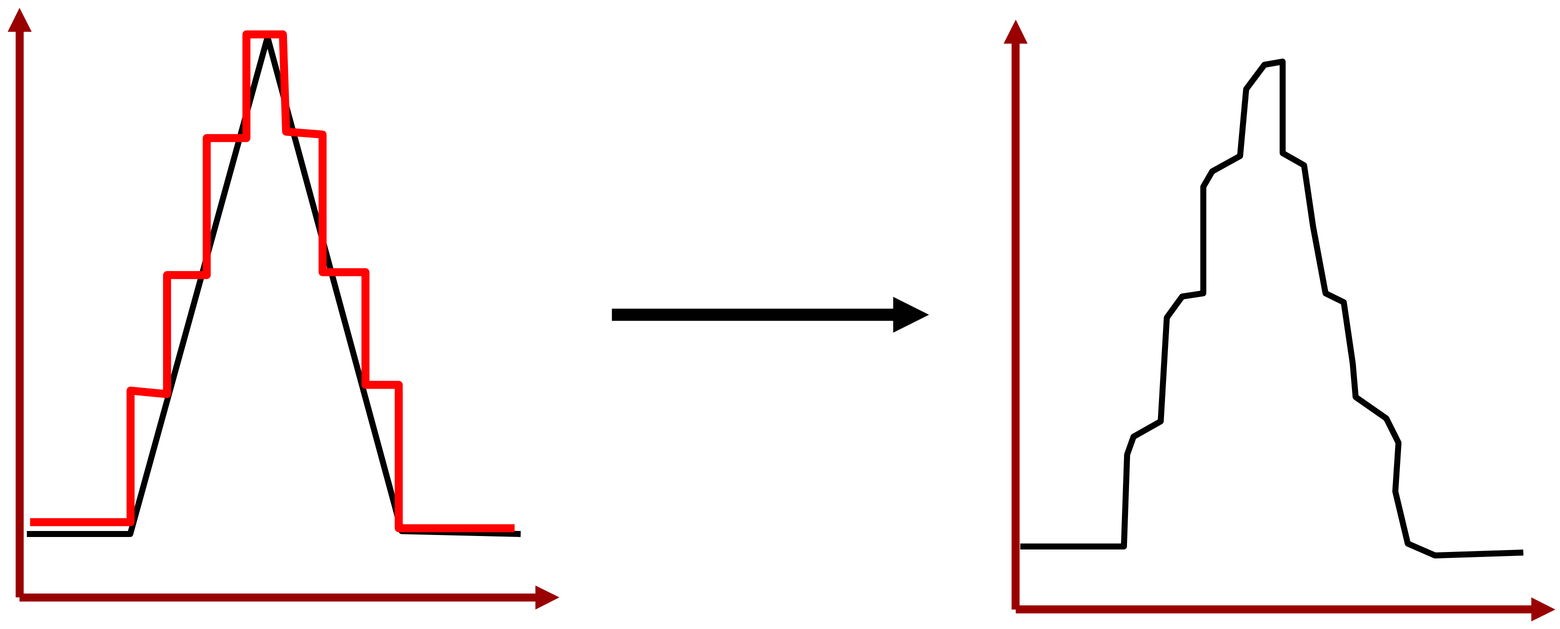
If we know the system response to one single pulse, we can predict the output for a 'complicated' chain of pulses.

Time-domain characterisation of LTI systems



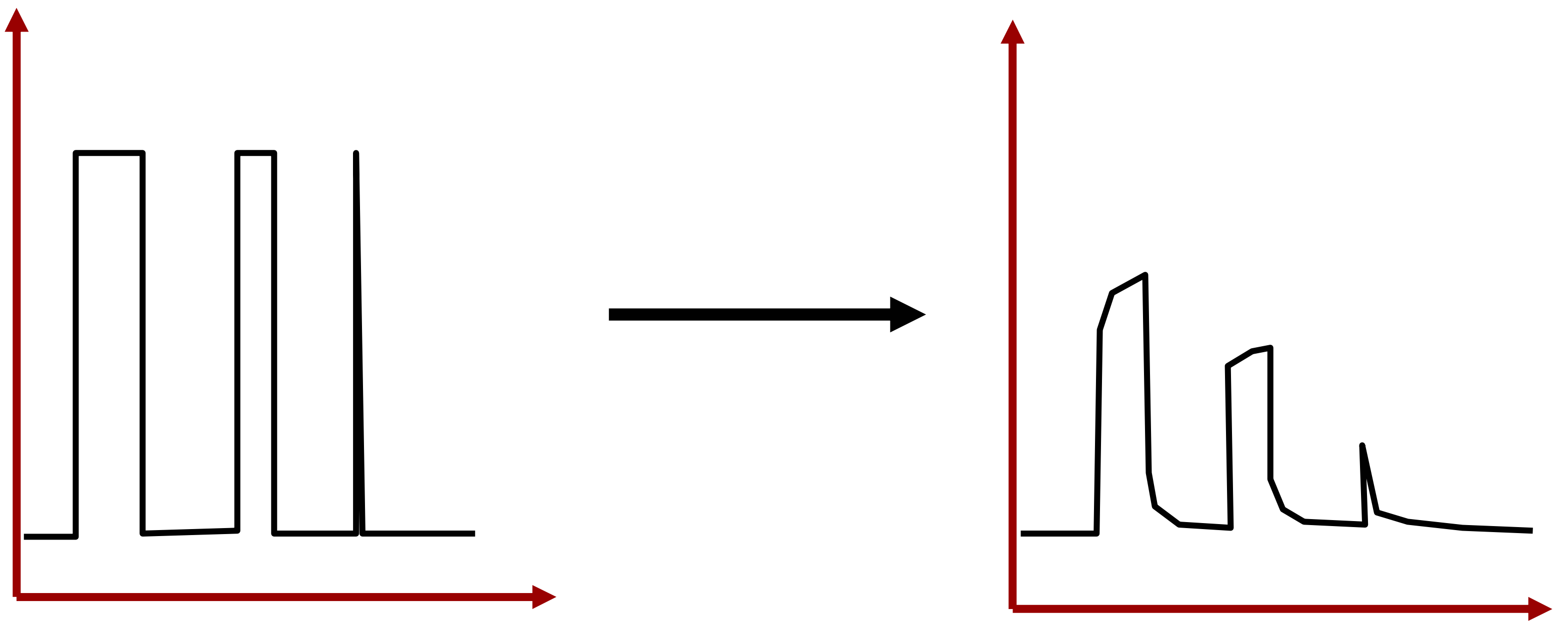
Time-domain characterisation of LTI systems

pulses of 1/7 ms



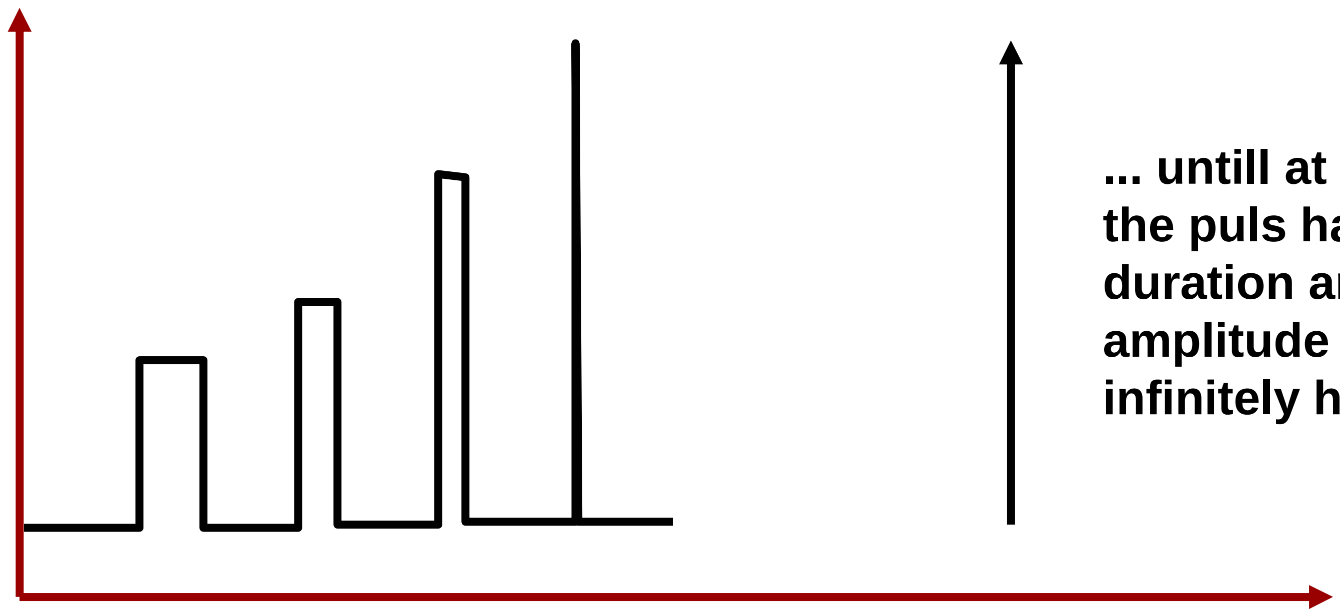
Time-domain characterisation of LTI systems

relationship between pulse duration and output amplitude:



the shorter the puls the less energy it contains,
the smaller will be the amplitude of the output signal

Time-domain characterisation of LTI systems

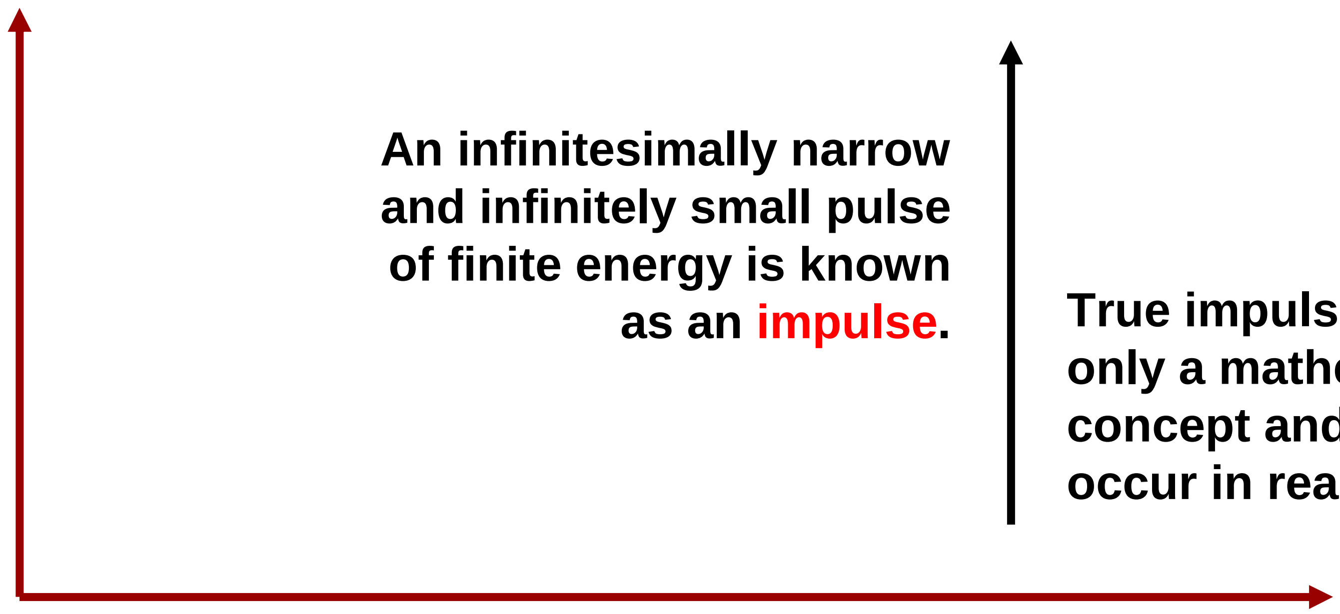


... untill at some point the puls has no duration and its amplitude is infinitely high

In order not to loose energy of the output signal we increase the amplitude of the input pulse ...

Time-domain characterisation of LTI systems

relationship between puls duration and output amplitude:



Time-domain characterisation of LTI systems

IMPORTANT:

- **With an impulse response ANY output of an LTI system can be predicted by the input.**

- **Therefor: In the time-domain LTI systems are completely characterised by their impulse response.**

Time-domain characterisation of LTI systems

Concept of CONVOLUTION

- **True impulses cannot be used in signal processing.**
- **Convolution is used to calculate the time response of an LTI system on a given pulse response.**

Convolution requires thorough knowledge of calculus.

Time-domain characterisation of LTI systems

The relationship between the
impulse response and the frequency response

impulse response and frequency response are two different points of view to look at the same thing: the response of a system

We should actually be able e.g. to determine the impulse response from the frequency domain.

This is what we will do in the following.

Time-domain characterisation of LTI systems

Suppose we know the frequency response to a system and want to know what its output is to an impulse. Remember how to do that?

Output amplitude (f) = Input amplitude (f) X Amplitude response (f)

And for the phase spectrum?

Output phase (f) = Input phase (f) + Phase response (f)

Time-domain characterisation of LTI systems

What is the phase spectrum of an impulse?

Answer: 0

What is the amplitude spectrum of an impules?

Answer: constant (k)

Output amplitude (f) = k X Amplitude response (f)

Output phase (f) = 0 + Phase response (f)

Time-domain characterisation of LTI systems

conclusion

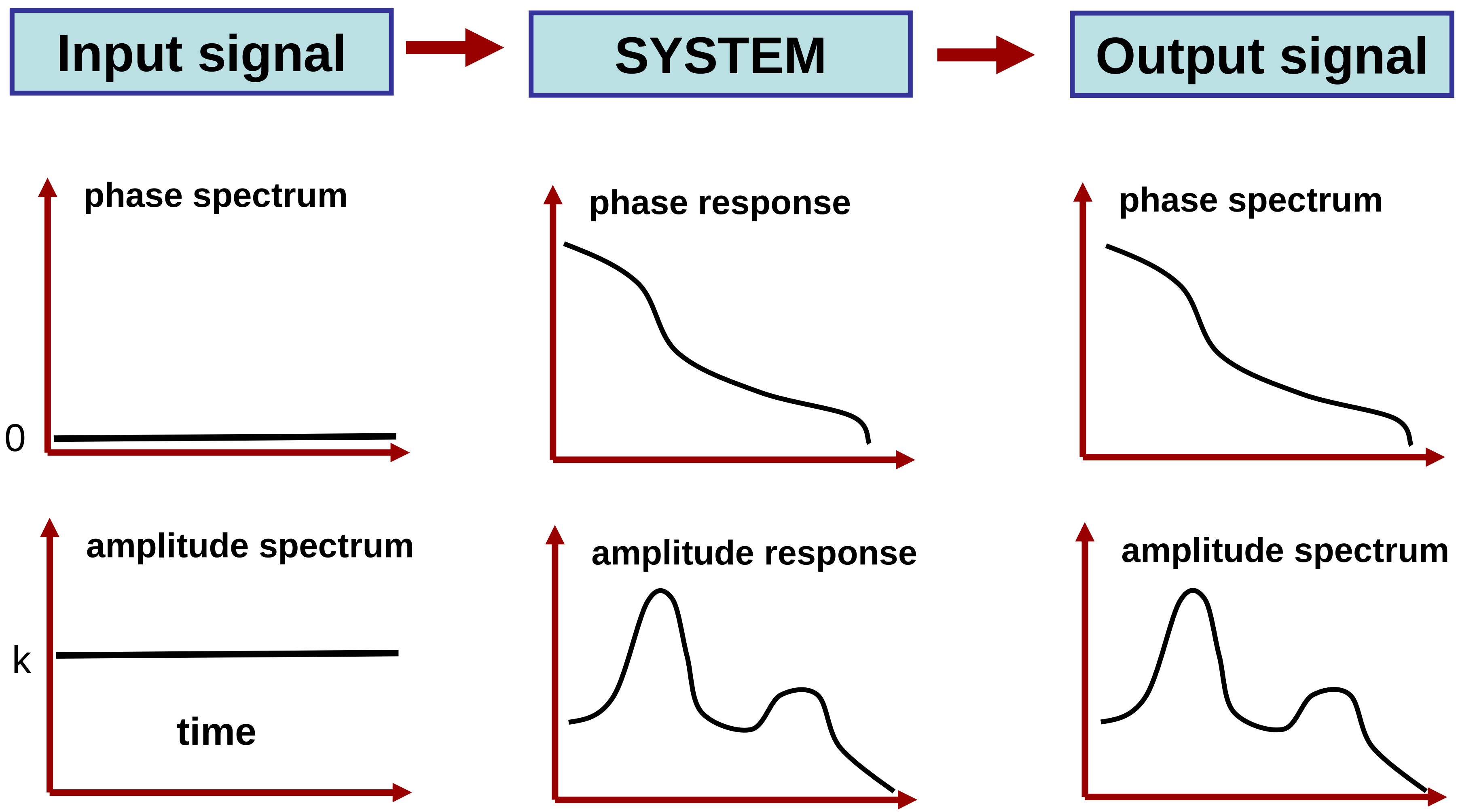
Output phase (f) = Phase response (f)

→ The phase spectrum of the impulse response of a system is simply the phase response of the system

Output amplitude (f) = k X Amplitude response (f)

→ The spectrum of a system impulse response is simply the amplitude response of the system.

Time-domain characterisation of LTI systems



Examples of Simple Systems

To get some idea of typical systems (and their properties), consider the electrical circuit example:

$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

which is a **first order**, **CT differential** equation.

Examples of **first order**, **DT difference** equations:

$$y[n] = x[n] + 1.01y[n-1]$$

where *y* is the monthly bank balance, and *x* is monthly net deposit

$$v[n] - \frac{RC}{RC+k} v[n-1] = \frac{k}{RC+k} f[n]$$

which represents a discretised version of the electrical circuit

Example of second order system includes:

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = x(t)$$

System described by **order** and **parameters** (*a*, *b*, *c*)

First Order Step Responses

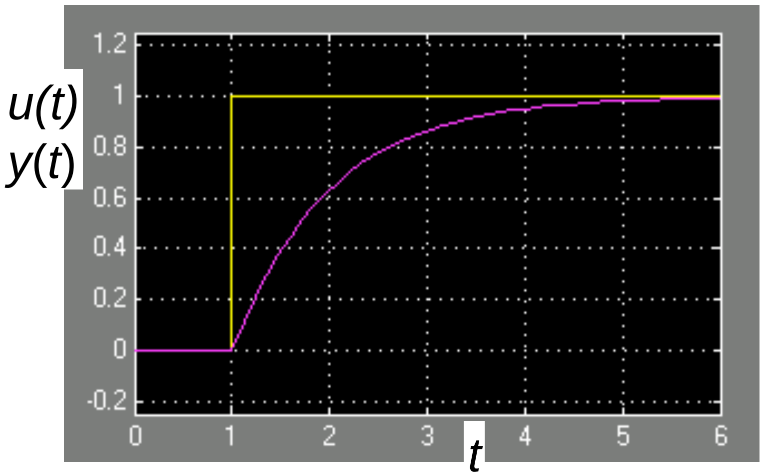
People tend to visualise systems in terms of their responses to simple input signals (see Lecture 4…)

The dynamics of the output signal are determined by the dynamics of the system, if the input signal has no dynamics

Consider when the input signal is a step at *t, n* = 1, *y*(0) = 0

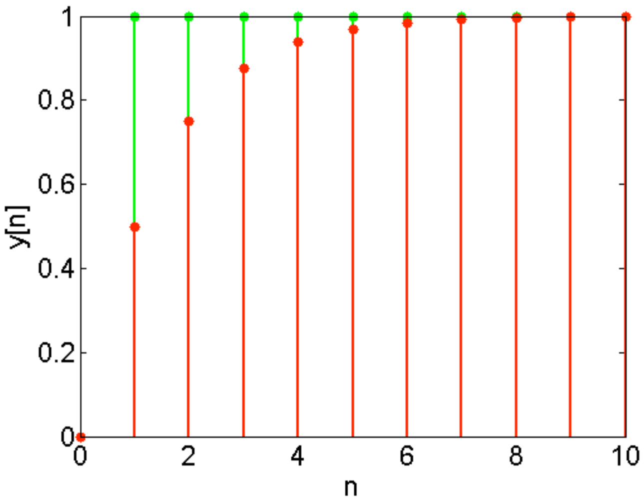
First order CT differential system

$$\frac{dy(t)}{dt} + ay(t) = u(t-1)$$



First order DT difference system

$$y[n](1+ak) - y[n-1] = ku[n-1]$$

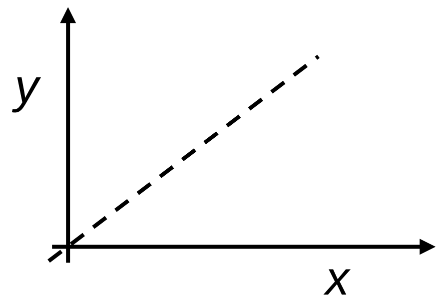


System Linearity

The most important property that a system possesses is **linearity**

It means allows any system response to be analysed as the sum of simpler responses (convolution)

Simplistically, it can be imagined as a line



Specifically, a linear system must satisfy the two properties:

1 Additive: the response to $x_1(t)+x_2(t)$ is $y_1(t) + y_2(t)$

2 Scaling: the response to $ax_1(t)$ is $ay_1(t)$ where $a \in \mathbb{C}$

Combined: $ax_1(t)+bx_2(t) \rightarrow ay_1(t) + by_2(t)$

E.g. Linear $y(t) = 3*x(t)$ why?

Non-linear $y(t) = 3*x(t)+2$, $y(t) = 3*x^2(t)$ why?

(equivalent definition for DT systems)

Bias and Zero Initial Conditions

Intuitively, a system such as:

$y(t) = 3*x(t)+2$

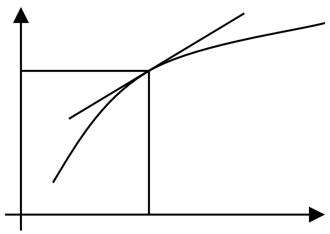
is regarded as being linear. However, it does not satisfy the scaling condition.

There are several (similar) ways to transform it to an equivalent linear system

Perturbations around operating value x^*, y^*

$$\varepsilon_x(t) = x(t) - x^*, \quad \varepsilon_y(t) = y(t) - y^*$$

$$\varepsilon_y(t) = 3*\varepsilon_x(t)$$



Linear System Derivative

$$\partial y(t) = 3\partial x(t)$$

Locally, these ideas can also be used to linearise a non-linear system in a small range

Linearity and Superposition

Suppose an input signal $x[n]$ is made of a linear sum of other (basis/simpler) signals $x_k[n]$:

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \cdots$$

then the (linear) system response is:

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \cdots$$

The basic idea is that if we understand how simple signals get affected by the system, we can work out how complex signals are affected, by expanding them as a linear sum

This is known as the superposition property which is true for linear systems in both CT & DT

Important for understanding **convolution** (next lecture)

Definition of Time Invariance

A system is time invariant if its behaviour and characteristics are fixed over time

We would expect to get the same results from an input-output experiment, if the same input signal was fed in at a different time

E.g. The following CT system is **time-invariant**

$$y(t) = \sin(x(t))$$

because it is invariant to a time shift, i.e. $x_2(t) = x_1(t-t_0)$

$$y_2(t) = \sin(x_2(t)) = \sin(x_1(t-t_0)) = y_1(x_1(t-t_0))$$

E.g. The following DT system is **time-varying**

$$y[n] = nx[n]$$

Because the **system parameter** that multiplies the input signal is time varying, this can be verified by substitution

$$x_1[n] = \delta[n] \Rightarrow y_1[n] = 0$$

$$x_2[n] = \delta[n-1] \Rightarrow y_2[n] = \delta[n-1]$$

System with and without Memory

A system is said to be memoryless if its output for each value of the independent variable at a given time is dependent on the output at only that same time (no system dynamics)

$$y[n] = (2x[n] - x^2[n])^2$$

e.g. a resistor is a memoryless CT system where *x*(*t*) is current and *y*(*t*) is the voltage

A DT system with memory is an accumulator (integrator)

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

and a delay

$$y[n] = x[n-1]$$

Roughly speaking, a memory corresponds to a mechanism in the system that retains information about input values other than the current time.

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n] \end{aligned}$$

System Causality

A system is causal if the output at any time depends on values of the output at only the present and past times. Referred to as non-anticipative, as the system output does not anticipate future values of the input

If two input signals are the same up to some point t_0/n_0, then the outputs from a causal system must be the same up to then.

E.g. The accumulator system is causal:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

because $y[n]$ only depends on $x[n]$, $x[n-1]$, …

E.g. The averaging/filtering system is non-causal

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

because $y[n]$ depends on $x[n+1]$, $x[n+2]$, …

Most physical systems are causal

System Stability

Informally, a stable system is one in which small input signals lead to responses that do not diverge

If an input signal is bounded, then the output signal must also be bounded, if the system is stable

$$\forall x : |x| < U \rightarrow |y| < V$$

To show a system is stable we have to do it for **all** input signals. To show instability, we just have to find one counterexample

E.g. Consider the DT system of the bank account

$$y[n] = x[n] + 1.01y[n-1]$$

when $x[n] = \delta[n]$, $y[0] = 0$

This grows without bound, due to 1.01 multiplier. This system is unstable.

E.g. Consider the CT electrical circuit, is stable if RC>0, because it dissipates energy

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

Invertible and Inverse Systems

A system is said to be **invertible** if distinct inputs lead to distinct outputs (similar to matrix invertibility)

If a system is invertible, an inverse system exists which, when **cascaded** with the original system, yields an output equal to the input of the first signal

E.g. the CT system is invertible:

$$y(t) = 2x(t)$$

because $w(t) = 0.5*y(t)$ recovers the original signal $x(t)$

E.g. the CT system is not-invertible

$$y(t) = x^2(t)$$

because distinct input signals lead to the same output signal

Widely used as a design principle:

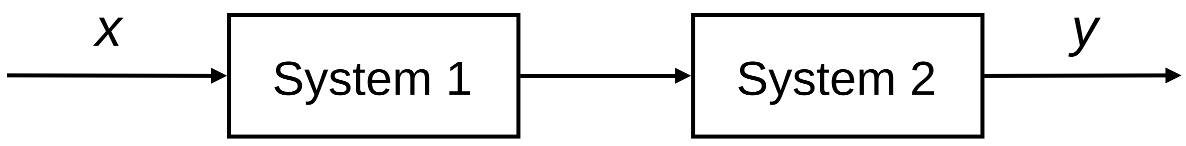
- Encryption, decryption
- System control, where the reference signal is input

System Structures

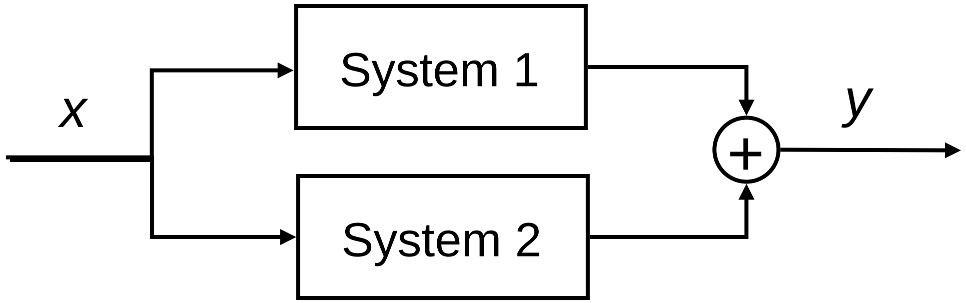
Systems are generally composed of components (sub-systems).

We can use our understanding of the components and their interconnection to understand the operation and behaviour of the overall system

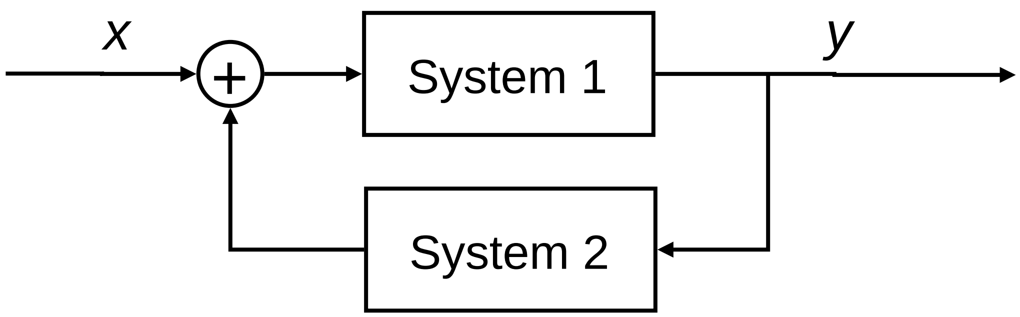
Series/cascade



Parallel



Feedback



Systems In Matlab

A system transforms a signal into another signal.

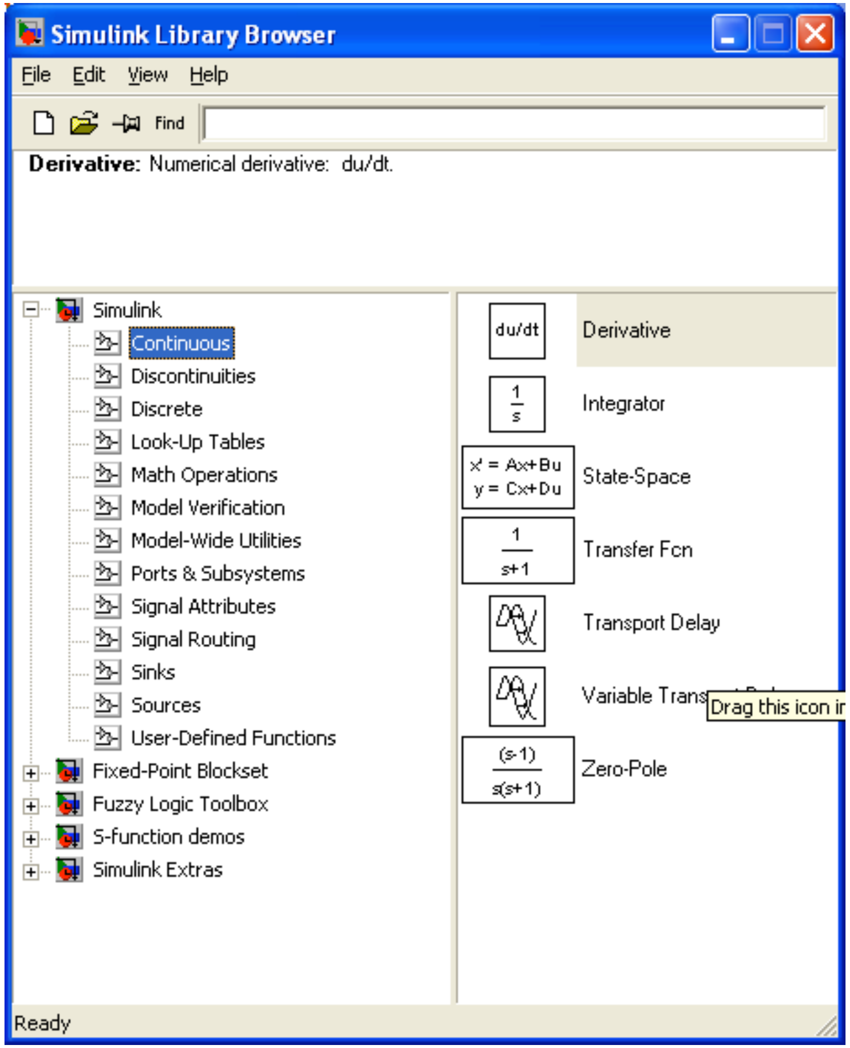
In Matlab a discrete signal is represented as an indexed vector.

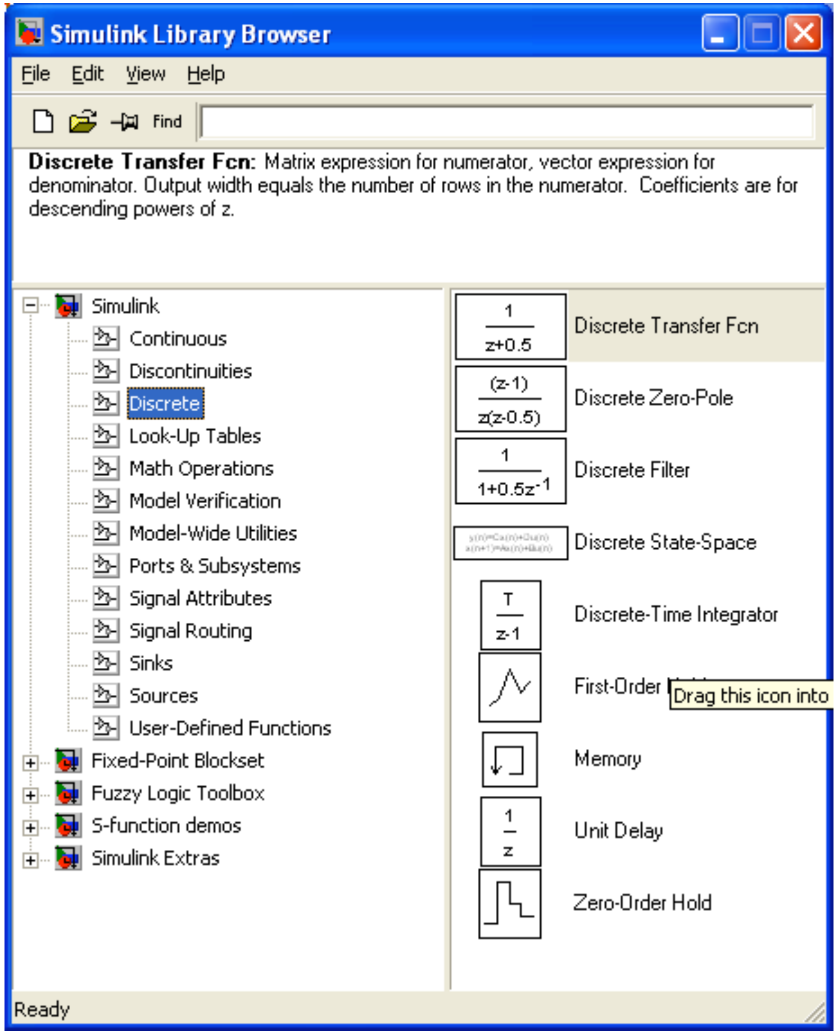
Therefore, a matrix or a for loop can be used to transform one vector into another

Example (DT first order system)

```
>> n = 0:10;
>> x = ones(size(n));
>> x(1) = 0;
>> y(1) = 0;
>> for i=2:11
      y(i) = (y(i-1) + x(i))/2;
end
>> plot(n, x, ‘x’, n, y, ‘.’)
```

System Libraries in Simulink





LTI system

- **Useful fact:** Given a sinusoid at the input, the output of a linear time-invariant (LTI) system is a sinusoid with the same frequency, but possibly with different phase and amplitude

  ⇨ Given an input that is described as a sum of sinusoids of certain frequencies, the output can be described as a sum of sinusoids with the same frequencies, but with possible phase and amplitude changes at each frequency

  - Key idea behind filtering

Discrete-time LTI systems

$x[n]$ - a discrete-time signal

- We want to rewrite $x[n]$ using unit impulse functions
  - More specifically, utilizing the sampling property

Define $a_k, k \in \mathbb{Z},$ where $a_k[n] = x[k]\ \delta[n-k]$

- We can rewrite $x[n] = \sum_{k \in \mathbb{Z}} a_k[n] = \sum_{k \in \mathbb{Z}} x[k]\ \delta[n-k]$
  - Superposition of scaled versions of time shifted unit impulses $\delta[n-k]$

Discrete-time LTI systems

- Why do we want to rewrite $x[n]$ in such a cumbersome way?!?

$$x \longrightarrow \boxed{S} \longrightarrow y = S(x)$$

- $x[n] = \sum_{k \in \mathbf{Z}} x[k] \;\; \delta[n-k]$

- Let $h = S(\delta)$, $v_k = D_k(\delta)$, and $w_k = S(v_k)$

$$\delta \longrightarrow \boxed{S} \longrightarrow h = S(\delta)$$

- $x = \sum_{k \in \mathbf{Z}} x[k] \;\; v_k \;\; , \;\; i.e., \;\; x[n] = \sum_{k \in \mathbf{Z}} x[k] \;\; v_k[n]$

- $\boxed{y[n] = \sum_{k \in \mathbf{Z}} x[k] \;\; h[n-k]}$

output is a sum of time-shifted versions of h with weights x[k]

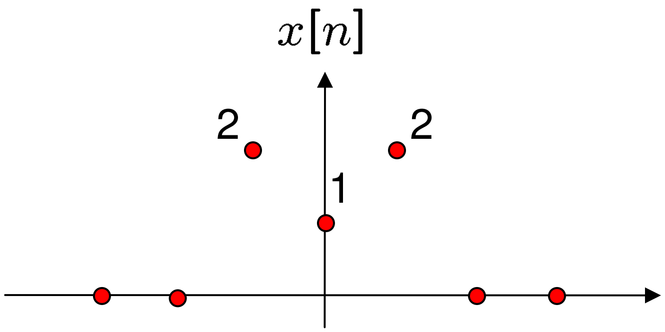
Discrete-time LTI systems

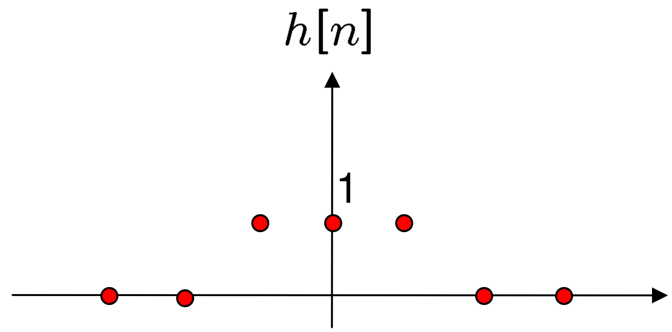
- A discrete-time LTI system is completely characterized by $h = S(\delta)$
  - Called **(unit) impulse response**

- $y[n] = \sum_{k \in \mathbb{Z}} x[k]\ h[n-k]$ is known as **convolution sum** or **superposition sum**
  - Operation is called the **convolution** of sequences $x[n]$ and $h[n]$
  - Notation: $y[n] = x[n] * h[n]$ or $y[n] = (x * h)[n]$

Discrete-time LTI systems

Example: $y[n] = \sum_{k \in \mathbf{Z}} x[k] \; h[n-k]$

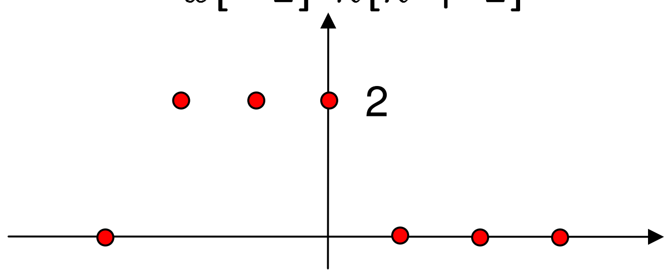




Evaluation of convolution sum: Method #1

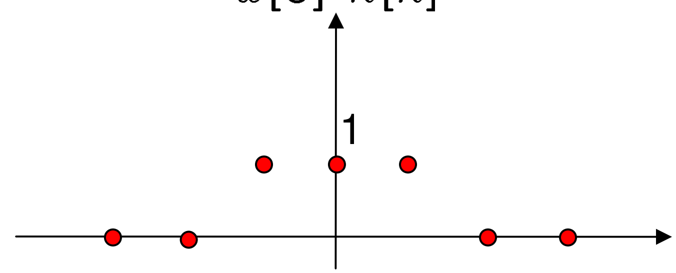
$$y[n] = \sum_{k \in \mathbf{Z}} x[k]\; h[n-k] = \sum_{k=-1}^{1} x[k]\; h[n-k]$$

$$x[-1]\; h[n+1]$$



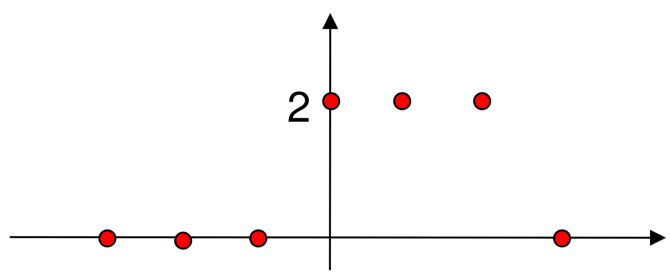
+

$$x[0]\; h[n]$$



+

$$x[1]\; h[n-1]$$



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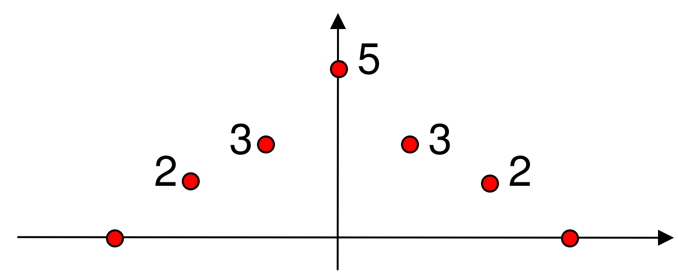
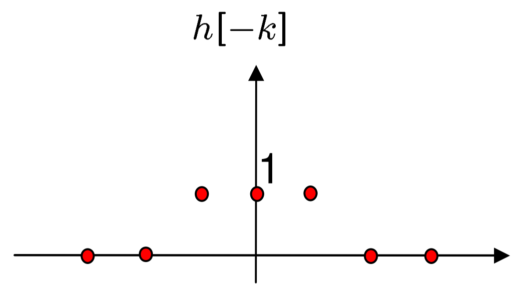
$$y[n] = \sum_{k=-1}^{1} x[k]\; h[n-k]$$



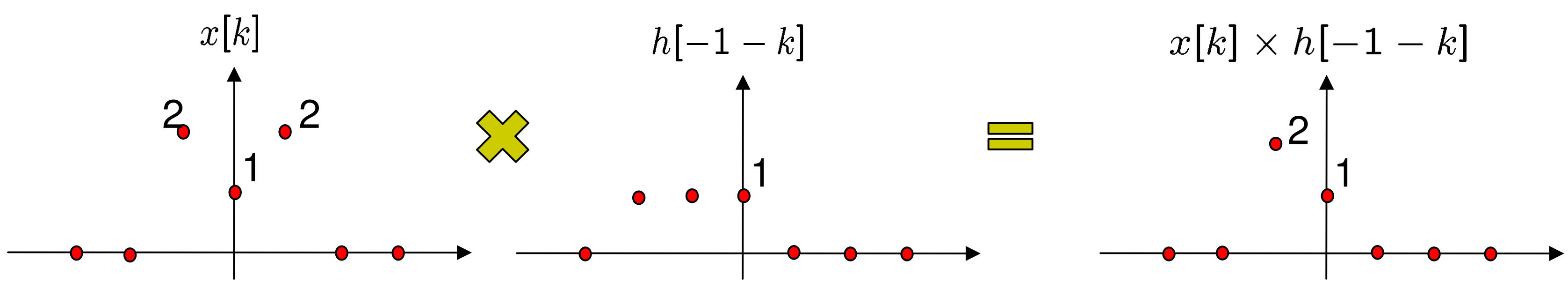
Evaluation of convolution sum: Method #2

$$y[n] = \sum_{k \in \mathbb{Z}} x[k]\ h[n-k]$$

- Take $h[-k]$
- Shift it to the **right** by n
- Take the product $x[k]\ h[n-k]$
- Take summation

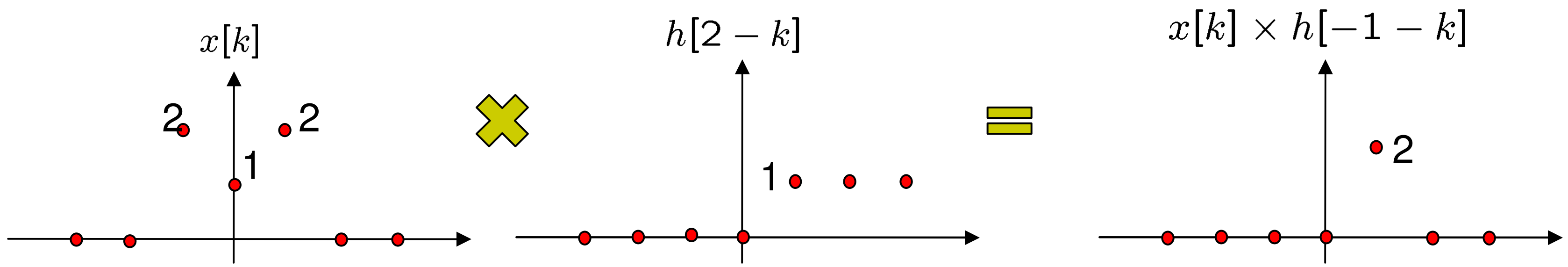


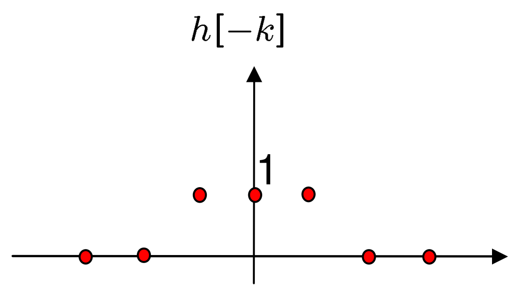
Example: y[-1]



Evaluation of convolution sum: Method #2

- Example: y[2]





Continuous-time LTI systems

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t)$$

- We can rewrite $x(t)$ using the unit impulse function

$$x(t) = \int_{\mathbb{R}} x(\tau)\ \delta(t-\tau)\ d\tau$$

- Continuous-time counterpart (after similar exercise)

$$y(t) = \int_{\mathbb{R}} x(\tau)\ h(t-\tau)\ d\tau$$

where $h = S(\delta)$ (called **(unit) impulse response**)

  - Integration called **convolution integral** or **superposition integral**
  - Notation: $y(t) = x(t) * h(t) \text{ or } y(t) = (x * h)(t)$
  - CT LTI system characterized by its impulse response

Procedure for evaluating convolution

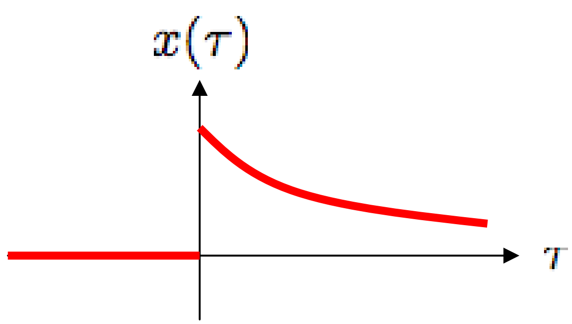
$$y(t) = \int_{\mathbb{R}} x(\tau)\ h(t-\tau)\ d\tau$$

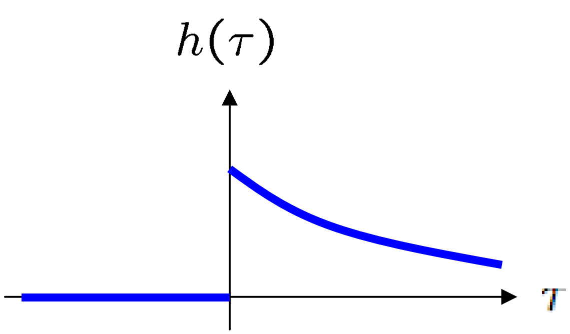
- Take $h(-\tau)$
- Shift it to the **right** by t
- Take the product $x(\tau)\ h(t-\tau)$
- Integrate over the "**intersection**"

Examples: $x(t) = e^{-at}\ u(t)\ ,\ a > 0$

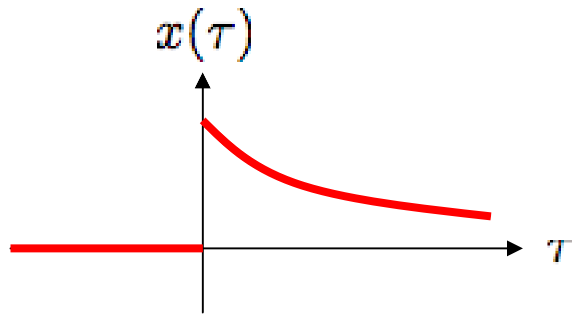
$h(t) = e^{-\beta t}\ u(t)\ ,\ \beta > 0$

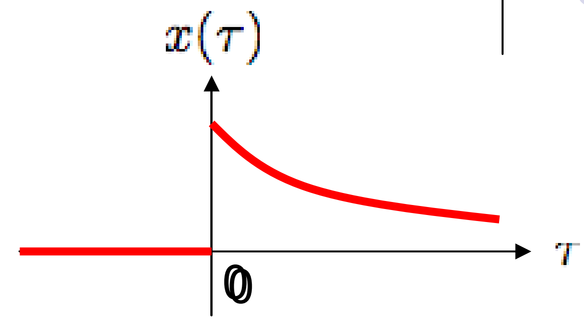
0

- 

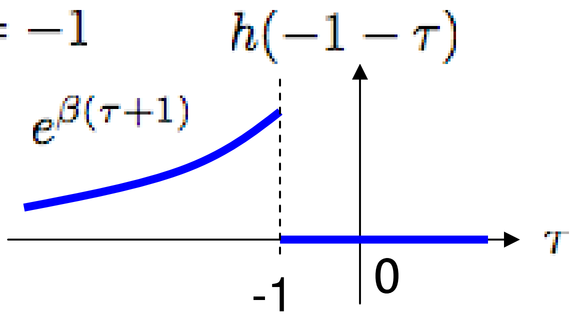


Procedure for evaluating convolution

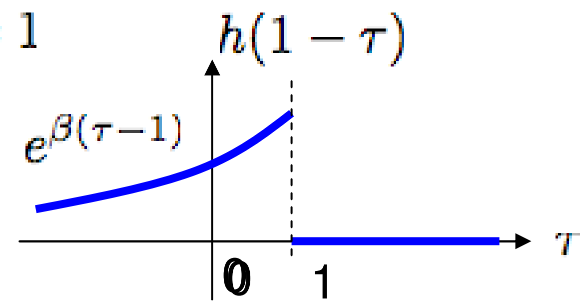


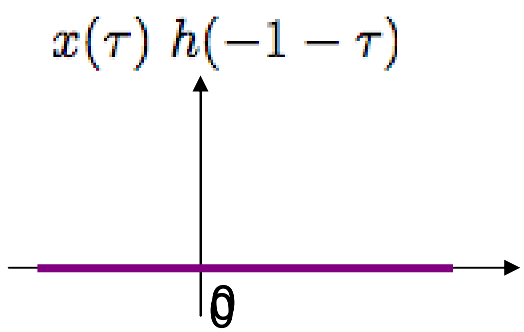


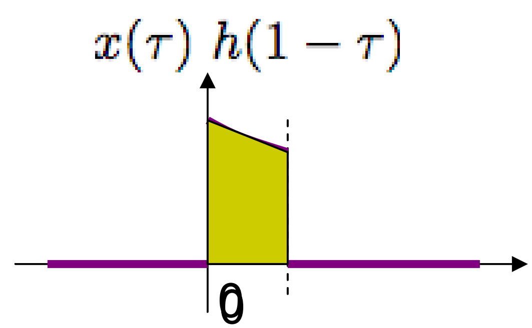
- $t = -1$



- $t = 1$







Procedure for evaluating convolution

Example: For t > 0, $$y(t) = \int_0^t e^{-\beta t}\ e^{-(\alpha-\beta)\tau}\ d\tau = \frac{e^{-\beta t} - e^{-\alpha t}}{\alpha - \beta}$$

Answer: $$y(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{e^{-\beta t} - e^{-\alpha t}}{\alpha-\beta} & \text{if } t > 0 \end{cases}$$

Properties of LTI systems

1. Commutative property

$$(x * h)[n] = (h * x)[n]$$
$$(x * h)(t) = (h * x)(t)$$

- If we interchange the roles of impulse response and input, the output remains the same

2. Distributive property

$$(x * (h_1 + h_2))[n] = (x * h_1)[n] + (x * h_2)[n]$$
$$(x * (h_1 + h_2))(t) = (x * h_1)(t) + (x * h_2)(t)$$

Properties of LTI systems

3. Associative property

$$(x * (h_1 * h_2))[n] = ((x * h_1) * h_2)[n]$$
$$(x * (h_1 * h_2))(t) = ((x * h_1) * h_2)(t)$$

$$\Rightarrow \quad y[n] = (x * h_1 * h_2)[n]$$
$$y(t) = (x * h_1 * h_2)(t)$$

- Impulse response of the cascade of two LTI systems is the convolution of their individual impulse responses
- **Order of the systems does not matter!**
- Can be generalized to the cascade of an arbitrary number of LTI systems
  - Require **both linearity and time invariance**

Properties of LTI systems

4. Invertibility of LTI systems
   - When S is LTI and invertible, there exists an inverse system which is LTI

5. Causality for LTI system

DT: $$y[n] = \sum_{k \in \mathbf{Z}} x[k]\ h[n-k] = \sum_{k \in \mathbf{Z}} h[k]\ x[n-k]$$

- Recall that in order for S to be causal, $y[n]$ should depend only on $x[m], m \leq n$
- $\boxed{h[k] = 0 \quad \text{for all } k < 0}$

CT: $$y(t) = \int_{\mathbb{R}} h(\tau)\ x(t-\tau)\ d\tau$$

- $\boxed{h(t) = 0 \quad \text{for all } t < 0}$

Properties of LTI systems

6. LTI systems with and without memory

- A system is **memoryless** if, for any two pairs of input-outputs (x_1, y_1) and (x_2, y_2) and given t_0, $x_1(t_0) = x_2(t_0)$ implies $y_1(t_0) = y_2(t_0)$

- DT: $y[n] = \sum_{k \in \mathbf{Z}} x[k]\ h[n-k]$
  - $\boxed{h[n] = K\ \delta[n] = h[0]\ \delta[n]}$
- CT: $y(t) = \int_{\mathbb{R}} x(\tau)\ h(t-\tau)\ d\tau$
  - $\boxed{h(t) = K\ \delta(t) \text{ for some constant } K}$
  - $y(t) = K\ x(t)$

Properties of LTI systems

7. Stability for LTI systems

- BIBO stability: every bounded input leads to a bounded output
- DT LTI system $\boldsymbol{S}$ (with impulse response $\boldsymbol{h}$) is stable iff (if and only if)

$$\boxed{\sum_{k \in \mathbf{Z}} |h[k]| < \infty}$$

  - Called "absolutely summable"

- CT LTI system $\boldsymbol{S}$ (with impulse response $\boldsymbol{h}$) is stable iff (if and only if)

$$\boxed{\int_{\mathbb{R}} |h(\tau)| \; d\tau < \infty}$$

  - Called "absolutely integrable"

Exploiting Superposition and Time-Invariance

$$x[n] = \sum_k a_k x_k[n] \xrightarrow{LinearSystem} y[n] = \sum_k a_k y_k[n]$$

Question: Are there sets of “basic” signals so that:

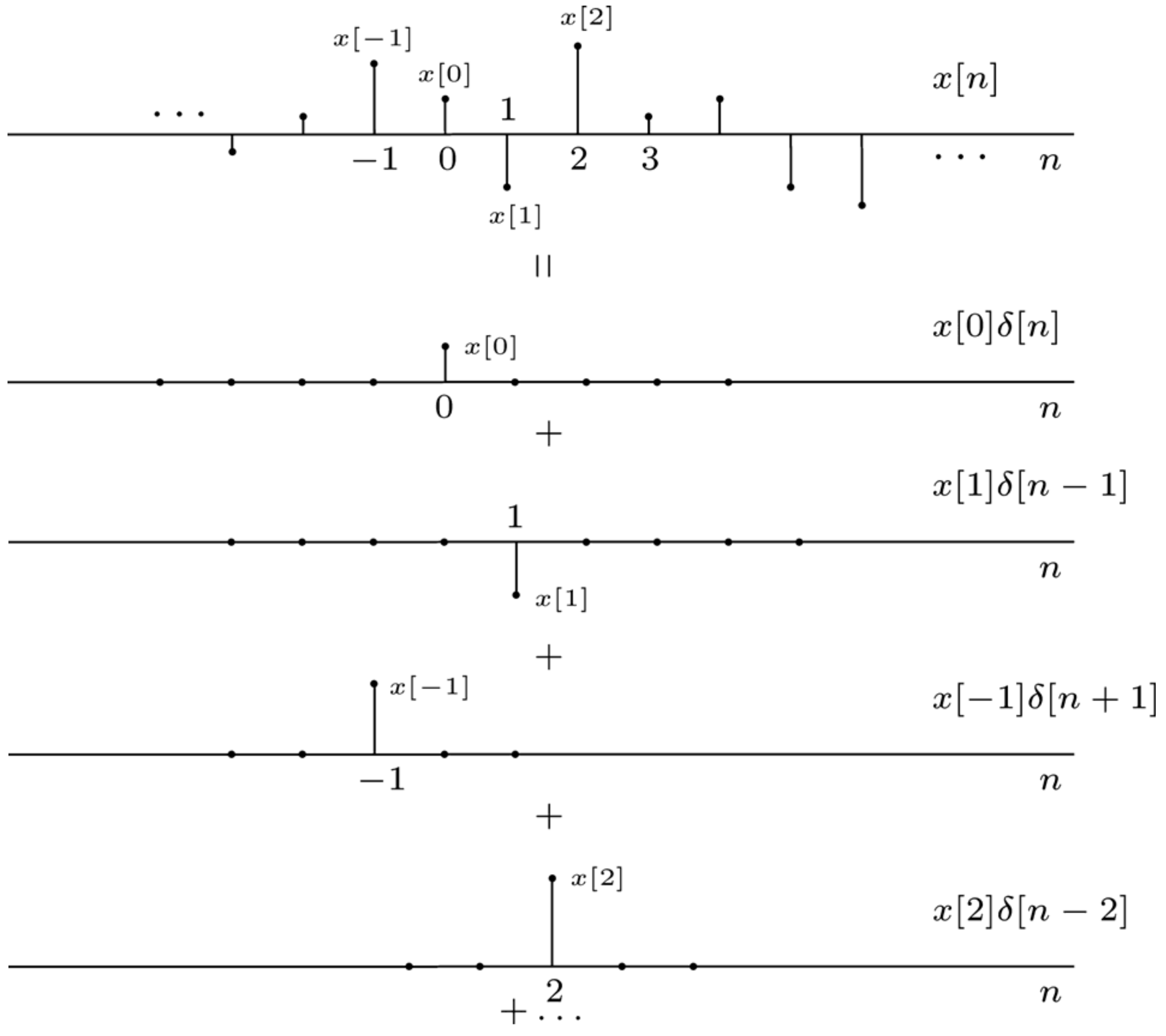
a) We can represent rich classes of signals as linear combinations of these building block signals.

b) The response of LTI Systems to these basic signals are both *simple* and *insightful*.

Fact: For LTI Systems (CT or DT) there are two natural choices for these building blocks

Focus for now:	DT	Shifted unit samples
	CT	Shifted unit impulses

Representation of DT Signals Using Unit Samples



That is ...

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$\Downarrow$$

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{Coefficients}} \, \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

The Sifting Property of the Unit Sample

$x[n] \longrightarrow$ $\boxed{\text{DT System}}$ $\longrightarrow y[n]$

- Suppose the system is **linear**, and define $h_k[n]$ as the response to $\delta[n - k]$:

$$\delta[n-k] \rightarrow h_k[n]$$

From superposition: $\Downarrow$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit sample response h*[*n*]*:*

$$\delta[n] \rightarrow h[n]$$

From TI:

$$\Downarrow$$

$$\delta[n-k] \rightarrow h[n-k]$$

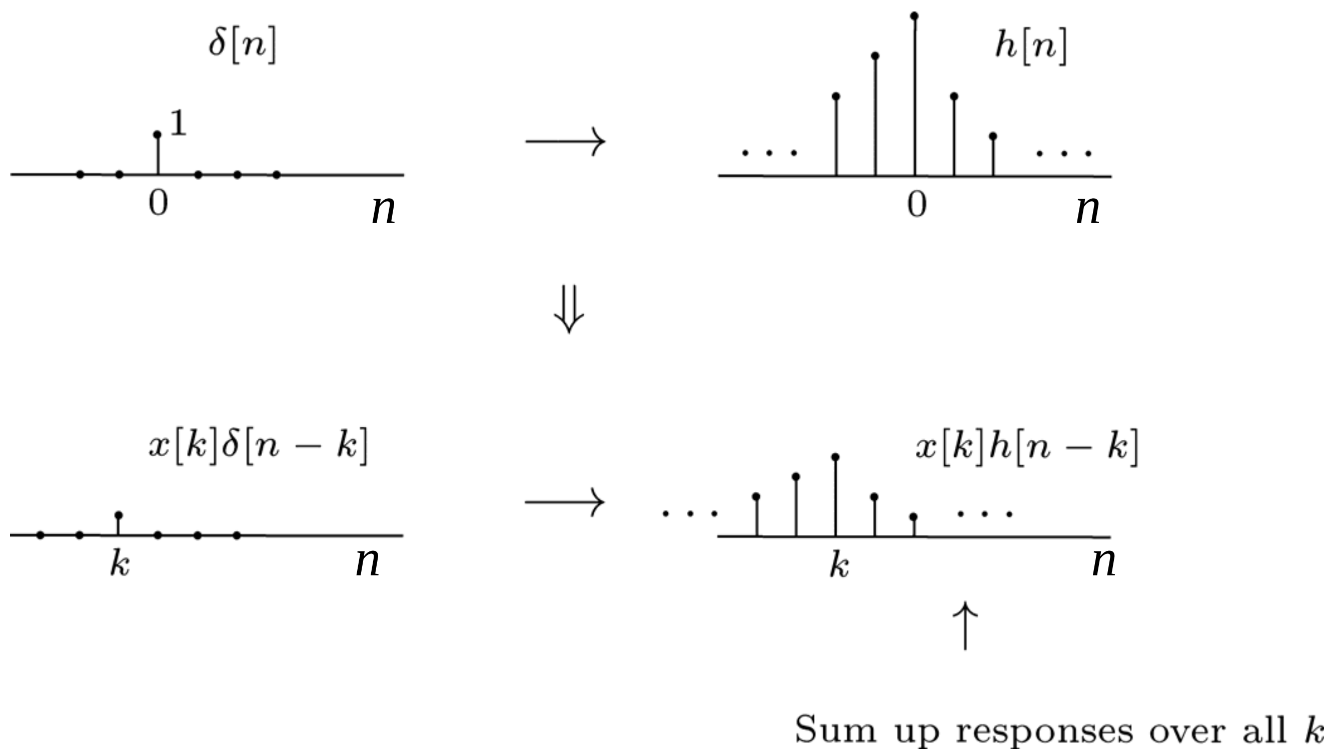
From LTI:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \underbrace{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{\text{Convolution Sum}}$$

Convolution Sum Representation of Response of LTI Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Interpretation

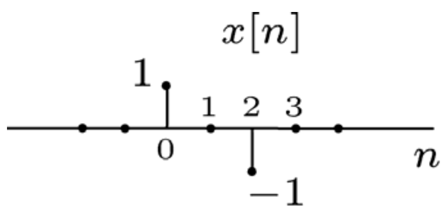


Visualizing the calculation of $y[n] = x[n] * h[n]$

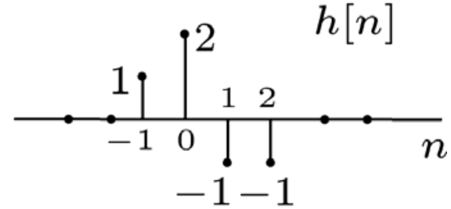
Choose value of *n* and consider it fixed

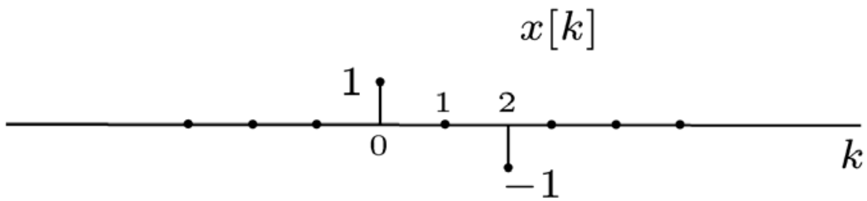
$$y[n] = \sum_{k=\infty}^{\infty} x[k]h[n-k]$$

View as functions of *k* with *n* fixed

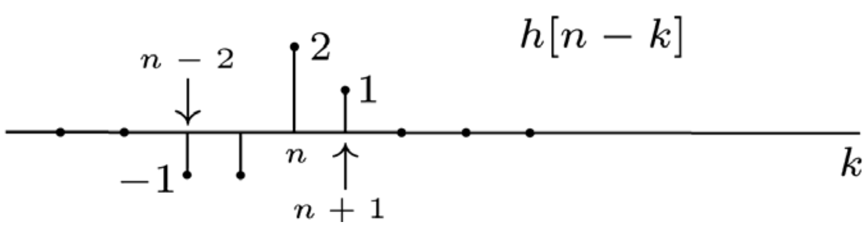


$\Downarrow$



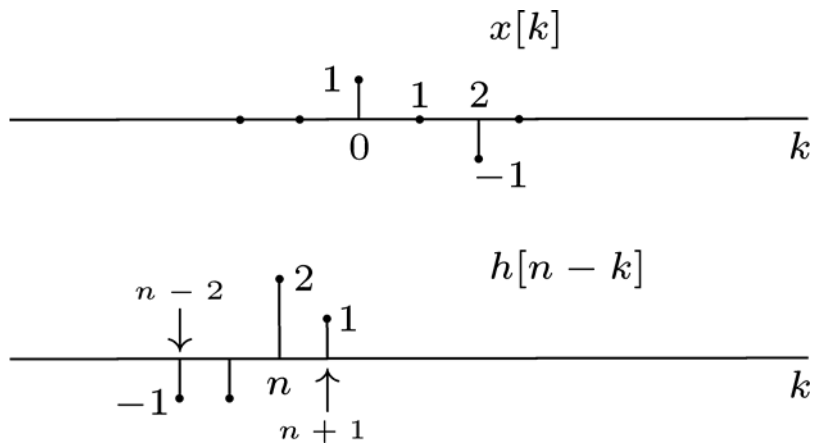


y[0] = ∑ prod of overlap for $n = 0$



y[1] = ∑ prod of overlap for $n = 1$

Calculating Successive Values: Shift, Multiply, Sum



$$\begin{aligned}
y[n] &= 0 \quad \text{for } n < \textbf{-1} \\
y[-1] &= \mathbf{1 \times 1 = 1} \\
y[0] &= \mathbf{0 \times 1 + 1 \times 2 = 2} \\
y[1] &= \mathbf{(-1) \times 1 + 0 \times 2 + 1 \times (-1) = -2} \\
y[2] &= \mathbf{(-1) \times 2 + 0 \times (-1) + 1 \times (-1) = -3} \\
y[3] &= \mathbf{(-1) \times (-1) + 0 \times (-1) = 1} \\
y[4] &= \mathbf{(-1) \times (-1) = 1} \\
y[n] &= 0 \quad \text{for } n > \mathbf{4}
\end{aligned}$$

Properties of Convolution and DT LTI Systems

1) A DT LTI System is *completely characterized* by its unit sample response

Ex. #1: $h[n] = \delta[n - n_0]$

There are *many* systems with this repsonse to $\delta[n]$

There is only *one* LTI System with this response to $\delta[n]$:

$$y[n] = x[n - n_0]$$

$$\Downarrow$$

$$\boxed{x[n] * \delta[n - n_0] = x[n - n_0]}$$

Ex. #2:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad \text{- An Accumulator}$$

Unit Sample response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

$$\Downarrow$$

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$$

The Commutative Property

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

<u>Ex</u>: Step response $s[n]$ of an LTI system

$$s[n] = \underset{\substack{\uparrow \\ \text{step input}}}{u[n]} * h[n] = \underset{\substack{\uparrow \\ \text{“input”}}}{h[n]} * \underset{\substack{\uparrow \\ \text{Unit Sample response of accumulator}}}{u[n]}$$

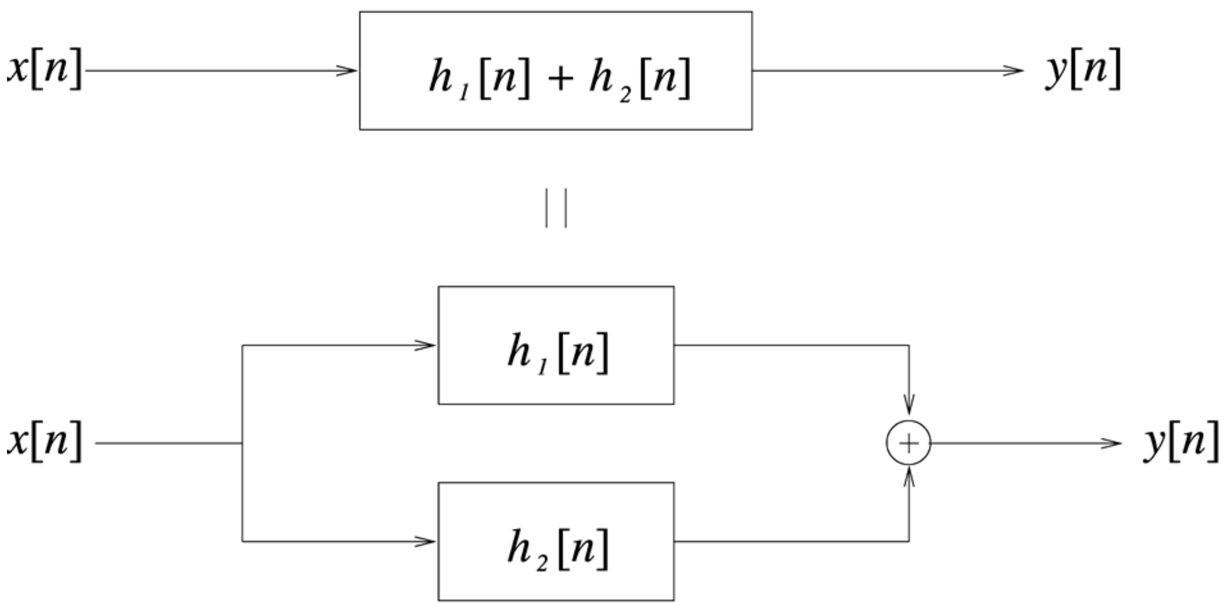
$$\Downarrow$$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

The Distributive Property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x * h_2[n]$$

Interpretation



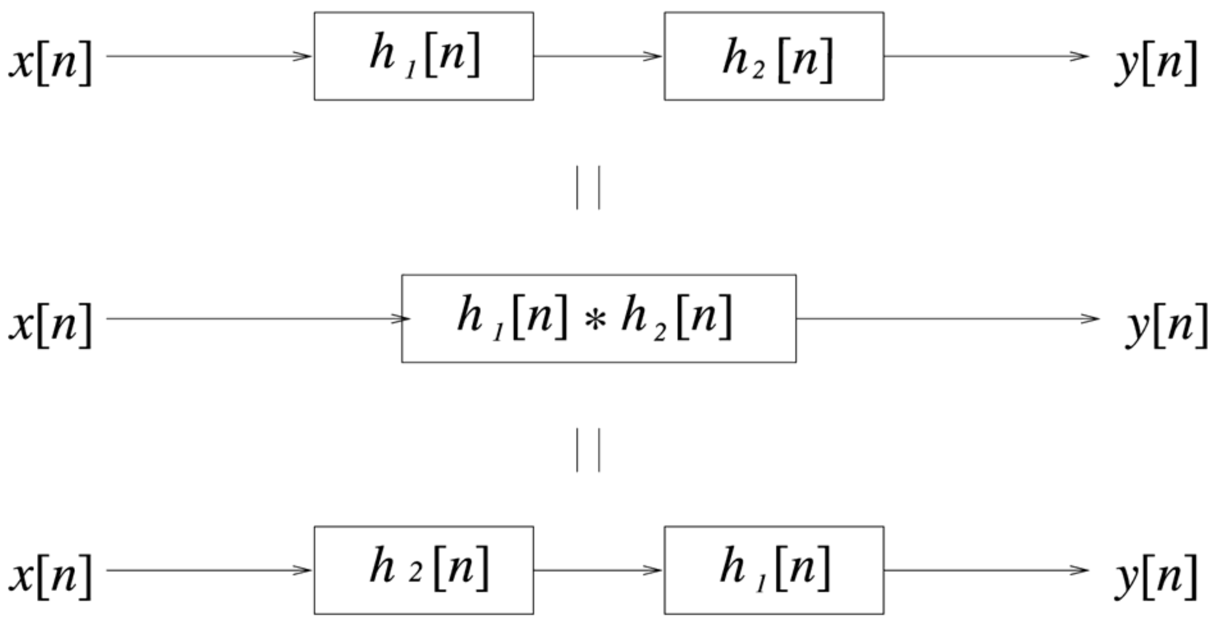
The Associative Property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

$$\text{(Commutativity)} \; \Big\|$$

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

Implication (Very special to LTI Systems)



Properties of LTI Systems

1) Causality $\Leftrightarrow$ $h[n] = 0 \text{ for all } n < 0$

2) Stability $\Leftrightarrow$ $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

Stationary and Ergodic Process

- strict-sense stationary (SSS)
- wide-sense stationary (WSS)

Gaussian

- SSS =>WSS ; WSS=>SSS
- Time average v.s ensemble average
- The ergodicity requirement is that the ensemble average coincide with the time average
- Sample function generated to represent signals, noise, interference should be ergodic

Time average v.s ensemble average

- Time average

mean: $\overline{x(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt$

mean-square: $\overline{x(t)^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)^2 dt$

variance: $\sigma_x^2 = \overline{x(t)^2} - \overline{x(t)}^2$

auto-correlation: $\phi_x(\tau) = \overline{x(t)x(t+\tau)}$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt$$

- ensemble average

$$\langle x(t_1)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x^{(i)}(t_1)$$

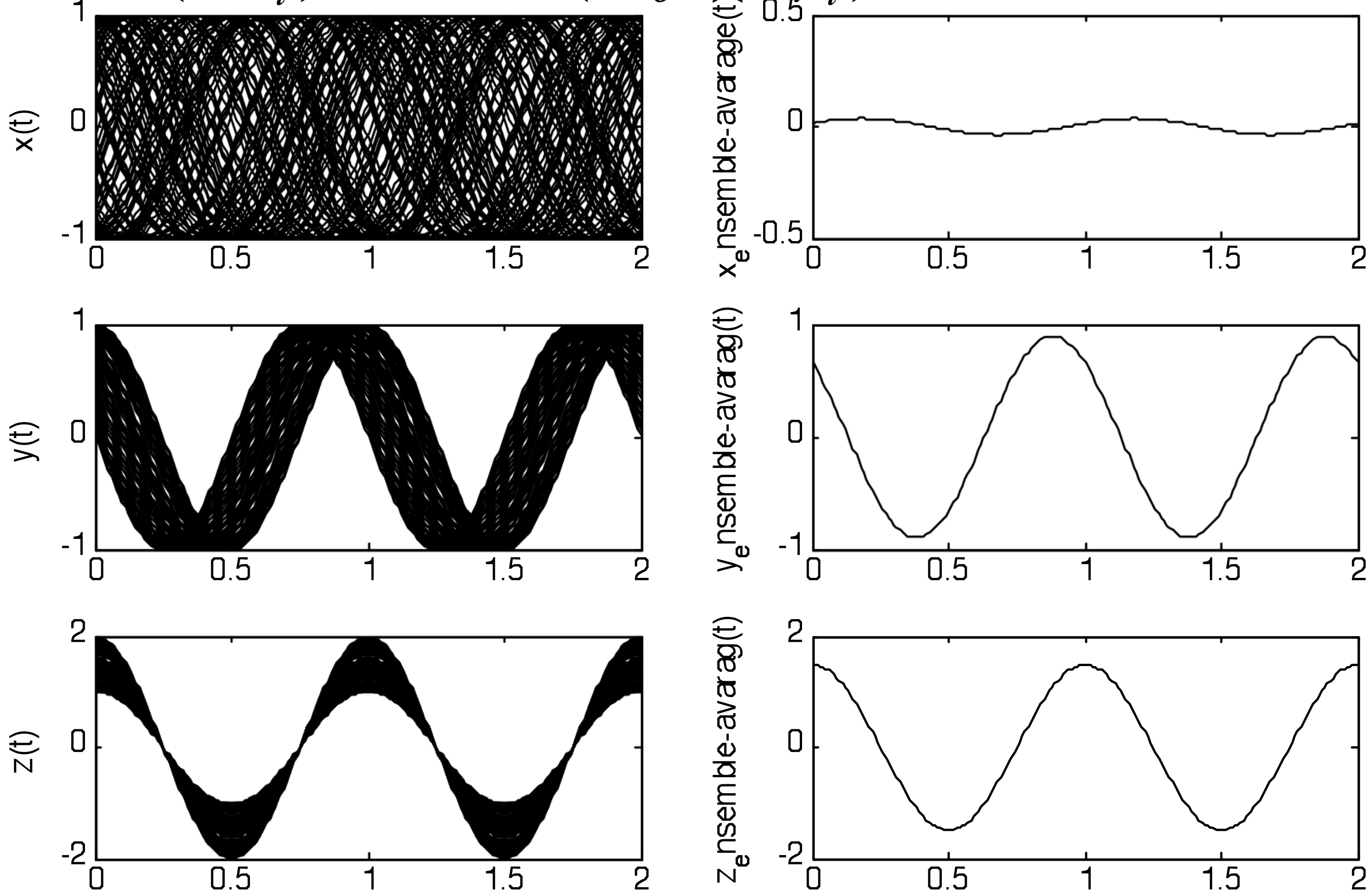
$$\langle x(t_1)^2\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x^{(i)}(t_1)^2$$

$$\sigma_x^2 = \langle x(t_1)^2\rangle - \langle x(t_1)\rangle^2$$

$$\langle x(t_1)x(t_2)\rangle = \lim_{T\to\infty} \frac{1}{N} \sum_{i=1}^{N} x^{(i)}(t_1)x^{(i)}(t_2)$$

Example 7.1 (N=100)

$$x(t,\xi_i) = A\cos(2\pi f t + \varphi_i)$$



$$x(t,\xi_i) = A(1+\mu_i)\cos(2\pi f t + \varphi_i)$$