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Stock Price Co-Movement and the Foundations of Pairs Trading

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Abstract

We study the theoretical implications of cointegrated stock prices on the profitability of pairs-trading strategies. If stock returns are fairly weakly correlated across time, cointegration implies very high Sharpe ratios. To the extent that the theoretical Sharpe ratios are "too large," our results suggest that either i) cointegration does not exist pairwise among stocks, and pairs-trading profits are a result of a weaker or less stable dependency structure among stock pairs, or ii) the serial correlation in stock returns stretches over considerably longer horizons than is usually assumed. Empirically, there is little evidence of cointegration, favoring the first explanation.

I. Introduction

Pairs trading is an investment strategy based on the notion of two stock prices "co-moving" with each other. If the two prices diverge, a long–short position can be used to profit from the expected future re-convergence of the prices. Although the pairs can be formed on fundamental similarities between firms, the modern incarnation of the strategy is typically based on statistical principles, picking pairs of stocks with share prices that have previously moved closely together according to some statistical measure. In a seminal study, Gatev, Goetzmann, and Rouwenhorst (GGR) (2006) documented strong and consistent excess returns for a simple statistical pairs-trading strategy, applied to the Center for Research in Security Prices (CRSP) universe of U.S. stocks.[1] In econometric terms,

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[1]Profitability of pairs-trading strategies has also been documented for other stock markets. For instance, Bowen and Hutchinson (2016) analyze pairs trading on the U.K. equity market and find results similar to those of GGR (2006). Jacobs and Weber (2015) analyze individual stock data from 34 international markets and find that pairs-trading profits appear to be a consistent feature across these markets. These studies also show that pairs-trading returns do not seem to be explained by traditional

the pairwise price patterns that give rise to pairs-trading profits are consistent with the existence of cointegration among stock prices, and the notion of price cointegration is often used to motivate why pairs trading might be profitable (e.g., GGR (2006), De Rossi, Jones, Lancetti, Jessop, and Sefton (2010), and Ardia, Gatarek, Hoogerheide, and van Dijk (2016)).

The current paper evaluates whether cointegration among stock prices is indeed a realistic assumption upon which to justify pairs trading. In particular, we derive the expected returns and Sharpe ratios of a simple pairs-trading strategy under the assumption of pairwise cointegrated stock prices, allowing for a flexible specification of the stochastic process that governs the individual asset prices. Our analysis shows that, under the typical assumption that stock returns only have weak and fairly short-lived serial correlations, price cointegration would result in extremely profitable pairs-trading strategies. In a cointegrated setting, a typical pairs trade might easily have an annualized Sharpe ratio greater than 10, for a single pair, ignoring any diversification benefits of trading many pairs simultaneously. Cointegration of stock prices therefore appears to deliver pairs-trading profits that are "too good to be true."

The existence of cointegration essentially implies that the deviations between two nonstationary series are stationary.[2] The speed at which the two series converge back toward each other after a given deviation depends on the short-run, or transient, dynamics in the two processes. If there are relatively long-lived transient shocks to the series, the two processes might diverge from each other over long periods, although cointegration ensures that they eventually converge. If the transient dynamics are short-lived, the two series must converge very quickly once they deviate from each other. In the latter case, most shocks to the series are of a permanent nature and therefore subject to the cointegrating restriction, which essentially says that any permanent shock must affect the two series in an identical manner.

To put cointegration in more economic terms, consider a simple example of two car manufacturers. If both of their stock prices are driven solely by a single common factor (e.g., the total (expected long-run) demand for cars), then the two stock prices could easily be cointegrated. However, it is more likely that the stock prices depend on firm-specific demands, which contain not only a common component but also idiosyncratic components. In this case, the idiosyncratic components of demands will cause deviations between the two stock prices, and price cointegration would require that the idiosyncratic demands only cause temporary changes in the stock prices. That is, cointegration imposes the strong restriction that any idiosyncratic effects must be of a transient nature, such that they do not cause a permanent deviation between the stock prices of different firms.

In the stock price setting considered here, most price shocks are usually thought to be permanent in nature. For instance, under the classical random-walk hypothesis, *all* price shocks are permanent. Although current empirical

factors such as market, size, value, momentum, and reversals. Do and Faff (2010) verify that pairs-trading profits persist in U.S. samples dating after those used by GGR (2006).

[2]This informal discussion implicitly assumes that the two processes are cointegrated with cointegration vector $(1,-1)$, but clearly the same basic intuition holds with a vector $(1,-\gamma)$, where the second series is multiplied by γ.

knowledge suggests that there are some transient dynamics in asset prices, these are usually thought to be small and short-lived. In this case, if two stock prices are cointegrated, there is very little scope for them to deviate from each other over long stretches of time. Thus, when a transient shock causes the two series to deviate, they will very quickly converge back to each other. Such quick convergence is, of course, a perfect setting for pairs trading and gives rise to the outsized Sharpe ratios implied by the theoretical analysis.

The theoretical analysis thus predicts that cointegration among stock prices leads to statistical arbitrage opportunities that are simply too large to be consistent with the notion that markets are relatively efficient and excess profits reasonably hard to achieve. Alternatively, the serial correlation in stock returns must be considerably longer-lived than is usually assumed, with serial dependencies stretching at least upward of 6 months. However, such long-lived transient dynamics imply a rather slow convergence of prices in pairs trades, at odds with the empirical evidence from pairs-trading studies (e.g., Engelberg, Gao, and Jagannathan (2009), Do and Faff (2010), and Jacobs and Weber (2015)).

The tension between traditional random-walk efficiency and stock price cointegration is not a new idea, as evidenced by remarks in Granger (1986). Granger's work was followed by many empirical studies of cointegration among stock prices, particularly for groups of international stock price indexes (e.g., Kasa (1992), Corhay, Tourani Rad, and Urbain (1993)). Richards (1995) provides a nice summary of this earlier literature and argues that there is no empirical evidence of cointegration among international stock indexes once appropriate econometric inference is conducted. Our current study contributes to this previous literature by explicitly quantifying the "profit opportunities" implied by pairwise cointegration of asset prices.

In the second part of the paper, we evaluate the extent to which there is any support in the data for the predictions of the cointegrated model. Our main empirical goal is to determine whether cointegration of stock prices is likely to exist for pairs of stocks when each of the two stocks in the pair is issued by a different firm. We refer to such pairs formed by stocks from two different firms as ordinary pairs. The analysis consists of two parts. First, we calculate empirical Sharpe ratios from the implementation of a pairs-trading strategy similar to that analyzed in GGR (2006). Second, we quantify the extent to which the estimated model parameters are at all close to satisfying the restrictions implied by cointegration.

The empirical analysis is based on stocks traded on the Stockholm Stock Exchange. A relatively unique feature of the Swedish stock market, namely the widespread use of listed A- and B-shares, gives rise to a very useful control group of stock pairs. A- and B-shares of a given company are traded openly on the same exchange, provide identical ownership fractions, and are claims to the exact same cash flow. The only difference between them is that A-shares give the holder more votes than the B-shares. Since A- and B-shares of a given firm are claims to the exact same dividends, their prices are likely to be highly correlated. In fact, as shown by Bossaerts (1988), one would expect the two prices to be cointegrated.[3]

[3]Bossaerts (1988) derives a general equilibrium model with cointegrated asset prices. However, the key assumption in his model is that the dividend processes are, in fact, themselves cointegrated,

The A–B pairs can therefore be seen as a form of control group, for which we would expect cointegration to hold.[4] If we find that the restrictions implied by cointegration are (much) further away from being satisfied for the ordinary pairs than they are for the control group of A–B stock pairs, we view this as reasonably convincing evidence that cointegration among ordinary pairs is unlikely. Again, we emphasize that our main empirical question is whether cointegration of prices is likely to exist among *ordinary* (i.e., not A–B) pairs of stocks. The results for the A–B pairs should be viewed as a form of calibration of the empirical methods, providing a reasonable set of benchmark estimates against which we can compare the results for the ordinary pairs.

There are two main findings from the empirical analysis. First, before-cost Sharpe ratios from trading A–B pairs are mostly in line with the predictions of the cointegrated model, and they are considerably higher than those that can be attained when trading ordinary pairs.[5] Second, the restrictions implied by cointegration are far from being satisfied for all the possible ordinary pairs, and the parameter estimates are uniformly closer to satisfying the cointegrating restrictions for all the A–B pairs than for any of the ordinary stock pairs.

The theoretical and empirical analysis together strongly suggest that cointegration is *not* a likely explanation for the profitability of pairs-trading strategies using ordinary pairs of stocks. Pairs trading is based on the idea of stock prices co-moving with each other and that deviations from this co-movement will be adjusted and reverted, such that prices eventually converge after deviating. The profitability of such strategies is consistent with cointegration, but cointegration is not a necessary condition for pairs trading to work. Instead, it is quite likely that pairs-trading profits arise because asset prices on occasion move together over shorter time spans. This could, for instance, be due to fundamental reasons, such as a common and dominant shock affecting all stocks in a given industry. This view is supported by findings in Engelberg et al. (2009) and Jacobs and Weber (2015), who document that (quick) convergence of pairs is more likely when the divergence is caused by macroeconomic news rather than by firm-specific news. One could, of course, always claim that such stories are consistent with stocks *occasionally* being cointegrated but, since cointegration is defined as a long-run property, such statements make little sense.

which in turn implies cointegration among the price processes. The result is therefore not particularly surprising, since cointegration among dividends effectively implies that, in the long-run, certain asset combinations will be claims to the same cash flows. That is, from a long-run perspective, the cointegrated assets are essentially cash-flow equivalent.

[4]Pairs trading of A–B pairs likely occurs and is fully consistent with a setting in which the pairs traders act as arbitrageurs who enforce the arbitrage relationship between the stocks, as suggested by Bossaerts's (1988) model. Such trading need not lead to outsized profits, because the A and B prices track each other very closely and the scope for making large monetary returns is likely limited. Our empirical results are consistent with this claim; in Section III.D, we derive theoretical results that explain how this behavior of A–B prices relates to the main theoretical results presented in this study.

[5]We also provide detailed discussions on the effect of transaction costs in both the theoretical and the empirical parts of the paper. In particular, we show that transaction costs tend to mostly eliminate the returns from A–B pairs trading.

In conclusion, cointegration of stock prices, for pairs of stocks with claims to different cash flows, is unlikely for the simple reason that it would provide unrealistically large statistical arbitrage opportunities. The analysis highlights the strength of a cointegrating relationship in a setting with very weak short-run dynamics and essentially shows that one cannot expect cointegration of stock prices unless there is a "mechanical" relationship that links the two assets, as in the A–B share case.

The remainder of the paper is as follows: Section II sets up a model of cointegrated stock prices, and Section III derives the main theoretical predictions for pairs-trading returns. The empirical analysis is conducted in Section IV, and Section V concludes. Technical proofs and some supplemental material are found in the Appendix.

II. A Model of Cointegrated Stock Prices

We start by formulating a very general time-series model for stock returns. We assume that the returns on a given pair of stocks follow a bivariate vector moving average (VMA) process with a possibly infinite lag length. Such a process is often referred to as a linear process. It follows from the Wold decomposition (e.g., Wold (1938), Brockwell and Davis (1991)) that any well-behaved covariance stationary process can be represented as a (vector) moving average process. Imposing a VMA structure is therefore a very weak assumption. At the same time, as we will illustrate in detail, this representation allows for a very simple and clear analysis of cointegration in the corresponding price processes. In the interest of generality, the model is formulated for a k-dimensional vector of cointegrated prices, with $k=2$ corresponding to the standard pairs-trading setting.

A. A VMA Representation of Stock Returns and Stock Prices

Let y_t be a $k\times 1$ vector of (log-) stock prices, and let the first difference of y_t, $\Delta y_t = y_t - y_{t-1}$, represent the corresponding vector of (log-) returns. The returns are assumed to satisfy

$$\Delta y_t = \mu + u_t, \tag{1}$$

where μ is a constant vector and u_t is a stochastic process that follows an infinite VMA process,

$$u_t = C(L)\epsilon_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}, \tag{2}$$

with $\epsilon_t \equiv \text{iid}(0,\Sigma)$ and Σ a positive definite covariance matrix. u_t and ϵ_t are $k\times 1$ vector processes and c_j, $j=0,1,2,\ldots$, are $k\times k$ coefficient matrices. $C(L)=\sum_{j=0}^{\infty} c_j L^j$, where L is the lag operator, and $C(1)=\sum_{j=0}^{\infty} c_j$. In order to justify the Beveridge and Nelson (BN) (1981) decomposition that we will use, the sum in $C(1)$ needs to converge sufficiently fast. A sufficient condition is given by $\sum_{j=0}^{\infty} j||c_j|| < \infty$ (Phillips and Solo (1992)). In order to avoid degenerate cases, it is also assumed that at least one element in the $k\times k$ matrix $C(1)$ is nonzero. This specification of u_t represents a stationary ($I(0)$) mean-zero vector process with a long-run covariance matrix $\Omega = C(1)\Sigma C(1)'$.[6] In order to make the system identifiable, we impose the normalization $c_0 = I$.

The price vector y_t is obtained by summing up over the returns, Δy_t,

$$y_t = y_0 + \mu t + \sum_{i=1}^{t} u_i, \tag{3}$$

where y_0 represents an initial condition. This is a VMA representation of a unit-root nonstationary ($I(1)$) process.

B. Cointegration in a VMA Process

The VMA representation allows for a very simple and intuitive analysis of cointegration. Using the BN (1981) decomposition, we can write

$$u_t = C(L)\epsilon_t = C(1)\epsilon_t + \tilde{\epsilon}_{t-1} - \tilde{\epsilon}_t, \tag{4}$$

where

$$\tilde{\epsilon}_t = \tilde{C}(L)\epsilon_t = \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t-j} \tag{5}$$

and

$$\tilde{c}_j = \sum_{s=j+1}^{\infty} c_s. \tag{6}$$

The $I(1)$ price process, y_t, can therefore be written as

$$\begin{aligned} y_t &= y_0 + \mu t + C(1)\sum_{i=1}^{t} \epsilon_i + \sum_{i=1}^{t} (\tilde{\epsilon}_{i-1} - \tilde{\epsilon}_i) \\ &= \mu t + C(1)\sum_{i=1}^{t} \epsilon_i - \tilde{\epsilon}_t + (y_0 + \tilde{\epsilon}_0), \end{aligned} \tag{7}$$

using the fact that $\sum_{i=1}^{t}(\tilde{\epsilon}_{i-1} - \tilde{\epsilon}_i) = \tilde{\epsilon}_0 - \tilde{\epsilon}_t$. The representation of the price process in equation (7) shows that the price can be written as the sum of four components: i) a deterministic trending component (corresponding to the equity premium), ii) a nonstationary ($I(1)$) stochastic martingale component, iii) a transitory ($I(0)$ stationary) "noise" component ($\tilde{\epsilon}_t$), and iv) an initial conditions component.

Cointegration of a vector $I(1)$ process implies that there exists a linear combination, $\beta' y_t$, which is $I(0)$ stationary for some $\beta \neq 0$. The $I(1)$ component in equation (7) is given by the martingale process, $C(1)\sum_{i=1}^{t} \epsilon_i$. If $\beta \neq 0$ is a cointegrating vector for y_t, it must hold that β eliminates the martingale component of y_t (i.e., $\beta' C(1) = 0$). Typically, it is also assumed that the deterministic trend is eliminated through cointegration, such that $\beta' \mu = 0$, and we will maintain this

[6] In the usual notation of stochastic processes, $I(1)$ denotes a process integrated of order 1 (i.e., a unit-root process) and $I(0)$ denotes a covariance stationary process (the first difference of an $I(1)$ process).

assumption throughout the paper.[7] That is, if β is a cointegrating vector, it follows from equation (7) that

$$(8)\quad \beta' y_t \;=\; \beta'\mu t + \beta' C(1)\sum_{i=1}^{t}\epsilon_i - \beta'\tilde{\epsilon}_t + \beta'(y_0+\tilde{\epsilon}_0) \;=\; -\beta'\tilde{\epsilon}_t + \beta'(y_0+\tilde{\epsilon}_0).$$

The cointegrated combination of y_t is made up of a transitory ($I(0)$) stochastic component and the initial condition. Pairs-trading strategies are based on standardized price processes (total return indexes), initiated at some pre-specified value, and with little loss of generality we therefore set $\beta'(y_0+\tilde{\epsilon}_0)=0$.[8]

C. Implicit Restrictions in the Cointegrated Model

The cointegrated model specified previously is stated in very general terms, relying essentially only on the assumption that returns follow a linear process. In the bivariate case ($k=2$) with cointegrating vector $\beta=(1,-1)'$, which would be the typical pairs-trading setting, the cointegrating relationship leads to some implicit restrictions on the model. Later, we will use these restrictions to empirically evaluate the presence of cointegration.

First, there are restrictions on the VMA coefficients. Denote the moving average coefficient matrices, for each lag j, as

$$(9)\qquad c_j \;=\; \begin{bmatrix} \psi_{11,j} & \psi_{12,j} \\ \psi_{21,j} & \psi_{22,j} \end{bmatrix},$$

with $c_0=I$. Define $\psi_{kl}=\sum_{j=1}^{\infty}\psi_{kl,j}$, for $k,l=1,2$, and it follows that

$$(10)\quad C(1) \;\equiv\; \sum_{j=0}^{\infty} c_j \;\equiv\; I+\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \;=\; \begin{bmatrix} 1+\psi_{11} & \psi_{12} \\ \psi_{21} & 1+\psi_{22} \end{bmatrix}.$$

If $\beta=(1,-1)'$ is a cointegrating vector, then $\beta' C(1)=0$ implies that

$$(11)\qquad \psi_{21} \;=\; 1+\psi_{11} \quad \text{and} \quad \psi_{12} \;=\; 1+\psi_{22}.$$

Second, there are restrictions on the long-run covariance matrix of returns. Let $\Gamma_j \equiv \mathrm{E}[(\Delta y_t-\mu)(\Delta y'_{t+j}-\mu)]$ denote the jth autocovariance of the returns Δy_t. The long-run covariance matrix of Δy_t is then defined as

$$(12)\qquad \Omega \;=\; \sum_{j=-\infty}^{\infty}\Gamma_j \;=\; \Gamma_0+\sum_{j=1}^{\infty}\left(\Gamma_j+\Gamma'_j\right).$$

In the VMA model, $\Omega = C(1)\Sigma C(1)'$, and under cointegration

$$(13)\quad \beta'\Omega \;=\; \beta'\left(C(1)\Sigma C(1)'\right) \;=\; \left(\beta' C(1)\right)\Sigma C(1)' \;=\; 0,$$

[7] Allowing for a nonzero deterministic trending component in the cointegrated combination implies that the linear combination $\beta' y_t$ is $I(0)$ stationary around a deterministic trend, rather than around a constant. Such a specification seems quite removed from the general idea of pairs trading, and indeed seems quite unlikely to occur in any empirical situation.

[8] That is, for $\beta=(1,-1)$, this implies that the two standardized price processes are initiated at the same value. Imposing $\beta'(y_0+\tilde{\epsilon}_0)=0$ has little impact on the derivations, but without this restriction one would need to explicitly subtract the initial state from the current one in certain expressions.

where the last equality follows from $\beta' C(1) = 0$. If $\beta = (1, -1)'$, this implies that

$$(14) \quad \beta'\Omega = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{21} \\ \omega_{21} & \omega_{22} \end{bmatrix} = \begin{bmatrix} \omega_{11} - \omega_{21} & \omega_{21} - \omega_{22} \end{bmatrix} = 0,$$

such that all the elements in the long-run covariance matrix must be identical in this case.

III. Return Properties of a Pairs-Trading Strategy

A pairs-trading strategy for a given pair of stocks is usually defined along the following lines: If the difference between the (standardized) prices of stock 1 and stock 2 exceeds a given threshold, a short position is taken in the stock with the higher price and a long position in the stock with the lower price. The long and short positions are of identical magnitude, resulting in a zero-cost strategy. The threshold is defined in terms of the unconditional standard deviation of the observed difference between the two price processes. A 2-standard-deviation difference is a standard trigger of a pairs trade. The position is closed either after a given amount of time or after the two prices converge. In the theoretical analysis that follows, we restrict ourselves to fixed holding periods, such that the position always closes after a given number of days. The joint price process used for measuring divergence is defined as the total return indexes for the two stocks, initiated at some prior date.

These conditions extend naturally to the formal setting considered here, with y_t interpreted as a total return series for the stocks; for simplicity, we continue to refer to y_t as the price process. If y_t is a bivariate price process with cointegrating vector $\beta = (1, -1)'$, the *change* in $\beta' y_t = y_{1,t} - y_{2,t}$ represents the return on a pairs-trading strategy triggered by a decline in the price of stock 1 relative to the price of stock 2. If the price of stock 1 was instead higher than that of stock 2, then the pairs trade would take on the negative position, $-\beta$. In the analysis that follows, without loss of generality, we define a pairs trade as taking on a position β, with the implicit understanding that if the price spread is reversed, the opposite position would be used. More generally, for a k-dimensional price process y_t with cointegration vector β, the change in $\beta' y_t$ represents the return on a generalized pairs-trading strategy involving k stocks. Such strategies represent a natural extension of the pairs-trading idea, as pointed out by GGR (2006), and the main results that follow are derived for a general k-dimensional price process with arbitrary cointegration vector β. However, we focus the discussion on the standard bivariate case, where $\beta = (1, -1)'$.

A. The Finite Lag Case

If we assume that Δy_t follows a finite order VMA process, such that $u_t = C(L)\epsilon_t = \sum_{j=0}^{q} c_j \epsilon_{t-j}$, with $q < \infty$, we can calculate explicit results for the returns on the pairs-trading strategy where the holding period, p, is identical to the lag length, q. In particular, Theorem 1 derives explicit expressions for the conditional moments of a pairs-trading strategy in this case:

Theorem 1. Suppose $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ is a $k \times 1$ dimensional returns process, with $\epsilon_t \equiv \text{iid}(0, \Sigma)$, $C(L) = \sum_{j=0}^{q} c_j L^j$, $q < \infty$, and $C(1) \neq 0$. The corresponding price process is given by y_t and the q-period returns on the pairs-trading strategy is defined as $r_{t \to t+q} \equiv \sum_{j=1}^{q} \Delta\beta' y_{t+j} = \beta' y_{t+q} - \beta' y_t$. If y_t is cointegrated with cointegration vector β, the following results hold for the returns on the pairs-trading strategy:

i. The time t conditional expected q-period return is given by

$$\text{E}_t\left[r_{t \to t+q}\right] = -\beta' y_t. \tag{15}$$

ii. The time t conditional variance of the q-period return is given by

$$\text{var}_t\left(r_{t \to t+q}\right) = \text{var}\left(\beta' y_t\right), \tag{16}$$

where $\text{var}(\beta' y_t)$ is the unconditional variance of $\beta' y_t$.

iii. The time t conditional Sharpe ratio for the q-period return is given by

$$\text{SR}_t\left(r_{t \to t+q}\right) \equiv \frac{\text{E}_t\left[r_{t \to t+q}\right]}{\sqrt{\text{var}_t\left(r_{t \to t+q}\right)}} = \frac{-\beta' y_t}{\sqrt{\text{var}\left(\beta' y_t\right)}}. \tag{17}$$

The results highlight several important points.

1. In the bivariate pairs-trading case, where $\beta = (1, -1)'$, the conditional expected q-period returns are exactly proportional to the deviation between the two prices, $\text{E}_t[r_{t \to t+q}] = -\beta' y_t = y_{2,t} - y_{1,t}$. That is, the larger the deviation between the prices, the greater the expected returns. Further, the conditional variance of the q-period pairs-trading returns is identical to the *unconditional* variance of the spread between the two price processes.

2. The VMA parameters, which govern the dynamics of the price processes, do not explicitly enter into the expected returns and variance formulas. In essence, the cointegrating relationship, along with the lag length in the model, pins down the speed of convergence over the next q periods in a cointegrated VMA model with q lags.[9]

3. Suppose that we observe a negative 2-standard-deviation outcome of the spread, $\beta' y_t$. That is, suppose that $-\beta' y_t = 2\sqrt{\text{var}(\beta' y_t)}$. In this case,

$$\begin{aligned} \text{SR}_t\left(r_{t \to t+q}\right) &= \frac{\text{E}_t\left[r_{t \to t+q}\right]}{\sqrt{\text{var}_t\left(r_{t \to t+q}\right)}} = \frac{-\beta' y_t}{\sqrt{\text{var}\left(\beta' y_t\right)}} \\ &= \frac{2\sqrt{\text{var}\left(\beta' y_t\right)}}{\sqrt{\text{var}\left(\beta' y_t\right)}} = 2. \end{aligned} \tag{18}$$

[9]If one were to calculate the expected returns over other periods than the q-period horizon used in Theorem 1, the answer would generally depend on the lag coefficients explicitly, as seen in Theorem C1 in Appendix C.

If q is measured in days and there are 250 trading days in the year, the annualized Sharpe ratio is

$$\text{SR}_{\text{ann}} = \sqrt{\frac{250}{q}} \times 2. \tag{19}$$

Table 1 reports annualized Sharpe ratios for different values of q. For instance, if returns follow a VMA(10) process in which the corresponding price processes are cointegrated and one puts on pairs trades with a 10-day holding period when the spread is 2 standard deviations wide, the strategy has an annualized Sharpe ratio of 10.

4. The results hold for general k-dimensional cointegrated price processes with cointegration vector β. Conditional on a given value of $\beta' y_t$, the expected returns and Sharpe ratios are unaffected by the dimension of the system (i.e., by the value of k).

TABLE 1

Properties of q-Period Pairs-Trading Returns

Table 1 presents annualized Sharpe ratios (SR_{ann}, equation (19)) of pairs-trading strategies where a trade is initiated by a 2-standard-deviation price spread and held open for q periods, and the returns are generated by a VMA(q) model.

q	1	5	10	25	50	125	250
SR_{ann}	32	14	10	6.3	4.5	2.8	2

The results in Theorem 1 provide a very clear picture of the return properties of a pairs-trading strategy when the returns follow a VMA of some finite order q and the holding period for the trading strategy is equal to q periods (days). For small to moderate values of q, such holding periods are quite sensible and realistic. However, as q increases, and in particular as $q \to \infty$, it is no longer feasible to consider holding periods that are equal to q days. Instead, we want to consider fixed holding periods and allow infinite values for the lag length q.

B. Fixed Holding Periods and the Infinite Lag Case

We start with deriving theoretical results for a holding period $p=1$, allowing for lag length $q=\infty$. As shown in Theorem C1 in Appendix C, for an arbitrary value of the lag length q (including $q=\infty$), the conditional expected pairs-trading return from t to $t+1$ is not solely a function of the distance between the two price processes, $\beta' y_t$, but also depends explicitly on the realizations of the previous shocks, ϵ_{t-j}, and the moving average (MA) coefficients, c_j. The simple mapping between the price difference, $\beta' y_t$, and the Sharpe ratio of the pairs-trading strategy, seen in Theorem 1, is therefore no longer present. That is, conditioning on the price difference is no longer sufficient to pin down the conditional Sharpe ratio for a given pairs trade.

In particular, as shown in Theorem C1, the 1-period conditional Sharpe ratio is given by

$$\text{(20)} \qquad \mathrm{SR}_{t,t\to t+1} \quad = \quad \frac{\beta' \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}}{\sqrt{\beta' \Sigma \beta}}.$$

This expression is not directly amenable to the analysis of pairs-trading strategies that are conditioned on a certain price divergence (i.e., $\beta' y_t$) between the two stocks.[10] The sequence $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ is not a directly observable quantity, and statements conditional on a specific realization of this sequence are not particularly useful. To get around this issue, we consider the notion of an *unconditional* pairs trade: At some arbitrary time t, an investor puts on a pairs trade *without* conditioning on the price difference or any other information. This is obviously not an attractive strategy, with an expected return equal to 0.[11] However, it enables us to think of the sequence $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ as a random, rather than a realized, quantity. Formally, given information at time t, $\sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}$ takes on a fixed (non-stochastic) value, which in turn delivers a fixed conditional Sharpe ratio. If one does not condition on information formally realized at time t, $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ is a random sequence and $\mathrm{SR}_{t\to t+1}$ a random variable. In particular, we can think of $\mathrm{SR}_{t\to t+1}$ as the time t stochastic Sharpe ratio facing the investor who puts on the unconditional pairs trade at time t. Had the investor observed $\{\epsilon_{t-j}\}_{j=0}^{\infty}$, the Sharpe ratio would have been a fixed number, but without this information, it is a random variable. Under more specific assumptions on the sequences c_j and ϵ_t, an explicit distribution can be derived for the stochastic Sharpe ratio:

Theorem 2. Suppose $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ is a bivariate returns process, with $\epsilon_t \equiv \mathrm{iid}\,\mathrm{N}(0,\Sigma)$, $C(L) = \sum_{j=0}^{\infty} c_j L^j$, and $C(1) \neq 0$. The corresponding price process is given by y_t, and the returns on the pairs-trading strategy is defined as $r_{t\to t+1} \equiv \Delta\beta' y_{t+1}$. Further, assume that the coefficients c_j can be written as

$$\text{(21)} \qquad c_j \quad = \quad h(j)\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $h(j)$ is a convergent series such that $H_\infty \equiv \sum_{j=1}^{\infty} h(j) < \infty$. In addition, define $H_\infty^{(2)} \equiv \sum_{j=1}^{\infty} h(j)^2$. If y_t is cointegrated with cointegration vector $\beta = (1,-1)$, the 1-period Sharpe ratio for the unconditional pairs-trading strategy is distributed according to

$$\text{(22)} \qquad \mathrm{SR}_{t\to t+1} \quad \sim \quad \mathrm{N}\left(0, \frac{H_\infty^{(2)}}{H_\infty^2}\right).$$

[10] As made clear in equation (C-14) in Appendix C, the conditional Sharpe ratio partly depends on the price difference $\beta' y_t$, but also on the sequence of previous shocks.

[11] To be clear, we define the pairs trade as always going long in stock 1 and short in stock 2, such that the position is given by $\beta = (1,-1)$. Thus, since the investor does not condition at all on the current prices, he is just as likely to put on a trade in the "wrong" direction (i.e., go long in stock 1 when it has increased in price relative to stock 2) as in the right direction. As seen in Theorem 2, the expected return is indeed equal to 0.

Theorem 2 provides the distribution of the Sharpe ratio under the assumption of normally distributed innovations and MA coefficients that are proportional to some function $h(j)$. The series $h(j)$ is assumed to be convergent. If $q<\infty$, this restriction is trivially satisfied and the same results hold, with H_∞ replaced by $H_q \equiv \sum_{j=1}^{q} h(j)$ and $H_\infty^{(2)}$ by $H_q^{(2)} \equiv \sum_{j=1}^{q} h(j)^2$. As is apparent from the definitions of H_∞^2 and $H_\infty^{(2)}$, the distribution of the Sharpe ratios is invariant to the overall scale of the lag coefficients (i.e., the values of a,b,c,d in equation (21)) and only depends on the relative weights attributed to each lag (i.e., the shape of the function $h(j)$). This result echoes that in Theorem 1, where the Sharpe ratio is completely invariant to the lag coefficients c_j.

How can one link the distribution of the stochastic Sharpe ratio in equation (22) with the actual fixed conditional Sharpe ratio from a conditional pairs-trading strategy? Suppose the conditional pairs-trading Sharpe ratio is monotonically increasing in the observed price deviation; that is, the larger $\beta' y_t$ is, the greater is the Sharpe ratio. In that case, the 2-standard-deviation outcome of the Sharpe ratio distribution should correspond to conditional pairs trades triggered by a 2-standard-deviation price divergence. That is, from the distribution of the Sharpe ratios for the *unconditional* pairs-trading strategy, we can infer the Sharpe ratios of the *conditional* pairs-trading strategy.

From Theorem C1 in Appendix C, it appears that the Sharpe ratio is increasing in the price deviation. It is not clear that the relationship is monotone, however, since the price difference is a function of the past shocks that also appear in the Sharpe ratio expression in equation (C-14). Therefore, we cannot say for certain that a 2-standard-deviation divergence between the price processes corresponds to a 2-standard-deviation outcome of the Sharpe ratio. However, it would be surprising if the speed of convergence did not increase in the size of the price deviation, and the simulation results that follow strongly suggest that this is indeed the case.

In Table 2, we report the 1-standard-deviation annual Sharpe ratio, from Theorem 2, for various parameterizations of $h(j)$. The daily 1-standard-deviation Sharpe ratio equals $\sqrt{H_\infty^{(2)}/H_\infty^2}$, and the corresponding annualized Sharpe ratio is $\sqrt{250}\sqrt{H_\infty^{(2)}/H_\infty^2}$. We also report the Sharpe ratios from simulated pairs trades triggered at either a 1- or 2-standard-deviation threshold ($\mathrm{SR}_{\mathrm{ann}}^1$ and $\mathrm{SR}_{\mathrm{ann}}^2$, respectively). Appendix A describes the details of the simulation procedure. As seen in Table 2, the simulated 1-standard-deviation Sharpe ratios ($\mathrm{SR}_{\mathrm{ann}}^1$) are very close to the corresponding 1-standard-deviation Sharpe ratios from the theoretical analysis ($\sqrt{250}\sqrt{H_q^{(2)}/H_q^2}$), and the simulated Sharpe ratios appear to grow linearly with the observed price difference, measured in standard deviations. Table 2 thus gives strong support to the conjecture that the Sharpe ratio is monotonically increasing in the price difference.

Table 2 reports Sharpe ratios for various specifications of $h(j)$ in the form

$$h(j) = \frac{1}{j^\gamma} \quad \text{or} \quad h(j) = \frac{(-1)^j}{j^\gamma}. \tag{23}$$

In all cases, the MA coefficients decline in absolute magnitude according to a power function or remain constant ($\gamma=0$). Since most of these specifications do

TABLE 2

Annualized Sharpe Ratios from 1-Period Pairs Trading

Table 2 presents annualized Sharpe ratios of pairs-trading strategies in which a trade is held open for 1 period and the returns are generated by a VMA(q) model. The $h(j)$ specifications describe the lag structure of the vector moving average (VMA) coefficients (see equation (21)), while $H_q \equiv \sum_{j=1}^{q} h(j)$ and $H_q^{(2)} \equiv \sum_{j=1}^{q} h(j)^2$. The $\mathrm{SR}_{\mathrm{ann}}^1$ and $\mathrm{SR}_{\mathrm{ann}}^2$ values correspond to the 1- and 2-standard-deviation strategies, respectively, and are calculated using simulated pairs trades. The columns labeled $\sqrt{250}\sqrt{H_q^{(2)}/H_q^2}$, $q=\{10,250,\infty\}$ indicate the theoretical 1-standard-deviation Sharpe ratios.

	$q=\infty$	$q=10$			$q=250$		
$h(j)$	$\sqrt{250}\sqrt{\frac{H_\infty^{(2)}}{H_\infty^2}}$	$\sqrt{250}\sqrt{\frac{H_{10}^{(2)}}{H_{10}^2}}$	$\mathrm{SR}_{\mathrm{ann}}^1$	$\mathrm{SR}_{\mathrm{ann}}^2$	$\sqrt{250}\sqrt{\frac{H_{250}^{(2)}}{H_{250}^2}}$	$\mathrm{SR}_{\mathrm{ann}}^1$	$\mathrm{SR}_{\mathrm{ann}}^2$
$1/j^2$	10.00	10.61	10.45	20.92	10.02	9.51	19.00
$1/j$		6.72	6.58	13.21	3.32	2.25	4.51
$1/j^{0.5}$		5.39	5.22	10.45	1.29	1.18	2.33
1		5.00	4.41	8.76	1.00	0.88	1.74
$(-1)^j/j^2$	20.00	20.11	20.03	40.09	20.00	19.94	39.90
$(-1)^j/j$	29.26	30.49	28.29	56.47	29.31	28.63	57.31

not result in finite $H_\infty = \sum_{j=1}^{\infty} h(j)$, results for finite q processes are also shown, setting $q=10$ or 250. In the convergent cases, restricting the MA process to only 10 lags leaves results almost identical to those in the MA(∞) case, for a given specification of $h(j)$. Most specifications in Table 2 result in very high annual Sharpe ratios. This is particularly true for the alternating series, $h(j)=(-1)^j/j^\gamma$, with Sharpe ratios of 40 and above for a 2-standard-deviation strategy. It is also clear that the Sharpe ratios become smaller as $h(j)$ declines slower. This makes intuitive sense since a more slowly declining $h(j)$ is associated with slower mean reversion in the model or, put differently, more long-lasting transient dynamics.

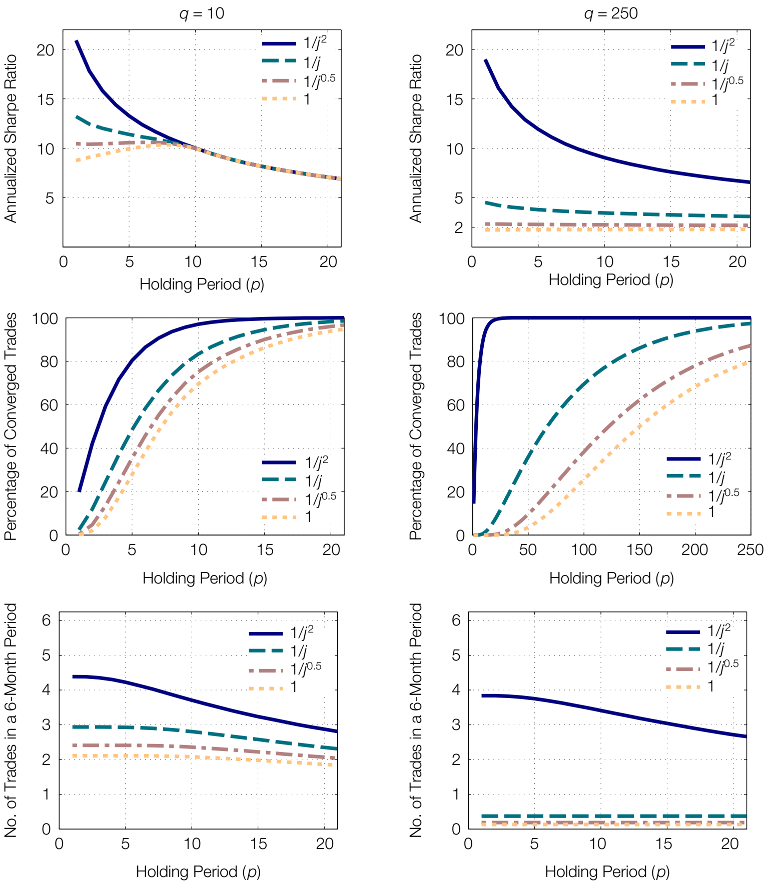
To some extent, the purpose of Table 2 is to evaluate whether there exists any "reasonable" parameterizations of c_j, which admit cointegration but do not result in Sharpe ratios that are too high. Note that Sharpe ratios of around 2 are generally in line with those that GGR (2006) document empirically for typical pairs-trading strategies, although a Sharpe ratio of 2 is probably still on the high side for an individual pair. The only parameterizations in Table 2 that result in annualized Sharpe ratios of around 2 for a 2-standard-deviation strategy are $h(j)=1/j^{0.5}$ and $h(j)=1$, with a lag length $q=250$. More quickly declining weights (a larger γ or a smaller maximal lag length q) result in Sharpe ratios that are considerably larger. Thus, judging only by the Sharpe ratios, it would seem that a lag structure spanning upward of a year ($q=250$) with MA coefficients that decline no faster than a rate of $1/j^{0.5}$ would be necessary to keep the Sharpe ratios within reasonable bounds.

Theorem 2 provides theoretical results for the case when the holding period is $p=1$. We provide simulation results for strategies when the pairs-trading position is closed after $p>1$ periods (days); the details of the simulation design are described in Appendix A. The top two graphs in Figure 1 show annualized Sharpe ratios generated by a 2-standard-deviation strategy for holding periods of up to a month ($p=21$). We present several VMA parameterizations ($q=\{10,250\}$ and different $h(j)=1/j^\gamma$ specifications). For $p=1$, the Sharpe ratios exactly correspond to the $\mathrm{SR}_{\mathrm{ann}}^2$ values from Table 2. Focusing first on the $q=10$ case, we see that the Sharpe ratios decline with increasing holding periods when $\gamma=2$ or 1. However, when $h(j)$ declines slowly ($\gamma=0.5$ or 0), Sharpe ratios

FIGURE 1

Properties of a 2-Standard-Deviation Strategy for Different Holding Periods

The graphs in Figure 1 present trade characteristics of pairs-trading strategies in which a trade is initiated by a 2-standard-deviation price spread and held open for p periods (on the horizontal axis) and returns are generated by a VMA(q) model ($q=10$ and $q=250$ for graphs on the left and right, respectively). The lines correspond to different $h(j)$ specifications (see the legends). The top pair of graphs presents annualized Sharpe ratios, the middle pair of graphs presents the percentage of converged trades within p periods, and the bottom pair of graphs presents the number of opened trades within 125 trading days (6 months). All characteristics are calculated using the simulation procedure described in Appendix A.



initially increase with the holding period. In these latter cases, since the convergence of the two prices is slower, holding the position for a few days leads to higher risk-adjusted returns. When $p=q=10$, Theorem 1 applies and the Sharpe ratio is the same for all $h(j)$ specifications. All in all, with $q=10$, the 2-standard-deviation strategy produces annual Sharpe ratios well above 5 for any $h(j)$ specification and holding periods of at least up to a month. Turning to the $q=250$ case, Table 2 demonstrates that the parameterizations with $h(j)=1/j^{0.5}$ and $h(j)=1$ lead to Sharpe ratios of around 2 for a 1-day holding period. The top right graph in Figure 1 shows that the Sharpe ratios do not change much with the holding period in these cases, staying around 2. Actually, we know from Theorem 1 that the

Sharpe ratio reaches exactly 2 when the holding period is $p = q = 250$ days. To summarize, considering longer holding periods ($p > 1$) does not change our conclusions from Theorem 2. A lag structure spanning upward of a year ($q = 250$), with MA coefficients that decline no faster than a rate $1/j^{0.5}$, would be necessary to keep the Sharpe ratios within reasonable bounds.

The middle two graphs in Figure 1 show the percentage of converged trades (where the two prices have converged) by period p in our simulation. We focus on the cases in which $q = 250$ (middle right graph) and $h(j)$ declines slowly ($h(j) = 1/j^{0.5}$ or $h(j) = 1$), as these parameterizations provide empirically reasonable Sharpe ratios of around 2. The convergence of pairs trades in these cases is quite slow (note that the scale for p is different in this graph compared to the others in Figure 1). Up to a holding period of 1 month (p up to 21), fewer than 1% of the trades converge. Even if we consider holding periods of up to 2 months (p up to about 40), only approximately 5% of the trades converge. This is at odds with the empirical evidence from pairs-trading studies, which suggests that pairs trading is a relatively fast strategy, with convergence of pairs often occurring within a month or so (e.g., Engelberg et al. (2009), Do and Faff (2010), and Jacobs and Weber (2015)).

The bottom two graphs in Figure 1 further illustrate that large values of q lead to a severe slowdown in pairs trading. The graphs show how often a new trade is opened in a given pair.[12] Specifically, the bottom graphs show the average number of trades per 6-month period (125 trading days); the 6-month period is chosen to align with the empirical summary statistics presented in Section IV. For $q = 10$, this number is typically around 2.5, depending somewhat on the holding period, p, and the shape of the function $h(j)$. That is, on average, a new pairs trade is put on approximately every 50 trading days. For $q = 250$, the trading frequency is much smaller, unless the MA coefficients decline very quickly at a rate $1/j^2$, with only approximately 0.25 trades in a given 6-month period or, equivalently, a new trade roughly every 500 days. To put these numbers in an empirical context, GGR (2006) find that a typical pair trades approximately twice in a 6-month period. This is similar to the trade frequency obtained for $q = 10$ but much more frequent than what is observed for $q = 250$ (unless $h(j) = 1/j^2$).

The previous results were derived under a VMA specification for the returns process. An alternative way of modeling stock returns is, of course, through a vector autoregressive (VAR) model. It is well known that stationary VAR models can be inverted into VMA models, and vice versa for invertible VMA models. One would therefore expect the two modeling approaches to yield similar pairs-trading implications. Appendix B derives a result similar to Theorem 1 but in a VAR setting. As is discussed in some detail in Appendix B, the implications for

[12] A pairs trade position is first opened when the spread between the two prices crosses the 2-standard-deviation trigger point. If the pair converges during the holding period of p days, the pair is eligible to trade again immediately after closing (i.e., a position will be opened again if the prices diverge beyond the 2-standard-deviation interval). If the pair does not converge during the holding period, then the pair does not become available for trading again until it has converged. That is, at any given point, at most one position can be open in the pair, and the pair must converge between each new trade. This trading rule essentially follows that of GGR (2006), apart from the fact that in GGR the positions are held until convergence instead of a fixed period.

pairs trading in a cointegrated VAR setting are indeed very similar to those derived in the VMA case.

C. Is Cointegration among Stock Prices Plausible?

Theorems 1 and 2 explicitly quantify the properties of the returns from a pairs-trading strategy in a cointegrated price system. Arguably, the most important determinant of the Sharpe ratio on the pairs-trading strategy is the maximal lag length, q, in the VMA process that governs the dynamics of the stock returns.[13] It is clear that the most outsized Sharpe ratios occur for small values of q. The parameter q can be viewed as the maximal lag length at which the returns exhibit any own- or cross-serial correlation. A value of $q = 250$ would suggest that the serial correlation in stock returns stretches back 1 year or, put differently, that it takes up to a year to fully incorporate news into stock prices after this news is initially revealed.

Cointegration becomes a very powerful concept when coupled with asset prices because the transitory component in asset prices is generally considered to be small and short-lived. This lack of short-run dynamics in stock prices puts strong bounds on the duration for which two cointegrated price processes can deviate from each other, and these bounds grow tighter as the temporal span of the lag effects (i.e., q) becomes smaller. In our view, the previous theoretical analysis therefore implies that either i) cointegration among stock prices does not exist (or is at least very unlikely), since the implied Sharpe ratios appear too large to be realistic, or ii) the serial correlation in stock returns stretches over considerably longer horizons than is usually assumed. The exact cutoff point for "too large a Sharpe ratio" is of course not precisely defined, but individual investment opportunities with Sharpe ratios above 3 or 4 should be few and far between, especially when they can be implemented as easily as with a pairs-trading strategy, which requires nothing more complicated than the ability to short-sell a stock.[14] Such a threshold would suggest that the dynamics in stock returns play out over at least 6 months, and more likely 12 months ($q = 250$), in order for cointegration to be realistic. However, such long-lasting transient dynamics appear to be at odds with the empirical evidence that pairs trading is a relatively fast strategy, with a new trade in a given pair every 2–3 months and convergence typically occurring within

[13]More generally, as seen in Theorem 2, Table 2, and Figure 1, the speed of the decline in the VMA lag coefficients (i.e., the shape of the $h(j)$ function) is the primary determinant of the profitability of pairs-trading strategies. In practice, distinguishing between an infinite-order VMA model with quickly declining lag coefficients and a finite-order VMA model with some maximal lag length, q, is essentially impossible. We therefore focus the discussion around the notion of a finite maximal lag length.

[14]There are examples of statistical arbitrage strategies that appear to deliver very high Sharpe ratios. For instance, both Nagel (2012) and Wahal and Conrad (2017) report annualized Sharpe ratios well above 5 for some strategies. Interestingly, both of these examples represent returns to some form of liquidity provision, which is nowadays closely connected with the ability to trade at low costs: Much of modern market making is conducted by high-frequency traders who specialize in trading with minimum frictions. As seen in the discussion on transaction costs in Section III.D, cointegration is most likely to be present when the two price processes co-move very closely, in which case the absolute level of returns from pairs trading is small and therefore highly sensitive to transaction costs. Since trading costs can arguably be decreased by investments in trading infrastructure, these large Sharpe ratios might be viewed partly as a return on the investment in trading infrastructure.

a month or so. Figure 1 shows that trading is much less frequent and that the convergence is considerably slower in the $q = 250$ case.

D. The Size of Price Deviations and Transaction Costs

The theoretical Sharpe ratios derived previously are all invariant to the overall scale of the price processes. That is, the Sharpe ratios correspond to pairs trades triggered by 2-standard-deviation price spreads, but the absolute size of that 2-standard-deviation spread does not enter into the formulas for the Sharpe ratios. However, once one starts considering transaction costs, the actual scale of the price processes becomes important. If trading costs shave off a fixed amount (i.e., a number of percentage points) from each trade, as is usually assumed, the overall level of returns becomes highly important. Corollary 1 shows what the actual price spreads would be under cointegration, given similar parameterizations to before.

Corollary 1. Suppose $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ is a bivariate returns process with $\epsilon_t \equiv \text{iid}(0, \Sigma)$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$, $C(L) = \sum_{j=0}^{q} c_j L^j$, and $C(1) \neq 0$. Assume that the coefficients c_j can be written as in equation (21). If the corresponding price process, y_t, is cointegrated with cointegration vector $\beta = (1, -1)$, then

$$\text{(24)} \quad \text{var}\left(\beta' y_t\right) = \text{var}\left(y_{1,t} - y_{2,t}\right) = (\sigma_{11} + \sigma_{22} - 2\sigma_{12}) \sum_{j=0}^{\infty} \left[\frac{\left(\sum_{s=j+1}^{\infty} h(s)\right)^2}{\left(\sum_{s=1}^{\infty} h(s)\right)^2} \right].$$

For ease of illustration, suppose that $\sigma_{11} = \sigma_{22}$, so that equation (24) can be written as

$$\text{(25)} \qquad \text{var}\left(y_{1,t} - y_{2,t}\right) = 2\sigma_{11}\left(1 - \rho_{12}\right) \sum_{j=0}^{\infty} \left[\frac{\left(\sum_{s=j+1}^{\infty} h\left(s\right)\right)^2}{\left(\sum_{s=1}^{\infty} h(s)\right)^2} \right],$$

where ρ_{12} is the correlation between the innovations to the two price processes. The formulation in equation (25) highlights the strong dependence between the variance of the price deviations, and the correlation between the innovations: As $\rho_{12} \uparrow 1$, $\text{var}\left(y_{1,t} - y_{2,t}\right) \downarrow 0$. A higher correlation implies that the two price processes are hit by more similar shocks, limiting the size of the deviations between the two processes, keeping all else equal.[15]

In order to get a sense of the actual scale of the price deviations, suppose that σ_{11} and σ_{22} are both equal to 4.5%. This is similar to the average daily stock return

[15] The function $h(s)$ also plays a role. As seen in equation (24), the less relative mass the function $h(s)$ has for large s, the smaller the variance of the price difference, keeping σ_{11}, σ_{22}, and σ_{12} fixed. That is, as discussed previously, limiting the short-run dynamics implies smaller deviations between the two price processes, keeping all else constant.

variance in our data and also in line with average daily variances for (large) U.S. stocks. Table 3 presents the 2-standard-deviation spread that would trigger a pairs trade for different ρ_{12} values. In the setting of Theorem 1, where the holding period is equal to the lag length ($p=q$), these numbers also represent the expected returns on the pairs trade (see equation (15)). For instance, with $h(j)=1/j$ and $q=10$, the expected returns over the 10-day holding period are equal to 6.0% and 1.9% for correlations $\rho_{12}=0.5$ and $\rho_{12}=0.95$, respectively. As seen from the result in Theorem 1, the Sharpe ratio of the trade is not affected by the correlation, since the variance of the returns also decreases as the correlation increases. However, if one takes trading costs into account, this invariance of the Sharpe ratio no longer holds.

TABLE 3

Theoretical 2-Standard-Deviation Spreads

Table 3 presents 2-standard-deviation spreads implied by the VMA(q) model. In particular, we report $2\sqrt{\text{var}(y_{1,t}-y_{2,t})}$ values calculated from equation (25) with $\sigma_{11}=0.045$, $\rho_{12}\in\{0.5,0.95\}$, $q\in\{10,250\}$, and various $h(j)$ specifications. All values in the table are expressed in percentages.

	$q=10$		$q=250$	
$h(j)$	$\rho_{12}=0.5$	$\rho_{12}=0.95$	$\rho_{12}=0.5$	$\rho_{12}=0.95$
$1/j^2$	4.63	1.46	4.87	1.54
$1/j$	5.98	1.89	15.46	4.89
$1/j^{0.5}$	7.14	2.26	28.74	9.09
1	8.32	2.63	38.85	12.28
$(-1)^j/j^2$	4.37	1.38	4.36	1.38
$(-1)^j/j$	5.28	1.67	5.10	1.61

As we discuss further in Section IV, it is reasonable to assume that the trading costs for a full round trip of a pairs trade are approximately 1.2 percentage points (120 basis points (bps)). Thus, the before-cost expected returns of 6.0% and 1.9% would reduce to 4.8 and 0.7% after costs, respectively. Since the variances of the returns are not affected by trading costs, the Sharpe ratios after costs decrease in proportion to returns, in this case by 20% and 65% for $\rho_{12}=0.5$ and $\rho_{12}=0.95$, respectively. Thus, transaction costs become increasingly important as $\rho_{12}\uparrow 1$.

Provided that the 2-standard-deviation spread that triggers a pairs trade is not too small, transaction costs have only a limited impact on the theoretical Sharpe ratios. Put differently, as long as the innovations to the two price processes are not too highly correlated, the theoretical results are only mildly affected by transaction costs. These findings also suggest a third interpretation of the results in this paper: If prices *are* cointegrated *and* have limited short-run dynamics, the correlation between the innovations must be near unity. In this case, although the theoretical Sharpe ratios are unaffected, the price deviations are small enough that they are likely difficult to profit substantially from, given transaction costs. As discussed in Section IV, prices for A- and B-shares issued by the same firm fit this description quite well, whereas ordinary (not A–B) pairs show much larger spreads and never exhibit very high correlations between the innovations.

E. Misclassification of Pairs

Our theoretical framework implicitly assumes that the investor has already identified pairs of stocks with cointegrated prices. In practice, even if cointegrated stock prices do exist, the investor would have to do some initial screening to find pairs of stocks with cointegrated prices. Such an empirical classification would naturally run the risk of misclassifying some pairs as cointegrated when they are, in fact, not.

In the theoretical analysis, we only consider the return properties of a single pairs trade, but clearly one can combine such trades into a portfolio of pairs trades. Specifically, suppose that some fraction, λ, of the pairs identified for pairs trading are not cointegrated, whereas the other pairs are cointegrated. The cointegrated pairs would result in pairs-trading returns that satisfy the results derived previously. However, the non-cointegrated pairs would likely perform considerably worse. For simplicity, suppose that the mean returns on the non-cointegrated pairs trades are equal to 0. In this case, assuming that the other properties of the non-cointegrated pairs are similar to those of the cointegrated pairs, the Sharpe ratio of the portfolio of pairs trades would drop by a fraction λ, compared to the case of a portfolio with no misclassified pairs.[16] If λ were equal to, say, 20% and the Sharpe ratio in the correctly classified pairs case were equal to 3, the investor's Sharpe ratio would drop to 2.4. This is a substantial performance deterioration to an actual investor but does not change the qualitative conclusions of the theoretical analysis. Therefore, unless the misclassification is of a very large order, upward of 50% or more, the potential effects on the Sharpe ratios are fairly limited in the sense that the overall conclusion of the theoretical analysis does not get overturned in any way.

IV. Empirical Analysis

The empirical analysis is based on stocks traded on the Stockholm Stock Exchange. Our initial sample consists of all stocks listed in the large cap segment of the Nasdaq Stockholm Exchange as of June 1, 2015, and the sample period is from Jan. 1995 to Dec. 2014.[17] We use data from Sweden because the widespread use of dual-class shares provides a useful control group of stock pairs for our analysis. Dual-class firms issue two types of shares, typically labeled as A- and B-shares, which provide identical ownership fractions in the underlying company and receive identical dividends but represent different voting rights. B-shares would typically provide one vote per share, while A-shares might provide 10 votes per share. Since they provide a claim to the same dividends, their prices are likely to be closely related.

[16]The mean-zero assumption would seem fairly conservative, essentially implying that no signal at all was identified when selecting the pair. The assumption that the Sharpe ratio is only affected through the mean returns implies that the variances and covariances are assumed identical for the cointegrated and non-cointegrated pairs. For short holding periods (e.g., 10 or 20 days), this does not seem an unreasonable assumption as the long–short position can be scaled up or down to some target variance; in theory, the variance of the spread between two nonstationary, non-cointegrated processes will increase over time, but over short horizons such effects are small. There seems to be little reason to assume that diversification benefits from non-cointegrated pairs would be worse than those from other cointegrated pairs, so the covariance assumption seems fairly innocuous.

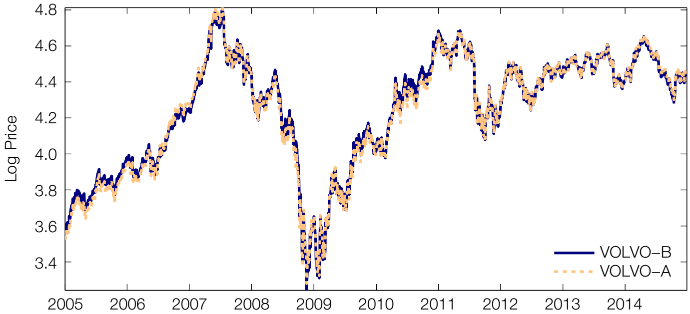
[17]The data are from the FinBas database, which is administered by the Swedish House of Finance and contains end-of-day stock prices, adjusted for corporate actions (e.g., stock splits and buybacks), from the Nordic stock exchanges. Further details are available at https://data.houseoffinance.se/finbas/finbasInfo.

Indeed, financial theory would suggest that the A- and B-share prices are cointegrated (as in Bossaerts (1988)). Also, as seen in the plot of the prices for the two classes of Volvo shares displayed in Figure 2, the A–B prices track each other almost perfectly. This visual evidence is certainly compelling and suggestive of cointegration. Based on the visual evidence and the theoretical motivations, we believe that the A–B prices are likely cointegrated, or at least as close to being cointegrated as one can practically find in terms of stock prices.[18] The empirical results for the A–B pairs are subsequently used as a form of calibration for our empirical methods. Specifically, the A–B pairs provide a benchmark for how closely the empirical results are expected to align with the theoretical predictions, if we are indeed observing cointegrated price pairs.[19]

Our empirical strategy is thus to investigate both ordinary pairs and A–B pairs, where by "ordinary" we refer to pairs in which the two stocks are issued by two different companies. The empirical analysis consists of two parts: First, we calculate empirical Sharpe ratios and other trade characteristics from the implementation of a pairs-trading strategy similar to that analyzed in GGR (2006). Second, we quantify how close the implicit restrictions of cointegration (discussed in Section II.C) are to being satisfied empirically. Both exercises provide direct evidence of how well the empirical results for the ordinary pairs approximate the

FIGURE 2
Log Price Series of Volvo A- and B-Shares

Figure 2 shows the (log-) price series for the two share classes of Volvo, one of the biggest Swedish companies, in the last 10 years of our sample (Jan. 2005 to Dec. 2014).



[18]As stressed in Section IV.B, it is empirically very difficult to irrefutably verify cointegration, but it is certainly hard to imagine stronger co-movement than that illustrated in Figure 2. If one studies a graph of the *difference* between the two prices, some persistent patterns are visible, but the overall scale of these price differences is clearly small. As a result, however, formal cointegration tests, which do not take into account the scale of the deviations vis-à-vis the original price series, can often *not* reject the null of no cointegration for the A–B pairs, despite the fact that the price co-movement for all A–B pairs in our sample look very similar to that illustrated in Figure 2. Nevertheless, we regard the A–B pairs as a relevant control group against which to compare the degree of co-movement in the ordinary pairs.

[19]To the extent that the A–B pairs are not perfectly cointegrated (see footnote 18), the comparison to A–B pairs is conservative in the sense that the results for other non-cointegrated pairs are less likely to be clearly separated from the A–B pairs. That is, one would be less likely to find evidence *against* cointegration of other pairs, compared to the case when the A–B pairs are perfectly cointegrated.

theoretical model predictions, in terms of both pairs-trading returns and actual coefficient restrictions implied by cointegration. In addition, we compare the results for the ordinary pairs and the for the A–B pairs. If the empirical trade characteristics and coefficient estimates of the A–B pairs are (much) more in line with the theoretical predictions of the cointegrated model than those of ordinary pairs, we view it as evidence that cointegration between prices of stocks issued by different companies is not likely.

Sweden is number one in the world in terms of the use of dual-class shares (La Porta, Lopez-de-Silanes, Shleifer, and Vishny (1998)). Some of the biggest and most well-known Swedish companies have dual-class shares, where both classes are publicly traded on the same exchange. Moreover, the voting premium on the high-voting A-shares is also among the lowest in the world, and it is lower than in the United States (Nenova (2003)). Before 1993, Swedish firms, apart from having A- and B-shares, could also have restricted and unrestricted share classes. As a consequence, many firms had four types of shares. Restricted and unrestricted shares not only differed in voting rights but also represented different cash-flow rights. Moreover, foreigners could only hold unrestricted shares. In Jan. 1993 the distinction between restricted and unrestricted share classes was abolished, leaving firms with only A- and B-shares (for a detailed analysis of the effects of this change, see Holmén (2011)). To avoid complications due to the differences between restricted and unrestricted shares, we begin our sample in Jan. 1995 to give the market enough time to incorporate the change in 1993. The end of the sample period is Dec. 2014, which gives us 20 years of daily data.

Table 4 provides a brief description of the sample as of June 1, 2015. There are 72 listed companies with a total market capitalization of 5,403 billion Swedish kronor (SEK). Most of these firms have dual-class shares. However, in many cases, only one of the share classes is listed on the exchange. We need to observe the price series of both classes for our analysis, and hence we need firms with both A- and B-shares listed. There are 21 such companies, representing 2,271 billion SEK, which is 42% of total market capitalization. We restrict the analysis to firms for which we observe all end-of-day prices (closing price or the average of the end-of-day bid and ask prices if the closing price does not exist) during our 20-year sample period. There are 24 such companies, representing 55% of total market capitalization (2,957 billion SEK). Of these 24 firms, 8 list both A- and B- shares, representing 30% of total market capitalization (1,898 billion SEK). Altogether, our sample includes 8 A–B pairs and 488 ordinary pairs. Panel B of Table 4 lists the top 10 Swedish firms in terms of market capitalization, 6 of which have both A- and B-shares listed on the exchange.

A. Empirical Sharpe Ratios

We start by calculating Sharpe ratios and other trade characteristics of pairs trading in our sample. The evaluation period runs from Jan. 1996 to Dec. 2014, since the first 12 months of the original 20-year sample have to be reserved for the first formation period.

GGR (2006) consider the following implementation of pairs trading: During a 12-month formation period, all possible stock pairs are formed, and for each pair the sum of squared deviations (SSD) in the standardized daily price series

TABLE 4
Sample Coverage

Table 4 describes the large cap firms on the Nasdaq Stockholm Exchange as of June 1, 2015. Panel A presents the number of firms and total market capitalization (in Swedish kronor (SEK)) for different subsets. Panel B presents the top 10 (in terms of market capitalization) firms.

Panel A. All Large Cap Firms

	No. of Firms	Market Cap (billion SEK)
Large cap firms	72	5,403
Dual-class firms with both classes listed	21	2,271
Firms in the sample	24	2,975
Dual-class firms in the sample	8	1,638

Panel B. Top 10 Firms

	A–B Listed	Market Cap (billion SEK)
Hennes & Mauritz AB	No	486
Nordea Bank AB	No	437
Ericsson, Telefonaktiebolaget LM	Yes	298
Atlas Copco AB	Yes	284
Investor AB	Yes	244
Svenska Handelsbanken AB	Yes	242
Skandinaviska Enskilda Banken AB	Yes	241
Volvo AB	Yes	232
Swedbank AB	No	227
TeliaSonera AB	No	217

of the two constituent stocks is calculated.[20] From all possible pairs, those with the lowest SSD are chosen for trading. During the trading period, defined as the 6-month period immediately following the 12-month formation period, a long–short position in a pair is opened whenever the standardized prices of the constituent stocks diverge by more than 2 standard deviations of the historical price difference observed over the formation period. In GGR's implementation, the position is closed when the two price series converge, or at the end of the trading period if convergence never occurs. Pairs that complete a round-trip are then available for trading again for that period. A typical trading portfolio is an equal-weighted combination of the top 5 or top 20 pairs with the lowest SSD. GGR repeat this 12–6 implementation cycle every month, effectively mimicking a hedge fund of separate managers whose implementation cycles are staggered by 1 month. The monthly return on the strategy is the equal-weighted monthly return across the six managers who are active in the given month. When calculating average returns and Sharpe ratios for this strategy, we follow GGR's "committed-capital" approach and assume that the fund has committed capital to each of the chosen pairs; when this capital is not invested in an open pair, it earns zero returns.

To be in line with our theoretical analysis, we modify GGR's (2006) strategy at one point: Instead of holding an open position in a pair until the two price series converge, the position is held for a fixed holding period (i.e., a fixed number of trading days). Otherwise, our strategy is exactly the same as the one described previously. In particular, while pairs are not held until convergence, after a pair has closed it is not eligible for trading again until it has converged; if it converged before closing, it is available for trading immediately upon closing.

[20]The standardized price series is the cumulative total return index scaled to start at 1 SEK at the beginning of the period. The scaling is done at the beginning of both the formation and the trading periods.

TABLE 5
Empirical Characteristics of Pairs-Trading Strategies

Table 5 presents Sharpe ratios and other characteristics of pairs-trading strategies in which each position is held open for a fixed number of trading days (5, 10, and 15 days in Panels A, B, and C, respectively). The first three columns correspond to strategies in which the top 5, 8, and 20 ordinary pairs (in terms of lowest sum of squared deviations (SSD)) are traded. The last two columns correspond to strategies in which the top 5 and top 8 A–B pairs (in terms of lowest SSD) are traded.

	Ordinary Pairs			A–B Pairs	
No. of Traded Pairs	5	8	20	5	8
Panel A. 5-Day Holding Period					
Annualized Sharpe ratio	1.32	1.60	1.89	2.87	2.91
Annualized Sharpe ratio (10 bps 1-way cost)	0.85	1.06	1.26	2.15	2.28
Annualized Sharpe ratio (30 bps 1-way cost)	−0.12	−0.08	−0.12	−0.27	0.04
Average trigger (%)	9.73	10.11	11.14	1.96	2.34
Trades per share in 6 months	1.36	1.33	1.21	2.21	1.81
Mean of per trade returns (%)	1.08	1.14	1.13	1.05	1.13
Standard deviation of per trade returns (%)	4.46	4.51	4.68	1.25	1.32
Annualized per trade Sharpe ratio	1.71	1.78	1.72	5.94	6.04
Panel B. 10-Day Holding Period					
Annualized Sharpe ratio	1.37	1.57	1.86	2.84	2.94
Annualized Sharpe ratio (10 bps 1-way cost)	1.00	1.16	1.35	2.07	2.27
Annualized Sharpe ratio (30 bps 1-way cost)	0.24	0.30	0.27	−0.24	0.11
Average trigger (%)	9.73	10.11	11.13	1.99	2.35
Trades per share in 6 months	1.32	1.29	1.17	1.99	1.64
Mean of per trade returns (%)	1.28	1.34	1.34	1.07	1.16
Standard deviation of per trade returns (%)	5.36	5.63	5.96	1.29	1.44
Annualized per trade Sharpe ratio	1.19	1.19	1.12	4.14	4.05
Panel C. 15-Day Holding Period					
Annualized Sharpe ratio	0.99	1.30	1.41	2.66	2.67
Annualized Sharpe ratio (10 bps 1-way cost)	0.68	0.96	1.01	1.89	2.00
Annualized Sharpe ratio (30 bps 1-way cost)	0.05	0.24	0.18	−0.20	0.13
Average trigger (%)	9.75	10.12	11.13	1.96	2.36
Trades per share in 6 months	1.28	1.26	1.14	1.78	1.49
Mean of per trade returns (%)	1.17	1.39	1.30	1.07	1.18
Standard deviation of per trade returns (%)	6.36	6.40	6.73	1.36	1.57
Annualized per trade Sharpe ratio	0.75	0.88	0.79	3.20	3.07

Table 5 reports Sharpe ratios and other characteristics of the pairs-trading strategies. To put the results into perspective, note that the OMXS30 index, which is the capitalization-weighted index of the 30 largest stocks of the Stockholm Stock Exchange, has an annualized Sharpe ratio of 0.37 over our sample period. We separately report results for trading ordinary and A–B pairs. The first three columns of Table 5 report the performance of strategies in which the top 5, 8, and 20 *ordinary pairs* are traded. The last two columns report the performance of strategies where the top 5 and top 8 *A–B pairs* are traded. Note that in the latter case, there is no pair selection going on, since the full set of 8 A–B pairs is available for trade all the time. Panels A, B, and C of Table 5 correspond to the cases in which the open positions are held for 5, 10, and 15 trading days, respectively.

The first row in each panel of Table 5 reports the before-cost Sharpe ratios for the committed-capital strategy. When trading ordinary pairs, all Sharpe ratios fall in the range of 1–1.9, which is in line with previous studies. GGR (2006) report only marginally higher Sharpe ratios when studying a different market (the United States) and a different time period (1962–2002). Do and Faff (2010), who study the same strategies as GGR using U.S. data, report Sharpe ratios similar to ours for the subperiods that overlap with our sample period. When trading A–B pairs, the Sharpe ratios fall in the range of 2.6–3.0. That is, before-cost Sharpe ratios from trading A–B pairs are considerably higher than can be attained when trading ordinary pairs.

We also consider the effect of transaction costs in Table 5. At least three types of costs emerge when implementing a pairs-trading strategy: commissions, short-selling fees, and the implicit cost of market impact (Do and Faff (2012)). The effect of commissions and market impact needs to be considered whenever a position is initiated or closed in a given stock. Since one complete pairs trade involves two round-trips, the associated transaction cost will be 4 times the per stock 1-way cost (commission plus market impact). Do and Faff (2012), who study the transaction costs associated with pairs trading in the United States, estimate the average 1-way cost to be approximately 30 bps in the period 1989–2009. Do and Faff (2012) also account for short-selling fees by including a constant loan fee of 1% per annum, payable over the life of a given pairs trade. We ignore the short-selling fees in our empirical implementation, since they are negligible compared to the other costs according to the estimates in Do and Faff (2012), and deduct 2 times the 1-way cost, both at the initiation and at the close of each pairs trade, to calculate the after-cost return corresponding to each strategy.[21]

The second and third rows in each panel of Table 5 report the after-cost Sharpe ratios for the committed-capital strategies. Specifically, we consider both a low-cost case, with a 1-way cost of 10 bps, as well as the 30 bps 1-way cost estimated by Do and Faff (2012). With a 10 bps 1-way cost, ordinary pairs earn Sharpe ratios between 0.7 and 1.4, while trading A–B pairs leads to Sharpe ratios of around 2. That is, when transaction costs are low, pairs trading still performs considerably better in the case of A–B pairs. However, when the 1-way cost is 30 bps, all Sharpe ratios decrease dramatically: For the ordinary pairs, the Sharpe ratios are never above 0.3, and for the A–B pairs, the Sharpe ratios are close to 0 or negative.[22] That is, when a realistic level of transaction costs is used, the returns from pairs trading are greatly reduced or extinguished. This is particularly true for the A–B pairs.

Table 5 also shows the average price deviation that triggers a pairs trade (i.e., the average 2-standard-deviation outcome of the price differences) as well as how often the pairs trade. For the ordinary pairs, the average trigger is approximately 10%.[23] For the A–B pairs, the average trigger is much smaller, at approximately

[21] Since short-selling fees are assumed to be proportional to the length of the trade, they are relatively small for strategies with short holding periods. For example, the 1% annual loan fee considered by Do and Faff (2012) translates into 2 bps for the strategy that holds each position for 5 trading days. The 1-way cost of 30 bps is realistic according to Do and Faff (2012), who consider typical institutional investors trading on the U.S. market. If actual transaction costs in Sweden are higher than those in the United States, the transaction-cost adjusted Sharpe ratios become even smaller.

[22] Schultz and Shive (2010) analyze a form of pairs trading in U.S. dual-class shares, which should be similar to the A–B shares in our sample, and find that such strategies can yield substantial returns. Their study accounts for bid–ask spreads but *not* other trading costs. Their results might therefore be most comparable to our low-transaction cost scenario; for instance, Jones (2002) reports average half-spreads of approximately 10 bps on Dow Jones stocks in 2000, which is in the middle of the sample period (1993–2006) used by Schultz and Shive (2010).

[23] This average opening trigger is larger than the average 5% trigger in GGR (2006) but quite similar to what Do and Faff (2012) find. They consider a number of different pair formations and

2%. On average, an ordinary pair trades 1.3 times per 6-month period, whereas the A–B pairs trade a bit more often, about twice every 6 months.

There are some key differences between the Sharpe ratios reported in the first row of each panel in Table 5 and the theoretical ones we study in Section III. The empirical (annualized) Sharpe ratios in Table 5 correspond to the monthly returns from trading a portfolio of pairs, where each "manager's" capital earns zero returns when not invested in a pair. On the other hand, in the theoretical analysis we study *per trade* Sharpe ratios from trading a *single pair*. In order to be closer to our theoretical analysis and to provide more details regarding the differences between A–B pairs and ordinary pairs, we therefore also report statistics for the per trade returns. The final three rows in each panel of Table 5 report the mean and standard deviation of all per trade returns, along with the per trade annualized Sharpe ratios, which are simply defined as the ratios of the average per trade returns to the standard deviation of the per trade returns, scaled to an annual basis. These per trade Sharpe ratios are thus direct empirical proxies for the theoretical Sharpe ratios derived earlier.

The per trade return statistics further emphasize that trading A–B pairs leads to considerably higher Sharpe ratios than trading ordinary pairs. For each holding period, the annualized Sharpe ratios for the A–B pairs are more than 3 times higher than the respective Sharpe ratios for the ordinary pairs. The mean return for a given pairs trade is approximately 1.2% for both ordinary and A–B pairs, and the differences in Sharpe ratios are almost entirely driven by a much greater variance in the ordinary pairs returns. Since the full round-trip transaction cost for a pairs trade is 120 bps (when the 1-way cost per stock is 30 bps), this level of transaction costs effectively wipes out almost all of the returns, as seen in the after-cost Sharpe ratios for the committed-capital strategy in Table 5 (row 3 in each panel).

Overall, the properties of the A–B pairs-trading returns line up reasonably well with the implications from the theoretical analysis. Before-cost Sharpe ratios are large (upward of 6.0 on a per trade basis), and not dissimilar from the theoretical ones obtained under cointegration. However, given the small spreads that trigger a trade and the subsequent small returns, the after-cost Sharpe ratios are close to 0, consistent with the discussion around Corollary 1 in Section III.D.[24] In practice, there are likely arbitrageurs that perform some form of pairs trading on the A–B pairs, essentially enforcing the no-arbitrage relationship between them. If these arbitrageurs are able to trade at relatively low transaction costs, such trading might be profitable, and these profits might provide a fair return on the investments in infrastructure needed to trade at low costs.

report an average opening spread of approximately 8% across all pairs over their sample period 1989–2009.

[24]The small spreads between the A–B prices suggest a very high correlation between the innovations to the two processes, such that the two processes are affected by almost identical shocks (see Section III.D). To empirically assess the correlations between innovations for the ordinary pairs and the A–B pairs, we use the innovations from VMA(5) estimations of the pairwise price processes and calculated their correlation for all the pairs in our sample. The correlations are all in the range 0.9–0.98 for A–B pairs, whereas they are all below 0.68 for ordinary pairs.

In contrast, the per trade Sharpe ratios for the ordinary pairs are much smaller than those for the A–B pairs, while the trigger spreads for the ordinary pairs are much larger than for the A–B pairs. Pairs trading in ordinary pairs is profitable before costs, but the after-cost Sharpe ratios for these pairs also approach 0 for reasonable transaction costs. This is consistent with ordinary pairs *not* being cointegrated, since under cointegration the large opening spreads should also translate into large expected returns, which should survive transaction costs quite easily (again, see the discussion in Section III.D).

B. Empirical Evidence of Cointegration

We finally turn to an empirical evaluation of pairwise cointegration of stock prices. There is a long econometric tradition of testing for cointegration among economic time series. However, the formal evaluation of long-run relationships, like cointegration, is fraught with difficulties, and several decades of empirical research have produced few solid empirical facts. Therefore, instead of attempting to provide a yes or no answer to the question of stock price cointegration using formal tests, we adopt a more gradual empirical strategy in which we try to quantify whether pairs of stocks are "close to" or "far from" being cointegrated. We use the implicit restrictions of cointegration discussed in Section II.C and analyze how close these restrictions are to being satisfied in the case of ordinary stock pairs. We compare the results from ordinary pairs to the control group of A–B pairs, which, at a minimum, should be *close to* cointegrated. The sample still consists of 8 A–B pairs and 488 ordinary pairs and the sample period is from Jan. 1995 to Dec. 2014.

The first restriction implied by cointegration, shown in equation (11), is on the long-run MA matrix, $C(1)$, and reads as

$$\psi_{12} \quad = \quad 1+\psi_{22} \quad \text{and} \quad \psi_{21} \quad = \quad 1+\psi_{11}. \tag{26}$$

To study this restriction, we estimate VMAs of orders $q \in \{1,3,5\}$ for each possible pair of stocks in our sample and then calculate

$$d_C \quad \equiv \quad \frac{(\psi_{12}-\psi_{22})+(\psi_{21}-\psi_{11})}{2}. \tag{27}$$

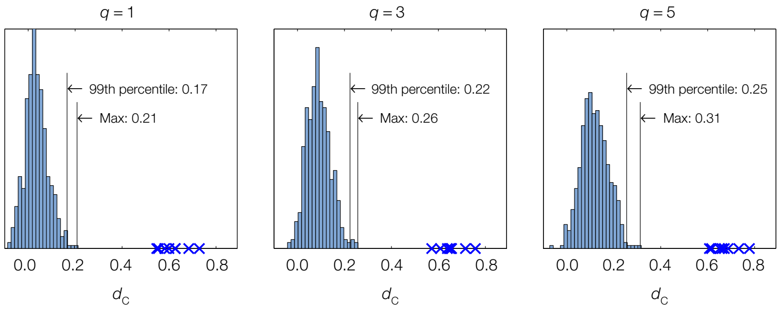
For a pair of cointegrated stock prices, the value of d_C should be close to 1. The histograms in each graph of Figure 3 describe the distribution of d_C for ordinary stock pairs. The $\times$ markers represent the A–B pairs in the sample. The ordinary pairs are clearly separated from the A–B pairs in all graphs of Figure 3. In the case of ordinary stock pairs, d_C is never above 0.31, while all d_C values lie between 0.5 and 0.8 in the case of the A–B pairs. The value of d_C does not actually reach 1 even for the A–B pairs, corroborating our earlier argument that the formal validation of cointegration is very hard as well as the possibility that the A–B pairs are not perfectly cointegrated. Our key empirical observation is that the coefficient estimates are much further away from satisfying the cointegrating restrictions for *all* the ordinary pairs than for *any* of the A–B pairs.

The second restriction we investigate is on the long-run covariance matrix of returns, Ω, shown in equation (14). Cointegration implies that all four elements

FIGURE 3

Cointegration Restrictions on the Vector Moving Average (VMA) Coefficients

The graphs in Figure 3 present the distribution of d_C from equation (27) in our sample. The histograms in each graph represent the distribution of d_C for ordinary stock pairs (488 pairs). The $\times$ markers correspond to the A–B pairs (8 pairs). The separate graphs show results for different lag orders of the estimated VMA(q) process with $q \in \{1,3,5\}$.



of Ω are identical. In order to evaluate this restriction, we first estimate the long-run covariance matrix for all possible pairs in the sample using the Newey–West (1987) estimator with lag lengths $l \in \{10,25\}$. The following statistic is subsequently calculated for each pair:

$$d_{\Omega} \equiv \left|\ln\left(\frac{\omega_{21}}{\omega_{11}}\right)\right| + \left|\ln\left(\frac{\omega_{22}}{\omega_{11}}\right)\right|. \tag{28}$$

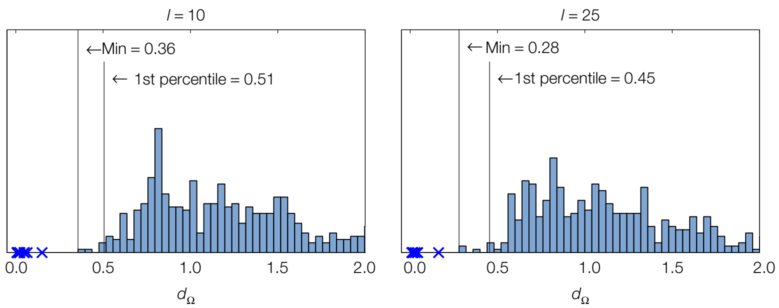
The value of d_{Ω} is 0 under cointegration and positive otherwise. The histograms in Figure 4 describe the distribution of d_{Ω} for ordinary stock pairs, while the $\times$ markers correspond to the A–B pairs. The ordinary pairs are clearly separated from the A–B pairs again, as d_{Ω} for *all* the ordinary pairs lies further away from satisfying the cointegrating restrictions than for *any* of the A–B pairs.

To be clear, Figures 3 and 4 show the values of d_C and d_{Ω}, respectively, for *all* ordinary pairs that can be formed in our sample of stocks. By any selection

FIGURE 4

Cointegration Restrictions on the Long-Run Covariance Matrix

The graphs in Figure 4 present the distribution of d_{Ω} from equation (28) in our sample. The histograms in each graph represent the distribution of d_{Ω} for ordinary stock pairs (488 pairs). The $\times$ markers indicate the A–B pairs (8 pairs). The separate graphs show results for different lag lengths, l, of the estimator with $l \in \{10,25\}$.



criterion, most of these pairs would clearly not be strong candidates for pairs trading, and clearly one should not expect *all* of these pairs to be cointegrated. Rather, the main message of Figures 3 and 4 is that *none* of these ordinary pairs are anywhere near as close to satisfying the cointegration restrictions as *any* of the A–B pairs. As emphasized previously, the A–B pairs are likely close to cointegrated (see also footnotes 18 and 19), and one can view the outcomes of the A–B pairs as a form of calibration result against which to compare the ordinary pairs. The clear separation between the ordinary pairs and the A–B pairs therefore strongly suggests that none of the ordinary pairs are cointegrated.

V. Conclusion

Cointegration of certain pairs of stock prices provides a seemingly simple and clear explanation for why pairs trading would be profitable. However, cointegration among stock prices turns out to be a very powerful concept. In a theoretical analysis, we show that cointegration among stock prices would imply annualized Sharpe ratios that could easily exceed 10, for a single pair, in a simple pairs-trading strategy. This result suggest either i) that cointegration among ordinary stocks is very unlikely or ii) that the transient component in stock prices is longer-lived (i.e., returns exhibit longer serial correlation) than is usually believed to be the case. In an empirical analysis, we attempt to distinguish between these two possible explanations and find strong evidence in favor of the first conjecture: Namely, stock prices are very rarely cointegrated.

In conclusion, our study suggests that cointegration is probably not the right starting point when trying to understand pairs trading. Instead, one needs to consider weaker concepts, and/or accept that pairs-trading opportunities are likely transient in nature and do not fit well into a model of a permanent long-run relationship as stipulated by cointegration.

Appendix A. Simulation Procedure

This appendix describes the procedure used to obtain the simulation-based results presented in the paper. Specifically, the simulations proceed according to following steps:

1. We start simulating values of Δy_t using the model in equation (1) and, at the same time, also create $y_t = \Delta y_t + y_{t-1}$ and $\beta' y_t$ values. For starting values, we use $\epsilon_t = 0$ for all $t \leq 0$ and $y_0 = 1$. The innovations, ϵ_t, are drawn from an iid normal distribution. For a given $h(j)$ function, the c_j coefficients are specified according to appendix equation (C-22), with $a = d = 0.1/H_\infty$. This parameterization keeps the long-run VMA matrix, $C(1)$, identical across different specifications of $h(j)$. As seen in Theorem 2, the distribution of the Sharpe ratios is, in fact, invariant to the values of a and d. This result is also confirmed for the simulated Sharpe ratios in simulations not reported here.

2. After simulating the first 10^5 periods, we estimate the unconditional standard deviation of the difference between the two price processes (i.e., $\sigma_{\beta' y} = \sqrt{\mathrm{var}(\beta' y_t)}$).

3. We continue to simulate Δy_t, y_t, and $\beta' y_t$ values for 10^6 more periods and use these realizations to estimate the number of initiated trades. A pairs trade is first initiated when $|\beta' y_t| > 2\sigma_{\beta' y}$. If the pair converges during p periods (the holding period), the pair is eligible to trade again immediately after the p periods. If the pair does not

converge during p periods, it becomes eligible for trading only after it has converged. Once the pair is eligible for trading, the next pairs trade is initiated when $|\beta' y_t| > 2\sigma_{\beta' y}$ again. We count the number of initiated trades over the 10^6 simulated periods and scale the result so that it corresponds to a 6-month interval (125 trading periods).

4. We continue simulating Δy_t, y_t, and $\beta' y_t$ values until the condition $r\sigma_{\beta' y} - \varepsilon < \beta' y_\tau < r\sigma_{\beta' y} + \varepsilon$ is met at a given time period, τ, at which point one particular pairs trade is initiated. We use $r \in \{-1, -2\}$, corresponding to 1- and 2-standard-deviation strategies, respectively, and we set $\varepsilon = 5 \times 10^{-4}$. We record the p-period return after the pairs trade is initiated, defined as $r_{\tau \to \tau+p} = \sum_{j=1}^{p} \Delta\beta' y_{\tau+j}$ for $p = 1, \ldots, 250$. We also record the number of periods needed until the two price processes converge (i.e., we find the first τ' such that $\beta' y_{\tau'} > 0$ and $\tau' > \tau$) and record $\tau' - \tau$.
5. We repeat step 4 until we have observations from $N = 10^6$ initiated pairs trades. The p-period returns from these N simulated pairs trades enable us to calculate the $\mathrm{E}_t[r_{t \to t+p}]$, $\mathrm{var}_t(r_{t \to t+p})$, and $\mathrm{SR}_t(r_{t \to t+p})$ values. The recorded time to convergence values enable us to produce the plots with the percentage of converged trades.

In Appendix B we also provide simulation results in a VAR setting. The procedure is the same as that described previously, with the only difference that the Δy_t values are simulated from the VAR model in Appendix equation (B-2).

Appendix B. Pairs-Trading Returns in a VAR Setting

We consider the VAR(1) case and focus on 1-period pairs-trading returns. Suppose the bivariate prices, $y_t = \left(y_{1,t}, y_{2,t}\right)'$, are generated according to a cointegrated VAR(1) process,

$$y_t = \Pi_1 y_{t-1} + \epsilon_t. \tag{B-1}$$

The vector error correction model (VECM) format is given by

$$\Delta y_t = \Pi y_{t-1} + \epsilon_t, \tag{B-2}$$

with $\Pi = \Pi_1 - I$. By standard results, if y_t is cointegrated with vector β, one can write $\Pi = \alpha\beta'$ for some vector $\alpha = (\alpha_1, \alpha_2)$, such that

$$\beta' y_t = (\beta' + \beta'\alpha\beta')\, y_{t-1} + \beta'\epsilon_t = (1 + \beta'\alpha)\, \beta' y_{t-1} + \beta'\epsilon_t \tag{B-3}$$

and

$$\Delta y_t = \alpha\beta' y_{t-1} + \epsilon_t. \tag{B-4}$$

For $\beta = (1, -1)$,

$$\beta'\alpha = \alpha_1 - \alpha_2 = -(\alpha_2 - \alpha_1) \equiv -\gamma. \tag{B-5}$$

Let $w_t \equiv \beta' y_t = y_{1,t} - y_{2,t}$ be the difference between the two (cointegrated) price processes, and define $\xi_t \equiv \beta'\epsilon_t$. The price difference w_t then follows an autoregressive process of order 1 (AR(1)),

$$w_t = (1 - \gamma) w_{t-1} + \xi_t. \tag{B-6}$$

Under cointegration, this process must be stationary, which holds for $|1 - \gamma| < 1$, or equivalently, $\gamma \in (0, 2)$.

The returns on the 1-period pairs-trading strategy is given by

$$\text{(B-7)} \qquad r_{t\to t+1} \equiv \Delta\beta' y_{t+1} = \beta'\alpha\beta' y_t + \beta'\epsilon_{t+1}.$$

The conditional expectation and conditional variance at time t are given by

$$\text{(B-8)} \qquad \begin{aligned} \mathrm{E}_t\left[r_{t\to t+1}\right] &= \mathrm{E}_t\left[\beta'\alpha\beta' y_t + \beta'\epsilon_{t+1}\right] = \beta'\alpha\beta' y_t = -\gamma\left(\beta' y_t\right), \\ \mathrm{var}_t\left(r_{t\to t+1}\right) &= \mathrm{var}\left(\beta'\epsilon_{t+1}\right) = \beta'\mathrm{var}\left(\epsilon_{t+1}\right)\beta = \beta'\Sigma\beta. \end{aligned}$$

The unconditional variance of $\beta' y_t$ is given by

$$\text{(B-9)} \qquad \begin{aligned} \mathrm{var}(\beta' y_t) &= \mathrm{var}\left((1+\beta'\alpha)\beta' y_{t-1}\right) + \mathrm{var}\left(\beta'\epsilon_t\right) \\ &= (1-\gamma)^2\mathrm{var}\left(\beta' y_{t-1}\right) + \beta'\Sigma\beta. \end{aligned}$$

By stationarity of the cointegrating relationship, $\mathrm{var}\left(\beta' y_t\right) = \mathrm{var}\left(\beta' y_{t-1}\right)$, and thus

$$\text{(B-10)} \qquad \mathrm{var}\left(\beta' y_t\right) = \frac{\beta'\Sigma\beta}{1-(1-\gamma)^2} = \frac{\beta'\Sigma\beta}{\gamma\left(2-\gamma\right)}.$$

The Sharpe Ratio for the pairs-trading strategy is then equal to

$$\text{(B-11)} \qquad \mathrm{SR}_{t,t\to t+1} = \frac{\mathrm{E}_t\left[r_{t\to t+1}\right]}{\sqrt{\mathrm{var}_t\left(r_{t\to t+1}\right)}} = -\frac{\gamma\left(\beta' y_t\right)}{\sqrt{\beta'\Sigma\beta}}.$$

If one sets the price deviation, $\beta' y_t$, equal to 2 standard deviations,

$$\text{(B-12)} \qquad -\beta' y_t = 2\sqrt{\mathrm{var}\left(\beta' y_t\right)} = 2\sqrt{\frac{\beta'\Sigma\beta}{\gamma\left(2-\gamma\right)}},$$

it follows that

$$\text{(B-13)} \qquad \mathrm{SR}_{t,t\to t+1} = -\frac{\gamma\left(\beta' y_t\right)}{\sqrt{\beta'\Sigma\beta}} = 2\frac{\gamma\sqrt{\dfrac{\beta'\Sigma\beta}{\gamma\left(2-\gamma\right)}}}{\sqrt{\beta'\Sigma\beta}} = 2\sqrt{\frac{\gamma}{\left(2-\gamma\right)}}.$$

In the VAR framework, the difference between the two cointegrated price processes follows an AR(1) process with the AR parameter given by $1-\gamma$. For γ close to 0, the difference between the two prices is almost nonstationary, and their convergence toward each other is slow. For γ close to 1, there is little persistence in the difference between the prices, and convergence happens quickly. The parameter γ thus plays a role here similar to that of q in the VMA setting. Appendix Table B1 shows the annualized Sharpe ratios across different values of γ for a 2-standard-deviation strategy with a 1-day holding period.

TABLE B1

Properties of Pairs Trading in the VAR(1) Model.

Table B1 presents annualized Sharpe ratios ($\mathrm{SR_{ann}}$) of pairs-trading strategies where a trade is initiated by a 2-standard-deviation price spread and held open for 1 period, and the returns are generated by a VAR(1) model. The parameter γ is determined by the parameters of the cointegrated VAR model and the difference between the two cointegrated price processes follows an AR(1) process with the AR parameter given by $1-\gamma$.

γ	0.008	0.01	0.02	0.05	0.10	0.20	0.50	0.99
$\mathrm{SR_{ann}}$	2.00	2.24	3.18	5.06	7.25	10.54	18.26	31.31

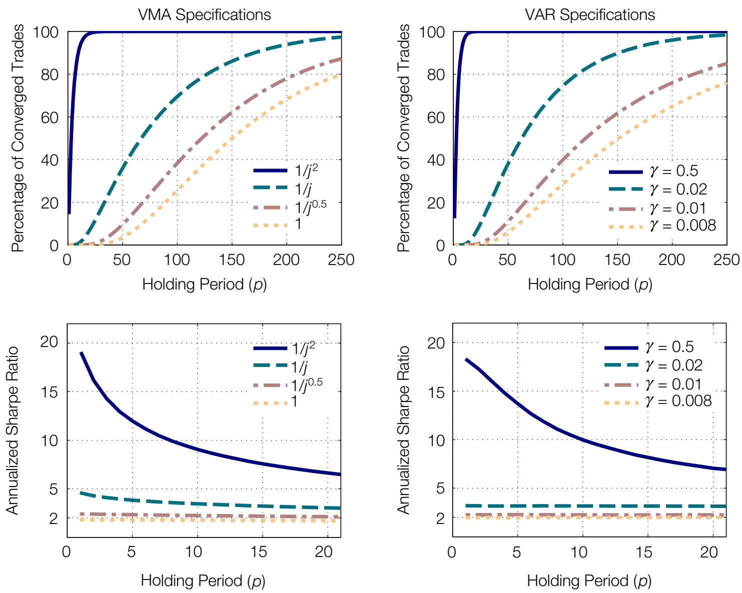
The results in the table highlight that the Sharpe ratio of the pairs-trading strategy quickly becomes very large as γ drifts away from 0.

In order to more directly compare the VMA and VAR formulations, Appendix Figure B1 shows the convergence properties and Sharpe ratios of a 2-standard-deviation pairs-trading strategy under VMA specifications (graphs on the left) and VAR specifications (graphs on the right). The results in the figure are based on simulations. For the VMA specifications, we pick $q=250$ and $h(j)=\{1,1/j^{0.5},1/j,1/j^{2}\}$; that is, we reproduce the same results as in the right column of Figure 1. For the VAR specifications, the $\gamma=\{0.008,0.01,0.02,0.5\}$ values are chosen such that the convergence properties of the trading strategy for each value is similar to the convergence properties of one of the VMA specifications. As shown in the top graphs of Figure B1, the percentage of converged trades shows a similar pattern for specific pairs of VMA and VAR specifications. The corresponding pairs-trading Sharpe ratios are shown in the two bottom graphs, and the main takeaway of Figure B1 is that VMA and VAR specifications with similar convergence patterns also have similar Sharpe ratios.

FIGURE B1

Properties of a 2-Standard-Deviation Strategy for Vector Moving Average (VMA) and Vector Autoregressive (VAR) Processes

The graphs in Figure B1 present trade characteristics of pairs-trading strategies in which a trade is initiated by a 2-standard-deviation price spread and held open for p periods (on the horizontal axis) and returns are generated by a VMA(250) model or VAR(1) model (for graphs on the left and right, respectively). The lines correspond to different $h(j)$ specifications in the VMA model and different values for γ in the VAR model (see the legends). The top pair of graphs presents the percentage of converged trades within p periods, and the bottom pair of graphs presents annualized Sharpe ratios. All characteristics are calculated using the simulation procedure described in Appendix A.



Appendix C. Proofs and Additional Theorems

Proof of Theorem 1. Under the presence of cointegration, the q-period pairs-trading returns are equal to

$$r_{t\to t+q} \equiv \sum_{k=1}^{q}\Delta\beta' y_{t+k} = -\beta'\sum_{k=1}^{q}\Delta\tilde{\epsilon}_{t+k} = -\beta'\left(\tilde{\epsilon}_{t+q}-\tilde{\epsilon}_{t}\right). \tag{C-1}$$

In the general VMA(∞) case, $\tilde{\epsilon}_t=\sum_{j=0}^{\infty}\tilde{c}_j\epsilon_{t-j}$, and

$$\begin{aligned}\tilde{\epsilon}_{t+q}-\tilde{\epsilon}_{t} &= \sum_{j=0}^{\infty}\tilde{c}_j\epsilon_{t+q-j}-\sum_{j=0}^{\infty}\tilde{c}_j\epsilon_{t-j}\\ &= \left(\tilde{c}_0\epsilon_{t+q}+\tilde{c}_1\epsilon_{t+q-1}+\cdots+\tilde{c}_{q-1}\epsilon_{t+1}+\tilde{c}_q\epsilon_t+\tilde{c}_{q+1}\epsilon_{t-1}+\cdots\right)\\ &\quad -\left(\tilde{c}_0\epsilon_t+\tilde{c}_1\epsilon_{t-1}+\tilde{c}_2\epsilon_{t-2}+\tilde{c}_3\epsilon_{t-3}+\cdots\right)\\ &= \tilde{c}_0\epsilon_{t+q}+\tilde{c}_1\epsilon_{t+q-1}+\cdots+\tilde{c}_{q-1}\epsilon_{t+1}\\ &\quad +\left(\tilde{c}_q-\tilde{c}_0\right)\epsilon_t+\left(\tilde{c}_{q+1}-\tilde{c}_1\right)\epsilon_{t-1}+\left(\tilde{c}_{q+2}-\tilde{c}_2\right)\epsilon_{t-2}+\cdots\\ &= \sum_{k=0}^{q-1}\tilde{c}_k\epsilon_{t+q-k}+\sum_{j=0}^{\infty}\left(\tilde{c}_{j+q}-\tilde{c}_j\right)\epsilon_{t-j}.\end{aligned} \tag{C-2}$$

Further, $\tilde{c}_j=\sum_{s=j+1}^{\infty}c_s$, which gives

$$\tilde{c}_{j+q}-\tilde{c}_j = \sum_{s=j+1+q}^{\infty}c_s-\sum_{s=j+1}^{\infty}c_s = -\left(c_{j+1}+c_{j+2}+\cdots+c_{j+q}\right) = -\sum_{k=1}^{q}c_{j+k} \tag{C-3}$$

and

$$\tilde{\epsilon}_{t+q}-\tilde{\epsilon}_{t} = \sum_{k=0}^{q-1}\tilde{c}_k\epsilon_{t+q-k}-\sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}. \tag{C-4}$$

That is,

$$r_{t\to t+q} = -\beta'\left(\tilde{\epsilon}_{t+q}-\tilde{\epsilon}_t\right) = -\beta'\left(\sum_{k=0}^{q-1}\tilde{c}_k\epsilon_{t+q-k}-\sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}\right). \tag{C-5}$$

The conditional variance at time t of $r_{t\to t+q}$ is thus equal to

$$\begin{aligned}\operatorname{var}_t\left(r_{t\to t+q}\right) &= \operatorname{var}_t\left(-\beta'\left(\sum_{k=0}^{q-1}\tilde{c}_k\epsilon_{t+q-k}\right)\right)\\ &= \beta'\sum_{k=0}^{q-1}\operatorname{var}\left(\tilde{c}_k\epsilon_{t+q-k}\right)\beta\\ &= \sum_{k=0}^{q-1}\left(\beta'\tilde{c}_k\Sigma\tilde{c}_k'\beta\right).\end{aligned} \tag{C-6}$$

Similarly, the conditional expectation at time t is given by

$$\begin{aligned}\mathrm{E}_t\left[r_{t\to t+q}\right] &= \mathrm{E}_t\left[-\beta'\left(\sum_{k=0}^{q-1}\tilde{c}_k\epsilon_{t+q-k}-\sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}\right)\right]\\ &= \beta'\left(\sum_{j=0}^{\infty}\left(\sum_{k=1}^{q}c_{j+k}\right)\epsilon_{t-j}\right).\end{aligned} \tag{C-7}$$

Note that (with initial conditions set to 0)

$$\text{(C-8)} \qquad \beta' y_t = -\beta'\tilde{\epsilon}_t = -\beta' \sum_{j=0}^{\infty} \tilde{c}_j \epsilon_{t-j} = -\beta' \sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty} c_s \epsilon_{t-j}.$$

Suppose now that we have a finite VMA(q) model. In that case, $c_j = 0$ for $j > q$. Thus,

$$\text{(C-9)} \qquad \begin{aligned} \beta' y_t &= -\beta' \sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty} c_s \epsilon_{t-j} = -\beta' \sum_{j=0}^{q-1}\sum_{s=j+1}^{q} c_s \epsilon_{t-j} \\ &= -\beta' \sum_{j=0}^{q-1}\sum_{k=1}^{q-j} c_{j+k} \epsilon_{t-j} = -\mathrm{E}_t\left[r_{t\to t+q}\right] \end{aligned}$$

since

$$\text{(C-10)} \qquad \mathrm{E}_t\left[r_{t\to t+q}\right] = \beta' \left(\sum_{j=0}^{\infty} \left(\sum_{k=1}^{q} c_{j+k} \right) \epsilon_{t-j} \right) = \beta' \left(\sum_{j=0}^{q-1}\sum_{k=1}^{q-j} c_{j+k} \epsilon_{t-j} \right).$$

The unconditional variance of $\beta' y_t$, in the VMA(q) case, is given by

$$\text{(C-11)} \quad \mathrm{var}\left(\beta' y_t\right) = \mathrm{var}\left(\beta'\tilde{\epsilon}_t\right) = \beta' \mathrm{var}\left(\sum_{j=0}^{q-1} \tilde{c}_j \epsilon_{t-j} \right) \beta = \sum_{j=0}^{q-1} \beta' \tilde{c}_j \Sigma \tilde{c}_j' \beta = \mathrm{var}_t\left(r_{t\to t+q}\right).$$

□

Theorem C1. Suppose $\Delta y_t = \mu + u_t = \mu + C(L)\epsilon_t$ is a $k \times 1$ dimensional returns process, with $\epsilon_t \equiv \mathrm{iid}(0,\Sigma)$, $C(L) = \sum_{j=0}^{\infty} c_j L^j$, and $C(1) \neq 0$. The corresponding price process is given by y_t, and the (1-period) returns on the pairs-trading strategy are defined as $r_{t\to t+1} \equiv \Delta\beta' y_{t+1}$. If y_t is cointegrated with cointegration vector β, the following results hold:

i) The time t conditional expected 1-period return is given by

$$\text{(C-12)} \qquad \mathrm{E}_t\left[r_{t\to t+1}\right] = -\beta' y_t - \beta' \sum_{j=0}^{\infty} \tilde{c}_{j+1} \epsilon_{t-j} = \beta' \sum_{j=0}^{\infty} c_{j+1} \epsilon_{t-j}.$$

ii) The time t conditional variance of the 1-period return is given by

$$\text{(C-13)} \qquad \mathrm{var}_t\left(r_{t\to t+1}\right) = \beta' \Sigma \beta.$$

iii) The time t conditional Sharpe ratio for the 1-period return is given by

$$\text{(C-14)} \quad \mathrm{SR}_{t\to t+1} = \frac{\mathrm{E}_t\left[r_{t\to t+1}\right]}{\sqrt{\mathrm{var}_t\left(r_{t\to t+1}\right)}} = \frac{-\beta' y_t - \beta' \sum_{j=0}^{\infty} \tilde{c}_{j+1} \epsilon_{t-j}}{\sqrt{\beta'\Sigma\beta}} = \frac{\beta' \sum_{j=0}^{\infty} c_{j+1} \epsilon_{t-j}}{\sqrt{\beta'\Sigma\beta}}.$$

Proof of Theorem C1. From the proof of Theorem 1,

$$\text{(C-15)} \qquad r_{t\to t+1} = -\beta'\left(\tilde{\epsilon}_{t+1} - \tilde{\epsilon}_t\right) = -\beta' \left(\tilde{c}_0 \epsilon_{t+1} - \sum_{j=0}^{\infty} c_{j+1} \epsilon_{t-j} \right),$$

where the last equality uses Appendix equation (C-4). The conditional variance at time t is then equal to

$$\text{(C-16)}\quad \operatorname{var}_t(r_{t\to t+1}) = \operatorname{var}_t\left(-\beta'\left(\tilde{c}_0\epsilon_{t+1} - \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}\right)\right) = \operatorname{var}_t(\beta'\tilde{c}_0\epsilon_{t+1}).$$

Further, $\tilde{c}_0 = \sum_{s=1}^{\infty} c_s = C(1) - I$ and $\beta' C(1) = 0$, resulting in

$$\text{(C-17)}\quad \operatorname{var}_t(r_{t\to t+1}) = \operatorname{var}_t(\beta'(C(1) - I)\epsilon_{t+1}) = \beta' \operatorname{var}(\epsilon_{t+1})\beta = \beta'\Sigma\beta.$$

The conditional expectation at time t is given by

$$\text{(C-18)}\quad \mathrm{E}_t[r_{t\to t+1}] = \mathrm{E}_t\left[-\beta'\left(\tilde{c}_0\epsilon_{t+1} - \sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}\right)\right] = \beta'\sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}.$$

Alternatively, by equation (8) with initial conditions equal to 0, $\beta' y_t = -\beta'\tilde{\epsilon}_t$ and

$$\begin{aligned}\text{(C-19)}\quad \mathrm{E}_t[r_{t\to t+1}] &= \mathrm{E}_t\left[-\beta'(\tilde{\epsilon}_{t+1} - \tilde{\epsilon}_t)\right] = \beta'\tilde{\epsilon}_t - \mathrm{E}_t\left[\beta'\tilde{\epsilon}_{t+1}\right]\\ &= -\beta' y_t - \mathrm{E}_t\left[\beta'\sum_{j=0}^{\infty}\tilde{c}_j\epsilon_{t-j+1}\right]\\ &= -\beta' y_t - \beta'\sum_{j=0}^{\infty}\tilde{c}_{j+1}\epsilon_{t-j}. \qquad \square\end{aligned}$$

Proof of Theorem 2. Note that

$$\begin{aligned}\text{(C-20)}\quad C(1) &= \sum_{j=0}^{\infty} c_j\\ &= I + \sum_{j=0}^{\infty} h(j)\begin{bmatrix} a & b\\ c & d\end{bmatrix} \equiv I + H_\infty\begin{bmatrix} a & b\\ c & d\end{bmatrix}\\ &= \begin{bmatrix} 1 + aH_\infty & bH_\infty\\ cH_\infty & 1 + dH_\infty\end{bmatrix}.\end{aligned}$$

Under cointegration with vector $\beta = (1, -1)'$, $C(1)$ must satisfy the restrictions in equation (11), which implies that the following parameter restrictions apply:

$$\text{(C-21)}\quad b = \frac{1}{H_\infty} + d, \quad \text{and} \quad c = \frac{1}{H_\infty} + a.$$

Thus, without loss of generality, we can write

$$\text{(C-22)}\quad c_j = h(j)\begin{bmatrix} a & b\\ c & d\end{bmatrix} = h(j)\begin{bmatrix} a & \frac{1}{H_\infty} + d\\ \frac{1}{H_\infty} + a & d\end{bmatrix}.$$

By Appendix Theorem C1,

$$\text{(C-23)}\quad \operatorname{var}_t(r_{t\to t+1}) = \beta'\Sigma\beta = \begin{bmatrix} 1 & -1\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}\begin{bmatrix} 1 - 1\end{bmatrix} = \sigma_{11} + \sigma_{22} - 2\sigma_{12}.$$

Using Appendix equation (C-18) in the proof of Appendix Theorem C1, the Sharpe ratio can then be written as

$$\text{(C-24)}\quad \mathrm{SR}_{t\to t+1} = \frac{\beta'\sum_{j=0}^{\infty} c_{j+1}\epsilon_{t-j}}{\sqrt{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}}.$$

The denominator in this Sharpe ratio is a constant. This reflects the fact that the time t variance of the pairs-trading returns is constant across time and does not depend on the current state of the price processes. Plugging in for the previous parameterization of c_j and $\beta=(1,-1)'$,

$$\text{(C-25)}\qquad \mathrm{SR}_{t\to t+1} = \frac{\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix} a & \frac{1}{H_\infty}+d \\ \frac{1}{H_\infty}+a & d\end{bmatrix}\sum_{j=0}^{\infty} h\,(j+1)\,\epsilon_{t-j}}{\sigma_{11}+\sigma_{22}-2\sigma_{12}} = \frac{\begin{bmatrix}-\frac{1}{H_\infty} & \frac{1}{H_\infty}\end{bmatrix}\sum_{j=1}^{\infty} h(j)\epsilon_{t-j+1}}{\sigma_{11}+\sigma_{22}-2\sigma_{12}}.$$

If ϵ_t is iid N$(0,\Sigma)$, it follows that $\sum_{j=1}^{\infty} h(j)\epsilon_{t-j+1}$ is also normally distributed with mean 0 and variance

$$\text{(C-26)}\qquad \mathrm{var}\left(\sum_{j=1}^{\infty} h(j)\epsilon_{t-j+1}\right) = \sum_{j=1}^{\infty} h(j)^2\Sigma \equiv \Sigma H_\infty^{(2)} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} H_\infty^{(2)},$$

with $H_\infty^{(2)} \equiv \sum_{j=1}^{\infty} h(j)^2$. The numerator in the Sharpe ratio in Appendix equation (C-25) is thus normally distributed with mean 0 and variance

$$\text{(C-27)}\qquad \mathrm{var}\left(\begin{bmatrix}-\frac{1}{H_\infty}\\ \frac{1}{H_\infty}\end{bmatrix}'\sum_{j=1}^{\infty} h(j)\epsilon_{t-j+1}\right) = \begin{bmatrix}-\frac{1}{H_\infty}\\ \frac{1}{H_\infty}\end{bmatrix}'\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} H_\infty^{(2)} \begin{bmatrix}-\frac{1}{H_\infty}\\ \frac{1}{H_\infty}\end{bmatrix} = \frac{H_\infty^{(2)}}{H_\infty^2}(\sigma_{11}+\sigma_{22}-2\sigma_{12}).$$

The Sharpe ratio is then distributed as

$$\text{(C-28)}\qquad \mathrm{SR}_{t\to t+1} \sim \frac{\mathrm{N}\left(0, \frac{H_\infty^{(2)}}{H_\infty^2}(\sigma_{11}+\sigma_{22}-2\sigma_{12})\right)}{\sqrt{\sigma_{11}+\sigma_{22}-2\sigma_{12}}} = \mathrm{N}\left(0, \frac{H_\infty^{(2)}}{H_\infty^2}\right). \qquad \square$$

Proof of Corollary 1. Using equation (8) and the definition of $\tilde{c}_j$ in equation (6),

$$\text{(C-29)}\qquad \mathrm{var}\,(\beta' y_t) = \mathrm{var}\,(\beta'\tilde{\epsilon}_t) = \left(\sum_{j=0}^{\infty} \beta'\tilde{c}_j\Sigma\tilde{c}_j'\beta\right) = \left(\sum_{j=0}^{\infty}\left(\sum_{s=j+1}^{\infty}\beta' c_s\right)\Sigma\left(\sum_{s=j+1}^{\infty} c_s'\beta\right)\right).$$

By the proof of Theorem 2, it must hold that $c_j = h(j)\begin{bmatrix} a & \frac{1}{H_\infty}+d \\ \frac{1}{H_\infty}+a & d \end{bmatrix}$. For $\beta=(1,-1)$, it follows that $\beta' c_s = h(s)\left[-\frac{1}{H_\infty} \quad \frac{1}{H_\infty}\right]$ and

$$
\begin{aligned}
\text{(C-30)} \quad \operatorname{var}(\beta' y_t) &= \begin{bmatrix} -\frac{1}{H_\infty} & \frac{1}{H_\infty} \end{bmatrix} \left(\sum_{j=0}^{\infty} \left[\left(\sum_{s=j+1}^{\infty} h(s) \right) \Sigma \left(\sum_{s=j+1}^{\infty} h(s) \right) \right] \right) \begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{H_\infty} & \frac{1}{H_\infty} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -\frac{1}{H_\infty} \\ \frac{1}{H_\infty} \end{bmatrix} \left(\sum_{j=0}^{\infty} \left[\left(\sum_{s=j+1}^{\infty} h(s) \right) \left(\sum_{s=j+1}^{\infty} h(s) \right) \right] \right) \\
&= \frac{1}{H_\infty^2} (\sigma_{11} + \sigma_{22} - 2\sigma_{12}) \sum_{j=0}^{\infty} \left[\left(\sum_{s=j+1}^{\infty} h(s) \right)^2 \right] \\
&= (\sigma_{11} + \sigma_{22} - 2\sigma_{12}) \sum_{j=0}^{\infty} \left[\frac{\left(\sum_{s=j+1}^{\infty} h(s) \right)^2}{\left(\sum_{s=1}^{\infty} h(s) \right)^2} \right],
\end{aligned}
$$

where the last equality follows from the definition $H_\infty \equiv \sum_{j=1}^{\infty} h(j)$. □

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